Sudan University of Science and Technology
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Toeplitz Projections and Corona with Localization and Toeplitz Algebra on the Bergman Space

مساقط تبوليتز وكورونا مع الموضعية وجبر تبوليتز على فضاء بيرجمان

A Thesis submitted in Fulfillment for the Degree of Ph.D in Mathematics

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Dedication

To my Family.

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Abstract

We study the operator commuting and essential commutant of analytic Toeplitz operators module the compact operators and Toeplitz operators in several complex variables and on the Bergman space of the until ball. We show the ordered groups and some exact sequences and the commutator ideal of the Toeplitz algebras of spherical isometries and on the Bergman spaces of the unit ball in the unitary space. We give the lower bounds in the matrix, new estimate for the vector-valued and matrix-valued H^p, Corona problems in the disk and polydisk and the codimension one conjecture. We also give the Toeplitz Corona theorems for the polydisk, the unit ball and the Douglas property for free functions. We discuss the characterizations of Toeplitz and Hankel operators with Toeplitz projections and Dixmier traces on the unit ball of the unitary space. We determine the locatization, compactness and Toeplitz algebra on the Bergman and Fock spaces.

الخلاصة

درسنا تبديلية المؤثر والمبدل الأساسي لمؤثرات تبوليتز التحليلية بمقياس مؤثرات التراص ومؤثرات تبوليتز في المتغيرات المركبة المتعددة وعلى فضاء بيرجمان لكرة الوحدة. أوضحنا الزمر المرتبة وبعض المتتاليات التامة والمثالي المبدل لجبريات تبوليتز لتساوي المسافات الكروية وعلى فضاءات بيرجمان لكرة الوحدة في الفضاء الواحدي. أعطينا الحديات السفلى في المصفوفة والتقدير الجديد لأجل مسائل كورونا قيمة–المتجه ومسائل كورونا H^p قيمة–المصفوفة في القرص المتعدد وتخمين البعد المصاحب الأوحد. ايضاً أعطينا مبرهنات كورونا تبوليتز لأجل القرص المتعدد وكرة الوحدة وخاصية دوقلاس لأجل الدوال الحرة. ناقشنا التشخيصات لمؤثرات تبوليتز وهانكل مع مساقط تبوليتز وآثار ديقسمير على كرة الوحدة للفضاء الواحدى. حددنا الموضعية والتراص وجبر تبوليتز على فضاءات بيرجمان وفوك.

Introduction

We show that an operator on H^2 of the disc commutes module the compacts with all analytic Toeplitz operators if and only if it is a compact perturbation of a Torplitz operator with symbol in $H^\infty + C$. Consequently, the essential commutant of the whole Toeplitz algebra is the algebra of Toeplitz operators with symbol in QC. Let S be the unit sphere in $\mathbb{C}^n$. We investigate the properties of Toeplitz operators on S, i.e., operators of the form $T_\phi f = P(\phi f)$ where $\phi \in L^\infty(S)$ and P denotes the projection of $L^2(S)$ onto $H^2(S)$. We determine how far the extensive one-variable theory remains valid in higher dimensions. We establish the spectral inclusion theorem, that the spectrum of T_ϕ contains the essential range of 4, and obtain a characterization of the Toeplitz operators among operators on $H^2(S)$ by an operator equation. A new extension is given principally with the objective of presenting a certain new class of C^*-algebras which have very interesting properties.

We give a lower boundfor the solution of the Matrix Corona Problem, which is pretty close to the best known upper bound $C \cdot \delta^{-n-1} \log \delta^{-2n}$ obtained recently by T. Trent. In particular, both estimates grow exponentially in n; the (only) previously known lower bound $C\delta^{-2}\ \log(\delta^{2n+1})$ grew logarithmically in n. Also, the lower boundis obtainedfor $(n + 1) \times n$ matrices, thus giving the negative answer to the so-called"co dimension one conjecture." Another important result is connecting left invertiblity in H^∞ and co-analytic orthogonal complements. We consider the matrix-valued H^p corona problem in the disk and polydisk. The result for the disk is rather well-known, and is usually obtained from the classical Carleson Corona Theorem by linear algebra. We provide a streamlined way of obtaining this result and allows one to get a better estimate on the norm of the solution. In particular, we were able to improve the estimate found in the recent work of Trent. Note that, the solution of the H^∞ matrix corona problem in the disk can be easily obtained from the H^2 corona problem either by factorization, or by the Commutant Lifting Theorem. The H^p corona problem in the polydisk was originally solved by Lin. The solution used Koszul complexes and was rather complicated because one had to consider higher order. $\bar{\partial}$-equations. Our proof is more transparent and it improves upon Lin's result in several ways. First, we deal with the more general matrix corona problem. Second, we were able to show that the norm of the solution is independent of the number of generators.

We show that if an operator A is a finite sum of finite products of Toeplitz operators on the Bergman space of the unit ball B_n, then A is compact if and only if its Berezin transform vanishes at the boundary. Let L_a^2 be the Bergman space of the unit disk and $\mathfrak{T}(L_a^2)$ be the Banach algebra generated by Toeplitz operators T_f, with $f \in L^\infty$. We compute the Dixmier trace of pseudo-Toeplitz operators on the Fock space. We find a formula for the Dixmier trace of the product of commutators of Toeplitz operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$.

A family $\{T_j\}_{j\in J}$ of commuting Hilbert space operators is said to be a spherical isometry if $\sum_{j\in J} T_j^* T_j = 1$ in the weak operator topology. We show that every commuting family $\mathcal{F}$ of spherical isometries has a commuting normal extension $\hat{\mathcal{F}}$. Moreover, if $\hat{\mathcal{F}}$ is minimal, then there exists a natural short exact sequence $0 \to C \to C^*(\mathcal{F}) \to C^*(\hat{\mathcal{F}}) \to 0$ with a completely isometric cross-section, where C is the commutator ideal in $C^*(\mathcal{F})$. We construct a Toeplitz projection for every regular A-isom-etry $T \in B(\mathcal{H})^n$ on a complex Hilbert space Hand use it to determine the essential commutant of the set of all analytic Toeplitz operators formed with respect to an essentially normal regular A-isometry. We show that the Toeplitz projection annihilates the compact operators if and only if Tpossesses no joint eigenvalues.

Trent and Wick establish a strong relation between the corona problem and the Toeplitz corona problem for a family of spaces over the ball and the polydisk. Their work is based on earlier work of Amar. Several of their lemmas are reinterpreted in the language of Hilbert modules, revealing some interesting facts and raising some questions about quasi-free Hilbert modules. A modest generalization of their result is obtained. The well known Douglas Lemma says that for operators A, B on Hilbert space that $AA^* - BB^* \gtrsim 0$ implies $B = AC$ for some contraction operator C. The result carries over directly to classical operator-valued Toeplitz operators by simply replacing operator by Toeplitz operator. Free functions generalize the notion of free polynomials and formal power series and trace back to the work of J. Taylor in the 1970s. They are of current interest, in part because of their connections with free probability and engineering systems theory.

Let L_a^2denote the Bergman space of the open unit ball B_n in $\mathbb{C}^n$, for $n > 1$. The Toeplitz algebra $\mathfrak{T}$ is the C^*-algebra generated by all Toeplitz operators T_f with $f \in L^\infty$. It was proved by D. Suárez that for $n = 1$, the closed bilateral commutator ideal generated by operators of the form $T_f\, T_g - T_g T_f$, where $f\, , g \in L^\infty$, coincides with $\mathfrak{T}$. We study compactness of operators on the Bergman space of the unit ball and the Bargmann-Fock space in $\mathbb{C}^n$ in terms of the behavior of their Berezin transforms. We show how a vanishing Berezin transform combined with certain (integral) growth conditions on an operator T are sufficient to imply that the operator is compact on the space in question. Let T_f denote the Toeplitz operator with symbol function f on the Bergman space $L_a^2(B;\, dv)$ of the unit ball in $\mathbb{C}^n$. It is a natural problem in the theory of Toeplitz operators to determine the norm closure of the set $\{T_f : f \in L^1(B;\, dv)\}$ in $B(L_a^2(B;\, dv))$.

The Contents

Chapter 1
Operators Commuting and Toeplitz Algebras

We show the image in the Calkin algebra of the Toeplitz operutors with symbol in $H^{\infty}+C$ is a maximal abelian algebra. These results lead to a characterization of automorphisms of the algebra of compact perturbations of the analytic Toeplitz operators. Particular attention is paid to the case where $\phi \in H^{\infty}(S)+C(S)$ where $C(S)$ denotes the algebra of continuous functions on S. Finally we describe a class of Toeplitz operators useful for providing counterexamples-in particular, Widom's theorem on the connectedness of the spectrum fails when $n > 1$.

Section (1.1): Toeplitz Operators Modulo the Compact Operators

We show that an operator on H^2 of the disc commutes module the compacts with all analytic Toeplitz operators if and only if it is a compact perturbation of a Toeplitz operator with symbol in $H^{\infty}+C$. Consequently, the essential commutant of the whole Toeplitz algebra is the algebra of Toeplitz operators with symbol in QC. The image in the Calkin algebra of the Toeplitz operutors with symbol in $H^{\infty}+C$ is a maximal abelian algebra. These results lead to a characterization of automorphisms of the algebra of compact perturbations of the analytic Toeplitz operators.

Johnson and Parrot [8] showed that if M is an abelian von Neumann algebra on a Hilbert space H, M' is its commutant, and $\ell\mathscr{b}(H)$ is the ideal of compact operators on H, then the essential commutant of $M$is $M'+\ell\mathscr{b}(H)$. Sarason [9] showed that a Toeplitz operator T_g on $H^2$of the unit circle commutes modulo the compacts with all analytic Toeplitz operators if and only if g is in $H^{\infty}+C$. Here C denotes the space of continuous functions on the unit circle. From this, Douglas [7] showed that the essential center of the Toeplitz algebra is the algebra of Toeplitz operators with symbol in $QC=H^{\infty}+C\cap\overline{H^{\infty}+C}$. Douglas [5] raised the natural question of which operators in $\ell(H^2)$ essentially commute with all Toeplitz operators.

We prove the following Theorem, which gives a complete answer to this question.

From this we get two immediate corollaries.

Corollary (1.1.1)[1]: The essential commutant of the Toeplitz algebra $\mathcal{F}(L^{\infty})$ is $\mathcal{F}(QC)$.

Corollary (1.1.2)[1]: The image $\mathcal{F}(H^{\infty}+C)$in the Calkin algebra is a maximal abelian algebra.

Corollary (1.1.1) gives the first concrete example of a maximal abelian algebra in the Calkin algebra which is not an image of a maximal abelian von Neumann algebra. (That the latter is an example is a consequence of [8].)

Let $\mathfrak{D}$ be the unit disc, and let $\partial\mathfrak{D}$ be its boundary. Let H^2 be the subspace of L^2. $L^2(\partial\mathfrak{D})$of those functions with all negative Fourier coefficients equal to zero. Let M_f denote the operator L^2 of multiplication by the L^{∞}function f. The Toeplitz operator T_f with symbol f is the compression of $M_f$to H^2. If $A$is a subset of L^{∞}, let $\mathcal{F}(A)$ be the norm closed algebra generated by $\{T_f : f \in A\}$. Let $\mathcal{L}(H^2)$ be the bounded operators on H^2, and let $\ell\mathscr{b}(H^2)$ be its ideal of compact operators. Let $\pi=\mathcal{L}(H^2)\rightarrow\mathcal{L}(H^2)/\ell\mathscr{b}(H^2)$ be the canonical homomorphism onto the Calkin algebra. If S is an operator, let $D(X)=XS-SX$ be the derivation on $\mathcal{L}(H^2)$ induced by S.

We show the following theorem, which is somewhat stronger than Theorem (1.1.7).

The proof will follow from a series of lemmas, but first we will outline the main ideas. If $T_zS - ST_z$ is not compact, we can take $h = z$. So for the remainder of the proof, We will assume that S essentially commutes with T_z.

It follows that S commutes modulo compacts with every Toeplitz operator with continuous symbol. We show that there is a subsequence A of the positive integers and a function f in L^∞, such that in the w^* topology on $\mathcal{L}(H^2)$,

$$T_f = \lim_{n\in A} T_{\bar{z}^n} S T_{z^n}.$$

We find a countable collection of disjoint closed intervals $\{\chi_n\}$ of the unit circle such that $\|T_{\chi_n}S - T_{\chi_n f}\| > \delta > 0$. Combining the two preceding results, we obtain functions P_n in H^∞ such that $\|D(T_{P_n})\| > \delta$ and so that the partial sums of the series $\sum P_n$are uniformly bounded. It follows that $T_{P_n}$and $D(T_{P_n})$converge strongly to zero, so we are able to extract a subsequence Γsuch that $h = \sum_{n\in\Gamma} P_n$is in $H^\infty$with the operators $D(T_{P_n})$almost mutually orthogonal. This will allow us to conclude that $D(T_h)$is not compact. By choosing the functions P_n so that their closed supports cluster at only one point, we ensure that h has only one discontinuity.

Let Σ be the net of inner functions ordered by divisibility. If ω is an inner function, let $\sigma_\omega = S - T_{\bar{\omega}} S T_\omega$. Let $\sigma_n = \sigma_{z^n}$, Consider the sequence $\{\sigma_n\}$ and the net $\{\sigma_\omega\}$. Both are norm bounded, and hence lie in a w^* compact set.

Consequently, they have w^* limit points.

Lemma (1.1.3)[1]: Let S be in $\mathcal{L}(H^2)$ with $T_zS - ST_z$ compact, and let S' be a w^*limit point of the sequence $\{\sigma_n\}$. Then,

(i) $T_gS - ST_g$is compact for all continuous functions g.

(ii)$S' = w^* \lim_{n\in A} \sigma_n$ for a subsequence A of $\mathbb{N}$.

(iii) $S - S' = T_f$ for some f in L^∞.

(iv) $S' - T_b S' T_b = \sigma_b$ fo r all continuous inner functions b.

(v)$w^* \lim_{n\in A} T_{\bar{z}^n} T_g S T_{z^n} = T_{gf}$ for all continuous functions g.

The operator S' is compact if and only if S is a compact perturbation of a Toeplitz operator. In this case, in the norm topology,

$$\lim_{n\to\infty} \sigma_n = S' \text{ and } \lim_{n\to\infty} T_{\bar{z}^n} T_g \bar{S} T_{z^n} = 0.$$

Let $S - T_{\bar{\omega}} S T_\omega$be compact for all inner functions $\omega \in \Sigma$, and let $S' = w^* \lim_{\omega\in A} \sigma_\omega$ for a subnet A of Σ. Then,

(iv') $S' - T_{\bar{\omega}} S T_\omega = \sigma_\omega$ for all inner functions ω.

(v') $w^* \lim_{n\in A} T_{\bar{\omega}} T_h S T_\omega = T_{hf}$ for all h in L^∞.

If $S = T_f + X$ is in the Toeplitz algebra, where X belongs to the commutator ideal of $\mathcal{F}(L^\infty)$, then

$$\lim_{\omega\in\Sigma} \sigma_\omega = X \text{ in norm.}$$

Proof. The set of operators $\{\sigma_n\}$ lies in the ball of radius $2\|S\|$, which is w^*compact and metrizable. Hence the w^* limit point S' can be taken to be the limit of a subsequence A of $\mathbb{N}$. Since $\pi(S)$ and $\pi(T_z)$ commute in the Calkin algebra, $\pi(S)$commutes with $\pi(T_{\bar{z}}) = \pi(T_z)^{-1}$. Hence S commutes modulo compacts with $T_{\bar{z}}$ and, therefore, also with $\mathcal{F}(C)$, the C^* algebra generated by T_z.

If K is compact, $\lim_{n\to\infty} T_{\bar{z}^n}KT_{z^n} = 0$ in norm since $T_{z_n} \to 0$ in the weak operator topology. Let b be a continuous inner function. Then,

$$S' - T_{\bar{b}}S'T_b = w^* \lim_{n\in A}(S - T_{\bar{z}^n}ST_{z^n}) - T_b(S - T_{\bar{z}^n}ST_{z^n})T_b$$
$$= w^* \lim_{n\in A}(S - T_{\bar{b}}S'T_b) - T_{\bar{z}^n}(S - T_{\bar{b}}ST_b)T_{z^n}.$$

But, $S - T_{\bar{b}}ST_b = T_{\bar{b}}(T_bS - ST_b)$ is compact, so the second term tends to zero in norm. So, $S' - T_{\bar{b}}S'T_b = S - T_{\bar{b}}S'T_b = \sigma_b$. In particular, for $b = z, S - S' = T_{\bar{z}}(S - S')T_z$. This is the functional equation determining the Toeplitz operators [2]. So there is a function f in L^∞ such that $S = T_f + S'$.

Now it follows that $T_f = w^* \lim_{n\in A} T_{\bar{z}^n}ST_{z^n}$. Let h_1, h_2 be functions in $H^\infty \cap C$. Then.

$$w^* \lim_{n\in A} T_{\bar{z}^n}T_{\bar{h}_1 h_2}ST_{z^n} = w^* \lim_{n\in A} T_{\bar{z}^n}T_{\bar{h}_1}\left[ST_{h_2} + D\left(T_{h_2}\right)\right]T_{z^n}$$
$$= w^* \lim_{n\in A}\left[T_{\bar{h}_1}T_{\bar{z}^n}ST_{z^n}T_{h_2} + T_{\bar{z}^n}T_{\bar{h}_1}D\left(T_{h_2}\right)T_{z^n}\right] = T_{\bar{h}_1}T_fT_{h_2}$$
$$= T_{\bar{h}_1 h_2 f}, \text{ since } D\left(T_{h_2}\right) \text{ is compact.}$$

Since $\left\{\bar{h}_1 h_2 : h_1, h_2 \in H^\infty \cap C\right\}$ is dense in C, it follows that

$$w^* \lim_{n\in A} T_{\bar{z}^n}T_gST_{z^n} = T_{gf}.$$

for all continuous functions g.

If S' is compact, then $S = T_f + S'$ and $\sigma_n = S' - T_{\bar{z}^n}S'T_{z^n}$. Hence by the above remarks, we have $\lim_{n\to\infty} \sigma_n = S'$ in norm.

The proofs of (iv’) and (v’) are identical to the calculations for (iv) and (v).

In (v’) we note that $\left\{\bar{h}_1 h_2 : h_1, h_2 \in H^\infty\right\}$is dense in L^∞. If $S = T_f + X$ with X in the commutator ideal of $\mathcal{L}(L^\infty)$, then $\sigma_\omega = X - T_{\bar{\omega}}XT_\omega$. By Douglas [5], for any $\epsilon > 0$, there is an inner function ω such that $\|XT_\omega\| < \epsilon$. Hence $\lim_{\omega\in\Sigma} \sigma_\omega = X$.

Let $\mathcal{pb}$ be the space of piecewise continuous functions in L^∞. Consider the function $F: \mathcal{pb} \to \mathcal{L}(H^\infty)$ defined by $F(\chi) = T_\chi S - T_{\chi f}$, where f is defined in terms of S as in Lemma (1.1.3). It is clear that if g_n are in $\mathcal{pb}$, uniformly bounded in the sup norm, such that $g_n \to g_0$ go pointwise, then $F(g_n) \to F(g_0)$ in the strong operator topology.
We have $F(1) = S - T_f = S'$, which is not compact unless S is a compact perturbation of a Toeplitz operator. Keep this function in mind to motivate the following lemma.

Lemma (1.1.4)[1]: Suppose $F: \mathcal{pb} \to \mathcal{L}(H^\infty)$ is a linear map such that:

(Pl) If g_n are in $\mathcal{pb}$ with $\|g_n\|_\infty \le M$ and $g_n \to g_0$ pointwise, then $w = \lim F(g_n) = F(g_0)$ in the weak operator topology;

(P2) $F(1)$ is not compact, say $\|\pi F(1)\| > \alpha > 0$;

(P3) if f, g are in $\mathcal{pb}$ and have disjoint closed supports, then $F(f)\, F(g)$ and $F(f)\, F(g)^*$are compact.

Then, there exist characteristic functions $\{\chi_n : n \ge 1\}$ in $\mathcal{pb}$ of disjoint closed support such that $\|F(\chi_n)\| > \alpha/4$. These sets can be chosen to cluster at only one point.

Consequently, there exist trigonometric polynomials $h_n$such that $\|h_n\| \le 2$, $\|h_n(1 - \chi_n)\| \le 2^{-n}$, and $\|F(h_n)\| > \alpha/4$.

Since χ_n is in $\mathcal{pb}$ it is the finite union of closed intervals. We can suppose, in fact, that χ_n is a closed interval if we change the constant to $\alpha/8$.

Let $\mathcal{s}$ be the collection of all characteristic functions χ of closed intervals such that $\|F(\chi)\| > \alpha/4$. Let

$$E = \bigcap_{n\geq 1}\left(\bigcup\{\chi \in \mathcal{s}: |\chi| \leq 1/n\}\right)^{cl}$$

Here $|\chi|$ is the linear measure of χ as a subset of the circle. We claim that E is nonempty. For if E is empty, then, since it is the intersection of nested closed sets, one of these sets is empty. That is, there exists an integer n such that $\chi| \leq 1/n$ implies that $\|F(\chi)\| \leq \alpha/4$.

Divide $\partial\mathcal{L}$ into an even number of closed intervals $\chi_i, i = 1, \dots, 2k$, which are disjoint except for their endpoints, so that $\partial\mathcal{L} = \cup\chi_i$. Each of the two collections $\{\chi_i: i\ is\ odd\}, \{\chi_i: i\ is\ even\}$ consist of mutually disjoint closed intervals. We have

$$\alpha < \|\pi F(1)\| \leq \frac{1}{2}\|\pi F(1) + F(1)^*\| + \frac{1}{2}\|\pi F(1) - F(1)^*\|.$$

Therefore, for $\epsilon = 1$ or -1, we have $\|\pi F(1) + \epsilon F(1)^*\| > \alpha$. Let $A = \pi(F(1) + \epsilon F(1)^*)$. Let $A_i = \pi(F(\chi_i) + \epsilon F(\chi_i)^*)$. By (P3), if χ_i, χ_j are disjoint, then $A_i, A_j = 0$. Hence $\{A_j: i\ odd\}$ (respectively, even) is a finite collection of normal, commuting, mutally annihilating operators. Therefore, by a simple estimate on the spectral radius, we get

$$\left\|\sum_{i\ odd} A_i\right\| = \max_i A_i \leq 2\max\|F(\chi_i)\| \leq \alpha/2.$$

The analogous inequality holds for i even. Now by the linearity of F, $\sum_{i-1}^{2k} A_i = A$.
So we have

$$\alpha < \|A\| = \left\|\sum A_i\right\| \leq \left\|\sum_{odd} A_i\right\| + \left\|\sum_{even} A_i\right\| \leq \alpha.$$

This contradiction shows that E is nonempty.

Let $x_0 \in E$. We proceed by induction. Suppose we have chosen disjoint characteristic functions $\chi_i \in \mathcal{s}, I, \dots, n$, with $\alpha_n = d(x_0, \cup\chi_i) >> 0$. Since $x_0 \in E$, we have

$$x_0 \in \left(\bigcup\{\chi \in \mathcal{s}: |\chi| \leq \alpha_n/3\}\right)^{cl}.$$

Hence, there exists μ in $\mathcal{s}, |\mu| \leq \alpha_n/3$ such that $d(x_0, \mu) < \alpha_n/3$. So, $d(\mu, \cup\chi_i) \geq \alpha_n/3$. If x_0 does not belong to μ, let $\chi_{n+1} = \mu$. Otherwise, set $\mu_t = \mu \cap \{x: d(x, x_0) \geq t\}$. Then $\mu_t \to \mu$ pointwise, so, by (PI), $F(\mu_t) \to F(\mu)$in the weak operator topology. Since the norm is lower semicontinuous in the weak operator topology, and $\|F(\mu)\| > \alpha/4$, we see that there exists a $t > 0$ such that $\|F(\mu_t)\| > \alpha/4$. Let $\chi_{n+1} = \mu_t$, Then $\alpha_{n+1} = d\left(x_0, \cup_{i=1}^{n+1}\chi_i\right) \geq t > 0$. It is clear that the sets $\{\chi_n\}$ cluster only at x_0 We remark that the sets μ_t may be the union of two intervals, say μ^+ and μ Then since $F(\mu_t) = F(\mu^+) + F(\mu^-)$, we can chose one of these with norm greater than $\alpha/8$.

Fix $\chi = \chi_n$, and choose continuous function g_i such that $0 \leq g_i \leq \chi$ and $g_i \to \chi$ pointwise. We argue as above to find an integer i such that $\|F(g_i)\| > \alpha/4$. Let k_j be the jth Fejer mean of g_i. Then $\|k_j - g_i\|_x$ tends to zero as j tends to infinity, so again by the above argument we choose an integer j such that $\|F(k_j)\| > \alpha/4$ and also $\|k_j - g_i\| < 2^{-n}$. For $\chi = \chi_n$, let $h_n = k_j$. We compute

$$\|h_n\| \leq \|g_i\| + 2^{-n} \leq 2,$$
$$\|h_n(1-\chi_n)\| \leq \|g_i(1-\chi_n)\| + 2^{-n} = 2^{-n}.$$

Hence we see that the functions h_n satisfy the requirements of the lemma.

Lemma (1.1.5)[1]: Let $\mathfrak{U}$ be a weakly closed subalgebra of $\mathcal{L}(H^2)$. Let S be an operator on H^2 and $D: \mathfrak{U} \to \mathcal{L}(H^2)$ be the derivation $D(a): aS - Sa$. Suppose there exist $\delta > 0, M > 0$, and elements a_n in $\mathfrak{U}$ such that $\|D(a_n)\| > \delta > 0$ and $\left\|\sum_{n\in J} a_n\right\| \leq M$ for all finite subsets J of $\mathbb{N}$. Then, there exists an element b in $\mathfrak{U}$ such that $D(b)$ is not compact.
Proof. We can assume that $D(a_n)$ is compact for all n. We claim that $a_n \to 0$ in the strong operator topology. If not, there is a unit vector h in H^2 such that $k_n = a_n h$ has $\|k_n\| \geq \delta$, for all n in an infinite set J. It is an elementary exercise, to show that we can find a finite subset J' of J so that $\left\|\sum_{n\in J} k_n\right\| > M$. This contradicts $\left\|\sum_{n\in J'} a_n\right\| \leq M$.

Let $\{z_n : n \geq 0\}$ be an orthonormal basis for H^2. Let R_n be the orthogonal projection onto the span of $\{z_0, \dots, z_n\}$. Now since $a_n \to 0$ strongly, we also have $D(a_n) \to 0$ strongly. Using this fact and the compactness of $D(a_n)$, we can inductively choose a subsequence Γ of $\mathbb{N}$ and corresponding projections $Q_k$which are finite dimensional and mutually orthogonal. These will be chosen so that for n_k in Γ, we have

(i) $\|Q_k\, D(a_{n_k}) Q_k\| > \delta,$

(ii) $\|D(a_{n_k})(I - Q_k)\| < 3^{-k}\delta,$

(iii) $\|R_k a_{n_k} R_k\| < 2^{-k}.$

If we have chosen $n_1, \dots, n_k$ and $Q_1, \dots, Q_k$, Let $Q = \sum Q_1$. Since Q is finite dimensional and $D(a_n) \to 0$ strongly, we can find an a_n such that $\|R_{k+1} a_n R_{k+1}\| < 2^{-k+1}$ and $\|D(a_n)Q\| < 3^{k-2}\delta$. Then since $D(a_n)$ is compact, we can choose Q_{k+1} orthogonal to Q, finite dimensional, so that (i) and (ii) are satisfied. Set $a_{n_{k+1}} = a_n$.

For convenience, we relabel so that $a_k = a_{n_k}$. Let $b_k = \sum_{n=1}^{k} a_n$, If $h_1, h_2$are in $R_n\, H$ and $k \geq l \geq n$, then

$$\left|\left((b_k - b_l)h_1, h_2\right)\right| \leq \sum_{l+1}^{k} |(a_i h_1, h_2)| \leq \|R_n a_i R_n\| \|h_1\| \|h_2\| \leq \sum_{l+1}^{k} 2^{-l} \|h_1\| \|h_2\|$$
$$< 2^{-l} \|h_1\| \|h_2\|.$$

Since $\bigcup_n R_n H^2$ is dense in H^2, and $\|b_k\| \leq M$ for every k, we conclude that the sequence $\{b_k\}$ converges weakly to an element b in $\mathfrak{U}$. Hence $D(b_k)$ converges weakly to $D(b)$. Therefore, since the $Q_n$are finite dimensional, we have

$$\lim_{k\to\infty} Q_n D(b_k) Q_n = Q_n D(b) Q_n \quad \text{in norm.}$$

Hence,

$$\|Q_n D(b) Q_n\| = \lim_{k\to\infty} \left\| Q_n \sum_{1}^{k} D(a_i) Q_n \right\| \geq \lim_{k\to\infty} \|Q_n D(a_n) Q_n\| - \sum_{i\neq n} \|Q_n D(a_i) Q_n\|.$$

But if $i \neq n$,

$$\|Q_n D(a_i) Q_n\| = \|Q_n [D(a_i)(I - Q_i)] Q_n\| < 3^{-i}\delta.$$

So,

$$\|Q_n D(b) Q_n\| \geq \lim_{k\to\infty} \delta - \sum 3^{-i}\delta < \delta/2.$$

This is true for all n, and the projections $\{Q_n\}$ are nonzero and mutually orthogonal. It follows that $D(b)$ is not compact.
Theorem (1.1.6)[1]: if an operator S in $\mathcal{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator, then there is a function $h \in H^\infty$ such that $T_h S - S T_h$ is not compact. The function h may taken to have at most one discontinuity.

Proof. We are now ready to complete the proof of our main theorem. Suppose S in $\mathcal{L}(H^2)$ is not the sum of a Toeplitz operator and a compact operator. We suppose that $T_zS - ST_z$ is compact, for otherwise we can take $h = z$. By Lemma (1.1.3), we choose a subsequence Λ of $\mathbb{N}$, and a function f in L^∞ such that

$$S' = S - T_f = w^* \lim_{n\in\Lambda} S - T_{\bar{z}^n} S T_{z^n}.$$

The operator S' is not compact, say $\|\pi(S')\| > \alpha > 0$.

We apply Lemma (1.1.4). to the map $F(g) = T_gS - T_{gf}$. Because of the remarks preceding Lemma (1.1.4), we need only show that F satisfies P3. Let f_1, f_2 be piecewise continuous with disjoint closed supports. Let g_1, g_2 be functions in C such that $g_i \equiv 1$ on the support of f_i, and $g_1g_2 = 0$. By Lemma (1.1.3), πT_{g_i} commutes with πS, and by [5], we have for every h in L^∞,

$$\pi(T_{g_i})\pi(T_h) = \pi(T_{g_ih}) = \pi(T_h)\pi(T_{g_i}).$$

It follows that

$$\pi(T_{g_i})\pi(F(f_i)) = \pi F(f_i) = \pi(F(f_i))\pi(T_{g_i}),$$

and the analogous relation holds for $\pi F(f_i)^*$. Thus we have

$$\pi F(f_i)\cdot \pi F(f_2) = \pi F(f_1)\cdot \pi T_{g_1}\cdot \pi T_{g_2}\cdot \pi F(f_2) = \pi F(f_i)\cdot \pi T_{g_1g_2}\cdot \pi F(f_2) = 0.$$

Hence $F(f_1)\,F(f_2)$ is compact, and similarly $F(f_1)\,F(f_2)^*$ is compact. So, from Lemma (1.1.4), there exist trigonometric polynomials h_n and characteristic functions χ_n of disjoint closed sets such that $\|h_n\| \le 2,\ \|h_2(1-\chi_n)\| \le 2^{-n}$, and $\|F(h_n)\| > \delta/4$.

We compute for h, h_n

$$T_{\bar{z}^k}T_hD(T_{z^k}) = T_{\bar{z}^k}T_h(T_{z^k}S - ST_{z^k}) = T_hS - T_{\bar{z}^k}T_hST_{z^k}.$$

From the derivation identity, we have

$$D(T_{z^kh}) = D(T_hT_{z^k}) = T_hD(T_{z^k}) + D(T_h)T_{z^k}.$$

We get

$$w^*\lim_{k\in\Lambda} T_{\bar{z}^k}D(T_{z^kh}) = w^*\lim_{n\in\Lambda}(T_hS - T_{\bar{z}^k}T_hST_{z^k}) + T_{\bar{z}^k}D(T_h)T_{z^k}$$
$$= T_hS - T_{hf} \text{ by lemma (1.1.3).}$$

By lower semicontinuity of the norm, and the inequality $\|T_{h_n}S - T_{k_nf}\| = \|F(h_n)\| > \delta/4$, we can choose an integer k in Λ such that $\|T_{\bar{z}^k}D(T_{z^kh_n})\| > \delta/4$ and $P_n = z^kh_n$ belongs to H^∞. We then have $\|D(T_{P_n})\| \ge \|T_{\bar{z}^k}D(T_{P_n})\| > \delta/4$ and

$$\|P_n\| = \|z^kh_n\| \le 2, and \|P_n(1-\chi_n)\| = \|z^kh_n(1-\chi_n)\| \le 2^{-n}.$$

Now, if J is a finite subset of $\mathbb{N}$, Iet $P_J = \sum_{n\in J} P_n$. Then

$$\|P_J\chi_m\|_\infty \le \sum_{n\in J}\|P_J\chi_m\| \le \|P_m\| + \sum_{n\ne m}\|P_n(1-\chi_n)\| \le 2 + \sum_n 2^{-n} = 3.$$

If $\mu = \sum_m \chi_m$, then $\|P_J\,\mu\| \le \sum\|P_n\,\mu\| \le \sum 2^{-n} = 1$. Hence $\|P_J\| \le 3$ for all finite subsets J.

Therefore we can apply Lemma (1.1.3), with $a_n = T_{P_n}$and $\mathfrak{U}\,\mathcal{F}(H^\infty)$. This gives us a function h in H^∞ such that $D(T_h)$ is not compact.

We have $T_h = w - \lim T_{b_k}$, where $b_k = \sum_{i=1}^{k} P_{n_i}$. The set $\{x: |P_n(x)| \ge 2^{-n}\}$ is contained in χ_n, and the sets $\{\chi_n\}$ cluster only at the point x_0. So the sequence $\{b_k\}$ converges uniformly on sets bounded away from x_0 to a function which is continuous except at x_0. Hence h is continuous except at x_0. This concludes the proof.

Theorem (1.1.7)[1]: An operator S on H^2 commutes module compacts with all analytic Toeplitz operators if and only if $S = T_g + k$, where g is in $H^\infty + C$ and K is compact.

Proof. We will now prove the results stated in the first section. If S commutes modulo the compacts with all analytic Toeplitz operators, it follows from Theorem (1.1.6). that S has the form $S = T_f + K$, where K is compact. Therefore, we see from Sarason [9] that f is in $H^\infty + C$.

If S is in the Toeplitz algebra, this result follows in a more elementary way. For then $S = T_f + X$, where X is in the commutator ideal of $\mathcal{F}(L^\infty)$. We have that $\sigma_\omega = T_{\overline{\omega}}(T_\omega S - ST_\omega)$ is compact for every inner function ω. So be Lemma (1.1.3),$X = \lim_{\omega \in \Sigma} \sigma_\omega$ in norm, and hence X is compact. We now apply Sarason's result as above.

To prove Corollary (1.1.1), we note that $\mathcal{F}(L^\infty)$ is generated by $\mathcal{F}(H^\infty)$ and $\mathcal{F}(\overline{H^\infty})$. Hence

$$\mathcal{F}(L^\infty)^{ec} = \mathcal{F}(H^\infty)^{ec} \cap \mathcal{F}(\overline{H^\infty})^{ec} = \mathcal{F}(H^\infty + C) \quad \cap \mathcal{F}(\overline{H^\infty} + C) \quad = \mathcal{F}(QC) \ .$$

Corollary (1.1.2). is immediate from Theorem (1.1.7). and the fact that $\pi\mathcal{F}(H^\infty + C)$ is abelian.

Corollary (1.1.8)[1]: If an operator S is not in $\mathcal{F}(H^\infty + C)$, then there is an inner function ω such that $ST_\omega - T_\omega S$ is not compact.

Proof. By Theorem (1.1.7), there is an analytic function h such that $ST_h - T_h S$ is not compact. A theorem of Marshall [11] shows that the linear span of the inner functions is norm dense in H^∞. The set of noncompact operators is open, so we can approximate h by a finite linear combination of inner functions $\sum \alpha, \omega,$ so that $\sum \alpha_i \left(ST_{\omega_i} - T_{\omega_i} S\right)$ is not compact.

If we take h to be continuous except at x_0, we can approximate h in norm by Blaschke products which are continuous except at x_0. So if S is not a Toeplitz operator plus a compact, we can find a Blaschke product b with zeros accumulating only at x_0 for which $ST_b - T_b S$ is not compact.

Definition (1.1.9)[1]: A derivation D of an algebra $\mathfrak{U}$ into itself is inner if $D(X) = XS - SX$ for some S in $\mathfrak{U}$.

Corollary (1.1.10)[1]: Every derivation D of $\mathcal{F}(H^\infty + C)$ into $\ell\mathscr{b}(H^2)$ is inner.

Proof. The operator D restricted to $\ell\mathscr{b}(H^2)$is a derivation of the compacts into themselves. It is well known [3] that every derivation on the compacts has the form $D(X) = XS - SX$ for some S in $\ell(H^2)$. If A is in $\mathcal{F}(H^\infty - C)$ and K is compact, then

$$D(AK) = AKS - SAK = (AKS - ASK) + (AS - SA)K = AD(K) + D(A)K$$
$$= AKS - ASK + \ D(A)K.$$

Therefore $D(A)K = (AS - SA)K$ f or every compact operator K. Hence $D(A) = AS - SA$. Since S commutes with all A in $\mathcal{F}(H^\infty + C)$ modulo the compacts, we have that S is in $\mathcal{F}(H^\infty + C)$ by 'Theorem (1.1.7).

An immediate consequence of this is the following.

Corollary (1.1.11)[1]: Every derivation of $\mathcal{F}(L^\infty)$ into the compact operators is of the form $D(X) = XS - SX$ with S in $\mathcal{F}(QC)$.

We consider the matrix-valued case. The operator algebra $\mathcal{F}(L^\infty) \otimes M_n$ acts $H^2 \otimes \mathbb{C}^n, M_n$ is the $n \times n$ matrix algebra over $\mathbb{C}$. See Douglas [6].

Corollary (1.1.12)[1]: An operator S in $\mathcal{L}(H^2 \otimes \mathbb{C}^n)$ commutes module the compacts with all operators in $\mathcal{F}(H^\infty) \otimes M_n$, if and only if $S = T_f \otimes I_n + K$, where f is in $H^\infty + C$ and K is compact.

Proof. Let δ_{ij} be the $n \times n$ matrix zero everywhere except for a 1 in the (i,j) entry. A simple computation of $D(T_h \otimes \delta_{ij})$ for h in H^∞shows that S has the desired form.

Corollary (1.1.13)[1]: An operator S in $(H^2 \otimes \mathbb{C}^n)$ is in the essential commutant of $\mathcal{F}(L^\infty) \otimes M_n$, if and only if $S = T_f \otimes I_n + K$, where f is in QC and K is compact.

Theorem (1.1.14)[1]: Let α be an automorphism of $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$. Then α is spatial, and has the factorization $\alpha = \alpha_1\alpha_2$, where

(i) $\alpha_1(T_h) = T_{h\circ b}$ or a Blaschke factor $b = \lambda(z-a)/(1-\bar{a}z)$, $|a| < 1$, and $|\lambda| = 1$. Moregenerally, $\alpha_1(A) = U_1 * AU_1$ for all A in $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$, where $U_1 * f = e(f \circ b)$for $f$in H^2 and e a unit vector in $H^2 \ominus bH^2$.

(ii) $\alpha_2(A) = U_2{}^* AU_2$, for a unitary U_2 in $\mathcal{F}(QC)$. So $U = T_g + K$, where g is unimodular in QC and K is compact.

Proof. Since $\ell\mathscr{b}(H^2)$ is the unique minimal closed two-sided ideal in $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$, we must have $\alpha\big(\ell\mathscr{b}(H^2) = \ell\mathscr{b}(H^2)\big)$. So, by a well known theorem [4], there is a unitary-operator W such that $\alpha(K) = W * KW$ for all K in $\ell\mathscr{b}(H^2)$. If A is in $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$,, then $(W * AW)(W * KW)W * AKW = \alpha(AK) = \alpha(A)\alpha(K) = \alpha(A)(A)W * KW$ for all compact operators K.

Hence $\alpha(A) = W * AW$.

There is a natural map from the automorphisms of $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$ onto the automorphisms of H^∞ by projecting into the Calkin algebra. The algebras $H^\infty, \mathcal{F}(H^\infty)$ and $\pi\mathcal{F}(H^\infty)$ are isometrically isomorphic as Banach algebras, so we can identify them here. The automorphisms of H^∞ are known [10] to be of the form $\alpha(h) = h \circ b$, where b is a conformal map of the disc onto itself.

The kernel of this map is the set of automorphisms α such that $\alpha(T_h) = T_h + k$, with k compact. Since α is spatial, it is induced by a unitary operator U. So U and U^* essentially commute with $\mathcal{F}(H^\infty)$. Hence U belongs to $\mathcal{F}(QC)$.

We now show that automorphisms of $\mathcal{F}(H^\infty)$ are spatial. Let b be a conformal automorphism of the disc. Let e be a unit vector in $H^2 \ominus bH^2$. Define an operator on H^2 by $U_1 * f = e(f \circ b)$ for f in H^2 Since e is in H^2, it follows that $e(f \circ b)$ is in H^2. A computation shows that $|db/dz| = |e|^2$. So $\|U_1 * f\|^2 = \int |e(f \circ b)|^2 \, dz = \int |f|^2 = b\left|\frac{db}{dz}\right| dz = \int |f|^2 dz = \|f\|_2^2$. The operator $U_1 *$ is clearly invertible, hence it is unitary on H^2. If h is in H^∞,

$$U_1 * T_h U_1 e(f \circ b) = U_1 * T_h f = e(fh \circ b) = (h \circ b)e(f \circ b) = T_{h\circ b} e(f \circ b).$$

Hence, $U_1 * T_h U_1 = T_{h\circ b}$.

Let α be an automorphism of $\mathcal{F}(H^\infty) + \ell\mathscr{b}(H^2)$. Then $\pi\alpha$ is an automorphism of H^∞. This lifts to a spatial automorphism of $\mathcal{F}(H^\infty)$, $\alpha_1(A) = U_1 * AU_1$. Let $\alpha_2 = \alpha_1^{-1}\alpha$.Since $\pi\alpha_2$is the identity, we have $\alpha_2(A) = U_2 * AU_2$, for some unitary U_2 in $\mathcal{F}(QC)$.

Note added in proof. Theorem (1.1.14) is also valid for the automorphisms of $\mathcal{F}(H^\infty) + C$.

Section (1.2): Several Complex Variables

For S be the unit sphere in $\boldsymbol{C}^n$. We investigate the properties of Toeplitz operators on S, i.e., operators of the form $T_\phi f = P(\phi f)$ where $\phi \in L^\infty(S)$ and P denotes the projection of $L^2(S)$ onto $H^2(S)$. We aim to determine how far the extensive one-variable theory remains valid in higher dimensions. We establish the spectral inclusion theorem, that the spectrum of T_ϕ contains the essential range of ϕ, and obtain a characterization of the Toeplitz operators among operators on $H^2(S)$ by an operator equation. Particular attention is paid to the case where $\phi \in H^\infty(S) + C(S)$ where $C(S)$ denotes the algebra of continuous functions on S. Finally we describe a class of Toeplitz operators useful for providing counterexamples-in particular, Widom's theorem on the connectedness of the spectrum fails when $n > 1$.

The Toeplitz operators on the classical Hardy space H^2 on the unit circle have been the object of much study. They are operators of the form $T_\phi f = P(\phi f)$ where $\phi \in L^\infty$ and P denotes the projection of L^2 onto H^2. An account of this theory, which is concerned mainly with describing the spectra of these operators, and with operator algebras generated by them, can be found in Douglas [5]. Recently there have been several important developments in function theory on the unit circle which are relevant to Toeplitz operators. For a survey see Sarason [22]; for more recent results, describing subalgebras of L^∞ containing H^∞, see Marshall [19] and Chang [15], [16].

We to study Toeplitz operators on the unit sphere in $\boldsymbol{C}^n$, in particular to determine how far the one-variable theory remains valid. In this context Toeplitz operators with continuous symbol have been studied by Coburn [l] and some related operators by Coifman, Rochberg, and Weiss [14].

We establish the spectral inclusion theorem, that the spectrum of T_ϕ contains the essential range of ϕ, answering a question of Coburn [13]. We also show that Toeplitz operators can be characterized among operators on $H^2$by an operator equation, we also study Toeplitz operators with symbol in $H^\infty + C$ and we consider a class of Toeplitz operators useful for providing counterexamples-in particular, Widom's theorem on the connectedness of the spectrum fails when $n > 1$. A large number of questions are left open.

We denote by B the open unit ball and by S the unit sphere in the n-dimensional complex Euclidean space, $\boldsymbol{C}^n$, of all ordered n-tuples $z = (z_1, \dots, z_n)$ of complex numbers z_i, with the usual inner product $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$, and the corresponding norm $\|z\| = \langle z, z\rangle^{1/2}$. We assume $n > 1$ unless otherwise stated. σ denotes surface area measure on S. We write L^∞ for $L^\infty(\sigma), L^2$for $L^2(\sigma)$. H^2 denotes the closure in L^2 of the polynomials in the coordinate functions $z_1, \dots, z_n$. We write C for $C(S)$, the algebra of all continuous functions on S.

If $f \in L^\infty, \mathcal{R}(f)$ denotes the essential range of f, i.e., the spectrum of f in L^∞.

If $f \in L^\infty$ then the Poisson integral of f gives a bounded harmonic function F on B, and F has radial boundary limits equal to f almost everywhere. This correspondence gives an isometry between L^∞ and the space of bounded harmonic functions on B with the supremum norm. Under this correspondence, the algebra of bounded analytic functions on B corresponds to a closed subalgebra H^∞ of L^∞.

If $\phi \in L^\infty$ we denote by T_ϕ the operator on the Hilbert space H^2 defined by $T_\phi f = P(\phi f)$ where P denotes the orthogonal projection of L^2 on H^2. T_ϕ is called the Toeplitz operator with symbol ϕ. If T is any operator on $H^2, \sigma(T)$denotes the spectrum of T and

$\sigma_e(T)$ denotes the essential spectrum of T, i.e., the set of $\boldsymbol{\lambda} \in \boldsymbol{C}$such that $T-\boldsymbol{\lambda}$ is not Fredholm [5]. $BL(H^2)$ denotes the algebra of all bounded linear operators on H^2.

We note two easily verified identities: $(T_\phi)^* = T$ - and $T_\phi T_\psi = T_{\phi\psi}$for $\phi \in L^\infty, \psi \in H^\infty$.

We answer a question of Coburn [13] by showing that for ϕ in L^∞ we have $\|T_\phi\| = \|\phi\|_\infty$. We obtain this result as a corollary of Theorem (1.2.1) which shows that the invertibility of a function, ϕ in L^∞ follows from the invertibility of T_ϕ in $BL(H^2)$. These results are known for $n = 1$ [5].

Before stating the result we make the remark that if ψ is a nonnegative measurable function on $\boldsymbol{C}$, if $z \in S$, and if $F(\zeta) = \psi\langle z, \zeta\rangle(\zeta \in \boldsymbol{C}^n)$, then $\int_S \; F d\sigma$ is independent of z. This follows since $d\sigma$ is a rotation-invariant measure.

Theorem (1.2.1)[12]: If ϕ is a function in L^∞ such that T_ϕ is invertible then ϕ is invertible in L^∞.

Proof. Suppose that T_ϕ is invertible. Then there exists an $\epsilon > 0$ such that $\|T_\phi f\|_2 \geq \epsilon\|f\|_2$ for all f in H^2. Then

$$\|\phi f\|_2 \geq \epsilon\|f\|_2 \text{for all } f \text{ in } H^2 \tag{1}$$

Now, by the remark above, $c_k = \int_S \; |1 + \langle z, \zeta\rangle|^{2k} \, d\sigma(\zeta)$ is independent of z in S. So, for any neighborhood U of z in S we have

$$c_k^{-1} \int_{S\backslash U} |1 + \langle z, \zeta\rangle|^{2k} \, d\sigma(\zeta) \to 0 \qquad \text{as } k \to \infty.$$

So, for any $g \in \; C$, it follows that

$$c_k^{-1} \int_S \; g(\zeta)|1 + \langle z, \zeta\rangle|^{2k} \, d\sigma(\zeta) \to g(z) \;\; as \; k \to \infty(z \in S). \tag{2}$$

Now, consider (1) for the particular case $f(z) = (1 + \langle z, \zeta\rangle)$where $\zeta \in S$. We have $\int_S \; |\phi z|^2 |1 + \langle z, \zeta\rangle|^{2k} d\sigma(z) \geq \epsilon^2 c_k$. So, for $g \in C, g \geq 0$ we conclude that

$$c_k^{-1} \int_S \; \int_S \; |\phi z|^2 g(\zeta)|1 + \langle z, \zeta\rangle|^{2k} d\sigma(z) d\sigma(\zeta) \geq \epsilon^2 \int_S \; g(\zeta) d\sigma(\zeta).$$

So

$$\int_S \; |\phi z|^2 g(z) d\sigma(z) \geq \epsilon^2 \int_S \; g(\zeta) d\sigma(\zeta)$$

Using Fubini's theorem and (2). Since this is true for all positive g in C it follows that $|\phi z| \geq \epsilon$ a.e., and so ϕ is invertible in L^∞.

As a corollary we obtain the spectral inclusion theorem:

Corollary (1.2.2)[12]: If ϕ is in L^∞, then $\mathcal{R}(\phi) = \; \sigma(M_\sigma) \subseteq \sigma(T_\phi)$.

(M_σ denotes the operator of multiplication by ϕ on L^∞).

Corollary (1.2.3)[12]: If ϕ is in L^∞, then $\|T_\phi\| = \|\phi\|_\infty$.

We now use similar methods to examine the mapping η from L^∞ to $BL(H^2)$ defined by $\eta(\phi) = T_\phi$. Let A be the closed algebra of $BL(H^2)$ generated by all the Toeplitz operators, T_ϕ, with $\phi \in L^\infty$. Let I be the closed ideal of A generated by operators of the form $T_{\phi\psi} - T_\phi T_\psi$where $\phi\psi$ are in L^∞.

Theorem (1.2.4)[12]: The mapping η_I induced from L^∞ to A/I by η is an isometric $*$-isomorphism.

Proof. The mapping η_I is clearly linear and contractive. The definition of I shows that it is multiplicative. To complete the proof we show that $\|T_\phi + I\| \geq \|\phi\|_\infty$for ϕ in L^∞ from which it follows that $\eta_I$is an isometry. To do this, it suffices to show that if p is a noncommuting polynomial in k variables and $\phi_1, \dots, \phi_k \in L^\infty$ then

$$\|P(T_{\phi_1}, \dots, T_{\phi_k})\| \geq \|P(\phi_1, \dots, \phi_k)\|_\infty$$

Assume $\|P(\phi_1, \dots, \phi_k)\|_\infty = 1$. Then there exists $\lambda_1, \dots, \lambda_k$ in the joint spectrum Of $(\phi_1, \dots, \phi_k)$ such that $|P(\lambda_1, \dots, \lambda_k)| = 1$. Let $\varepsilon > 0$. Then since $\sum_{j=1}^{k} |\phi_j - \lambda_i|^2$is not bounded below, the argument used in the proof of Theorem (1.2.1) shows that there exists $f \in h^2$ such that $\|f\|_2 = 1$ and $\int_S \sum_{j=1}^{k} |\phi_j - \lambda_i|^2 |f|^2 d\sigma < \epsilon^2$.

So for each j, $\|(\phi_j - \lambda_i) f\|_2 < \epsilon$ so $\|T_{\phi_i} f - \lambda_i f\|_2 < \epsilon$.

Then $\|P(T_{\phi_1}, \dots, T_{\phi_k})f - P(\lambda_1, \dots, \lambda_k)f\|$ can be made arbitrarily small by choosing ϵ small enough, whence $\|P(T_{\phi_1}, \dots, T_{\phi_k})\| \geq 1$.

When $n = 1$, I coincides with the commutator ideal of A[5]. It is not clear whether this holds in general, though I clearly contains the commutator ideal.

As a corollary to the theorem we obtain the following sharpening of the spectral inclusion theorem.

Corollary (1.2.5)[12]: If $\phi \in L^\infty$then $\mathcal{R}(\phi) = \sigma(M_\sigma) \subseteq \sigma_e(T_\phi)$.

Proof. By a result of Coburn [13], I contains the ideal $\mathcal{K}$ of all compact operators. Now $\sigma_e(T_\phi)$ is the spectrum of the coset of T_ϕ in $A/\mathcal{K}$ (since A is a C^*-algebra) and so contains the spectrum of T_ϕ in A/I, and hence by the theorem contains $\mathcal{R}(\phi)$.

We remark that for any $\phi \in L^\infty, \sigma(\phi)$is contained in the convex hull of $\mathcal{R}(\phi)$. In particular, if ϕ is real valued then $\sigma(T_\phi)$ is contained in the interval [ess inf ϕ, ess sup ϕ]. We conjecture that in fact $\sigma(T_\phi)$ is equal to this interval; this is true when $n = 1$[5].

Finally we identify the spectra when $\phi \in H^\infty$.

Proposition (1.2.6)[12]: If $\phi \in H^\infty$ then $\sigma_e(T_\phi)$ (and so also $\sigma(T_\phi)$) coincides with the spectrum of ϕ in H^∞.

Proof. We have to show that if T_ϕ is Fredholm then ϕ is invertible in H^∞. The corollary to Theorem (1.2.4) shows that ϕ is invertible in L^∞. Then $T_{\phi^{-1}}T_\phi = 1$ so the operator $H_{\phi^{-1}}: H^2 \to L^2$defined by $H_{\phi^{-1}}f = \phi^{-1}f - T_{\phi^{-1}}f$ has finite rank. Hence there is an integer m and complex numbers $a_1, \dots, a_m$,not all zero, with $H_{\phi^{-1}}f = 0$ where $f(z) = \sum_{k=0}^{m} a_k {z_1}^k$. So $\phi^{-1}f \in H^2$. That $\phi^{-1} \in H^\infty$ is a consequence of the following fact:

If $g \in L^\infty$ and $g(z)(z_1 - \lambda)$is in H^∞ for some $\lambda \in \boldsymbol{C}$ then $g \in H^\infty$.

We remark that Proposition (1.2.6) is false if $n = 1 - \sigma_e(T_z)$ is the unit circle.

It may be conjectured that the spectrum of ϕ in H^∞ coincides with $\mathcal{R}(\phi)$ —or equivalently, that if $\phi \in H^2$ and ϕ is invertible in L^∞ then $\phi^{-1} \in H^\infty$. (This is of course false when $n = 1$.) This would imply that any inner function is constant ($\phi \in H^2$ is inner if $|\phi| = 1$ a.e. on S). For a discussion of questions of this type see [20]. We merely remark here that the question of the existence of inner functions can be expressed in operator-theoretic form. For if $\phi \in H^2$then ϕ is inner if and only if T_ϕ is an isometry (if T_ϕ is an isometry then $T_\phi * T_\phi = 1$ so $T_{(1-|\phi|^2)} = 0$ so $(1 - |\phi|^2) = 0$). From this we can get a condition for ϕ to be inner in terms of its power series coefficients. Considering the case $n=2$ for simplicity, suppose $\phi(z_1,z_2)=\sum_{k,l=0}^{\infty}a_{kl}{z_1}^k{z_2}^l$. Using the fact that

$$\left\{\frac{1}{\pi 2^{1/2}}\left(\frac{(k+l+1)!}{k!\,l!}\right)^{1/2}\ {z_1}^k{z_2}^l\right\}$$

forms an orthonormal basis for H^2, we find that ϕ is inner if and only if, for all $m,n,r\in\mathbb{Z}$ with $r\geq 0$,

$$\sum_{k,l=0}^{\infty}a_{kl}\bar{a}_{k+m,l+n}\frac{(k+r)!\,l!}{(k+r+l+1)!}=\begin{cases}(r+1)^{-1}\text{if } m=n=0\\ 0 \qquad\qquad \text{otherwise,}\end{cases}$$

where a_{rs} is interpretated as 0 if $r<0$ or $s<0$. An equivalent condition is: for all $m,n\in\mathbb{Z}$ and almost all $\rho\in\ [0,I]$,

$$\sum_{k,l=0}^{\infty}a_{kl}\bar{a}_{k+m,l+n}\rho^k(1-\rho)^l=\begin{cases}1 \qquad \text{if } m=n=0\\ 0 \qquad \text{otherwise.}\end{cases}$$

So the inner function problem is equivalent to the problem of finding $\{a_{kl}\}$ to satisfy these conditions (apart from the trivial solution where a_{00} is the only nonzero term).

We end by showing that the Toeplitz operators are characterized among all operators on H^2 by the equation

$$\sum_{k=1}^{\infty}T_{\bar{z}_k}TT_{z_k}=T.$$

We first obtain a characterization of multiplication operators on L^2 in a more general setting.

Proposition (1.2.7)[12]: Let $M_1,\dots,M_n$ be commuting normal operators on a Hilbert space H with $\sum_{r=1}^{n}M_r*M_r=1$. Let $T\in BL(H)$ satisfy $\sum_{r=1}^{n}M_r*TM_r=T$. Then T commutes with $M_r,M_r*\ (r=1,\dots,n)$.

Proof. For any positive integer m and $f,g\in\ H$ we have

$$\langle Tf,g\rangle=\sum_{k_1+\cdots+k_n=m}\frac{m!}{k_1!\dots k_n!}\langle TM_1^{k_1}\dots M_n^{k_n}f,M_1^{k_1}\dots M_n^{k_n}g\rangle\,.$$

So

$$\begin{aligned}\langle(TM_1-M_1T)f,g\rangle\\ &=\sum_{k_1+\cdots+k_n=m}\frac{m!}{k_1!\dots k_n!}\langle TM_1^{k_1+1}\dots M_n^{k_n}f,M_1^{k_1}\dots M_n^{k_n}g\rangle\\ &-\sum_{k_1+\cdots+k_n=m}\frac{(m+1)!}{(k_1+1)!\,k_2!\dots k_n!}\langle TM_1^{k_1+1}\dots M_n^{k_n}f,\qquad M_1*M_1^{k_1+1}\dots M_n^{k_n}g\rangle\\ &-\sum_{k_2+\cdots+k_n=m+1}\frac{(m+1)!}{k_2!\dots k_n!}\langle TM_2^{k_2}\dots M_n^{k_n}f,\qquad M_1*M_2^{k_2}\dots M_n^{k_n}g\rangle\\ &=\sum_{k_2+\cdots+k_n=m}\frac{m!}{k_1!\dots k_n!}\langle TM_1^{k_1+1}\dots M_n^{k_n}f,\left(1-\frac{m+1}{k_1+1}M_1*M_1\right)M_1^{k_1}\dots M_n^{k_n}g\rangle\\ &-\sum_{k_2+\cdots+k_n=m+1}\frac{(m+1)!}{k_2!\dots k_n!}\langle TM_2^{k_2}\dots M_n^{k_n}f,M_1*M_2^{k_2}\dots M_n^{k_n}g\rangle.\end{aligned}$$

So

$$\|T\|^{-1}|\langle(TM_1-M_1T)f,g\rangle|$$

$$\leq \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1!\dots k_n!}\left\|M_1^{k_1+1}\dots M_n^{k_n}f\right\|\left\|\left(1-\frac{m+1}{k_1+1}M_1\right.\right.$$

$$\left.\left.*M_1\right)M_1^{k_1}\dots M_n^{n_n}g\right\|$$

$$+\sum_{k_2+\cdots+k_n=m+1}\frac{(m+1)!}{k_2!\dots k_n!}\left\|M_2^{k_2}\dots M_n^{k_n}f\right\|\left\|M_1*M_2^{k_2}\dots M_n^{k_n}g\right\|$$

$$\leq\left[\sum_{k_1+\cdots+k_n=m}\frac{(m+1)!}{(k_1+1)!\dots k_n!}\left\|M_1^{k_1+1}\dots M_n^{k_n}f\right\|^2\right.$$

$$\left.+\sum_{k_2+\cdots+k_n=m+1}\frac{(m+1)!}{k_2!\dots k_n!}\left\|M_2^{k_2}\dots M_n^{k_n}f\right\|^2\right]^{1/2}$$

$$\times\left[\sum_{k_1+\cdots+k_n=m}\frac{m!}{k_1!\dots k_n!}\frac{k_1+1}{(m+1)}\left\|\left(1-\frac{m+1}{k_1+1}M_1*M_1\right)M_1^{k_1}\dots M_n^{k_n}g\right\|^2\right.$$

$$\left.+\sum_{k_2+\cdots+k_n=m+1}\frac{(m+1)!}{k_2!\dots k_n!}\left\|M_1*M_2^{k_2}\dots M_n^{k_n}g\right\|^2\right]^{1/2}.$$

The expression in the first square brackets is equal to

$$\langle\sum_{p_1+\cdots+p_n=m+1}\frac{(m+1)!}{p_1!\dots p_n!}(M_1*M_1)^{p_1}\dots(M_n*M_n)^{p_n}f,f\rangle$$

$$=\langle(M_1*M_1+\cdots+M_n*M_n)^{m+1}f,f\rangle=\|f\|^2,$$

and the expression in the second square bracket can be similarly reduced to $[1/(m+1)](\|g\|^2-\|M_1g\|^2)$.

So $|\langle(TM_1-M_1T)f,g\rangle|\leq\left[1/(m+1)^{1/2}\right]\|f\|\|g\|\|T\|$. and since m is arbitrary, $TM_1-M_1T=0$. Similarly, T commutes with $M_2,\dots,M_n$ and with $M_1*,\dots,M_2*$.

We are indebted to S. C. Power for the following lemma:

Lemma (1.2.8)[12]: Let $T\in BL(H^2)$ satisfy $\sum_{k=1}^n T_{\bar z_k}TT_{z_k}=T$. Then there is $S\in BL(L^2)$ with$\|S\|=\|T\|$, $\sum_{k=1}^n M_{\bar z_k}SM_{z_k}=S$, and such that T is the compression of S to H^2.

Proof. Define $\psi: BL(L^2)\to\in BL(L^2)$ by $\psi(R)=\sum_{k=1}^n T_{\bar z_k}RT_{z_k}$. Then $\|\psi(R)\|\leq\|R\|$. Let $\tilde T$ be any operator on L^2 whose compression is T, with $\|\tilde T\|=\|T\|$. Let $S_m=(1/m)\sum_{r=1}^m\psi^r(\tilde T)$ and let S be a weak operator topology limit point of $\{S_m\}$. Then S has the required properties (easy to check).

Theorem (1.2.9)[12]: Let $T\in BL(H^2)$. Then T is T_ϕ for some $\phi\in L^2$ if and only if $\sum_{k=1}^n T_{\bar z_k}TT_{z_k}=T$.

Proof. That T_ϕ satisfies the equation is easy. Conversely if T satisfies the equation, and S is the operator given by Lemma (1.2.8), then Proposition (1.2.7) shows that S commutes with M_{z_k} and $M_{\bar z_k}(k=1,\dots,n)$. So $S=M_\phi$ for some $\phi\in L^\infty$and $T=T_\phi$.

Note that: (i) Proposition (1.2.7) and Lemma (1.2.8) together with the easily verified fact that $T_\psi = 0 \Longrightarrow \psi = 0$ provide an alternative proof of the fact that $\|T_\phi\| = \|\phi\|_\infty$ or every Toeplitz operator, T_ϕ.

(ii) One can define Toeplitz operators with σ replaced by any positive Bore1 measure on S; then Theorems (1.2.1), (1.2.4), (1.2.9) still hold, with essentially the same proofs.

We denote by $H^\infty + C$ the set of all functions $f \in L^\infty$ which can be expressed in the form $f = u + v$ where $u \in H^\infty$ and $v \in C$. Rudin [21] showed that $H^\infty + C$ is a closed subalgebra of L^∞. We consider Toeplitz operators with symbol belonging to this algebra; these are somewhat more tractable than general Toeplitz operators.

Coburn [13] studied Toeplitz operators with symbol in C. He showed that the commutator ideal of the closed algebra generated by $\{T_\phi : \phi \in C\}$ is precisely the algebra of all compact operators on H^∞, and coincides with the closed ideal generated by $\{T_{\phi\psi} - T_\phi T_\psi : \phi, \psi \in C\}$. His arguments are easily modified to show that if $\phi \in C$ then the operator $H_\phi : H^2 \to L^2$ given by $H_\phi f = \phi f - T_\phi f$ is compact. This implies that T_ϕ commutes modulo the compact operators with any Toeplitz operator, so the second sentence of this paragraph is true with C replaced by $H^\infty + C$. It then follows from Theorem (1.2.4) that the closed algebra generated by $\{T_\phi : \phi \in H^\infty + C\}$ is precisely the set of operators of the form $T_\phi + K$ where $\phi \in H^\infty + C$ and K is a compact operator on H^2. This extends [5].

We consider spectra. Let $\phi \in H^\infty + C$, and let $\sigma(\phi)$ denote its spectrum as an element of $H^\infty + C$. Then using the result of the above paragraph we have

$$\mathcal{R}(\phi) \subseteq \sigma_e(T_\phi) \subseteq \sigma(\phi).$$

It seems likely that $\sigma_e(T_\phi) = \sigma(\phi)$. We make the stronger conjecture that in fact $\mathcal{R}(\phi) = \sigma(\phi)$. This conjecture is equivalent to the assertion that if $\phi \in H^\infty + C$ and ϕ is invertible in L^∞ then $\phi^{-1} \in H^\infty + C$. It is easy to see that if $\phi \in H^\infty$ and ϕ is invertible in $H^\infty + C$ then $\phi^{-1} \in H^\infty$. So the above conjecture would imply the conjecture discussed after Proposition (1.2.6).

Another problem concerns connectedness of the spectra. For $n = 1, \sigma(T_\phi)$ and $\sigma_e(T_\phi)$ are always connected, for any $\phi \in L^\infty$[5]. For $n > 1$, if $\phi \in C$ then $\sigma_e(T_\phi)$ coincides with the range of ϕ and so is connected. If $\phi \in H^\infty$ then both $\sigma(T_\phi)$ and $\sigma_e(T_\phi)$ coincide with the spectrum of ϕ in H^∞, which is connected (because H^∞ contains no nontrivial idempotents). However, we show that there exists $\phi \in L^\infty$ with $\sigma_e(T_\phi)$ disconnected, and $\phi \in C$with $\sigma(T_\phi)$ disconnected. We conjecture that if $\phi \in H^\infty + C$then $\sigma_e(T_\phi)$is connected. If it were true that $\sigma_e(T_\phi) = \sigma(\phi)$ then this would follow since $\sigma(\phi)$ is connected (because $H^\infty + C$ has no nontrivial idempotents-for if $f \in H^\infty + C$ satisfies $f^2 = f$ then its Poisson integral F on B satisfies $F(z)^2 - F(z) \to 0$ as $d(z, S) \to 0$, so either $F(z) \to 0$ or $F(z) \to 1$ as $d(z, S) \to 0$, so either $f = 0$ or $f = 1$).

We conjecture also that the set $\sigma(T_\phi) \backslash \sigma_e(T_\phi)$ is discrete whenever $\phi \in H^\infty + C$ (it may even be true for all ϕ in L^∞). This would contrast with the case $n = 1$ where $\sigma(T_\phi)$ is obtained from $\sigma_e(T_\phi)$ by filling in holes. A class of functions for which this is true is described. We have not been able to prove it even for $\phi \in C$, but we can prove the following weaker assertion for $\phi \in H^\infty + C$.

First we need the solution of the $\bar{\partial}$-problem with bounds [18]:

Suppose $u_1, \dots, u_n$ are smooth bounded functions on B satisfying $\partial u_i/\partial \bar{z}_j = \partial u_j/\partial \bar{z}_i$ for $1 \le i < j \le n$. Then there exists f, smooth and uniformly continuous on B, with $u_i = \partial f/\partial \bar{z}_i$ on $B (1 \le i \le n)$.

Using this fact and the proof of [17] we obtain the following version of the solution of the second Cousin problem. If W is open, $A(W)$denotes the algebra of all continuous functions on $\bar{W}$ which are analytic on W.

Suppose $U_1, \dots, U_m$ is an open cover of $\bar{B}$. Suppose that for each i and j with $U_i \cap U_j \cap B \neq \emptyset$ we are given $\phi_{ij} \in A(U_i \cap U_j \cap B)$ such that $\phi_{ij}\phi_{ji} = 1$ and $\phi_{ij}\phi_{jk}\phi_{ki} = 1$ on $U_i \cap U_j \cap U_k \cap B$. Then there exists, for $i = 1, \dots, n$, an invertible element ϕ_i of $A(U_i \cap B)$ such that $\phi_{ij} = {\phi_i}^{-1}\phi_j$.

For the proof of Theorem (1.2.12) it will be convenient to use a somewhat different version of $H^\infty + C$, defined on B rather than S. We denote by H the set of all functions f on B which can be expressed as $f = u + \underline{v}$ where $u \in H^\infty$ and v is uniformly continuous. H can also be described as the set of functions f of the form $g + h$ where g is the Poisson integral of a function in $H^\infty + C$ and h is continuous on $\bar{B}$, vanishing on S. It is clear from this that H is a (uniformly) closed algebra of functions on B, and the mapping $g + h \to g$ is an algebra homo-morphism of H onto $H^\infty + C$.

Lemma (1.2.10)[12]: Let $x \in S$, let U be an open set containing x, and let f be a bounded analytic function on $U \cap B$. Then there exists $g \in H$ and an open set V containing x, with $g = f$ on $V \cap B$.

Proof. Let V be any open set with $x \in V$ and $\bar{V} \subseteq U$. Let h be smooth with support in U and $h = 1$ on V. Define u_i on U by $u_i = \partial(fh)/\partial \bar{z}_i = f(\partial h/\partial \bar{z}_i)$ $(i = 1, \dots, n)$ and extend to B by $u_i = 0$ on $B \backslash U$. Then u_i are smooth on B, and bounded, and $\partial u_i/\partial \bar{z}_j = \partial u_i/\partial \bar{z}_i$ so $u_i = \partial \phi/\partial \bar{z}_i$ where ϕ is smooth and uniformly continuous on B. Then $(\partial/\partial \bar{z}_i)(\phi - fh) = 0$ so $\phi - fh \in H^\infty$, so $fh = \phi - (\phi - fh) \in H$.

Since $fh = f$ on V, this proves the lemma.

Lemma (1.2.11)[12]: Let f be an invertible element of $H^\infty + C$. Then there exist invertible elements u of H^∞ and v of C with $f = uv$.

Proof. Denote the Poisson integral of f also by f. Then f is an invertible element of H. It is clearly enough to find u invertible in H^∞ and v, uniformly continuous on B with $f = uv$ on B.

Let $x \in S$. We claim that there is an open set V with $x \in V$ such that $f = pq$ on $V \cap B$ where p is bounded analytic on $V \cap B$ and q continuous on $\overline{V \cap B}$.

To prove this, write $f = \phi + \psi, f^{-1} = \phi_1 + \psi_1$ where $\phi, \phi_1 \in H^\infty$ and $\psi, \psi_1 \in C(\bar{B})$with $\psi(x) = 0, \psi_1(x) = 0$. Since $(\phi + \psi)(\phi_1 + \psi_1) = 1$ there is an open neighborhood U of x with $|\phi\phi_1 - 1| < \frac{1}{2}$ on $U \cap B$. So ϕ^{-1} is bounded on $U \cap B$, and by Lemma (1.2.10) we can find an open neighborhood V of x and $g \in H$ with $\|g\| < \frac{1}{2}$ and $g = \phi^{-1}\psi$ on $V \cap B$. Then $\log(1 + g) \in H$ so $1 + g = e^{\sigma + r}$on B where $\sigma \in H^\infty, \tau \in C(\bar{B})$. Then on $V \cap B$ we have

$$f = \phi(1 + \phi^{-1}) = \phi(1 + g) = (\phi e^\sigma)e^\tau,$$

which proves the claim.

Thus we can cover $\bar{B}$ by open sets $V_1, \dots, V_m$ such that on $V_k \cap B$ we have $f = p_k + q_k$ where $p_k \in H^\infty(V_k \cap B)$ and $q_k \in C(\overline{V \cap B})$. On $V_i \cap V_j \cap B$ define $\phi_{ij} = p_i {p_j}^{-1} =$

${q_i}^{-1}q_i \in A(V_i \cap V_j \cap B)$. Then the second Cousin problem yields ϕ_i invertible in $A(V_i \cap B)$ with $\phi_{ij} = \phi_i {\phi_j}^{-1}$.

Then on $V_i \cap B, f = u_i v_i$ where $u_i = {\phi_j}^{-1} p_i \in H^\infty(V_i \cap B)$, $v_i = \phi_i q_i \in C(\overline{V_i \cap B})onV_i \cap V_j \cap B$, $u_i {u_j}^{-1} = p_i {p_j}^{-1} {\phi_i}^{-1} \phi_i = \phi_{ij} {\phi_{ij}}^{-1} = 1$, so $u_i = u_j$, and similarly $v_i = v_j$. Thus we can define $u \in H^\infty$ and $v \in C(\bar{B})$ by $u = u_i$ and $v = v_i$ on $V_i \cap B$, and then $f = uv$ as required.

Theorem (1.2.12)[12]: Suppose $\phi \in H^\infty + C$and $\lambda \notin \sigma(\phi)$. Then the Fredholm operator $T_\phi - \lambda$ has index zero.

Proof. We may assume $\lambda = 0$; then f is invertible in $H^\infty + C$.
By Lemma (1.2.11), $f = uv$ with u invertible in H^∞ and v invertible in C. Then $T_f = T_v T_u$. T_u is invertible and T_v has index zero [13]. So T_f has index zero.

We conclude with another problem. As mentioned above, the methods of Coburn [13] show that if $\phi \in C$ then H_ϕ, defined by $H_\phi f = f - T_\phi f$, is a compact operator from H^2 to L^2. One may ask: for what $\phi \in L^\infty$ is H_ϕ compact? It is clear that the set A of such ϕ is a closed subalgebra of L^∞, containing $H^\infty + C$. In the case $= 1, A = H^\infty + C$. In general, the largest C^*-algebra contained in A is the algebra $QC = L^\infty \cap VMO$ (this follows from [14], which is applicable to the ball as the authors of [14] point out VMO is the space of functions of vanishing mean oscillation). We show that, when $n > 1, QC$ is not contained in $H^\infty + C$, so that $A \neq H^\infty + C$. It is natural to ask whether $A = H^\infty + QC$, especially since a recent theorem of Chang [16] asserts that, when $n = 1$, any closed subalgebra of L^∞ containing H^∞ is of the form $H^\infty +$ some C^*-algebra. However, we have not been able to show that $H^\infty + QC$ is closed, nor that its closure is an algebra, nor that the closed algebra it generates is A. A related problem is to describe the largest C^*-algebra contained in $H^\infty + C$. When $n = 1$ it is QC[22], but since $QC \nsubseteq H^\infty + C$ this is false for $n > 1$. Note that $H^\infty + C$ is not the smallest closed algebra containing H^∞ properly, in contrast to the case $n = 1$.

For the sake of simplicity we assume that $n = 1$. It is clear that the ideas and results can be extended to the general case of $n > 1$ with minor alternations in the proofs.

We make use of the following parametrization of the unit sphere:

$$z = (z_1, z_2) = \left(\rho^{1/2} e^{i\theta}, (1-\rho)^{1/2} e^{i\psi}\right)(0 \leq \rho \leq 1, 0 \leq \theta, \psi < 2\pi).$$

For this set of coordinates the measure becomes:

$$d\sigma = \frac{1}{2} d\rho\, d\theta\, d\psi.$$

The standard basis for H^2 is then given by:

$$e_{nm} = \frac{1}{\pi 2^{1/2}} \left[\frac{(n+m+1)!}{n!\,m!}\right]^{1/2} \rho^{n/2}(1-\rho)^{m/2} e^{in\theta} e^{im\psi} (n, m \geq 0).$$

We consider Toeplitz operators, T_ϕ, where the symbol ϕ depends on only the coordinate ρ, i.e.,

$$\phi(z_1, z_2) = \phi\left(\rho^{1/2}, e^{i\theta}, (1-\rho)^{1/2} e^{i\psi}\right) = g(\rho) \;\; where \; g \in L^\infty[0,1].$$

It is obvious that this type of symbol cannot occur when $n = 1$ and it is amongst Toeplitz operators of this class that we discover some differences between properties in the cases $n = 1$ and $n > 1$. We examine particularly the properties of the spectra of such Toeplitz operators and this sheds a little light on questions raised above.

First we note when a symbol of this form will be in $H^\infty + C$.

Proposition (1.2.13)[12]: Let ϕbe a symbol of the above form. Then $\phi \in H^\infty + C$ if and only if g continuous on $[0, 1]$.

Proof. One implication is clear. Conversely if $\phi \in H^\infty + C$, write $\phi = u + v$ with u in H^∞, v in C. Let $p(z_1, z_2) = q(|z_1|^2)$be orthogonal to $H^\infty (q \in L^\infty[0, 1])$, i.e., $\int_0^1 q(\rho)d\rho = 0$. Then $\int_S updσ = 0$, i.e., $\int_S (\phi - v)qd\rho\, d\theta\, d\psi = 0$. Let $\tilde{v}\rho = \int_0^{2\pi}\int_0^{2\pi} v(\rho, \theta, \psi)\, d\theta\, d\psi, 0 \le \rho \le 1$. Then

$$\int_0^1 [g(\rho) - \tilde{v}\rho]\, q(\rho)d\rho = 0.$$

This is true for all such q. Hence $g(\rho) - \tilde{v}\rho$ is constant. But $\tilde{v}$ is continuous and so g is continuous.

The proposition justifies the remark that $QC(= L^\infty \cap VMO)$ is not contained in $H^\infty + C$. For it is not hard to see that if g is in VMO on $[0, 1]$ then ϕ is in VMO. Then we have ϕ in QC, but not necessarily in $H^\infty + C$.

Proposition (1.2.14)[12]: Let ϕ be a symbol of the above form. Then T_ϕ is a diagonal operator.

Proof. We have

$$T_\phi e_{nm} = P\left\{\frac{1}{\pi 2^{1/2}}\left[\frac{(n+m+1)!}{n!\,m!}\right]^{1/2} \rho^{n/2}(1-\rho)^{m/2}e^{in\theta}e^{im\psi}\, g(\rho)\right\}$$
$$= \left[\frac{(n+m+1)!}{n!\,m!}\int_0^1 \rho^n(1-\rho)^m\, g(\rho)d\rho\right]e_{nm} \quad = \lambda_{nm}e_{nm}$$

where

$$\lambda_{nm} = \frac{(n+m+1)!}{n!\,m!}\int_0^1 \rho^n(1-\rho)^m\, g(\rho)d\rho.$$

So T_ϕ has a countable set of eigenvalues which are essentially convex combinations of values in the range of g. Note that if g is continuous then the set of limit points of $\{\lambda_{nm}: n, m \ge 0\}$ is easily seen to be the range of g, which shows that the essential spectrum of T_ϕ is connected. However, the following theorem shows that this need not always be the case.

Theorem (1.2.15)[12]: There exists

(i) a symbol $\phi \in C$ for which $\sigma(T_\phi)$ is disconnected, and

(ii) a symbol $\psi \in L^\infty$ for which $\sigma_e(T_\psi)$ is disconnected.

Proof. Let ϕ be a symbol of the above form where g is continuous and as a suitable "nonconvex" range, e.g., let $g(\rho) = e^{2\pi i\rho}$. As noted above it is clear that the set of limit points of $\{\lambda_{nm}: n, m \ge 0\}$ is just the unit circle. Since $\lambda_{00} = 0$ it follows that $\sigma(T_\phi)$ is disconnected, which gives (i).

For (ii), let $g(\rho) = \rho^i = e^{i\log\rho}$. We have $\psi(z_1, z_2) = g(|z_1|^2) = g(\rho)$. In this case,

$$\lambda_{nm} = \frac{(n+m+1)!}{n!\,m!}\int_0^1 \rho^{n+i}(1-\rho)^m\, d\rho = \frac{\Gamma(n+i+1)}{n!}\frac{(n+m+1)!}{\Gamma(n+m+i+2)},$$

where the gamma function is given by $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ $(z \in C, Re\, z > 0)$. Now, by Stirling's formula, for large n

$$\frac{\Gamma(n+i+1)}{n!} \sim \frac{(2\pi)^{1/2}(n+i)^{n+i+(1/2)}e^{-n-i}}{(2\pi)^{1/2}n^{n+(1/2)}e^{-n}} \qquad = n^i\left(1+\frac{i}{n}\right)^{n+i+(1/2)} e^{-i} \sim n^i$$
$$= e^{i\log n}.$$
So the set of limit points of $\Gamma(n+i+1)/n!$ is the unit circle. If n is fixed then as $m \to \infty$ the set of limit points of $(n+m+1)!/\Gamma(n+m+i+2)$ is the unit circle and so the set of limit points of $\{\lambda_{nm}: n,m \geq 0\}$ (for this fixed n) is the circle of radius $|\Gamma(n+i+1)|/n!$ and center 0. Also, as $n \to \infty$, $|\lambda_{nm}| \to 1$ uniformly in m.

Hence the set of limit points of $\{\lambda_{nm}: n,m \geq 0\}$ is the union of circles of radius $|\Gamma(n+i+1)|/n!$ $(n = 0,1,\dots)$, center 0, together with their limiting circle, the unit circle. This is $\sigma_e(T_\psi)$ which is thus disconnected
$$\left(\frac{|\Gamma(n+i+1)|}{n} = \prod_{k=n+1}^{\infty} \frac{k}{(k^2+1)^{1/2}}\right).$$
Note that:(i) As noted in above, $\phi(T_\phi)$ and $\sigma_e(T_\phi)$ are both connected for any symbol ϕ in L^∞ when $n = 1$.

(ii) Although we do not have $\sigma_e(T_\phi)$ connected for all ϕ it is not hard to show that $\sigma_e(T_\phi)$ is always contained in the convex hull of a connected subset of itself for any symbol ϕ of the above form. Moreover, if g is continuous at 0 and 1, $\sigma_e(T_\phi)$ is connected.
The remark made after the proof of Theorem (1.2.12) is sufficient to give that the operator $H_\phi f = \phi f - T_\phi f$ from H^2 to L^2 is compact when ϕ is in VMO. In the case of these special symbols it is easy to see directly that H_ϕ is compact if and only if $g \in VMO$ on $[0,1]$.

A possible generalization of the described class of symbols are those of the form:
$$\phi(z_1,z_2) = \phi\left(\rho^{1/2}, e^{i\theta}, (1-\rho)^{1/2}e^{i\psi}\right) = g\left(\rho, e^{i(\theta-\psi)}\right)$$
where g is in L^∞on $[0,1] \times \mathbb{T}$ (where $\mathbb{T}$ is the unit circle). Then T_ϕ need no longer be diagonal but has the weaker property that T_ϕ is reduced by the $(m+1)$-dimensional subspace of H^2 spanned by ${z_1}^m, {z_1}^{m-1}z_2, \dots, {z_2}^m$for each $m \geq 0$. From this it follows that $\sigma(T_\phi)\backslash\sigma_e(T_\phi)$is discrete. This class of symbols has the advantage of being invariant under rotation (which the first type does not). An alternative characterization is
$$\phi(z_1,z_2) = \phi\left(e^{i\theta}z_1, e^{i\theta}z_2\right)\text{for all } \theta,$$
and this easily generalizes to C^n for $n > 2$.

Section (1.3): Ordered Groups

The classical of Toeplitz operators and their associated $\boldsymbol{C}^*$-algebras is an elegant and important area of modern mathematics. For this reason many (e.g. Douglas, Singer, Howe, Devintaz) have sought to extend this theory to a more general setting. A new extension is given principally with the objective of presenting a certain new class of C^*-algebras which have very interesting properties- to each partially ordered group G we associate a C^*-algebra $\mathcal{F}(G)$, its Toeplitz algebra $\mathcal{F}(G)$ has a certain universal property which may be useful in general C^*-algebra theory, particularly K-theory. Often $\mathcal{F}(G)$ contains a simple C^*-algebra as a closed ideal and this is analyzable in terms of G.

The classical Toeplitz algebra $\mathcal{F}(Z)$ associated to the ordered group of integers Z appears in two guises in the literature:

(i) $\mathcal{F}(Z)$ is the $\boldsymbol{C}^*$-algebra generated by the Toeplitz operators with continuous symbol.
(ii) $\mathcal{F}(Z)$ is the $\boldsymbol{C}^*$-algebra (unique to $*$isomorphism) generated by a non-unitary isometry (Coburn [26]).

It is principally in its second guise that we are interested in generalization $\mathcal{F}(Z)$. However in analyzing the Toeplitz algebra of a general partially ordered group we need to extend many results of the classical Toeplitz operator theory.

We construct $\mathcal{F}(G)$ and show that the functor $G \to \mathcal{F}(G)$ is "continuous", a result very important for the sequel, we also specialize to the case where the ordering on G is total. In this situation $\mathcal{F}(G)$ is representable as a hereditary C^*- subalgebra of a certain crossed product C^*-algebra got by an action α of G on an abelian C^*-algebra $\mathcal{J}_1(G)$. By showing that $\mathcal{J}_1(G)$ is G-prime and calculating Connes' spectrum $\Gamma(\alpha)$ in the case that G is finitely generated, and then extending by "continuity" to the general case, we show that $\mathcal{F}(G)$ is prime, and is isomorphic to any C^*-algebra generated by a non-unitary semigroup of isometries over G. We then identify a certain simple ideal in $\mathcal{F}(G)$ ("usually" not type I) which plays a somewhat similar role in the general theory to that played by the ideal of compact operators in the classical case. A number of the results here generalize some results of Douglas [27]. However Douglas confines himself to ordered subgroups of $\boldsymbol{R}$, and our methods are completely different from his. Then, we return to general partially ordered groups. Here we extend many results of the classical theory, for example we show that generalized analytic Toeplitz operators have connected spectra. For each partially ordered group G we exhibit a very explicit and useful irreducible representation of $\mathcal{F}(G)$ (faithful if G is totally ordered). Finally we show that a number of our results are best possible, and that for a totally ordered group $G, \mathcal{F}(G)$ has simple commutator ideal iff G is (isomorphic to) an ordered subgroup of $\boldsymbol{R}$ (sufficiency is due to Doglas [27]).

For G be a discrete abelian group and $\leq$ a partial ordering on G. For $S \subseteq G$ we write S^+ for the set of all x in S such that $0 \leq x$. We call $(G, \leq)$ a partially ordered group if $G = G^+ - G^+$ and $x \leq y$ implies that $x + z \leq y + 2$ $(x, y \in G)$. If I is a subgroup of G such that $I = I^+ - I^+$ we call I a partially ordered subgroup of G. Then of course $(I, \leq)$ is itself a partially ordered group. If $\leq$ is a total ordering on G (i.e. for all $x, y \in G$ we have $x \leq y$ or $y \leq x$) then we refer to $(G, \leq)$ simply as an ordered group. In this case any subgroup I of G is a (partially) ordered subgroup, since $G = G^+ \cup (-G^+)$ implies that $I = I^+ \cup (-I^+) = I^+ - I^+$. Thus $(I, \leq)$ is an ordered group.

Suppose now only that G is a discrete abelian group and M is a subset of G such that

$$0 \in M, M + M \subseteq M, M \cap (-M) = 0 \ \ and \ G = M - M.$$

In this case we call M a cone in G and we define $x \leq_M y$ to mean that $y - x \in M$ for x, y in G. It is easily checked that $(G, \leq_M)$ is a partially ordered group with $G^+ = M$. We shall often use (G, M) to refer to $(G, \leq_M)$. If $(G, \leq)$ is a partially ordered group then G^+ is a cone and $(G, \leq_G) = (G, \leq)$. Moreover $(G, \leq)$ is an ordered group iff $G = G^+ \cup (-G^+)$.

If H is a closed subspace of a Hilbert space K we shall let S_H denote the compression to H of the bounded linear operator S on K.

If G is a partially ordered group and B a unital C^*-algebra, a semigroup of isometries in B (relative to G) is a map $\beta: G^+ \to B$ such that each $\beta(x)$ is an isometry, i.e. $\beta(x)^*\beta(x) = 1$ for all $x \in G^+$, and $\beta(x+y) = \beta(x)\beta(y)$ for all $x, y \in G^+$. (This implies that $\beta(0) = 1$.) if $B = B(H)$, C^*-algebra of all bounded linear operators on the Hilbert space H, we call a pair (K, π)a unitary lifting of β if K is a Hilbert space containing H as a closed subspace, $\pi: G \to B(K)$ is a homo-morphism of G into the group of unitaries of $B(K)$, H is invariant for all $\pi(x), x \in G^+$, and $\beta(x) = (\pi(x))_H$ for such x. The following is the basic result concerning unitary liftings and will be used a number of times below.

Theorem (1.3.1)[23]: (Ito). Let G be a partially ordered group and $\beta: G^+ \to B(H)$ a semigroup of isometries on the Hilbert space H. Then β admits a unitary lifting (K, π).

For a proof, see Suciu [37]. (Note that there it is only shown that $\beta(x)$ is a compression of $\pi(x)$- i.e. invariance of H is not stated. However an elementary 2×2 operator matrix argument shows that if an ismoetry is the compression of a unitary it is in fact a restriction of the unitary. Hence we can conclude that H above is invariant for all $\pi(x), x \in G^+$.

Here is another result that we will be using a number of times. It is well known and follows easily from the von Neumann inequality, see e.g. Suciu [37].

(For G a discrete abelian group, $G^\wedge$ will denote the dual group of G considered as a compact abelian group. If T denotes the circle group and $x \in G$, we let ε_x or $\varepsilon(x)$ denote the evaluation homomorphism from $G^\wedge$ to T defined by $\varepsilon_x(y) = y(x), y \in G^\wedge$.)

Lemma (1.3.2)[23]: if $\pi: G \to B$ is a homomorphism from an abelian group into the group of unitaries of a unital C^*- algebra B then there is a unique $*$- homomorphism $\beta: CG^\wedge \to B$ such that $\beta(\varepsilon_x) = \pi(x), x \in G$.

We need one more preliminary concept for the construction we are about to undertake: let A be a C^*-algebra with identity element 1 and let A_1, A_2 be C^*-subalgebras such that $1 \in A_1 \cap A_2$. We say that A is a free product of A_1 and A_2 and we write $A = A_1 * A_2$ if for every unital C^*-algebra B each pair of unital $*$- homomorphisms $\beta_j: A_j \to B$ $(j = 1, 2)$ have a unique extension to a $*$- homomorphism $\beta: A \to B$. Any two unital C^*-algebras admit a free product (Brown [24]). By the way, since any two free products of A_1 and A_2 are canonically $*$-isomorphic, we can talk about the free product. $A_1 \cup A_2$ generates $A_1 * A_2$.

We can now define the Toeplitz algebra and show it has a certain universal property.

Let G be a partially ordered group. Let p denote the projection $(1, 0)$ in C^2, and let I be the closed ideal in $C^2 * C\ (G^\wedge)$ generated by all $\varepsilon_x p - p\varepsilon_x p, \in G^+$. If π denotes the quotient map from $C^2 * C\ (G^\wedge)$ to $C^2 * C\ (G^\wedge)/I$ then we set $\mathcal{F}(G) = \pi(p)(Im(\pi))\pi(p)$. Thus $\mathcal{F}(G)$ is a unital C^*-algebra ($\pi(p)$ is the identity element). We call $\mathcal{F}(G)$ the Toeplitz algebra of G. We define he canonical semigroup of isometries $V = V^G: G^+ \to \mathcal{F}(G)$ by $V_x = \pi(\varepsilon_x)\pi(p)$. It is readily verified that V is in fact a semigroup of isoometries generating $\mathcal{F}(G)$.

Theorem (1.3.3)[23]: Let G be a partially ordered group and $\beta: G^+ \to B$ a semigroup of ismoetries in a unital C^*-algebra B. Then there is a unique $*$- homomorphism $\beta^*: \mathcal{F}(G) \to B$ such that $\beta^* V = \beta$.

Proof. We may assume without loss of generality that B is a C^*-subalgebra of $B(H)$ for some Hilbert space H such that $1 = id_H \in B$. By the Theorem (1.3.1) the semigroup of isometries $\beta: G^+ \to B(H)$ has a unitary lifting (K, π). There exists y_2 a unique unital $*$-homomorphism from $C = C(G^\wedge)$ to $B(K)$ such that $y_2(\varepsilon_x) = \pi(x), x \in G$, by Lemma (1.3.2). let $Q \in B(K)$ be the projection onto H and $y_1: C^2 \to B(K)$ the unital $*$-homomorphism such that $y_1(p) = Q$ where $p = (1,0) \in C^2$. Let y denote the unique $*$-homomorphism extending y_1 and y_2 to $C^2 * C \to B(K)$. Since $y(p) = Q$ and $y(\varepsilon_x) = \pi(x)$ we have $y(\varepsilon_x p - p\varepsilon_x p) = \pi(x)Q - Q\pi(x)Q = 0$ for all $x \in G^+$ (H is invariant for $\pi(x), x \in G^+$. Thus $y(1) = 0$ where I is the closed ideal in $C^2 * C$ generated by all $\varepsilon_x p - p\varepsilon_x p, x \in G^+$. It now follows that the map $\beta^*: \mathcal{F}(G) \to B(H)$ defined by $\beta^*(a + I) = y(a)_H$for $a + I \in \mathcal{F}(G)$, is a well- defined $*$- homomorphism. Also $\beta^*(V_x) = \beta^*(\varepsilon_x p + 1) = y(\varepsilon_x p)_H = \pi(x)_H = \beta(x)$ for all $x \in G^+$, so $Im(\beta^*) \subseteq B$. Thus $\beta^*: \mathcal{F}(G) \to B$ is a $*$- homomorphism such that $\beta^* V = \beta$.

Uniqueness of β^* is trivial, since the $V_x(x \in G^+)$ generate $\mathcal{F}(G)$.

If A is C^*-algebra we let $K(A)$ denote its commutator ideal, i.e. the closed ideal generated by all $ab - ba$ $(a, b \in A)$. $K(A)$ is the smallest closed ideal Jin A such that A/J is abelian. If $\beta: A \to B$ is $*$- homo-morphism of C^*-algebras then $\beta\big(K(A)\big) \subseteq K(B)$, with equality if β is surjective.

If $\varphi: G_1 \to G_2$is a homomorphism of partially ordered groups we say that φ is positive if $\varphi(G_1^+) \subseteq G_2^+$ (equivalently $x \le y$ in $G_1 \Longrightarrow \varphi(x) \le \varphi(y)$ in G_2). We let $\tilde{\varphi}: G_1^+ \to G_2^+$ be the restriction of φ. There is a unique $*$- homomorphism $\varphi^*: \mathcal{F}(G_1) \to \mathcal{F}(G_2)$ such that $\varphi^* V^{G_1} = V^{G_2}\tilde{\varphi}$ (simply take $\varphi^* = (V^{G_2}\tilde{\varphi})^*$ - it is clear that $V^{G_2}\tilde{\varphi}: G_1^+ \to \mathcal{F}(G_2)$ is a semigroup of ismoetries). We thus get covariant functors $G \to \mathcal{F}(G)$ and $G \to K\big(\mathcal{F}(G)\big)$.

For G is a partially ordered group we define $q_x = q_x^G = 1 - V_x V_x^* (x \in G^+)$. Since V_x is an isometry, q_x is a projection (in $K\big(\mathcal{F}(G)\big)$). If $x \le y$ then $q_x \le q_y$ (because: $y = x + z$ for some $z \in G^+ \Longrightarrow V_y = V_x V_z \Longrightarrow V_y V_y^* = V_x V_z V_z^* V_x^*$ since $V_z V_z^* \le 1$. Hence $q_y = 1 - V_y V_y^* \ge 1 - V_x V_x^* = q_x$). We are now going to show that $q_x \le q_y \Longrightarrow x \le y$. To do this we consider a certain representation of $\mathcal{F}(G)$ that will be very important later. Before doing this, a useful remark: if $\varphi: G_1 \to G_2$ is a positive homomorphism of partially ordered groups then $\varphi^*(q_x) = q_{\varphi(x)}$ for all $x \in G_1^+$.

Preposition (1.3.4)[23]: If G is a partial ordered group and $x, y \in G^+$ then $x \le y$ if and only if $q_x \le q_y$.

Proof. Let $H^2 = H^2(G)$ be the closed linear span in $L^2(G^\wedge)$ for all $\varepsilon_x (x \in G^+)$. Define $U_x \in B(H^2)$ by $U_x f = \varepsilon_x f, x \in G^+$. The map $U: G^+ \to B(H^2), x \longmapsto U_x$, is easily seen to be a semigroup of isometries: thus U^* maps $\mathcal{F}(G)$ to $B(H^2)$ and $U^* V = U$. Now the projections $Q_x = 1 - U_x U_x^*$ on H^2 satisfy the relations $Q_x(\varepsilon_y) = 0$ for $x \le y$ and $Q_x(\varepsilon_y) = \varepsilon_y$for $x \nleq$

$y\ (x, y \in G^+)$. Hence $x \leq y$ iff $Q_x \leq Q_y$. But $U^* q_x = Q_x$, so $x \leq y \Rightarrow q_x \leq q_y \Rightarrow Q_x \leq Q_y \Rightarrow x \leq y$.

Note in passing that it is now easy to see that $V: G^+ \to \mathcal{F}(G)$ is injective $\left(V_x = V_y \Rightarrow q_x = q_y \Rightarrow x = y\right)$.

Given any partially group G the map $\varepsilon: G^+ \to C(G^\wedge), x \longmapsto \varepsilon_x$, is a semigroup of isometries (actually of course the ε_x are unitaries) so we have the induced map $\varepsilon^*: \mathcal{F}(G) \to C(G^\wedge)$. Since $\varepsilon_x (x, \in G^+)$ generate $C(G^\wedge)$ (by the Stone- Weierstrass theorem), ε^* is surjective.

Theorem (1.3.5)[23]: If G is a partially ordered group then $\ker(\varepsilon^*) = K\big(\mathcal{F}(G)\big)$ and the map $\mathcal{F}(G)/K\big(\mathcal{F}(G)\big) \to C(G^\wedge), a + K\big(\mathcal{F}(G)\big) \longmapsto \varepsilon^*(a)$, is $*$-isomorphism.

Proof. All we have to show is that $K\big(\mathcal{F}(G)\big) = \ker(\varepsilon^*)$.

Since $\mathcal{F}(G)/\ker(\varepsilon^*)$is abelian. $\mathrm{Ker}(\varepsilon^*) \supseteq K\big(\mathcal{F}(G)\big)$. The map $\pi: G \to \frac{\mathcal{F}(G)}{K(\mathcal{F}(G))}, x - y \longmapsto V_y * V_x + K\big(\mathcal{F}(G)\big)$, (for $x, y \in G^+$) is a well- defined homomorphism into the unitaries of $\mathcal{F}(G)/K\big(\mathcal{F}(G)\big)$, so be Lemma (1.3.2) there exists a unique *-homomorphism $\gamma: C(G^\wedge) \to \frac{\mathcal{F}(G)}{K(\mathcal{F}(G))}$ such that $\gamma\big(\varepsilon_{x-y}\big) = \pi(x-y) = V_y * V_x + K\big(\mathcal{F}(G)\big)(x, y \in G^+)$. If δ is the$*$-homomorphism $\frac{\mathcal{F}(G)}{K(\mathcal{F}(G))} \to C(G^\wedge), a + K\big(\mathcal{F}(G)\big) \longmapsto \varepsilon^*(a)$ then, $\gamma\delta\left(V_x + K\big(\mathcal{F}(G)\big)\right) = \gamma\varepsilon^*(V_x) = \gamma(\varepsilon_x) = \pi(x) = V_x + K\big(\mathcal{F}(G)\big)(x \in G^+) \Rightarrow \gamma\delta = \mathrm{id}$(since $V_x + K\big(\mathcal{F}(G)\big)$generate $\mathcal{F}(G)/K\big(\mathcal{F}(G)\big)$). Hence $a \in \ker(\varepsilon^*) \Rightarrow \delta\left(a + K\big(\mathcal{F}(G)\big)\right) = \varepsilon^*(a) = 0 \Rightarrow a + K\big(\mathcal{F}(G)\big) = \gamma\delta\left(a + K\big(\mathcal{F}(G)\big)\right) = 0 \Rightarrow a \in K\big(\mathcal{F}(G)\big)$. Thus $K\big(\mathcal{F}(G)\big) = \ker(\varepsilon^*)$.

Our next result, showing that the functor $G \to \mathcal{F}(G)$ is “continuous”, i.e. preserves direct limits, is interesting in its own right and plays a crucial role in the development of the theory. First we need to make a somewhat technical remark about direct limits in the category of partially ordered groups. Let $\left(\varphi_{ij}: G_i \to G_j\right)_{i \leq j}$ be a direct system of partially ordered groups (indexed by I) with direct limit G and natural maps $\left(\varphi^i: G_i \to G\right)_i$. if $x \in G_i, y \in G_j$, and$\varphi^i(x) = \varphi^i(y)$ then there exists $k \in I, k \geq i, j$ and $\varphi_{ik}(x) = \varphi_{jk}(y)$. This detail is needed in the proof that follows. The way to see it is to construct one example of a direct limit G and natural maps $\left(\varphi^i\right)_i$ satisfying it. Then it follows from an elementary diagram chase that every direct limit and system of natural maps for $\left(\varphi_{ij}: G_i \to G_j\right)_{i \leq j}$ has the above property. Here is a sketch of how to construct the required limit: Define an equivalence relation $\sim$ on the disjoint union of the sets $G_i (i \in I)$ by setting $(i, x) \sim (j, y)$ if there exists $k \in I, k \geq i, j$, such that $\varphi_{ik}(x) = \varphi_{jk}(y)$. Let $[i, x]$ denote the equivalence class of (i, x) and G be the set of all equivalence classes. Define the map $\varphi^i: G_i \to G$ by $\varphi^i(x) = [i, x]$. There is a unique operation on G making G an abelian group and all the maps φ^i homomorphisms. Define G^+ to be the union of all the sets $\varphi^i(G_i^+)(i \in I)$. Then $G^+$is a cone in G and it is easily checked that the partially ordered group (G, G^+) is a direct limit of the direct system $\left(\varphi_{ij}: G_i \to G_j\right)_{i \leq j}$ in the category of all partially ordered group (with maps

φ^i as natural maps). Clearly $x \in G_i, y \in G_j,$ and $\varphi^i(x) = \varphi^j(y)$ imply that there exists $k \in I, k \geq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$.

Theorem (1.3.6)[23]: Let the partially ordered group G be the direct limit of the direct system of partially ordered groups $(\varphi_{ij}: G_i \to G_j)_{i \leq j}$. Then $\mathcal{F}(G)$ is the direct limit of the direct system of C^*-algebras $\left((\varphi_{ij})^*: \mathcal{F}(G_i) \to \mathcal{F}(G_j)\right)_{i \leq j}$.

More explicitly if $(\varphi^i: G_i \to G)_i$ are natural maps for G then $(\varphi^i)^*: \mathcal{F}(G_i) \to \mathcal{F}(G_j)_i$ are natural maps for $\mathcal{F}(G)$.)

Proof. Let I be the index set for the direct system $(\varphi_{ij}: G_i \to G_j)_{i \leq j}$. Given B a C^*-algebra and $\beta^i: \mathcal{F}(G_i) \to B$ $*$-homomorphism such that $\beta^j(\varphi_{ij})^* = \beta^i$ for all $i \leq j$ in I, we must show that there is a unique $*$-homomorphism $\beta: \mathcal{F}(G) \to B$ such that $\beta(\varphi^i)^* = \beta^i (i \in I)$. By replacing B by the C^*-subalgebra $C = \left(\bigcup\{\beta^i(\mathcal{F}(G_i)): i \in I\}\right)^-$ if necessary, we may assume without loss of generality that B is unital and that all maps β^i are unital. Let $V^i: G_i^+ \to \mathcal{F}(G_i)$ and $V: G^+ \to \mathcal{F}(G)$ be the canonical maps and recall that $(\varphi_{ij})^\sim$ and $(\varphi^i)^\sim$ are the restrictions to the positive cones of φ_{ij} and φ^i respectively. We have

(i) $\quad (\varphi_{ij})^* V^i = V^j (\varphi_{ij})^\sim (i \leq j)$
(ii) $\quad (\varphi^i)^* V^i = V^j (\varphi^i)^\sim (i = I)$
(iii) $\quad \beta^i (\varphi_{ij})^* = \beta^i (i \leq j)$.

Let $x \in G_i, y \in G_j$and suppose that $\varphi^i(x) = \varphi^j(y)$. Then there exists $k \in I, k \geq i, j$ such that $\varphi_{ik}(x) = \varphi_{jk}(y)$ implies $\beta^i V_x^i = \beta^k(\varphi_{ik})^* V_x^i$ (by (iii)) $= \beta^k V^k(\varphi_{ik}(x))$ (by (i)) $= \beta^k V^k\left(\varphi_{jk}(y)\right) = \beta^k(\varphi_{jk})^* V_y^i$ (by (i) again) $= \beta^j V_y^i$ (by (iii) again). Thus $\varphi^i(x) = \varphi^j(y)$ implies $\beta^i V_x^i = \beta^j V_y^i$.

We define the map $\beta: G^+ \to B$ by setting $\beta\left(\varphi^i(x)\right) = \beta^i V_x^i$. This is well defined from the above calculation and from the fact that G^+ is the union of all the sets $\varphi^i(G_i^+)(i \in I)$. Now β is clearly a semigroup of isometries, so $\beta^*: \mathcal{F}(G) \to B$ is a $*$-homomorphism and $\beta^* V = \beta$. Also, by (ii), $\beta^*(\varphi^i)^* V^i = \beta^* V(\varphi^i)^\sim = \beta(\varphi^i)^\sim = \beta^i V^i$ (by the definition of β), which implies $\beta^*(\varphi^i)^* V^i = \beta^i V^i$, so $\beta^*(\varphi^i)^* = \beta^i$ (since $V_x^i (x \in G_i^+)$ generate $\mathcal{F}(G_i)$).

Finally suppose that $y: \mathcal{F}(G) \to B$ were another $*$-homorphism such that $y(\varphi^i)^* = \beta^i (i \in I)$. We must show that $y = \beta^*$. But $y^V(\varphi^i)^\sim = y(\varphi^i)^* V^i$ (by (ii)) $= \beta^i V^i = \beta(\varphi^i)^\sim$, so $yV = \beta$ (since G^+ is the union of the sets $(\varphi^i)(G_i^+)(i \in I)$). Thus $yV = \beta = \beta^* V \Longrightarrow y = \beta^*$ (since $V_x(x \in G^+)$ generate $\mathcal{F}(G)$.

To prove deeper results about te Toeplitz algebra $\mathcal{F}(G)$ one needs to specialize G. Specifically one needs to assume that G is totally ordered. That this assumption is not merely convenient, but actually necessary to get our results, is shown below. Our technique is to represent $\mathcal{F}(G)$ as a hereditary C^*- algebra, and to use some powerful results of the theory of crossed products C^*- algebra, and to use some powerful results of the theory of crossed products to analyse $\mathcal{F}(G)$.

Let G be an ordered group. We define an action α of G on $\ell^\infty(G)$ by setting $(\alpha_x f)(y) = f(y-x)$for all $f \in \ell^\infty(G)$ and all $x, y \in G$. It is clear that $\alpha_x \in \text{Aut}(\ell^\infty(G))$ and that the map $G \to \text{Aut}(\ell^\infty(G)), x \longmapsto \alpha_x$, is a homomorphism. We define p_0 to be the characteristic function of the set G^+ as a subset of G^+ and $p_x = \alpha_x(p_0)$ for each $x \in G$. Thus p_x is the characteristic function of the set $x + G^+$.we call p_x the projection determined by x. Clearly $x \leq y$ iff $p_x \ \geq p_y$. If $x \vee y$ denotes the maximum of x and y then $p_x p_y = p_{x \vee y}$. It follows that the closed linear span $\mathcal{J}(G)$ of all projections $p_x(x \in G)$ is a C^*-subalgebra of $\ell^\infty(G)$. Put $\mathcal{J}_1(G) = \mathcal{J}(G) + \boldsymbol{C1}$ $(1 \in \mathcal{J}(G) \Leftrightarrow G = 0)$ and let $\mathcal{J}_0(G)$ denote the closed linear span of all $p_x - p_y(x, y \in G)$. $\mathcal{J}_0(G)$ is clearly a closed ideal in $\mathcal{J}(G)$. Since each α_x maps $\mathcal{J}_1(G)$ into itself we get by restriction a homomorphism $\alpha: G \to \text{Aut}(\mathcal{J}_1(G)), x \longmapsto \alpha_x$, i.e. $(\mathcal{J}_1(G), G, \alpha)$ is a C^*- dynamical system. Clearly $\mathcal{J}(G)$ and $\mathcal{J}_0(G)$ are G- invariant ideals in $\mathcal{J}_1(G)$. Recall that one can regard $\mathcal{J}_1(G)$ as C^*-subalgebra of the crossed product C^*-algebra $\mathcal{J}_1(G) \times_\alpha G$ and that $\mathcal{J}_1(G)$ contains the identity element of $\mathcal{J}_1(G) \times_\alpha G$. Also if $\delta: G \to \mathcal{J}_1(G) \times_\alpha G$ is the canonical homomorphism into the unitaries then we have $\alpha_x(f) = \ \delta_x f \delta_x^* (f \in \mathcal{J}_1(G), x \in G)$. $\mathcal{J}_1(G) \times_\alpha G$ is generated by $\mathcal{J}_1(G)$ and $\delta(G)$. If J is a closed G- invariant ideal of $\mathcal{J}_1(G)$ then the C^*-subalgebra of $\mathcal{J}_1(G) \times_\alpha G$ generated by all $f\delta_x(f \in J, x \in G)$ is in fact a closed ideal of $\mathcal{J}_1(G) \times_\alpha G$ which is $*$-isomorphic to the crossed product $J \times_\alpha G$. We can, and do, therefore regard $J \times_\alpha G$ as this ideal in $\mathcal{J}_1(G) \times_\alpha G$. Note that $J \subseteq J \times_\alpha G$.

Now we define $\mathcal{A}(G) = p_0(\mathcal{J}_1(G) \times_\alpha G)p_0$. Thus $\mathcal{A}(G)$ is a hereditary C^*-subalgebra of $\mathcal{J}_1(G) \times_\alpha G$ with identity element p_0. Define $W: G^+ \to \mathcal{A}(G)$ by setting $W_x = p_0 \delta_x p_0$. W is a semigroup of isometries. This is immediate from the fact $p_0 \delta_x p_0 = \delta_x p_0$ for all $x \in G^+ \big(p_0 \delta_x p_0 \delta_x^* = (\chi_{G_+})(\chi_{x+G^+}) = \chi_{G^+ \cap (x+G^+)} = \chi_{x+G^+} = \delta_x p_0 \delta_x^* \Longrightarrow p_0 \delta_x p_0 = \delta_x p_0\big)$. Thus we have the induced $*$- homomorphism $W^*: \mathcal{F}(G) \longrightarrow \mathcal{A}(G)$ with $W^* V = W$. Since $\mathcal{J}_1(G) \times_\alpha G$ is generated by p_0 and all $\delta_x(x \in G^+)$ one can easily show that $W_x(x \in G^+)$ generate $\mathcal{A}(G)$. This implies that W^* is surjective. We are going to see in a moment that W^* is in fact a $*$-isomorphism, but first we need a lemma which shows that $\mathcal{J}_1(G)$ has an interesting universal property.

Lemma (1.3.7)[23]: Let G be a non-zero ordered group, $\mu: G \to B$ a homomorphism into the unitaries of a unital C^*-algebra B, and q a projection in B such that $\mu(x)q = q\mu(q)q$ for all x in G^+. Then there is a unique unital $*$-homomorphism $y: \mathcal{J}_1(G) \to B$ such that $y(p_x) = \mu(x)q\mu(x)^*$ for all x in G.

Proof. Uniqueness is obvious, we show existence.Let Γ denote the linear span of all $p_x(x \in G)$, so Γ is a dense $*$-subalgebra of $\mathcal{J}(G)$. Put $q_x = \mu(x)q\mu(x)^*$and note that if $x \leq y$ then $q_x \geq q_y$, since $q_x q_u = \mu(x)q\mu(y-x)q\mu(y)^* = \mu(x)\mu(y-x)q\mu(y)^*$ (as $y - x \in G^+) = q_y$. Thus $q_x q_y = q_{x \vee y}$. Now let $x^1, \dots, x^n \in G$ and let $p_i = p_{x^i}(i = 1, \dots, n)$. We show that $\|\lambda_1 p_1 + \dots + \lambda_n p_n\| \geq \|\lambda_1 q_1 + \dots + \lambda_n q_n\|(\lambda_1, \dots, \lambda_n \in \boldsymbol{C})$.

We may assume (by re-indexing if necessary) that $x^1, \dots, x^n$, and hence that $p_1 \geq \dots \geq p_n$and$q_1 \geq \dots \geq q_n$. Therefore the projections $p_1 - p_2, p_2 - p_3, \dots, p_{n-1} - p_n, p_n$ are pairwise orthogonal, as are the projections $q_1 - q_2, q_2 - q_3, \dots, q_{n-1} - q_n, q_n$. Let $\nu_i = \lambda_1 + \dots + \lambda_i (i = 1, \dots, n)$. Then we have $\lambda_1 p_1 + \dots + \lambda_n p_n = \nu_1(p_1 - p_2) + \dots + \nu_{n-1}(p_{n-1} - p_n) + \nu_n p_n$ and correspondingly $\lambda_1 q_1 + \dots + \lambda_n q_n = \nu_1(q_1 - q_2) + \dots + \nu_{n-1}(q_{n-1} - q_n) + \nu_n q_n$. Since $p_x = p_y$ implies $x = y$ and so $q_x = q_y$, we now deduce that

$$\begin{aligned} \|\lambda_1 p_1 + \dots + \lambda_n p_n\| &= \max\{|\nu_i| : p_i - p_{i+1} \neq 0 \ (i = 1, \dots, n-1) or\ p_n \neq 0\} \\ &\geq \max\{|\nu_i| : q_i - q_{i+1} \neq 0 \ (i = 1, \dots, n-1) or\ q_n \neq 0\} \\ &= \|\lambda_1 q_1 + \dots + \lambda_n q_n\|. \end{aligned}$$

It is now routine algebra to check that the map $\gamma: \Gamma \to B$ defined by setting $\gamma(\lambda_1 p_1 + \dots + \lambda_n p_n) = \lambda_1 q_1 + \dots + \lambda_n q_n$ is a (norm- decreasing) $*$-homomorphism, and so extends to a $*$-homomorphism $\gamma: \mathcal{J}(G) \to B$. Finally we extend γ to initial $*$-homomorphism $\gamma: \mathcal{J}_1(G) \to B$by setting $\gamma(1) = 1$.

Theorem (1.3.8)[23]: If G is an ordered group then the canonical map $W^*: \mathcal{F}(G) \longrightarrow \mathcal{A}(G)$ is a $*$- isomorphism.

Proof. We already know that W^* is a surjective, so we just have to show injectivity.

Now we can regard $\mathcal{F}(G)$ as a C^*- subalgebra of $B(H)$ for some Hilbert space H and with $id_H = 1 \in \mathcal{F}(G)$. Also we may assume $G \neq 0$. By Theorem (1.3.1) the semigroup of isometries $V: G^+ \to B(H), x \mapsto V_x$, admits a unitary lifting (K, π). Thus π is a homomorphism from G into the unitary operators on the Hilbert space K, and if Qdenotes the projection of K onto its subspace H we have $\pi(x)Q = Q\pi(x)Q$ for all $x \in G^+$, since H is invariant for such $\pi(x)$. Also $V_x = \pi(x)_H (x \in G^+)$. By Lemma (1.3.7) there exists a unique unital $*$- homomorphism $y: \mathcal{I}_1(G) \to B(K)$ such that $y(p_x) = \pi(x)Q\pi(x)^* (x \in G)$.

We now claim that y, π, K is a covariant representation of the C^*-dynamical system $(\mathcal{I}_1(G), G, \alpha)$. All we need to do to see this is to show that $y(\alpha_x(f)) = \pi(x) y(f) \pi(x)^*$ for all $f \in \mathcal{I}_1(G)$ and all $x \in G$. By using the fact that 1 and all the projections $p_x (x \in G)$ have closed linear span $\mathcal{I}_1(G)$, it clearly suffices to show the above equation for f of the form $f = p_y$. But

$$\begin{aligned} \gamma\left(\alpha_x\left(p_y\right)\right) &= \gamma(p_{x+y}) = \pi(x+y)\, Q\pi(x+y)^* = \pi(x)\, \pi(y)\, Q\pi\,(y)^*\, \pi\,(x)^* \\ &= \pi(x)\gamma(p_y)\, \pi\,(x)^*. \end{aligned}$$

Thus since (γ, π, K) is a covariant representation it induces a unique $*$-homomorphism $y^\sim: \mathcal{I}_1(G) \times_\alpha G \to B(K)$ extending γ such that $\gamma^\sim(\delta_x) = \pi(x) (x \in G)$. It is now easily verified that the map $\mu: \mathcal{A}(G) \to B(H), a \to \gamma^\sim(a)_H$, is a $*$-homomorphism. However $\mu(W_x) = \mu(\delta_x p_0) = \left(\gamma^\sim(\delta_x)\gamma(p_0)\right)_H = (\pi(x)Q)_H = V_x \in \mathcal{I}(G) (x \in G^+)$so $\mathrm{Im}(\mu) \subseteq \mathcal{I}(G)$. Thus we can regard μ as a $*$-homomorphism from $\mathcal{A}(G)$ to $\mathcal{I}(G)$. Again since $\mu(W_x) = V_x$ we have $\mu W^*(V_x) = V_x (x \in G^+)$, so $\mu W^* = \mathrm{id}_{\mathcal{I}(G)}$, thus W^* is injective.

We state now a result of Power [35] that we will need for the next theorem Power defines a C^*-algebra C of operators on the Hilbert space K to be inner with respect to a close subspace H of K if $\mathrm{id}_K \in C$ and C is generated by its elements T such that $T(H) \subseteq H$. If this is the case and C is commutative, and B is the C^*-subalgebra of $B(H)$ generated by all

$T_H(T \in C)$ then $T \in K\left(C^*(C \cup \{Q\})\right)$ (Q is the projection of K on H) implies $T_H \in K(B)$ (see [35]).

Theorem (1.3.9)[23]: If G is an ordered group then $K\big(\mathcal{A}(G)\big) = p_0\big(K(\mathcal{I}_1(G) \times_\alpha G)\big)p_0$is a full hereditary C^*-subalgebra of $K(\mathcal{I}_1(G) \times_\alpha G)$.

Proof. Let $Z = \mathcal{I}_1(G) \times_\alpha G$. Since $\mathcal{A}(G)$ is a C^*-subalgebra of Z, $K\big(\mathcal{A}(G)\big) \subseteq K(Z)$, and since $\mathcal{A}(G) = p_0 Z p_0, K\big(\mathcal{A}(G)\big) \subseteq p_0 K(Z) p_0$.

Now regard Z as a C^*-subalgebra of $B(K)$ for some Hilbert space K with $\mathrm{id}_K = 1 \in Z$, and let $H = p_0(K)$. Since $\delta_x p_0 = p_0 \delta_x p_0 (x \in G^+)$, H is an invariant subspace for these δ_x, so the commutative C^*-subalgebra C of $B(K)$ generated by all $\delta_x(x \in G^+)$ is inner with respect to H. Let B be the C^*-subalgebra of $B(H)$ generated by all $T_H(T \in C)$. By Power's result mentioned above $T \in K(Z)$ implies $T_H \in K(B)$ (since $Z = C^*(C \cup \{p_0\})$). Now the map $: \mathcal{A}(G) \to B, T \mapsto T_H$, is easily seen to be a $*$-isomorphism. Thus $T \in p_0 K(Z) p_0$ implies $T \in K(Z)$, and $T \in \mathcal{A}(G)$ implies $T_H \in K(B)$, which implies $\beta^{-1}(T_H) \in K\big(\mathcal{A}(G)\big)$. We have therefore $p_0 K(Z) p_0 = K\big(\mathcal{A}(G)\big)$, so $K\big(\mathcal{A}(G)\big)$ is hereditary C^*-subalgebra of $K(Z)$.

Finally we show $K\big(\mathcal{A}(G)\big)$ is full in $K(Z)$, i.e. the closed ideal J in $K(Z)$ generated by $K\big(\mathcal{A}(G)\big)$ is $K(Z)$ itself.

This is because J contains $p_0 - p_x = p_0 - W_x W_x^*(x \in G^+)$, and therefore $p_0\delta_x - \delta_x p_0 = (p_0 - \delta_x p_0 \delta_x^*)\delta_x = \big(p_0 - p_y\big)\delta_x \in J$. Hence $Z/J$is abelian (it is generated by commutating normal elements), so $J \supseteq K(Z) \Longrightarrow J = K(Z)$.

As a consequence of a theorem of Brown [25] and Theorem (1.3.9) above it follows that if $K(\mathcal{I}_1(G) \times_\alpha G)$is seperable (e.g. if G is countable) then $K\big(\mathcal{A}(G)\big)$ and $K(\mathcal{I}_1(G) \times_\alpha G)$ are stably isomorphic.

Although we shall not be using it, we record here the interesting fact that for G an ordered group $K(\mathcal{I}_1(G) \times_\alpha G) = \mathcal{I}_0(G) \times_\alpha G$. (Proof. Let $Z = \mathcal{I}_1(G) \times_\alpha G$ and $J = \mathcal{I}_0(G) \times_\alpha G$. J is a closed ideal of Z generated as a C^*-algebra by all $f\delta_x(f \in \mathcal{I}_0(G), x \in G)$. Now $(p_0 - p_x)\delta_x = (p_0 - \delta_x p_0 \delta_x^*)\delta_x = p_0\delta_x - \delta_x p_0 \in K(Z)$, so $\big(p_x - p_y\big)\delta_z = \big(p_0 - p_y\big)\delta_z - (p_0 - p_x)\delta_z \in K(Z)$, thus $f\delta_x \in K(Z)$ for all $f \in \mathcal{I}_0(G)$, and all $z \in G$. Hence $J \subseteq K(Z)$. Also $p_0\delta_x - \delta_x p_0 = (p_0 - p_x)\delta_x \in J$ implies Z/J is abelian, so $J \supseteq K(Z\ )$.

Recall that a subgroup I of a partially ordered group G is an ideal of G if $I = I^+ - I^-$ and $0 \le x \le y \in I$ implies $x \in I$ $(x \in G)$. G is said to be simple if 0 and G are its only ideals. All ordered subgroups of $\mathbf{R}$ with the usual order relation are simple. For $n = 2, 3, \ldots$ the group $\mathbf{Z}^n$ with the lexicographic order $((a_1, \ldots, a_n) < (b_1, \ldots, b_n)$if $a_1 < b_1$ or $a_1 = b_1, \ldots ,\ a_i = b_i$, and $a_{i+1} < b_{i+1})$ is a non-simple ordered group.

If I is an ideal in a partially ordered group G, and φ is the quotient map from G to G/I then $\varphi(G^+)$ is a cone in the quotient group G/I. We call the partially ordered group $\big(G/I, \varphi(G^+)\big)$ the quotient partially ordered group of G by I. Of course φ is a positive homomorphism from G to G/I. If G is totally ordered, so are I and G/I.

Lemma (1.3.10)[23]: If G s a finitely generated non- zero ordered group then G contains a non- zero simple ideal I.

Proof. Let K be the rank of G. Note that an ordered group is necessary torsion- free. Thus if I_1 is a proper ideal in G, then G/I_1 is non- zero and so has positive rank. This implies rank $(I_1) =$ rank$(G) -$ rank $\left(\frac{G}{I_1}\right) < rank(K) = k$.

We show the result by induction on k. Suppose it is true for all ranks $< k$. If G has no proper ideals then there is nothing to prove (take $I = G$). Otherwise G contains a non-zero ideal I_1 with rank $(I_1 < k)$. By the induction hypothesis I_1 contains a non- zero simple ideal I. I is then an ideal in G, thus completing the induction.

If G is any ordered group let $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$ $(x \in G)$. Let

$$F(G) = \{x \in G\text{: for all } y \in G, y > 0, \text{there exists } n \in \mathbf{N}, |x| \leq ny\}.$$

Using the triangle inequality $|x + y| \leq |x| + |y|$ one easily sees that $F(G)$ is a simple ideal in G. Hence if I is any non- zero ideal of $G, F(G) \subseteq I$ (since if $F(G)$ is non- zero then there exists $x \in F(G), x > 0$, and there exists $y \in I, y > 0$, so that if z is their minimum, then $0 < z \in I \cap F(G)$ implies $I \cap F(G)$ is a non- zero ideal of $F(G)$, so $I \cap F(G) = F(G)$, thus $F(G) \subseteq I$). Of course $F(G)$ might be just the zero ideal. In fact it is for $G = \mathbf{Z}^{\infty}$, the direct sum of countably infinitely many copies of **Z**with the lexicographic order: $(a_1, a_2, \dots) < (b_1, b_2, \dots)$if $a_1 < b_1$ or if $a_1 = b_1, \dots, a_i = b_i$, and $a_{i+1} < b_{i+1}$ for some integer i.

The point of Lemma (1.3.10) can be rephrased as follows: if$G$ is a non- zero finitely generated group, then $F(G)$is non- zero. (Proof: By Lemma (1.3.10), G contains a non- zero simple ideal I. Let $x \in G, x > 0$, and let I_x be the set of y in G such that for some positive integer $n, |y| \leq nx$. Then I_x is a non- zero ideal in $G(I_x \neq 0 \; as \; x \in I_x)$. Now $I \cap I_x$ is a non- zero ideal in I, so $I \cap I_x = I$, thus $I \subseteq I_x$. hence $y \in I$ implies $|y| \leq nx$ for some $n \in$ **N**. Thus we have shown$I \subseteq F(G)$, and since $F(G) \subseteq I$ by our earlier remarks, $F(G) = I \neq 0$.)

If G is a non-zero finitely generated ordered group we let $F\mathcal{J}(G)$ denote the closed linear span of all $p_x - p_y\big(x, y \in G, x - y \in F(G)\big)$.

Lemma (1.3.11)[23]: If G is a non-zero finitely generated ordered group then $F\mathcal{J}(G)$ is the smallest non- zero G-invariant closed ideal $\mathcal{J}_1(G)$.

Proof. If $x, y, z \in G$ with $x - y \in F(G)$ then $z \vee x - z \vee y \in F(G)$. Hence $p_z\big(p_x - p_y\big) = p_{z \vee x} - p_{y+z} \in F\mathcal{J}(G)$, so it is clear that $F\mathcal{J}(G)$ is an ideal in $\mathcal{J}_1(G)$. Since $\alpha_z\big(p_x - p_y\big) = p_{x+z} - p_{y+z}$, it is trivial that $F\mathcal{J}(G)$ is G-invariant. As $F(G)$ is non- zero, $p_0 - p_x \neq 0$ for some $x \in F(G)$, so $F\mathcal{J}(G)$ is non- zero.

Now let J be a non- zero G- invariant closed ideal in $\mathcal{J}_1(G)$. We have to show that $F\mathcal{J}(G) \subseteq J$. By replacing J by $J \cap \mathcal{J}_0(G)$ if necessary, we may assume that $J \subseteq \mathcal{J}_0(G)$ (the reason that $J \cap \mathcal{J}_0(G)$ is non- zero is the easily checked fact that $\mathcal{J}_0(G)$ is an essential closed ideal in $\mathcal{J}_1(G)$).

Put $I=\{x\in G: p_0-p_x\in J\}$. If $x,y\in I$ then p_0-p_x and $\alpha_z(p_0-p_y\in J)$ implies $p_0-p_x+p_x-p_{x+y}=p_0-p_{x+y}\in J$, so $x+y\in I$. Also $x,y\in G$ and $0\leq x\leq y\in I$ implies $0\leq p_0-p_x\leq p_0-p_y\in J$, so $p_0-p_x\in J$, thus $x\in I$. Thus I is an ideal of G.

We define Γ to be the linear span of 1 and all $p_x(x\in G)$. In the terminology of Goodearl [29], Γ is a dense ultramatricial $*$-subalgebra of the AF-algebra $\mathcal{J}_1(G)$. Hence $(J\cap\Gamma)^-=J$ and $J\cap\Gamma$ is the linear span of its projections (see [29]). since J is a non- zero, there is a non- zero projection p in $J\cap\Gamma$. Hence there exists $x^1,\dots,x^n\in G$ determining projections $p_1,\dots,p_n$ in $\mathcal{J}_1(G)$ such that $p=\lambda_1p_1+\cdots+\lambda_np_n$ for some $\lambda_1,\dots,\lambda_n\in\mathbf{C}$. Recalling a detail from the proof of Lemma (1.3.7) we may assume that $x^1\leq\cdots\leq x^n$ (by re-indexing if necessary) and then we have $p=v_1(p_1-p_2)+\ldots+v_{n-1}(p_{n-1}-p_n)+v_np_n$ for some $v_i\in\mathbf{C}$. Of course the projections $p_1-p_2,\dots,p_{n-1}-p_n,p_n$ are pairwise orthogonal. Now if $\boldsymbol{v_n}$is non- zero then $p_n\in\mathcal{J}_0(G)$ (since $,p_1-p_2,\dots,p_{n-1}-p_n\in\mathcal{J}_0(G)$) and this is easily seen to be impossible. Thus we have $v_n=0$ and of course each $v_i=0$ or 1. In short p is a sum of k pairwise orthogonal projections $p_{u^j}-p_{v^j}$ with $u^j\leq v^j$ in G. Thus $p\geq p_{u^j}-p_{v^j}\geq 0$ and $p\in J$ implies $p_{u^j}-p_{v^j}\in J$, so $\alpha(-u^j)(p_{u^j}-p_{v^j})=p_0-p_{v^j-u^j}\in J$, so $v^i-u^j\in I$. Since $p\neq 0$, $v^j\neq u^j$ for some j, we have $I\neq 0$. Hence $F(G)\subseteq I$, so if $x\in F(G)$ then $p_0-p_x\in J$ as $x\in I$. More generally if $x,y\in G$ and $x-y\in F(G)$ then $p_0-p_{x-y}\in J$ implies $\alpha_y(p_0-p_{x-y})=p_y-p_x\in J$. Hence $F\mathcal{J}(G)\subseteq J$.

We do not state the strongest possible forms of these results, just versions sufficient for our purposes.

Let (A,G,α) be a C^*-dynamical systems with A a separable abelian C^*-algebra and G a countable discrete abelian group. A is G-prime if every two non- zero G- invariant closed ideals of A have non- zero intersection. A is G-simple if 0 and A are its only G-invariant closed ideals. The Arveson spectrum$\mathrm{sp}(\alpha)$ of (A,G,α) is the set of all $\rho\in G^\wedge$ for which there exists a sequence of unit vectors f_n in A such that $\|\alpha_x(f_n)-\rho(x)f_n\|$ converges to 0 as $n\to\infty(x\in G)$. A useful fact: the annihilator $\mathrm{sp}(\alpha)^\perp=\{x\in G:\alpha_x=\mathrm{id}_A\}$. The connes spectrum $\Gamma(\alpha)$ of (A,G,α) is the intersection of all $\mathrm{sp}(\alpha|J)$ where J runs over all non- zero G-invariant closed ideals of A, and $\mathrm{sp}(\alpha|J)$ is the Arveson spectrum of the C^*-dynamical system (J,G,α)got by restriction of α_x to $J(x\in G)$. $\Gamma(\alpha)$ is a closed subgroup of $G^\wedge$, so $\Gamma(\alpha)^\perp=0$ then $\Gamma(\alpha)=G^\wedge$.

The following two important results will be needed:

Result 1. If A is G-prime and $\Gamma(\alpha)=G^\wedge$ then $A\times_\alpha G$ is primitive.

Result 2. If A is G-simple and $\Gamma(\alpha)=G^\wedge$ then $A\times_\alpha G$ is simple.

See Pedersen [33] and [34].

Lemma (1.3.12)[23]: Let G be a finitely generated ordered group. Then $\mathcal{J}_1(G)\times_\alpha G$ is primitive and $F\mathcal{J}(G)\times_\alpha G$ is simple.

Proof. If $G=0$ then $\mathcal{J}_1(G)\times_\alpha G=\mathbf{C}$ and $F\mathcal{J}(G)\times_\alpha G=0$, so there is nothing to prove. So we may suppose that G is non- zero, and hence $F(G)$ is non- zero. If J_1 and J_2 are non- zero G-invariant closed ideals of $\mathcal{J}_1(G)$, then by Lemma (1.3.11) $F\mathcal{J}(G)\subseteq J_1\cap J_2$ and $F\mathcal{J}(G)$ is a non- zero G-invariant closed ideal in $\mathcal{I}_1(G)$. Thus $\mathcal{I}_1(G)$ is G-prime, and since $J_1 \subseteq J_2$ implies $\mathrm{sp}(\alpha|J_1) \subseteq \mathrm{sp}(\alpha|J_2)$ we have $\Gamma(\alpha) = \mathrm{sp}(\alpha|F\mathcal{I}(G))$.

Hence $\Gamma(\alpha)^{\perp} = \mathrm{sp}(\alpha|F\mathcal{I}(G))^{\perp} = \{x \in G: \alpha_x = \mathrm{id}\}$. Let $x \in \Gamma(\alpha)^{\perp}$ and $y \in F(G)\,, y > 0$. Then $\alpha_x(p_0 - p_y) = p_x - p_{x+y} = p_0 - p_y$ (since $p_0 - p_y \in F\mathcal{I}(G)$). Now if $z < t$ in G and $[z,t) = \{u \in G: z \le u < t\}$ then $p_z - p_t = \chi_{[z,t)}$. Thus $\chi_{(x,x+y)} = \chi_{(0,y)}$, so $x = 0$, and therefore $\Gamma(\alpha)^{\perp} = 0$, implying that $\Gamma(\alpha) = G^{\wedge}$. By result 1 above, $\mathcal{I}_1(G) \times_{\alpha} G$ is primitive. Of course from that we have just shown, it is clear that $(F\mathcal{I}(G), G, \alpha)$ is G-simple (since any G-invariant closed ideal of $F\mathcal{I}(G)$ is one of $\mathcal{I}_1(G)$ also) and the Connes spectrum $\Gamma(\alpha) = G^{\wedge}$ for $(F\mathcal{I}(G), G, \alpha)$ also. Hence by result 2 above $F\mathcal{I}(G) \times_{\alpha} G$ is simple. (Note that $F\mathcal{I}(G) \times_{\alpha} G$ is non- zero since it contains $F\mathcal{I}(G)$, and this is opn- zero).

If G is an ordered group we let $\mathcal{FI}(G)$ denote the closed ideal in $\mathcal{I}(G)$ generated by all $q_x = 1 - V_0 V_x^* (x \in F(G)^+)$. Clearly $\mathcal{FI}(G) \subseteq K(\mathcal{I}(G))$, and $\mathcal{FI}(G) = K(\mathcal{I}(G))$ if $F(G) = G$.

Lemma (1.3.13)[23]: If G is a finitely generated ordered group then $\mathcal{I}(G)$ is primitive and $\mathcal{FI}(G)$ is simple.

Proof. $\mathcal{A}(G)$ is a hereditary C^*-subalgebra of $\mathcal{I}_1(G) \times_{\alpha} G$, so $\mathcal{A}(G)$ is primitive as $\mathcal{I}_1(G) \times_{\alpha} G$ is. Let $J = p_0(F\mathcal{I}(G) \times_{\alpha} G)p_0$. Then J is a closed ideal in $\mathcal{A}(G) = p_0(\mathcal{I}_1(G) \times_{\alpha} G)p_0$ (since $F\mathcal{I}(G) \times_{\alpha} G$ is a closed ideal $\mathcal{I}_1(G) \times_{\alpha} G$), and J is a hereditary C^*-subalgebra of the simple C^*-algebra $F\mathcal{I}(G) \times_{\alpha} G$, so J is simple.

Now let $W^*: \mathcal{FI}(G) \to \mathcal{A}(G)$ be the conical $*$-iomorphism. Since $\mathcal{A}(G)$ is primitive, so is $\mathcal{I}(G)$. Also $W^*(\mathcal{FI}(G)) = J$, so $\mathcal{FI}(G)$ is simple. (To see that $W^*(\mathcal{FI}(G)) = J$, note $W^*(q_x) = W^*(1 - V_* V_x^*) = p_0 - W_* W_x^* = p_0 - \delta_x p_0 \delta_x^* = p_0 - p_x$. Thus $x \in (G)^+$ implies $W^*(q_x) \in p_0 \mathcal{FI}(G) p_0 \subseteq J$ which implies $W^*(\mathcal{FI}(G)) \subseteq J$. If $G = 0$ then $\mathcal{FI}(G) = 0$, so $J = 0$, thus $W^*(\mathcal{FI}(G)) = J$. If G is non- zero, then $F(G)$ has a positive element x, so $q_x \neq 0$, so $W^*(q_x) \neq 0$, so $W^*(\mathcal{FI}(G)) = J$ by simplicity of J.)

Recall that C^*-algebra A is prime if every two non- zero closed ideals of A have non-zero intersection. Every primitive C^*-algebra is prime (we are about to use this fact in a moment) and the converse holds for separable C^*-algebra. (The non- separable case is an open question, see Pedersen [33].)

Let $\beta: G^+ \to B$ be a semigroup off isometries in the unital C^*-algebra B, over the partially ordered group G. We say that β is nonunitary if $\beta(x)$ is non- unitary for all $x > 0, x \in G$. The following lemma will be generalized immediately in Theorem (1.3.15) below.

Lemma (1.3.14)[23]: Let G be a finitely generated ordered group and $\beta: G^+ \to B$ a nonunitary semigroup of isometries in a unital C^*-algebra B. Then the unique $*$-homomorphism $\beta^*: \mathcal{I}(G) \to B$ such that $\beta^* V = \beta$ is injective.

Proof. Let $J = \ker(\beta^*)$. If J is non- zero then G is non- zero ($G = 0$ implies $\mathcal{I}(G) = 0$, so $J = 0$), so $F(G)$ is non- zero, thus $\mathcal{FI}(G)$ is non- zero. Hence $J \cap \mathcal{FI}(G)$ is non- zero (as $\mathcal{I}(G)$ is primitive and therefore prime). As $\mathcal{FI}(G)$ is simple, $J \cap \mathcal{FI}(G) = \mathcal{FI}(G)$, so $\mathcal{FI}(G) \subseteq J$. Now there exists $x \in F(G), x > 0$, so we have $q_x \in J$, thus $0 = \beta^*(q_x) =$

$\beta^*(1 - V_x V_x^*) = 1 - \beta(x)\beta(x)^*$, which implies that $\beta(x)$ is unitary. Since β is nonunitary this is impossible, so J cannot be non- zero. Thus β^* is injective.

Theorem (1.3.15)[23]: Let G be an ordered group and $\beta: G^+ \to B$ a nonunitary semigroup of isometries in a unital C^*-algebra B. Then $\beta^*: \mathcal{J}(G) \to B$ is injective.

Proof. Let I be the set of finite non-empty subsets of G, ordered by set inclusion (i.e. $i \le j \; iff \; i \subseteq j$). Thus I is a direct set. For $i \in I$ let G_i be the subgroup of G generated by i, and let $\varphi^i: G_i \to G$ be the inclusion homomorphism. Likewise for $i \le j$ in I let $\varphi_{ij}: G_i \to G_j$ be the inclusion homomorphism. Of course all the G_i are ordered groups and the maps φ^i and φ_{ij} are positive. Since G is the union of all $G_i (i \in I)$ it is easily checked that G is the direct limit (in the category of all partially ordered groups) of the direct system $\left(\varphi_{ij}: G_i \to G_j\right)_{i \le j}$ with the maps φ^i as natural maps. By Theorem (1.3.6) $\mathcal{J}(G)$ is the direct limit (in the category of C^*-algebras) of the direct system $\left(\left(\varphi_{ij}\right)^*: \mathcal{J}(G_i) \to \mathcal{J}\left(G_j\right)\right)_{i \le j}$ with the maps $\left(\varphi^i\right)^*: \mathcal{J}(G_i) \to \mathcal{J}\left(G_j\right)$ as natural maps. Let $A_i = \left(\varphi^i\right)^*\left(\mathcal{J}(G_i)\right)(i \in I)$. Then $\mathcal{J}(G) = (\bigcup\{A_i: i \in I\})^-$, since $\mathcal{J}(G)$ is the direct limit.

Let $\psi^i: G_i^+ \to G^+$ be the restriction of φ^i. Now $\beta\psi^i: G_i \to B$ is a semigroup of isometries over G_i and $\beta\psi^i$ is a nonunitary, since $\beta\psi^i(x)$ were a unitary then $\psi^i(x) = 0$ implies $x = 0$ $\left(\psi^i(x) = x\right)$. Hence by Lemma (1.3.14), $\left(\beta\psi^i\right)^*$ is injective. Let $V^i: G_i^+ \to \mathcal{J}(G_i)$ and $V: G^+ \to \mathcal{J}(G)$ be the canonical maps. The $\beta^*\left(\varphi^i\right)^* V^i = \beta^* V \psi^i = \beta\psi^i$, so $\beta^*\left(\varphi^i\right)^* = \left(\beta\psi^i\right)^*$. Thus β is an isometry on each $A_i: i \in I$, implying that β^* is an isometry on $\mathcal{J}(G) = (\bigcup\{A_i: i \in I\})^-$.

Theorem (1.3.16)[23]: Let G be an ordered group. Then $\mathcal{J}(G)$ is prime.

Proof. We retain the notation of the proof of Theorem (1.3.15). Let J be a non- zero closed ideal of $\mathcal{J}(G)$. Then $J \cap A_i$ is non- zero for some $i \in I$. (For otherwise letting π be the quotient map from $\mathcal{J}(G)$ to $\mathcal{J}(G)/J, \pi$ is isometric on each C^*-algebra A_i, so π is isometric on $\mathcal{J}(G) = (\bigcup\{A_i: i \in I\})^-$, thus $J = \ker(\pi) = 0$.) Thus if J_1 and J_2 are non- zero closed ideals of $\mathcal{J}(G)$ then (since I is directed) $J_1 \cap A_i$ and $J_2 \cap A_i$ are non- zero closed ideals in some A_i. Now $\left(\varphi^i\right)^*: \mathcal{J}(G_i) \to \mathcal{J}\left(G_j\right)$ is injective since $\left(\varphi^i\right)^* = \left(V\psi^i\right)^*$ and $V\psi^i: G^+ \to \mathcal{J}(G)$ is a nonunitary semigroup of isometries (which implies $\left(V\psi^i\right)^*$is injective by Theorem (1.3.15)). Hence $A_i = \left(\varphi^i\right)^*\left(\mathcal{J}(G_i)\right)$ is $*$-isomorphic to $\mathcal{J}(G_i)$, so A_i is primitive (by Lemma (1.3.13)), and therefore prime. It follows that $(J_1 \cap A_i) \cap (J_2 \cap A_i)$ is non- zero, so $J_1 \cap J_2$ is non- zero. Thus $\mathcal{J}(G)$ is prime.

We included Theorem (1.3.16) here since one can derive it so easily given one has set up the machinery to prove Theorem (1.3.15). Actually however we will show that $\mathcal{J}(G)$ is primitive (for G an ordered group) by exhibiting explicitly a faithful irreducible representation of $\mathcal{J}(G)$.

Theorem (1.3.17)[23]: If G is an ordered group then $\mathcal{FJ}(G)$ is simple.

Proof. Let J be a non- zero closed ideal of $\mathcal{FJ}(G)$ and let π be the quotient map from $\mathcal{J}(G)$ to $\mathcal{J}(G)/J$. Let the map $\beta: G^+ \to \mathcal{J}(G)/J$ be defined by setting $\beta(x) = \pi(V_x)$ (i.e. $\beta = \pi V$).

β is clearly a semigroup of isometries and $\beta^* = \pi$. Suppose β were nonunitary. Then $\pi = \beta^*$ is injective, so $J = 0$. Thus β is not nonunitary, and so there is an element $x \in G, x > 0$, such that $\beta(x)$ is a unitary. If $y \in F(G)^*$ then $y \leq nx$ for some $n \in \mathbf{N}$, so $nx = y + z$ for some $z \in G^+$. Hence $\beta(x)^n = \beta(y)\beta(z) = \beta(z)\beta(y)$, so $\beta(y)$ is invertible as $\beta(x)^n$ is. Thus $\pi(q_y) = \pi(1 - V_y V_y^*) = 1 - \beta(y)\beta(y)^* = 0$, so $\mathcal{FJ}(G) \subseteq \ker(\pi)$ thus $\mathcal{FJ}(G) = J$.

Of course $\mathcal{FJ}(G)$ is non- zero iff $F(G)$ is non- zero.

Corollary (1.3.18)[23]: (Douglas, [27]). If G is an ordered subgroup of $\mathbf{R}$ (usual order) then $K(\mathcal{J}(G))$ is simple.

Proof. In this case $F(G) = G$. Hence $\mathcal{J}(G)/\mathcal{FJ}(G)$ is abelian (as $1 - V_x V_x^* \in \mathcal{FJ}(G)$ for all $x \in G^+$), so $\mathcal{FJ}(G) \supseteq K(\mathcal{J}(G))$ and we know already that $\mathcal{FJ}(G) \subseteq K(\mathcal{J}(G))$, so $\mathcal{FJ}(G) = K(\mathcal{J}(G))$.

This result is attributed to Douglas because for G an order subgroup of $\mathbf{R}, V: G^+ \to \mathcal{J}(G)$ is a nonunitary one- parameter semigroup of isometries in his terminology, and the corollary follow from [27]. The techniques used by Douglas to prove this result are completely different from ours.

We return to partially ordered groups. We exhibit an irreducible representation of the Toeplitz algebra as a C^*-algebra of generalized "Toeplitz" operators (this representation is faithful for ordered groups). This involves our deriving a theory of such operators. The results and many of the proofs are closely analogous to the classical special case $G = \mathbf{Z}$, although there are some interesting differences. Perhaps the most remarkable fact here is that so much of the classical theory extends in such generality.

Let G be a partially ordered group, T the circle group, and recall that $\varepsilon(x): G^\wedge \leftrightarrow T$is the evaluation homomorphism $\varepsilon(x)(y) = y(x) (x \in G, y \in G^\wedge)$as is well known $(\varepsilon(x))_{x \in G}$ forms an orthonormal basis for the Hilbert space $L^2 = L^2(G^\wedge)$, and letting P_G denote their linear span, it follows from the Stone- Weierstrass theorem that this $*$-subalgebra of $C(G^\wedge)$ is dense in $C(G^\wedge)$ in the sup-norm topology. The elements of P_G are called the trigonometric polynomials (relative to G). Denote by $H^2 = H^2(G)$ the closed subspace of L^2 having orthonormal basis $(\varepsilon(x))_{x \in G^+}$, and let $P \in B(L^2)$ be the projection onto H^2. If $\varphi \in L^\infty = L^\infty(G^\wedge)$ we define $T_\varphi \in B(H^2)$ by setting $T_\varphi(f) = P(\varphi f)$. T_φ is the Toeplitz operator with symbol φ (relative to G). The map $L^\infty \to B(H^\infty), \varphi \longmapsto T_\varphi$, is easily seen to be linear and norm- decreasing. Also $T_{\varphi^*} = T_{\varphi^-}$.

If $G = \mathbf{Z}$ (with the usual ordering) then of course H^2 is the usual Hardy space and we get the classical Toeplitz operators.

If G is a partially ordered group and F is finite non- empty subset of G, then there exists $x \in G^+$such that $x \geq y (y \in F)$. (Proof: if $F = \{x^1, \dots, x^n\}$ then each $x^i = y^i - z^i$ with $y^i, z^i \in G^+$. Take $x = y^1 + \cdots + y^n$.) This is used in the next easy but useful lemma. (Both this result and the next lemma will be often used tacitly.)

Lemma (1.3.19)[23]: If G is a partially ordered group and $\varphi \in P_G$ then $\varepsilon(x)\varphi \in H^2(G)$ for some x in G^+.

Proof. $\varphi = \lambda_1\varepsilon(y^1) + \cdots + \lambda_n\varepsilon(y^n)$ for some $y^1, \dots, y^n \in G$, and some $\lambda_1, \dots, \lambda_n \in \mathbf{C}$. Choose $x \in G^+$ such that $x \geq -y^1, \dots, -y^n$. Then $\lambda_1\varepsilon(x + y^1) + \cdots + \lambda_n\varepsilon(x + y^n) \in H^2$.

As in the proof above shall often drop explicit reference to G when referring to the spaces $L^2(G^\wedge), H^2(G)$ and $L^\infty(G^\wedge)$.

If E is a subset of $\mathbf{C}$ then $\text{hull}(E)$ denotes its closed convex hull.

Theorem (1.3.20)[23]: Let G be a partially ordered group. If $\varphi \in L^\infty(G^\wedge)$ then $\|T_\varphi\| = \|\varphi\|_\infty$ and $\text{Sp}(\varphi) \subseteq \text{Sp}(T_\varphi) \subseteq \text{hull}(\text{Sp}(\varphi))$. ($\text{Sp}(\varphi)$ is the essential range of φ.)

Proof. Let $M_\varphi \in B(L^2)$ be the multiplication defined by $M_\varphi(f) = \varphi f$. Now the map $L^\infty \to B(L^2), \varphi \mapsto M_\varphi$, is an isometric $*$-homo-morphism, so $\text{Sp}(M_\varphi) = \text{Sp}(\varphi)$. Let $S = \{\overline{\varepsilon(x)}f : x \in G^+, f \in H^2\}$. Then $S^- = L^2$, since $(P_G)^- = L^2$and$L_2 \subseteq S$. Suppose that T_φ is bounded below, so for some $\mu > 0, \|T_\varphi\| \geq \mu\|f\|(f \in H^2)$. Then $\|M_\varphi f\varepsilon(x)^-\| = \|\varphi f\| \geq \|P(\varphi f)\| = \|T_\varphi(f)\| \geq \mu\|f\| = \mu\|\overline{\varepsilon(x)f}\|$. Hence $\|M_\varphi(g)\| \geq \mu\|g\|(g \in L^2)$. As $S^- = L^2$. Thus for any $\varphi \in L^2, \text{Sp}(\varphi) = \text{Sp}(M_\varphi) \subseteq \text{Sp}(T_\varphi)$. Hence $\|T_\varphi\| \geq r(T_\varphi) \geq r(M_\varphi) = \|\varphi\|_\infty$, so $\|T_\varphi\| = \|\varphi\|_\infty$. But an isometric $*$-linear map $\rho: A \to B$ from an abelian C^*-algebra A to another C^*-algebra B has the property that $\text{Sp}(\rho(a)) \subseteq \text{Hull}(\text{Sp}(a))$ for all $a \in A$(Douglas [5]). Hence $(A = L^\infty,\ B = B(H^2), \rho(\varphi) = T_\varphi)\text{Sp}(T_\varphi) \subseteq \text{Hull}(\text{Sp}(\varphi))(\varphi \in L^\infty)$.

For G a partially ordered group let $H^\infty = H^\infty(G)$be the set of all $\varphi \in L^\infty$ such that $\varphi \in H^2$. Then H^∞ is a closed subalgebra of the Banach algebra L^∞ (since for $\varphi \in L^\infty$ we have $\varphi \in H^\infty$iff $\varphi H^2 \subseteq H^2$).

Preposition (1.3.21)[23]: Let G be a partially ordered group and $\varphi, \psi \in L^\infty(G^\wedge)$. If φ^- or $\psi \in H^\infty(G)$ then $T_{\varphi\psi} = T_\varphi T_\psi$.

Proof. If $\psi \in H^\infty$then $\psi H^2 \subseteq H^2$implies $T_\varphi T_\psi(f) = T_\varphi P(\psi f) = T_\varphi(\psi f) = P(\varphi\psi f) = T_{\varphi\psi}(f)(f \in H^2)$, so $T_{\varphi\psi} = T_\varphi T_\psi$. If on the other hand $\varphi^- \in H^\infty$ then $(T_{\varphi\psi})^* = T_{\psi^-\varphi^-} = T_{\psi^-}T_{\varphi^-}$ (by what we just shown) $= T_{\varphi^*}T_{\psi^*}$, thus $T_{\varphi\psi} = T_\varphi T_\psi$.

If G is a partially ordered group then we denote by $\mathcal{J}^r(G)$ the C^*-subalgebra of $B(H^2)$ generated by all $T_\varphi(\varphi \in C(G^\wedge))$. We call $\mathcal{J}^r(G)$ the reduced Toeplitz algebra of G. For $x \in G^+$, let U_x be the isometry $T_{\varepsilon(x)}$ and Q_x be the projection $1 - U_xU_x^*$, and let U denote the map $G^+ \to \mathcal{J}^r(G),\ x \mapsto U_x$. U is a semigroup of isometries and $x \leq y$ in G^+ is equivalent to $Q_x \leq Q_y$ in $K(\mathcal{J}^r(G))$ (as we saw already in the proof of Preposition (1.3.4)). these projections Q_x commute. Finally since $(P_G)^- = C(G^\wedge)$ it is easily checked (using (Lemma (1.3.19)) that $U_x(x \in G^+)$ generated $\mathcal{J}^r(G)$.

Lemma (1.3.22)[23]: Let G be a partially ordered group and let J be the linear span in $\mathcal{J}^r(G)$ of all $T_{\varphi_1}T_{\varphi_2} \dots T_{\varphi_n} - T_{\varphi_1} - \varphi_n(\varphi_1, \dots, \varphi_n \in P_G)$. Then $S \in J$ implies that $S = SQ_x$ for some $x \in G^+$.

Proof. If S_1 and S_2 are in J and $S_j = S_jQ_{x_j}$; then $(S_1 + \lambda S_2)Q_{x_1+x_2} = S_1Q_{x_1}Q_{x_1+x_2} + \lambda S_2Q_{x_2}Q_{x_1+x_2} = S_1Q_{x_1} + \lambda S_2Q_{x_2} = S_1 + \lambda S_2(\lambda \in \mathbf{C})$. This calculation shows that is suffices to prove the theorem for S of the form $S = T_{\varphi_n}T_{\varphi_{n-1}} \dots T_{\varphi_1} - T_{\varphi_n \dots \varphi_1}(\varphi_1, \dots, \varphi_n \in$

P_G). However since P_G is the linear span of all $\varepsilon(x)(x \in G)$ it follows that we may, again without loss of generality, assume each $\varphi_i = \varepsilon(y^i)$ for some y^i in G. In this case choose $x \in G^+$ such that $x \geq$ all the elements $-y^i,\ -(y^1 + y^2),\ \dots, -(y^1 + \cdots + y^n)$. Then

$$\begin{aligned} SU_x &= T_{\varepsilon(y^n)} T_{\varepsilon(y^{n-1})} \dots T_{\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} \\ &= T_{\varepsilon(y^n)} T_{\varepsilon(y^{n-1})} \dots T_{\varepsilon(y^2)\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} = \cdots \\ &= T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} - T_{\varepsilon(y^n)\dots\varepsilon(y^1)\varepsilon(x)} = 0 \end{aligned}$$

(true by Preposition (1.3.21) and since $\varepsilon(y^i) \dots\ \varepsilon(y^1)\varepsilon(x) \in H^\infty, i = 1, \dots, n$). Thus $SU_x = 0$, so $SU_x U_x^* = S(1 - Q_x) = 0$, implying $S = SQ_x$.

Theorem (1.3.23)[23]: If G is a partially ordered group then

(i) $(Q_x)_{x\in G^+}$ is an approximate unit for $K(\mathcal{J}^r(G))$.
(ii)if $\in L^\infty(G^\wedge)$, then $T_\varphi \in K(\mathcal{J}^r(G))$ if and only if $\varphi = 0$.

Proof. (i). $(G^+, \leq)$ is a direct set, so $(Q_x)_{x\in G^+}$ is a net. Let J be defined as in Lemma (1.3.22). If $\varphi, \varphi_1, \dots, \varphi_n \in P_G$ then $T_\varphi(T_{\varphi_1} T_{\varphi_2} \dots T_{\varphi_n} - T_{\varphi_1\dots\varphi_n}) = T_\varphi T_{\varphi_1} \dots T_{\varphi_n} - T_{\varphi\varphi_1\dots\varphi_n} + T_{\varphi\varphi_1\dots\varphi_n} - T_\varphi T_{\varphi_1\dots\varphi_n}$ is in J. Hence J^- is a closed ideal in $\mathcal{J}^r(G)$. By Lemma (1.3.22), $S \in J$ implies $S = SQ_x$ for some $\in G^+$, so we have $\lim_y TQ_y = T(T \in J^-)$. Thus $(Q_x)_{x\in G^+}$ is an approximate unit for J^-. Since all $Q_x \in K(\mathcal{J}^r(G)), J^- \subseteq K(\mathcal{J}^r(G))$, and since all $Q_x \in J^-, K(\mathcal{J}^r(G))/J^-$ is abelian, implying that $J^- \supseteq K(\mathcal{J}^r(G))$. Thus $J^- = K(\mathcal{J}^r(G))$ and $(Q_x)_{x\in G^+}$ is an approximate unit for $K(\mathcal{J}^r(G))$.

(ii). Let $\varphi \in L^\infty$ and $T_\varphi \in K(\mathcal{J}^r(G))$. Then $T_\varphi = \lim_x T_\varphi Q_x$, so $0 = \lim_x T_\varphi U_x U_x^*$, thus $0 = \lim_x \|T_\varphi U_x\| = \lim_x \|T_{\varphi\varepsilon(x)}\| = \lim_x \|\varphi\varepsilon(x)\|_\infty = \|\varphi\|_\infty$, so $0 = \varphi$.

Part (ii) of the above theorem generalizes the classical result that 0 is the only compact Toeplitz operator (relative to $G = \mathbf{Z}$) . $K(\mathcal{J}^r(G))$ is $K(H^2(\mathbf{Z}))$., the ideal of all compact operations on $H^2(\mathbf{Z})$.

Corollary (1.3.24)[23]: If $\phi, \psi \in C(G^\wedge)$ then $T_\phi T_\psi - T_{\phi\psi} \in K(\mathcal{J}^r(G))$.

Proof. Since $(P_G)^- = C(G^\wedge)$ it suffices to show the result for $\phi, \psi \in P_G$. But this case is obvious from the proof of Theorem (1.3.23).

Theorem (1.3.25)[23]: Let G be a partially ordered group.

(i) The map

$$C(G^\wedge) \to \mathcal{J}^r(G)/\mathcal{J}^r(G)\ \phi \mapsto T_\phi + K(\mathcal{J}^r(G))$$

is a $*$- isomorphism.

(ii) If $S \in \mathcal{J}^r(G)$ then there exists unique $\phi \in C(G^\wedge)$ and unique $K \in K(\mathcal{J}^r(G))$ such that $S = T_\phi + K$.

Proof. Let ρ denote the map in (i). Then ρ is clearly $*$-linear and by Corollary (1.3.24)ρ is multiplicative. ρ is injective by Theorem (1.3.23) and surjective since $T_\phi(\varphi \in C(G^\wedge))$ generate $\mathcal{J}^r(G)$. This proves (i), (ii) follows immediately from (i).

Lemma (1.3.26)[23]: Let H_0 be a dense linear submanifold of a Hilbert space H, and $(S_\lambda)_{\lambda\in\Lambda}$ a net in $B(H)$ such that $\lim_\lambda(S_\lambda, f, g)$ exists for all $f, g \in H_0$ and that there is a positive number μ such that $|(S_\lambda, f, g)| \le \mu\|f\|\|g\|(\lambda \in \Lambda, f, g \in H)$. Then there exists $S \in B(H)$ such that $S \in B(H)$ such that $S = \lim_\lambda S_\lambda$in the weak operator topology on $B(H)$.

For a proof, see Halmos [30].

Lemma (1.3.27)[23]: Let G be a discrete abelian group and suppose that the matrix $(a_{x,y})_{x,y\in G}$of $S \in B(L^2(G^\wedge))$ with respect to the orthonormal basis $(\varepsilon(x)_{x\in G})$ is a Laurent matrix (i.e. $a_{x+z,y+z} = a_{x,y}(x, y, z \in G)$). Then S is a multiplication, $S = M_\varphi$ for some $\varphi \in L^\infty(G^\wedge)$.

(Explicitly: $a_{x,y} = \left(S(\varepsilon(y)), \varepsilon(x)\right)$.)

For a proof, see Murphy [32].

Theorem (1.3.28)[23]: Let G be a partially ordered group, and let $S \in B(H^2(G))$. Then S is a Toeplitz operator (relative to G) if and only if $U_x^*SU_x = S(x \in G^+)$.

Proof. If $S = T_\varphi$ for some $\varphi \in L^\infty$ then $U_x^*SU_x = T_{\overline{\varepsilon(x)}}T_\varphi T_{\varepsilon(x)} = T_{\overline{\varepsilon(x)}\varphi\varepsilon(x)} = T_\varphi = S$ (as $\varepsilon(x) \in H^\infty$). Conversely suppose that $U_x^*SU_x = S(x \in G^+)$. Define $S_x \in B(L^2)$ be setting $S_x(f) = \overline{\varepsilon(x)}\, SP\varepsilon(x)f$, for $x \in G^+$, and note that $\|S_x\| \le \|S\|$. Also for $f, g \in H^2$,$(S_x\, f, g) = \left(\overline{\varepsilon(x)}\, SP\varepsilon(x)f, g\right) = (U_x^*SU_xf, g) = (Sf, g)$.

Now let $\varphi_1, \varphi_2, \in P_G$ and put $\mu_x = (S_x\varphi_1, \varphi_2)$. We show that the net $(\mu_x)_{x\in G^+}$converges by showing that there exists $x_0 \in G^+$such that $\mu_x = \mu_{x_0}$ for $x \ge x_0$. Certainly there exists $x_0 \in G^+$ such that $\varphi_1, \varphi_2, \in \overline{\varepsilon(x_0)}H^2$. Let $\psi_j = \varepsilon(x_0)\varphi_j$, so $\psi_j \in H^\infty$. Now if $x \ge x_0$ then$\mu_x = \left(S_x\overline{\varepsilon(x_0)}\psi_1, \overline{\varepsilon(x_0)}\psi_2\right) = (S\varepsilon(x - x_0)\psi_1, \varepsilon(x - x_0)\psi_2) = \left(S_{x-x_0}\psi_1, \psi_2\right) = (S\psi_1, \psi_2)$(as $\psi_1\psi_2 \in H^2 = \mu_{x_0}$. Since $(P_G)^- = L^2$ it follows from Lemma (1.3.26) that there exists $T \in B(L^2)$ such that $T = \lim_x S_x$ in the weak operator toplogy. Let $(a_{x,y})_{x,y\in G}$ be the matrix of T relative to the basis $(\varepsilon(x))_{x\in G}$ of L^2. If $y, z \in G$ and $x \in G^+$, then

$$a_{y+x,z+x} = \left(T\varepsilon(z)\varepsilon(x), \varepsilon(y)\varepsilon(x)\right) = \lim_t\left(S_t\varepsilon(x)\varepsilon(z), \varepsilon(x)\varepsilon(y)\right) = \lim_t\left(S_{t+x}\varepsilon(z), \varepsilon(y)\right) = \lim_t\left(S_t\varepsilon(z), \varepsilon(y)\right)$$

(since $\lim_t \alpha_{t+x} = \lim_t \alpha_t$). Thus $a_{y+x,z+x} = a_{y,z}$, and one can now immediately extend this eqation to arbitrary $x \in G$ since $G = G^+ - G^+$. Hence by Lemma (1.3.26), $T = M_\varphi$ for some $\varphi \in L^\infty$. clearly for $f, g \in H^2$, $\left(T_\varphi f, g\right) = (\varphi f, g) = (Tf, g) = \lim_x(S_xf, g) = (Sf, g)$as $(S_xf, g) = (Sf, g)$.Thus $S = T_\varphi$.

The next preposition is important- it shows that $H^\infty(G)$ displays "analytic behaviour".

Preposition (1.3.29)[23]: Let G be a partially ordered group. If φ and $\bar{\varphi} \in H^\infty(G)$ then $\varphi \in$ **C**1.

Proof. If $x \in G^+$ and $\overline{\varepsilon(x)} \in H^2$ then - $x \in G^+$, so $x = 0$. Now $\varphi, \bar{\varphi} \in H^\infty$and $x \in G^+, x > 0$ implies $0 = \left(\bar{\varphi}, \overline{\varepsilon(x)}\right) = \int \bar{\varphi}\varepsilon(x)\ as\ \overline{\varepsilon(x)} \in (H^2)^\perp$. Hence $\int \bar{\varphi}\varepsilon(x) = 0$, i.e. $\left(\varphi, \varepsilon(x)\right) = 0$. But $\left(\varphi, \varepsilon(x)\right) = 0\ for\ x \in G\backslash G^+$ also, since $\varphi \in H^\infty$. Thus $\varphi \in \mathbf{C}\varepsilon(0) = \mathbf{C}1$.

If G is a partially ordered group and $\varphi \in H^\infty(G)$ we say that T_φ is an analytic Toeplitz operator (relative to G). Of course T_φ is subnormal (it is restriction of M_φ). All analytic Toeplitz operators commute. The map $H^\infty \to B(H^2)\,, \varphi \mapsto T_\varphi$, is an isometric algebra isomorphism onto the closed subalgebra of all analytic Toeplitz operators.

Theorem (1.3.30)[23]: Let G be a partially ordered group.

(i) If $S \in B\left(H^2(G)\right)$ then S is an analytic Toeplitz operator (relative to G) if and only if $U_x S = S U_x (x \in G^+)$.
(ii) The analytic Toeplitz operators relative to G form a maximal commutative subalgebra of $B\left(H^2(G)\right)$.
(iii) If $\varphi \in H^\infty$ then $\mathrm{Sp}\left(T_\varphi\right) = \mathrm{Sp}_{H^\infty}(\varphi)$.
(iv) Every analytic Toeplitz operator has connected spectrum.

Proof.

(i) if S is an analytic Toeplitz operator then $SU_x = U_x S$ since U_x are analytic Toeplitz operators. Conversely if $SU_x = U_x (x \in G^+)$ then $U_x^* S U_x = S$, so $S = T_\varphi$ for some $\varphi \in L^\infty$by Theorem (1.3.28).

Now for $x, y \in G^+$, $\left(\varphi, \varepsilon(x-y)\right) = \left(\varphi\varepsilon(y), \varepsilon(x)\right) = \left(T_\varphi U_y \varepsilon(0), \varepsilon(x)\right) = \left(U_y T_\varphi \varepsilon(0), \varepsilon(x)\right) = \left(T_\varphi \varepsilon(0), \varepsilon(x-y)\right)$. Thus if $x - y \notin G^+$, then $\left(\varphi, \varepsilon(x-y)\right) = 0$. So $\varphi \in H^\infty$. This proves (i), and (ii) follows immediately from this.

(iii) Let A be the maximal commutative subalgebra of B(H^2) of all analytic Toeplitz operators. Then $\mathrm{Sp}_A\left(T_\varphi\right) = \mathrm{Sp}\left(T_\varphi\right)$ for $\varphi \in H^\infty$. But $\mathrm{Sp}_A\left(T_\varphi\right) = \mathrm{Sp}_{H^\infty}(\varphi)$ since the map $H^\infty \to A\,, \varphi \mapsto T_\varphi$, is an isomorphism. This proves (iii).

(iv) Let X be the character space of H^∞. If $\varphi \in H^\infty$ is an idempotent then $\varphi = \varphi^2$, thus $= \bar{\varphi} \in H^\infty$, so $\varphi \in \mathbf{C}1$ by Proposition (1.3.29). Thus $\varphi = 0$ or 1. Since H^∞ thus has no non- trivial idempotents it follows from the Shilov Idempotent Theorem that X is connected. Now if $\varphi \in H^\infty$ and $\varphi^\wedge$ denotes its Gelfand transform then $\mathrm{Sp}_{H^\infty}(\varphi) = \varphi^\wedge(\chi)$ is connected . i.e. $Sp\left(T_\varphi\right)$ is connected.

Theorem (1.3.31)[23]: If G is a partially ordered group then $\mathcal{J}^r(G)$ and $K\left(\mathcal{J}^r(G)\right)$ are irreducible algebras on $H^2(G)$. Moreover if $G \neq 0$ then $\lim_x Q_x = 1$ in the strong operator topology on $B(H^2(G))$.

Proof. $\mathcal{J}^r(G)$ is irreducible iff its commutant $B = \mathbf{C}1$ iff $0, 1$ are the only projections in B (since B is a von Newmann algebra). Now for $\in B, QU_x = U_x Q(x \in G^+)$, so Q is an analytic Toeplitz operator, thus $\mathrm{Sp}(Q)$ is connected. Thus if Q is a projection, then $\mathrm{Sp}(Q) = \{0\}$ or $\{1\}$, so $Q = 0$ or 1. Hence $B = \mathbf{C}1$ and $\mathcal{J}^r(G)$ is irreducible on H^2.

If $G = 0$ then $\dim(H^2) = 1$ so $K(\mathcal{J}^r(G))$ is irreducible on H^2. So we suppose that G is non-zero. Let $M = \left(K(\mathcal{J}^r(G))H^2\right)^-$. If $M = 0$ then $(\mathcal{J}^r(G)) = 0$, so $Q_x = 0(x \in G^+)$, thus $G^+ = 0$ which implies $G = 0$. Thus $M \neq 0$. Since M reduces $\mathcal{J}^r(G), M = H^2$. If $f \in K(\mathcal{J}^r(G))H^2$ then $f = T_1 f_1 + \cdots + T_n f_n$ for some $T_1, \ldots, T_n \in K(\mathcal{J}^r(G))$ and some $f_1, \ldots, f_2$ in H^2.

Thus $\lim_x Q_x f = \lim_x Q_x T_1 f_1 + \cdots + \lim_x Q_n T_n f_n = T_1 f_1 + \cdots + T_n f_n = f$, because $T_j = \lim_x Q_x T_j$ $(j = 1, \ldots n)$. Hence $f = \lim_x Q_x f$ $(f \in H^2)$ as $K(\mathcal{J}^r(G))H^2$ is dense in H^2 , so $\lim_x Q_x = 1$ in the strong operator topology on $B(H^2)$.

Now suppose that N is an invariant closed subspace of H^2 for $K(\mathcal{J}^r(G))$, and $f \in N, T \in \mathcal{J}^r(G)$. Then $Tf = \lim_x TQ_x f$ *is in* N, since $TQ_x f \in N(x \in G^+)$. Thus N is an invariant subspace for $\mathcal{J}^r(G)$, so $N = 0$ or H^2. We have thus shown $K(\mathcal{J}^r(G))$ is irreducible on H^2.

Recall that if G is a partially ordered group then the map $U: G^+ \to \mathcal{J}^r(G)$ is a semigroup of isometries, so it induces unique $*$- homo-morphism $U^*: \mathcal{J}(G) \to \mathcal{J}^r(G)$. Since $U_x(x \in G^+)$ generate $\mathcal{J}^r(G), U^*$ is onto. We can thus regard U^* as an irreducible representation of $\mathcal{J}(G)$ on $H^2(G)$. This representation is not always faithful as the next example shows:

For Example, $M = N\backslash\{1\}$ is a cone on $\mathbf{Z}$. Thus $(\mathbf{Z}, M)$ is a partially ordered group. Note that 2 and 3 are not comparable for the partial order $\leq_M$. Let $G_1 = (\mathbf{Z}, M)$ and $G_2 = (\mathbf{Z}, N)$. The identity map $\varphi: G_1 \to G_2$ is a positive homomorphism, so it induces a $*$-homomorphism $\varphi^*: \mathcal{J}(G_1) \to \mathcal{J}(G_2)$ and this is surjective since φ is surjective. Hence the restriction $\varphi^*: K(\mathcal{J}(G_1)) \to K(\mathcal{J}(G_2))$ is surjective, and thus non- zero. If $K(\mathcal{J}(G_1))$ were simple then this restriction map φ^* would be a $*$-isomorphism, so $2 \leq_N 3$ implies $q_2 \leq q_3$ in $(\mathcal{J}(G_2))$, so $\varphi^*(q_2) \leq \varphi^*(q_3)$, implying $q_2 \leq q_3$ in $K(\mathcal{J}(G_1))$, so $2 \leq_M 3$, which is false. Thus $K(\mathcal{J}(G_1))$ is not simple. However it is easily seen that all Q_x^1 are of finite rank $(x \in M)$, so $K(\mathcal{J}^r(G_1)) \subseteq K(\mathcal{J}^r(G_2))$ as $(Q_x^1)_{x \in M}$ are an approximate unit for $K(\mathcal{J}^r(G_1))$ and since $K(\mathcal{J}(G_1))$ is irreducible on $H^2(G_1)$ we therefore deduce that $K(\mathcal{J}^r(G_1)) = K(H^2(G_1))$. In particular $K(\mathcal{J}^r(G_1))$ is simple. Since $K(\mathcal{J}(G_1))$ is not simple, the map $U^*: \mathcal{J}(G_1) \to T^r(G_1)$ is not injective.

Theorem (1.3.32)[23]: If G is an ordered group then the canonical map $U^*: \mathcal{J}(G) \to \mathcal{J}^r(G)$ is a faithful irreducible representation of $\mathcal{J}(G)$ on $H^2(G)$.

Proof. The map $U: G^+ \to \mathcal{J}^r(G)$ is a nonunitary semigroup of isometries, so by Theorem (1.3.15) U^* is injective.

The idea is to show that a number of the stronger results we proved earlier are in fact "best possible". For example if G is torsion- free partially ordered group for which $K(\mathcal{J}(G))$ is simple we show G is isomorphic to an ordered subgroup of **R** (cf. Corollary (1.3.18)) one can interpret Theorem (1.3.15) as saying there is essentially only on candidate for the title "Toeplitz algebra" if G is an ordered group. More specifically it implies that if B is any C^*-algebra generated by a nonunitary semigroup of isometries β over G then $\beta^*: \mathcal{J}(G) \to B$ is a $*$-isomorphism. We show that this result characterizes the ordered groups amongst the torsion- free partially ordered groups.

If G is an abelian group we call a cone M in G maximal if M is not contained in any other cone of G. A simple application of Zorn's Lemma implies that every cone of G is contained in a maximal cone of G. The following elementary result is probably known, but we include a proof for the sake of completeness.

Lemma (1.3.33)[23]: If G is a torsion- free abelian group and M a cone of G then M is a maximal cone of G if and only if $(G, \leq_M)$ is an ordered group (i.e. $\leq_M$ is a total ordering).

Proof. It is trivial that if $(G, \leq_M)$ is totally ordered, then M is maximal (this does not require G to be torsion- free). Suppose conversely that M is maximal. First, let $x \in G\{0\}$ such that $nx \in M$ for some positive integer n. We show that $x \in M$: define $N = \{y + mx: y \in M, m \in N\}$. Clearly $0 \in N, N + N \subseteq N$ and $G = N - N$. Suppose that $z \in N \cap (-N)$, so that $z = y_1 + m_1 x = -y_2 - m_2 x$ for some $y_1, y_2 \in M$, and some $m_1, m_2 \in \mathbf{N}$. Thus $0 \leq_M n(y_1 + y_2) = (m_1 + m_2)nx \leq_M 0$, so $n(y_1 + y_2) = 0 = -(m_1 + m_2)nx$ implying that $y_1 + y_2 = 0 = (m_1 + m_2)x$ (since G is torsion- free), thus $y_1 = y_2 = 0$ (since $y_1, y_2 \geq_M 0$), and $m_1 + m_2 = 0$ implies $m_1, m_2 = 0$. Hence $z = 0$, implying that $N \cap (-N) = 0$. Thus N is a cone, and $N \supseteq M$ implies $N = M$, so $x \in M$.

Now suppose only that $x \in G \backslash (-M)$. Again let $N = \{y + mx: y \in M, m \in \mathbf{N}\}$and again we have $0 \in N, N + N \subseteq N$ and $G = N - N$. If $z \in N \cap (-N)$ then $z = y_1 + m_1 x = -y_2 - m_2 x$ for some $y_1, y_2 \in M$, and some $m_1, m_2 \in \mathbf{N}$. So $y_1 + y_2 = -(m_1 + m_2)x$ thus $n(-x) \in M$ where $n = m_1 + m_2$. If $n > 0$ then by the earlier part of this proof, $-x \in M$, so $x \in -M$, which is false. Thus $n = 0$ implies $m_1 = m_2 = 0$, so $y_1 + y_2 = 0$, which implies $y_1, y_2 = 0$ (since $0 \leq_M y_1, y_2$), so $z = 0$. We therefore have $N \cap (-N) = 0$, thus N is a cone, and since $M \subseteq N, M = N$. Thus $x \in M$. We have shown that $G = M \cup (-M)$ i.e. $(G, \leq_M)$ is totally ordered.

The hypothesis that G is torsion- free is necessary in Lemma (1.3.33), since ordered groups are torsion- free.

Theorem (1.3.34)[23]: Let $(G, \leq)$ be a torsion- free partially ordered group such that for every unital C^*-algebra B and every nonunitary semigroup of isometrics $\beta: G^+ \to B, \beta^*$ is injective. Then $(G, \leq)$ is an ordered group.

Proof. G^+ is contained in a maximal cone M, so if φ denotes id_G then φ is a positive homomorphism from $G_1 = (G, G^+)$ to $G_2 = (G, M)$, and so induces a $*$-homomorphism φ^* from $\mathcal{J}(G_1)$to $\mathcal{J}(G_2)$. Since φ is surjective, so is φ^*. The semigroup of isometries $\beta: G_1^+ \to T(G_2), x \to \varphi^*(V_x)$, is nonunitary (for if $\beta(x)$ is unitary then $\varphi^*(V_x) = V_{\varphi(x)}$ is unitary, so $\varphi(x) = 0$, i.e. $x = 0$). Hence $\beta^* = \varphi^*$ is injective. Thus if $x, y \in G^+$ then we may suppose

$x \leq_M y$ (since (G, M) is totally ordered). Hence $q_x \leq q_y$ in $K(\mathcal{J}(G_2))$, so $\varphi^*(q_x) \leq \varphi^*(q_y)$, implying that $q_x \leq q_y$ in $K(\mathcal{J}(G_2))$, so $x \leq y$. More generally if $x, y \in G$ then there exists $z \in G^+$ such that $x, y \in z$ imply $z - y, z - x \in G^+$, so $z - y, z - x$ are comparable in $(G, \leq)$, thus x, y are comparable in $(G, \leq)$. Thus $(G, \leq)$ is totally ordered.

Theorem (1.3.35)[23]: If G is a torsion- free partially ordered group for which $K(\mathcal{J}(G))$ is simple then G is order isomorphic to an ordered subgroup of **R**.

Proof. (Two partially ordered groups G_1, G_2 are order isomorphic if there exists a bijective map $\psi: G_1 \to G_2$ such that ψ and ψ^{-1} are positive homeomorphisms.)

First we show that G is simple: Let I be an ideal of G, and let $i: I \to G$ and $\varphi: G \to G/I$ be the inclusion and quotient homeomorphisms respectively. Since φ is surjective, so is $\varphi^*: \mathcal{J}(G) \to \mathcal{J}(G/I)$, and hence also the restriction map $\varphi^*: K(\mathcal{J}(G)) \to K(\mathcal{J}(G/I))$. As $K(\mathcal{J}(G))$ is simple this restriction map φ^* is zero or injective. In the first case we have $K(\mathcal{J}(G/I)) = 0$, so $G/I = 0$, so $G = I$. In the second case for $x \in I^+, \varphi^*(q_x) = q_{\varphi(x)} = q_0 = 0$, so $q_x = 0$, so $x = 0$. Thus $I^+ = 0$ implying that $I = 0$. This shows G is simple.

Now we show G is totally ordered: Using the same trick as in the proof of Theorem (1.3.34), there is a maximal cone M containg G^+. Let $\psi = \mathrm{id}_G, G_1 = (G, G^+)$, and $G_2 = (G, M)$. Thus ψ is a positive homomorphism from $G_1$to G_2. As ψ is surjective, so is $\psi^*: \mathcal{J}(G_1) \to \mathcal{J}(G_2)$, implying that the restriction map $\psi^*: K(\mathcal{J}(G_1)) \to K(\mathcal{J}(G_2))$ is zero or injective (again we are using the simplicity of $K(\mathcal{J}(G_1))$). In the first case $K(\mathcal{J}(G_2)) = 0$ which implies $G = 0$, so G is order isomorphic to the ordered subgroup 0 of **R**. In the second case if $x, y \in G^+$ we may suppose that $x \leq_M y$, so$q_x \leq q_y$ in $K(\mathcal{J}(G_2))$, i.e. $\varphi^*(q_x) \leq \varphi^*(q_y)$, so $q_x \leq q_y$ in $K(\mathcal{J}(G_1))$, thus $x \leq y$. This implies that $(G, \leq)$is totally ordered.

Thus G is a simple ordered group, so $G = F(G) = \{x \in G: \text{for all } y > 0,\ |x| \leq ny \text{ for some } n \in \mathbf{N}\}$.(for if $x \in G$ and $x > 0$ then $I_x = \{z \in G: |z| \leq nx \text{ for some } n \in \mathbf{N}\}$ is a non- zero ideal of G, so $I_x = G$.) Thus G is an Archimedean group in the terminology of Rudin [36], and by a well know result G is order group isomorphic to an ordered subgroup of **R** ([36]).

One can thus summarize Theorem (1.3.35) and Douglas' result (Corollary (1.3.18)) as follows: for G a torsion- free partially ordered group $K(\mathcal{J}(G))$ is simple iff G is a simple ordered group iff G is order isomorphic to an ordered subgroup of **R**.

Recall that a C^*-algebra A is elementary if there is a Hilbert space H such that A is $*$-isomorphic to $K(H)$.

Theorem (1.3.36)[23]: Let G be a torsion- free partially order group. Then $K(\mathcal{J}(G))$ is elementary if and only if G is order isomorphic to 0 or **Z**.

Proof. If $G = 0$ then $\mathcal{J}(G) = \mathbf{C}$, so $K(\mathcal{J}(G)) = 0 = K(H)$ for $H = 0$. If $G = \mathbf{Z}$then by Theorem (1.3.32), $K(\mathcal{J}(G))$ is $*$-isomorphic to $K(\mathcal{J}^r(G))$. but $K(\mathcal{J}^r(G)) = K(H^2(G))$ since all $Q_x (x \in G^+)$are finite rank, and $K(\mathcal{J}^r(G_1))$ is irreducible on $H^2(G)$.

Conversely, suppose that $K(\mathcal{I}(G))$ is elementary and that $\beta: K(\mathcal{I}(G)) \to K(H)$ is a $*$-isomprphism for some Hilbert space H. Then in particular $K(\mathcal{I}(G))$ is simple, so we may assume that G is an ordered subgroup of **R**, and without loss of generally suppose also $G \neq 0$. Since all $\beta(q_x)(x \in G^+)$ have finite rank in $K(H)$ we may choose $x \in G, x > 0$, such that the rank of $\beta(q_x)$ is minimal. Then x is a smallest positive element of G. Hence $G = \mathbf{Z}x$, so G is order isomorphic to **Z**.

Chapter 2
Lower Bounds with New Estimate and Matrix Valued

We show the following theorem: given $\delta, 0 < \delta < 1/3$ and $n \in \mathbb{N}$ there exists an $(n+1) \times n$ inner matrix function $F \in H^{\infty}_{(n+1)\times n}$ such that $I \geq F^*(z)F(z) \geq \delta^2 I \ \forall z \in D$, but the norm of any left inverse for F is at least $\left[\frac{\delta}{1-\delta}\right]^{-n} \geq \left(\frac{3}{2}\delta\right)^{-n}$. We improve an estimate of Fuhrmann and Vasyunin in the vector valued corona theorem. We illustrate that the norm of the solution of the H^2 corona problem in the polydisk $\mathbb{D}^n$ grows at most proportionally to $\sqrt{n}$. Our approach is based on one that was originated by Andersson. In the disk it essentially depends on Green's Theorem and duality to obtain the estimate. In the polydisk we use Riesz projections to reduce the problem to the disk case.

Section (2.1): The Matrix Corona Theorem and the Codimension One Conjecture

The Operator Corona Problem is the problem of finding a necessary and sufficient condition for a bounded operator-valued function $F \in H^{\infty}_{E_* \to E}$ to have a left inverse in H^{∞}, i.e. a function $G \in H^{\infty}_{E_* \to E}$ such that

(B) $$G(z)F(z) \equiv I \ \forall z \in \mathbb{D}.$$

If $\dim E < \infty$we call it the *Matrix* Corona Problem.

The equations of type (B) are sometimes called in the literature the Bezout equations, and "B" here is for Bezout. The simplest necessary condition for (B) is

$$(C) F^*(z)F(z) \geq \delta^2 I\,, \qquad \forall z \in \mathbb{D} \qquad (\delta > 0)$$

(the tag "C" is for Carleson).

If the condition (C) implies (B), we say that the Operator (Matrix) Corona Theorem holds.

The Operator Corona Theorem plays an important role in different areas of analysis, in particular in Operator Theory (angles between invariant subspaces, unconditionally convergent spectral decompositions, see [44], [45], [52], [53]), as well as in Control Theory and other applications.

Let us discuss the cases when the Operator Corona Theorem holds.

The first case is $\dim E = 1, \dim E_* = n < \infty$. In this case $F = [f_1, f_2, \dots, f_n]^T, G = [g_1, g_2, \dots, g_n]$and it is simply the famous Carleson Corona Theorem [40], see also [42], [45].

Later, using the ideas from the T. Wolff's proof of the Carleson Corona Theorem, M. Rosenblum [47], V. Tolokonnikov [50] and Uchiyama [55] independently proved that the Operator Corona Theorem holds if $\dim E = 1, \dim E_* = \infty$.

Then, using simple linear algebra argument, P. Fuhrman [41] and V. Vasyunin [50] independently proved that the Operator Corona Theorem holds if $\dim E < \infty, \dim E_* \leq \infty$. This theorem is now commonly referred to as the Matrix Corona Theorem.

A trivial observation: if $F(z)E = E_* \forall z \in \mathbb{D}$, then the left invertibility (C) implies the invertibility of $F(z)$, and so we can simply put $G = F^{-1}$. So in this case the Operator Corona Theorem holds as well.

As for the general Operator Corona Theorem, it was shown by the author ([51], see also [52]) that it fails in the general case $\dim E = +\infty$.

Let us discuss the codimension one conjecture first. As we just mentioned above, the Operator Corona Theorem fails if $\dim E = \infty$, but it holds if $F(z)E = E_* \forall z \in \mathbb{D}$. So, what happens if the operators $F(z)$ are "almost" onto, namely if $\operatorname{codim}(F(z)E = 1 \forall z \in \mathbb{D}$? It was conjectured by N. Nikolski and the author that in this case the Operator Corona Theorem holds. It was also conjectured that the Matrix Corona Theorem holds for all $(n+1)\times n$ matrix-valued functions in $H^\infty_{(n+1)\times n}$ with uniform (independent on n) estimates. The Operator Corona Theorem holds in the matrix case by Fuhrmann–Vasyunin, the question is only in uniform estimates.

Besides the naïve reason that the codimension one case is very close to the case of invertible operator-function, there were some more deep facts that lead to the codimension one conjecture. Namely, see Lemma (2.1.4) and Theorem (2.1.9) below, with each $(n+1)\times n$ matrix function in H^∞one can canonically associate an $(n+1)\times n$ matrix (vector) with the same best norm of a left inverse.

Since in $n+1$ case it is possible to obtain the estimate in the Corona Theorem independent of n, it seemed reasonable to propose the codimension one conjecture, at least in its matrix version.

It is interesting that the above canonically associated vector (the so called co-analytic complement) plays an important role in the construction.

Other important results in are Theorem (2.1.9) and Corollary (2.1.6) which clarify the role of co-analytic orthogonal complements in the Operator Corona Problem.

The second problem we are dealing with is the problem of estimates in the matrix case. It was shown by V. Vasyunin [50] (see also [54]) that if $\dim E = n < \infty$ and we normalize the function $F \in H^\infty_{E\to E_*}$ as

$$I \geq F^*(z)F(z) \geq \delta^2 I\,, \qquad \forall z \in \mathbb{D},$$

then one can always find a left inverse $G \in H^\infty_{E\to E_*}, GF \equiv I$, such that $\|G\|_\infty \leq C(n,\delta)$, where

$$C(n,\delta) = C\sqrt{n}\cdot\delta^{-2n}\log\delta^{2n}.$$

Recently, T. Trent [54] was able to improve this estimate with

$$C(n,\delta) = C\cdot\delta^{-n-1}\log\delta^{-2n},$$

but in both cases $C(n,\delta)$ grows exponentially in n.

On the other hand, it had been shown in [51], see also [52] that $C(n,\delta)$ cannot be uniformly (in n) bounded, namely that for any sufficiently small $\delta > 0$ one can find $F \in H^\infty_{E\to E_*}, \dim E = n$, satisfying $I \geq F^*(z)F(z) \geq \delta^2 I\,, \forall z \in \mathbb{D}$, and such that any left inverse $G \in H^\infty_{E\to E_*}$ satisfies the inequality

$$\|G\|_\infty \geq c(n,\delta) = C\delta^{-2}\log(\delta^2 n + 1)$$

so $c(n,\delta)$ grows logarithmically in n. From this estimates it is easy to get
that the operator Corona Theorem fails in the general case $\dim E = \infty$.

We prove the lower bound $[\delta/(1-\delta)\,]^{-n}$ which is quite close to Trent's estimate. so his estimate is probably very close to a sharp one. This estimate is obtained for $(n+1)\times n$matrices, so it disproves the codimension one conjecture (its matrix version).

Theorem (2.1.1)[38]: Given$\delta, 0 < \delta < 1/3$ and $n \in \mathbb{N}$ there exists an$(n+1)\times n$ inner matrix function$F \in H^\infty_{(n+1)\times n}$such that

$$I \geq F^*(z)F(z) \geq \delta^2 I\,, \qquad \forall z \in \mathbb{D},$$

but the norm of any left inverse for$F$is at least$[\delta/(1-\delta)\,]^{-n} \geq (\frac{3}{2}\delta)^{-n}$.

Recall that a function $F \in H^\infty_{E\to E_*}$is called inner if operators $F(z)$ are isometries a.e. on $\mathbb{T}$, and an outer if FH^2_E is dense in $H^2_{E_*}$. We will call F co-inner (resp. co-outer) if F^T is inner (resp. outer).

Let us recall that any $F \in H^\infty_{E\to E_*}$admits an inner-outer factorization

$F = F_i F_0$, where $F_i \in H^\infty_{E_1 \to E_*}$ is inner and $F_0 \in H^\infty_{E \to E_1}$is outer. Let us recall also that the inner part F_i (resp. the outer part F_0) is unique up to a constant unitary factor on the right (resp. on the left).

Let us also recall that any z-invariant subspace $M(zM \subset M)$ of H^2_E can be represented as $\Theta H^2_{E_1}$, where E_1 is an auxiliary Hilbert space and $\Theta \in H^\infty_{E_1 \to E}$ is an inner function.

Definition (2.1.2)[38]: We say that an operator valued function $v \in H^\infty_{E \to E_*}$ has a bounded co-analytic (orthogonal) complement if there exists a function $v \in H^\infty_{E_1 \to E_*}$such that $\ker v^T(z) = V(z)E_1$ a.e. on $\mathbb{T}$. The function $\bar{V}$ is called a co-analytic (orthogonal) complement of v.

We will usually skip the word orthogonal, and simply say co-analytic complement.

The reason for the word co-analytic complement is the that the equality $\ker v^T(z) = V(z)E_1$ can be rewritten as

$$\bar{V}(z)E_1 = (v(z)E)^\perp \text{ a.e.on } \mathbb{T}.$$

(Take the inner part of V) we can always assume that the function $V$is inner. Moreover, by taking the outer part of V^T we can always assume that V is also co-outer. So, when we say the co-analytic complement we usually mean $\bar{V}$, where $V$is an inner and co-outer function. Lemma (2.1.3) below states that such inner and co-outer function is unique up to a constant unitary factor on the right.

That if the function v is inner (i.e. operators $v(z)$ are isometries a.e. on $\mathbb{T}$), then the operators $W(z)$, where $W = [v\bar{V}] \in H^\infty_{E \oplus E_1 \to E_*}$ defined by

$$W(z)(e \oplus e_1) = v(z)e + \bar{V}(z)e_1 \quad , e \in E \, , e_1 \in E_1$$

are unitary a.e. on $\mathbb{T}$.

Lemma (2.1.3)[38]: Let $v \in H^\infty_{E \to E_*}$have a bounded co-analytic complement $\bar{V}$, where $V \in H^\infty_{E_1 \to E_*}$ is an inner and co- outer function such that

$$\ker v^T(z) = V(z)E_1 \quad \text{a.e. on } \mathbb{T}.$$

Then

$$\{f \in H^2_{E_*} : v^T f \equiv 0 \qquad\qquad \text{a.e. on } \mathbb{T}\} = VH^2_{E_1}.$$

Moreover, any other such inner and co-outer function $V_2 \in H^\infty_{E_2 \to E_*}$ satisfies $V_2 = V\,U$, where $U: E_2 \to E_1$ is a constant unitary operator.

Proof. Denote

$$M := \{f \in H^2_{E_*} : v^T f \equiv 0 \quad \text{a.e. on } \mathbb{T}\}..$$

Clearly, $VH^2_{E_1} \subset M$. Since $zM \subset M$,

$$M = \tilde{V} H^2_{\tilde{E}} \, ,$$

where $\tilde{V} \in H^\infty_{\tilde{E} \to E_*}$is an inner function.

Since $H^2_{E_1} \subset M = \tilde{V} H^2_{\tilde{E}}$, the preimage $\tilde{V}^{-1}(VH^2_{E_1})$ is a z-invariant subspace of $H^2_{\tilde{E}}$. So it can be represented as $\tilde{V}^{-1}(VH^2_{E_1}) = UH^2_{E_U}$, where $U \in H^\infty_{E_U \to \tilde{E}}$ is an inner function and $E_U$is an auxiliary Hilbert space. Therefore $VH^2_{E_1} = \tilde{V}UH^2_{E_U}$, so the space $E_U$can be identified with E_1 and the function Vcan be factorized as $V = \tilde{V}\,U$. Therefore $V(z)E_1 \subset \tilde{V}(z)E_2$ a.e. on $\mathbb{T}$, and so

$$\ker v(z)^T = V(z)E_1 \subset \tilde{V}(z)\tilde{E} \subset \ker v(z)^T \text{a.e.on } \mathbb{T}.$$

This implies that $V(z)E_1 = \tilde{V}(z)\tilde{E}$and hence $U(z)E_1 = \tilde{E}$a.e. on $\mathbb{T}$. This means that the function U takes unitary values a.e. on $\mathbb{T}$, so U^T is also an inner function. Since V is a co-outer function, Umust be a constant (unitary operator).

The following well-known lemma asserts that a matrix valued function always has a bounded co-analytic complement.

Let $F \in H^\infty_{m\times n}$be a matrix valued function. Since any minor of Fbelongs to H^∞, if it is non-zero on a set of positive measure in $\mathbb{T}$, then it is non-zero a.e. on $\mathbb{T}$. Recalling that the rank of a matrix $A$is a maximal $k$such that there exists a non-zero minor of order k, we can conclude rank $F(z) \equiv Const$ a.e. on $\mathbb{T}$. We will call this constant the rank of F(and denote rank F).

Lemma (2.1.4)[38]: Let v be an $n \times m(m < n)$matrix-value d function in $H^\infty = H^\infty_{m\times n}$, and let rank $v = n - r$ a.e. on $\mathbb{T}$.

Then $v$has a bounded co-analytic complement $\bar{V}$, where $V \in H^\infty_{m\times r}$ is an inner and co-outer function.

Proof. Consider the equation $v^t f \equiv 0$. Standard linear algebra argument implies that there exist vector functions $f_1, f_2, \dots, f_r$with entries in the meromorphic Nevanlina class $\mathcal{N} := \{f/g : f, g \in H^\infty\}$such that for almost all $z \in \mathbb{T}$

$$\ker v^T(z) = \mathcal{L}\{f_1(z), f_2(z), \dots, f_r(z)\}.$$

Multiplying all entries by the product ofall denominators, we obtain that all vector functions $f_1, f_2, \dots, f_r$can be chosen to have entries in H^∞.

Consider subspace $M \subset H^2_n$consisting of all vectors $f \in H^2_n$ satisfying $v^T f \equiv 0$. Clearly $zM \subset M$, so Mcan be represented as $M = VH^2_{r'}$, where $V$is an $n \times r'$inner function. Clearly, $r' \le r$and $V(z)\mathbb{C}r' \subset \ker v(z)^T$a.e. on $\mathbb{T}$.

Since $f_1, f_2, \dots, f_r \in M = VH^2_r$, we have the opposite inclusion

$$\ker v^T(z) \subset V(z)\mathbb{C}r' \qquad \text{a.e. on } \mathbb{T}.$$

Both inclusions together imply that $r' = r$ and

$$\ker v^T(z) = V(z)\mathbb{C}r \qquad \text{a.e. on } \mathbb{T}.$$

The rest follows from Lemma (2.1.3).

The theorem below shows relation between co-analytic complements and Corona Problem. Let us mention, that to prove Theorems (2.1.1) and (2.1.10) one can use a weaker version of it, namely the fact that the best possible norms of the left inverses of $F$and Vcoincide (provided that a co-analytic complement exists). Note, that such simple version was first proved by V. Peller and [46]. However, it is always nice to have a complete understanding, so we present the theorem in full generality.

Remark (2.1.5)[38]: Note, that in the matrix case $(\dim E, \dim E_* < \infty)$ a simple dimension/rank counting shows that the equality

$$V(z)E_1 = \ker F^T(z) \forall z \in \mathbb{D},$$

and the same equality a.e. on $\mathbb{T}$ are equivalent.

In the general operator valued case it is not known whether one implies the other. Numerous counterexamples in the theory of analytic range functions, see [43] lead one to suspect that none of the implications holds.

The following corollary gives necessary and sufficient condition for the Operator Corona Theorem to be true in the case of finite codimension.

Corollary (2.1.6)[38]: Let $F \in H^\infty_{E\to E_*}$be an operator-valued function satisfying

$$F(z)^* F(z) \ge \delta^2 I \qquad \forall z \in \mathbb{D},$$

for some $\delta > 0$. Assume that at a point $z \in \mathbb{D}$ (or on a set of positive measure in $\mathbb{T}$) $\operatorname{codim}(F(z)E) = n < \infty,\ n \ne 0$. Then the following statements are equivalent:

(i) F is left invertible in H^∞, i.e. there exists $G \in H^\infty_{E_*\to E}$such that $GF \equiv I$;

(ii) There exists function $V \in H^\infty_{\mathbb{C}^n \to E_*}$ satisfying $V(z)^*V(z) \geq \tilde{\delta}^2 I,\ \forall z \in \mathbb{D}, \tilde{\delta} > 0$, such that

$$V(z)\mathbb{C}^n = \ker F^T(z) \qquad \forall z \in \mathbb{D} \text{ (and a.e. on } \mathbb{T})$$

Moreover, the function V always can be chosen to be inner, and if both F and V are inner, the best norms of the left inverses for F and V coincide.

Proof. Consider the inner-outer factorizeation of $F, F = F_i F_0$. Note, that

$$F_i^*(z)F_i(z) \geq \tilde{\delta}^2 I, \qquad \forall z \in \mathbb{D},$$

where $\tilde{\delta} = \delta/\|F\|_\infty$. The condition about the codimension of $F(z)E$ implies that F(and therefore F_i) is not invertible in H^∞. Therefore, by Theorem (2.1.9) we have (i) ⇒ (ii).

Since the Operator Corona theorem holds for functions in $H^\infty_{\mathbb{C}^n \to E_*}$,

Condition $V(z)^*V(z) \geq \tilde{\delta}^2 I$ implies that $V$is left invertible. Since $\bar{V}_i$ is a co-analytic complement of V, Theorem (2.1.9) implies that (ii) ⇒ (i).

To prove Theorem (2.1.9) we will need the following two well known results.

Recall, that given $\Phi \in L^\infty_{E \to E_*}$, Hankel and Toeplitz operators H_Φ and $T_\Phi$with symbol Φ are defined as

$$H_\Phi : H^2_E \to \left(H^2_{E_*}\right)^\perp H_\Phi f := P_-(\Phi f)'$$
$$H_\Phi : H^2_E \to H^2_{E_*} H_\Phi f := P_+(\Phi f),$$

where $P_+$and $P_-$are orthogonal projections onto $H^2$and $(H^2)^\perp$ respectively.

Theorem (2.1.7)[38]: (Arveson [39], Sz.-Nagy–Foias [49]). Let F $\in H^\infty_{E \to E_*}$. The following two statements are equivalent:

(i) The function $F$is left invertible in H^∞, i.e. there exists $G \in H^\infty_{E_* \to E}$such that $GF \equiv I$;

(ii)The Toeplitz operator $T_{\bar{F}}$ is left invertible, that is

$$\inf_{f \in H^2_E, \|f\|=1} \|T_{\bar{F}} f\| := \delta > 0.$$

Moreover, the best possible norm of a left inverse G is exactly $1/\delta$.

This theorem also can be found in the Monograph [44].

[44] states that F is right invertible in H^∞ if and only if T_{F^*} is left invertible: applying it to F^T we get the statement of Theorem (2.1.7). Similarly, the theorem in [49] states that F is left invertible in H^∞ if and only if $T_{F^\#}$ is left invertible, where $F^\#(z) := F(\tilde{z})$. Again, applying this theorem to $\overline{F(\tilde{z})}$ we get Theorem (2.1.7).

Lemma (2.1.8)[38]: Let $F \in H^\infty_{E \to E_*}$, and let $F(\xi)$ be an isometry a.e. on $\mathbb{T}$. Then Toeplitz operator T_F is left invertible if and only if $\|H_F\| < 1$ (H_F is the Hankel operator). Moreover, $\|H_F\|^2 = 1 - \delta^2$, where

$$\delta := \inf_{f \in H^2_E, \|f\|=1} \|T_F f\|$$

Proof. Clearly $Ff = T_F f + H_F f$ for $f \in H^2_E$, so

$$\|f\|^2 = \|Ff\|^2 = \|T_F f\|^2 + \|H_F f\|^2,$$

and the lemma follows immediately.

Theorem (2.1.9)[38]: Let $F \in H^\infty_{E \to E_*}$ be an inner function. Assume that F is left invertible, but not right invertible in H^∞. Then Fhas an co-analytic complement $\bar{V}$, where $V \in H^\infty_{E_1 \to E_*}$ is an inner and co-outer function. Moreover

$$V(z)E_1 = \ker F^T(z) \forall z \in \mathbb{D},$$

the function V is also left invertible, and the best possible norms of left inverses for $F$and Vcoincide.

Proof. Let $G \in H^\infty_{E_*\to E}$ be a left inverse of F, that is $GF \equiv I$. Then $F^TG^T \equiv I$, and for $\mathcal{P} \in H^\infty_{E_*\to E_*}$, $\mathcal{P} := G^TF^T$ we have

$$\mathcal{P}^2 := G^T(F^TG^T)F^T = G^TF^T = \mathcal{P},$$

so $\mathcal{P}(z)$ is a projection a.e. on $\mathbb{T}$ (and $\forall z \in \mathbb{D}$). Since

$$\mathcal{P}(z)G^T(z)e = G(z)^TF(z)^TG(z)^Te = G(z)^Te \quad \forall e \in E_*,$$

we can conclude that $\mathcal{P}(z)E_* = G(z)^TE$a.e. on $\mathbb{T}$, as well as $\forall z \in \mathbb{D}$. Since G^T is left invertible, $\ker \mathcal{P}(z) = \ker F(z)^T$a.e. on $\mathbb{T}$ (and $\forall z \in \mathbb{D}$). So, for the complementary projection $Q(z), Q := I - \mathcal{P}$ we have

$$Q(z)E_* = \ker F(z)^T \qquad \text{a.e. on } \mathbb{T}(\text{and } \forall z \in \mathbb{D}).$$

Note, that $Q(z) \not\equiv 0$ because F is not right invertible in H^∞. Taking V to be the inner part of Q we see that F has a co-analytic complement.

Now let us prove the rest of the theorem. Since F is left invertible,
Theorem (2.1.7) implies that

$$\inf_{f\in H^2_E, \|f\|=1} \|T_{\bar{F}}f\| =: \delta > 0.$$

By Lemma (2.1.8) $\|H_{\bar{F}}\|^2 = 1 - \delta^2$ (recall that $F(\xi)$ is an isometry a.e. on $\mathbb{T}$).

Consider the operator valued function $W = [\bar{F}, V] \in H^\infty_{E\oplus E_1\to E_*}$,

$$W(z)(e \oplus e_1) = \bar{F}(z)e + V(z)e_1, \quad e \in E, e_1 \in E_1, z \in \mathbb{T},$$

where $V \in H^\infty_{E_1\to E_*}$ is the inner and co-outer function from the definition of the co-analytic complement. Recallthat the function Wtakes unitary values a.e. on $\mathbb{T}$.

Consider Hankel operator H_W. Its symbol is obtained from $\bar{F}$ by adding an analytic block, so H_W differs from $H_{\bar{F}}$ by a zero block. Hence

$$\|H_W\|^2 = \|H_{\bar{F}}\|^2 = 1 - \delta^2$$

and the Toeplitz operator T_W is left invertible.

Let us show that the adjoint operator $T_W^* = T_{W^*}$has trivial kernel.
Indeed, let $T_{W^*}f = 0, f \in H^2_{E_*}$. Then

$$W^*f = \begin{bmatrix} F^Tf \\ V^*f \end{bmatrix} \in \left(H^2_{E\oplus E_1}\right)^\perp.$$

Since $F \in H^\infty_{E\to E_*}$, this implies $F^Tf \equiv 0$. By Lemma (2.1.3) $f = Vg, g \in H^2_E$, thus $g \equiv 0$ and therefore $f \equiv 0$. So $\ker T_{W^*} = \{0\}$.
The operator $T_W$is left invertible and $\ker T_W^* = \{0\}$, that means $T_W$is invertible. By Lemma (2.1.8)$\|T_{W^*}\|^2 = 1 - \delta^2$. Since W^*differs from V^* by an analytic block, Hankel operators H_{V^*} and H_{W^*}differ by a zero block, thus

$$\|H_{V^*}\|^2 = \|H_{W^*}\|^2 = \|H_{W^*}\|^2 = 1 - \delta^2.$$

By the Nehari theorem $\|H_{V^*}\| = \text{dist}_{L^\infty}(V^*, H^\infty)$, and since the transposition does not change the norm in an operator valued $L^\infty$and $\bar{V} = (V^*)^T$,

$$\|H_{\bar{V}}\| = \text{dist}_{L^\infty}(\bar{V}, H^\infty) = \text{dist}_{L^\infty}(V^*, H^\infty) = \|H_{V^*}\| = \sqrt{1 - \delta^2}.$$

Therefore, by Theorem (2.1.7) the Toeplitz operator $T_{\bar{V}}$ is left invertible, and thus $V$is left invertible in H^∞.
Main construction and the proof of Theorem (2.1.1).

For $\delta < 1/3$ let $\alpha > 0$ be a small number such that

$$\frac{\delta}{1-\alpha} =: \delta' < \frac{1}{3}.$$

Define $a := \delta/(1 - \delta')$, so $a/(1 + a) = \delta'$. Note, that $a < 1/2$.

Consider $(n + 1) \times n$matrix $F \in H^\infty$,

$$F = \begin{pmatrix} \varphi_1(z) & 0 & 0 & 0 & \cdots & 0 \\ -a & \varphi_2(z) & 0 & 0 & \cdots & 0 \\ 0 & -a & \varphi_3(z) & 0 & \cdots & 0 \\ 0 & 0 & -a & \varphi_4(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \varphi_n(z) \\ 0 & 0 & 0 & 0 & \cdots & -a \end{pmatrix},$$

where φ_k are some inner functions to be chosen later. Another way to describe F is to say that its columns F_K can be represented as

$$F_k = \varphi_k(z)e_k - ae_{k+1},$$

where $e_k, k = 1, 2, \ldots, n+1$ is the standard basis in $\mathbb{C}^{n+1}$.

It is easy to check that the $n+1$ dimensional vector-function (column)

$$V = \left(\frac{1-a^2}{1-a^{2n+2}}\right)^{1/2} \begin{pmatrix} a^n \\ a^{n-1}\varphi_1 \\ a^{n-2}\varphi_1\varphi_2 \\ \ldots \\ a\varphi_1\varphi_2 \ldots \varphi_{n-1} \\ \varphi_1\varphi_2 \ldots \varphi_n \end{pmatrix}$$

is inner and co-outer, and that $\bar{V}$ is orthogonal to the columns of F. Let us now assume that φ_n has zero, for example that $\varphi_n(0) = 0$. Then $\|V(0)\| = a^n\sqrt{(1-a^2)/(1-a^{2n+2})} < a^n\sqrt{1+a^2}$, so the norm of a left inverse to the column V is at least $\sqrt{1+a^2}/a^n$.

It is easy to see that $\|F\|_\infty \leq 1 + a$(1 comes from the main diagonal, a from the one below).

Suppose, that we can pick the inner functions φ_k such that

$$\|F(z)e\| \geq (1-\alpha)a\|e\|, \qquad \forall z \in \mathbb{D}, \forall e \in \mathbb{C}^n,$$

where $\alpha > 0$ is from above.

Then for the inner part F_i of F

$$\|e\| \geq \|F_i(z)e\| \geq \frac{(1-\alpha)a}{1+a}\|e\| = \delta\|e\|, \forall z \in \mathbb{D}, \forall e \in \mathbb{C}^n.$$

Theorem (2.1.7) implies that the norm of an analytic left inverse to F_i is at least $\sqrt{1+a^2}/a^n$. Since

$$\delta' = \frac{\delta}{1-\alpha} \to \delta\,, a = \frac{\delta'}{1-\delta'} \to \frac{\delta}{1+\delta} \qquad as\ \alpha \to 0\,,$$

and α can be chosen arbitrarily small, we can construct a function F_i such that the norm of any H^∞ left inverse is at least $[\delta/(1-\delta)]^{-n}$ and the theorem is proved.

To construct functions φ_k, pick an integer N such that $(2a)^N < \alpha a$. Put $\varphi_1(z) = z, \varphi_2(z) = z^N, \varphi_3(z) = z^{N^2}, \ldots, \varphi_n(z) = z^{N^{n-1}}$. In other words, $\varphi_1(z) \equiv z, \varphi_{n+1}\varphi_n^N$. Then for any fixed z the inequality

$$\alpha a \leq \varphi_k(z) \leq 2a$$

holds for at most one $k = k(z)$ (it may happen that for a some z it does not hold for any k). Fix $z \in \mathbb{D}$, and let $k = k(z)$ be such number. Then in the matrix $F(z)$ below

$$F(z)=\begin{pmatrix} \varphi_1 & & & & & & & \\ -a & \varphi_2 & & & & \mathbf{0} & & \\ & -a & \ddots & & & & & \\ & & \ddots & \varphi_{k-1} & & & & \\ 0 & 0 & \dots & -a & \varphi_k & 0 & \dots & 0 \\ & & & & -a & \varphi_{k+1} & & \\ & \mathbf{0} & & & & -a & \ddots & \varphi_n \\ & & & & & & \ddots & \\ & & & & & & & -a \end{pmatrix}$$

in both, upper left and lower right blocks, the main diagonal dominates the other non-trivial one. Therefore, both the upper left and the lower right blocks are invertible with the estimates of the norm of the inverse $1/a$and $a^{-1}(1-\alpha)^{-1}$ respectively. So, if we delete the kth row, we get a (block diagonal) invertible operator with the norm of the inverse at most $a^{-1}(1-\alpha)^{-1}$.

Since adding an extra row does not spoil left invertibility, we conclude that $F(z)$ is left invertible and

$$\|F(z)e\| \geq (1-\alpha)a\|e\|, \qquad \forall e \in \mathbb{C}^n$$

Theorem (2.1.10)[38]: Given $\delta, 0 < \delta < 1/3$ there exists an inner operator valued function $F \in H^{\infty}_{E\to E_*}(\dim E = \infty)$ such that

$$I \geq F^*(z)F(z) \geq \delta^2 I \qquad \forall z \in \mathbb{D},$$

the codimension of $F(z)E$ is E_* is one $\forall z \in \mathbb{D}$, but F does not have a left inverse in H^∞. Moreover, the function Fcan be chosen such that $\operatorname{codim}(F(z)E) = 1$ a.e. on $\mathbb{T}$, or such that $F(z)$ is onto a.e. on $\mathbb{T}$.

Proof.

Consider an operator-valued function F,

$$F = \begin{pmatrix} \varphi_1(z) & 0 & 0 & 0 & \cdots & 0 & \cdots \\ -a & \varphi_2(z) & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -a & \varphi_3(z) & 0 & \cdots & 0 & \cdots \\ 0 & 0 & -a & \varphi_4(z) & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \ddots & \varphi_n(z) & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -a & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix},$$

which is an infinite dimensional analogue of the function F constructed previously. Here $a, \varphi_1, \dots, \varphi_n, \dots$ are exactly the same as above. We can show that

$$(1+a)^2 I \geq F^*(z)F(z) \geq (1-\alpha)^2 a^2 I \geq \delta^2 I.$$

Note, that for $z \in \mathbb{D}$ operator $F(z)$ is not onto: subspace $(F(z)E)^\perp$ is spanned by the vector

$$\bar{V} := \begin{pmatrix} 1 \\ \bar{\varphi}_1/\bar{a} \\ \bar{\varphi}_1\bar{\varphi}_2/\bar{a}^2 \\ \vdots \\ \bar{\varphi}_1\bar{\varphi}_2 \dots \bar{\varphi}_n/\bar{a}^n \\ \vdots \end{pmatrix}$$

On the other hand, it is easy to see that the operators $F(z)$ are invertible a.e. on $\mathbb{T}$ (the main diagonal dominates), so according to Corollary (2.1.6), F does not have a left inverse. To get the statement about inner function, it is sufficient to take the inner part of F.

To prove that it is possible to pick Fsuch that $F(z)E$has codimension 1 a.e. on $\mathbb{T}$, let us first notice that for the vector Vabove

$$\lim_{|z|\to 1-} \|V(z)\| = \infty.$$

Therefore it is possible to find a simply connected domain $D \subset \mathbb{D}$ with C^∞-smooth boundary, which touches $\mathbb{T}$ exactly at one point, such that

$$\int_{\partial D} \log\|V(z)\| \cdot |dz| = +\infty.$$

Since for such domains the harmonic measure is equivalent to the arclength $|dz|$, we can replace $|dz|$by the harmonic measure, and still get $+\infty$ in the integral. Note also, that everywhere on ∂Dexcept the point $\partial D \cap \mathbb{T}$, the codimension of $F(z)E$is 1.

So, if $\omega: \mathbb{D} \to D$ is a conformal mapping, then for $F_1 := F \circ \omega,\ V_1 := V \circ \omega$ we have that for all $z \in \operatorname{clos}\mathbb{D}$ except one point on $\mathbb{T}$

$$(F_1(z)E)^\perp = \operatorname{span}\bar{V}_1(z).$$

Notice, that also

$$\lim_{r\to 1-} \int_{r\mathbb{T}} \log\|V_1(z)\| \cdot |dz| = \int_{\mathbb{T}} \log\|V_1(z)\| \cdot |dz| = +\infty.$$

But it is easy to see that F_1 is not left invertible in H^∞. Indeed, if F is left invertible in H^∞, Corollary (2.1.6) asserts that there exist a vector function $v \in H^\infty_{E_*} = H^\infty_{\mathbb{C}\to E_*}$such that

$$(F(z)E)^\perp = \operatorname{span}\bar{v}(z)$$

for all $z \in \mathbb{D}$ and a.e. on $\mathbb{T}$. Any such function must be represented as $v(z) = u(z)V_1(z)$ where $u$is a scalar function. Since $\|V_1(z)\| \ge 1$ the function u must be bounded, and since both v and V_1 are holomorphic in $\mathbb{D}$, the function $u$also must be holomorphic. Therefore $u \in H^\infty$.

We discuss some open problem concerning the Operator Corona Problem. The ultimate problem is to find a local necessary and sufficient condition for left invertibility of $F \in H^\infty_{E\to E_*}$. But this is probably hopeless, so there are several problems that seem to be more tractable.

Does there exist $\delta > 0$ (close to 1) such that for any $F \in H^\infty_{E\to E_*}$the inequality

$$I \ge F(z)^*F(z) \ge \delta^2 I$$

implies that there exists $G \in H^\infty_{E\to E_*}$such that $GF \equiv I$?

The counterexample constructed works only for $\delta < \frac{1}{3}$, the method from [51] gives counterexample for $\delta < 1/\sqrt{2}$.

As Theorem (2.1.10) asserts, if $F \in H^\infty_{E\to E_*}$is invertible, there exists a left invertible $G \in H^\infty_{E_1\to E_*}$such that $\ker F(z)^T = V(z)E_1$ a.e. on $\mathbb{T}$.

Suppose $F \in H^\infty_{E\to E_*}$satisfies

$$F(z)^*F(z) \ge \delta^2 I \qquad \forall z \in \mathbb{D},$$

and suppose we know that there exists $VH^\infty_{E_1\to E_*}$such that $\ker F(z)^T = V(z)E_1$ a.e. on $\mathbb{T}$ also satisfying the Corona Condition

$$V(z)^*V(z) \ge \varepsilon^2 I\ .$$

Does this imply that F is left invertible in H^∞?

Section (2.2): Vector Valued Corona Problem

We give an improved estimate in the vector valued corona theorem for the unit disk. Our proof will be split into two parts: first, we will resurrect the operator corona theorem from the 1970s, and then we will use a Hilbert space argument, based on a version of Wolff's ideas. Since we use Hilbert space methods, the Riesz representation theorem replaces the usual $\bar{\partial}$-techniques.

In 1962 Carleson determined when a finitely generated ideal in $H^\infty(D)$ is actually all of $H^\infty(D)$, by giving a function theory condition on the generators. Namely, $\mathcal{J}(f_1, \dots, f_m) = H^\infty(D)$ if and only if there exists an $\varepsilon > 0$, so that $\sum_{i=1}^{m} |f_i(z)|^2 \geq \varepsilon^2$ for all $z \in D$. Actually, more was shown, as a bound for the size of solutions was given. If $\{f_i\}_{i=1}^{m} \subset H^\infty(D)$ such that $1 \geq \sum_{i=1}^{m} |f_i(z)|^2 \geq \varepsilon^2$ for all $z \in D$,then there exists a $B(\varepsilon, m) < \infty$ and there exist $\{g_i\}_{i=1}^{m} \subset H^\infty(D)$ with $\sum_{i=1}^{m} f_i g_i = 1$ and $\sum_1^m |g_i(z)|^2 \leq B(\varepsilon, m)$ for all $z \in D$.

Spawned numerous investigations in many directions, but we will mention only those most closely related to our work [40]. For the scalar case above, Hormander introduced $\bar{\partial}$-methods, culminating in Wolff's surprising proof of the corona theorem. (See [42].) Rosenblum [47] and, independently, Tolokonnikov [66] removed the dependency on m. This was fine-tuned by Uchiyama [55] to yield the estimate (for ε small)

$$B(\varepsilon) \leq \frac{C}{\varepsilon^2} \ln \frac{1}{\varepsilon^2}, \qquad C \text{ a universal constant.}$$

For the operator version of the corona problem on the unit disk, Fuhrmann and later Vasyunin (see [41], [45]) considered $F \in B\left(H^2(D)^{(\infty)},\ H^2(D)^{(n)}\right)$, with $S^{(n)}F = FS^{(\infty)}$. Here S denotes the unilateral shift and n is finite. So F can be viewed as an $n \times \infty$. matrix with analytic Toeplitz operators for entries. Vasyunin showed that the hypothesis $I \geq F(z)F(z)^* \geq \varepsilon^2 I$ for all $z \in D$ and $\varepsilon > 0$, enables one to get an estimate for the solution to $F(z)G(z) = I$ of the form

$$G(z)G(z)^* \leq C(n, \varepsilon)^2 I \qquad for\ all\ z \in D,$$

where $C(n, \varepsilon) \leq C\sqrt{n}(1/\varepsilon^{2n}) \ln(1/\varepsilon^{2n})$ and C is a universal constant. If the best theoretical bound for $n = 1, C(1, \varepsilon) = B(\varepsilon) \approx 1/\varepsilon^2$, is proven, the estimate of Vasyunin would be $C(n, \varepsilon) \leq C\sqrt{n}(1/\varepsilon^{2n})$ (for small ε). Later an important result of Treil [67] showed that $C(n, \varepsilon) \nearrow \infty$as $n \nearrow \infty$. Thus the existence of a lower bound for $F(z)F(z)^*$does not guarantee the existence of a $G(z)$ with $F(z)\, G(z) = I$, when $n = \infty$. See Nikolskii [45]and the article by Sz.-Nagy, [69]. Our estimate will remove the ''$\sqrt{n}$'' and, also, replace ε^{2n} by ε^{n+1} and thus give an improvement of Vasyunin's estimate. However, for the case $n = 1$, it just reproduces the estimate of Uchiyama.

We will consider operators in $B\left(H^2(D)^{(m)}, H^2(D)^{(n)}\right)$, where m and n are finite and $n < m$. Since our estimates will be independent of m, the case $m = \infty$will follow from a routine argument (see Nikolskii [45]). In part of our argument, we will be factoring certain projections. This is where the finiteness of m is used.

F will stand for the multiplication operator acting from $L^2(\partial D)^{(m)}$ into $L^2(\partial D)^{(n)}$ by $\left(F\underline{u}\right)(e^{it}) = \left[f_{ij}(e^{it})\right]_{i,j=1}^{n,m} \underline{u}(e^{it})\ a.e.\ t \in [-\pi, \pi]$, where $f_{ij} \in H^\infty(D)$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. For $z \in \overline{D}, F(z)$ will denote the $n \times m$ matrix $\left[f_{ij}(z)\right]_{i,j=1}^{n,m}$, giving a linear transformation in $B(\mathbb{C}^m,\mathbb{C}^n)$. T_F will denote the analytic Toeplitz operator matrix, gotten by restricting F to $H^2(D)^m$.

Our estimates are based on the following two theorems. Let $\{f_{ij}(e^{it})\}_{i,j=1}^{n,m} \subset H^\infty(D)$ and let $F = [M_{f_{ij}}] \in B(L^2(\partial D)^{(m)}, L^2(\partial D)^{(n)})$.

Theorem (2.2.1)[54]: If

$$I \geq F(z)F(z)^* \geq \delta^2 I \qquad for\ z \in D,$$

Then

$$T_F T^*{}_F \left(2C\left[\frac{1}{\varepsilon^{n+1}}\ln\frac{1}{\varepsilon^{2n}}\right]\right)^{-2} I.$$

[Here $C = 2\sqrt{e} + 2\sqrt{2}e$ and $0 < \varepsilon^2 < \frac{1}{e}$.]

Theorem (2.2.2)[54]: If

$$T_F T^*{}_F \geq \delta^2 I\ ,$$

then there exists

$$\{g_{ij}\}_{i,j=1}^{m,n} \subset H^\infty(D),$$

so that

$$T_F T_G = I \qquad and\ \ \|T_G\| \leq \frac{1}{\delta}\ .$$

Note that

$$\|T_G\| = \sup_{z\in D}\|G(z)\|_{B(\mathbb{C}^m,\mathbb{C}^n)}\ ,$$

so Theorems (2.2.1) and (2.2.2) together constitute the vector valued corona theorem with the improved bound. The conclusion of Theorem (2.2.2) is the desired ‘‘corona theorem’’ conclusion, but the hypothesis for Theorem (2.2.2) is an operator theory one, as contrasted with the pointwise function theoretic assumption in Theorem (2.2.1). For this reason, Theorem (2.2.2) is referred to as an ‘‘operator corona’’ theorem. Various versions of operator corona theorems were proved in the 1970s. See Arveson [39], Sz.-Nagy and Foias [65], and Schubert [64]. More recent versions include Katsoulis et al. [63], Helton [60], Ball and Trent [58], and Agler and McCarthy [56]. An expository account of the control theory approach is given in Trent [68]. Thus we will not include a proof of Theorem (2.2.2). However, we will make two comments. First, the exact constant occurs in Theorem (2.2.2), so finding the optimal estimates in the corona theorem resides in the proof of Theorem (2.2.1). Second, if the scalar corona problem is similarly posed over, say the bidisk, then for a finite set $\{f_i\}_{j=1}^m$ of bounded analytic functions on $D \times D$, Theorem (2.2.1) holds for a constant depending on m and ε. See Lin [62] and Li [61]. Therefore the corona problem for the bidisk reduces to the question of whether Theorem (2.2.2), an operator corona theorem, holds for $H^2(D \times D)$. For further comments on this see Trent [68] and Ball et al. [57].

Our proof of Theorem (2.2.1) will require the standard Littlewood–Paley type lemma.

Lemma (2.2.3)[54]: Let ϕ be $C^{(2)}$ in a neighborhood of $\overline{D}$. Then

$$\phi(0) = \int_{-\pi}^{\pi} \phi(e^{it})d\sigma - \frac{1}{4\pi}\int_D \Delta\phi(z)\ln\frac{1}{|z|^2}\ dm\ (z).$$

The proof of Lemma (2.2.3) follows from Green’s theorem. See, for example, Garnett [42]. Note the special case that $\phi(z) = p_0(z)\overline{q(z)}$, where p_0 and q are analytic polynomials and $p_0(0) = 0$. Then the above equality says that

$$\langle p_0, q\rangle_{\partial D} = \int_D p_0'(z)\overline{q'(z)} \ln\frac{1}{|z|^2}\frac{dm(z)}{\pi}.$$

To just check the Littlewood–Paley inner product result, consider terms of the form $z^n\overline{z}^m, n$ and m nonnegative integers, and $n + m \geq 1$. If $n \neq m$ we clearly get 0 on both sides. If $n + m \geq 1$ we need only check that

$$1 = \int_D n^2|z|^{2n-2} \ln\frac{1}{|z|^2}\frac{dm(z)}{\pi}.$$

This is easily verified, using polar coordinates.

We require Lemma (2.2.3), which replaces the Hardy space inner product on ∂D with an essentially equivalent inner product on D, to utilize the necessary condition that $\varepsilon^2 I \leq F(z)F(z)^*$ for all $z \in D$.

Also, we use the standard observation that if Theorem (2.2.1) is proven for $\{f_{ij}\}_{i,j=1}^{n,m}$ analytic on D and across ∂D and satisfying the hypothesis, then a compactness argument gives the general case. Again details of this (but with a Mordell type compactness argument) may be found in Nikolskii [45].

To derive a conclusion like $AA^* \geq \delta^2 I$, where $A \in B(\mathcal{K}, \mathcal{H})$, we need the following well-known lemma.

Lemma (2.2.4)[54]: Let $A \in B(\mathcal{K}, \mathcal{H})$, where $\mathcal{K}$ and $\mathcal{H}$ are Hilbert spaces. Suppose that for all h in a dense subset of $\mathcal{H}$, there exists a $u_h \in \mathcal{K}$ with $Au_h = h$ and $\|u_h\|_{\mathcal{K}} \leq \frac{1}{\delta}\|h\|_{\mathcal{H}}$. Then

$$AA^* \geq \delta^2 I_{\mathcal{H}}.$$

(The converse holds, but we do not use it.)

Proof. By completeness of $\mathcal{K}$ and $\mathcal{H}$, we may assume that for every $h \in \mathcal{H}$, there exists a $u_h \in \mathcal{K}$ with $Au_h = h$ and $\|u_h\|_{\mathcal{K}} \leq \frac{1}{\delta}\|h\|_{\mathcal{H}}$. Thus the range of A is closed, so the range of A^* is closed and $\ker A^* = \{0\}$. Since the smallest solution to $Ax = h, v_h$, belongs to $(\ker A)^\perp = \overline{\operatorname{ran} A^*} = \operatorname{ran} A^*$, we have $v_h = A^*q$ for some q in $\mathcal{H}$ and $\|A^*q\|_{\mathcal{K}} = \|u_h\|_{\mathcal{K}} \leq \|u_h\|_{\mathcal{K}} \leq \frac{1}{\delta}\|h\|_{\mathcal{H}}$. Now since AA^* is one-to-one and onto, thus invertible, we have $AA^*q = h$ and $q = (AA^*)^{-1}h$.

Thus $\|A^*(AA^*)^{-1}h\|_{\mathcal{K}} \leq \frac{1}{\delta}\|h\|_{\mathcal{H}}$ so $\langle (AA^*)^{-1}h, h\rangle_{\mathcal{H}} \leq (1/\delta^2)\langle h, h\rangle$ or $(AA^*)^{-1} \leq (1/\delta^2)I$. Thus

$$I_{\mathcal{H}} = (AA^*)^{\frac{1}{2}}(AA^*)^{-1}(AA^*)^{\frac{1}{2}} \leq \frac{1}{\delta^2}AA^*$$

and we're done.

Notice that if $A \in B(H)$ and $AA^* \geq \delta^2 I$ with $\delta > 0$, then $P_{\operatorname{ran} A^*} = A^*(AA^*)^{-1}A$ and $P_{\ker A} = I_H - A^*(AA^*)^{-1}A$. Here P_N denotes the orthogonal projection onto N, where N is a closed subspace of H.

The proof of Theorem (2.2.1) proceeds as follows:

Assume that $\{f_{ij}\}_{i,j=1}^{n,m}$ are analytic functions in a neighborhood of $\overline{D}$ and that $F(z) = [f_{ij}(z)]$ satisfies $I \geq F(z)F(z)^* \geq \varepsilon^2 I$ for all $z \in \overline{D}$. To reach our conclusion we use Lemma (2.2.4). Let $\underline{h} \in H^2(\partial D)^{(m)}$ be an m-vector of analytic polynomials. We must find a $\underline{u}_h \in H^2(\partial D)^{(n)}$ with $F\underline{u}_h = \underline{h}$and $\|\underline{u}_h\| \le (C/\varepsilon^{n+1})\ln(1/\varepsilon^{2n})$, with $C = 2[2\sqrt{e} + 2\sqrt{2}e]$.

Now all solutions of $F\underline{x} \quad = \underline{h}$ in $L^2(\partial D)^{(m)}$have the form

$$\underline{x} = F^*(FF^*)^{-1}\underline{h} - P_{\ker F}\underline{k} \qquad for\ \underline{k} \in L^2(\partial D)^{(m)}.$$

To get $\underline{u}_h \in H^2(\partial D)^{(n)}$, we must find $\underline{k} \in L^2(\partial D)^{(m)}$ so that

$$\langle F^*(FF^*)^{-1}\underline{h} - P_{\ker F}\underline{k}, \underline{\bar{p}_0}\rangle = 0 \qquad for\ \underline{p}_0 \in H_0^2(D)^{(n)}$$

Or

$$\langle F^*(FF^*)^{-1}\underline{h}, \underline{\bar{p}_0}\rangle = \langle \underline{k}, P_{\ker F}\underline{\bar{p}_0}\rangle \qquad for\ \underline{p}_0 \in H_0^2(D)^{(n)} \quad (1)$$

Obviously,

$$\|\underline{x}\| \le \frac{1}{\delta}\|\underline{h}\| + \|\underline{k}\|.$$

For the existence and norm estimate of a $\underline{k} \in L^2(\partial D)^{(m)}$ satisfying (1), we apply the Riesz representation theorem. By considering the linear functional which sends $P_{\ker F}\overline{p}_0 \mapsto \langle \underline{\bar{p}}_0, F^*(FF^*)^{-1}h\rangle$, we require that

$$\left|\langle F^*(FF^*)^{-1}\underline{h} - , \underline{\bar{p}_0}\rangle\right| \le \frac{C}{\varepsilon^{n+1}}\ln\frac{1}{\varepsilon^{2n}}\|\underline{h}\|\|P_{\ker F}\overline{p}_0\| \qquad (2)$$

for all $\underline{p}_0 \in H_0^2(D)^{(n)}$ and $C = 2\sqrt{e} + 2\sqrt{2}e.$ The remainder of the proof consists in establishing (2).

We will need three lemmas.

Lemma (2.2.5)[54]: For $z \in \overline{D}, P_{\ker F(z)} = \frac{Q(z)Q(z)^*}{\det(F(z)F(z)^*)}$, where $Q(z) \in B\left(\mathbb{C}^{\binom{m}{n+1}}, \mathbb{C}^m\right)$ and the entries of $Q(z)$ are analytic functions in D.

Proof. Fix any $z \in \overline{D}$,. Let $\underline{f_i} + \left(f_{i,1}(z), \dots, f_{i,m}(z)\right)$ for $i = 1, \dots, n$. Let $\{\underline{e}_j\}_{j=1}^m$denote the standard orthonormal basis for $\mathbb{C}^m$, so $\underline{f_i} = \sum_{j=1}^m f_{i,j}(z)\underline{e}_j$. Also $\{\underline{u}_j\}_{j=1}^{m-n}$ will denote any orthonormal basis for $\mathbb{C}^m \ominus span\left\{\underline{\overline{f}}_i\right\}_{i=1}^n$. Note that our hypothesis that $F(z)F(z)^* \ge \varepsilon^2 I$ and $m\backslash n$ shows that then $\left\{\underline{\overline{f}}_i\right\}_{i=1}^n$ are linearly independent.

We will be using elementary computations with forms. See [59]. For $1 \le k \le m, \Pi_k$ will denote the set of increasing k-tuples of $\{1,2,\dots,m\}$. Then for $\lambda \in \Pi_k$ with $\lambda = (\lambda_1, \dots, \lambda_k), \lambda_1 < \lambda_2 < \dots < \lambda_k$, and $\lambda_j \in \{1,2,\dots,m\}$,

$$\underline{e}_\lambda = \underline{e}_{\lambda_1} \wedge \underline{e}_{\lambda_2} \wedge \dots \underline{e}_{\lambda_k}.$$

It is not hard to see that $\{\underline{e}_\lambda\}_{\lambda \in \Pi_k}$ can be identified with the standard orthonormal basis for $\mathbb{C}^{\binom{m}{n}}$ in a natural way.

Define $Q(z)^* \in B\left(\mathbb{C}^n, \mathbb{C}^{\binom{m}{n+1}}\right)$, formally by

$$\left(Q(z)\right)^*(\underline{w}) = \underline{w} \wedge \underline{\overline{f}}_1 \wedge \dots \wedge \underline{\overline{f}}_n.$$

By our hypothesis $m \ge n$. If $m = n$, then $Q(z)^* = 0$. Of course, in this case $\ker F(z) = \{0\}$. Clearly, as a function of $z \in D, Q(z)^*$ is conjugate analytic.
Denote the form $\underline{\overline{f}}_1 \wedge \dots \wedge \underline{\overline{f}}_n$ by $\overline{\mathcal{F}}$ and denote $Q(z)$ by Q.

Now for $\underline{w} \in \mathbb{C}^{\binom{m}{n+1}}$,

$$Q(\underline{w}) = \sum_{j=1}^{m-n} \langle Q(\underline{w}), u_j \rangle \underline{u}_j = \sum_{j=1}^{m-n} \langle \underline{w}, Q^*(\underline{u}_j) \rangle \underline{u}_j = \sum_{j=1}^{m-n} \langle \underline{w}, u_j \wedge \overline{\mathcal{F}} \rangle \underline{u}_j.$$

Also, $Q^*\left(\underline{\overline{f}}_j\right) = \underline{\overline{f}}_j \wedge \underline{\overline{f}}_1 \dots \wedge \underline{\overline{f}}_n = 0$ for $j = 1, \dots, n$(see [59]), so $Q^*F(z)^* = 0$ and $F(z)Q = 0$. Thus $ranQ \subset \ker F(z)$.

We claim that $ranQ = \ker F(z)$. Let $\underline{y} \in \ker F(z) \ominus ranQ$. Then $\underline{y} \in (ran\, Q)^{\perp} = \ker Q^*$, so $0 = Q^*\underline{y} = \underline{y} \wedge \underline{\overline{f}}_1 \wedge \dots \wedge \underline{\overline{f}}_n$ and thus $\underline{y} \in sp\left\{\underline{\overline{f}}_1, \dots, \underline{\overline{f}}_n\right\}$. See [59]. Hence $\underline{y} = \sum_{j=1}^{n} \alpha_j \underline{\overline{f}}_j$ and $\underline{y} \in \ker F(z)$. So, if $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, then $0 = F(z)\underline{y} = F(z)F(z)^*\underline{\alpha}$. But $F(z)F(z)^*$ is invertible, so $\underline{\alpha} = \underline{0}$ and $\underline{y} = \underline{0}$.

We must show that

$$\frac{QQ^*}{\det(F(z)F(z)^*)} = P_{\ker F(z)}.$$

Since $ran\, Q = \ker F(z)$, we need only show that $Q/\sqrt{\det(F(z)F(z)^*)}$ is a partial isometry. That is $Q^*QQ^* = \det(F(z)F(z)^*)Q^*$.

For $\underline{w} \in \mathbb{C}^n$,

$$Q^*QQ^*(\underline{w}) = Q^*Q(\underline{w} \wedge \overline{\mathcal{F}}) = Q(\underline{w} \wedge \overline{\mathcal{F}}) \wedge \overline{\mathcal{F}} = \left[\sum_p \langle \underline{w} \wedge \overline{\mathcal{F}}, \underline{u}_p \wedge \overline{\mathcal{F}} \rangle \underline{u}_p\right] \wedge \overline{\mathcal{F}}.$$

Recall that $\{\underline{u}_p\}_{p=1}^{m-n}$ is an orthonormal basis for $\mathbb{C}^m \ominus ran\, F(z)^*$.

Let $\underline{w} = \underline{u}_j$, then since $\langle \underline{u}_j \wedge \overline{\mathcal{F}}, \underline{u}_p \wedge \overline{\mathcal{F}} \rangle = \delta_{jp} \det(F(z)F(z)^*)$ we're done. This last observation follows, since if A is an $m \times n$ matrix with $m \geq n$ and $\underline{a}_1, \dots, \underline{a}_n$are the columns of A, then $\underline{a}_1 \wedge \dots \wedge \underline{a}_n = \sum_{\mathcal{K} \in \pi_n} \det(E_{\mathcal{K}}A)\, \underline{a}_{\mathcal{K}}$. Here $E_{\mathcal{K}}A$ is $n \times n$ and $E_{\mathcal{K}}$ is the identity on $\{e_j\}_{j \in \mathcal{K}}$.

Thus

$$\langle \underline{a}_1 \wedge \dots \wedge \underline{a}_n, \underline{b}_1 \wedge \dots \wedge \underline{b}_n \rangle = \sum_{\mathcal{K} \in \pi_n} \det(E_{\mathcal{K}}A)\, \overline{\det(E_{\mathcal{K}}B)}$$

$$= \sum_{\mathcal{K} \in \pi_n} \det(E_{\mathcal{K}}A) \det(B^*E_{\mathcal{K}}) = \sum_{\mathcal{K} \in \pi_n} \det(B^*E_{\mathcal{K}}A) = \det(B^*A)\,.$$

Again, details can be found in [59].

We will need two additional lemmas. The first of these can be viewed as an extension of the Littlewood–Paley lemma, Lemma (2.2.3), and seems to be due to Uchiyama. See Nikolskii [45]. We include the proof for convenience.

Lemma (2.2.6)[54]: Assume $a \in C^2(\overline{D}), a \geq 0$ and $\Delta a \geq 0$. Then for p an analytic polynomial, we have

$$\int_D \Delta a|p|^2 \ln \frac{1}{|z|^2} \leq e\|a\|_\infty \int_{\partial D} |p|^2 d\sigma.$$

Proof. A computation gives us that for $t > 0$

$$\Delta(e^{ta}|p|^2) = te^{ta}\Delta a|p|^2 + 4e^{ta}|ta_z p + p'|^2 \geq t\Delta a|p|^2.$$

Thus by Lemma (2.2.3),

$$\int_D \Delta a|p|^2 \ln\frac{1}{|z|^2} \le \frac{1}{t}\int_D \Delta(e^{ta}|p|^2)\ln\frac{1}{|z|^2}\frac{dm}{4}(z) \qquad = \frac{\pi}{t}\int_{\partial D} e^{ta}|p|^2 d\sigma$$

$$\le \frac{\pi}{t}e^{t\|a\|_\infty}\int_{\partial D}|p|^2 d\sigma.$$

Letting $t = 1/\|a\|_\infty$. completes the proof.

We will apply the previous lemma to suitable choices for $a(z)$. Recall that

$$I \ge F(z)F(z)^* \ge \varepsilon^2 I \qquad for\ all\ z \in \overline{D},$$

Lemma (2.2.7)[54]:

(i) $\Delta[tr(F(z)F(z)^*)^{-1}] = 4\,tr[(F(z)F(z)^*)^{-1}F(z)F'(z)^* \times (F(z)F(z)^*)^{-1} \times F'(z)F(z)^*(F(z)F(z)^*)^{-1}] - 4\,tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker(F(z)} \times F'(z)^*(F(z)F(z)^*)^{-1}\right]$

(ii) $\Delta\left[\ln\frac{\det(F(z)F(z)^*)}{\varepsilon^{2n}}\right] = 4\,tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker F(z)}F'(z)^*\right]$

(iii) $\Delta[\det(F(z)F(z)^*)^{-1}] = 4\,[(\det(F(z)F(z)^*))^{-1} \times |tr\,(F(z)F(z)^*)^{-1}F(z)F^*(z)^*|^2] - 4\left[\det(F(z)F(z)^*)^{-1} \times tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker F(z)}F'(z)^*\right]\right].$

Proof. Using the resolvent identity, it is easy to see that for $z \in D$,

$$\partial_z[(F(z)F(z)^*)^{-1}] = (F(z)F(z)^*)^{-1}F'(z)F(z)^*(F(z)F(z)^*)^{-1}$$

and similarly for $\overline{\partial_z}$. Now let Ω be a simply connected region containing $[\varepsilon^2, 1]$ and with $\partial\Omega$ a simple smooth Jordan curve traced counter-clockwise. Let g be analytic in a neighborhood of $\overline{\Omega}$. By the Riesz functional calculus and the above derivative equation, we get that

$$\partial_z tr[g(F(z)F(z)^*)] = tr[g'(F(z)F(z)^*)F'(z)F(z)^*].$$

Applying this to $g(z) = \log z\ , -\pi < \operatorname{Arg} z < \pi$, we get

$$\partial_z tr[\ln(F(z)F(z)^*)] = tr[(F(z)F(z)^*)^{-1}F'(z)F(z)^*].$$

Using the product rule (which is easily established for our case) yields

$$\begin{aligned}
&\Delta\, tr[\ln(F(z)F(z)^*)] \\
&= 4\,tr[-(F(z)F(z)^*)^{-1}F(z)F'(z)^*(F(z)F(z)^*)^{-1}F'(z)F(z)^*] \\
&+ 4\,tr\,[(F(z)F(z)^*)^{-1}F'(z)F'(z)^*] \\
&= 4\,tr[-(F(z)F(z)^*)^{-1}F'(z)F(z)^*(F(z)F(z)^*)^{-1}F(z)F'(z)^*] \\
&+ tr\,[(F(z)F(z)^*)^{-1}F'(z)F'(z)^*] \\
&= 4\,tr\,[(F(z)F(z)^*)^{-1}F'(z)[1 - F(z)^*(F(z)F(z)^*)^{-1}F(z)]F'(z)^*] \\
&= 4\,tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker(F(z)}F'(z)^*\right].
\end{aligned}$$

Now since

$$\ln\left[\frac{\det(F(z)F(z)^*)}{\varepsilon^{2n}}\right] = \ln\left[\det\left[\frac{(F(z)F(z)^*)}{\varepsilon^2}\right]\right] = tr\ \left[\ln\left[\frac{(F(z)F(z)^*)}{\varepsilon^2}\right]\right],$$

(ii) follows.

From above

$$\overline{\partial_z}\ln\left[\det\left(\frac{F(z)F(z)^*}{\varepsilon^2}\right)\right] = tr\ [(F(z)F(z)^*)^{-1}F(z)F'(z)^*].$$

But

$$\overline{\partial_z}\ln\left[\frac{\det(F(z)F(z)^*)}{\varepsilon^{2n}}\right]=\frac{\overline{\partial_z}\det(F(z)F(z)^*)}{\det(F(z)F(z)^*)}\,.$$

Thus

$$\overline{\partial_z}\det(F(z)F(z)^*)=\det(F(z)F(z)^*)\ \ tr\ [(F(z)F(z)^*)^{-1}F(z)F'(z)^*].$$

So

$$\begin{aligned}\Delta[(\det(F(z)F(z)^*)^{-1})]&\\ &=4\,\partial_z\big[-(\det(F(z)F(z)^*))^{-2}\det(F(z)F(z)^*)\\ &\times tr[(F(z)F(z)^*)^{-1}F(z)F'(z)^*]\big]\\ &=4\partial_z\big[-(\det(F(z)F(z)^*))^{-1}\times tr[(F(z)F(z)^*)^{-1}F(z)F'(z)^*]\big]\\ &=4\,[\det(F(z)F(z)^*)^{-2}\det(F(z)F(z)^*)\\ &\times tr|[(F(z)F(z)^*)^{-1}F(z)F'(z)^*]|^2]\\ &-4\left[\det(F(z)F(z)^*)^{-1}\times tr\ \left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker(F(z))}F'(z)^*\right]\right].\end{aligned}$$

and (iii) holds.

For (i),

$$\begin{aligned}\Delta\, tr[(F(z)F(z)^*)^{-1}]\quad &=4\,\partial_z\big[tr[-(F(z)F(z)^*)^{-1}F(z)F'(z)^*(F(z)F(z)^*)^{-1}]\big]\\ &=4\,tr\,[(F(z)F(z)^*)^{-1}F'(z)F'(z)^*(F(z)F(z)^*)^{-1}\\ &\times F(z)F'(z)^*(F(z)F(z)^*)^{-1}]\\ &+4\,tr[-(F(z)F(z)^*)^{-1}F'(z)F'(z)^*(F(z)F(z)^*)^{-1}]\\ &+4\,tr\,[(F(z)F(z)^*)^{-1}F(z)F'(z)^*(F(z)F(z)^*)^{-1}\\ &\times F'(z)F(z)^*(F(z)F(z)^*)^{-1}]\\ &=-4\,tr\,\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker(F(z)}F'(z)^*(F(z)F(z)^*)^{-1}\right]\\ &+4\,tr[(F(z)F(z)^*)^{-1}F(z)F'(z)(F(z)F(z)^*)^{-1}\\ &\times F'(z)F(z)^*(F(z)F(z)^*)^{-1}].\end{aligned}$$

This completes the proof.

The remainder of our argument for Theorem (2.2.1) consists of establishing (2). That is, for $\underline{h}\in H^2(\partial D)^{(m)}$ with analytic polynomial entries and for any $\underline{p_0}\in H_0^2(\partial D)^{(n)}$ with analytic polynomial entries that vanish at 0, we claim that for $C=\left(2\sqrt{e}+2\sqrt{2}e\right)$

$$\left|\langle F^*(FF^*)^{-1}\underline{h},\underline{\overline{p}}_0\rangle\right|\le\frac{C}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\|\underline{h}\|\left\|P_{\ker F}\underline{\overline{p}}_0\right\|\,. \tag{2}$$

By Lemma (2.2.5),

$$\left\|P_{\ker F}\underline{\overline{p}}_0\right\|^2=\langle P_{\ker F}\underline{\overline{p}}_0,\underline{\overline{p}}_0\rangle=\int_{\partial D}\langle\frac{\mathcal{Q}(e^{it})\mathcal{Q}(e^{it})^*}{\det(F(e^{it})F(e^{it})^*)}\overline{\underline{p}_0(e^{it})},\overline{\underline{p}_0(e^{it})}\ \rangle_{\mathbb{C}^n}\ d\sigma(t)$$

$$\ge\int_{\partial D}\left\|\mathcal{Q}^*(e^{it})\overline{\underline{p}_0(e^{it})}\right\|^2d\sigma(t).$$

Denote $\mathcal{Q}^*(z)\overline{\underline{p}_0(e^{it})}$ by $\overline{\underline{k}_0(z)}$. Note that $\overline{\underline{k}_0}$ has entries that vanish at 0 and are co-analytic in a neighborhood of $\overline{D}$. Thus it suffices to show that

$$\left|\langle F^*(FF^*)^{-1}\underline{h},\underline{\overline{p}}_0\rangle\right|\le\frac{C}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\|\underline{h}\|\|\underline{k}_0\|\,. \tag{3}$$

By Lemma (2.2.3), the left hand side of (3) becomes

$$\langle F^*(FF^*)^{-1}\underline{h},\underline{\overline{p}}_0\rangle = \frac{1}{4\pi}\int_D \Delta\left[\langle F(z)^*(F(z)F(z)^*)^{-1}\underline{h}(z),\overline{\underline{p}_0(z)}\rangle\right] \times \ln\frac{1}{|z|^2}\, dm(z)$$
$$= \frac{1}{\pi}\int_D \partial_z\Big[\langle\big(I - F(z)^*(F(z)F(z)^*)^{-1}F(z)\big)F'(z)^*$$
$$\times (F(z)F(z)^*)^{-1}\underline{h}(z),\overline{\underline{p}_0(z)}\rangle\Big]\ln\frac{1}{|z|^2}\, dm(z)$$
$$= \frac{1}{\pi}\int_D \partial_z\langle P_{\ker F(z)}F'(z)^*(F(z)F(z)^*)^{-1}\underline{h}(z),\overline{\underline{p}_0(z)}\rangle \times \ln\frac{1}{|z|^2}\, dm(z).$$

Let dA denote $\frac{dm}{\pi}$. This last term becomes, by Lemma (2.2.5),

$$\int_D \partial_z\langle Q^*(z)F'(z)^*\frac{(F(z)F(z)^*)^{-1}}{\det(F(z)F(z)^*)}\underline{h}(z),\overline{\underline{k}_0(z)}\rangle\ln\frac{1}{|z|^2}\, dA(z). \qquad (4)$$

Computing the ∂_z-derivative, we get that (4) is equal to the sum of the following four terms:

$$\text{(i)} = \int_D \langle Q^*(z)F'(z)^*\frac{(F(z)F(z)^*)^{-1}}{\det(F(z)F(z)^*)}\underline{h}(z),\overline{\underline{k}_0'(z)}\rangle\ln\frac{1}{|z|^2}\, dA(z),$$

$$\text{(ii)} = \int_D \langle Q^*(z)F'(z)^*\frac{(F(z)F(z)^*)^{-1}}{\det(F(z)F(z)^*)}\underline{h}'(z),\overline{\underline{k}_0(z)}\rangle\ln\frac{1}{|z|^2}\, dA(z),$$

$$\text{(iii)} = -\int_D \langle Q^*(z)F'(z)^*\frac{\begin{pmatrix}(F(z)F(z)^*)^{-1}F'(z)F(z)^*\\ \times (F(z)F(z)^*)^{-1}\end{pmatrix}}{\det(F(z)F(z)^*)}\underline{h}(z),\overline{\underline{k}_0(z)}\rangle \times \ln\frac{1}{|z|^2}\, dA(z),$$

and, suppressing the z in the inner product,

$$\text{(iv)} = -\int_D \langle Q^*F'^*(FF^*)^{-1}\left[\frac{\det(FF^*)}{(\det(FF^*))^2}tr[(FF^*)^{-1}F'F^*]\right]\underline{h}(z),\overline{\underline{k}_0}\rangle \times \ln\frac{1}{|z|^2}\, dA(z).$$

Note that the calculation of $\partial_z(\det(F(z)F(z)^*))$ is done in the proof of Lemma (2.2.7). We will be using the fact that for $A \in B(H)$ and $\underline{x} \in H$, then

$$\|A\underline{x}\|^2 \leq \|A\|^2\|\underline{x}\|^2 = \|A^*A\|\|\underline{x}\|^2 \leq tr\,(A^*A)\|\underline{x}\|^2.$$

In addition, we use the fact that if $A \geq 0$ and if $0 \leq P \leq cI$, then $tr(PAP) = trA^{\frac{1}{2}}P^2A^{\frac{1}{2}} \leq trc^2A = c^2trA$.

By Cauchy–Schwartz in both the inner product and the measure and then by Lemma (2.2.3), we get, suppressing the z in a portion of the next line,

(i)

$$\leq \left[\iint_D tr\left[(FF^*)^{-1}F'\frac{QQ^*}{(\det(FF^*))^2}F'^*(FF^*)^{-1}\right] \times \|\underline{h}(z)\|^2\ln\frac{1}{|z|^2}\, dA(z)\right]^{\frac{1}{2}}\|\underline{k}_0\|$$
$$\leq \left[\iint_D tr\left[(F(z)F(z)^*)^{-1}F'(z)\frac{P_{\ker F(z)}}{(\det(F(z)F(z)^*))}F'(z)^*(F(z)F(z)^*)^{-1}\right]\right.$$
$$\left.\times \|\underline{h}(z)\|^2\ln\frac{1}{|z|^2}\, dA(z)\right]^{\frac{1}{2}}\|\underline{k}_0\|.$$

So

$$(i) \leq \left[\frac{1}{\varepsilon^{2n}}\frac{1}{\varepsilon^{2}}\int_D \; tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker F(z)}F'(z)^*\right]\right.$$

$$\left.\times \left\|\underline{h}(z)\right\|^2 \ln\frac{1}{|z|^2}\; dA(z)\right]^{\frac{1}{2}} \left\|\underline{k}_0\right\|.$$

Appealing to Lemmas (2.2.6) and (2.2.7) with $a(z) = \ln\left(\det\frac{(F(z)F(z)^*)}{\varepsilon^{2n}}\right)$, we get

$$(i) \leq \left[\frac{e}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\right]^{\frac{1}{2}} \left\|\underline{h}\right\|\left\|\underline{k}_0\right\| \leq \sqrt{e}\,\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\left\|\underline{h}\right\|\left\|\underline{k}_0\right\|.$$

An entirely analogous argument applies to (ii).

For (iii), we use two Cauchy–Schwartz estimates and Lemmas (2.2.4) and (2.2.7) with $a(z)$as before and with $b(z) = tr((F(z)F(z)^*)^{-1}) + (\frac{1}{\varepsilon^2})\, a(z)$. Then $\|a\|_\infty \leq \ln(\frac{1}{\varepsilon^{2n}})and\|b\|_\infty \leq \frac{n}{\varepsilon^2} + \left(\frac{1}{\varepsilon^2}\right)\ln\left(\frac{1}{\varepsilon^{2n}}\right)$. We have

$$\frac{1}{4}\Delta a = tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{\ker F(z)}F'(z)^*\right]$$

And

$$\frac{1}{4}\Delta b \geq \frac{\left(tr\begin{bmatrix}(F(z)F(z)^*)^{-1}F(z)F'(z)^*(F(z)F(z)^*)^{-1} \\ \times F'(z)F(z)^*(F(z)F(z)^*)^{-1}\end{bmatrix}\right)}{\det\left((F(z)F(z)^*)\right)}\; in\; \overline{D}.$$

So

$$\text{(iii)} \leq \left[\int_D \; \frac{\left\|Q^*(z)F'(z)^*(F(z)F(z)^*)^{-\frac{1}{2}}\right\|^2}{det(F(z)F(z)^*)}\left\|\underline{h}(z)\right\|^2 ln\frac{1}{|z|^2}\; dA(z)\right]^{\frac{1}{2}}$$

$$\times\left[\int_D \; \frac{\left\|(F(z)F(z)^*)^{-\frac{1}{2}}F'(z)F(z)^*(F(z)F(z)^*)^{-1}\right\|^2}{det(F(z)F(z)^*)} \times \left\|\underline{k}_0(z)\right\|^2 ln\frac{1}{|z|^2}\; dA(z)\right]^{\frac{1}{2}}$$

$$\leq \left[\int_D \; tr\left[(F(z)F(z)^*)^{-1}F'(z)P_{ker\,F(z)}F'(z)^*\right]\left\|\underline{h}(z)\right\|^2 ln\frac{1}{|z|^2}\; dA(z)\right]^{\frac{1}{2}}$$

$$\times\left[\int_D \; \frac{\left(tr\begin{bmatrix}(F(z)F(z)^*)^{-1}F(z)F'(z)^*(F(z)F(z)^*)^{-1} \\ \times F'(z)F(z)^*(F(z)F(z)^*)^{-1}\end{bmatrix}\right)}{det\left((F(z)F(z)^*)\right)} \times \left\|\underline{k}_0(z)\right\|^2 ln\frac{1}{|z|^2}\; dA(z)\right]^{\frac{1}{2}}$$

$$\leq \left[e\left(ln\frac{1}{\varepsilon^{2n}}\right)\right]^{\frac{1}{2}}\left[\frac{e}{\varepsilon^2}\left(\frac{n}{\varepsilon^2} + \frac{1}{\varepsilon^{2n}}ln\frac{1}{\varepsilon^{2n}}\right)\right]^{\frac{1}{2}}.$$

If $0 < \varepsilon^2 < \frac{1}{e}$, then $\ln(1/\varepsilon^2) > 1$. So $\frac{n}{\varepsilon^2} \leq (1/\varepsilon^{2n})\ln(1/\varepsilon^{2n})$.

Then

$$\text{(iii)} \leq e\sqrt{2}\,\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right).$$

To handle (iv), we again begin with two applications of Cauchy–Schwartz. Use Lemmas (2.2.6) and (2.2.7) with $a(z)$ as before and with $c(z) = (\det(F(z)F(z)^*))^{-1} + (1/\varepsilon^{2n})\ln[\det((F(z)F(z)^*)/\varepsilon^{2n}]$.

Then

$$\|c\|_\infty \le \frac{1}{\varepsilon^{2n}} + \frac{1}{\varepsilon^{2n}} ln\left(\frac{1}{\varepsilon^{2n}}\right) \le \frac{2}{\varepsilon^{2n}} \ln\left(\frac{1}{\varepsilon^{2n}}\right)$$

And

$$\frac{1}{4}\Delta c \ge \frac{|tr[(F(z)F(z)^*)^{-1}F'(z)F(z)^*]|^2}{\det(F(z)F(z)^*)} \ on\ \overline{D}.$$

We compute that

$$\text{(iv)} \le \left[\int_D \frac{\left\|(F(z)F(z)^*)^{-\frac{1}{2}}F'(z)Q(z)\right\|^2}{det(F(z)F(z)^*)} \left\|\underline{k}_0(z)\right\|^2 ln\frac{1}{|z|^2}\, dA(z)\right]^{\frac{1}{2}}$$

$$\times \left[\int_D \left\|(F(z)F(z)^*)^{-\frac{1}{2}}\right\|^2 \frac{|tr[(F(z)F(z)^*)^{-1}F'(z)F(z)^*]|^2}{det(F(z)F(z)^*)}\right.$$

$$\left.\times \left\|\underline{h}(z)\right\|^2 ln\frac{1}{|z|^2}\, dA(z)\right]^{\frac{1}{2}} \le \left[e\left(ln\frac{1}{\varepsilon^{2n}}\right)\right]^{\frac{1}{2}}\left[\frac{1}{\varepsilon^2} e \frac{2}{\varepsilon^{2n}} ln\left(\frac{1}{\varepsilon^{2n}}\right)\right]^{\frac{1}{2}} \|\underline{h}\|\|\underline{k}_0\|$$

$$\le e\sqrt{2}\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\|\underline{h}\|\|\underline{k}_0\|.$$

Finally, combining the four estimates, we deduce that whenever $0 < \varepsilon^2 < \frac{1}{e}$ and $\varepsilon^2 I \le F(z)F(z)^* \le I$ for all $z \in D$, then

$$\left|\langle F^*(FF^*)^{-1}\underline{h}, \underline{\overline{p}}_0\rangle\right| \le \left(2\sqrt{e} + 2\sqrt{2}e\right)\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\|\underline{h}\|\|\underline{k}_0\|.$$

Therefore

$$T_F T_{F^*} \ge \left(2\left[2\sqrt{e} + 2\sqrt{2}e\right]\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right)\right)^{-2} I.$$

We have completed the proof of Theorem (2.2.1).

Combining Theorems (2.2.1) and (2.2.2), under the above hypothesis on $F(z)$ we get estimates for analytic solutions, $G(z)$, of $F(z)G(z) = I$ for all $z \in D$ of the form

$$\sup_{z\in D}\|G(z)\|_{B(\mathbb{C}^m,\mathbb{C}^n)} \le 2\left[2\sqrt{e} + 2\sqrt{2}e\right]\frac{1}{\varepsilon^{n+1}}\ln\left(\frac{1}{\varepsilon^{2n}}\right).$$

We end with two remarks:

(i) We apply similar techniques to obtain an improvement in the current bounds in Theorem (2.2.1) for the polydisk and remove the dependency on m.

(ii) If we were just interested in a proof of the vector-valued corona theorem without worrying about best bounds, the computational Lemmas (2.2.6) and (2.2.7) could be omitted. Then we would get a direct Hilbert space proof of the vector-valued corona theorem.

Also, we note that if F is $1 \times m$, then Lemma (2.2.5) has a simple direct proof.

Section (2.3): Corona Problem in the Disk and Polydisk

The classical Carleson Corona Theorem, see [40], states that if functions $f_j \in H^2(\mathbb{D})$are such that $\sum_{j=1}^{\infty}\left|f_j\right|^2 \ge \delta^2 > 0$ then there exist functions $g_j \in H^2(\mathbb{D})$ such that $\sum_{j=1}^{\infty} g_j f_j = 1$. This is equivalent to the fact that the unit disk $\mathbb{D}$ is dense in the maximal ideal space of the algebra H^∞, but the importance of the Corona Theorem goes much beyond the theory of maximal ideals of H^∞.

The Corona Theorem, and especially its generalization, the so called Matrix (Operator) Corona Theorem play an important role in operator theory (such as the angles between invariant subspaces, unconditionally convergent spectral decompositions, computation of spectrum, etc.). The Matrix Corona Theorem says that if $F \in H^\infty(\mathbb{D}; E_* \to E)$ is a bounded analytic function whose values are operators from a Hilbert space E_*, $\dim E_* < +\infty$, to another Hilbert space E such that

$$F^*(z)F(z) \geq \delta^2 I > 0, \qquad \forall z \in \mathbb{D}, \qquad (C)$$

then F has a bounded analytic left inverse $G \in H^\infty(\mathbb{D}; E_* \to E), GF \equiv I$. We should emphasize that the requirement $\dim E_* + \infty$is essential here. It was shown in [51], see also [52] or [38], that the Operator Corona Theorem fails if $\dim E_* = +\infty$. Note also that the above condition (C) is necessary for the existence of a bounded left inverse.

The classical Carleson Corona Theorem is a particular case of the matrix one: one just needs to consider F being the column $F = (f_1, f_2, \dots, f_n)^T$. It also worth noticing that the Matrix Corona Theorem follows from the classical one. Using a simple linear algebra argument Fuhrmann, see [41], was able to get the matrix version ($\dim E_*$, $\dim E < +\infty$) of the theorem from the classical result of Carleson. Later, using ideas from Wolff's proof of the Corona Theorem, M. Rosenblum, V. Tolokonnikov and A. Uchiyama, see [47], [50], [55], independently extended the Corona Theorem to infinitely many functions f_k. Using their result, V. Vasyunin was able to get the Operator Corona Theorem in the case $\dim E_* < +\infty, \dim E = +\infty$.

Since the Corona Theorem turns out to be very important in operator theory, there were some attempts to prove it using operator methods. While these attempts were not completely successful, some interesting relations were discovered. In particular, it was shown that a function $F \in H^\infty = H^\infty(\mathbb{D}; E_* \to E)$ is left invertible in H^∞if and only if the Toeplitz operator $T_{\overline{F}}$ is left invertible; here $\overline{F}$ denotes the complex conjugate of the matrix F.

Let us recall that given an operator function $\Phi \in L^\infty(\mathbb{T}; E_* \to E)$, the Toeplitz operator $T_\Phi: H^2(E_*) \to H^2(E)$ with symbol Φ is defined by

$$T_\Phi f := P_+(\Phi f),$$

where P_+ is the Riesz Projection (orthogonal projection onto H^2).

Considering the adjoint operator $\left(T_{\overline{F}}\right)^* = T_{\overline{F}^*} = T_{F^T}$one can conclude from here that F is left invertible in H^∞if and only if the Toeplitz operator $T_{F^T}: H^2(E) \to H^2(E_*)$ is right invertible. Since F^T is an analytic function

$$T_{F^T} f = F^T f, \qquad \forall f \in H^2(E).$$

and F is left invertible in H^∞if and only if for any $g \in H^2(E_*)$ the equation

$$F^T f = g \qquad (5)$$

has a solution $g \in H^2(E)$ satisfying the uniform estimate $\|f\|_2 \leq C\|g\|_2$.

The result that condition (C) implies (if $\dim E_* < +\infty$) left invertibility of the Toeplitz operator $T_{\overline{F}}$, or equivalently the solvability of Eq. (5), is called the Toeplitz Corona Theorem. In the case of the unit disk $\mathbb{D}$ one can easily deduce the Matrix Corona Theorem from the Toeplitz Corona Theorem by using the Commutant Lifting Theorem.

The main result is the Toeplitz Corona Theorem for the polydisk, see Theorem (2.3.2) below. To simplify the notation we used F instead of F^T, so the condition (C) is replaced by the condition $FF^* \geq \delta^2 I$. While in the polydisk it is not known how to get the Corona Theorem from the Toeplitz Corona Theorem (the Commutant Lifting Theorem for the polydisk is currently not known) the result seems to be of independent interest. In a particular case when F from Theorem (2.3.2) is a row vector (a $1 \times n$ matrix) this theorem was proved by Lin, see [62] or [74]. His approach involved using the Koszul complex to write down the $\bar{\partial}$-equations. Unfortunately, in several variables, unlike the one-dimensional case, higher order equations appear in addition to the $\bar{\partial}$-equation so the computation become quite messy. It is not clear how to use his technique to get the result in the matrix case we are treating here since the Fuhrmann–Vasyunin trick of getting the matrix result from the result for a column (row) vector does not work to solve the Toeplitz Corona Theorem.

We use tools from complex differential geometry to solve $\bar{\partial}$-equations on holomorphic vector bundles. In doing this we are following the ideas of Andersson, see [71] or [72], which in turn go back to Berndtsson.

To solve the $\bar{\partial}$-equation he uses a Hörmander type approach with weights and a modification of a Bochner–Kodaira–Nakano–Hörmander identity from complex geometry. While our approach is more along the lines of T. Wolff's proof and does not require anything more advanced than Green's formula.

We first use our technique to get an estimate in the Toeplitz Corona Theorem in the disk:

Theorem (2.3.1)[70]: Let $F \in H^\infty(\mathbb{D}; E \to E_*)$, $\dim E_* = r < +\infty$, such that $\delta^2 I \leq FF^* \leq I$ for some $0 < \delta^2 \leq \frac{1}{e}$. For $1 \leq p \leq \infty$ if $g \in H^p(\mathbb{D}; E_*)$ then the equation

$$Ff = g$$

has an analytic solution $f \in H^p(\mathbb{D}; E)$ with the estimate

$$\|f\|_p \leq \left(\frac{C}{\delta^{r+1}} \log \frac{1}{\delta^{2r}} + \frac{1}{\delta}\right) \|g\|_p \tag{6}$$

with $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e} \approx 8.38934$.

For the $p = 2$ case the above result with a different constant C was obtained recently using a different method by Trent [54]. The constant he obtained was $C = 2\sqrt{e} + 2\sqrt{2e} \approx 10.9859$..

The result for all p can be obtained from the case $p = 2$ via the Commutant Lifting Theorem, but we present here a simple direct proof.

Note, that we do not assume $\dim E < +\infty$ here.

Using a simple modification of our proof in one dimension we are also able to get the following result in the polydisk:

Theorem (2.3.2)[70]: Let $F \in H^\infty(\mathbb{D}^n; E \to E_*)$, $\dim E_* = r < +\infty$, such that $\delta^2 I \leq FF^* \leq I$ for some $0 < \delta^2 \leq \frac{1}{e}$. For $1 < p < \infty$ if $g \in H^p(\mathbb{D}^n; E_*)$ then the equation

$$Ff = g$$

has an analytic solution $f \in H^p(\mathbb{D}^n; E)$ with the estimate

$$\|f\|_p \leq \left(\frac{nCC(p)^n}{\delta^{r+1}} \log \frac{1}{\delta^{2r}} + \frac{1}{\delta}\right) \|g\|_p \tag{7}$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e} \approx 8.38934$, and $C(p) = \frac{1}{\sin\left(\frac{\pi}{p}\right)}$ the norm of the (scalar) Riesz projection from $L^p(\mathbb{T})$ onto $H^p(\mathbb{D})$. For $p = 2$ the estimate can be improved to

$$\|f\|_2 \le \left(\frac{\sqrt{n}C}{\delta^{r+1}}\log\frac{1}{\delta^{2r}} + \frac{1}{\delta}\right)\|g\|_2 \tag{8}$$

With $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e} \approx 8.38934$

We will start with proving Theorem (2.3.1) for $p = 2$, we set up the main estimate needed to prove the theorem, we discuss a version of the Carleson Embedding Theorem and its analogue for functions defined on holomorphic vector bundles, which will be later used to prove the main estimates, we perform computation of some derivatives and Laplacians that will be used in the estimates. We also construct there subharmonic functions to be used in the embedding theorems. Then we deal with the main estimate for $p = 2$; and explain how to use the construction for other p and we treat the case of the polydisk for $p = 2$ and finally we treat the case of general p.

To prove Theorem (2.3.1) for $p = 2$, for a given $g \in H^2 := H^2(E_*)$ with $\|g\|_2 = 1$, we need to solve the equation

$$Ff = g, f \in H^2(E) \tag{9}$$

with the estimate $\|f\|_2 \le C = C(\delta, r)$. By a normal families argument it is enough to suppose that F and g are analytic in a neighborhood of $\mathbb{D}$. Any estimate obtained in this case can be used to find an estimate when F is only analytic on $\mathbb{D}$. Since $\delta^2 I \le FF^* \le I$, it is easy to find a non-analytic solution f_0 of (9),

$$f_0 := \Phi g := F^*(FF^*)^{-1}g.$$

To make f_0 into an analytic solution, we need to find $v \in L^2(E)$ such that $f := f_0 - v \in H^2$ and $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$. Then

$$Ff = F(f_0 - v) = Ff_0 - Fv = g,$$

and we are done. The standard way to find such v is to solve a $\bar{\partial}$-equation with the condition $v(z) \in \ker F(z)$ insured by a clever algebraic trick. This trick also admits a “scientific” explanation, for one can get the desired formulas by writing a Koszul complex. What we do essentially amounts to solving the $\bar{\partial}$-equation $\bar{\partial}v = \bar{\partial}f_0$ on the holomorphic vector bundle $\ker F(z)$. We mostly follow the ideas of Andersson found in [71]. He used ideas from complex differential geometry to solve the corona problem by finding solutions to the $\bar{\partial}$-equation on holomorphic vector bundles.

Since our target audience consists of analysts, all differential geometry will be well hidden. Our main technical tool will be Green’s formula

$$\int_{\mathbb{T}} u\, dm - u(0) = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta u \log\frac{1}{|z|} dx\, dy \tag{10}$$

Instead of the usual Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ it more convenient for us to use the “normalized” one $\tilde{\Delta} := \frac{1}{4}\Delta\bar{\partial}\partial = \partial\bar{\partial}$. If we denote by μ the measure defined by

$$d\mu = \frac{2}{\pi}\log\frac{1}{|z|}\, dx\, dy,$$

then Green’s formula can be rewritten as

$$\int_{\mathbb{T}} u\, dm - u(0) = \int_{\mathbb{D}} \tilde{\Delta}u\, d\mu. \tag{11}$$

To find the function v we will use duality. We want $f_0 - v \in H^2(E)$, therefore the equality

$$\int_{\mathbb{T}} \langle f_0, h\rangle dm = \int_{\mathbb{T}} \langle v, h\rangle\, dm$$

must hold for all $h \in (H^2)^{\perp}$. Using Green's formula we get

$$\int_{\mathbb{T}} \langle f_0, h\rangle dm = \int_{\mathbb{T}} \langle \Phi g, h\rangle\, dm = \int_{\mathbb{D}} \partial\bar{\partial}[\langle \Phi g, h\rangle]\, d\mu = \int_{\mathbb{D}} \partial\big[\langle \bar{\partial}\Phi g, h\rangle\big]\, d\mu$$

Here we used the harmonic extension of h, so h is anti-analytic and $h(0) = 0$. The functions $\Phi \coloneqq F^*(FF^*)^{-1}$ and g are already defined in the unit disk $\mathbb{D}$.

Now the critical moment: let $\Pi(z) \coloneqq P_{\ker F(z)}$ be the orthogonal projection onto $\ker F(z)$, $\Pi = I - F^*(FF^*)^{-1}F$.Direct computation shows that $\tilde{\partial}\Phi = \Pi(\partial\Phi)^*(FF^*)^{-1}$, so $\Pi\bar{\partial}\Phi = \bar{\partial}\Phi$. Therefore, if we define a vector-valued function ξ on $\mathbb{D}$ by $\xi(z) \coloneqq \Pi(z)h(z)$, then

$$\int_{\mathbb{D}} \partial\big[\langle \bar{\partial}\Phi g, h\rangle\big]\, d\mu = \int_{\mathbb{D}} \partial\big[\langle \bar{\partial}\Phi g, \Pi\, h\rangle\big]\, d\mu = \int_{\mathbb{D}} \partial\big[\langle \bar{\partial}\Phi g, \xi\rangle\big]\, d\mu =: L(\xi)$$
$$= L_g(\xi). \qquad (12)$$

Note, that $L = L_g$ is a conjugate linear functional, i.e. $\bar{L}$ (defined by $\bar{L}(\xi) \coloneqq \overline{L(\xi)}$) is a linear functional. Suppose we are able to prove the estimate

$$|L(\xi)| \le C(r,\delta)\|\xi\|_2\,, \qquad \forall \xi = \Pi h, h \in H^2(E)^{\perp}. \qquad (13)$$

Then (by a Hilbert space version of the Hahn–Banach Theorem, which is trivial) L can be extended to a bounded linear functional on $L^2(E)$, so there exists a function $v \in L^2(E), \|v\|_2 \le C,$ such that

$$L(\xi) = \int_{\mathbb{T}} \langle v, \xi\rangle dm\,, \quad \forall \xi = \Pi h, h \in H^2(E)^{\perp}.$$

Replacing v by Πv we can always assume without loss of generality that $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}$, so $Fv = 0$. By the construction

$$\int_{\mathbb{T}} \langle v, h\rangle dm = \int_{\mathbb{T}} \langle v, \Pi h\rangle dm = L(\Pi h) = \int_{\mathbb{T}} \langle \Phi g, h\rangle dm\,, \quad \forall\, h \in H^2(E)^{\perp},$$

so $f\colon f_0 - v \coloneqq \Phi g - v \in H^2(E)$ is the analytic solution we want to find. It satisfies the estimate

$$\|f\|_2 \le \|f_0\|_2 + \|v\|_2 \le \frac{1}{\delta}\|g\|_2 + C(r,\delta)\|g\|_2.$$

Therefore, Theorem (2.3.1) would follow from the following proposition

Proposition (2.3.3)[70]: Under the assumptions of Theorem (2.3.1) the linear functional L defined by (12) satisfies the estimate

$$|L(\xi)| \le C(r,\delta)\|\xi\|_2\,, \qquad \forall \xi = \Pi h, h \in H^2(E)^{\perp}$$

With

$$C(r,\delta) = \frac{C}{\delta^{r+1}}\log\frac{1}{\delta^{2r}}\,,$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e}$.

In what follows we will need the following simple technical lemma that is proved by direct computation.

Lemma (2.3.4)[70]: For Π and Φ defined above we have

$$\partial\Pi = -F^*(FF^*)^{-1}F'\Pi,$$
$$\bar{\partial}\Phi = \Pi(F')^*(FF^*)^{-1},$$

and

$$\partial\bar{\partial}\Phi = \partial\Pi(F')^*(FF^*)^{-1} - (\bar{\partial}\Phi)F'\Phi = \partial\Pi\bar{\partial}\Phi + (\partial\Pi)^*\Phi F'\Phi.$$

Corollary (2.3.5)[70]: For the projection Π defined above we have

$$\Pi\partial\Pi = 0\,, (\partial\Pi)\Pi = \partial\Pi\,, (\bar{\partial}\Pi)\Pi = 0\,, \Pi\bar{\partial}\Pi = \bar{\partial}\Pi.$$

The above identities are well-known in complex differential geometry, but we can easily get them from Lemma (2.3.4). Namely, since Π is the orthogonal projection onto $\ker F$ we have $F\Pi = 0$. Taking the adjoint we get $\Pi F^* = 0$ which implies $\Pi\partial\Pi = 0$. The second identity is trivial, and the last two are obtained from the first two by taking adjoints.

As is well known, Carleson measures play a prominent role in the proof of the Corona theorem, both in Carleson's original proof and in T. Wolff's proof and subsequent modifications. It is also known to the specialists, that essentially all Carleson measures can be obtained from the Laplacian of a bounded subharmonic function. We will need the following well-known theorem, see [45], which was probably first proved by Uchiyama.

Theorem (2.3.6)[70]: (Carleson Embedding Theorem). Let φ be a non-negative, bounded, subharmonic function. Then for any $f \in H^2(E)$

$$\int_{\mathbb{D}} \tilde{\Delta}\varphi(z)\|f(z)\|^2 d\mu \le e\|\varphi\|_\infty \|f\|_2^2\,.$$

Here $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}\; dx\, dy\,, and\ \tilde{\Delta} = \frac{1}{4}\Delta = \partial\bar{\partial}.$

Proof. Because of homogeneity, we can assume without loss of generality that $\|\varphi\|_\infty = 1$. Direct computation shows that

$$\tilde{\Delta}\left(e^{\varphi(z)}\|f(z)\|^2\right) = e^{\varphi}\tilde{\Delta}\varphi\|f\|^2 + e^{\varphi}\|\partial\varphi f + \partial f\|^2 \ge \tilde{\Delta}\varphi\|f\|^2.$$

Then Green's formula implies

$$\int_{\mathbb{D}} \tilde{\Delta}\varphi\|f\|^2\, d\mu \le \int_{\mathbb{D}} \tilde{\Delta}(e^{\varphi}\|f\|^2)\, d\mu = \int_{\mathbb{T}} e^{\varphi}\|f\|^2\, dm - e^{\varphi(0)}\|f(0)\|^2$$
$$\le e\int_{\mathbb{T}} \|f\|^2\, dm = e\|f\|_2^2.$$

Remark (2.3.7)[70]: It is easy to see, that the above Lemma implies the embedding $\int_{\mathbb{D}} \|f\|^2\, d\mu \le C\int_{\mathbb{T}} \|f\|^2\, dm$ (with $C = e$) for all analytic functions f . Using the function $4/(2-\varphi)$) instead of e^{φ} it is possible to get the embedding for harmonic functions with the constant $C = 4$. We suspect the constants e and 4 are the best possible for the analytic and harmonic embedding respectively. We cannot prove that, but it is known that 4 is the best constant in the dyadic (martingale) Carleson Embedding Theorem.

We will need a similar embedding theorem for functions of form $\xi = \Pi h, h \in H^2(E)^\perp$. Such functions are not analytic or harmonic, so the classical Carleson Embedding Theorem does not apply. As a result, the proof is more complicated, and the constant is significantly worse.

Recall that $\Pi(z) = P_{\ker F(z)}$ is the orthogonal projection onto $\ker F(z), \Pi - F^*(FF^*)^{-1}F,$ and that $d\mu = \frac{2}{\pi}\log\frac{1}{|z|}\; dx\, dy.$

Lemma (2.3.8)[70]: Let φ be a non-negative, bounded, subharmonic function in $\mathbb{D}$ satisfying

$$\tilde{\Delta}\varphi(z) \ge \|\partial\Pi(z)\|^2\,, \qquad \forall z \in \mathbb{D},$$

and let $K = \|\varphi\|_\infty$. Then for all ξ of the form $\xi = \Pi h, h \in H^2(E)^\perp$

$$\int_{\mathbb{D}} \tilde{\Delta}\varphi(z)\|\xi(z)\|^2\, d\mu\,(z) \le eKe^K\|\xi\|_2^2$$

and

$$\int_{\mathbb{D}} \left\|\bar{\partial}\xi\right\|^2 d\mu \le (1 + eKe^K)\|\xi\|_2^2 \,.$$

Proof. Let us take an arbitrary non-negative bounded subharmonic function φ and compute $\tilde{\Delta}(e^{\varphi}\|\xi\|^2)$. Corollary (2.3.5) implies that $\Pi\partial\Pi = 0$ and $\partial\Pi\Pi = \partial\Pi$. Therefore, using $\partial h = 0$ we get $\partial\xi = \partial(\Pi h) = \partial\Pi h + \Pi\partial h = \partial\Pi h = \partial\Pi\xi$, and so

$$\langle \partial\xi, \xi\rangle = \langle\, \partial\xi, \Pi\xi\rangle = \langle\, \partial\Pi h, \Pi\xi\rangle = 0.$$

Therefore

$$\partial(e^{\varphi}\|\xi\|^2) = e^{\varphi}\partial\varphi\|\xi\|^2 + e^{\varphi}\langle\partial\xi,\xi\rangle + e^{\varphi}\langle\, \xi, \bar{\partial}\xi\rangle = e^{\varphi}\partial\|\xi\|^2 + e^{\varphi}\langle\, \xi, \bar{\partial}\xi\rangle.$$

Taking $\bar{\partial}$ of this equality (and again using $\langle\, \xi, \bar{\partial}\xi\rangle = 0$) we get

$$\tilde{\Delta}(e^{\varphi}\|\xi\|^2) = e^{\varphi}\left(\tilde{\Delta}\varphi\|\xi\|^2 + \left\|\bar{\partial}\varphi\xi + \bar{\partial}\xi\right\|^2 + \langle\xi, \tilde{\Delta}\xi\rangle\right).$$

To handle $\langle\xi, \tilde{\Delta}\xi\rangle$we take the ∂ derivative of the equation $\langle\, \xi, \partial\xi\rangle = 0$ to get

$$\langle\partial\xi, \partial\xi\rangle + \langle\xi, \bar{\partial}\partial\xi\rangle = 0,$$

and therefore $\langle\xi, \tilde{\Delta}\xi\rangle = -\|\partial\xi\|^2 = -\|(\partial\Pi)\xi\|^2$. Since $\varphi \ge 0$

$$\begin{aligned}\int_{\mathbb{D}} &\left(\tilde{\Delta}\varphi\|\xi\|^2 - \|(\partial\Pi)\xi\|^2\right) d\mu \\ &\le \int_{\mathbb{D}} \left(\tilde{\Delta}\varphi\|\xi\|^2 - \|(\partial\Pi)\xi\|^2 + \left\|\bar{\partial}\varphi\xi + \bar{\partial}\xi\right\|^2\right) e^{\varphi}\, d\mu \\ &= \int_{\mathbb{T}} e^{\varphi}\|\xi\|^2\, dm; \qquad (14)\end{aligned}$$

the equality is just Green's formula (recall that $\xi(0) = 0$). In the last inequality replacing φ by $t\varphi, t > 1$ we get

$$\int_{\mathbb{D}} \left(t\tilde{\Delta}\varphi\|\xi\|^2\right) - \|(\partial\Pi)\xi\|^2\, d\mu \le \int_{\mathbb{T}} e^{t\varphi}\|\xi\|^2\, dm \le e^{tK}\|\xi\|_2^2.$$

Now we use the inequality$\tilde{\Delta}\varphi \ge \|(\partial\Pi)\|^2$. It implies $\tilde{\Delta}\varphi\|\xi\|^2 - \|\partial\Pi\xi\|^2 \ge 0$, and therefore

$$(t-1)\int_{\mathbb{D}} \tilde{\Delta}\varphi\|\xi\|^2\, d\mu \le e^{tK}\|\xi\|_2^2.$$

Hence

$$\int_{\mathbb{D}} \tilde{\Delta}\varphi\|\xi\|^2\, d\mu \le \min_{t>1}\frac{e^{tK}}{t-1}\|\xi\|_2^2 = eKe^{tK}\|\xi\|_2^2$$

(minimum is attained at $t = 1 + 1/K$), and thus the first statement of the lemma is proved.

To prove the second statement, put $\varphi \equiv 0$ in (14) (we do not use any properties of φ except that $\varphi \ge 0$ in (14)) to get

$$\int_{\mathbb{D}} \left(\left\|\bar{\partial}\xi\right\|^2 - \|(\partial\Pi)\xi\|^2\right) d\mu = \int_{\mathbb{T}} \|\xi\|^2\, dm = \|\xi\|_2^2 \,.$$

But the second term can be estimated as

$$\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2\, d\mu \le \int_{\mathbb{D}} \tilde{\Delta}\varphi\|\xi\|^2\, d\mu \le eKe^K\|\xi\|_2^2 \,,$$

and therefore

$$\int_{\mathbb{D}} \left\|\bar{\partial}\xi\right\|^2 d\mu \le (1 + eKe^K)\|\xi\|_2^2 \,.$$

There will be points in the proof where we would like to invoke Carleson's Embedding Theorem. To do so we will need a non-negative, bounded, subharmonic function. We construct the necessary subharmonic functions so they will be available when we finally estimate the integral in question. We define the two functions used and collect their relevant properties. First, we recall a basic fact that will aid in showing that the functions we construct are subharmonic.

Lemma (2.3.9)[70]: Let $A(t)$ be a differentiable $n\times n$ matrix-valued function. Define the function $f(t)=\det\big(A(t)\big)$. Then

$$f'(t)=\det(A(t))\,tr\,\big(A^{-1}(t)A'(t)\big).$$

Proof. Fix a point t and for brevity of notation let us use A instead of $A(t)$. Since $A(\cdot)$ is differentiable

$$\det(A(t+h))=\det(A+A'h+o(h))=\det A\det(I+A^{-1}A'h+o(h))$$
$$=\det A\prod(1+h\mu_k+o(h)),$$

where μ_k are the eigenvalues of $A^{-1}(t)A'(t)$. Expanding this product we have

$$\prod(1+h\mu_k+o(h))=1+h\sum\mu_k+o(h)=1+htr(A^{-1}A')+o(h).$$

Then

$$\det(A(t+h))=\det(A)+h\det(A)\,tr\,(A^{-1}A')+o(h),$$

which implies the desired formula for the derivative.

Define the function $\varphi=tr\,(\log(\delta^{-2}FF^*))=\log(\delta^{-2n}\det(FF^*))$

Then a straight forward application of the above lemma gives

$$\tilde{\Delta}\varphi=\partial\bar{\partial}\varphi=\partial[tr((FF^*)^{-1}F(F')^*)]=tr\,[((FF^*)^{-1}F'\Pi(F')^*)]$$

with the last line following by substitution of Π. For another approach to this computation see [54]. Using the identities $\Pi^2=\Pi, tr(AB)=tr(BA)$, and recalling that

$$\partial\Pi=-F^*(FF^*)^{-1}F'\Pi$$

we get

$$\tilde{\Delta}\varphi=tr[(FF^*)^{-1}F'\Pi(F')^*]=tr[F^*(FF^*)^{-1}F'\Pi\Pi(F')^*(FF^*)^{-1}F]=tr[\partial\Pi(\partial\Pi)^*]$$
$$\geq\|\partial\Pi\|^2$$

with the last inequality following since $tr\,[AA^*]\geq\|A\|^2$. This function will play a prominent role in the estimation of certain integrals. We should also note that

$$0\leq\varphi\leq K:=\log\frac{1}{\delta^{2n}}.$$

We will also need another function to help in the estimation of the linear functional L in question. Let $\lambda=tr[(FF^*)^{-1}]$. A simple computation gives,

$$\tilde{\Delta}\lambda=tr[\Phi^*(F')^*(FF^*)^{-1}F'\Phi]-tr[(FF^*)^{-1}F'\Pi(F')^*(FF^*)^{-1}]$$
$$\geq tr[\Phi^*(F')^*(FF^*)^{-1}F'\Phi^*]-\delta^2tr[\partial\Pi(\partial\Pi)^*].$$

Now we define the function $\psi=\lambda+\delta^2\varphi$. Then, recalling that $\Phi=(F)^*(FF^*)^{-1}$ we get

$$\tilde{\Delta}\psi\geq tr[\Phi^*(F')^*(FF^*)^{-1}F'\Phi]=tr[\Phi F'\Phi(\Phi(F')\Phi)^*]\geq[\Phi F'\Phi]^2.$$

So ψ is subharmonic and $0\leq\psi\leq\frac{n}{\delta^2}+\frac{1}{\delta^2}\log\frac{1}{\delta^{2n}}$. We should note that the assumption $0<\delta^2\leq\frac{1}{e}$ implies $\log\delta^{-2}\geq 1$. This gives

$$0\leq\psi\leq L:=\frac{2}{\delta^2}\log\frac{1}{\delta^{2n}}\ .$$

Now we need to estimate $L(\xi)$. Computing ∂ of the inner product we get

$$L(\xi) = \int_{\mathbb{D}} \partial\big[\langle \bar{\partial}\Phi g, \xi\rangle\big]\, d\mu = \int_{\mathbb{D}} \langle \partial\bar{\partial}\Phi g, \xi\rangle\, d\mu + \int_{\mathbb{D}} \langle \bar{\partial}\Phi g', \xi\rangle\, d\mu + \int_{\mathbb{D}} \langle \bar{\partial}\Phi g, \bar{\partial}\xi\rangle\, d\mu$$
$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

We need to estimate each of the above integrals as closely as possible. Each integral has a term involving derivatives of Π, g and ξ. The idea is to separate the integralsusing Cauchy–Schwarz, giving one derivative to each term.

We now estimate the first integral. Recalling that $\partial\bar{\partial}\Phi = \partial\Pi\bar{\partial}\Phi + (\partial\Pi)^*\Phi F'\Phi$ we get

$$I = \int_{\mathbb{D}} \langle \partial\bar{\partial}\Phi g, \xi\rangle\, d\mu = \int_{\mathbb{D}} \big\{\langle \partial\Pi\bar{\partial}\Phi g, \xi\rangle + \langle (\partial\Pi)^*\Phi F'\Phi g, \xi\rangle\big\}\, d\mu.$$

Since $(\partial\Pi)^*\Pi = 0$ we have $(\partial\Pi)^*\xi = 0$, and so $\langle \partial\Pi\bar{\partial}\Phi g, \xi\rangle = 0$. Therefore

$$I = \int_{\mathbb{D}} \langle (\partial\Pi)^*\Phi F'\Phi g, \xi\rangle\, d\mu = \int_{\mathbb{D}} \langle \Phi F'\Phi g, (\partial\Pi)\xi\rangle\ d\mu,$$

and the Cauchy–Schwarz inequality implies

$$|\mathrm{I}| \leq \left(\int_{\mathbb{D}} \|\Phi F'\Phi g\|^2\, d\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2\, d\mu\right)^{1/2}.$$

To estimate the second factor we use Lemma (2.3.9). Recall that the function

$$\varphi = \log(\delta^{-2n}\det(FF^*)),$$

constructed above satisfies the inequalities

$$\tilde{\Delta}\varphi \geq \|\partial\Pi\|^2\ , and\ 0 \leq \varphi \leq K \coloneqq \log\delta^{-2n} \tag{15}$$

Therefore, Lemma (2.3.9) implies

$$\int_{\mathbb{D}} \|(\partial\Pi)\xi\|^2\, d\mu \leq eKe^K\|\xi\|_2^2 = e\delta^{-2n}\log\delta^{-2n}\|\xi\|_2^2\,.$$

To estimate the first factor, notice that the function ψ constructed satisfies

$$\tilde{\Delta}\psi \geq \|\Phi F'\Phi\|^2\ ,\quad and\ 0 \leq \psi \leq L \coloneqq 2\delta^{-2}\log\delta^{-2n}\,.$$

Then the Carleson Embedding Theorem (Theorem (2.3.6)) implies

$$\int_{\mathbb{D}} \|\Phi F'\Phi g\|^2\, d\mu \leq eL\|g\|_2^2 = 2e\delta^{-2}\log\delta^{-2n}\|g\|_2^2\,,$$

and thus

$$|\mathrm{I}| \leq \sqrt{KL}\|\xi\|_2\|g\|_2 = \frac{\sqrt{2e}}{\delta^{n+1}}\log\delta^{-2n}\|\xi\|_2\|g\|_2\,.$$

Now we estimate II. By the Cauchy–Schwarz inequality, we have

$$|\mathrm{II}| \leq \int_{\mathbb{D}} \big|\langle \bar{\partial}\Phi g', \xi\rangle\big|\, d\mu \leq \left(\int_{\mathbb{D}} \big\|\bar{\partial}\Phi\big\|^2\|\xi\|^2\, d\mu\right)^{1/2} \left(\int_{\mathbb{D}} \|g'\|^2\, d\mu\right)^{1/2}.$$

Observe that $\tilde{\Delta}\|g\|^2 = \|g'\|^2$ since g is holomorphic. So, applying Green's Theorem to the second factor we get

$$\int_{\mathbb{D}} \|g'\|^2\, d\mu = \int_{\mathbb{T}} \|g\|^2 dm - \|g(0)\|^2 \leq \|g\|_2^2\,.$$

To estimate the first integral, notice, that

$$\|\Phi\|^2 = \|\Phi^*\Phi\| = \|(FF^*)^{-1}\| \leq \delta^{-2}$$

(recall that $\Phi = F^*(FF^*)^{-1}$). Since $\bar{\partial}\Phi = -(\partial\Pi)^*\Phi$, we can estimate

$$\big\|\bar{\partial}\Phi\big\|^2 = \big\|\big(\bar{\partial}\Phi\big)^*\bar{\partial}\Phi\big\| = \|\Phi^*\partial\Pi(\partial\Pi)^*\Phi\| \leq \|\partial\Pi(\partial\Pi)^*\| \cdot \|\Phi\|^2 \leq \delta^{-2}\|\partial\Pi\|^2.$$

Therefore (see (15)), $\|\bar{\partial}\Phi\|^2 \le \delta^{-2}\tilde{\Delta}\varphi$, where $\varphi = \log(\delta^{-2n}\det(FF^*))$ is the subharmonic function constructed before. Applying Lemma (2.3.9) we get

$$\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|\xi\|^2 \, d\mu \le \delta^{-2} \int_{\mathbb{D}} \tilde{\Delta}\varphi \|\xi\|^2 \, d\mu \le \delta^{-2} eKe^K \|\xi\|_2^2 \,,$$

where $K = \log \delta^{-2n}$, see (15). Joining the estimates together, we get

$$|\mathrm{II}| \le \delta^{-1}\sqrt{eK}e^{K/2}\|g\|_2\|\xi\|_2 \le \delta^{-1}\sqrt{e}Ke^{K/2}\|g\|_2\|\xi\|_2 = \frac{\sqrt{2}}{\delta^{n+1}}\log\delta^{-2n}\,\|g\|_2\|\xi\|_2$$

(since $\delta^2 \le \frac{1}{e}$, the value of K satisfies $K^{1/2} \le K$).

Finally moving on to integral III. Using Cauchy–Schwarz, we have

$$|\mathrm{III}| \le \int_{\mathbb{D}} \left|\bar{\partial}\Phi g, \bar{\partial}\xi\right| \quad d\mu \le \left(\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|g\|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2 \, d\mu\right)^{1/2} .$$

As we already have shown above, $\|\bar{\partial}\Phi\|^2 \le \delta^{-2}\tilde{\Delta}\varphi$. The Carleson Embedding Theorem (Theorem (2.3.6)) implies

$$\int_{\mathbb{D}} \|\bar{\partial}\Phi\|^2 \|g\|^2 \, d\mu \le \delta^{-2} \int_{\mathbb{D}} \tilde{\Delta}\varphi \|g\|^2 \, d\mu \le \delta^{-2} eK \|g\|_2^2 \,.$$

Using Lemma (2.3.9) we can estimate

$$\int_{\mathbb{D}} \|\bar{\partial}\xi\|^2 \, d\mu \le (1 + eKe^K)\|\xi\|_2^2 \le (e^{-1} + e)Ke^K \|\xi\|_2^2 \,.$$

Here we are using the fact that $K \ge 1$ for $\delta^2 \le 1/e$. Combining the estimates, we get

$$|\mathrm{III}| \le \sqrt{1+e^2}Ke^{K/2}\|g\|_2\|\xi\|_2 = \frac{\sqrt{1+e^2}}{\delta^{n+1}}\log\delta^{-2n}\,\|g\|_2\|\xi\|_2 \,.$$

Joining the estimates for I, II, III we get

Proposition (2.3.10)[70]: Under the assumptions of Theorem (2.3.1) the linear functional L defined by (12) satisfies the estimate

$$|L(\xi)| \le C(r,\delta)\|\xi\|_2 \,, \qquad \forall \xi = \Pi h, h \in H^2(E)^\perp \,,$$

With

$$C(r,\delta) = \frac{C}{\delta^{r+1}}\log\frac{1}{\delta^{2r}} \,,$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e}$.

Proposition (2.3.10) is just a restatement of Proposition (2.3.3), and this then proves Theorem (2.3.1) for the case of $p = 2$.

Note, that the constant C is a bit better than the constant $\frac{2\sqrt{2e}}{2\sqrt{2e}} + 2\sqrt{e} \approx 10.9859$ obtained by Trent in [54].

Now we indicate how we can use the H^2 result to figure out the H^p result. We can use much of the same approach as in the $H^2(E)$ case. Our goal is to solve the equation

$$Ff = g, f \in H^p(E)$$

for the given $g \in H^p(E_*)$, with $\|g\|_p = 1$, and furthermore we want the estimate $\|f\|_p \le C$. Again we will have the obvious non-analytic solution to the problem

$$f_0 := \Phi g := F^*(FF^*)^{-1}g.$$

To make this into an analytic solution we will need to find a function $v \in L^p(E)$ such that $f_0 - v \in H^p$ and $v(z) \in \ker F(z)$. This will be accomplished by duality. As in the $H^2(E)$ case we need

$$\int_{\mathbb{T}} \langle f_0, h\rangle\, dm = \int_{\mathbb{T}} \langle v, h\rangle\, dm$$

to hold for all $h \in H^p(E)^\perp = H_0^q(E)$ (this uses the standard duality of H^p spaces see [42] or [45]). Again we can ensure that $v \in \ker F(z)$ since $\bar{\partial}\Phi = \Pi\bar{\partial}\Phi$. So we need to get an estimate on the linear functional

$$L(\xi) = L_g(\xi) = \int_{\mathbb{D}} \partial\left[\bar{\partial}\Phi g, \xi\right] d\mu$$

with $\xi = \Pi h$ and $h \in H^p(E)^\perp$. If we can then prove that

$$|L(\xi)| \leq C\|\xi\|_q$$

then by the Hahn–Banach Theorem and duality in L^p spaces with values in a Hilbert space we would have the existence of a function $v \in L^p(E)$ with $\|v\|_p \leq C$, such that

$$L(\xi) = \int_{\mathbb{T}} \langle v, \xi\rangle\, dm, \qquad \forall \xi = \Pi h, h \in H^p(E)^\perp.$$

Then replacing v by Πv we can assume without loss of generality that $v(z) \in \ker F(z)$a.e. on $\mathbb{T}$. But then the construction would give,

$$\int_{\mathbb{T}} \langle v, h\rangle\, dm = \int_{\mathbb{T}} \langle v, \Pi h\rangle\, dm = L(\Pi h) = \int_{\mathbb{T}} \langle \Phi g, h\rangle\ dm, \qquad \forall h \in H^p(E)^\perp,$$

so $v - f_0 \in H^p(E)$. So we only need to show how to prove the estimate

$$|L(\xi)| \leq C\|\xi\|_q .$$

The main idea is to use the L^2 result we just proved. Namely, if we replace g by $\tilde{g} = \varphi^{-1}g$and ξ by$\tilde{\xi} = \bar{\varphi}\xi$, where $\varphi$is an appropriate (scalar) outer function, then

$$L_g(\xi) = L_{\tilde{g}}(\tilde{\xi}).$$

Suppose we are able to find the outer function φsuch that $\|\tilde{g}\|_2\|\tilde{\xi}\|_2 \leq \|g\|_p\|\xi\|_q$.
Then, since φ is analytic, $\tilde{g} \in H^2(E)$ and

$$\tilde{\xi} \in K \coloneqq \operatorname{clos}_{L^2}\{\Pi h : h \in H^2(E)^\perp\}.$$

Therefore we can apply the L^2 result we have proved before to get

$$\begin{aligned} \left|L_g(\xi)\right| = \left|L_{\tilde{g}}(\tilde{\xi})\right| &\leq \frac{C}{\delta^{n+1}} \log \delta^{-2n}\, \|\tilde{g}\|_2\|\tilde{\xi}\|_2 \\ &\leq \frac{C}{\delta^{n+1}} \log \delta^{-2n}\, \|g\|_p\|\xi\|_q. \end{aligned} \tag{16}$$

To find the function φ we need to consider the cases $p < 2$ and $p > 2$ separately.

First look at the case $p < 2$. Consider the outer part of g, i.e. a scalar-valued outer function g_{out} such that

$$|g_{\text{out}}(z)| = \|g(z)\| \qquad a.e. on\ \mathbb{T}.$$

Define

$$\tilde{g}(z) = (g_{\text{out}})^{p/2-1}(z)g(z) \qquad and$$
$$\tilde{\xi}(z) = (\tilde{g}_{\text{out}})^{1-p/2}(z)\xi(z).$$

Then $\|\tilde{g}\|_2 = \|g\|_p^{p/2}$, and computation using Hölder's Inequality gives and$\|\tilde{\xi}\|_q \leq \|\xi\|_q\|g\|_p^{1-p/2}$,where$\frac{1}{p} + \frac{1}{q} = 1$. Therefore $\|\tilde{g}\|_2\|\tilde{\xi}\|_2 \leq \|g\|_p\|\xi\|_q$ and the main inequality (16) is proved.

The case when $p > 2$ is analogous, except in this case we need to construct a scalar outer function ξ_{out} such that

$$|\xi_{\text{out}}(z)| = \|\xi(z)\| \text{ a.e. on } \mathbb{T}.$$

Note, that here we cannot say that ξ_{out} is the outer part of ξ, because ξ is neither holomorphic nor antiholomorphic. So, a little more explanation is needed.

First of all recall that we assumed (without loss of generality) that F is an analytic function in a slightly bigger disk than $\mathbb{D}$, so the projection $\Pi = I - F^*(FF^*)^{-1}F$ is real analytic on the unit circle $\mathbb{T}$. Second, we only need to estimate the functional L on a dense set, so we can assume that the test function h is a trigonometric polynomial in $(H^2)^{\perp}$. Therefore the function $\xi = \Pi h$ is real analytic on $\mathbb{T}$, and so $\int_{\mathbb{T}} \log\|\xi(z)\|\, dm(z) > -\infty$ which guarantees existence of the outer function ξ_{out}.

Similarly to the above reasoning for the case $p < 2$ define for our case $p > 2$ $(q < 2)$,

$$\tilde{\xi} := \left(\bar{\xi}_{\text{out}}\right)^{q/2-1}\xi \qquad and$$
$$\tilde{g} := (\xi_{\text{out}})^{1-q/2}g,$$

where $1/p + 1/q = 1$.Then $\|\tilde{\xi}\|_2 = \|\xi\|_q^{q/2}$,and applying Hölder inequality to $\tilde{g}$we get $\|\tilde{g}\|_2 \le \|g\|_p\|\xi\|_q^{1-q/2}$(note, that the computations are the same as in the case $1 < p < 2$ if we interchange p with q and g with ξ). Then again $\|\tilde{g}\|_2\|\tilde{\xi}\|_2 \le \|g\|_p\|\xi\|_q$, so (16) holds.

The main estimate (16) implies (via duality) the solution of the H^p corona problem for $1 < p \le \infty$.

The case $p = 1$ requires just a little more work since L^1 is not the dual of L^∞, and a bounded linear functional on $L^\infty$is generally a measure. Namely, the main estimate (16) implies that L is a bounded conjugate-linear functional, and by Hahn–Banach Theorem it can be extended to a bounded conjugate-linear functional on $L^\infty(E)$. Since any bounded linear functional on $L^\infty$is a bounded linear functional on the space of continuous functions on the unit circle, there exists a vector-valued measure ν such that

$$L(\xi) = \int_{\mathbb{T}} \langle d\nu, \xi\rangle.$$

Without loss of generality one can replace ν with $\Pi\nu$, then

$$\int_{\mathbb{T}} \langle d\nu, h\rangle = \int_{\mathbb{T}} \langle d\nu, \xi\rangle = L(\Pi h) = \int_{\mathbb{T}} \langle f_0, h\rangle\, dm.$$

Then rewriting this, and treating f_0 dm as a vector-valued measure we have

$$\int_{\mathbb{T}} \langle (f_0 dm - d\nu), h\rangle = 0$$

for any anti-analytic polynomial h. Then applying the F. & M. Riesz Theorem, see [45], we can conclude that the measure $f_0 dm - d\nu$ is absolutely continuous with respect to Lebesgue measure, and moreover it is an analytic measure meaning $f_0 dm - d\nu = (f_0 - \nu)dm$ with $f_0 - \nu \in H^1(E)$ (The F. & M. Riesz Theorem is usually stated for scalar measures, but applying it to the "coordinate" of the measure with respect to some orthonormal basis, one can easily see that it holds for measures with values in a separable Hilbert space as well).

We will be considering operator- and vector-valued functions on the polydisk $\mathbb{D}^n$. We begin with the $H^2(E)$ case. The general goal from previous has not changed. We want, for a given $F \in H^\infty(\mathbb{D}^n; E \to E^*)$ and $g \in H^2 \coloneqq H^2(\mathbb{D}^n; E_*)$ with $\|g\|_2 = 1$, to solve the equation

$$Ff = g, \qquad f \in H^2(\mathbb{D}^n; E) \tag{17}$$
with the estimate $\|f\|_2 \leq C$. Again by a normal families argument it is enough to suppose that F and g are analytic in a neighborhood of $\mathbb{D}^n$ because any estimate obtained can be used to get an estimate when F is only analytic in $\mathbb{D}^n$. It is still easy to find a non-analytic solution f_0 of (17),
$$f_0 := \Phi g := F^*(FF^*)^{-1}g.$$
because we have $\delta^2 I \leq FF^* \leq I$. We will again need to find a $v \in L^2(\mathbb{T}^n; E)$ such that $f := f_0 - v \in H^2(\mathbb{D}^n; E)$ with $v(z) \in \ker F(z)$ a.e. on $\mathbb{T}^n$. Our approach is straightforward reduction to the one variable case, unfortunately this approach will not yield a proof of the H^∞Corona problem on the polydisk since the projections are not bounded when $p = \infty$.

We will denote a point in $\mathbb{D}^n$ or $\mathbb{T}^n$ by $\boldsymbol{z} = (z_1, z_2, \dots, z_n)$. We will use the symbol $\boldsymbol{z}_j$ for $\boldsymbol{z}$without the coordinate z_j and, slightly abusing notation, we can then write $\boldsymbol{z} = (\boldsymbol{z}_j, z_j) = (z_j, \boldsymbol{z}_j)$.

Let $H_j^p = H_j^p(\mathbb{D}^n; E)$ be a subspace of $L^p(\mathbb{T}^n; E)$ consisting of all functions analytic in z_j, i.e.
$$H_j^p(\mathbb{D}^n; E) := \{f \in L^p(\mathbb{T}^n; E) : f(\boldsymbol{z}_j, \cdot) \in H^p(\mathbb{D}; E) \text{for almost all } \boldsymbol{z}_j \in \mathbb{T}^{n-1}\}. \tag{18}$$
Lemma (2.3.11)[70]: Any $h \in H^2(\mathbb{D}^n; E)^\perp$can be written as $h = \sum_{j=1}^n h_j$ with$h_j \in H_j^2(\mathbb{D}^n; E)^\perp$.

Proof. Let $P_j := P_{H_j^2}$ be the orthogonal projection onto $H_j^2 := H_j^2(\mathbb{D}^n; E)$. We can decompose h in the following way:
$$h = P_1 h + (I - P_1)h = h^1 + h_1 h_1 \in H_1^2(\mathbb{D}^n; E)^\perp, h^1 = P_1 h.$$
Similarly,
$$h^1 = P_2 h^1 + (I - P_2)h^1 = h^2 + h_2 h_2 \in H_2^2(\mathbb{D}^n; E)^\perp, h^2 = P_2 P_1 h.$$
Continuing the procedure we get
$$h^{k-1} = P_k h^{k-1} + (I - P_k)h^{k-1} = h^k + h_k h_k \in H_k^2(\mathbb{D}^n; E)^\perp, \qquad h^k = P_k \dots P_2 P_1 h.$$
Combining everything we get
$$h = h_1 + h_2 + \cdots h_n + h^n, h^n = P_n P_{n-1} \dots P_1 h$$
which proves the lemma, because the assumption $h \in H^2(\mathbb{D}^n; E)^\perp$implies that $h^n = P_n \dots P_2 P_1 h = 0$.

We also are going to need an analogue of Lemma (2.3.11) dealing with the decomposition of functions on the holomorphic vector bundle ΠH^2, i.e. for the functions of the form $\xi = \Pi h, h \in H^2(\mathbb{D}^n; E)^\perp$. To state this lemma we need some auxiliary definitions. Let
$$K(\mathbb{D}^n; E) := \operatorname{clos}\big(\Pi(H^2(\mathbb{D}^n)^\perp)\big). \tag{19}$$
and
$$K_j(\mathbb{D}^n; E) := \operatorname{clos}\left(\Pi\big(H_j^2(\mathbb{D}^n)^\perp\big)\right), \qquad \forall j = 1,,,, n, \tag{20}$$
Lemma (2.3.12)[70]: Let X be a subspace of a Hilbert space H, and let Π be some orthogonal projection in H. Then $\operatorname{Ran}\Pi = \Pi H$ is decomposed into the orthogonal sum
$$\Pi H = \operatorname{clos}(\Pi X) \oplus (X^\perp \cap \Pi H).$$
Proof. The proof is a simple exercise in functional analysis.

Define the subspaces
$$\mathcal{Q}(\mathbb{D}^n; E) := H^2 \cap \Pi L^2 \ , \mathcal{Q}_j(\mathbb{D}^n; E) := H_j^2 \cap \Pi L^2. \tag{21}$$

Applying the above lemma to $H = L^2$ and $X = (H^2)^{\perp}$ or $X = \left(H_j^2\right)^{\perp}$we get the following result.

Corollary (2.3.13)[70]: The subspace $\Pi L^2 = \Pi L^2(\mathbb{D}^n; E), n = 1, 2, 3, \ldots$ admits the orthogonal decompositions

$$\Pi L^2 = K \oplus Q \ , \Pi L^2 = K_j \oplus Q_j$$

with the subspaces $K \coloneqq K(\mathbb{D}^n; E), K_j \colon= K_j(\mathbb{D}^n; E), Q \coloneqq Q(\mathbb{D}^n; E)$ and $Q_j \coloneqq Q_j(\mathbb{D}^n; E)$defined by (2.3.12), (2.3.13) and (2.3.14), respectively.

Remark (2.3.14)[70]: Note, that the orthogonal projections P_{K_j} and P_{Q_j} are essentially "onevariable" operators. Namely, to perform the projection P_{Q_j} on the function $\xi \in \Pi L^2$we simply need to perform for each $\boldsymbol{z}_j \in \mathbb{T}^{n-1}$ (recall that $\boldsymbol{z} = \left(z_j\ , \boldsymbol{z}_j\right)$) the "one-variable" projection $P_{Q(\boldsymbol{z}_j)}$ onto the subspace

$$Q\left(\boldsymbol{z}_j\right) \coloneqq H^2(\mathbb{D}; E) \cap \Pi\left(\cdot, \boldsymbol{z}_j\right) L^2(\mathbb{D}; E) \subset H^2 = H^2(\mathbb{D}; E),$$

and similarly for the projection P_{K_j}.

Indeed, if

$$\xi^1\left(\cdot, \boldsymbol{z}_j\right) \coloneqq P_{Q(\boldsymbol{z}_j)} \xi\left(\cdot, \boldsymbol{z}_j\right) \text{ for almost all } \boldsymbol{z}_j \in \mathbb{T}^{n-1},$$

then clearly

$$\xi^1\left(\cdot, \boldsymbol{z}_j\right) \in H^2(\mathbb{D}; E) \cap \Pi\left(\cdot, \boldsymbol{z}_j\right) L^2(\mathbb{T}) \text{ for almost all} \boldsymbol{z}_j \in \mathbb{T}^{n-1},$$

so $\xi^1 \in H^2(\mathbb{D}^n; E) \cap \Pi L^2(\mathbb{D}^n; E)$. Moreover, for $\xi_1 \colon= \xi - \xi^1$ and any $\eta \in H^2(\mathbb{D}^n; E) \cap \Pi L^2$

$$\int_{\mathbb{T}} \left\langle \xi_1\left(z_j, \boldsymbol{z}_j\right), \eta\left(z_j, \boldsymbol{z}_j\right)\right\rangle \, dm\left(z_j\right) = 0 \ \text{ for almost all} \boldsymbol{z}_j \in \mathbb{T}^{n-1} ,$$

and integrating over other variables $\boldsymbol{z_k}$ we get that $\xi_1 \perp \eta$.

The following two lemmas says that in many respects the projection P_{Q_j} behaves like the projection $1 - P_j$ from Lemma (2.3.11).

Lemma (2.3.15)[70]: Let $H^2 = H^2(\mathbb{D}^2; E)$ and let Q and $Q_{\boldsymbol{j}}, j = 1,2$, be the subspaces as defined above in (21). Then for the orthogonal projections P_{Q_j} onto the subspaces $Q_{\boldsymbol{j}}$ we have

$$P_{Q_1} P_{Q_2} = P_{Q_2} P_{Q_1} = P_Q.$$

Proof. It follows from the definition of Q and $Q_{\boldsymbol{j}}$ and from the inclusion $H^2 \subset H_j^2$ that

$$Q = \Pi L^2 \cap H^2 \subset \Pi L^2 \cap H_j^2 = Q_{\boldsymbol{j}}$$

we can conclude that for $\xi \in Q$ we have $P_{Q_j} \xi = \ \xi, j \ = \ 1, 2$.

Since by Corollary (2.3.13) we have the orthogonal decomposition $\Pi L^2 = K \oplus Q$, to prove the lemma we need to show that the equalities $P_{Q_2} P_{Q_1} \xi = 0, P_{Q_1} P_{Q_2} \xi = 0$ hold for all $\xi \in K$. Clearly, it is sufficient to prove only one, say the first as the second can be obtained by interchanging indices.

Consider the orthogonal decomposition of $\xi \in K$,

$$\xi = P_{K_1} \xi + P_{Q_1} \xi =: \xi_1 + \xi^1.$$

To prove that $P_{Q_2} P_{Q_1} \xi = 0$ we need to show that $\xi^1 \in K_2$.

By definition $\xi^1 \perp K_1 \colon= \operatorname{clos}\left(\Pi((H_1^2)^{\perp})\right)$, and since $\Pi((H_1^2)^{\perp}) \supset (H_1^2)^{\perp} \cap \Pi L^2$, we can conclude that

$$\xi^1 \perp (H_1^2)^{\perp} \cap \Pi L^2.$$

We know that $\xi, \xi_1 \in K$ ($\xi_1 \in K$ because $K_1 \subset K$), so $\xi^1 \in K$. By Corollary (2.3.13),
$$\xi^1 \perp Q := H^2 \cap \Pi L^2.$$
Combining the above two orthogonality relations we get
$$\xi^1 \perp ((H_1^2)^\perp + H^2) \cap \Pi L^2,$$
and since in the bidisk $H_2^2 \subset (H_1^2)^\perp + H^2$, we get that
$$\xi^1 \perp \Pi L^2 \cap H_2^2 =: Q_2$$
i.e. that $\xi^1 \in K_2$.

We get the following lemma.

Lemma (2.3.16)[70]: On $\Pi L^2 := \Pi L^2(\mathbb{T}^n; E)$ we have
$$P_{Q_k} P_{Q_j} = P_{Q_j} P_{Q_k} = P_{Q_k \cap Q_j} = P_{H_{jk}^2 \cap \Pi L^2} \qquad \forall 1 \le k, j \le n,$$
where $H_{jk}^2(\mathbb{D}^n) := H_j^2(\mathbb{D}^n) \cap H_k^2(\mathbb{D}^n)$. Furthermore, this implies
$$P_{Q_1} \dots P_{Q_n} = P_{Q_n} \dots P_{Q_1} = P_{H^2 \cap \Pi L^2}.$$

One can think of the space $H_{jk}^2(\mathbb{D}^n)$ as the space of functions in $L^2(\mathbb{T}^n)$ which are, upon fixing the other variables, holomorphic in both the jth and kth variable.

Proof. The first part of the lemma follows immediately from Lemma (2.3.15), because we can just "freeze" all variables except z_j and z_k. Namely, to perform the projection P_{Q_j} on the function $\xi \in \Pi L^2$ we simply need to perform for each $\boldsymbol{z}_j \in \mathbb{T}^{n-1}$ (recall that $\boldsymbol{z} = (z_j, \boldsymbol{z}_j)$) the "one variable" projection $P_{Q(\boldsymbol{z}_j)}$ onto the subspace
$$Q(\boldsymbol{z}_j) := H^2(\mathbb{D}; E) \cap \Pi(\cdot, \boldsymbol{z}_j) L^2(\mathbb{D}; E) \subset H^2 = H^2(\mathbb{D}; E),$$
see Remark (2.3.14).

To prove the second statement of the lemma let us notice that a product of commuting orthogonal projections is an orthogonal projection. Therefore $P = P_{Q_1} P_{Q_2} \dots P_{Q_n}$ is an orthogonal projection.

Since for $\xi \in H^2(\mathbb{D}^n; E) \cap \Pi L^2 = Q \subset Q_j$
$$P_{Q_j}\xi = \xi \qquad \forall j = 1, 2, \dots, n,$$
we can conclude that
$$Q = H^2(\mathbb{D}^n; E) \cap \Pi L^2 \subset \operatorname{Ran} P.$$
On the other hand, since the projections P_{Q_j} commute and $\operatorname{Ran} P_{Q_j} = H_j^2 \cap \Pi L^2$
$$\operatorname{Ran} P \subset H_j^2 \cap \Pi L^2 = Q_j \forall j = 1, 2, \dots, n,$$
So
$$\operatorname{Ran} P \subset \bigcap_{j=1}^{n} Q_j = \bigcap_{j=1}^{n} H_j^2 \cap \Pi L^2 = H^2 \cap \Pi L^2 = Q.$$
Therefore,Ran $P = Q$, i.e. P is the orthogonal projection onto Q.

We can now move onto proving Lemma (2.3.17).

Lemma (2.3.17)[70]: Let $\xi \in K$, then $\xi = \sum_{j=1}^n \xi_j$ with $\xi_j \in K_j$ for $j = 1, \dots, n$ and
$$\|\xi\|_2^2 = \sum_{j=1}^{n} \|\xi_j\|_2^2.$$

To prove Lemma (2.3.17) we will need a few other lemmas. The first one is a simple fact about the geometry of a Hilbert space.

Proof. We will follow the argument in Lemma (2.3.11). For $\xi \in K$ consider the orthogonal decomposition

$$\xi = P_{K_1}\xi + P_{Q_1}\xi =: \xi_1 + \xi^1, \qquad \xi_1 \in K_1(\mathbb{D}^n; E).$$

Since $\xi_1 \perp \xi^1$,

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi^1\|_2^2.$$

Decomposing ξ^1 as

$$\xi^1 = P_{K_2}\xi^1 + P_{Q_2}\xi^1 =: \xi_2 + \xi^2, \quad \|\xi^1\|_2^2 = \|\xi_2\|_2^2 + \|\xi^2\|_2^2$$

we get the decomposition of ξ

$$\xi = \xi_1 + \xi_2 + \xi^2 , \xi_j \in K_j , \xi^2 = P_{Q_2}P_{Q_2}\xi,$$

and

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi_2\|_2^2 + \|\xi^2\|_2^2.$$

Repeating the procedure of decomposing on each step ξ^k using $P_{K_{k+1}}$we finally obtain

$$\xi = \xi_1 + \xi_2 + \dots + \xi_n + \xi^n , \xi_j \in K_j, j = 1,2, \dots, n, \xi^n = P_{Q_n} \dots P_{Q_2}P_{Q_1}\xi,$$

and

$$\|\xi\|_2^2 = \|\xi_1\|_2^2 + \|\xi_2\|_2^2 + \dots + \|\xi_n\|_2^2 + \|\xi^n\|_2^2.$$

But according Lemma (2.3.16)$\xi^n = 0$, so the lemma is proved.

The idea of the proof of the H^2 corona for the polydiskis quite simple, we want to reduce everything to one-variable estimates. In the one-variable case we defined the functional L on functions of the form Πh where $h \in \left(H^2(\mathbb{D})\right)^{\perp}$by

$$L(\xi) = \int_{\mathbb{D}} \partial\left[\langle \bar{\partial}\Phi g, \xi\rangle\right] \, d\mu,$$

where $d\mu = \frac{2}{\pi}\log\frac{1}{|z|} \, dxdy$, see (12). We have also proved (see Proposition (2.3.3)) that the functional L is bounded in the L^2 norm on clos$\left\{\Pi h : h \in \left(H^2(\mathbb{D})\right)^{\perp}\right\}$ (this is the one-variable analogue of the space K defined for the polydisk).

For the polydisk, define (conjugate linear) functionals L_j on K_j by

$$L_j(\xi) = \int_{\mathbb{T}^{n-1}} L_{g(\cdot,\boldsymbol{z}_j)}\left(\xi(\cdot, \boldsymbol{z}_j)\right) \, dm_{n-1}(\boldsymbol{z}_j).$$

Since $\xi(\cdot, \boldsymbol{z}_j) \in K$ for almost all $\boldsymbol{z}_j \in \mathbb{T}^{n-1}$ if $\xi \in K_j$ (see Remark (2.3.7)) the functional L_j are well defined and bounded, $\|L_j\| = \|L\|$. Note also, that on a dense set of ξ of the form $\xi = \Pi h, h \in \left(H_j^2\right)^{\perp}$we can represent

$$L_j(\xi) = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{D}} \partial_j\left[\langle \bar{\partial}_j\Phi g, \xi\rangle\right] \, d\mu(z_j) \, dm_{n-1}(\boldsymbol{z}_j).$$

Define a conjugate linear functional $\mathbf{L}$ on K by decomposing $\xi \in K$ as

$$\xi = \xi_1 + \xi_2 + \dots + \xi_n , \xi_j \in K , j = 1, 2, \dots, n \qquad (22)$$

and putting

$$\mathbf{L}(\xi) \coloneqq \sum_{j=1}^{n} L_j(\xi_j).$$

We will show later that the functional $\mathbf{L}$ is well defined, i.e. that it does not depend on the choice of decomposition of ξ(note that by Lemma (2.3.17) one can always find at least one such decomposition).

Assuming for now that $\mathbf{L}$ is well defined, let us prove Theorem (2.3.2) for $p = 2$. First of all, by Lemma (2.3.17) any function $\xi \in K$ can be decomposed as

$$\xi = \sum_{j=1}^{n} \xi_j \,, \quad \text{where} \xi_j \in K, \quad \text{and} \sum_{j=1}^{n} \left\|\xi_j\right\|^2 = \|\xi\|^2.$$

Therefore, using the fact that $\left\|L_j\right\| = \|L\|$we get for $\xi \in K$

$$\|\mathbf{L}(\xi)\| \leq \sum_{j=1}^{n} \left\|L_j\right\| \cdot \left\|\xi_j\right\| = \|L\| \sum_{j=1}^{n} \left\|\xi_j\right\| \leq \|L\|\sqrt{n}\left(\sum_{k=1}^{n} \left\|\xi_j\right\|^2\right)^{1/2} = \sqrt{n}\|L\| \cdot \|\xi\| \,,$$

So

$$\|\mathbf{L}\| \leq \sqrt{n}\|L\| \ \leq \frac{\sqrt{n}C}{\delta^{r+1}} \log \frac{1}{\delta^{2r}},$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2}e \approx 8.38934$is the constant from Theorem (2.3.2).

Take $h \in \left(H_j^2\right)^{\perp}$, and decompose it according to Lemma (2.3.11) as

$$h = \sum_{j=1}^{n} h_j \ , h_j \in \left(H_j^2\right)^{\perp}.$$

Denote

$$\xi \coloneqq \Pi h \,, \xi_j = \Pi h_j.$$

Repeating the reasoning with the Green’s Formula from the one-variable case we can easily show that

$$\int_{\mathbb{T}^n} \langle \Phi g, h_j \rangle \ dm_n(\boldsymbol{z}) = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{D}} \partial_j \left[\langle \bar{\partial}_j \Phi g, \xi_j \rangle\right] \ d\mu\left(z_j\right) \ dm_{n-1}\left(\boldsymbol{z}_j\right) = L_j\left(\xi_j\right),$$

so

$$\int_{\mathbb{T}^n} \langle \Phi g, h \rangle \ dm_n(\boldsymbol{z}) = \mathbf{L}(\Pi h) = \mathbf{L}(\xi).$$

By the Hilbert space version of the Hahn–Banach Theorem the linear functional $\bar{\mathbf{L}}$ can be extended to a bounded functional on all of L^2, i.e., we can find $v \in L^2 = L^2(\mathbb{T}^n; E)$ such that

$$\mathbf{L}(\xi) = \int_{\mathbb{T}^n} \langle v, \xi \rangle \ dm_n(\boldsymbol{z}) = \qquad \forall \xi \in K.$$

Replacing v by Πv if necessary, one can assume without loss of generality that $v(\boldsymbol{z}) \in \operatorname{Ran} \Pi(\boldsymbol{z}) = \ker F(\boldsymbol{z})$ a.e. on $\mathbb{T}^n$, so $Fv \equiv 0$ on $\mathbb{T}^n$. Since by the construction

$$\int_{\mathbb{T}^n} \langle v, h \rangle \ dm_n(\boldsymbol{z}) = \int_{\mathbb{T}^n} \langle v, \Pi h \rangle \ dm_n(\boldsymbol{z}) = \mathbf{L}(\Pi h) = \int_{\mathbb{T}^n} \langle \Phi g, h \rangle \ dm_n(\boldsymbol{z}) \forall h \in H^2(\mathbb{D}^n; E)^{\perp},$$

the function $f \coloneqq f_0 - v \coloneqq \Phi g - v$ is analytic. Since $Fv = 0$, it satisfies $Ff = Ff_0 = g$, so f is the analytic solution we want to find.

Let us consider first the case of the bidisk $\mathbb{D}^2$. To show that **L** is well defined in this case, it is sufficient to show that if

$$0 = \xi_1 + \xi_2 \,, \xi_j \in K_j$$

then $L_1(\xi_1) + L_2(\xi_2) = 0$ (simply take the difference of two representations of the same function in K). This holds if and only if

$$L_1(\xi) = L_2(\xi) \forall \xi \in K_1 \cap K_2.$$

Thus, the following lemma shows that **L** is well defined in the case of bidisk $\mathbb{D}^2$.

Lemma (2.3.18)[70]: Let $\xi \in K_1 \cap K_2 \subset \Pi L^2(\mathbb{T}^2; E)$. Then

$$L_1(\xi) = L_2(\xi)$$

Proof. The proof of this lemma is really nothing more than repeated applications of Green's Formula, and using that $K_1 \cap K_2 = \operatorname{clos}\left(\Pi \overline{H^2}\right)$ where H^2 are the functions which are anti-holomorphic in both variables.

To see that $K_1 \cap K_2 = \operatorname{clos}\left(\Pi \overline{H^2}\right)$ we use Lemma (2.3.12). Since $(K_1 \cap K_2)^{\perp} = \mathcal{Q}_1 + \mathcal{Q}_2 = K_1^{\perp} + K_2^{\perp} = (H_1^2 + H_2^2) \cap \Pi L^2$, then by Lemma (2.3.12) we have the result.

By density we can work with ξ of the form $\xi = \Pi h$ with h anti-holomorphic in both variables. So applying Green's Formula twice gives

$$L_1(\xi) = \int_{\mathbb{T}} \int_{\mathbb{D}} \partial_1 \left[\langle \bar{\partial}_1 \Phi g, \xi \rangle \right] \, d\mu(z_1) \, dm(z_2) = \int_{\mathbb{T}} \int_{\mathbb{T}} \langle \Phi g, \xi \rangle \, dm(z_1) \, dm(z_2)$$

$$= \int_{\mathbb{T}} \int_{\mathbb{D}} \partial_2 \left[\langle \bar{\partial}_2 \Phi g, \xi \rangle \right] \, d\mu(z_2) \, dm(z_1) = L_2(\xi).$$

Since this result holds on a dense set of ξ, and the functionals L_1 and L_2 are continuous we have the result for all $\xi \in K_1 \cap K_2$.

For the polydisk the lemma has the following important corollary

Corollary (2.3.19)[70]: Let $\xi \in K_j \cap K_k \subset L^2(\mathbb{T}^n; E)$. Then

$$L_j(\xi) = L_k(\xi).$$

Proof. To prove the corollary one needs to apply Lemma (2.3.18) to the bidisk in variables z_j and z_k and then integrate the obtained equality over $\mathbb{T}^{n-2}$ (with respect to Lebesgue measure in all other variables).

Now we are ready to prove that **L** is well defined. To prove this it is sufficient to show for any representation of

$$0 = \sum_{j=1}^{n} \xi_j\,, \qquad \xi_j \in K_j \tag{23}$$

the equality

$$\sum_{j=1}^{n} L_j(\xi_j) = 0$$

holds.

We will use induction in n. The case $n = 2$ is already settled, so let us assume the functional **L** is well defined for the polydisk $\mathbb{D}^{n-1}$. It follows from (23) that

$$\xi_n \in K_n \cap (K_1 + K_2 + \cdots + K_{n-1}) = (K_1 \cap K_n) + (K_2 \cap K_n) + \cdots + (K_{n-1} \cap K_n),$$

so ξ_n can be represented as

$$\xi_n = \sum_{j=1}^{n-1} \eta_j\,, \qquad \eta_j \in K_j \cap K_n\,, \qquad j = 1, 2, \dots n-1.$$

On the other hand we know that $\xi_n = \sum_{j=1}^{n-1} \xi_j$. Using the induction hypothesis and integrating it over $\mathbb{T}$ with respect to $dm\,(z_n)$ we obtain that

$$\sum_{j=1}^{n-1} L_j(\eta_j) = -\sum_{j=1}^{n-1} L_j(\xi_j).$$

Since $\eta_j \in K_j \cap K_n$, Corollary (2.3.19) implies that $L_j(\eta_j) = L_n(\eta_j)$. Therefore

$$L_n(\xi_j) = \sum_{j=1}^{n-1} L_n(\eta_j) = \sum_{j=1}^{n} L_j(\eta_j) = -\sum_{j=1}^{n-1} L_j(\xi_j),$$

and so $= -\sum_{j=1}^{n} L_j(\xi_j) = 0$.

A simple idea of proving the H^p corona problem in the polydisk is to try to mimic the proof of the H^2 case. However, there is a much easier way: just use objects which are already defined, and modify the crucial estimates.

First of all notice, that replacing the Corona data F and g by $F(r\boldsymbol{z})$ and $g(r\boldsymbol{z})$, $r < 1$ and using the standard normal families argument one can assume without loss of generality (as long as we are getting the same uniform estimates on the norm of the solution) that both F and G are holomorphic in a slightly bigger polydisk. So we can always assume that, for example, the right hand side g is not only in H^p, but is also bounded, smooth, etc.

As in the H^2 case we first construct a smooth solution $f_0 \coloneqq \Phi g$, where $\Phi \coloneqq F^*(FF^*)^{-1}$, of the equation $Ff = g$ and then correct it to be analytic. To do that it is sufficient to show that the conjugate linear functional $\mathbf{L}$ introduced in the previous is L^q bounded, $\frac{1}{p} + \frac{1}{q} = 1$, i.e. that

$$|\mathbf{L}(\xi)| \le C\|\xi\|_q$$

for all ξ of form $\xi = \Pi h$, where h is a trigonometric polynomial in $H^2(\mathbb{D}^n; E)^\perp$.

If this estimate is proved, the linear functional $\bar{\mathbf{L}}$can be extended by the Hahn– Banach Theorem to a linear functional on L^q , so there will exist a function $v \in L^p(\mathbb{T}^n; E)$, $\|v\|_p = \|\mathbf{L}\|_p$ such that

$$\mathbf{L}(\xi) = \int_{\mathbb{T}^n} \langle v, \xi \rangle \, dm_n(\boldsymbol{z}) \qquad\qquad \forall \xi = \Pi h, h \in H^2(\mathbb{D}^n; E)^\perp \cap \text{Pol}.$$

Again, replacing v by Πv we can always assume without loss of generality that $v(\boldsymbol{z}) \in \text{Ran}\Pi(z) = \ker F(z)$ a.e. on $\mathbb{T}^n$. As in the previous, decomposing h as

$$h = \sum_{j=1}^{n} \xi_j \ , \xi_j \in H_j^2$$

(h is a trigonometric polynomial, so we can use Lemma (2.3.11) here), we can show that

$$\int_{\mathbb{T}^n} \langle \Phi g, h \rangle \, dm_n(\boldsymbol{z}) = \mathbf{L}(\Pi h) = \mathbf{L}(\xi)$$

so

$$\int_{\mathbb{T}^n} \langle v, h \rangle \, dm_n(\boldsymbol{z}) = \int_{\mathbb{T}^n} \langle v, \Pi h \rangle \, dm_n(\boldsymbol{z}) = \mathbf{L}(\Pi h) = \int_{\mathbb{T}^n} \langle \Phi g, h \rangle \, dm_n(\boldsymbol{z}),$$

for all $h \in H^2(\mathbb{D}^n; E)^\perp \cap$ Pol. Therefore, the function $f = f_0 - v = \Phi g - v$ is analytic, and it clearly solves the equation $Ff = g$ (on $\mathbb{T}^n$, and therefore on $\mathbb{D}^n$).

Let us introduce some notation. Denote

$$K^q \coloneqq \text{clos}(\Pi((H^p)^\perp)) \subset \Pi L^q, Q^q \coloneqq H^q \cap L^q,$$

so for K and Q introduced in the previous $K = K^2$ and $Q = Q^2$. Let also

$$H_j^q = H_j^q(\mathbb{D}^n; E) \coloneqq \{f \in L^q(\mathbb{T}^n; E) \colon f(\cdot, \boldsymbol{z}^j) \in H^q(\mathbb{D}; E)\}$$

be the spaces of functions analytic in variable z_j, and let

$$K_j^q \coloneqq \text{clos}(\Pi(H_j^q(\mathbb{D}^n; E)^\perp)) \subset \Pi L^q(\mathbb{T}^n; E), \qquad Q_j^q := H_j^q(\mathbb{D}^n; E) \cap \Pi L^q(\mathbb{T}^n; E).$$

To estimate the functional $\mathbf{L}$ we need the following analogue of Lemma (2.3.17)

Let us show how this lemma implies the estimate for $\mathbf{L}$. We have proved the L^p bound for the functional L (in the one-variable case),

$$|L(\xi)| \leq C(r,\delta)\|\xi\|_q \,, C(r,\delta) = \frac{1}{\delta^{r+1}}\log\frac{1}{\delta^{2r}}\,,$$

where $C = \sqrt{1+e^2} + \sqrt{e} + \sqrt{2e}$. That would imply the same estimates for the functionals L_j on $L^q(\mathbb{T}^n; E)$, so applying Lemma (2.3.22) we get

$$|L(\xi)| \leq C(r,\delta)\sum_{j=1}^{n}\left\|\xi_j\right\|_q \leq C(r,\delta)\|\xi\|_q\sum_{j=1}^{n}C(q)^j \leq C(r,\delta)nC(q)^n\|\xi\|_q.$$

Recalling that $C(p) = C(q)$ we get the desired estimate of the solution.

There is a little detail here as the functional $\mathbf{L}$ was defined initially only on K^2. So formally, if $q < 2$ (i.e. if $p > 2$) the functional is not defined on K^q. However this is not a big problem and the simplest way of dealing with it is to use the standard approximation arguments. Since the polynomials in $\left(H_j^2\right)^\perp \cap$ Pol are dense in $\left(H_j^p\right)^\perp$,the functions of form $\Pi h, h \in H_j^2 \cap$ Pol are dense in K_j^q. So, approximating functions ξ_j from Lemma (2.3.22) by functions of this form, we will get the desired estimate. Note, that we are estimating $\mathbf{L}(\xi)$ on a dense set of functions $\xi = \Pi h, h \in (H^2)^\perp \cap$ Pol, so we do not need it be formally defined on K^q.

The main step in proving Lemma (2.3.22) is the following result that states that in the one-variable case the norm of the orthogonal projections P_K and P_Q in L^q is the same as the norm of the Riesz projection P_+in L^q. See [73] for the norms of P_+in L^p.

Lemma (2.3.20)[70]: Let $H^2 = H^2(\mathbb{D}; E)$ and let $K, Q \subset H^2$ be the subspaces defined above in (19) and (21). Then for $1 < q < \infty$

$$\|P_K\xi\|_q \leq C(q)\|\xi\|_q \,, \left\|P_Q\xi\right\|_q \leq C(q)\|\xi\|_q \quad \forall\xi \in \Pi L^2 \cap \Pi L^q,$$

where $C(q) = 1/\sin(\pi/q)$ is the norm of the Riesz Projection P_+ in L^q (or in $L^p, 1/p + 1/q = 1$).

Note that since $\Pi L^2 \cap \Pi L^q$ is dense in ΠL^q, the projections P_K and P_Q extend to bounded operators on ΠL^q.

Proof. Take $\xi \in \Pi L^2 \cap \Pi L^q$ and decompose it as

$$\xi = P_K\xi + P_Q\xi =: \xi_K + \xi_Q.$$

Since Q is a z-invariant subspace of $H^2(\mathbb{D}; E)$, by the Beurling–Lax theorem, see [44], it can be represented as $Q = \Theta H^2(\mathbb{D}; E_*)$, where $\Theta \in H^\infty(E_* \to E)$ is an inner function (i.e. $\Theta(z)$ is an isometry a.e. on $\mathbb{T}$) and E_* is an auxiliary Hilbert space. So ξ_Q can be represented as

$$\xi_Q = \Theta\eta \,, \eta \in H^2(E_*) \cap H^q(E_*).$$

By duality

$$\left\|\xi_Q\right\|_q = \|\eta\|_q = \sup_{\substack{h\in L^p\cap L^2: \\ \|h\|_q=1}}\left|\int_{\mathbb{T}}\langle\eta, h\rangle\, dm\right|.$$

Let $h_+ = P_+h$. Since $\eta \in H^2$

$$\int_{\mathbb{T}} \langle \eta, h\rangle\, dm = \int_{\mathbb{T}} \langle \eta, h_+\rangle\, dm = \int_{\mathbb{T}} \langle \Theta\eta, \Theta h_+\rangle\, dm = \int_{\mathbb{T}} \langle \xi_Q, \Theta h_+\rangle\, dm$$
$$= \int_{\mathbb{T}} \langle \xi, \Theta h_+\rangle\, dm;$$
the second equality holds because Θ is an isometry a.e. on $\mathbb{T}$, and the last one holds because $\xi_K \in K \perp \Theta h_+$.Therefore, since $\|h_+\|_p \le C(p)\|h\|_p$we can conclude
$$\left|\int_{\mathbb{T}} \langle \eta, h\rangle\, dm\right| \le \left|\int_{\mathbb{T}} \langle \xi, \Theta h_+\rangle\, dm\right| \le \|\xi\|_q \|h_+\|_p \le C(p)\|\xi\|_q\|h\|_p$$
so $\|\xi_Q\|_q \le C(p)\|\xi\|_q$. Thus we get the desired estimate for the norm of P_Q.

Since $P_K + P_Q = I$ we can estimate the norm of P_K by $C(p) + 1$ for free. Note, that unlike the case of Hilbert spaces, complementary projections in Banach spaces do not necessarily have equal norms. So, to get rid of the 1 some extra work is needed.

It is easy to see that $\bigcup_{n>0} \bar{z}K = \{0\}$, so the decomposition $\Pi L^2 = K \oplus Q$ implies that the set
$$\bigcup_{n>0} \bar{z}^n Q = \bigcup_{n>0} \bar{z}^n \Theta\, H^2(E_*)$$
is dense in ΠL^2. Thus $\Pi L^2 = \Theta L^2$, and since Θ is an isometry a.e. on $\mathbb{T}$ we can conclude that $K = \Theta(H^2(E)^\perp)$. Therefore we can represent ξ_K as
$$\xi_K = \Theta\eta, n \in H^2(E_*)^\perp \cap L^q(E_*).$$
Performing the same calculations as in the case of ξ_Q, only using $h_- = P_- h, P_- = I - P_+$instead of h_+we get the estimate $\|P_K\|_{L^q} \le \|P_-\|_{L^q}$. But the isometry τ,
$$\tau(z)^k = z^{-k-1}\,, k \in \mathbb{Z}$$
interchanges H^2 and $(H^2)^\perp$, and sinceτ is an isometry in all L^p, we conclude the $\|P_-\|_{L^q} \le \|P_+\|_{L^q}$.

Corollary (2.3.21)[70]: Let $H^2 = H^2(\mathbb{D}^n; E)$ and let $K_j, Q_j \subset H^2$ be the subspaces defined in (20) and (21). Then for $1 < q < \infty$and $1 \le j \le n$ we have
$$\left\|P_{K_j}\xi\right\|_q \le C(q)\|\xi\|_q, \left\|P_{Q_j}\xi\right\|_q \le C(q)\|\xi\|_q \qquad \forall\xi \in \Pi L^2 \cap \Pi L^q,$$
where $C(q) = 1/\sin\left(\frac{\pi}{q}\right)$ is the norm of the (one- dimensional) Riesz Projection P_+in L^q (or in $L^p, 1/p + 1/q = 1$).

Proof. This corollary follows directly from Lemma (2.3.20). Since by Remark (2.3.14) we can view P_{K_j} and P_{Q_j} as “one-variable” operators. Then we “freeze” all variables except the z_j variable and apply Lemma (2.3.20) and then integrate in the “frozen” variables.

It only remains to prove Lemma (2.3.22).

Lemma (2.3.22)[70]: Any function $\xi \in K^q$ can be decomposed as
$$\xi = \sum_{j=1}^{n} \xi_j\ , \xi_j \in K_j^q\ , \left\|\xi_j\right\|_q \le C(q)^j \|\xi\|_q\,,$$

where $C(q) = 1/\sin(\pi, q)$ is the norm of the scalar Riesz Projection $P_+$from $L^q(\mathbb{T})$onto $H^q(\mathbb{D})$ (note that $C(p) = C(q)$ for $1/p + 1/q = 1$).

Proof. The proof is almost the same as the proof of Lemma (2.3.17), only here we cannot use the fact that the $P_{K_j}$are orthogonal projections. However, according to Corollary (2.3.21) the projections P_{K_j} are bounded, and this allows the proof to go through.

Take $\xi \in K^q$. Repeating the proof of Lemma (2.3.17) we can write

$$\xi = P_{K_1}\xi + P_{\mathcal{Q}_1}\xi =: \xi_1 + \xi^2.$$

By Corollary (2.3.21) we have that $\xi_1 \in K_1^q$ with $\|\xi_1\|_q \le C(q)\|\xi\|_q$ and $\|\xi^1\|_q \le C(q)\|\xi\|_q$.

Decomposing ξ^1 in the same manner we have

$$\xi^1 = P_{K_2}\xi^1 + P_{\mathcal{Q}_2}\xi^1 =: \xi_2 + \xi^2,$$

So

$$\xi = \xi_1 + \xi_2 + \xi^2\,, \qquad \xi_j \in K_j^q, \quad \xi^2 = P_{\mathcal{Q}_2}P_{\mathcal{Q}_1}\xi.$$

Corollary (2.3.21) applied twice gives $\|\xi_2\|_q \le C(q)\|\xi^1\|_q \le C(q)^2\|\xi\|_q$, and thus $\|\xi_j\|_q \le C(q)^j\|\xi\|_q$. Continuing this decompo-sition at each step we find

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n + \xi^n\,, \qquad \xi_j \in K_j^q, \quad \xi^n = P_{\mathcal{Q}_n} \ldots P_{\mathcal{Q}_2}P_{\mathcal{Q}_1}\xi.$$

and $\|\xi_j\|_q \le C(q)^j\|\xi\|_q$ by applying Corollary (2.3.21) j times. Finally, by Lemma (2.3.16) $P_{\mathcal{Q}_n} \ldots P_{\mathcal{Q}_1} = 0$ on the dense set $K^q \cap K^2$.

Chapter 3
Toeplitz and Hankel Operators with Toeplitz Algebras

We show that the closed bilateral ideal of $\mathfrak{T}(L_a^2)$ generated by operators of the form $T_fT_g - T_gT_f$ coincides with $\mathfrak{T}(L_a^2)$. We generalize an earlier work of Helton-Howe for the usual trace of the anti-symmetrization of Toeplitz operators.

Section (3.1): Bergman Space of the Unit Ball

We show that if an operator A is a finite sum of finite products of Toeplitz operators on the Bergman space of the unit ball B_n, then A is compact if and only if its Berezin transform vanishes at the boundary. For $n = 1$ the result was obtained by Axler and Zheng in 1997.

Let B_n be the open unit ball in $\mathbf{C}^n$. We write $L^2(B_n)$ for the Hilbert space of the square integrable functions defined on B_n. The inner product is defined in the standard way; this means that for any $f, g \in L^2(B_n)$ we define $\langle f, g\rangle = \int_{B_n} f(z)\overline{g(z)}dv\,(z)$,where $dv(z)$ is the normalised volume measure on the unit ball B_n.

The Hilbert space $L^2(B_n)$ contains, as a closed submanifold, the space of square integrable holomorphic functions. This space, called the Bergman space, will be denoted with the symbol $H^2(B_n)$. By a standard theorem we have that the orthogonal projection P, from $L^2(B_n)$ to $H^2(B_n)$, is a bounded linear operator. It is well known that there exists a function $K: B_n \times B_n \to \mathbf{C}$, analytic with respect to the first entry and conjugate analytic with respect to the second entry, such that, for every $f \in L^2(B_n)$ and every $w \in B_n$, we have

$$Pf(w) = \int_{B_n} f(z)\overline{K(z,w)}dv\,(z).$$

The function K, called the Bergman Reproducing Kernel, can be written explicitly:

$$K(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1}},$$

where $\langle z,w\rangle = \sum_{j=1}^{n} z_j\overline{w_j}$. If $\varphi \in L^\infty(B_n)$ we can construct the so called Toeplitz operator T_φ where, by definition, $T_\varphi = PM_\varphi$. The symbol M_φ stands for the standard multiplication operator. Therefore we can write

$$(T_\varphi g)w = \int_{B_n} \varphi(z)g(z)\overline{K_w(z)}dv\,(z).$$

We shall show that if an operator $S \in B(H^2(B_n))$ can be written as $S = \sum_{j=1}^{m}\prod_{k=1}^{m_j} T_{u_{j,k}}$, where $u_{j,k} \in L^\infty(B_n)$ then S is compact if and only if $\lim_{|a|\to 1} \tilde{S}(a) = 0$, where, by definition, we have $\tilde{S}(a) = \langle Sk_a, k_a\rangle$. We study a family of operators which plays a central role in our proof and we set up the basic construction, we prove an important inequality which is a generalization of an inequality proven by S. Axler in the case of one complex variable. Then we complete the proof of the main theorem. In the last we show that many well-known results of this type are consequences of our result.

We need to point out some special features of B_n. Let $\mathrm{Aut}(B_n)$ be the group of all biholomorphic maps of B_n of into B_n. It is well-known that $\mathrm{Aut}(B_n)$ is generated by the unitary operators on $\mathbf{C}^n$ and the involutions of the form

$$\psi_a(z) = \frac{a - P_a z - (1-|a|^2)^{1/2}Q_a z}{1-\langle z,a\rangle}$$

where $a \in B_n, P_a$, is the orthogonal projection into subspace generated by a, and Q_a is the projection into the orthogonal complement, that is, $Q_a = 1 - P_a$. We remind the reader that we have the following equation

$$K(\phi(z), \phi(w))(J_{\mathbf{C}}\phi)z\overline{(J_{\mathbf{C}}\phi)w} = K(z, w)$$

where $\phi: B_n \to B_n$ is a biholomorphism and K, as usual, is the reproducing kernel and the symbol $J_{\mathbf{C}}\phi$ denotes the complex Jacobian of the transformation. In the following we shall indicate with the symbol k_w the normalised reproducing kernel, that is $k_w(z) = |k_w|_2^{-1} k_w(z) = \|k_w\|_2^{-1} K(z, w)$. A direct calculation shows that the identity $|J_{\mathbf{C}}\psi_a(z)|^2 = |k_a(z)|^2$ holds for any $z, a \in B_n$.

Proposition (3.1.1)[75]: On $B_n, k_a(\psi_a(z))k_a(z) \equiv 1$.

With the family of automorphisms above defined we construct a family of operators in $B(L^2(B_n))$ which will play an important role in our proof. For any $a \in B_n$ we define the operator $U_a: H^2(B_n) \to H^2(B_n)$ as $(U_a f) = f \circ \psi_a \cdot k_a$, that is, for any $\zeta \in B(n)$we have $(U_a f)\zeta = f(\psi_a(\zeta)) \cdot k_a(\zeta)$. In the following Propositions we state and prove the main properties of the U_a's.

Lemma (3.1.2)[75]: For any $a \in B_n$ the operator U_a is a self-adjoint, idempotent isometry.

Proof. We start proving that U_a is idempotent. Let $f \in H^2$ then $U_a^2 f = U_a(U_a f)$and this implies that $U_a f = U_a(f \circ \psi_a \cdot k_a) = (f \circ \psi_a \circ \psi_a) \cdot (k_a \circ \psi_a) \cdot k_a$. Then we have $U_a f = f \cdot (k_a \circ \psi_a) \cdot k_a$, therefore $U_a^2 f = f$. Now we prove that U_a is an isometry. In fact, if f is an element of $H^2(B_n)$ then

$$\begin{aligned}
\langle U_a f, U_a f \rangle &= \int_{B_n} |f \circ \psi_a(w)|^2 |k_a(w)|^2 dv(w) \\
&= \int_{B_n} |f \circ \psi_a \circ \psi_a(z)|^2 \left|k_a(\psi_a(z))\right|^2 |J_R \psi_a(z)| \quad dv(z) \\
&= \int_{B_n} |f \circ \psi_a \circ \psi_a(z)|^2 \left|k_a(\psi_a(z))\right|^2 |J_C \psi_a(z)|^2 dv(z) \\
&= \int_{B_n} |f \circ \psi_a \circ \psi_a(z)|^2 \left|k_a(\psi_a(z))\right|^2 |k_a(z)|^2 dv(z) \\
&= \int_{B_n} |f \circ \psi_a \circ \psi_a(z)|^2 \left|k_a(\psi_a(z)) \cdot k_a(z)\right|^2 dv(z) = \int_{B_n} |f(z)|^2 dv(z) \\
&= \langle f, f \rangle
\end{aligned}$$

As before we have used the fact that $k_a \circ \psi_a \cdot k_a \equiv 1$ on B_n. Finally, we prove that U_a is a self-adjoint operator. Let f and g be elements of $H^2(B_n)$, then

$$\langle U_a f, g\rangle = \int_{B_n} (U_a f)(w)\overline{g(w)}\,dv(z) = \int_{B_n} (f\circ\psi_a)(w)\cdot k_a(w)\overline{g(w)}\,dv(z)$$
$$= \int_{B_n} (f\circ\psi_a\circ\psi_a)(z)\cdot(k_a\circ\psi_a)(z)\overline{g(\psi_a(z))}|J_R\psi_a(z)|\,dv(z)$$
$$= \int_{B_n} f(z)(k_a\circ\psi_a)(z)\overline{g(\psi_a(z))}|J_R\psi_a(z)|^2\,dv(z)$$
$$= \int_{B_n} f(z)(k_a\circ\psi_a)(z)\overline{g(\psi_a(z))}|k_a(z)|^2\,dv(z)$$
$$= \int_{B_n} f(z)(k_a\circ\psi_a\cdot k_a)(z)\overline{g(\psi_a(z))k_a(z)}\,dv(z) = \int_{B_n} f(z)\overline{U_a g(z)}\,dv(z)$$
$$= \langle f, U_a g\rangle.$$

For any A in $B(H^2(B_n))$ we define the operator A_z as $A_z = A_z A U_z$, and we denote with the symbol $\tilde{A}$ the Berezin Transform of the operator A. This function is defined by $\tilde{A}(z) = \langle Ak_z, k_z\rangle$. Now we can state the following.

Theorem (3.1.3)[75]: Let A be an operator in $B(H^2(B_n))$ which can be written as $A = \sum_{j=1}^{m}\prod_{k=1}^{m_j} T_{u_{j,k}}$ where $u_{j,k}\in L^2(B_n)$. Then the following are equivalent:

(i) A is compact;

(ii)$\tilde{A}(z)\to 0$ as $z\to\partial B_n$;

(iii) $A_z 1\to 0$ weakly as $z\to\partial B_n$;

(iv) $\|A_z 1\|_p \to 0$ as $z\to\partial B_n$ for any $p>1$.

The proof of the stated theorem involves some technicalities. In order to make our work clearer we prove some basic lemmas and we shall prove the theorem later.

We start with a simple Lemma which clarifies the relationship between the Berezin Transform and the maps $\{\psi_z\}$.

Lemma (3.1.4)[75]: If C is a bounded operator on $L^2(B_n)$ and $z\in\mathbf{C}^n$, then $\tilde{C}\circ\psi_z = \widetilde{C_z}$.

Proof. Suppose that $C\in B(L^2(B_n))$ and $z,w\in\mathbf{C}^n$. If f is in $L^2(B_n)$ then we have

$$\langle f, U_z K_w\rangle = \langle U_z f, K_w\rangle = (U_z f)w = (f\circ\psi_z)(w)k_z(w) = \langle f, \overline{k_z(w)}K_{\psi(w)}\rangle.$$

Since this equality holds for any f in $L^2(B_n)$, thus we have $U_z K_w = \overline{k_z(w)}K_{\psi(w)}$. If we rewrite the last equation as

$$\|K_w\|(U_z\|K_w\|^{-1}K_w) = \|K_{\psi(w)}\|\overline{K_z(w)}\left(\|K_{\psi(w)}\|^{-1}K_{\psi(w)}\right),$$

it follows that $U_z k_w = \left(\|K_w\|^{-1}\|K_{\psi(w)}\|K_{\psi(w)}\right)$. Since the operator U_z is unitary we can assert that $\|K_w\|_2^{-1}\|K_{\psi(w)}\|_2\left|\overline{K_z(w)}\right| = 1$.

We can write

(i) $\tilde{C}\circ\psi_z(w) = \tilde{C}(\psi_z(w))$

(ii) $= \langle Ck_{\psi_z(w)}, k_{\psi_z(w)}\rangle$

(iii) $= \langle CU_z k_w, U_z k_w\rangle$

(iv) $= \langle U_z C U_z k_w, k_w\rangle$

(v) $= \langle C_z k_w, k_w\rangle$

(vi) $= \widetilde{C_z}(w)$.

We observe that in (ii) we only use the definition of the Berezin transform and in (iii) we use the fact that $U_z k_w = a k_{\psi_z(w)}$ where a is a complex number of modulus 1.

Finally, we observe that the above calculations imply that $\tilde{C} \circ \psi_z = \tilde{C}_z$, and we are done.

Lemma (3.1.5)[75]: For every $u \in L^\infty(B_n)$ and for every $z \in B_n$we have $U_z T_u U_z = T_{u\circ\psi_z}$.

Proof. Since U_z is an idempotent operator it is enough to prove that $U_z T_u = T_{u\circ\psi_z} U_z$. We start computing $T_u U_z$. Let f be an element of $H^2(B)$, then we have

$$\langle T_u U_z f, k_w\rangle = \langle T_u\big((f \circ \psi_z)k_z\big), k_w\rangle = \langle PM_u\big((f \circ \psi_z)k_z\big), k_w\rangle = \langle\big(u\big((f \circ \psi_z)k_z\big), k_w\big)\rangle$$
$$= \int_{B_n} u(\eta)(f \circ \psi_z)(\eta)k_z(\eta)\overline{k_w(\eta)}dv(\eta).$$

Now we calculate $U_z T_{u\circ\psi_z}$. Let f be an element of $H^2(B)$, then we have

$$\langle U_z T_{u\circ\psi_z} f, k_w\rangle = \langle T_{u\circ\psi_z} f, U_z k_w\rangle = \langle PM_{u\circ\psi_z} f, U_z k_w\rangle = \langle (u \circ \psi_z)f, (k_w \circ \psi_z)k_z\rangle$$
$$= \int_{B_n} f(\eta)(u \circ \psi_z)(\eta)\overline{(k_w \circ \psi_z)(\eta)k_z(\eta)}dv(\eta).$$

To show the equality we use the substitution $\eta = \psi_z(\beta)$. If we call the last integral we have written in the above list of equalities $\mathcal{A}$, then we have

$$\mathcal{A} = \int_{B_n} (f \circ \psi_z)(\beta)u(\beta)\overline{k_w(\beta)(k_z \circ \psi_z)(\beta)}|(J_R\psi_z)\beta|dv(\beta)$$
$$= \int_{B_{(n)}} (f \circ \psi_z)(\beta)u(\beta)\overline{k_w(\beta)(k_z \circ \psi_z)(\beta)}|k_z(\beta)|^2 dv(\beta)$$
$$= \int_{B_n} (f \circ \psi_z)(\beta)u(\beta)\overline{k_w(\beta)(k_z \circ \psi_z)(\beta)k_z(\beta)}k_z(\beta)dv(\beta)$$
$$= \int_{B_n} (f \circ \psi_z)(\beta)u(\beta)\overline{k_w(\beta)}k_z(\beta)dv(\beta)$$
$$= \int_{B_n} (f \circ \psi_z)(\beta)u(\beta)k_z(\beta)\overline{k_w(\beta)}dv(\beta).$$

Then it is clear that the claimed identity holds.

Before we state the next result we need to introduce a new operator. For any $f \in H^2(B_n)$ we define $U_{\mathcal{R}} f \in H^2(B_n)$ by $(U_{\mathcal{R}} f)w = f(-w)$.

If we denote with the symbol $\mathcal{J}_{c,t}: B \to [0, \infty)$ the function that acts in the following way:

$$\mathcal{J}_{c,t}(z) = \int_{B_n} \frac{(1 - |w|)^t dv(w)}{(1 - \langle z, w\rangle)^{n+1+t+c}},$$

it is possible to prove (see [79]) that for $c < 0$ and $t > -1$ the function $\mathcal{J}_{c,t}$ is bounded on B_n.

The following proposition is necessary because we are going to apply Schur's Test (see [77]), when we give a complete proof of the main Theorem.

Lemma (3.1.6)[75]: Given $p \in \mathcal{R}$, with $0 < p - 1 < (n + 1)^{-1}$, and $A \in B\big(H^2(B_n)\big)$ then

$$\int_{B_n} |U_{\mathcal{R}} A U_{\mathcal{R}} K_z(w)| \|K_w\|_2^\varepsilon dv(w) \le K(z, z)^{\varepsilon/2} \left(\sup_{z\in B} \|A_{-z} 1\|_q\right) \left(\sup_{z\in B} \left|\mathcal{J}_{a,b}(z)\right|^{1/p}\right)$$

and

$$\int_{B_n} |U_{\mathcal{R}} A U_{\mathcal{R}} K_z(w)| \|K_z\|_2^\varepsilon dv(w) \le K(w, w)^{\varepsilon/2} \left(\sup_{w\in B} \|A_{-w}^* 1\|_q\right) \left(\sup_{w\in B} \left|\mathcal{J}_{a,b}(w)\right|^{1/p}\right)$$

Where: $2(p-1)/p < \varepsilon < 2/(n+1)p$, $a = (p-1)(n+1) - (n+1)\varepsilon p/2$ and $b = -(n+1)\varepsilon p/2$ and $p^{-1} + q^{-1} = 1$. Moreover, with such a choice the quantity

$$\sup_{z\in B}\left|\mathcal{J}_{a,b}(z)\right|^{1/p} = \sup_{z\in B}\left|\mathcal{J}_{(p-1)(n+1)-(n+1)\varepsilon p/2,-(n+1)\varepsilon p/2}(z)\right|^{1/p}$$

is finite.

Proof. We prove the first inequality and we shall show that the second one is an easy consequence of the first. We observe that

$$\begin{aligned} U_{\mathcal{R}}AU_{\mathcal{R}}K_z &= \|K_z\|_2 U_{\mathcal{R}}AU_{\mathcal{R}}k_z = \|K_z\|_2 U_{\mathcal{R}}AU_{-z}1 = \|K_z\|_2 U_{\mathcal{R}}U_{-z}A_{-z}1 \\ &= \|K_z\|_2 U_{\mathcal{R}}\big((A_{-z}1)\circ\psi_{-z}\cdot k_{-z}\big) = \|K_z\|_2\big((A_{-z}1)\circ\psi_{-z}\circ\mathcal{R}\big)\cdot k_z \\ &= \big((A_{-z}1)\circ\psi_{-z}\circ\mathcal{R}\big)\cdot K_z. \end{aligned}$$

Then we can write the left term of the first inequality as

$$\|K_z\|_2\int_{B_n}\left|\big((A_{-z}1)\circ\psi_{-z}(-w)\big)\right||k_z(-w)|\|K_w\|_2^{\varepsilon}dv(w).$$

and, of course, this is the same as

$\|K_z\|_2\int_{B_n}\left|\big((A_{-z}1)\circ\psi_{-z}(-w)\big)\right||k_z(-w)|\|K_w\|_2^{\varepsilon}dv(w).$

In the last evaluation we used the fact that $\|k_w\|_2 = \|k_{-w}\|_2$. Now we use the substitution $w = \psi_{-z}(\lambda)$ and we obtain

$$\|K_z\|_2\int_{B_n}|(A_{-z}1)(\lambda)|\left|k_{-z}\big(\psi_{-z}(\lambda)\big)\right|\left\|K_{\psi_{-z}}(\lambda)\right\|_2^{\varepsilon}|(J_{\mathcal{R}}\psi_{-z})\lambda|dv(\lambda).$$

Since we have the identity $|(J_{\mathcal{R}}\psi_{-z})\lambda| = |k_{-z}(\lambda)|^2$. then we can write the last integral as

$$\|K_z\|_2\int_{B_n}|(A_{-z}1)(\lambda)|\left|k_{-z}\big(\psi_{-z}(\lambda)\big)\right|\left\|K_{\psi_{-z}(\lambda)}\right\|_2^{\varepsilon}|k_{-z}(\lambda)|^2dv(\lambda).$$

If we write $\left\|K_{\psi_{-z}(\lambda)}\right\|_2^{\varepsilon}$ as $\left\|K_{\psi_{-z}(\lambda)}\right\|_2^{\varepsilon-1}$, we observe that

$$K\big(-\psi_z(\lambda),-\psi_z(\lambda)\big) = K\big(\psi_z(\lambda),\psi_z(\lambda)\big) = K(\lambda,\lambda)|(J_C\psi_{-z})\lambda|^{-2} = K(\lambda,\lambda)|k_{-z}(\lambda)|^{-2}$$

and if we take the square root of both sides, then we have

$$\begin{aligned} \left\|K_{\psi_{-z}(\lambda)}\right\|_2\left|k_{-z}\big(\psi_{-z}(\lambda)\big)\right||k_{-z}(\lambda)|^2 &= \|K_\lambda\|_2|k_{-z}(\lambda)|^{-1}\left|k_{-z}\big(\psi_{-z}(\lambda)\big)\right||k_{-z}(\lambda)|^2 \\ &= \|K_\lambda\|_2\left|k_{-z}\big(\psi_{-z}(\lambda)\big)k_{-z}(\lambda)\right| = \|K_\lambda\|_2 \end{aligned}$$

This implies that the last integral is equal to

$$\|K_z\|_2\int_B|(A_{-z}1)(\lambda)|\|K_\lambda\|_2^{\varepsilon}|k_{-z}(\lambda)|^{1-\varepsilon}dv(\lambda)$$

and, if we use the non-normalised reproducing kernel, we obtain

$$\|K_z\|_2\int_{B_n}|(A_{-z}1)(\lambda)|\|K_\lambda\|_2^{\varepsilon}\|K_z\|_2^{\varepsilon-1}|K_{-z}(\lambda)|^{1-\varepsilon}dv(\lambda)$$

In other words, we have obtained

$$\|K_z\|_2^{\varepsilon}\int_{B_n}|(A_{-z}1)(\lambda)|\|K_\lambda\|_2^{\varepsilon}|K_{-z}(\lambda)|^{1-\varepsilon}dv(\lambda).$$

If we apply Holder's Inequality with $q^{-1} + p^{-1} = 1$ and we call the last integral $\mathcal{I} = \mathcal{I}(z)$, we have

$$\mathcal{I}(z) \le \|K_z\|_2^{\varepsilon}\|A_{-z}1\|_q\left(\int_{B_n}\|K_\lambda\|_2^{p\varepsilon}|K_{-z}(\lambda)|^{p(1-\varepsilon)}dv(\lambda)\right)^{1/p}.$$

In order to complete the proof we need to study the integral in the right hand side. Using the fact that we know explicitly the kernel, we have

$$\int_{B_n} \|k_\lambda\|_2^{p\varepsilon} |K_{-z}(\lambda)|^{p(1-\varepsilon)} dv(\lambda) = \int_{B_n} \frac{dv(\lambda)}{(1-|\lambda|^2)^{(n+1)p\varepsilon/2} |1-\langle -z,\lambda\rangle|^{(n+1)p(1-\varepsilon)}}.$$

In order to apply the theorem we define $t = -(n+1)p\varepsilon/2$. Because we need $t > -1$, then we conclude that $2/(n+1)p > \varepsilon$. We also want to write $(n+1)p(1-\varepsilon)$ as $n+1+t+c$ and this gives us $c = (n+1)(p-1) - p\varepsilon\ (n+1)/2$. We also want that $c < 0$ so we conclude that $2(p-1)/p < \varepsilon$. We can summarise our condition on ε in the double inequality $2(p-1)/p < \varepsilon < 2(n+1)/p$. It is clear that we can find such an ε if and only if $(p-1) < (n+1)^{-1}$ and this is possible because the only condition on p is that $2 > p > 1$.

So far we have proved the first inequality. To prove the second inequality it is enough to proceed as before, starting with A^* and using the fact that $(A^*K_w)(z) = \overline{(AK_z)w}$.

We remark that the last lemma implies the following

Corollary (3.1.7)[75]: If D denotes the unit disk in C and dA the Lebesgue area measure and S is a bounded operator on $H^2(D)$ then, for any $\alpha \in (1, 3/2)$ and $\beta \in (2(\alpha-1)\alpha^{-1}, 2\alpha^{-1})$, there exists a constant $c_{\alpha,\beta} < \infty$ such that, if $\alpha^{*-1} + \alpha^{-1} = 1$, we have

$$\int_D \frac{|SK_z(w)|}{(1-|w|^2)^{\beta/2}}\, dA(w) \le \frac{c_{\alpha,\beta}\|S_z 1\|_{\alpha^*}}{(1-|w|^2)^{\beta}}$$

for all $z \in D$ and

$$\int_D \frac{|SK_z(w)|}{(1-|w|^2)^{\beta/2}}\, dA(w) \le \frac{c_{\alpha,\beta}\|S_w^* 1\|_{\alpha^*}}{(1-|w|^2)^{\beta}}$$

for all $w \in D$.

This proposition has been proved, in the special case where $\alpha = 6/5$ and $\beta = 1/2$, in [76]. Finally, we prove the following

Proposition (3.1.8)[75]: Let A be an operator in $B\big(H^2(B_n)\big)$ which can be written as $A = \sum_{j=1}^{m} \prod_{k=1}^{m_j} T_{u_{j,k}}$, where $u_j, k \in L^\infty(B_n)$. Then, for every $q \in (1, \infty)$, $\sup_{z\in B} \|A_z 1\|_q < \infty$.

Proof. We can assume that $A = \prod_{k=1}^{m} T_{u_j}$. Using Lemma (3.1.5) we have that $A_z = \prod_{k=1}^{m} T_{u_j \circ \psi_z}$. Since $P_q : L^q(B) \to H^q(B)$is a bounded operator, then there exists a constant $c_q > 0$ such that, for every $f \in L^q(B)$, $\|P_q f\|_q \le c_q \|f\|_q$. This implies that, for any $f \in L^q(B)$, $\|T_u f\|_q = \|P_q M_u f\|_q \le c_q \|M_u f\|_q \le c_q \|u\|_\infty \|f\|_q$. Since $\|u \circ \psi_z\|_\infty = \|u\|_\infty$, we obtain $\|T_{u\circ\psi_z} f\|_\infty \le c_q \|u\|_\infty \|f\|_q$. Then we can conclude that $\|A_z 1\|_q \le \prod_{k=1}^{m} \|T_{u_k \circ \psi_z}\|_q \le c_q^m \prod_{k=1}^{m} \|u_k\|_\infty$. Therefore we are done because our estimate is independent of $z \in B_n$.

We give the complete proof of the main theorem, and we remind that the main result can be stated as follows:

Theorem (3.1.9)[75]: Let A be an operator in $B\big(H^2(B^n)\big)$ which can be written as $A = \sum_{j=1}^{m} \prod_{k=1}^{m_j} T_{u_{j,k}}$, where $u_{j,k} \in L^\infty(B^n)$. Then the following are equivalent:

(i) A is compact;

(ii)$\tilde{A}(z) \to 0$ as $z \to \partial B_n$;

(iii) $A_z 1 \to 0$ weakly as $z \to \partial B_n$;

(iv) $\|A_z 1\|_p \to 0$ as $z \to \partial B_n$ for any $p > 1$.

Proofs.i $\Longrightarrow$ ii. Because A is compact and $k_z \to 0$ weakly in $H^2(B)$ as $z \to \partial B$ then a standard theorem about compact operators implies that $\|Ak_z\|_2 \to 0$ as $z \to \partial B_n$and the Cauchy-Schwarz inequality implies that $|\tilde{A}(z)| = |\langle Ak_z, k_z\rangle| \le \|Ak_z\|_2$. Therefore we see that $\tilde{A}(z) \to 0$ as $z \to \partial B_n$.

(ii) $\Longrightarrow$ (iii). We suppose that (ii) holds and we want to show that $A_z 1 \to 0$ weakly in $L^2(B)$ as $z \to \partial B$. We know that we can construct an orthonormal basis for $H^2(B_n)$using just polynomials in n variables, so it is enough to show that $\left(A_z 1 \omega^{\alpha^*}\right) \to 0$ as $z \to \partial B_n$, where $\alpha^*$is a multindex and $\omega = \omega_1^{\alpha_1^*} \dots \omega_n^{\alpha_n^*}$.

We start by observing that since $\tilde{A}(\psi_z(\omega)) = \widetilde{A_z}(\omega) = \langle A_z k_\omega, k_\omega\rangle$ and $k_\omega(w) = (1-|\omega|^2)^{(n+1)/2} \sum_{\alpha \in N^n} (\overline{\omega^\alpha} \cdot w^\alpha)/\gamma_\alpha$, we obtain

$$\tilde{A}(\psi_z(\omega)) = (1-|\omega|^2)^{n+1} \sum_{\alpha,\beta \in N^n} \frac{\langle A_z w^\alpha, w^\beta\rangle}{\gamma_\alpha \gamma_\beta} \overline{\omega^\alpha} w^\beta .$$

If we multiply both sides by $\overline{\omega^{\alpha^*}}/(1-|\omega|^2)^{n+1}$ we obtain that

$$\int_{rB_n} \frac{\tilde{A}(\psi_z(\omega))\overline{\omega^{\alpha^*}}}{(1-|\omega|^2)^{n+1}}\, d\nu(\omega) = \sum_{\alpha,\beta \in N^n} \frac{\langle A_z w^\alpha, w^\beta\rangle}{\gamma_\alpha\gamma_\beta} \int_{rB_n} \overline{\omega^{\alpha^*+\alpha}\omega^\beta}\, d\nu(\omega).$$

This implies that

$$\int_{rB_n} \frac{\tilde{A}(\psi_z(\omega))\overline{\omega^{\alpha^*}}}{(1-|\omega|^2)^{n+1}}\, d\nu(\omega) = r^{2|\alpha^*|+2n} \sum_{\alpha \in N^n} \frac{\langle A_z w^\alpha, w^{\alpha^*+\alpha}\rangle}{\gamma_\alpha} r^{2|\alpha|}.$$

Observe that the left hand side goes to zero as $|z| \to 1$ because the domain of integration is bounded and the function goes to zero since (ii) holds. Therefore for any fixed $r \in (0,1)$ the left side of the equality has limit zero as $|z| \to 1$. Now we divide the left and right hand sides by r^{2n+2}. Then, on the right hand side, we obtain

$$\frac{\langle A_z 1, w^{\alpha^*}\rangle}{\gamma_{\underline{0}}} + \sum_{\alpha \in N^n \setminus \underline{0}} \frac{\langle A_z w^\alpha, w^{\alpha^*+\alpha}\rangle}{\gamma_\alpha} r^{2|\alpha|}$$

and we can conclude that, for any fixed $r \in (0,1)$,

$$\frac{\langle A_z 1, w^{\alpha^*}\rangle}{\gamma_{\underline{0}}} + \sum_{\alpha \in N^n \setminus \underline{0}} \frac{\langle A_z w^\alpha, w^{\alpha^*+\alpha}\rangle}{\gamma_\alpha} r^{2|\alpha|} \to 0$$

as $z \to \partial B_n$. We also observe that

$$\left| \sum_{\alpha \in N^n \setminus \underline{0}} \frac{\langle A_z w^\alpha, w^{\alpha^*+\alpha}\rangle}{\gamma_\alpha} r^{2|\alpha|} \right| \le \|A\| \left(\sum_{|\alpha|=1}^{n} r^{2|\alpha|} + \sum_{|\alpha|>n}^{\infty} r^{2|\alpha|} \right).$$

To complete the proof we need to analyse $\sum_{|\alpha|>n}^{\infty} r^{2|\alpha|}$. To study this function we define $P_\ell(n) = \{z_1{}^{\alpha_1} \dots z_n{}^{\alpha_n} : \alpha_1 \dots \alpha_n = \ell\}$ and we denote with the symbol $a_\ell(n)$ the cardinality of $P_\ell(n)$. We have $\|A\|\left(\sum_{|\alpha|=1}^{n} r^{2|\alpha|} + \sum_{|\alpha|>n}^{\infty} r^{2|\alpha|}\right) = \|A\|\left(\sum_{|\alpha|=1}^{n} r^{2|\alpha|} + \sum_{k>n}^{\infty} a_\ell(n) r^{2k}\right)$.

Claim (3.1.10)[75]: If $P_\ell(n)$ and $a_\ell(n)$ are defined as above then

$$* \qquad \limsup_{\ell\to\infty}\frac{a_{\ell+1}(n)}{a_\ell(n)}<\infty.$$

Proof. We use induction and we observe that if $n=1$ then $a_\ell(n)$ for all $\ell\geq 1$ so, for $n=l$, the claim is true. Since, from $m>0, P_m(n+1)=\cup_{j=0}^{m}P_j(n)z_{n+1}^{m-j}$ then it follows that $a_{m+1}(n+1)=a_{m+1}(n)+a_m(n+1)$ and this implies that

$$\frac{a_{\ell+1}(n+1)}{a_\ell(n+1)}\leq\frac{a_{\ell+1}(n)}{a_\ell(n)}+1.$$

Therefore, using induction, we can conclude that (*) is true. Now we observe that the Claim, together with the Ratio test, implies that there exists an $R>0$ such that $\sum_{k>n}a_\ell(n)r^{2k}<\infty$ if $r\in[0,R)$. This implies that if r is small enough, then, for all $z\in B_n$, the series is less than $\varepsilon>0$, where e is arbitrary. Therefore we have

$$\overline{\lim_{z\to\partial B_n}}\left|\langle A_z1,w^{\alpha^*}\rangle\right|<\varepsilon.$$

Since ε is an arbitrary positive number, the inequality implies that $\left|\langle A_z1,w^{\alpha^*}\rangle\right|\to 0$ as $z\to\partial B_n$ and this proves our claim.

iii $\Longrightarrow$ iv. We want to prove that $A_z\to 0$ weakly in $H^2(B_n)$ as $z\to\partial B_n$ implies that $\|A_z1\|_2\to 0$ as $z\to\partial B_n$. If $r\in(0,1)$, we can write

$$\|A_z1\|_2=\int_{B_n}|A_z1(w)|^2dv(w)=\int_{B_n\backslash rB_n}|A_z1(w)|^2dv(w)+\int_{rB_n}|A_z1(w)|^2dv(w)$$

$$\leq v(B_n\backslash rB_n)^{1/2}\|A_z1\|_4^2+\int_{rB_n}|A_z1(w)|^2dv(w)$$

It is clear that we can choose r close enough to 1 in order to make the first term on the right smaller than δ. In fact we have shown that $\|A_z1\|$ is bounded independent of z. We observe that a sequence of holomorphic functions which goes weakly to zero is going to zero in norm on compacta and this shows that the second term on the right hand side goes to zero as z goes to the boundary and this completes our proof. Now we assume that $p\in(2,\infty)$. Then we have

$$\|A_z1\|_p\leq\|A_z1\|_2^{1/p}\|A_z1\|_{2p-p}^{(p-1)/p}.$$

By our hypothesis the first term on the right hand side has limit zero as $z\to\partial B_n$ and the second is bounded independent of z. Therefore, by Proposition (3.1.8), we conclude that $\|A_z1\|_p\to 0$ as $z\to\partial B^n$. For $1<p<2$ we observe that $\|A_z1\|_p\leq\|A_z1\|_2$ and we are done.

iv $\Rightarrow$ i. We suppose that $\|A_z1\|_p\to 0$ as $z\to\partial B_n$, for every $q\in(1,\infty)$and we want to conclude that the operator is compact. Since the operator $U_{\mathcal{R}}$ is invertible, A is compact if and only if $U_{\mathcal{R}}AU_{\mathcal{R}}=A_{\mathcal{R}}$ is compact. We are going to show that $U_{\mathcal{R}}AU_{\mathcal{R}}$ is compact. We observe that for any $f\in H^2(B_n)$ and for any $w\in B_n$ we have

$$(U_{\mathcal{R}}AU_{\mathcal{R}}f)w=\langle U_{\mathcal{R}}AU_{\mathcal{R}}f,K_w\rangle=\langle f,U_{\mathcal{R}}A^*U_{\mathcal{R}}K_w\rangle=\int_{B_n}f(z)\overline{(U_{\mathcal{R}}A^*U_{\mathcal{R}}K_w)(z)}dv(z)$$

$$=\int_{B_n}f(z)\overline{(U_{\mathcal{R}}AU_{\mathcal{R}}K_z)(w)}dv(z).$$

Observe that equation (iv) is a consequence of

$$(U_{\mathcal{R}}A^*U_{\mathcal{R}}K_w)(z) = \langle U_{\mathcal{R}}A^*U_{\mathcal{R}}K_w, K_z\rangle = \langle K_w, U_{\mathcal{R}}AU_{\mathcal{R}}K_z\rangle = \overline{\langle U_{\mathcal{R}}AU_{\mathcal{R}}K_z, K_w\rangle}$$
$$= \overline{(U_{\mathcal{R}}AU_{\mathcal{R}}K_z)(w)}.$$

For any $t \in (0,1)$ we define the operator $A_{[t]}$on $H^2(B_n)$ by

$$\big((A_R)_{[t]}f\big)w = \int_{tB_n} f(z)(A_R K_z)(w)d\nu(w),$$

in other words, this integral operator has kernel

$$\mathcal{K}_{[t]}(z,w) = \mathcal{X}_{tB_n}(z)(A_R K_z)(w),$$

where the symbol $\mathcal{X}_{tB_n}$ stands for the characteristic function of the set $tB_n = \{\zeta \in \mathrm{C}^n: |\zeta| < t\}$. We observe that the operator $A_{[t]}$ is Hilbert-Schimdt for any $t \in [0,1)$, in fact

$$\int_{B_n}\int_{B_n} \left|(A_R K_z)(w)\mathcal{X}_{tB(n)}(z)\right|^2 d\nu(z)d\nu(w) \le \|A_R\| \int_{tB_n} \|K_z\|_2^2\, d\nu(z)$$
$$= \|A_R\| \int_{tB_n} \frac{d\nu(z)}{(1-|z|^2)^{n+1/2}} < \infty.$$

To show that A_R is compact, it is enough to prove that

$$\lim_{t\to 1^-} \left\|A_R - (A_R)_{[t]}\right\| = 0.$$

We observe that the operator $A_R - (A_R)_{[t]}$, is an integral operator whose kernel is given by the function$\mathcal{K}_{[1-t]}(z,w) = \mathcal{X}_{(B\setminus tB)}(z)(A_R K_z)(w)$. To estimate the norm of this operator we use the Schur Test. We choose, as test function, the function $K(z,z)^{\varepsilon/2}$. If we choose p such that $0 < (p-1) < (n+1)^{-1}$, and q such that $p^{-1} + q^{-1} = 1$, and $\varepsilon \in (2(p-1)p^{-1}, 2(n+1)^{-1}p^{-1})$ then we can apply Lemma (3.1.6), and if we denote the function $\mathcal{X}_{B_n\setminus tB_n}$ by the symbol $\mathrm{G}(t,\cdot)$ we see that

$$\int_{B_n} |G(t,z)A_R K_z(w)|\|K_w\|_2^{\varepsilon} d\nu(w) \le \|K_z\|_2^{\varepsilon} \sup_{z\in B_n\setminus tB_n} \|A_{-z}1\|_q \sup_{w\in B_n} \left|\mathcal{J}_{a,b}(z)\right|^{1/p}$$

and

$$\int_{B_n} |G(t,z)A_R K_z(w)|\|K_z\|_2^{\varepsilon} d\nu(w) \le \|K_w\|_2^{\varepsilon} \sup_{w\in B_n\setminus tB_n} \|A_{-w}^*1\|_q \sup_{w\in B_n} \left|\mathcal{J}_{a,b}(w)\right|^{1/p}.$$

If we choose a and b as in Lemma (3.1.6) then we obtain $\sup_{w\in B_n} \left|\mathcal{J}_{a,b}(w)\right|^{1/p} < \infty$ and our hypothesis on the behaviour of $\|A_{-w}^*1\|_q$ as w goes to ∂B_n implies that $\left|A_R - (A_R)_{[t]}\right| \to 0$ as $t \to 1^-$. Therefore we have proved that $U_{\mathcal{R}}AU_{\mathcal{R}} = A_R$ is compact and this implies that A is compact.

Observe that in the case when there is only a single operator, the main result can be stated in a very simple form, in fact.

Corollary (3.1.11)[75]: Let $f \in L^\infty(B_n)$. Then T_f is compact if and only if $\tilde{f}(z) \to 0$ as $z \to \partial B_n$.

We remind that for any $f \in L^\infty(B_n)$, the Hankel operator $H_f: H^2(B_n) \to H^2(B_n)^\perp$is defined by $H_f g = (I-P)(fg)$, for any $g \in L^2(B_n)$. The next result has been proved, in the case where the symbol is holomorphic, by Stroethoff.

Corollary (3.1.12)[75]: Let $f \in L^\infty(B_n)$. Then the following are equivalent:

(i) H_f is compact;

(ii) $\|H_f K_z\|_2 \to 0$ as $z \to \partial B_n$;

(iii) $\|f \circ \psi_z - P(f \circ \psi_z)\|_2 \to 0$ as $z \to \partial B_n$.

The proof is the same as in the case of the disk and we refer the reader to [l] for details. Using this result it is also possible to recover the main Theorem in [78].

Section (3.2): Bergman Space Coincides with Its Commutator Ideal

If $0 < p \leq \infty$ let $L^p = L^p(\mathbb{D}, \mathrm{d}A)$, where $\mathbb{D}$ is the open unit disk and $\mathrm{d}A(z) = (1/\pi)dxdy$, with $z = x + \mathrm{i}y$, is the normalized area measure on $\mathbb{D}$. The Bergman space L_a^p is formed by the analytic functions in L^p. If $1 < p < \infty$ then

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(\omega)}{(1 - \bar{\omega} z)^2} \mathrm{d}A(\omega)$$

is a bounded projection from L^p onto L_a^p.. This is the usual Bergman projection. For $a \in L^\infty$ let $M_a: L^p \to L^p$ be the operator of multiplication by a and $P_a = PM_a$. Then $\|P_a\| \leq C_p \|a\|_\infty$, where C_p is the norm of P acting on L^p. The Toeplitz operator $T_a: L_a^p \to L_a^p$ is the restriction of P_a to the space L_a^p. If B is a Banach space, we will write $\mathfrak{L}(B)$ for the algebra of all bounded operators on B and $\mathfrak{T}(L_a^p)$ for the closed subalgebra of $\mathfrak{L}(L_a^p)$ generated by $\{T_a: a \in L^\infty\}$.

If A is a Banach algebra, its commutator ideal $\mathfrak{C}A$ is the closed bilateral ideal generated by elements of the form $[x, y] \overset{\text{def}}{=} xy - yx$, with $x, y \in A$. It is clear that $\mathfrak{C}A$ is the smallest closed ideal of A such that $A/\mathfrak{C}A$ is a commutative Banach algebra. There is an extensive literature on commutator ideals and abelianizations of Toeplitz algebras acting on the Hardy space H^2. See ([45]). We only have a handful of results for Toeplitz algebras of operators on L_a^2. Probably the most relevant on the subject are [13], [84] and [82].

If H is a Hilbert space of dimension greater than one then $\mathfrak{C}\mathfrak{L}(H) = \mathfrak{L}(H)$. Although this situation is very unusual for Toeplitz algebras, the purpose is to prove the following

Theorem (3.2.1)[81]: The Toeplitz algebra on L_a^2coincides with its commutator ideal.

In [83] it is shown that if $\phi(z) = \exp(i \log|z|^{-2})$ then the semicommutator $T_{\overline{\phi}\phi} - T_{\overline{\phi}} T_\phi$ is a nontrivial scalar multiple of the identity. Analogously, it could happen that there are two simple functions $a, b \in L^\infty$ such that $T_a T_b - T_b T_a$is easily seen to be invertible. This would immediately prove Theorem (3.2.1). Since I was unable to find such functions or even prove their existence, the proof here is considerably more complicated.

For $z \in \mathbb{D}$ let $\varphi_z(\omega) = \frac{(z - \omega)}{(1 - \bar{z}\omega)}$, the special automorphism of $\mathbb{D}$ that interchanges 0 and z. The pseudo-hyperbolic metric is defined by $\rho(z, \omega) = |\varphi_z(\omega)|$for $z, \varphi \in \mathbb{D}$. It is well known that ρ is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. We will also use that

$$\rho(z, \omega) \geq \frac{\rho(z, u) - \rho(u, w)}{1 - (z, u)(u, w)} \text{for all } z, \omega, u \in \mathbb{D}.$$

If $0 < r < 1$ write $K(z, r) \overset{\text{def}}{=} [\omega \in \mathbb{D}: \rho(\omega, z) \leq r]$ for the closed ball of center z and radius r with respect to ρ. A sequence $S = \{z_n\}$ in $\mathbb{D}$ will be called separated if $\inf_{i \neq j} \rho(z_i, z_j) > 0$. Although I have not found the next result in its present form, it is a well known feature of separated sequences. We sketch here a proof.

Lemma (3.2.2)[81]: Let S be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $S = S_1 \cup \dots \cup S_N$ such that for every $1 \le i \le N$: $\rho(z,\omega) > \sigma$ for all $z \ne \omega$ in S_i.

Proof. Since S is separated, there is some positive integer N depending only on σ and the degree of separation of S, such that $K(z,\sigma) \cap S$ has no more than N points for every $z \in \mathbb{D}$. Let $S_i \subset S$ be a maximal sequence such that $\rho(z,\omega) > \sigma$ for every $z,\omega \in S_1$ with $z \ne \omega$. The maximality implies that $S \subset \bigcup_{z \in S_1} K(z,\sigma)$.

If $S = S_1$ we are done. Otherwise suppose that $n \ge 2, S_1, \dots, S_{n-1}$ are chosen and $S \backslash (S_1 \cup \dots \cup S_{n-1}) \ne 0$. Let $S_n \subset S \backslash (S_1 \cup \dots \cup S_{n-1})$ be a maximal sequence such that $\rho(z,\omega) > \sigma$ for every $z,\omega \in S_n$ with $z \ne \omega$. By the maximality at the previous steps, if $z \in S_n$ there is some $z_i \in S_i$ such that $z \in K(z_i,\sigma)$ for every $1 \le i \le n-1$.

Therefore $\{z, z_1, \dots, z_{n-1}\} \subset K(z,\sigma) \cap S$, and consequently $n \le N$.

Lemma (3.2.3)[81]: For $1 \le k \le m$ let $\{a_j^k\}_{j \ge 1}$ be sequences in the unit ball of L^∞ such that supp $a_j^k \subset (\alpha_j, r)$, where $K(\alpha_j, r) \cap K(\alpha_j, r) = \emptyset$ if $i \ne j$. Suppose that $1 < p < \infty$ and $\{R_j\}_{j \ge 1}$ is a bounded sequence in $\mathfrak{L}(L^p)$. If $f \in L^p$ is such that $\sum_{j \ge 1} M_{a_j^m} R_j f \in L^p$ then

$$\left\| \sum_{j \ge 1} P_{a_j^1} \dots P_{a_j^m} R_j f \right\|_p \le C_p^m \left\| \sum_{j \ge 1} M_{a_j^m} R_j f \right\|_p,$$

where C_p is the norm of the projection P acting on L^p.

Proof. Write $Q_j = P_{a_j^2} \dots P_{a_j^{m-1}} P$ for all $j \ge 1$ and $S = \sum_{j \ge 1} M_{a_j^1} Q_j M_{a_j^m} R_j$.

Then $\|Q_j\| \le C_p^{m-1}$ and for $f \in L^p$ we have

$$\|Sf\|_p^p = \left\| \sum_{j \ge 1} M_{a_j^1} Q_j M_{a_j^m} |R_j f \right\|_p^p = \sum_{j \ge 1} \left\| M_{a_j^1} Q_j M_{a_j^m} R_j f \right\|_p^p \le C_p^{(m-1)p} \sum_{j \ge 1} \left\| M_{a_j^m} R_j f \right\|_p^p$$

$$= C_p^{(m-1)p} \left\| \sum_{j \ge 1} \left(M_{a_j^m} R_j \right) f \right\|_p^p. \qquad (1)$$

If the last quantity is finite then $Sf \in L^p$ and the sums $S_n f = \sum_{j=1}^n M_{a_j^1} Q_j M_{a_j^m} R_j f$ converge to Sf in L^p-norm when $n \to \infty$. Therefore

$$\left\| \sum_{j \ge 1} P_{a_j^1} \dots P_{a_j^m} |R_j f \right\|_p^p = \lim_n \left\| \sum_{j \ge 1} P_{a_j^1} \dots P_{a_j^m} R_j f \right\|_p^p = \lim_n \|P S_n f\|_p^p \le C_p \|Sf\|_p^p.$$

The lemma follows combining this inequality with (1).

Corollary (3.2.4)[81]: Taking $R_j = 1$ for every j in Lemma (3.2.3) we obtain

$$\left\| \sum_{j \ge 1} P_{a_j^1} \dots P_{a_j^m} \right\|_{\mathfrak{L}(L^p)} \le C_p^m.$$

Proof. By the lemma,

$$\left\| \sum_{j \ge 1} P_{a_j^1} \dots P_{a_j^m} f \right\|_p \le C_p^m \left\| \sum_{j \ge 1} M_{a_j^m} \right\|_p \le C_p^m \left\| M_{\left(\sum_{j \ge 1} a_j^m\right)} f \right\|_p \le C_p^m \|f\|_p$$

for every $f \in L^p$.

The next result is a particular case of Lemma 4.2.2 in [85].

Lemma (3.2.5)[81]: If $t > -1, c$ is real and

$$F_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|\omega|^2)^t}{|1-z\overline{\omega}|^{2+t+c}} \mathrm{d}A(\omega) \qquad z \in \mathbb{D},$$

then $F_{c,t}$ is bounded when $c < 0$ and $|F_{c,t}(z)| \le C(1-|z|^2)^{-c}$ when $c > 0$.

Lemma (3.2.6)[81]: Let $0 < r < 1$ and $\{\alpha_j\}_{j\ge1} \subset \mathbb{D}$ such that $K(\alpha_j, r) \cap K(\alpha_j, r) = \emptyset$ if $i \ne j$. If $r < R < 1$ and $0 < \beta < 1$ then

$$\int_{\mathbb{D}} \sum_j \left[\chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,\mathcal{R})}(\omega)\right] \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2} \mathrm{d}A(\omega)$$
$$\le c_\beta(\mathcal{R})(1-|z|^2)^{-\beta}, \qquad (2)$$

where $c_\beta(R) \to 0$ when $R \to 1$.

Proof. If $z \in K(\alpha_j, r)$ and $\omega \in \mathbb{D}\setminus K(\alpha_j, R)$ then

$$\rho(\omega, z) \ge \frac{\rho(\omega,\alpha_j) - \rho(\alpha_j, z)}{1-\rho(\alpha_j,z)\rho(\omega,\alpha_j)} > \frac{R-r}{1-Rr} = \delta,$$

where $\delta = \delta(R) \to 1$ when $R \to 1$. Therefore $\mathbb{D}\setminus K(\alpha_j, R) \subset \mathbb{D}\setminus K(z,\delta)$ and

$$\sum_j \chi_{K(\alpha_j,r)}(z)\,\chi_{D\setminus K(\alpha_j,\mathcal{R})}(\omega) \le \sum_j \chi_{K(\alpha_j,r)}(z)\,\chi_{D\setminus K(z,\delta)}(\omega).$$

Hence, the integral in (2) is bounded by

$$\sum_j \chi_{K(\alpha_j,r)}(z) \int_{\mathbb{D}} \chi_{D\setminus K(z,\delta)}(\omega) \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2} \mathrm{d}A(\omega)$$
$$= \sum_j \chi_{K(\alpha_j,r)}(z) \int_{|v|>\delta} \frac{(1-|\varphi_z(v)|^2)^{-\beta}}{|1-z\overline{v}|^2} \mathrm{d}A(v)$$
$$\le \int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}} (1-|z|^2)^{-\beta}\, \mathrm{d}A(v), \qquad (3)$$

where the equality comes from the change of variables $v = \varphi_z(\omega)$ and the inequality because $K(\alpha_j, r)$ are pairwise disjoint. Pick some $p = p(\beta) > 1$ satisfying simultaneously the conditions $p\beta < 1$ and $p(2-\beta) < 2$. If $p^{-1} + q^{-1} = 1$, Holder's inequality gives

$$\int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}} \mathrm{d}A(v) \le \left(\int_{\mathbb{D}} \frac{(1-|v|^2)^{-p\beta}}{|1-z\overline{v}|^{2p(1-\beta)}} \mathrm{d}A(v)\right)^{1/p} (1-\delta^2)^{1/q}.$$

Since $2p(1-\beta) = 2 - p\beta + [p(2-\beta) - 2] < 2 - p\beta$,then Lemma (3.2.5) says that the last expression is bounded by $C_\beta(1-\delta^2)^{1/q}$, where C_β depends only on β. Going back to (3) we see that the integral in (2) is bounded by

$$C_\beta(1-\delta(\mathcal{R})^2)^{1/q(\beta)}(1-|z|^2)^{-\beta},$$

proving the lemma.

Lemma (3.2.7)[81]: Let $0 < r < 1$ and $\alpha_j \in \mathbb{D}, j \ge 1$, such that $K(\alpha_j, r)$ are pairwise disjoint. Suppose that $\mathcal{R} \in (r, 1)$ and $a_j, A_j \in L^\infty$ are functions of norm ≤ 1 such that

$$\operatorname{supp} a_j \subset K(\alpha_j, r) \text{ and } \operatorname{supp} A_j \subset D \backslash K(\alpha_j, \mathcal{R}).$$

Then $\sum_{j\geq 1} M_{a_j} P M_{A_j}$ is bounded on L^p for every $1 < p < \infty$, with norm bounded by some constant $k_p(\mathcal{R}) \to 0$ when $\mathcal{R} \to 1$.

Proof. Write

$$\Phi(z,\omega) = \sum_{j\geq 1} \chi_{K(\alpha_j,r)}(z)\, \chi_{D\backslash K(\alpha_j,\mathcal{R})}(\omega) \frac{1}{|1-\overline{\omega} z|^2}.$$

Let $f \in L^p$. Since $\|a_j\|_\infty, \|A_j\|_\infty \leq 1$ for all j, then

$$\left| \left(\sum_{j\geq 1} M_{a_j} P M_{A_j} f \right)(z) \right| = \left| \sum_{j\geq 1} a_j(\omega) \int_{\mathbb{D}} A_j(\omega) f(\omega) \frac{\mathrm{d}A(\omega)}{(1-\overline{\omega} z)^2} \right|$$

$$\leq \int_{\mathbb{D}} \Phi(z,\omega) |f(\omega)| \mathrm{d}A(\omega).$$

Taking $h(z) = |1-|z|^2|^{-1/pq}$, where $p^- + q^{-1} = 1$, Lemma (3.2.6) asserts that

$$\int_{\mathbb{D}} \Phi(z,\omega) h(\omega)^q \mathrm{d}A(\omega) \leq C_{p_{-1}}(\mathcal{R}) h(z)^q$$

and Lemma (3.2.5) implies that there is some $C > 0$ such that

$$\int_{\mathbb{D}} \Phi(z,\omega) h(z)^p \mathrm{d}A(\omega) \leq C h(\omega)^p.$$

By Schur's theorem ([85], p. 42) the integral operator with kernel $\Phi(z,\omega)$ is bounded on L^p and its norm is bounded by $\left(C_{p_{-1}}(\mathcal{R})\right)^{1/q} C^{1/p} \to 0$ as $\mathcal{R} \to 1$.

Let $a_j^i, b_j \in L^\infty, j \geq 1$ and $1 \leq i \leq m$, be functions of norm at most 1 supported on $K(\alpha_j, r)$, where the pseudo-hyperbolic disks are pairwise disjoint.

By Lemma (3.2.2) for any $\sigma \in (r,1)$ there is some $n = n(\sigma) \geq 1$ and a partition of the positive integers $\mathbb{N} = N_1 \cup \ldots \cup N_n$ such that

$$\rho(\alpha_i, \alpha_j) > \sigma \qquad \text{for } i \neq j, i, j \in N_k, 1 \leq k \leq n.$$

Lemma (3.2.8)[81]: If $1 < p < \infty$ then

$$\sum_{1\leq k\leq n} \left[\left(\sum_{j\in N_k} P_{a_j^1} \ldots P_{a_j^m} \right) P_{\left(\sum_{i\in N_k} b_i\right)} \right] \to \sum_{j\geq 1} P_{a_j^1} \ldots P_{a_j^m} P_{b_j} \tag{4}$$

in operator norm when $\sigma \to 1$.

Proof. Write $B_j = \sum_{\substack{i\in N_k \\ i\neq j}} b_i$ when $j \in N_k$ for some $1 \leq k \leq n$. Since $P_{\left(\sum_{i\in N_k} b_i\right)} = P_{b_j} + P_{B_j}$, the first term in (4) can be decomposed as

$$\sum_{k=1}^{n} \left[\sum_{j\in N_k} P_{a_j^1} \ldots P_{a_j^m} P_{b_j} + \sum_{j\in N_k} P_{a_j^1} \ldots P_{a_j^m} P_{B_j} \right] = S_1 + S_2,$$

where

$$S_1 = \sum_{k=1}^{n} \sum_{j\in N_k} P_{a_j^1} \ldots P_{a_j^m} P_{b_j} = \sum_{j\geq 1} P_{a_j^1} \ldots P_{a_j^m} P_{b_j}$$

and

$$S_2 = \sum_{k=1}^{n} \sum_{j \in N_k} P_{a_j^1} \dots P_{a_j^m} P_{B_j} = \sum_{j \geq 1} P_{a_j^1} \dots P_{a_j^m} P_{B_j}.$$

Let $f \in L^p$. By Lemmas (3.2.3) and (3.2.7)

$$\|S_2 f\|_p \leq C_p^m \sum_{j \geq 1} \left\| M_{a_j^m} P_{B_j} f \right\|_p. \tag{5}$$

If $\omega \in \text{supp} B_j$ for $j \in N_k$ with $1 \leq k \leq n$, then there is $i \neq j$ in N_k such that $\omega \in K(\alpha_j, r)$. Then

$$\rho(\omega, \alpha_j) \geq \frac{\rho(\alpha_j, \alpha_i) - \rho(\omega, \alpha_i)}{1 - \rho(\alpha_j, \alpha_a)\rho(\omega, \alpha_i)} > \frac{\sigma - r}{1 - \sigma r} = \mathcal{R}(\sigma),$$

meaning that $\text{supp} B_j \subset \mathbb{D} \backslash K\left(\alpha_j, \mathcal{R}(\sigma)\right)$. Since $\mathcal{R}(\sigma) \to 1$ when $\sigma \to 1$, (5) and Lemma (3.2.7) prove (4).

Corollary (3.2.9)[81]: Under the conditions of Lemma (3.2.8),

$$\sum_{1 \leq k \leq n} \left[\left(\sum_{j \in N_k} T_{a_j^1} \dots T_{a_j^m} \right) T_{\left(\sum_{i \in N_k} b_i\right)} \right] \to \sum_{j \geq 1} T_{a_j^1} \dots T_{a_j^m} T_{b_j} \tag{6}$$

and

$$\sum_{1 \leq k \leq n} \left[T_{\left(\sum_{i \in N_k} b_i\right)} \left(\sum_{j \in N_k} T_{a_j^1} \dots T_{a_j^m} \right) \right] \to \sum_{j \geq 1} T_{b_j} T_{a_j^1} \dots T_{a_j^m} \tag{7}$$

in operator norm when $\sigma \to 1$.

Proof. We obtain (6) by restricting the operators of (4) to L_a^p. To prove (7) use (6) with

$$\sum_{1 \leq k \leq n} \left[\left(\sum_{j \in N_k} T_{\overline{a}_j^m} \dots T_{\overline{a}_j^1} \right) T_{\left(\sum_{i \in N_k} \overline{b}_i\right)} \right]$$

acting on L_a^q and then take adjoints.

Proposition (3.2.10)[81]: Let $1 < p < \infty$ and $c_j^1, \dots, c_j^i, a_j, b_j, d_j^1, \dots, d_j^m \in L^\infty$ be functions of norm ≤ 1 supported on $K(\alpha_j, r)$ for $j \geq 1$, where $K(\alpha_j, r) \cap K(\alpha_i, r) = \emptyset$ if $i \neq j$. Then

$$\sum_{j \geq 1} T_{c_j^1} \dots T_{c_j^l} \left(T_{a_j} T_{b_j} - T_{b_j} T_{a_j} \right) T_{d_j^1} \dots T_{d_j^m} \in \mathfrak{CT}(L_a^p).$$

Proof. For $r < \sigma < 1$ decompose $\mathbb{N} = N_1 \cup \dots \cup N_n$as in the paragraph that precedes Lemma (3.2.8). By Corollary (3.2.9),

$$\sum_{1 \leq k \leq n} \left[T_{\left(\sum_{j \in N_k} a_j\right)} T_{\left(\sum_{i \in N_k} b_i\right)} - T_{\left(\sum_{i \in N_k} b_i\right)} T_{\left(\sum_{j \in N_k} a_j\right)} \right] \to \sum_{j \geq 1} \left(T_{a_j} T_{b_j} - T_{b_j} T_{a_j} \right)$$

in operator norm when $\sigma \to 1$. Since the first operators belong to the commutator ideal, so does their limit. Thus,

$$\sum_{j \in F} \left(T_{a_j} T_{b_j} - T_{b_j} T_{a_j} \right) \in \mathfrak{CT}(L_a^p)$$

for any subset $F \subset \mathbb{N}$. In particular, this hold for $F = N_k, 1 \le k \le n$. Then

$$\sum_{1\le k\le n}\left[\left(\sum_{j\in N_k}\left(T_{a_j}T_{b_j}-T_{b_j}T_{a_j}\right)T_{\left(\sum_{i\in N_k} d_i^1\right)}\right)\right] \in \mathfrak{C}\mathfrak{T}\left(L_a^p\right),$$

and since (6) says that the above operators converge to

$$\sum_{j\ge 1}\left(T_{a_j}T_{b_j}-T_{b_j}T_{a_j}\right)T_{d_j^1}$$

when $\sigma \to 1$, this operator is also in $\mathfrak{C}\mathfrak{T}\left(L_a^p\right)$. Clearly, the same holds if the sum is over any set $F \subset \mathbb{N}$. We can repeat this process $m-1$ more times using (6) and then l times using (7) to obtain the desired result.

Let $a \in L^\infty$ be a real-valued function such that $a(\omega) \ge \delta > 0$ for every $\omega \in \mathbb{D}$. Then T_a is self-adjoint and

$$\langle T_a f, f\rangle = \int_{\mathbb{D}} a|f|^2 \mathrm{d}A \ge \delta \int_{\mathbb{D}} |f|^2 \mathrm{d}A = \delta\|f\|_2^2$$

for every $f \in L_a^2$. Therefore T_a is invertible. Theorem (3.2.1) will be proved by constructing a function a as above such that $T_a \in \mathfrak{C}\mathfrak{T}(L_a^2)$.

We need to summarize several basic features of Toeplitz operators. If $a, b \in L^\infty$ then $T_a T_b = T_{ab}$ when $\overline{a} \in H^\infty$ or $b \in H^\infty$. If $z \in \mathbb{D}$ then $U_z f = (f \circ \varphi_z)\varphi_z'$defines a unitary self-adjoint operator on L_a^2.Therefore, if $a \in L^\infty$ and $f, g \in L_a^2$,

$$\langle U_z T_a U_z f, g\rangle = \langle T_a U_z f, U_z g\rangle = \langle a(f\circ\varphi_z)\varphi_z', (g\circ\varphi_z)\varphi_z'\rangle = \langle (a\circ\varphi_z) f, g\rangle,$$

where the last equality comes from changing variables inside the integral. Thus

$$U_z T_{a_1} \dots T_{a_n} U_z = U_z T_{a_1} U_z \dots U_z T_{a_1} U_z = T_{a_1\circ\varphi_z} \dots T_{a_n\circ\varphi_z} \tag{8}$$

for $a_j \in L^\infty, 1 \le j \le n$. By diagonal operator we always mean diagonal with respect to the orthonormal basis $\left\{\sqrt{n+1}z^n\right\}_{n\ge 0}$.

A straightforward calculation with polar coordinates shows that if $a \in L^\infty$is a radial function (i.e. $a(z) = a(|z|)$), then T_a is diagonal with n-entry

$$\lambda_n(a) = \int_0^1 a\left(t^{\frac{1}{2}}\right)(n+1)t^n \mathrm{d}t. \tag{9}$$

If $\chi_r$denotes the characteristic function of the ball $\{|\omega| \le r\}$, where $0 < r < 1$, then (9) yields $T_{\chi_r}\omega^n = r^{2(n+1)}\omega^n$.

Lemma (3.2.11)[81]: Let $a \in L^\infty$ be a radial function and $0 < r < 1$. Then

$$T_{\chi_r}T_a = T_{\chi_r(\omega)a(\omega/r)}.$$

Proof. The operator $T_{\chi_r(\omega)a(\omega/r)}$ is diagonal, and its n-entry is

$$\int_0^1 \chi_{[0,r]}\left(t^{1/2}\right)a\left(\frac{t^{1/2}}{r}\right)(n+1)t^n\mathrm{d}t. = \int_0^{r^2} a\left(\frac{t^{1/2}}{r}\right)(n+1)t^n\mathrm{d}t$$

$$= r^{2n+2}\int_0^1 a\left(u^{1/2}\right)(n+1)u^n\mathrm{d}t,$$

where the last equality comes from the change of variables $u = t/r^2$. By (9) $T_{\chi_r}T_a$ is also diagonal and has the same entries.

A simple calculation shows that if $n \geq 1$ then $\langle T_{\overline{\omega}}\omega^n, \omega^k\rangle = \langle \omega^n, \omega^{k+1}\rangle = \langle (n/n+1)\omega^{n-1}, \omega^k\rangle$. A recursive argument then gives that for every nonnegative integer k,

$$T_{\overline{\omega}^k}\omega^n = \frac{(n+1-k)}{n+1}\omega^{n-k} \qquad if\ n \geq k$$

and $T_{\overline{\omega}^k}\omega^n = 0$ if $n < k$. Thus

$$T_{\overline{\omega}^k}T_{\chi_r}\omega^n = r^{2(n+1)}\left(\frac{n+1-k}{n+1}\right)\omega^{n-k} \qquad if\ n \geq k,$$

and since $T_{\chi_r}T_{\omega^k}\omega^n = r^{2(n+k+1)}\omega^{n+k}$ then

$$\begin{aligned}\left(T_{\overline{\omega}^k}T_{\chi_r}\right)\left(T_{\chi_r}T_{\omega^k}\right)\omega^n &= r^{4(n+k+1)}\left(\frac{n+1}{n+k+1}\right)\omega^n \\ &= r^{4k}T_{\chi_{r^2}}T_{\overline{\omega}^k}T_{\omega^k}\omega^n, \qquad (10)\end{aligned}$$

where the second equality comes from the limit case $r = 1$ in the first equality and from $T_{\chi_{r^2}}\omega^n = r^{4(n+1)}\omega^n$. Since T_{χ_r} and $T_{\omega^k}T_{\overline{\omega}^k}$ are diagonal, they commute, and since $T_{\chi_r}^2 = T_{\chi_{r^2}}$ then

$$T_{\chi_r}T_{\omega^k}T_{\overline{\omega}^k}T_{\chi_r} = T_{\chi_r}^2T_{\omega^k}T_{\overline{\omega}^k} = T_{\chi_{r^2}}T_{\omega^k}T_{\overline{\omega}^k}. \qquad (11)$$

By (10), (11) and Lemma (3.2.11),

$$\begin{aligned}S_k \overset{\text{def}}{=} \left[T_{\omega^k\chi_r}, T_{\overline{\omega}^k\chi_r}\right] &= T_{\chi_{r^2}}\left(T_{\omega^k}T_{\overline{\omega}^k} - r^{4k}T_{\overline{\omega}^k}T_{\omega^k}\right) \\ &= T_{\chi_{r^2}}T_{\omega^k}T_{\overline{\omega}^k} - T_{\chi_{r^2}|\omega|^{2k}}. \qquad (12)\end{aligned}$$

Let $P_0 \in \mathfrak{L}(L_a^2)$ be the operator $P_0f = f(0)$. Straightforward evaluations on the basis $\{z^n\}_{n\geq 0}$ give the following identities

$$T_\omega T_{\overline{\omega}} = T_{1+\log|\omega|^2}, T_{\omega^2}T_{\overline{\omega}^2} = T_{1+2\log|\omega|^2} + P_0 \ \text{ and } T_{\chi_{r^2}}P_0 = r^4P_0. \quad (13)$$

Then

$$\begin{aligned}2S_1 - S_2 \underset{\overline{\overline{}}}{by\ (3.5)} T_{\chi_{r^2}}\left(2T_\omega T_{\overline{\omega}} - T_{\omega^2}T_{\overline{\omega}^2}\right) + T_{\chi_{r^2(|\omega|^4-2|\omega|^2)}} \underset{\overline{\overline{}}}{by\ (3.6)} T_{\chi_{r^2(1+|\omega|^4-2|\omega|^2)}} \\ - r^4P_0 = T_{\chi_{r^2(1-|\omega|^2)^2}} - r^4P_0. \qquad (14)\end{aligned}$$

Since $2S_1 - S_2, T_{\chi_r}$and $P_0$are diagonal operators, they commute. Consequently

$$P_0T_{\chi_r}T_\omega = T_{\chi_r}P_0T_\omega = 0,$$

which together with Lemma (3.2.11) and (14) gives

$$T_{\chi_{r\overline{\omega}}}(2S_1 - S_2)T_{\chi_r\omega} = T_{\overline{\omega}}T_{\chi_r}(2S_1 - S_2)T_{\chi_r}T_\omega = T_{\chi_{r^4}\left(1-\frac{|\omega|^2}{r^4}\right)^2|\omega|^2}. \quad (15)$$

If $\alpha \in \mathbb{D}$ then (8), (12) and (15) yield

$$\begin{aligned}&T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}\left(2\left[T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}\right] - \left[T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha^2}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha^2}\right]\right)T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha} \\ &\qquad = U_\alpha T_{\chi_{r\overline{\omega}}}(2S_1 - S_2)T_{\chi_r\omega}U_\alpha = T_{(\chi_{r^4}\circ\varphi_\alpha)(1-|\varphi_\alpha|^2/r^4)^2|\varphi_\alpha|^2}. \qquad (16)\end{aligned}$$

Suppose that $0 < r < 1$ and $\{\alpha_j\} \subset \mathbb{D}$ is a sequence such that $K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ for $i \neq j$. Since $(\chi_{r^4} \circ \varphi_\alpha)(\omega) = \chi_{K(\alpha,r^4)}(\omega)$, the characteristic function of $K(\alpha, r^4)$, then

$$A(\omega) \overset{\text{def}}{=} \sum_{j\geq 1}\chi_{r^4}\left(\varphi_{\alpha_j}(\omega)\right)\left(1 - \frac{\left|\varphi_{\alpha_j}(\omega)\right|^2}{r^4}\right)^2\left|\varphi_{\alpha_j}(\omega)\right|^2$$

is in L^∞ with $\|A\|_\infty \leq 1$. In conjunction with (16), Proposition (3.2.10) tells us that

$$T_A = \sum_{j\geq 1} T_{\left(\chi_{r^4}\circ\varphi_{\alpha_j}\right)\left(1-\left|\varphi_{\alpha_j}(\omega)\right|^2/r^4\right)^2\left|\varphi_{\alpha_j}(\omega)\right|^2} \in \mathfrak{CT}(L_a^2). \tag{17}$$

When $\omega \in \mathbb{D}$ satisfies $r^4/4 < \rho(\omega, \alpha_j) \leq (3/4)r^4$ for some α_j we have

$$\left(1 - \frac{\left|\varphi_{\alpha_j}(\omega)\right|^2}{r^4}\right)^2 \left|\varphi_{\alpha_j}(\omega)\right|^2 \geq \left(1 - \frac{3^2 r^8}{4^2 r^4}\right)^2 \frac{r^8}{4^2} \geq \frac{r^8}{2^8},$$

meaning that

$$A(\omega) \geq \left(\frac{r}{2}\right)^8 \text{ when } \omega \in K\left(\alpha_j, \left(\frac{3}{4}\right)r^4\right)\setminus K\left(\alpha_j, \frac{r^4}{4}\right) \text{for some} \alpha_j\,. \tag{18}$$

Lemma (3.2.12)[81]: Given $0 < \sigma < 1$ there is a separated sequence $\{\alpha_j\}$ in $\mathbb{D}$ such that every $z \in \mathbb{D}$ is in $K(\alpha_j, 3\sigma/4)\setminus K(\alpha_j, \sigma/4)$ for some α_j.

Proof. Take a sequence $\{\alpha_j\} \subset \mathbb{D}$ such that $\rho(\alpha_i, \alpha_j) > \sigma/100$ if $i \neq j$ and

$$\rho\left(\{\alpha_j\}_{j\geq 1}, \omega\right) \leq \frac{\sigma}{8} \text{for every } \omega \in \mathbb{D}. \tag{19}$$

For an arbitrary $z \in \mathbb{D}$ write $\beta_j = \varphi_z(\alpha_j)$. The conformal invariance of ρ implies that $\{\beta_j\}_{j\geq 1}$ satisfies (19). We claim that there is some β_j such that $\frac{\sigma}{4} < |\beta_j| \leq (3/4)\sigma$. Otherwise

$$\rho\left(\frac{\sigma}{2}, \{\beta_j\}_{j\geq 1}\right) \geq \rho\left(\frac{\sigma}{2}, \mathbb{D}\setminus\left\{\frac{\sigma}{4} < |\omega| \leq \frac{3}{4}\sigma\right\}\right) = \rho\left(\frac{\sigma}{2}, \left\{\frac{\sigma}{4}, \frac{3\sigma}{4}\right\}\right) \geq \frac{\frac{\sigma}{4}}{1 - \frac{\sigma}{4}\cdot\frac{\sigma}{2}} > \frac{\sigma}{4}.$$

This contradicts (19) with respect to $\{\beta_j\}_{j\geq 1}$ for $\omega = \sigma/2$. If $\frac{\sigma}{4} < \{\beta_{j_0}\} \leq (3/4)\sigma$ then

$$\rho(\alpha_{j_0}, z) = \rho\left(\varphi_z(\alpha_{j_0}), \varphi_z(z)\right) = \rho(\beta_{j_0}, 0) = |\beta_{j_0}| \in \left(\frac{\sigma}{4}, \frac{3\sigma}{4}\right],$$

and since $z \in \mathbb{D}$ is arbitrary, the lemma follows.

Returning to our construction, fix $0 < r < 1$ and suppose that $S = \{\alpha_j\}_{j\geq 1}$ is a sequence satisfying Lemma (3.2.12) for $\sigma = r^4$. Since S is separated, by Lemma (3.2.2) we can decompose $S = S_1 \cup \ldots \cup S_N$, where for each $1 \leq k \leq N, K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ if $\alpha_i, \alpha_j \in S_k$ with $i \neq j$. For $1 \leq k \leq N$ write

$$A_k(\omega) = \sum_{\alpha_j\in S_k} \chi_{r^4}\left(\varphi_{\alpha_j}(\omega)\right)\left(1 - \frac{\left|\varphi_{\alpha_j}(\omega)\right|^2}{r^4}\right)^2 \left|\varphi_{\alpha_j}(\omega)\right|^2.$$

Then $\|A_k\|_\infty \leq 1$ and (17) says that $T_{A_k} \in \mathfrak{CT}(L_a^2)$. Consequently

$$\sum_{k=1}^N T_{A_k} = T_{\left(\sum_{k=1}^N A_k\right)} \in \mathfrak{CT}(L_a^2).$$

In addition, (18) says that for every $1 \leq k \leq N$,

$$A_k(\omega) \geq \left(\frac{r}{2}\right)^8 \text{ when } \omega \in K\left(\alpha_j, \left(\frac{3}{4}\right)r^4\right)\setminus K\left(\alpha_j, \frac{r^4}{4}\right) \text{for some} \alpha_j \in S_k,$$

and since Lemma (3.2.12) asserts that

$$\mathbb{D} = \bigcup_{1\le k\le N} \bigcup_{\alpha_j \in S_k} K\left(\alpha_j, \left(\frac{3}{4}\right) r^4\right) \backslash K\left(\alpha_j, \frac{r^4}{4}\right)$$

Then $\sum_{k=1}^{N} A_k(\omega) \ge \left(\frac{r}{2}\right)^8$ for every $\omega \in \mathbb{D}$. This completes the construction and proves Theorem (3.2.1).

Section (3.3): Dixmier Traces on the Unit Ball of $\mathbb{C}^n$

We study the Dixmier trace of a class of Toeplitz and Hankel operators on the Hardy and weighted Bergman spaces on the unit ball of $\mathbb{C}^d$. We give a brief account of the problem and explain some motivations. Consider the Bergman space $L_a^2(D)$of holomorphic functions on the unit disk D in the complex plane. For a bounded function f let T_f be the Toeplitz operator on $L_a^2(D)$. It is a well-known that for a holomorphic function f the commutator $[T_f^*, T_f]$ is of trace class and the trace is given by the square of the Dirichlet norm of f,

$$\mathrm{tr}[T_f^*, T_f] = \int_D |f'(z)|^2 dm(z),$$

which is one of the best known M¨obius invariant integrals. This formula actually holds for Toeplitz operators on any Bergman space on a bounded domain with the area measure replaced any reasonable measure [88]. There is a significant difference between Toeplitz operators on the unit disk and on the unit ball $B = B^d$ in $\mathbb{C}^d, d > 1$. Let $\mathcal{L}^p$ be the Schatten - von Neumann class of p-summable operators. The commutator $[T_f^*, T_f]$on the weighted Bergman space, say for holomorphic functions f in a neighborhood of the closed the unit disk, is in the Schatten - von Neumann class $\mathcal{L}^p$, for $p > \frac{1}{2}$and is zero if it is in $\mathcal{L}^p$, for $p \le \frac{1}{2}, \frac{1}{2}$ being called the cut-off; on the Hardy space $[T_f^*, T_f]$can be in any Schatten - von Neumann class $\mathcal{L}^p$, for $p > 0$; see [98] and [99] for the case of Hardy space and [87] for the case of weighted Bergman space. However for $d > 1$, it is in $\mathcal{L}^p$, for $p > d$, with $p = d$ being the cut off, both on the weighted Bergman spaces and on the Hardy space. Thus no trace formula was expected for the commutators. Nevertheless Helton and Howe [96] were able to find an analogue of the previous formula. They showed, for smooth functions $f_1, \dots, f_{2d}$ on the closed unit ball, that the anti-symmetrization $[T_{f_1}, T_{f_2}, \dots T_{f_{2d}}]$ of the $2d$ operators $T_{f_1}, T_{f_2}, \dots T_{f_{2d}}$is of trace class and found that

$$\mathrm{tr}[T_{f_1}, T_{f_2}, \dots T_{f_{2d}}] = \int_B df_1 \wedge df_2 \dots \wedge df_{2d}.$$

On the other hand, we observe that $[T_f, T_g]$ is, for smooth functions f and g, in the Macaev class $\mathcal{L}^{d,\infty}$ (which is an analogue of the Lorentz space $\mathcal{L}^{d,\infty}$), thus the product of d such commutators $[T_{f_1}, T_{g_1}]\, [T_{f_2}, T_{g_2}] \dots [T_{f_d}, T_{g_d}]$ is in $\mathcal{L}^{1,\infty}$ and hence has a Dixmier trace. One of our goals is to prove the following formula for the Dixmier trace of this product of commutators:

$$\mathrm{tr}_\omega[T_{f_1}, T_{g_1}] \dots [T_{f_d}, T_{g_d}] = \int_S \{f_1, g_1\} \dots \{f_d, g_d\}.$$

Here $\{f, g\}$ is the Poisson bracket of f and g; its restriction to the boundary S of B depends only on the boundary values of f and g and can be expressed in terms of the boundary CR

operators. This can be viewed as a generalization of the Helton-Howe theorem. We apply our result also to Hankel operators and obtain a formula for the Dixmier trace of the d-th power of the square modulus of the Hankel operators $H_{\overline{f}}^{*}H_{\overline{f}}$for holomorphic functions f. This provides a boundary $\mathcal{L}^{d,\infty}$ result for the Schattenvon Neumann $\mathcal{L}^{p}(p > d)$ properties of the square modulus of the Hankel operators (see [89], [91] and [101]).

There has been an intensive study of Dixmier trace and residue trace of pseudo-differential operators, mostly on compact manifolds where the analysis is relatively easier, see e.g. [90] and [93], thus the Toeplitz operators on Hardy spaces on the boundary of a bounded strictly pseudo-convex domains can be treated using the techniques developed there. The Hankel and Toeplitz operators on Bergman spaces, generally speaking, behaves rather differently from those on Hardy space, and the result of Howe [97] roughly speaking proves that Toeplitz operators of certain class can be treated similarly as in Hardy space case (also called the de Monvel - Howe compactification [92]). Our result can thus be viewed a generalization of the compactification to weighted Bergman spaces and an application of the [94] ideas of computing Dixmier traces. In particular our Theorem (3.3.5) are closely related to the results in [90] where the residue trace of pseudo-differential operators of certain class is computed; here we use the Weyl transforms and they differ from pseudo-differential operators of lower order, so that Theorem (3.3.5) can also be obtained from [90] provided one proves the the lower order terms are of trace class.

We will study the Dixmier trace for Toeplitz operators on a general strongly pseudo-convex domain [95].

Let $dm(z)$ be the Lebesgue measure on $\mathbb{C}^d$ and consider the weighted measure

$$d\mu_{\upsilon} = C_{\upsilon}(1-|z|^2)^{\upsilon-d-1}dm(z),$$

where C_{υ} is the normalizing constant to make $d\mu_{\upsilon}$ a probability measure and $\upsilon > d$. We let $\mathcal{H}_{\upsilon}$ be the corresponding Bergman space of holomorphic functions on B. We will also consider the Hardy space of square integrable functions on S which are holomorphic on B. This can be viewed as the analytic continuation of $\mathcal{H}_{\upsilon}$ at $\upsilon = d$. Thus we assume that $\upsilon \geq d$.

Let f be a bounded smooth function on $\overline{B}$, the closure of B. The Toeplitz operator T_f on $\mathcal{H}_{\upsilon}$ with symbol f is defined by

$$T_f g = P(fg)$$

where P is the Bergman or the Hardy projection for $\upsilon > d$ and $\upsilon = d$, respectively.

As was shown by Howe [97] there is a more flexible and effective way of studying the spectral properties of Toeplitz operators with smooth symbol, by using the theories of representations of the Heisenberg group and of pseudo-differential operators. We will adopt that approach. See [97] and [100]. So let $H_n = \mathbb{C}^d \times T$ be the Heisenberg group as in loc. cit.. The Heisenberg group has an irreducible representation, ρ, on the Fock space $\mathcal{F}$ consisting of entire functions f on $\mathbb{C}^d$ such that

$$\int_{\mathbb{C}^d} |f(z)|^2 e^{-\pi|z|^2}\, dm(z) < \infty.$$

The action of the Heisenberg group is explicitly given as follows. For $w \in \mathbb{C}^d$ viewed as an element in H_d,

$$\rho(w)f(w') = e^{-\frac{\pi}{2|w|^2}+\pi w'.\overline{w}} f(w'-w),$$

where $w'.\overline{w}$ is the Hermitian inner product on $\mathbb{C}^d$. The action of T is given by the change of variables.

Identifying the Lie algebra $\mathfrak{h}$ of the Heisenberg group with $\mathbb{R}^{2n} \oplus \mathbb{R}$ and thus $\mathbb{R}^{2n}$ with a subspace of the Lie algebra we get an action of $\mathbb{R}^{2n}$ as holomorphic differential operators on $\mathcal{F}$, which extends from $\mathfrak{h}$ to the whole enveloping algebra $\mathfrak{U}(\mathfrak{h})$ and which will also be denoted by ρ. In particular, taking the basis elements $\partial_j = \partial/\partial w_j$ and $\overline{\partial}_j = \partial/\partial \overline{w}_j$ of $\mathbb{R}^{2n}$ we have

$$\rho(\partial_j)f(w) = -\partial_j f(w), \quad \rho(\overline{\partial}_j)f(w) = \pi w_j f(w). \qquad (20)$$

Let, following the notation in [97], $\Delta \in \mathfrak{U}(\mathfrak{h})$ be the element

$$\Delta = \frac{1}{2}\left(\partial_j \cdot \overline{\partial}_j + \overline{\partial}_j \cdot \partial_j\right).$$

Then $\rho(\Delta)$ acts on $\mathcal{F}$ as a diagonal self-adjoint operator [97], under the orthogonal basis $\{w^\alpha, \alpha = (\alpha_1, \dots, \alpha_d)\}$, viz

$$\rho(\Delta)w^\alpha = -\pi\left(|\alpha| + \frac{d}{2}\right)w^\alpha. \qquad (21)$$

Let $F(z)$be a function on $\mathbb{C}^d$ (viewed as a function on the Heisenberg group). The Weyl transform $\rho(F)$ of F is defined by

$$\rho(F) = \int_{\mathbb{C}^d} F(w)\rho(w)dm(w).$$

To understand the operator theoretic properties of $\rho(F)$ we will need the Fourier transform of F. Let $\hat{F}$ be the (symplectic-) Fourier transform of F

$$\hat{F}(w') = 2^{-d}\int_{\mathbb{C}^d} F(w)e^{\pi i \operatorname{Im} w'.\overline{w}}dm(w),$$

and $F * G$ the twisted symplectic convolution

$$F * G(w) = \int_{\mathbb{C}^d} F(z)G(w-z)e^{\pi i \operatorname{Im} w.\overline{z}}dm(z).$$

We recall that

$$\widehat{F * G} = F * \hat{G}$$

and

$$\rho(F)\rho(G) = \rho(F * G)$$

for appropriate class of functions. A well-known theorem of Calderón-Vaillancourt states that if $\hat{F}$ and all its derivatives are bounded then $\rho(F)$ can be defined as a bounded operator on $\mathcal{F}$.

We will need a finer class of symbols introduced by Howe. Let

$$\mathcal{PT}(m,\mu) = \left\{F \in S^*(\mathbb{C}^d) : \left|\partial^\alpha \bar{\partial}^\beta \hat{F}\right| \le C_{\alpha\beta}(1+|w|)^{m-\mu(|\alpha|+|\beta|)}\right\}$$

and

$$\mathcal{PT}_{\text{rad}}(m,\mu) = \left\{F \in \mathcal{PT}(m,\mu) : \hat{F} = \left(1 - g(|w|)\right)\psi\left(\frac{w}{|w|}\right)|w|^m + D_1, D_1 \in \mathcal{PT}(m-\mu,\mu)\right\}.$$

Here g is a smooth function on $\mathbb{R}$ such that $0 \le g(t) \le 1$ on $\mathbb{R},, g(t) = 0$ for $|t| \ge 2$ and $g(t) = 1$ for $0 \le t \le 1$.

For $F \in \mathcal{PT}_{\text{rad}}(m,\mu)$ we will call

$$\sigma_m(F) := \psi\left(\frac{w}{|w|}\right)|w|^m \qquad (22)$$

its principal symbol. It can be obtained, up to the factor $|w|^m$, by

$$\psi(w) = \lim_{t\to\infty} t^{-m}\hat{F}(tw), \quad w \in S.$$

Following Howe [97], we will call $\rho(F), F \in \mathcal{PT}(m,\mu)$, a pseudo-Toeplitz operator of order m and smoothness μ. One has [97]

$$F \in \mathcal{PT}(m_1,\mu), G \in \mathcal{PT}(m_2,\mu) \Longrightarrow F * G \in \mathcal{PT}(m_1+m_2,\mu). \quad (23)$$

We will realize the Toeplitz operators T_f on $\mathcal{H}_\upsilon$ for f on B (or on S for the Hardy space) as Weyl transforms $\rho(F)$ of certain symbols F on $\mathbb{C}^d$. First we notice that

$$e_\beta := \left(\frac{(\upsilon)_{|\beta|}}{\beta!}\right)^{\frac{1}{2}} z^\beta$$

form an orthonormal basis of $\mathcal{H}_\upsilon$, and so do

$$E_\beta := \left(\frac{1}{\pi^{|\beta|}\beta!}\right)^{\frac{1}{2}} w^\beta$$

for $\mathcal{F}$. (Here $(\upsilon)_j := \upsilon(\upsilon+1)\ldots(\upsilon+j-1)$ is the usual Pochhammer symbol.) Thus the map

$$U: e_\beta \to E_\beta \quad (24)$$

is an unitary operator. First we will find the action of the elementary Toeplitz operators T_{z^α} under the intertwining map U.

Lemma (3.3.1)[86]: The operator $UT_{z^\alpha}U^*$ on $\mathcal{F}$ is given by

$$UT_{z^\alpha}U^* = \rho(z)^\alpha \rho\left(\pi^{|\alpha|}\left(\nu - \frac{d}{2} - \frac{1}{\pi}\Delta\right)_{|\alpha|}\right)^{-1/2} \quad (25)$$

This can be proved by direct computation. Indeed we have

$$T_{z^\alpha} e_\beta = \left(\frac{(\beta)_\alpha}{(\upsilon+|\beta\theta|)_{|\alpha|}}\right)^{\frac{1}{2}} e_{\beta+\alpha},$$

and the right hand side (25) can be easily computed by (20) and (21).

By using the previous Lemma we have then the following result which was proved by Howe [97] in the case when $\upsilon = d+1$; the general case of $\upsilon \geq d$ is essentially the same.

Proposition (3.3.2)[86]: Let $f \in C^\infty(S)$ and let $\tilde{f}$ be a C^∞ extension to B and $T_{\tilde{f}}$ the Toeplitz operator on $\mathcal{H}_\upsilon$. Then under the unitary equivalence of $\mathcal{H}_\upsilon$ and the Fock space $\mathcal{F}$ on $\mathbb{C}^d$, the Toeplitz operators are pseudo-Toeplitz operators with radial asymptotic limits $\mathcal{PT}_{\text{rad}}(0,1)$. More precisely, there exists $F \in \mathcal{PT}_{\text{rad}}(0,1)$ such that $UT_{\tilde{f}}U^* = \rho(F)$, and $f(\zeta) = \lim_{t\to\infty} \tilde{F}(t\zeta)$ for each $\zeta \in S$.

Recall that the Schatten-von Neumann class $\mathcal{L}^p, p \geq 1$, consists of compact operators T such that the eigenvalues $\{\mu n\}$ of $|T| = (T^*T)^{\frac{1}{2}}$ are p-summable, $\sum \mu_n^p < \infty$. In particular $\mathcal{L}^2$ is the Hilbert-Schmidt class, $\mathcal{L}^1$ the trace class and $\mathcal{L}^\infty$ are the compact operators. For $1 < p < \infty, 1 \leq q \leq \infty$, the Macaev class $\mathcal{L}^{p,q}$ is obtained by the real interpolation between $\mathcal{L}^1$ and $\mathcal{L}^\infty$. However, we will need the Macaev class $\mathcal{L}^{1,\infty}$, which consists of all compact operators such that, if $\mu_1 \geq \mu_2 \geq \ldots$,

$$\sum_{n=1}^{N} \mu_n = O(\log N).$$

There exists a linear functional on the space $\mathcal{L}^{1,\infty}$ that resembles the usual trace, called the Dixmier trace. Its definition is rather involved and we refer to [95] for details. Let

$C_b(\mathbb{R}_+)$ be the space of bounded continuous functions on $\mathbb{R}_+$ and $C_0(\mathbb{R}_+)$ the subspace of functions vanishing at ∞. Let ω be a positive linear functional on the quotient space $C_b(\mathbb{R}_+)/C_0(\mathbb{R}_+)$ such that $\omega(1)=1$. For a positive compact operator $T\in\mathcal{L}^{1,\infty}$ with eigenvalues $\{\mu_n\}$, extend μ_n to a step function on $\mathbb{R}_+$ and let $M_T(\lambda)$ be its Cesaro mean, which is a bounded continuous function on $\mathbb{R}^+$. The Dixmier trace of T is then defined by

$$\operatorname{tr}_\omega T=\omega(M_T).$$

It is then extended to all of $\mathcal{L}^{1,\infty}$ be linearity. In particular it is bounded and vanishes on trace class operators. The fact that we will need is that

$$\operatorname{tr}_\omega T=\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\mu_n(T)$$

if T is a positive operator and if the right hand side exists.

Lemma (3.3.3)[86]: For any $c\ge 0$the operator $\left(c-\rho(\Delta)\right)^{-d}=\rho(c\delta_0-\Delta)^{-d}$ is in the Macaev class $\mathcal{L}^{1,\infty}$.

Proof. It follows from (21) that the eigenvalues of $\left(c-\rho(\Delta)\right)^{d}$ are $\left(c+\pi\left(m+\frac{d}{2}\right)\right)^{d}, m=0,1,\ldots$,each of multiplicity

$$d_m:=\dim\{w^\alpha,|\alpha|=m\}=\binom{d+m-1}{d-1}\approx m^{d-1}.$$

The partial sums thus satisfy

$$\sum_{m\le N}\left(c+\pi\left(m+\frac{d}{2}\right)\right)^{-d}d_m\approx\sum_{m\le N}\left(c+\pi\left(m+\frac{d}{2}\right)\right)^{-d}m^{d-1}\approx\log N,$$

completing the proof.

Proposition (3.3.4)[86]: Let $F\in\mathcal{PT}(-2d,1)$. Then the Weyl transform $\rho(F)$ is in the Macaev class $\mathcal{L}^{1,\infty}$.

Proof. By (3.5.6) in [97],

$$\hat{\Delta}=\frac{\pi^2}{4}|w|^2, \tag{26}$$

so $-\Delta\in\mathcal{PT}(2,1)$, whence by (23) $(-\Delta)^{*d}\in\mathcal{PT}(2,1)$ and $(-\Delta)^{*d}*F\in\mathcal{PT}(0,1)$. By the Calderón-Vaillancourt theorem [97], the corresponding Weyl transform, $\rho(-\Delta)^d\rho(F)$, is bounded. Hence by the previous lemma $\rho(F)\in\mathcal{L}^{1,\infty}$, since the Macaev class $\mathcal{L}^{1,\infty}$ is an ideal.

Theorem (3.3.5)[86]: Let $F\in\mathcal{PT}_{\mathrm{rad}}(-2d,1)$ with the principal symbol $\sigma_{-2d}(\hat{F})$ as defined in (22). Then the Dixmier trace $\operatorname{tr}_\omega\rho(F)$ is independent of ω and is given by

$$\operatorname{tr}_\omega\rho(F)=\frac{\pi^d}{4^d}\int_S\hat{\sigma}_{-2d}(F)(w)$$

where $\int_S$ is the normalized integral over the unit sphere.

Proof. The proof is quite similar to that of Connes [94] for pseudo-differential operators on compact manifolds. Namely, by [97] and the definition of $\mathcal{PT}_{\mathrm{rad}}$, the Dixmier trace $\operatorname{tr}_\omega\rho(F)$ depends only on the leading symbol of $\sigma_{-2d}(\hat{F})$ and defines a positive measure on the unit sphere S in $\mathbb{C}^d$. By the unitary invariance of $\rho(F)$ the measure has to be a constant multiple of the area measure. To find the constant we
note that the symbol of $c\delta_0-\Delta, c>0$, is absolutely elliptic in the sense of (4.2.20) in [97], and thus by pp. 246–247 in [97] we can construct $F_0\in\mathcal{PT}_{\mathrm{rad}}(-2d,1)$ such that $\rho(F_0)=$

$\left(c-\rho(\Delta)\right)^{-d}$. The eigenvalue of $\rho(F_0)$ on the space of all m-homogeneous polynomials is, by the proof of Lemma (3.3.3),

$$\frac{1}{\left(c+\pi\left(m+\frac{d}{2}\right)\right)^{d}}.$$

Its Dixmier trace exists and is

$$\operatorname{tr}_\omega \rho(F_0)=\left(\frac{1}{\pi^D}\right).$$

On the other hand the principal symbol $\sigma_{-2d}(F_0)$ is the constant function $(4/\pi^2)^d|w|^{-2d}$ by the definition (cf. (26)), whose integration over the sphere is $(4/\pi^2)^d$. This completes the proof.

To apply our result to Toeplitz operators we need to introduce some more notation. We let

$$\partial_j^b=\partial_j-\bar z_j R,\ \ \bar\partial_j^b=\bar\partial_j-z_j\bar R\ ,$$

be the boundary Cauchy-Riemann operators [79], where $R=\sum_{j=1}^d z_j\partial_j$ is the holomorphic radial derivative. As vector fields they are linearly dependent, to wit,

$$\sum_{j=1}^d z_j\partial_j^b=0\,,\quad \sum_{j=1}^d \bar z_j\bar\partial_j^b=0. \tag{27}$$

Definition (3.3.6)[86]: We define a bracket $\{f,g\}_b$ for smooth functions f and g on S by

$$\{f,g\}_b:=\sum_{j=1}^d\left(\partial_j^b f\bar\partial_j^b g-\bar\partial_j^b f\partial_j^b g\right)$$

and call it the boundary Poisson bracket.

Lemma (3.3.7)[86]: Let F and G be two functions in $\mathcal{PT}_{\mathrm{rad}}(0,\mu)$ with principal symbols

$$\sigma_0(F)(z)=f\left(\frac{z}{|z|}\right)\ ,\sigma_0(G)(z)=g\left(\frac{z}{|z|}\right)$$

for f and g in $C^\infty(S)$. Then the principal symbol of $F*G-G*F$ is given by

$$\sigma_{-2}(F*G-G*F)(z)=\frac{4}{\pi}\{f,g\}_b\left(\frac{z}{|z|}\right)|z|^{-2}.$$

Proof. By the general result for the symbol calculus for pseudo-Toeplitz operators, cf. (2.2.5) in [97], we have $F*G-G*F\in\mathcal{PT}_{\mathrm{rad}}(-2\mu,\mu)$ with the principal symbol

$$\sigma_{-2}(F*G-G*F)(z)=\frac{4}{\pi}\{\sigma_0(F),\sigma_0(G)\}(z),$$

where $\{\cdot,\cdot\}$ is the ordinary Poisson bracket in complex coordinates

$$\{\Psi,\Phi\}:=\sum_{j=1}^d\left(\partial_j\Psi\bar\partial_j\Phi-\partial_j\Phi\bar\partial_j\Psi\right).$$

The function $\sigma_{-2}(F*G-G*F)(z)$ is positive homogeneous degree of -2. We need only to compute it for $z\in S$. We write the radial derivative as

$$R=-E+\frac{N}{2}\quad ,E:=\frac{1}{2}(\bar R-R)\ ,\quad N:=\bar R+R,$$

E being the Reeb vector field, which is well-defined on S, and N being the outward unit normal vector field on S. The vector field $\partial_j^b - \bar{z}_j E$ is thus a well-defined vector field on S, and for any function $\Phi(z) = \phi\left(\frac{z}{|z|}\right)$ we have

$$\partial_j \Phi(z) = \left(\partial_j^b + \bar{z}_j R\right)\Phi(z) = \left(\partial_j^b - \bar{z}_j E + \frac{\bar{z}_j}{2} N\right)\Phi(z) = \left(\partial_j^b - \bar{z}_j E\right)\phi(z),$$

since $N\Phi(z) = 0$ by homogeneity. Similarly $\bar{\partial}_j^b \Phi = \left(\partial_j^b + z_j E\right)\phi$on S. From this it follows that for $z \in S$

$$\begin{aligned}\{\sigma_0(F), \sigma_0(G)\}(z) &= \sum_{j=1}^{d}\Big(\left(\partial_j^b f(z) - \bar{z}_j E f(z)\right)\left(\bar{\partial}_j^b g(z) + z_j E g(z)\right) \\ &\quad - \left(\bar{\partial}_j^b f(z) - \bar{z}_j E f(z)\right)\left(\partial_j^b g(z) + z_j E g(z)\right)\Big) = \{f, g\}_b,\end{aligned}$$

by using (27).

Theorem (3.3.8)[86]: Let $f_1, g_1, \dots, f_d, g_d$ be smooth functions on $S, \tilde{f}_1, \tilde{g}_1, \dots, \tilde{f}_d, \tilde{g}_d$ their smooth extensions to B and $T_{\tilde{f}_1}, T_{\tilde{g}_1}, \dots, T_{\tilde{f}_d}, T_{\tilde{g}_d}$the associated Toeplitz operators on $\mathcal{H}_\upsilon$ for $\upsilon \geq d$. Then the product $\prod_{j=1}^{d}\left[T_{\tilde{f}_j}, T_{\tilde{g}_j}\right]$ is in the Macaev class and its Dixmier trace is given by

$$\mathrm{tr}_\omega \prod_{j=1}^{d}\left[T_{\tilde{f}_j}, T_{\tilde{g}_j}\right] = \int_S \prod_{j=1}^{d}\{f_j, g_j\}_b .$$

Proof. The proof is straightforward from the preceding lemma, formula (2.2.5) in [97] and Theorem (3.3.5).

We apply our result to Hankel operators with anti-holomorphic symbols. Let f be a holomorphic function in a neighborhood of B and $H_{\bar{f}} g = (I - P)\bar{f} g, \quad g \in \mathcal{H}_\upsilon$the Hankel operator. Then

$$\left[T_{\bar{f}}, T_f\right] = \left[T_f^*, T_f\right] = \left|H_{\bar{f}}\right|^{2d} = H_{\bar{f}}^* H_{\bar{f}}.$$

Corollary (3.3.9)[86]: Let f be as above. Then the Hankel operator is in $\mathcal{L}^{2d,\infty}$, equivalently the commutator $\left[T_{\bar{f}}, T_f\right]$ is in $\mathcal{L}^{d,\infty}$ and we have

$$\mathrm{tr}_\omega \left|H_{\bar{f}}\right|^{2d} = \mathrm{tr}_\omega\left(\left|T_{\bar{f}}, T_f\right|^d\right) = \int_S \left(|\nabla f|^2 - |Rf|^2\right)^d.$$

Notice that $H_{\bar{f}}$ is in the Schatten class $\mathcal{L}^p$ for $p > 2d$ and that its Schatten norm is

$$\left\|H_{\bar{f}}\right\|_p^p \approx \int_B (1 - |z|^2)^p \left(|\nabla f|^2 - |Rf|^2\right)^{\frac{p}{2}} dm(z);$$

see [89] and [101] for the Bergman space case ($\upsilon = d + 1$) and Hardy space ($\upsilon = d$). Our result formula provides thus a limiting result of the above estimates, and it is interesting to note that estimate has an equality as its limit for $p \to 2n$.

Chapter 4
Essential Commutants and Exact Sequences

We show that the space of Toeplitz operators associated to $\mathcal{F}$ is completely isometric to the commutant of the minimal normal extension $\hat{\mathcal{F}}$. Applications of these results are given for Toeplitz operators on strictly pseudoconvex or bounded symmetric domains. We deduce an essential version of the classical Hartman–Wintner spectral inclusion theorem, give a new proof of Johnson and Parrot's theorem on the essential commutant of abelian von Neumann algebras for separable Hilbert spaces and construct short exact sequences of Toeplitz algebras.

Section (4.1): Analytic Toeplitz Operators

For B_n, be the unit ball of C^n. The n-dimensional vector space over the complex field C, $S_n = \partial B_n$, is the boundary of B_n. We use σ to denote the unique rotation-invariant probability measure on S_n. The Lebesgue spaces $L_2(S_n, d\sigma)$have their customary meaning. $H^2(S_n)$ denotes the Hardy space which is the subspace of $L^2(S_n, d\sigma)$. For $\varphi \in L^\infty(S_n), T_\varphi$and H_φ, denotes the Toeplitz operator and the Hankel operator respectively. If $N \subset L^\infty(S_n)$, $\mathcal{F}(N)$ be the norm closed algebra generated by $\left|T_f : f \in N\right|. \mathcal{L}\left(H^2(S_n)\right)$ denotes all linearly bounded operators on $H^2(S_n)$, and let $\mathcal{K}$ be its ideal of compact operators. For $A, B \in \mathcal{L}\left(H^2(S_n)\right)$, if $[A, B] = AB - BA \in \mathcal{K}$, then we say that A essentially commutes with B . The set of operators which essentially commutes with any operator in $\mathcal{T}(N)$ is defined as the essential cornmutant of $\mathcal{T}(N)$, marked by $E_c\mathcal{T}(N)$. Let

$$\mathcal{A}_c = \left|f \in L^\infty(S_n): T_f \in E_c\mathcal{T}\left(H^\infty(S_n)\right)\right|, \mathcal{A} = \left|f \in L^\infty(S_n): H_f \text{is compact}\right|.$$

When $n = 1$, Davidsson[1] showed $E_c\mathcal{T}(H^\infty) = \mathcal{T}\mathcal{A}_c = \mathcal{T}\mathcal{A} = \mathcal{T}(H^\infty + C), \mathcal{A}_c = \mathcal{A} = H^\infty + C$ in 1977. When $n > 1$, $\mathcal{A}_c \supset \mathcal{A} \supsetneq H^\infty(S_n) = +C(S_n)$[12].

Recently, Guo Kunyu has obtained $E_c\mathcal{T}\left(H^\infty(S_n)\right) = \mathcal{T}\mathcal{A}_c$, and naturally put forward the conjecture: $\mathcal{A}_c = \mathcal{A}$!

We will show that the conjecture is true.

For convenience, $L^2(S_n, d\sigma), H^2(S_n)$ ... will be simply written as L^2, H^2 ... .

We take an inner function $\eta, \eta(0) = 0$; then $\eta^k (k = 0, \pm 1, \pm 2 \ldots)$ are orthogonal to each other in L^2. So we have $\lim_{k\to\infty} \int f\eta^k d\sigma = 0$, for $\forall f \in L$[103].

Fix $S \in E_c\mathcal{T}(H^\infty)$, and put $\sigma_k = S - T_\eta^{-k} S T_\eta^k$. Then there exists a subsequence which is w^*-converges. We still write it as $|\sigma_k|$, and mark $S' = w^* - \lim_k \sigma_k$.

Lemma (4.1.1)[102]: For S, S' and η, the following are true :

(i) $S - S' = T_f$ for some $f \in L^\infty$;

(ii) $w^* - \lim_k T_\eta^{-k} T_h S T_\eta^k = T_{hf}$ for any $h \in C(S)$;

(iii) $[T_h, S]$ is compact for any $h \in C(S)$;

(iv) there is a nautural number subsequence $\{m_k\}$ such that $w^* - \lim_k T_\mu^{-k} T_\eta^m T_h S T_\eta^m . T_\mu^k = T_{hf}$ for any inner function μ and any $h \in C(S)$.

Lemma (4.1.2)[102]: Define a linear map $F: L^\infty \to \mathcal{I}(H^2)$ by $F(h) = T_h S - T_{fh}$. For $h \in L^\infty$

(i) If $\|h_m\|_\infty$, are uniformly bounded and $h_m \to h$, a . e. $d\sigma$, then $w^* - \lim_{m\to\infty} F(h_m) = F(h)$

(ii) If h_1, h_2 are measuruble characteristic functions and the distance between their supports is greater than zero, then $F(h_1)F(h_2), F(h_1)F^*(h_2)$ and $F^*(h_2)F(h_1)$ are compact.

Argument of Lemma (4.1.1) and Lemma (4.1.2) are respectively similar to Davedson's Lemma (4.1.1) and Lemma (4.1.2), so the proof is omitted here.

The following lemma is similar to Lemma (4.1.2) of [104] .

Lemma (4.1.3)[102]: There are natural numbers K_n (only depending on n) . If $M > 0$, then the sphere S_n has a division $\{V_l\}$, satisfying

(i) $\{V_l\}$ is a finite class ; every V_l is a closed set, $S_n = \cup_l \ V_l$;

(ii) $|V_l| \leq M$, in which $|V_l|$ demotes the diameter of V;

(iii) $V_l \cap V_{l'} \subset \partial V_l \cap \partial V_{l'}, l \neq l', \partial V$ denotes the boundary of V;

(iv) $\{V_l\}$ can be divided into K_n subclasses such that each subclass consists of mutually disjoint sets V_1.

Let g be a bounded measurable characteristic function on S_n, the support of g is still marked by g. We use χ to denote the characteristic function with a closed support on S. In the following, χg denotes the characteristic function as well as the support of χg.Let$\mathcal{E}_m = \left\{\chi g: |\chi| \leq \frac{1}{m}\right\}, C_n = \sup_{\chi \leq \frac{1}{m}} \|F(\chi g)\|$. Obviously. $C_m \geq C_{m+1}$.

Let $\pi: \mathcal{J}(H^2) \to \mathcal{J}(H^2)/\mathcal{K}$ be the canonical homomorphism onto the Calkin algebra. It is easy to see that if $\left\|\pi\big(F(g)\big)\right\| > \alpha > 0$, then $\lim_{m\to\infty} C_m = C(\alpha) > 0$.

Lemma (4.1.4)[102]: If $\left\|\pi\big(F(g)\big)\right\| > \alpha > 0$, then there exists a characteristic function sequence $\{\chi_m\}$ with mutually disjoint closed support such that $\|F(\chi_m g)\| > \frac{C(\alpha)}{2}$. So there are continuous function sequences $\{h_m\}$ such that $\|h_m\|_\infty \leq 1 + \frac{1}{2^m}$, $\|h_m(1-\chi_m g)\|_\infty \leq \frac{1}{2^m}$ and $\|F(h_m)\| > \frac{C(\alpha)}{2}$.

Proof. Similar to Davidson's[1], we can find a characteristic function sequence $\{\chi_m\}$with a mutually disjoint closed support such that $\|F(\chi_m g)\| > \frac{C(\alpha)}{2}$. Fix m such that $\chi_m g$ is a measurable set in S_n. For every natural number k, there exists a closed set $F_k \subset \chi_m g, \sigma(\chi_m g - F_k) < \frac{1}{2k}$. Let g_k be the characteristic function of F_k. Then there exists a continuous functions sequence $\{h'_i\}$ such that $0 \leq h'_i \leq g_k + \frac{1}{2^m}$ and $h'_i \to g_k$, a.e. $d\sigma$. By Egoroff 's theorem, there exists a σ-measurable set $E_k \subset S_n$, such that $\sigma(S_n - E_k) < \frac{1}{2k}$ and $\{h'_i\} \to g_k$uniformly on E_k. Hence there is $h'_k \in \{h'_i\}$ such that $|h'_k - g_k| < \frac{1}{k}$, $\forall z \in E_k$, and $\sigma\left(|h'_k - \chi_m g| \geq \frac{1}{k}\right) \leq \sigma(\chi_m g - F_k) + \sigma(S_n - E_k) < \frac{1}{k}$. So $h'_k \to \chi_m g$have according to the a measure. Thus there exists a subsequence $\left\{h'_{k_j}\right\}$ such that $h'_{k_j} \to \chi_m g$ a.e. $d\sigma, 0 \leq h'_{k_j} \leq g_{k_j} + \frac{1}{2^m} \leq \chi_m g + \frac{1}{2^m}$, and $\left\|h'_{k_j}(1-\chi_m g)\right\|_\infty \leq \left\|h'_{k_j}\left(1 - g_{k_j}\right)\right\|_\infty \leq \frac{1}{2^m}$.

According to quality (i) of function F in Lemma (4.1.2), there must $k_{j(m)}$ such that $\left\|F\left(h'_{k_{j(m)}}\right)\right\| > \frac{C(\alpha)}{2}$. Put $h_m = h'_{k_{j(m)}}$, that is our request. The proof is finished.

Lemma (4.1.5)[102]: Let $h \in C(S_n), \varepsilon > 0$. Then there are inner functions μ and $G \in H^\infty$ such that $|\mu(\zeta)h(\zeta) - G(\zeta)| \leq \varepsilon$ a.e. $d\sigma$ (c.f. Theorem 5.2 of [103]) .

Lemma (4.1.6)[102]: Let $T \in \mathcal{I}(H^2)$, $\mathcal{U}$ a weakly closed subalgebra of $\mathcal{I}(H^2), \{A_m\} \subset \mathcal{U}$. Suppose that there exist $\delta > 0, M > 0$ such that $\|[A_m, T]\| > \delta$ and $\left\|\sum_{m\in J} A_m\right\| \leq M$ for all finite subsets J of number set N. Then there exists an element b in $\mathcal{U}$ such that $[b, T]$ is not compact (see Lemma 3 of [1]) .

Theorem (4.1.7)[102]: Let g be a a-measurable characteristic function, $F(g) = T_g S - T_{gf}, S \in E_c\mathcal{T}(H^\infty)$, where f is as in Lemma (4.1.1) . Then $F(g)$ is compact.

Proof. Suppose that $F(g)$ is not a compact. Then $\left\|\pi\big(F(g)\big)\right\| > \alpha > 0$. By Lemma (4.1.4), there exists a continuous functions sequence $\{h_m\}$ such that $\|h_m\|_\infty \leq 1 + \frac{1}{2^m}$, $\|h_m(1 - \chi_m g\|_\infty \leq \frac{1}{2^m}$, and $\|F(h_m)\| > \frac{C(\alpha)}{2}$, where $\{\chi_m\}$ are characteristic functions with mutually disjoint closed supports. By Lemma (4.1.5), for each $h_m, \varepsilon_m = \frac{1}{2^m}$, there exists an inner function μ_m and $f_m \in H^\infty$ such that $|\mu_m(\zeta)h_m(\zeta) - f_m(\zeta)| \leq \varepsilon_m$, a.e. $d\sigma$. By Lemma (4.1.1), (iv), for μ_m there exists a number sub-sequence $\{l_k\}, l_{k+1} > l_k$ such that $w^* - \lim_k T_{\mu_m}^{-k} T_\eta^{-l}.T_{h_m} S T_n^l.T_{\mu_m}^k = T_{h_m f}$. Hence

$$w^* - \lim_k T_{\mu_m \eta^{-l}}^{-k}\left[T_{\mu_m \eta^l h_m}^k, S\right] = T_{h_m} S - T_{h_m f} + w^* - \lim_k T_{\mu_m \eta^{-l}}^{-k}\left[T_{h_m}, S\right].$$

$T_{\mu_m}^{-k}$(since $\left[T_{h_m}, S\right]$ is compact, the latter limit is zero) $=T_{h_m} S - T_{h_m f} = F(h_m)$. It follows that the norm is lower semicontinuous on w^*-topology so that there exists $k(m)$ such that $\left\|\left[T_{\mu_m \eta^l k_m}^{k(m)}, S\right]\right\| > \frac{C(\alpha)}{2}$.

But $\left\|\mu_m^{k(m)} \eta^l k_m h_m - \mu_m^{k(m)-1} \eta^{l_{k_m}} f_m\right\|_\infty \leq \varepsilon_m;$

Hence $\left\|\left[T_{\mu_m}^{k(m)-1} \eta^{l_{k_m}} f_m, S\right]\right\| > \frac{C(\alpha)}{2} - 2\|S\|_{\varepsilon_m}$.

Put $\varphi_m = \mu_m^{k(m)-1} \eta^{l_{k_m}} f_m$. Then $\varphi_m \in H^\infty, \|\varphi_m\|_\infty \leq 1 + 2_{\varepsilon_m}$and $\|\varphi_m(1 - \chi_m g)\|_\infty \leq \left\|\varphi_m - \mu_m^{k(m)} \eta^{l_{k(m)}} h_m\right\|_\infty + \|h_m(1 - \chi_m g)\|_\infty \leq 2_{\varepsilon_m}$.

For any finite subsets J of Z_+. put $\varphi_J = \sum_{m\in J} \varphi_m$. Then $\left\|\varphi_J \chi_m g\right\|_\infty \leq \|\varphi_m\|_\infty + \sum_{l\neq m} \|\varphi_l(1 - \chi_l g)\|_\infty \leq 1 + 2_{\varepsilon_m} + 2\sum_{l\neq m} \varepsilon_1 \leq 3$, $\left\|\varphi_J\big(1 - \sum_m \chi_m g\big)\right\|_\infty \leq \sum_m \|\varphi_m(1 - \chi_m g)\|_\infty \leq 2\sum \varepsilon_m \leq 2$. Thus

$$\left|\varphi_J(z)\right| \leq \begin{cases} 2, & z \in 1 - \sum_m \chi_m g \ , \\ 3, & z \in \sum_m \chi_m g \ . \end{cases}$$

So we have $\left\|\varphi_J\right\|_\infty \leq 3$.

Put $\mathcal{U} = \mathcal{T}(H^\infty), A_m = T_{\varphi_m}$. By Lemma (4.1.6), there exists $\varphi \in H^\infty$ such that $\left[T_\varphi, S\right]$ is not compact. But $S \in E_c\mathcal{T}(H^\infty)$; a contradiction. Hence $F(g)$ is compact. The proof is completed.

Corollary (4.1.8)[102]: For $h \in L^\infty, F(h) = T_h S - T_{hf}$ is compact .

Proof. Because h can be approximated uniformly by finite linear combinations of characteristic functions, $F(h)$ can be approximated by compact operators in operator norm. Therefore $F(h)$ is compact. The proof is thus completed.

Theorem (4.1.9)[102]: Let $S \in \mathcal{J}(H^2)$. Then S essentially commutes with all analytic Tieplitz operators if and only if $S = T_f + K$, where $f \in L^\infty$ such that Hankel operator H_f is compact, and $K \in \mathcal{K}$.

Proof. The "if" part is obvious. Only the "only if" part needs to be proved.

First take $h = 1$. Then $F(1) = S - T_f = K$ iscompact. Hence $S = T_f + K$, and $K \in \mathcal{K}$.And then take $h = \bar{f} \in L^\infty, F(h) = T_{\bar{f}}S - T_{|f|^2} = T_{\bar{f}}T_f - T_{|f|^2} + T_{\bar{f}}K = -H_f^* H_f + T_{\bar{f}}K$ is compact. So $H_f^* H_f K$ is compact. It follows that H_f is compact. The proof is thus completed.

According to Theorem (4.1.9), we obtain the result that $E_c\mathcal{T}(H^\infty), \mathcal{T}(\mathcal{A}_c) = \mathcal{T}(\mathcal{A})$, and $\mathcal{A}_c = \mathcal{A}$.

Since $H^\infty \subset H^\infty + C \subset \mathcal{A}$, we have the following corollaries.

Corollary (4.1.10)[102]: $E_c\mathcal{T}(H^\infty) = E_c\mathcal{T}(H^\infty + C) = E_c\mathcal{T}(\mathcal{A})$.

Corollary (4.1.11)[102]: Sequence $(0) \to \mathcal{K} \to \mathcal{T}(\mathcal{A}) \overset{\tau}{\to} \mathcal{A} \to (0)$ is short exact, where $\tau: E_c\mathcal{T}(H^\infty) \to L^\infty, for\ S \in E_c\mathcal{T}(H^\infty), S = T_f + k, f \in L^\infty,\ k \in \mathcal{K}$, sutisfying $\tau(S) = f$.

Section (4.2): Toeplitz Algebras of Spherical Isometries

A spherical isometry on a Hilbert space $\mathcal{H}$ is a commuting family $S = \{T_j\}_{j\in J}$of bounded operators on $\mathcal{H}$ such that $\sum_{j\in J} T_j^* T_j = 1$, To each spherical isometry one can associate its set of Toeplitz-type operators consisting of all solutions of the operator equation

$$\sum_{j\in J} T_j^* X T_j = X.$$

One defines in a similar way Toeplitz operators associated to arbitrary commuting families of spherical isometries.

We apply some operator space techniques in order to construct exact sequences for C^*-algebras generated by spherical isometries or by their associated Toeplitz-type operators. We make use of the more or less known fact that the set $\mathcal{T}(\mathcal{F})$ of all Toeplitz operators associated to an arbitrary family $\mathcal{F}$ of commuting spherical isometries is an injective operator system, which means that it is the range of a completely positive unital mapping $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$with $\Phi \circ \Phi = \Phi$. This fact, when combined with a certain basic property of such projections proved in [115] , enables us to show (see Theorem (4.2.10)) that $\mathcal{F}$ admits a commuting normal extension $\hat{\mathcal{F}}$ on some Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$ and that if $\hat{\mathcal{F}}$ is minimal then there exists a $*$-representation π from the C^*-algebra generated by $\mathcal{T}(\mathcal{F})$ onto the commutant of $\hat{\mathcal{F}}$ in $B(\hat{\mathcal{H}})$ for which the compression map $\rho(X) = P_{\mathcal{H}} X|_{\mathcal{H}}$ is a complete isometric cross whose range coincides with$\mathcal{T}(\mathcal{F})$. When we restrict π to the C^*-algebra $C^*(\mathcal{F})$ generated by $\mathcal{F}$ in $\mathcal{B}(\mathcal{H})$ we obtain a $*$-representation of $C^*(\mathcal{F})$ onto $C^*(\hat{\mathcal{F}})$ whose kernel coincides with the commutator ideal of $C^*(\mathcal{F})$. We also show that any operator in the commutant of $\mathcal{F}$ has a unique norm-preserving extension to an operator in the commutant of $\hat{\mathcal{F}}$.

It turns out that several classes of Toeplitz operators on various Hardy spaces can be realized as common fixed points of some commuting families of completely positive mappings induced by spherical isometries. In this sense, apart from the well-known Brown-Halmos characterization of Toeplitz operators on the unit circle (see [2]) there is a similar result due to A.M. Davie and N. Jewell [12] for the case of Toeplitz operators on the unit sphere in $\mathbb{C}^n$ where the unilateral shift is replaced by the Szego $n-$tuple. Similar characterizations also hold for Toeplitz operators on Hardy spaces on ordered groups [23] . The method we shall develop here allows us to enlarge considerably the class of Toeplitz operators that admit such characterization. We show that if $\Omega \subset \mathbb{C}^n$ is either a bounded strictly pseudoconvex domain or a bounded symmetric domain and m is any Borel probability measure on the Shilov boundary of Ω then the Toeplitz operators on the corresponding Hardy space $H^2(m)$ are indeed the fixed points of a certain spherical isometry.

As a consequence we obtain exact sequences and spectral inclusion theorems for operators of the type mentioned above.

We recall for later use the following by-product of the proof of Theorem 3.1 in [115] , which is also stated as Lemma 6.1.2 in [118] .

Theorem (4.2.1)[105]: Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$. Then for all $X, Y \in \mathcal{B}(\mathcal{H})$ we have

$$\Phi(\Phi(X)Y) = \Phi\big(X\Phi(Y)\big) = \Phi\big(\Phi(X)\Phi(Y)\big).$$

In [115] and [118] this result is used to show that the range of Φ is completely isometric to a C^*-algebra , where the multiplication is defined by the rule

$$\Phi(X) \circ \Phi(Y) = \Phi\big(\Phi(X)\Phi(Y)\big)$$

for every $X, Y \in \mathcal{B}(\mathcal{H})$. We shall recover the latter result as a consequence of Theorem (4.2.2) below. Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$ and let $\varepsilon = \text{Ran}\,\Phi$ denote its range. We denote by $C^*(\mathcal{E})$ the unital C^*-algebra generated by ε in $\mathcal{B}(\mathcal{H})$. Let

$$\Phi_0: C^*(\mathcal{E}) \to \mathcal{B}(\mathcal{H})$$

denote the restriction of Φ to $C^*(\mathcal{E})$. Then, according to the Stinespring dilation theorem (see Theorem 5.2.1 in [118]), there exist a Hilbert space $\mathcal{K}$, a bounded operator $V: \mathcal{H} \to \mathcal{K}$ and a unital $*$-representation

$$\pi: C^*(\mathcal{E}) \to \mathcal{B}(\mathcal{K})$$

such that $\Phi_0(X) = V^*\pi(X)V$ for all $X \in C^*(\mathcal{E})$. Thus the diagram

$$\begin{array}{ccc} C^*(\mathcal{E}) & \overset{\pi}{\longrightarrow} & \mathcal{B}(\mathcal{K}) \\ \Phi_0 \downarrow & & \downarrow \rho \\ \varepsilon & \overset{\iota}{\rightarrow} & \mathcal{B}(\mathcal{H}) \end{array}$$

is commutative, where $\rho(X) = V^*XV$ whenever $X \in \mathcal{B}(\mathcal{H})$, and ι is the inclusion map. We shall assume that π is minimal in the sense that $\mathcal{K}$ is the smallest invariant subspace for π containing the range of V. Now, under these conditions, we can state the following theorem which will be very useful for our study of Toeplitz algebras.

Theorem (4.2.2)[105]: Let $\Phi, \Phi_0, \mathcal{E}, C^*(\mathcal{E}), \mathcal{K}, V$ and π be as above. Then $\text{Ker}\,\Phi_0 = \text{Ker}\,\pi$and the mapping

$$\rho: \pi(C^*(\mathcal{E}\,) \to \mathcal{B}(\mathcal{H})$$

defined by $\rho\big(\pi(X)\big) = V^*\pi(X)V\ for\ X \in C^*(\mathcal{E})$ is a complete isometry whose range equals $\varepsilon = \text{Ran}\,\Phi$. Moreover, if $\text{Ran}\,\Phi$ is σ-weakly closed, then $\pi\big(C^*(\mathcal{E})\big)$ is also σ-weakly closed, hence a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$ and the map ρ defined above is a σ-weak homeomorphism.

Proof. First of all, one can easily see that $\text{Ker}\,\Phi_0$ is an ideal in $C^*(\mathcal{E})$. Indeed, Theorem (4.2.1) implies that $\text{Ker}\,\Phi_0$ is invariant under multiplication by elements in $\mathcal{E}$ and then use the fact that, since E is selfadjoint, the C^*-algebra $C^*(\mathcal{E})$ is the closed linear span of all finite products of elements from $\mathcal{E}$ and the identity. Now, in order to prove the equality of the two kernels, we fix $T \in C^*(\mathcal{E})$ such that $\Phi_0(t) =$ and let $X, Y \in C^*(\mathcal{E})$ and $\xi, \eta \in \mathcal{H}$ be arbitrary. Then

$$(\pi(T)\pi(X)V\xi, \pi(Y)V\eta) = (V^*\pi(Y^*TX)V\xi, \eta) = (\Phi_0(Y^*TX)\xi, \eta) = 0$$

because Ker Φ_0 is an ideal. Since π is minimal, this shows that $\pi(T) = 0$. Since the other inclusion is trivial, the equality of the two kernels is proved.

We now show that the mapping ρ is a complete isometry. First, we see that, since $\Phi_0 = \rho \circ \pi$ and $Ker\, \pi = Ker\Phi_0$ it follows that ρ is one-to-one. Moreover, since $\Phi_0^2 = \Phi_0$, we have that $\rho \circ \pi \circ \rho \circ \pi = \rho \circ \pi$ hence $\pi \circ \rho$ is the identity on $\pi\big(C^*(\mathcal{E})\big)$. It then follows, since both π and ρ are completely contractive, that ρ is actually completely isometric. The last assertion of the theorem follows easily from the previous one and the separate weak$*$-continuity of the multiplication on a von Neumann algebra. The proof of the theorem is completed.

As we have mentioned above, this result offers an alternate proof of Theorem 3.1 in [115] .

Theorem (4.2.3)[105]: Let $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a completely positive and completely contractive mapping such that $\Phi^2 = \Phi$. Then RanΦ is completely isometric with a unital C^*-algebra where the product is defined by the rule

$$\Phi(X) \circ \Phi(Y) = \Phi\big(\Phi(X)\Phi(Y)\big)$$

$for\ all\ X, Y \in \mathcal{B}(\mathcal{H}).$

Proof. Using the notation in Theorem (4.2.2), we see that the above defined product is precisely the one induced from $\pi\big(C^*(\mathcal{E})\big)$ via the complete isometry ρ.

Remark (4.2.4)[105]: It is a well-known fact that if A is a unital C^*-algebra and $\theta: A \to \mathcal{B}(\mathcal{H})$ is a unital completely isometric mapping, then there exists a $*$-homomorphism

$$\pi : C^*\big(\theta(A)\big) \to A$$

such that $\pi \circ \theta = id_A$, (see Theorem 4.1 in [114]). Therefore, in the case when the mapping Φ in Theorem (4.2.2) is unital and assuming Theorem (4.2.3) one can immediately see that the mapping Φ_0 appearing in Theorem (4.2.2) becomes a $*$-homomorphism when its range is endowed with the multiplication defined in Theorem (4.2.3). This offers a shorter proof of Theorem (4.2.2); however the line we took in that theorem gives simultaneously the isomorphism in Theorem (4.2.3) and a spatial representation for that algebraic structure.

We shall apply Theorem (4.2.2) to the study of the C^*- algebra generated by a commuting family of spherical isometries.

Definition (4.2.5)[105]: A commuting family $S = \{T_j\}_{j\in J}$ of bounded operators on a Hilbert space $\mathcal{H}$ is said to be a spherical isometry if

$$\sum_{j\in J} T_j^*T_j = 1$$

in the weak operator topology.

For instance, if m is any probability Borel measure on the unit sphere S^{2n-1} in $\mathbb{C}^n$, and $\mathcal{H} \subset L^2(m)$ is a jointly invariant subspace for the multiplication operators $\{M_{z_1}, \dots, M_{z_n}\}$ on $L^2(m)$, then their restrictions $\{T_{z_1}, \dots, T_{z_n}\}$ to $\mathcal{H}$ form a spherical isometry. Of particular interest is the case when m is the normalized area measure and $\mathcal{H}$ is the $L^2(m)$-closure of all analytic polynomials (the Hardy space $H^2(S^{2n-1})$) in which case $\{T_{z_1}, \dots, T_{z_n}\}$ is called the Szego n-tuple on $H^2(S^{2n-1})$.

If $\{T_j\}_{j\in J}$ is an arbitrary family of operators on $\mathcal{H}$ satisfying the above equation in particular a spherical isometry then a completely positive unital, hence completely contractive mapping $\phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be defined by the formula

$$\phi(X) = \sum_{j\in J} T_j^* X T_j$$

and is also obvious that ϕ is σ-weakly continuous.

The object of study in what follows is a commuting family of spherical isometries $\mathcal{F} = \{S_\alpha\}_{\alpha\in\Gamma}$ on some Hilbert space $\mathcal{H}$. This means that if $\alpha \in \Gamma$ then $S_\alpha = \{T_{j,\alpha}\}_{j\in J_\alpha}$ is a spherical isometry and the union $\bigcup_{\alpha\in\Gamma} S_\alpha$ is a commutative set of operators.

Definition (4.2.6)[105]: Given a commuting family $\mathcal{F} = \{S_\alpha\}_{\alpha\in\Gamma}$ of spherical isometries on $\mathcal{H}$ we define, using the notations above, the space $\mathcal{T}(\mathcal{F})$ of all $\mathcal{F}$-Toeplitz operators to be the set of all operators $X \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{j\in J_\alpha} T_{j,\alpha}^* X T_{j,\alpha} = X$$

for all $\alpha \in \Gamma$.

In other words, $\mathcal{T}(\mathcal{F})$ is the set of all common fixed points of the completely positive mappings associated to each spherical isometry from $\mathcal{F}$. It is obvious that $\mathcal{T}(\mathcal{F})$ contains the commutant of $\mathcal{F}$ in particular it contains all the sets S_α for $\alpha \in \Gamma$. We shall construct a completely positive projection Φ on this space which will play a crucial role for our study of Toeplitz algebras associated to spherical isometries. We need the following lemma which is a particular case of a more general result proved in [110] . However for completeness we shall give below a direct proof.

Lemma (4.2.8)[105]: Let $\{\phi_\alpha\}_{\alpha\in\Gamma}$ be a set of commuting completely positive unital and σ-weak continuous mappings acting on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there exists a completely positive mapping $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ whose range is precisely the set

$$\{X \in \mathcal{B}(\mathcal{H}): \phi_\alpha(X) = X, \alpha \in \Gamma\}$$

and such that $\Phi^2 = \Phi$.

Proof. Let S denote the semigroup of all finite products of elements from the set $\{\phi_\alpha\}_{\alpha\in\Gamma}$. Each element $s \in S$ corresponds to a completely positive unital and σ−weak continuous mapping $\psi_S: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ which is a finite product of ϕ_α's.

It is obvious that the fixed point set of $\{\alpha_\alpha\}_{\alpha\in\Gamma}$ is the same as that of $\{\psi_s\}_{s\in S}$.

We thus obtain an action

$$\gamma: S \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

defined by

$$\gamma(s, X) = \psi_s(X)$$

for all $s \in S$ and $X \in \mathcal{B}(\mathcal{H})$. Since S is commutative, a well-known result of Dixmier [116] shows that S is amenable, which means that there exists a state μ on the C^*-algebra $\ell^\infty(S)$ of all bounded complex functions on S which is invariant under all translations with elements from S. More precisely, if $t \in S$ and $L_t: \ell^\infty(S) \to \ell^\infty(S)$is defined by $L_t(f)(s) = f(ts)$ for $s \in S$ then $\mu(L_t f) = \mu(f)$ for all $f \in \ell^\infty(S)$.

Now, given $T \in \mathcal{B}(\mathcal{H})$, for each pair of vectors $\xi, \eta \in \mathcal{H}$ define $[\xi, \eta]_T = \mu(\gamma(\cdot, T)\xi, \eta)$ and observe that this is a bounded sesquilinear map therefore there exists an operator that we shall denote by $\Phi(T)$ in $B(\mathcal{H})$ such that

$$(\Phi(T)\xi, \eta) = [\xi, \eta]_T$$

for all $\xi, \eta \in \mathcal{H}$. It is now a matter of routine to verify that the mapping $T \longmapsto \Phi(T)$is completely positive. It is straightforward to see that if $T \in \mathcal{B}(\mathcal{H})$ is such that $\psi_s(T) = T$ for all $s \in S$ then $\Phi(T) = T$ as well.

We will show now that $\psi_s\big(\Phi(T)\big) = \Phi(T)$ for all $T \in B(\mathcal{H})$. In order to see that, recall that all the mappings ψ_s are σ-weakly continuous, which means that for any $s \in S$ there exists a norm continuous mapping ψ_s^*on the space $\mathcal{G}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ such that $\mathrm{tr}(\psi_s(T)L) = \mathrm{tr}\big(T\psi_s^*(L)\big)$ for all $T \in B(\mathcal{H})$and $L \in \mathcal{G}_1(\mathcal{H})$. Moreover one can see that the mapping Φ satisfies the identity $\mathrm{tr}(\Phi(T)L) = \mu(\mathrm{tr}(\gamma(\cdot, T)L)$ for all $T \in B(\mathcal{H})$and $L \in \mathcal{G}_1(\mathcal{H})$. It then follows that for any $t \in S$ we have

$$\mathrm{tr}\big(\psi_t(\Phi(T)L)\big) = \mathrm{tr}\big(\Phi(T)\psi_s^*(L)\big) = \mu(\mathrm{tr}\big(\gamma(\cdot, T)\psi_t^*(L)\big) = \mu(\mathrm{tr}\big(\gamma(t\,\cdot, T)(L)\big)$$

and because μ is an invariant mean, this last term equals $\mu(\mathrm{tr}\big(\gamma(\cdot, T)(L)\big)$ which is equal to $\mathrm{tr}\big(\Phi(T)(L)\big)$ for all $T \in B(\mathcal{H})$and $L \in \mathcal{G}_1(\mathcal{H})$. This shows that indeed $\mathrm{tr}\psi_t(T) = \Phi(T)$ for all $T \in B(\mathcal{H})$, hence $\Phi^2 = \Phi$ as well. The proof of this lemma is completed.

We also need the following lemma.

Lemma (4.2.9)[105]: Suppose $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra and $\mathcal{M}'$ denotes its commutant. Let $\mathcal{R} = \{T_j\}_{j \in J}$ be a family of operators in $\mathcal{M}$ such that for all $X \in \mathcal{M}$

$$\sum_{j \in J} T_j^* X T_j = X$$

in the $\sigma-$weak topology. Then $\mathcal{R} \subset \mathcal{M} \cap \mathcal{M}'$.

We have the following result.

Theorem (4.2.10)[105]: Let $\mathcal{F} = \{S_\alpha\}_{\alpha \in \Gamma}$ be a commuting family of spherical isometries on some Hilbert space $\mathcal{H}$ with $S_\alpha = \{T_{j,\alpha}\}_{j \in J_\alpha}$ for each $\alpha \in \Gamma$ and let $\mathcal{T}(\mathcal{F})$ be the space of all $\mathcal{F}$-Toeplitz operators (see Definition (4.2.6) above). Let also $C^*\big(\mathcal{T}(\mathcal{F})\big)$denote the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathcal{T}(\mathcal{F})$. Then we have:

(i) There exists a completely positive unital mapping $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\Phi^2 = \Phi$ and whose range coincides with $\mathcal{T}(\mathcal{F})$.

(ii) There exist a Hilbert space $\widehat{\mathcal{H}}$ containing $\mathcal{H}$ and a commuting family $\hat{\mathcal{F}} = \{\hat{S}_\alpha\}_{\alpha \in \Gamma}$of normal spherical isometries on $\widehat{\mathcal{H}}$ with $\hat{S}_\alpha = \{\hat{T}_{j,\alpha}\}_{j \in J_\alpha}$ which leaves $\mathcal{H}$ invariant and whose restriction to $\mathcal{H}$ coincides with $\mathcal{F}$ in other words the family $\mathcal{F}$ is subnormal.

(iii) Suppose that the normal extension $\hat{\mathcal{F}}$ is minimal, i.e. $\hat{\mathcal{F}}$ is the smallest reducing subspace for $\hat{\mathcal{F}}$ containing $\mathcal{H}$. Then there exists a unital $*$-representation

$$\pi: C^*\big(\mathcal{T}(\mathcal{F})\big) \to \mathcal{B}\big(\widehat{\mathcal{H}}\big)$$

such that:

(3a) $\pi\big(T_{j,\alpha}\big) = \hat{T}_{j,\alpha}$ for all $\alpha \in \Gamma$ and $j \in J_\alpha$.

(3b) If $\mathcal{P}_{\mathcal{H}}$ is the orthogonal projection of $\widehat{\mathcal{H}}$ onto $\mathcal{H}$ then

$$\Phi(X) = \mathcal{P}_{\mathcal{H}}\pi(X)|_{\mathcal{H}}$$

for every $X \in C^*\big(\mathcal{T}(\mathcal{F})\big)$.

(3c) The image $\pi\Big(C^*\big(\mathcal{T}(\mathcal{F})\big)\Big)$ of π coincides with the commutant in $\mathcal{B}\big(\widehat{\mathcal{H}}\big)$ of the C^*-algebra $C^*\big(\hat{\mathcal{F}}\big)$ generated by $\hat{\mathcal{F}}$ in $\mathcal{B}\big(\widehat{\mathcal{H}}\big)$.

(3d) The mapping

$$\rho: \pi\left(C^*(\mathcal{T}(\mathcal{F}))\right) \to \mathcal{B}(\mathcal{H})$$

defined by $\rho(\pi(X)) = \mathcal{P}_{\mathcal{H}}\pi(X)|_{\mathcal{H}}\ for\ X \in C^*(\mathcal{T}(\mathcal{F}))$ is a complete isometry onto the space $\mathcal{T}(\mathcal{F})$ of all $\mathcal{F}$ -Toeplitz operators such that $\pi \circ \rho$ is the identity on $\pi\left(C^*(\mathcal{T}(\mathcal{F}))\right)$. Therefore the short exact sequence

$$0 \to \operatorname{Ker}\pi \hookrightarrow C^*(\mathcal{T}(\mathcal{F})) \xrightarrow{\pi} \pi\left(C^*(\mathcal{T}(\mathcal{F}))\right) \to 0$$

has a completely isometric cross. Moreover, $\operatorname{Ker}\pi$ coincides with the closed two-sided ideal of $C^*(\mathcal{T}(\mathcal{F}))$ generated by all operators of the form $XY - \Phi(XY)\ with\ X, Y \in \mathcal{T}(\mathcal{F})$.

(3e) If $C^*(\mathcal{F})$ denotes the unital C^*-algebra generated by $\mathcal{F}$ in $\mathcal{B}(\mathcal{H})$then $\Phi(C^*(\mathcal{F})) = C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. Moreover the kernel of the restriction of π to $C^*(\mathcal{F})$ coincides with the closed ideal of $C^*(\mathcal{F})$ generated by all the commutators $XY - YX$ with $X, Y \in \mathcal{T}(\mathcal{F}) \cap C^*(\mathcal{F})$ hence it coincides with the commutator ideal $\mathcal{C}$ of $C^*(\mathcal{F})$. Therefore we have a short exact sequence

$$0 \to \mathcal{C} \hookrightarrow C^*(\mathcal{F}) \xrightarrow{\pi} C^*(\hat{\mathcal{F}}) \to 0$$

for which the restriction of ρ to $C^*(\hat{\mathcal{F}})$ is a completely isometric cross.

(3f) An operator $X \in \mathcal{B}(\mathcal{H})$ belongs to the commutant of $\mathcal{F}$ if and only if both X and $X * X$ belong to the space $\mathcal{T}(\mathcal{F})$ of $\mathcal{F}$ -Toeplitz operators. In this case there exists a unique operator $\hat{X}$ in the commutant of $\hat{\mathcal{F}}$ which leaves $\mathcal{H}$ invariant and whose restriction to $\mathcal{H}$ coincides with X. Moreover the map $X \mapsto \hat{X}$ is norm preserving.

Proof. For each $\alpha \in \Gamma$ let $\phi_\alpha: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the completely positive σ-weakly continuous mapping associated to the spherical isometry S_α, so for all $X \in \mathcal{B}(\mathcal{H})$we have $\phi_\alpha(X) = \sum_{j \in J_\alpha} T_{j,\alpha}^* X T_{j,\alpha}$. It follows that $\mathcal{T}(\mathcal{F})$ is precisely the set of common fixed points of the commuting family of mappings $\{\phi_\alpha\}_{\alpha\in\Gamma}$. Therefore we can apply Lemma (4.2.8) to infer the existence of an idempotent completely positive mapping $\Phi: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ whose range is precisely $\mathcal{T}(\mathcal{F})$. This proves item (i).

Let Φ as in item (i) and let Φ_0 denote its restriction to $C^*(\mathcal{T}(\mathcal{F}))$. Denote by $\pi: C^*(\mathcal{T}(\mathcal{F})) \to \mathcal{B}(\hat{\mathcal{H}})$ the minimal Stinespring dilation of Φ_0. Therefore there exists an isometry $V: \mathcal{H} \to \hat{\mathcal{H}}$ such that

$$\Phi_0(X) = V^* \pi(X) V$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. We see that we are precisely in the situation of Theorem (4.2.2) above, and moreover the range of Φ is also σ-weakly closed because it is the set of all common fixed points of a family of σ-weakly continuous mappings. The conclusion that follows from Theorem (4.2.2) is that the mapping

$$\rho: \pi: \left(C^*(\mathcal{T}(\mathcal{F}))\right) \to \mathcal{B}(\mathcal{H})$$

defined by $\rho(\pi(X)) = V^*\pi(X)V\ for\ X \in C^*(\mathcal{T}(\mathcal{F}))$ is a complete isometry onto the space of all $\mathcal{F}$ -Toeplitz operators and that the image of π is a von Neumann subalgebra of $\mathcal{B}(\hat{\mathcal{H}})$. Let

$$\hat{T}_{j,\alpha} = \pi(T_{j,\alpha})$$

for all $\alpha \in \Gamma$ and $j \in J_\alpha$ and let also denote $\hat{S}_\alpha = \{\hat{T}_{j,\alpha}\}_{j\in J_\alpha}$ and let $\hat{\mathcal{F}} = \{\hat{S}_\alpha\}_{\alpha\in\Gamma}$. Our next aim is to show that each family $\hat{S}_\alpha$ is a spherical isometry and that

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\pi(X)\hat{T}_{j,\alpha} = \pi(X)$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. For this purpose, fix $\alpha \in \Gamma$ and observe that $\sum_{j\in F} \hat{T}^*_{j,\alpha}\hat{T}_{j,\alpha} \leq 1$ for each finite subset $F \subset J_\alpha$. Therefore

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\hat{T}_{j,\alpha} \leq 1.$$

Now, let $X \in C^*(\mathcal{T}(\mathcal{F}))$ and let $F \subset J_\alpha$ be a finite set. Then we see that

$$\rho\left(\sum_{j\in F} \hat{T}^*_{j,\alpha}\pi(X)\hat{T}_{j,\alpha}\right) = \sum_{j\in F} \hat{T}^*_{j,\alpha}\Phi(X)T_{j,\alpha}.$$

Taking weak$*$-limits in both sides, using the fact that ρ is a weak$*$-homeomorphism and using that ρ is isometric we infer that

$$\sum_{j\in J_\alpha} \hat{T}^*_{j,\alpha}\pi(X)\hat{T}_{j,\alpha} = \pi(X)$$

for all $X \in C^*(\mathcal{T}(\mathcal{F}))$. In particular it follows that $\hat{S}_\alpha$ is indeed a spherical isometry. Moreover, using Lemma (4.2.9) we infer that all $\hat{T}_{j,\alpha}$ belong to the center of $\pi\left(C^*(\mathcal{T}(\mathcal{F}))\right)$, in particular they are commuting normal operators. Since $T_{j,\alpha} = V^*\hat{T}_{j,\alpha}V$ and both S_α and $\hat{S}_\alpha$ are spherical isometries it is easy to see that $\hat{T}_{j,\alpha}V\,\mathcal{H} \subset V\mathcal{H}$ for all $\alpha \in \Gamma$ and $j \in J_\alpha$. This shows that the family $\mathcal{F}$ is subnormal, which proves item (ii).

We will show now that $\hat{\mathcal{F}}$ is the minimal normal extension of $\mathcal{F}$. For this purpose let $\mathcal{K}$ be the smallest reducing subspace for $\pi(C^*(\mathcal{F}))$ containing $V\mathcal{H}$. Let $\pi_\mathcal{K}: C^*(\mathcal{F}) \to \mathcal{B}(\mathcal{K})$ the $*$-representation defined by $\pi_\mathcal{K}(X) = \mathcal{P}_\mathcal{K}\pi(X)|_\mathcal{K}$ where $\mathcal{P}_\mathcal{K}$denotes the orthogonal projection of $\hat{\mathcal{H}}$ onto $\mathcal{K}$. We will show that the map defined by $\rho_\mathcal{K}(\pi(X)) = \mathcal{P}_\mathcal{K}\pi(X)|_\mathcal{K}$ is a $*$-isomorphism of $\pi\left(C^*(\mathcal{T}(\mathcal{F}))\right)$ onto the commutan $\pi_\mathcal{K}(C^*(\mathcal{F}))'$ of $\pi_\mathcal{K}(C^*(\mathcal{F}))$.

It is clear that $\rho_\mathcal{K}$is a completely positive and completely contractive mapping. It takes values in $\pi_\mathcal{K}(C^*(\mathcal{F}))'$ because each $\hat{T}_{j,\alpha}$ is in the center of $\pi\left(C^*(\mathcal{T}(\mathcal{F}))\right)$ and because the space $\mathcal{K}$ is reducing for all $\hat{T}_{j,\alpha}$. Let $\rho_\mathcal{H}: \pi_\mathcal{K}(C^*(\mathcal{F}))' \to \mathcal{B}(\mathcal{H})$ be defined by $\rho_\mathcal{H}(V) = V^*YV$ for $Y \in \pi_\mathcal{K}(C^*(\mathcal{F}))'$. Then it is obvious that its image is included in $\mathcal{T}(\mathcal{F})$ and moreover $\rho = \rho_\mathcal{H}\rho_\mathcal{K}$ where $\rho: \pi\left(C^*(\mathcal{T}(\mathcal{F}))\right) \to \mathcal{B}(\mathcal{H})$ was defined above as $\rho(Y) = V^*Y\,V$ for $Y \in \pi\left(C^*(\mathcal{T}(\mathcal{F}))\right)$. Recall now that we already proved that ρ is completely isometric which implies that the mapping $\rho_\mathcal{K}$ is completely isometric. Therefore in order to show that $\rho_\mathcal{K}$is onto, it suffices to show that the mapping $\rho_\mathcal{H}$ is one-to-one. Suppose therefore that $Y \in \pi_\mathcal{K}(C^*(\mathcal{F}))'$ is such that $\rho_\mathcal{H}(Y) = V^*YV = 0$. In order to show that $Y = 0$ it suffices, because $\pi_\mathcal{K}$is a minimal dilation of Φ restricted to $C^*(\mathcal{F})$ and $\pi_\mathcal{K}(C^*(\mathcal{F}))'$ is abelian and Y is in its commutant, to show that for any two finite families $\{T_{j_1,\alpha_1}, \ldots, T_{j_m,\alpha_m}\}$and $\{T_{i_1,\beta_1}, \ldots, T_{i_n,\beta_n}\}$ we have $V^*\hat{T}^*_{j_1,\alpha_1} \ldots \hat{T}^*_{j_m,\alpha_m} Y \hat{T}^*_{i_1,\beta_1} \ldots \hat{T}^*_{i_n,\beta_n} V = 0$ and the latter equality follows immediately from the fact that $V\mathcal{H}$ is invariant for all $\hat{\mathcal{F}}$. This shows that $\rho_{\mathcal{H}}$ is indeed one-to-one hence $\rho_{\mathcal{K}}$ is onto. Since, by a well known result of Kadison [122], any completely isometric surjective unital mapping between two C^*−algebras is multiplicative it follows that $\rho_{\mathcal{K}}$ is indeed a $*$-isomorphism of $\pi\left(C^*\left(\mathcal{T}(\mathcal{F})\right)\right)$ onto $\pi_{\mathcal{K}}\left(C^*(\mathcal{F})\right)'$ in particular the space $\mathcal{K}$ is invariant under $\pi\left(C^*\left(\mathcal{T}(\mathcal{F})\right)\right)$. Since π is minimal, this shows that in fact we have that $\mathcal{K} = \widehat{\mathcal{H}}$. In particular this shows that $\pi_{\mathcal{K}}(X) = \pi(X)$ for all $X \in C^*(\mathcal{F})$. Moreover, the fact that $\operatorname{Ker}\pi$ is the ideal generated by all operators of the form $XY - \Phi(XY)$ with $X, Y \in \mathcal{T}(\mathcal{F})$ follows by an easy induction argument on the length of an arbitrary product of elements from $\mathcal{T}(\mathcal{F})$ using the fact that $\operatorname{Ker}\pi = \operatorname{Ker}\Phi_0$ which equals $\left\{X - \Phi(X) : X \in C^*\left(\mathcal{T}(\mathcal{F})\right)\right\}$ together with Theorem (4.2.1). This completes the proof of (3a), (3b), (3c) and (3d).

In order to prove (3e) we show first that $\Phi\left(C^*(\mathcal{F})\right) = C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. Since $\Phi^2 = \Phi$ it is enough to show that $\Phi\left(C^*(\mathcal{F})\right) \subset C^*(\mathcal{F})$. This inclusion follows easily from the fact that since $\pi_{\mathcal{K}}\left(C^*(\mathcal{F})\right)$ is abelian, Φ takes any finite product of $T_{j,\alpha}$'s and $T_{j,\alpha}^*$'s into a permutation of the same product having all the $T_{j,\alpha}^*$'s at the left and all the $T_{j,\alpha}$'s at the right.

Now we can easily prove that the kernel of $\pi_{\mathcal{K}}$ coincides with the ideal of $C^*(\mathcal{F})$generated by all commutators $XY - YX$ with $X, Y \in C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. First, since $\pi_{\mathcal{K}}\left(C^*(\mathcal{F})\right)$ is commutative, we have that any such commutator is in $\operatorname{Ker}\pi_{\mathcal{K}}$. Let us denote by Φ_{00} the restriction of Φ to $C^*(\mathcal{F})$.We see from the proof of Theorem (4.2.2) that $\operatorname{Ker}\Phi_0 = \operatorname{Ker}\pi$ therefore $\operatorname{Ker}\pi_{\mathcal{K}} = \operatorname{Ker}\Phi_{00}$ as well. On the other hand, since $\Phi_{00}^2 = \Phi_{00}$ we see that $\operatorname{Ker}\Phi_{00} = \operatorname{Ran}(I - \Phi_{00})$. Now, if $X \in C^*(\mathcal{F})$ is a finite product of $T_{j,\alpha}$'s and $T_{j,\alpha}^*$'s it becomes obvious from the above description of $\Phi_{00}(X)$ that $X - \Phi_{00}(X)$belongs to the ideal generated by all commutators $XY - YX$ with $X, Y \in C^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F})$. This completes the proof of (3e).

An alternate proof of the fact that $\operatorname{Ker}\pi_{\mathcal{K}}$ coincides with the commutator ideal in $C^*(\mathcal{F})$ can be based on Bunce' characterization of multiplicative functional on C^*-algebras generated by commuting hyponormal operators in terms of their joint approximate point spectrum (see [113]). Indeed, in our case, it can be easily shown that for any $\{T_1, \dots, T_m\} \subset \mathcal{F}$, we have that $\sigma_{ap}(T_1, \dots, T_m)$equals $\sigma_{ap}\left(\pi\left(T_1, \ \dots, \pi(T_m)\right)\right)$ where σ_{ap} stands for the joint approximate spectrum.

We now prove (3f). If $X \in \mathcal{B}(\mathcal{H})$ is such that X commutes with all operators from $\mathcal{F}$ then obviously X and $X * X$ belong to $\mathcal{T}(\mathcal{F})$. Suppose now that $X \in \mathcal{B}(\mathcal{H})$is such that both X and $X * X$ belong to $\mathcal{T}(\mathcal{F})$. If $\hat{X} = \pi(X)$ then $\hat{X}$ commutes with all the normal extensions from $\hat{\mathcal{F}}$ and $V^*\hat{X}V = X$. Moreover $\|\hat{X}\| = \|X\|$ so all we need to show is that $\hat{X}V\mathcal{H} \subset V\mathcal{H}$. For this purpose, we observe that since $X^*X \in \mathcal{T}(\mathcal{F})$ then

$$X^*X = V^*\pi(X^*X)V = V^*\hat{X}^*\hat{X}V.$$

Therefore if $\xi \in \mathcal{H}$ then

$$\|V^*\hat{X}V\xi\| = \|X\xi\| = \|\hat{X}V\xi\|$$

which implies that indeed $\hat{X}V\mathcal{H} \subset V\mathcal{H}$. This finishes the proof of (3f) and the proof of the theorem as well.

Let us note that both in the case of the unilateral shift S on $H^2(\mathbb{T})$, and in the more general case of the Szego n-tuple all the assertions from (ii) and (iii) are well known. We should refer here to the pioneering work of L.A. Coburn on the C^*-algebra generated by a single isometry; see [26] and [26]. In particular, in the first case (3d) and (3e) are classical results due to L. Coburn and R.G. Douglas, and moreover, the kernel of π at (3d) is the corresponding commutator ideal; see Chapter VII in [5]. In the case of the Szegó n-tuple on the unit sphere in $\mathbb{C}^n$ (3e) is proved in [13] , and (3d) appears in [12] . We mention also that in both these cases the commutator ideal appearing in (3e) was shown to coincide with the ideal of all compact operators on the corresponding Hardy space.

For the case of a commuting family of isometries, the existence of a commuting unitary extension was proved in [121] . In this case, the commutant lifting at (3f) is quite straightforward once we assume Ito's result. These results hold true even on Banach spaces, see [117] . Exact sequences similar to that in (3e) for finite families of commuting isometries have been studied in [112] .

C^*-algebras generated by isometric representations of commuting semigroups have been studied mainly for semigroups of positive elements in ordered abelian groups, see for example [111] , [27] ,[23] and [107] for a study based on crossed products by endomorphism's.

For the case of a single finite spherical isometry the existence of a normal extension along with a commutant lifting theorem were proved in [108] ; see also [109] for alternate proofs.

The following general framework is frequently used when dealing with Toeplitz operators on Hardy spaces. Let K be a compact Hausdorff space and let $C(K)$denote the Banach algebra of all complex-valued continuous functions on K. Let $A \subset C(K)$ be a norm-closed subalgebra containing the constants and separating the points of K. Such algebras are called function algebras or uniform algebras (see [119]). Let us consider a Borel probability measure m on K and let $supp(m)$ be its closed support. The generalized Hardy space $H^2(m)$associated to A is the $L^2(m)$ closure of A. For any function $\varphi \in L^\infty(m)$ the Toeplitz operator $T_\phi: B(H^2(m)) \to B(H^2(m))$ is defined by $T_\phi h = P_{H^2(m)}(\varphi h)$ for $h \in H^2(m)$ where $P_{H^2(m)}$ is the orthogonal projection of $L^2(m)$onto $H^2(m)$. We shall also consider the usual multiplication operators M_ϕ defined on $L^2(m)$ by $M_\phi f = \phi f$ for all $f \in L^2(m)$. Let $H^\infty(m)$ denote the intersection $H^2(m) \cap L^\infty(m)$ which is
a weak$*$-closed subalgebra of $L^\infty(m)$. If $B \subset L^\infty(m)$ is any unital subalgebra, we shall denote by $\mathcal{T}(\mathcal{B})$ the C^*-subalgebra of $B(H^2(m))$ generated by all Toeplitz operators T_ϕ with $\phi \in \mathcal{B}$ and by $\mathcal{C}(\mathcal{B})$ the closed ideal in $\mathcal{T}(\mathcal{B})$ generated by all operators of the form $T_\phi T_\psi = T_{\phi\psi}$ for arbitrary $\phi, \psi \in \mathcal{B}$.

For our purposes we need to introduce the following definition. We shall say that a finite family of functions $F = \{\phi_1, \ldots, \phi_n\} \subset C(K)$ is a spherical multifunction if

$$\sum_{j=1}^{n} \left|\phi_j(x)\right|^2 = 1$$

for every $x \in K$.

We are now able to state our main result.

Theorem (4.2.11)[105]: Let K be a compact Hausdorff space and let $A \subset C(K)$ be a unital norm-closed subalgebra. Suppose there exists a family $\{F_\alpha\}_{\alpha\in\Gamma}$ of spherical multifunctions in $C(K)$ where each F_α is of the form $F_\alpha = \{\phi_{j,\alpha}\}_{j\in J_\alpha}$ with each $\phi_{j,\alpha} \in A$ and such that for each pair of distinct points $x, y \in K$ there exist an index $\alpha \in \Gamma$ and an index $j \in J_\alpha$ such that $\phi_{j,\alpha}(x) \neq \phi_{j,\alpha}(y)$. Then for any Borel probability measure m on K the following assertions hold true.

(i) A bounded operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if it satisfies the following equations:

$$\sum_{j\in J_\alpha} T^*_{\phi_{j,\alpha}} X T_{\phi_{j,\alpha}} = X$$

for all $\alpha \in \Gamma$ (in the case when A is the disc algebra and m is the Lebesgue measure on the unit circle we then retrieve the classical result of Brown-Halmos by taking as spherical multifunction $\phi(z) = z$).

(ii) A bounded operator $X \in B(H^2(m))$ is of the form $X = T_\psi$ for some $\psi \in H^\infty(m)$if and only if it commutes with $T_{\phi_{j,\alpha}}$ for all $\alpha \in \Gamma$ and all $j \in J_\alpha$, if and only if it commutes with all T_ϕ with $\phi \in A$. The map $\psi \longmapsto T_\psi$ is a Banach algebra isometric isomorphism between $H^\infty(m)$ and the commutant of the family $\{T_\phi : \phi \in A\}$. Moreover this commutant is a maximal abelian subalgebra of $B(H^2(m))$ and hence, for every $\phi \in B(H^2(m))$ we have that $\sigma(T_\phi) = \sigma_{H^2(m)}(\phi)$.

(iii) There exists a short exact sequence of C^*-algebras

$$0 \to \mathcal{C}(L^\infty(m)) \hookrightarrow \mathcal{T}(L^\infty(m)) \xrightarrow{\pi} L^\infty(m) \to 0$$

such that $\pi(T_\phi) = \phi$ for all $\phi \in L^\infty(m)$. In particular the spectral inclusion essran$(\phi) \subset \sigma(T_\phi)$ holds true (in the case of the unit circle this is the classical theorem of Hartman and Wintner [120]) and we also have that

$$\sigma(T_\phi) \subset \text{conv}(\text{essran}(\phi))$$

where conv denotes the convex hull.

(iv) There exists a short exact sequence

$$0 \to \mathcal{C}(C(K)) \hookrightarrow \mathcal{T}(C(K)) \xrightarrow{\pi} C(supp\ (m)) \to 0$$

such that $\pi(T_\phi) = \phi$ on $supp(m)$. Moreover, in this case $\mathcal{C}(C(K))$ coincides always with the closed ideal in $\mathcal{T}(C(K))$ generated by all commutators $T_\phi T_\psi - T_\psi T_\phi$ with $\phi, \psi \in C(K)$.

Proof. Let us denote, for each $\alpha \in \Gamma$ and each $j \in J_\alpha$ by $T_{j,\alpha}$ the Toeplitz operator with symbol $\phi_{j,\alpha}$. Since each tuple $\{\phi_{j,\alpha}\}_{j\in J_\alpha}$ is a spherical multifunction it follows easily that in this case $S_\alpha = \{T_{j,\alpha}\}_{\alpha\in\Gamma}$ is a spherical isometry and that $\mathcal{F} = \{S_\alpha\}_{\alpha\in\Gamma}$ is a commuting family of spherical isometries in $B(H^2(m))$. The separation property imposed on these spherical multifunctions implies via the Stone- Weierstrass theorem that the C^*-algebra generated in $C(K)$ by the union of all families F_α with $\alpha \in \Gamma$ equals $C(K)$ itself. In turn this implies that the set $\hat{\mathcal{F}}$of all the corresponding multiplication operators $M_{\phi_{j,\alpha}}$ on $L^2(m)$ is the minimal normal extension of $\mathcal{F}$. Therefore, using Theorem (4.2.10) we infer that every operator $X \in B(H^2(m))$ satisfying equations (i) is the compression of a bounded operator Y in the commutant of all operators $M_{\phi_{j,\alpha}}$ therefore Y commutes with all multiplication operators M_ϕ with $\phi \in C(K)$ which implies that Y itself is a multiplication operator with some function $\psi \in L^\infty(m)$ which shows that X is a Toeplitz operator i.e. $X = T_\psi$. Conversely, any Toeplitz operator obviously satisfies these equations because $H^2(m)$ is invariant for all operators $M_{\phi_{j,\alpha}}$. This completes the proof of (i). Now, the proofs of (ii), (iii) and (iv) follow easily from the previous remarks combined with Theorem (4.2.10) (the last assertion at (iii) follows from the well-known fact that for any bounded operator T we have that $conv(\sigma(T)) \subset W(T)$ with equality for normal operators, where $W(T)$ stands for the numerical range of T; the case of the unit circle is due to A. Brown and P.R. Halmos [2]). As a remark, it can be shown, on the same lines of the proof of Theorem (4.2.10), or using the results from [113] , that if the C^*-algebra generated by $H^\infty(m)$ in $L^\infty(m)$ coincides with $L^\infty(m)$ (equivalently if $H^\infty(m)$ separates the points in the maximal ideal space of $L^\infty(m)$) then $\mathcal{C}(L^\infty(m))$ coincides with the closed ideal of $\mathcal{T}(L^\infty(m))$ generated by all commutators $T_\phi T_\psi - T_\psi T_\phi$ with $\phi, \psi \in L^\infty(m)$ (for instance this is the case when K is the unit circle, A is the disc algebra and m is the Lebesgue measure on K; see [5]).

We also remark that a description of the character space of the quotient

$$\mathcal{T}(L^\infty(m))/\mathcal{C}(L^\infty(m))$$

valid for Hardy spaces over any function algebra was given in [125] .

Here follow two general examples of function algebras satisfying the hypotheses of Theorem (4.2.11). We emphasize that this result holds for every Borel probability measure on the corresponding Shilov boundaries. Toeplitz operators on such domains have been studied see [126] and [127] .

Example (4.2.12)[105]: Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain and let $A(\Omega)$ be the algebra of all continuous functions on its closure and holomorphic on Ω. Let $K = \partial\Omega$ be the topological boundary of Ω which coincides in this case with the Shilov boundary of $A(\Omega)$ and let A be the set of all restrictions to $\partial\Omega$ of all functions from $A(\Omega)$. It follows from an embedding theorem for such domains (see Theorem 3 in [124]) that there exist a natural number $N > 1$ and functions $f_1, \dots, f_N$ in $A(\Omega)$ such that the function $F: \partial\Omega \to \mathbb{C}^N$ defined by $F(x) = (f_1(x), \dots, f_N(x))$is one-to-one and takes $\partial\Omega$ into the unit sphere in $\mathbb{C}^N$. This shows that it is a separating spherical multifunction for A and hence Theorem (4.2.11) applies in this case. In particular this applies to any bounded domain with C^2 boundary in the complex plane. For the case of finitely connected domains in $\mathbb{C}$ with analytic boundary, exact sequences of the form (iii) and (iv) were constructed in [106] .

Example (4.2.13)[105]: Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain containing the origin and such that $e^{i\theta}\zeta \in \Omega$ whenever $\zeta \in \Omega$ and $\theta \in \mathbb{R}$. Let $A(\Omega)$ be as in the previous example. Let $K \subset \partial\Omega$ be the Shilov boundary of $A(\Omega)$ and let $\gamma = \max\{|\zeta| : \zeta \in \overline{\Omega}\}$. It is then known that $K = \{z \in \partial\Omega : |z| = \gamma\}$ (see Theorem 6.5 in [123]). Therefore the function $F(z) = z/\gamma$ is an imbedding of K into the unit sphere in $\mathbb{C}^n$ hence Theorem (4.2.11) applies in this case as well, taking A to be the algebra of all restrictions to K of functions from $A(\Omega)$. In particular we obtain that, given any Borel probability measure m on K, an operator $X \in B(H^2(m))$ is a Toeplitz operator if and only if

$$\sum_{j=1}^{n} T_{z_j}^* X T_{z_j} = \gamma^2 X.$$

Section (4.3): Toeplitz Projections

A result of K. Davidson [1] from 1977, answering a question of R. Douglas, shows that the essential commutant $\mathcal{T}_a^{ec}$of the set $\mathcal{T}_a = \{T_f ; f \in H^\infty(\mathbb{T})\} \subset B(H^2(\mathbb{T}))$ of all analytic Toeplitz operators on the Hardy space $H^2(\mathbb{T})$ of the unit circle is given by

$$\mathcal{T}_a^{ec} = \{T_f + K ; f \in H^\infty(\mathbb{T}) + C(\mathbb{T}) \; and \; K \in \mathcal{K}(H^\infty(\mathbb{T}))\},$$

where $\mathcal{K}(\mathcal{H})$ denotes the set of all compact operators on a given Hilbert space $\mathcal{H}$. Itwas observed by X. Ding and S. Sun [135] that the result of Davidson remains true on the Hardy space $H^2(\mathbb{S})$ of the unit sphere $\mathbb{S} = \partial\mathbb{B}_n$ in dimension $n > 1$ when the symbol algebra

$H^\infty(\mathbb{T}) + C(\mathbb{T})$ is replaced by the closed subalgebra $\mathcal{S} = \{f \in L^\infty(\mathbb{S}); H_f \textit{ is compact}\} \subset L^\infty(\mathbb{S})$, that is,

$$\mathcal{T}_a^{ec} = \{T_f + K; f \in \mathcal{S} \text{ and } K \in \mathcal{K}(H^2(\mathbb{S}))\}.$$

It is well known that $H^2(\mathbb{S}) + C(\mathbb{S}) \subsetneq \mathcal{S}$ is a proper subalgebra in every dimension $n > 1$(see [12]) and that therefore the higher dimensional version of Davidson's result fails if the algebra S is replaced by the smaller algebra $H^2(\mathbb{S}) + C(\mathbb{S})$.

In [134] the above results were extended to Toeplitz operators formed with respect to a quite general class of subnormal tuples on arbitrary Hilbert spaces containing, as a very particular case, Toeplitz operators on strictly pseudoconvex domains in $\mathbb{C}^n$.

Let $A \subset C(K)$ be a closed subalgebra of the Banach algebra of all $\mathbb{C}$-valued continuous functions on a compact subset $K \subset \mathbb{C}^n$such that Acontains at least the polynomials. A subnormal tuple $T \in \mathcal{B}(\mathcal{H})^n$ is called an A-isometry [136] if the spectrum of the minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$of T is contained in the Shilov boundary $\partial_A$of A and if A is contained in the restriction algebra $\mathcal{R}_T$of T. In this setting concrete T-Toeplitz operators are defined as compressions $T_f = \mathcal{P}_{\mathcal{H}}\Psi_U(f)|_{\mathcal{H}}$, where $\Psi_U: L^\infty(\mu) \to \mathcal{B}(\widehat{\mathcal{H}})$ is the L^∞-functional calculus of $U$and $f \in L^\infty(\mu)$, while abstract T-Toeplitz operators are defined as those operators $X \in \mathcal{B}(\mathcal{H})$ which satisfy the Brown–Halmos condition

$$T_\theta^* X T_\theta = X$$

for all μ-inner functions θ.

By results of A .Athavale [108] and T. Ito [121] the $A(\mathbb{B}_n)$-isometries on a given Hilbert space hare precisely the spherical isometries on $\mathcal{H}$, that is, the commuting tuples $T \in \mathcal{B}(\mathcal{H})^n$satisfying the identity $\sum_{1\le i\le n} T_i^* T_i = 1_{\mathcal{H}}$ and the class of $A(\mathbb{D}^n)$-isometries on $\mathcal{H}$is given by the commuting tuples of isometries on $\mathcal{H}$. For any strictly pseudoconvex or symmetric domain $D \subset \mathbb{C}^n$, the tuple $T_z = (T_{z_1}, \dots, T_{z_n}) \in \mathcal{B}(H^2(\sigma))^n$on the Hardy space $H^2(\sigma)$formed with respect to the canonical probability measure σon the Shilov boundary of the domain algebra $A(D) = \{f \in C(\overline{D});\ f|_D \in \mathcal{O}(D)\}$ is an example of an $A(D)$-isometry. Finally, every commuting tuple $N \in \mathcal{B}(\mathcal{H})^n$of normal operators on a Hilbert space $\mathcal{H}$is a $C(\sigma(N))$-isometry.

Under a suitable regularity condition on T, which is satisfied in all the above examples and which is needed to apply results of Aleksandrov [129] on the existence of sufficiently many μ-inner functions, it follows that the set $\mathcal{T}(T)$ of abstract T-Toeplitz operators is given by the compressions

$$\mathcal{T}(T) = \mathcal{P}_{\mathcal{H}}(U)'|_{\mathcal{H}}$$

of the operators in the commutant $(U)' = W^*(U)'$of the von Neumann algebra generated by U, while by the very definition, the concrete T-Toeplitz operators are given by the compressions of all operators in $W^*(U)$.

It follows from results of B.Prunaru [105] on families of spherical isometries that there is a completely positive unital projection $\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ onto the set $\mathcal{T}(T)$ of all abstract T-Toeplitz operators [134] .We give a much more direct and straightforward construction of Toeplitz projections Φ_T. We use the properties of these projections to improve the main result of [134] on the essential commutant of analytic Toeplitz operators and to extend a number of classical results on Toeplitz operators to our general setting.

After constructing Toeplitz projections, we show that every operator Sin the essential commutant of the analytic Toeplitz operators associated with an essentially normal regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$ is a compact perturbation of the Toeplitz operator $\Phi_T(S)$. Thus we improve a corresponding result obtained in [134] under the additional condition that Tpossesses no joint eigenvalues. We obtain complete characterizations of the essential commutant of essentially normal regular A-isometries and give, as a direct application, a new proof of a classical theorem of Johnson and Parrot [8] on the essential commutant of abelian von Neumann algebras in the case of separable Hilbert spaces. We show that the Toeplitz projection associated with an arbitrary regular A-isometry annihilates the compact operators if and only if Tpossesses no joint eigenvalues. We conclude that the Toeplitz calculus associated with a regular A-isometry T with empty point spectrum satisfies the essential version of the Hartman–Wintner spectral inclusion theorem and that the semi-commutator ideal of Toeplitz algebras $\mathcal{T}_{\mathcal{B}}$generated by arbitrary symbol algebras $\mathcal{B}$necessarily contains every compact operator in $\mathcal{T}_{\mathcal{B}}$.

Let $T \in \mathcal{B}(\mathcal{H})^n$be a subnormal tuple on a complex Hilbert space $\mathcal{H}$, that is, acommuting tuple that can be extended to a commuting tuple of normal operators on a larger Hilbert space. We denote by $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$ the minimal normal extension of Twhich is unique up to unitary equivalence [131] , and fix a scalar spectral measure $\mu$for U. The measure $\mu$is a positive regular Borel measure on the normal spectrum $\sigma_n(T) = \sigma(U)$of T. By the spectral theorem for normal tuples there is an isomorphism of von Neumann algebras $\Psi_{\mathrm{U}}: L^\infty(\mu) \to W^*(U) \subset \mathcal{B}(\widehat{\mathcal{H}})$ extending the polynomial calculus of U. The restriction algebra

$$\mathcal{R}_T = \{f \in L^\infty(\mu); \Psi_{\mathrm{U}}(f)\mathcal{H} \subset \mathcal{H}\} \subset L^\infty(\mu)$$

is a weak* closed subalgebra. For $f \in L^\infty(\mu)$, we define the T-Toeplitz operator with symbol $f$as the compression

$$T_f = \mathcal{P}_{\mathcal{H}} \Psi_{\mathrm{U}}(f)|_{\mathcal{H}}.$$

Toeplitz operators of this form will be called concrete T-Toeplitz operators in the sequel.

Let $A \subset C(K)$ be a unital closed subalgebra of the Banach algebra of all $\mathbb{C}$-valued continuous functions on a compact subset $K \subset \mathbb{C}^n$ such that A contains at least the co-ordinate functions. Then a subnormal tuple $T \in \mathcal{B}(\mathcal{H})^n$ as above is called an A-isometry if $\sigma_n(T)$ is contained in the Shilov boundary $\partial_A$of $A$and $A|_{\partial_A} \subset \mathcal{R}_T$. Here the Shilov boundary $\partial_A \subset K$is the smallest closed set such that $\|f\|_{\infty,K} = \|f\|_{\infty,\partial_A}$for every $f \in A$and we regard the scalar spectral measure $\mu$of Uas a positive measure on ∂_Avia trivial extension. Since $\mathcal{R}_T \subset L^\infty(\mu)$ is weak*closed and contains A, it also contains the dual algebra

$$H_A^\infty(\mu) = \bar{A}^{w^*} \subset L^\infty(\mu).$$

The unimodular elements in $H_A^\infty(\mu)$, that is, the elements of the set

$$I_\mu = \{\theta \in H_A^\infty(\mu); |\theta| = 1\ \mu - \text{almost everywhere on } \partial_A\}$$

will be called μ-inner functions. In [129] Aleksandrov gives a sufficient condition for $H_A^\infty(\mu)$ to contain a rich supply of μ-inner functions. The triple (A, K, μ) is called regular in the sense of Aleksandrov if, for every function $\phi \in C(K)$ with $\phi > 0$ on K, there is a sequence (ϕ_k) of functions in $A$with $|\phi_k| < \phi$on $K$and $\lim_{k\to\infty} |\phi_k| = \phi\mu$-almost everywhere on K. It follows from the results of Aleksandrov that the regularity of the triple (A, K, μ) implies that the set $I_\mu \subset H_A^\infty(\mu)$ of μ-inner functions generates $L^\infty(\mu)$ as a von Neumann algebra, that is, $L^\infty(\mu) = W^*(I_\mu)$(Corollary 2.5 in [133]). We call $T \in \mathcal{B}(\mathcal{H})^n$ a regular A-isometry if $T$is an A-isometry and the triple (A, K, μ) is regular in the sense of Aleksandrov. It was observed by Aleksandrov [129] that, for every regular positive measure μ on the Shilov boundary of the domain algebra $A(D)$ of a strictly pseudoconvex or symmetric domain $D \subset \mathbb{C}^n$, the triple $(A(D), \overline{D}, \mu)$ is regular.

Let $T \in \mathcal{B}(\mathcal{H})^n$ be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Since $L^1(\mu)$ is separable, its dual unit ball $B_{L^\infty(\mu)} = \{f \in L^\infty(\mu); \|f\|_{L^\infty(\mu)} \le 1\}$ equipped with the relative weak*topology of $L^\infty(\mu) = L^1(\mu)'$ is a compact metrizable space. Hence $B_{L^\infty(\mu)}$ and its subset I_μ consisting of all μ-inner functions are separable metrizable spaces in the relative weak*topology. For any countable weak∗dense subset $I \subset I_\mu$, the von Neumann algebra generated by I in $L^\infty(\mu)$ satisfies

$$W^*(I) = W^*(I_\mu) = L^\infty(\mu).$$

Let us fix any sequence $(\theta_k)_{k\ge1}$ in I_μ with the property that

$$W^*(\{\theta_k : k \ge 1\}) = L^\infty(\mu).$$

For $r \ge 0$, the norm-closed ball $\mathcal{B}_r = \{X \in \mathcal{B}(\widehat{\mathcal{H}}); \|X\| \le r\}$ equipped with the relative topology of the weak∗topology of $\mathcal{B}(\widehat{\mathcal{H}})$ is a compact Hausdorff space. For $X \in \mathcal{B}(\widehat{\mathcal{H}})$, the averages

$$\Phi_{U,k}(X) = \frac{1}{k^k} \sum_{1\le i_1,\dots,i_k\le k} \Psi_U\left(\theta_k^{i_k} \cdot \dots \cdot \theta_k^{i_1}\right)^* X \Psi_U\left(\theta_k^{i_1} \cdot \dots \cdot \theta_k^{i_k}\right) \in \mathcal{B}(\widehat{\mathcal{H}})$$

form a sequence $\left(\Phi_{U,k}(X)\right)_k$ in $\mathcal{B}_{\|X\|}$. Since by Tychonoff's theorem the topological product $\prod_{X\in\mathcal{B}(\widehat{\mathcal{H}})} \mathcal{B}_{\|X\|}$ is compact and since convergence in the product topology is equivalent to componentwise convergence, there is a subnet $\left(\Phi_{U,k_\alpha}\right)_\alpha$ of the sequence $\left(\Phi_{U,k}\right)_k$ such that the weak*limits

$$\Phi_U(X) = w^* - \lim_\alpha \Phi_{U,k_\alpha}(X) \in \mathcal{B}(\widehat{\mathcal{H}})$$

exist simultaneously for every $X \in \mathcal{B}(\widehat{\mathcal{H}})$. Each choice of such a subnet yields a well-defined map $\Phi_U : \mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\widehat{\mathcal{H}})$ with the properties that will be deduced.

Theorem (4.3.1)[128]: The mapping

$$\Phi_U : \mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\widehat{\mathcal{H}}), \qquad X \mapsto \Phi_U$$

constructed above is a completely positive unital projection with

$$\operatorname{ran}(\Phi_U) = (U)'.$$

Proof. Obviously, the mappings

$$\Phi_{U,k} : \mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\widehat{\mathcal{H}}), \qquad X \mapsto \Phi_{U,k}(X)$$

are completely positive and unital. Since, for each $N \in \mathbb{N}$, weak*convergence for a net in $\mathcal{B}(\widehat{\mathcal{H}}^N)$ identified with the space $M\left(N, \mathcal{B}(\widehat{\mathcal{H}})\right)$ of all $N \times N$ matrices over $\mathcal{B}(\widehat{\mathcal{H}})$ is equivalent to coefficient wise weak*convergence in $\mathcal{B}(\widehat{\mathcal{H}})$ and since the set of all positive operators on a Hilbert space is weak∗closed, it follows that

$$\Phi_U : \mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\widehat{\mathcal{H}}), \qquad X \mapsto w^* - \lim_\alpha \Phi_{U,k_\alpha}(X)$$

is completely positive and unital. By construction the mappings $\Phi_{U,k}$, and hence also Φ_U, act as the identity operator on the commutant $(U)' = W^*(U)'$. To complete the proof, it suffices to show that $\operatorname{ran}(\Phi_U) \subset (U)'$.

For $1 \le j \le k$ and $i = (i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k) \in \{1, \dots, k\}^{k-1}$, we use the abbreviation

$$\mathcal{R}_{ij} = \Psi_U\left(\prod_{\substack{v=1\\ v\neq j}}^{k} \theta_v^{i_v}\right).$$

Note that, for $X \in \mathcal{B}(\widehat{\mathcal{H}}), k \geq 1$and $1 \leq j \leq k$, the estimates

$$\begin{aligned}&\left\|\Psi_U(\overline{\theta_j})\Phi_{U,k}(X)\Psi_U(\theta_j) - \Phi_{U,k}(X)\right\| \\ &\quad \leq \frac{1}{k^k}\left\|\sum_i \mathcal{R}_{i_j}^*\left(\sum_{\mu=1}^{k} \Psi_U(\bar{\theta}_j^{\mu+1})X\Psi_U(\theta_j^{\mu+1}) - \Psi_U(\bar{\theta}_j^{\mu})X\Psi_U(\theta_j^{\mu})\right)\mathcal{R}_{i_j}\right\| \\ &\quad \leq \frac{k^{k-1}}{k^k}\, 2\|X\| = \frac{2\|X\|}{k}\end{aligned}$$

hold. Hence for $j \geq 1$ and $X \in \mathcal{B}(\widehat{\mathcal{H}})$, we obtain

$$\Psi_U(\overline{\theta_j})\Phi_U(X)\Psi_U(\theta_j) = w^* - \lim_{\alpha} \Psi_U(\overline{\theta_j})\Phi_{U,k_\alpha}(X)\Psi_U(\theta_j) = \Phi_U(X),$$

or equivalently, $\Phi_U(X)\Psi_U(\theta_j) = \Psi_U(\theta_j)\Phi_U(X)$. It follows that

$$\Phi_U(X) \in W^*(\{\Psi_U(\theta_j); j \geq 1\})' = W^*(U)' = (U)'$$

for all $X \in \mathcal{B}(\widehat{\mathcal{H}})$. This observation completes the proof.

A projection onto the space of all Toeplitz operators on the Hardy space of the unit circle was constructed by Arveson in [39] using a generalized limit argument. In [105] Prunaru used invariant means to construct a completely positive unital projection onto the set of Toeplitz operators associated with a commuting family of spherical isometries. In our setting, a projection onto the set of all abstract T-Toeplitz operators is obtained by compressing $\Phi_U$to $\mathcal{H}$.

For $X \in \mathcal{B}(\mathcal{H})$, we denote by $\tilde{X} = X \oplus 0 \in \mathcal{B}(\widehat{\mathcal{H}})$its trivial extension to $\widehat{\mathcal{H}}$. Then for $k \geq 1$ and $X \in \mathcal{B}(\mathcal{H})$, the operators

$$\Phi_{T,k}(X) = \frac{1}{k^k}\sum_{1\leq i_1,\ldots,i_k\leq k} T^*_{\theta_k^{i_k}\cdot\ldots\cdot\theta_1^{i_1}} X T_{\theta_1^{i_1}\cdot\ldots\cdot\theta_k^{i_k}} \in \mathcal{B}(\mathcal{H})$$

are the compressions of the corresponding operators $\Phi_{U,k}(\tilde{X})$, that is,

$$\Phi_{T,k}(X) = \mathcal{P}_{\mathcal{H}}\Phi_{U,k}(\tilde{X})\big|_{\mathcal{H}}((k \geq 1, X \in \mathcal{B}(\mathcal{H})).$$

As before we denote by I_μ the set of all μ-inner functions θin $H^\infty A(\mu)$ and write

$$\mathcal{T}(T) = \{X \in \mathcal{B}(\mathcal{H}); T_\theta^* X T_\theta = X \; for \; all \; \theta \in I_\mu\}$$

for the set of all abstract T stract T-Toeplitz operators on $\mathcal{H}$.

Corollary (4.3.2)[128]: The mapping

$$\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad X \mapsto \Phi_T(X) = w^* - \lim_{\alpha} \Phi_{T,k_\alpha}(X) = \mathcal{P}_{\mathcal{H}}\Phi_U(\tilde{X})\big|_{\mathcal{H}}$$

is a well-defined completely positive unital projection with

$$\operatorname{ran}(\Phi_T) = \mathcal{T}(T).$$

Proof. Since the compression mapping $\mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\mathcal{H}), X \mapsto \mathcal{P}_{\mathcal{H}}X|_{\mathcal{H}}$, is weak∗continuous, completely positive and unital, it follows that

$$w^* - \lim_{\alpha} \Phi_{T,k_\alpha}(X) = \mathcal{P}_{\mathcal{H}}\Phi_U(\tilde{X})\big|_{\mathcal{H}}$$

for $X \in \mathcal{B}(\mathcal{H})$ and that the map $\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $X \mapsto \mathcal{P}_{\mathcal{H}}\Phi_U(\tilde{X})\big|_{\mathcal{H}}$, is completely positive and unital. Since $\Phi_{T,k}(X) = X$for each abstract T-Toeplitz operator $X \in \mathcal{T}(T)$ and every $k \geq 1$, it follows that $\Phi_T(X) = X$ for $X \in \mathcal{T}(T)$. Using Theorem (4.3.1), we obtain that

$$T_\theta^* \Phi_T(X) T_\theta = \mathcal{P}_\mathcal{H} \Psi_U(\theta)^* \mathcal{P}_\mathcal{H} \Phi_U(\bar{X}) \Psi_U(\theta)|_\mathcal{H} = \mathcal{P}_\mathcal{H} \Psi_U(\theta)^* \Phi_U(\bar{X}) \Psi_U(\theta)|_\mathcal{H} = \mathcal{P}_\mathcal{H} \Phi_U(\bar{X})|_\mathcal{H} = \Phi_T(X)$$

for every operator $X \in \mathcal{B}(\mathcal{H})$ and each μ-inner function $\theta \in I_\mu$. Hence $\operatorname{ran}(\Phi_T) = \mathcal{T}(T)$, and the proof is complete.

As a direct application of Theorem (4.3.1) and Corollary (4.3.2) we obtain a natural description of the abstract T-Toeplitz operators.

Corollary (4.3.3)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$. Then we have

$$\mathcal{T}(T) = \mathcal{P}_\mathcal{H} (U)'|_\mathcal{H}.$$

Proof. By Corollary (4.3.2) and Theorem (4.3.1)we have $\mathcal{T}(T) \subset \mathcal{P}_\mathcal{H}(U)'|_\mathcal{H}$. Conversely, if $X \in (U)'$and $\theta \in I_\mu$is a μ-inner function, then

$$T_\theta^* (\mathcal{P}_\mathcal{H} X|_\mathcal{H}) T_\theta = \mathcal{P}_\mathcal{H} \Psi_U(\theta)^* \mathcal{P}_\mathcal{H} X \Psi_U(\theta)|_\mathcal{H} = \mathcal{P}_\mathcal{H} \Psi_U(\theta)^* X \Psi_U(\theta)|_\mathcal{H} = \mathcal{P}_\mathcal{H} X|_\mathcal{H}.$$

Hence also the reverse inclusion $\mathcal{P}_\mathcal{H}(U)'|_\mathcal{H} \subset \mathcal{T}(T)$ holds.

Let $\Phi_U$and $\Phi_T$be defined as above. Then

$$\hat{\pi}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widehat{\mathcal{H}}), \qquad X \mapsto \Phi_U(\bar{X})$$

defines a completely positive linear mapping with $\Phi_T(\mathrm{X}) = \mathcal{P}_\mathcal{H} \hat{\pi}(X)|_\mathcal{H}$ for all $X \in \mathcal{B}(\mathcal{H})$ and $\operatorname{ran}(\hat{\pi}) \subset (U)'$. To see that equality holds here, we need some more preparations. Note that

$$\mathcal{J}_U = \Psi_U(I_\mu) \subset W^*(U)$$

defines an abelian semigroup of unitary operators with $W^*(\mathcal{J}_U) = W^*(U)$. The minimality of Uas a normal extension of Timplies that

$$\widehat{\mathcal{H}} = \bigvee V^* \mathcal{H}; V \in \mathcal{J}_\mathrm{U}\,.$$

To see this it suffices to observe that the space on the right-hand side is invariant under $W^*(\mathcal{J}_U) = W^*(U)$.

Corollary (4.3.4)[128]: The compression mapping

$$\varrho: (U)' \to \mathcal{T}(T), X \longmapsto \mathcal{P}_\mathcal{H} X|_\mathcal{H}$$

defines a completely isometric linear isomorphism with inverse given by

$$\mathcal{T}(T) \to (U)', \qquad X \longmapsto \hat{\pi}(X).$$

Proof. We know from Corollary (4.3.3) that $\varrho$is well-defined and surjective. As a compression mapping $\varrho$is completely contractive. Since

$$\langle X V^* h, W^* k \rangle = \langle \varrho(X) W h, V k \rangle$$

for all $X \in (U)', V, W \in \mathcal{J}_U$ and $h, k \in \mathcal{H}$, the remarks preceding the corollary imply that$\varrho$is injective. The observation that

$$\varrho(\hat{\pi}(X)) = \Phi_T(X) = X$$

for all $X \in \mathcal{T}(T)$ shows that $\mathcal{T}(T) \to (U)', X \longmapsto \hat{\pi}(X)$, defines the inverse of the bijection $\varrho: (U)' \to \mathcal{T}(T)$. Since also $\hat{\pi}$ is completely contractive as a composition of completely contractive mappings, it follows that ϱ is completely isometric.

The restriction of $\hat{\pi}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widehat{\mathcal{H}})$to the C^*-algebra $C^*(\mathcal{T}(T))$ generated by all abstract T-Toeplitz operators is even a C^*-algebra homomorphism.

Theorem (4.3.5)[128]: The restriction

$$\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))} C^*(\mathcal{T}(T)) \to \mathcal{B}(\widehat{\mathcal{H}})$$

is the minimal Stinespring dilation of the completely positive unital projection

$$C^*(\mathcal{T}(T)) \to C^*(\mathcal{T}(T)), X \mapsto \Phi_T(X).$$

For $X \in \mathcal{B}(\mathcal{H})$ and $Y \in C^*(\mathcal{T}(T))$, we have

$$\hat{\pi}(XY) = \hat{\pi}(X)\hat{\pi}(Y).$$

Proof. We know that $\hat{\pi}: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\widehat{\mathcal{H}})$, and hence also its restriction π, are completely positive maps. To prove that π is a homomorphism of C^*-algebras, it suffices to check its multiplicativity. Fix operators $X \in \mathcal{B}(\mathcal{H})$ and $Y \in \mathcal{T}(T)$. Since $\operatorname{ran}(\hat{\pi}) \subset W^*(U)'$, it follows that

$$\langle \hat{\pi}(XY)V^*h, k\rangle = \lim_{\alpha} \langle V^*\Phi_{U,k_\alpha}(\widetilde{XY})h, k\rangle$$

for $V \in \mathcal{I}_U$ and $h \in \mathcal{H}, k \in \widehat{\mathcal{H}}$. Applying Corollary (4.3.4) to the operator $Y \in \mathcal{T}(T)$, we obtain the identity

$$\Psi_U(\theta)^*\widetilde{XY}\Psi_U(\theta)h = \Psi_U(\theta)^* X \mathcal{P}_{\mathcal{H}} \Psi_U(\theta)\hat{\pi}(Y)h$$

for $\theta \in I_\mu$and $h \in \mathcal{H}$. Using the definition of $\hat{\pi}(X) = \Phi_U(\bar{X})$, we find that

$$\langle \hat{\pi}(XY)V^*h, k\rangle = \langle V^*\hat{\pi}(X)\hat{\pi}(Y)h, k\rangle = \langle \hat{\pi}(X)\hat{\pi}(Y)V^*h, k\rangle$$

for $\theta \in I_\mu$and $h \in \mathcal{H}, k \in \widehat{\mathcal{H}}$. By the remarks preceding Corollary (4.3.4) it follows that $\hat{\pi}(XY) = \hat{\pi}(X)\hat{\pi}(Y)$.

Inductively one obtains that

$$\hat{\pi}(X_1 \cdot \ldots \cdot X_r) = \hat{\pi}(X_1) \cdot \ldots \cdot \hat{\pi}(X_r)$$

holds for any finite number of operators $X_1 \cdot \ldots \cdot X_r \in \mathcal{T}(T)$. Since $C^*(\mathcal{T}(T))$ is the norm-closed linear span of products of this type and since $\hat{\pi}$ is norm-continuous, the multiplicativity of $\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))}$ follows.

Using the definition of $\pi(1_{\mathcal{H}}) = \Phi_U(1_{\mathcal{H}} \oplus 0_{\mathcal{H}^\perp})$, one easily finds that $\pi(1_{\mathcal{H}})$ acts as the identity operator on $\mathcal{H}$. Since

$$\pi(1_{\mathcal{H}})V^*h = V^*\pi(1_{\mathcal{H}})h = V^*h$$

for all $\theta \in \mathcal{I}_U$ and $h \in \mathcal{H}$, it follows that $\pi(1_{\mathcal{H}}) = 1_{\widehat{\mathcal{H}}}$. As an application of Corollary (4.3.4) one obtains that $\pi(T_f) = \Psi_U(f)$ for all $f \in L^\infty(\mu)$. Hence the minimality of Uimplies that $\pi$is the minimal Stinespring dilation of $\Phi_T|_{C^*(\mathcal{T}(T))}$. To see that $\hat{\pi}$possesses the additional multiplicativity property claimed in the theorem, it suffices to observe that

$$\hat{\pi}(XY_1 \cdot \ldots \cdot Y_r) = \hat{\pi}(X)\hat{\pi}(Y_1) \cdot \ldots \cdot \hat{\pi}(X_r) = \hat{\pi}(X)\hat{\pi}(Y_1 \ldots Y_r)$$

for $X \in \mathcal{B}(\mathcal{H}), Y_1, \ldots, Y_r \in \mathcal{T}(T)$, and to use the norm-continuity of $\hat{\pi}$.

For $Y \in (U)'$, we define the Toeplitz operator $T_Y \in \mathcal{T}(T)$ with symbol Yas the compression $T_Y = \mathcal{P}_{\mathcal{H}} Y|_{\mathcal{H}}$. In the particular case that $Y = \Psi_U(f)$ with $f \in L^\infty(\mu)$ we obtain that $T_Y = T_f$is the Toeplitz operator with symbol f.

Corollary (4.3.6)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$ be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$and scalar spectral measure $\mu \in M^+(\partial_A)$. Let $\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and $\pi: C^*(\mathcal{T}(T)) \to \mathcal{B}(\widehat{\mathcal{H}})$ be defined as before.

(i) For $X \in \mathcal{T}(T)$, the operator $\pi(X)$ is the unique element in $(U)'$ with $X = T_{\pi(X)}$. For $Y \in (U)'$, we have $\pi(T_Y) = Y$.

(ii) For $Y \in (U)'$ and $f \in L^\infty(\mu)$, we have

$$\|T_Y\| = \|Y\| \qquad \textit{and } \|T_f\| = \|f\|_{L^\infty(\mu)};$$

(iii) For $Y_{ij} \in (U)', 1 \le i \le r, 1 \le j \le s$, we have

$$\Phi_T\left(\sum_{i=1}^{r}\prod_{j=1}^{S} T_{Y_{ij}}\right) = T_{\sum_{i=1}^{r}\prod_{j=1}^{S} Y_{ij}}.$$

Proof. Part (i) and part (ii) follow immediately from Corollary (4.3.4). Since by Theorem (4.3.5) the restriction $\pi = \hat{\pi}|_{C^*(\mathcal{T}(T))}$is a C^*-algebra homomorphism, we obtain that

$$\Phi_T\left(\sum_{i=1}^{r}\prod_{j=1}^{S} T_{Y_{ij}}\right) = \mathcal{P}_{\mathcal{H}}\left(\sum_{i=1}^{r}\prod_{j=1}^{S} \pi\left(T_{Y_{ij}}\right)\right)\Bigg|_{\mathcal{H}} = T_{\sum_{i=1}^{r}\prod_{j=1}^{S} Y_{ij}}$$

for $Y_{ij} \in (U)'$as in part (iii).

Since the C^*-algebra $C^*(\mathcal{T}(T))$ is the norm-closure of the set of all finite sums of finite products of Toeplitz operators of the form T_Y with $Y \in (U)'$, part(iii) of Corollary (4.3.6) shows in particular that the action of any Toeplitz projection $\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined as above is uniquely determined by T on the Toeplitz C^*-algebra $C^*(\mathcal{T}(T))$.

If $W^*(U)$ is a maximal abelian W^*-algebra, or equivalently, $W^*(U) = W^*(U)'$, then the abstract and concrete Toeplitz operators coincide, that is,

$$\mathcal{T}(T) = \{T_f;\ f \in L^\infty(\mu)\}.$$

This can be seen as a generalization of the classical Brown–Halmos characterization [2] of the Toeplitz operators on $H^2(\mathbb{T})$.

Let $T \in \mathcal{B}(\mathcal{H})^n$be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$and scalar spectral measure $\mu \in M^+(\partial_A)$.We denote by $\Phi_T: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ a Toeplitz projection defined as above. Recall that Φ_T is the compression

$$\Phi_T(U) = \mathcal{P}_{\mathcal{H}}\Phi_U(\bar{X})|_{\mathcal{H}}(X \in \mathcal{B}(\mathcal{H}))$$

of a projection $\Phi_U: \mathcal{B}(\widehat{\mathcal{H}}) \to \mathcal{B}(\widehat{\mathcal{H}})$ with $\operatorname{ran}(\Phi_U) = (U)'$ and that

$$\pi: C^*(\mathcal{T}(T)): \to \mathcal{B}(\widehat{\mathcal{H}}), \qquad X \mapsto \Phi_U(X)$$

is the minimal Stinespring dilation of the completely positive and unital mapping $\Phi_T|_{C^*(\mathcal{T}(T))}$. We denote by

$$\mathcal{T}_a(T) = \{T_f;\ f \in H_A^\infty(\mu)\} \subset \mathcal{B}(\mathcal{H})$$

the weak*closed subalgebra consisting of all analytic Toeplitz operators. Our next aim is to calculate the essential commutant $\mathcal{T}_a(T)^{ec}$of the set of all analytic Toeplitz operators.

Lemma (4.3.7)[128]: Suppose that $M \subset \mathcal{H}$is a closed reducing subspace for $\mathcal{T}(T)$. Then

$$\Phi_T(\mathrm{X}) = \Phi_T\left((P_M X|_M) \oplus (P_{M^\perp} X|_{M^\perp})\right)$$

for every operator $X \in \mathcal{B}(\mathcal{H})$.

Proof. We denote by $\mathcal{M}$the set of all operators $X \in \mathcal{B}(\mathcal{H})$ with the property that $XM \subset M^\perp$and $XM^\perp \subset M$. Fix an operator $X \in \mathcal{M}$. Then $T_\theta^* X T_\theta \in \mathcal{M}$for all μ-inner functions $\theta \in I_\mu$and hence also $\Phi_T(\mathrm{X}) \in \mathcal{M}$ (see Corollary (4.3.2)). On the other hand, the space $M$is reducing for the operator $\Phi_T(\mathrm{X}) \in \mathcal{T}(T)$. Therefore $\Phi_T(\mathrm{X}) = 0$ and the assertion follows.

Let $S \in \mathcal{T}_a(T)^{ec}$be arbitrary. It follows from Corollary (4.3.4) that $Y_S = \hat{\pi}(S)$ is the unique operator in $(U)'$with $\Phi_T(S) = \mathcal{P}_{\mathcal{H}} Y_S|_{\mathcal{H}}$. Our aim is to show that, under suitable conditions on T, the operator S is a compact perturbation of an abstract T-Toeplitz operator. Since $\Phi_T(\mathrm{S}) \in \mathcal{T}(T)$, it suffices to show that

$$S - \Phi_T(\mathrm{S}) \in \mathcal{K}(\mathcal{H}).$$

To prove this, we shall use the map

$$F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H})\,, f \mapsto T_f S - \mathcal{P}_{\mathcal{H}}\left(Y_S \Psi_U(f)\right)\big|_{\mathcal{H}}.$$

Note first that $S - \Phi_T(\mathrm{S}) = F(1)$and that

$$F(f) = T_f S - \Phi_T(\mathrm{S}) T_f$$

for every function $f \in H_A^\infty(\mu)$. It clearly suffices to find conditions which ensure that the whole image of Fconsists of compact operators. Since F is continuous linear, we only need to show that Fmaps the characteristic function χ_ω of each Borel set $\omega \subset \partial_A$ into $\mathcal{K}(\mathcal{H})$. We begin with a very modest first step.

Lemma (4.3.8)[128]: For every point $z \in \partial_A$, the operator $F(\chi_{\{z\}})$is compact.

Proof. We may suppose that $\mu(\{z\}) > 0$, since otherwise $F(\chi_{\{z\}}) = 0$. As shown in [133] the regularity of Timplies that $\chi_{\{z\}} \in H_A^\infty(\mu)$. Exactly as in [133]), it follows that the eigenspace $H_z$of Tassociated with the joint eigenvalue $z$coincides with the eigenspace of Uassociated with z, that is,

$$\bigcap_{i=1}^{n} \operatorname{Ker}(z_i - T_i) = \bigcap_{i=1}^{n} \operatorname{Ker}(z_i - U_i)$$

and that $P_z = \Phi_U(\chi_{\{z\}})\big|_{\mathcal{H}} \in \mathcal{B}(\mathcal{H})$ is the orthogonal projection onto H_z. The space $H_z = P_z\mathcal{H}$is reducing for $\mathcal{T}(T)$, since

$$(\mathcal{P}_{\mathcal{H}}X|_{\mathcal{H}})P_z = \mathcal{P}_{\mathcal{H}}\Psi_U(\chi_{\{z\}})\mathcal{P}_{\mathcal{H}}X\big|_{\mathcal{H}} = P_z(\mathcal{P}_{\mathcal{H}}X|_{\mathcal{H}})$$

for all $X \in (U)'$. Let $S = (S_{ij})_{ij=1,2}$ be the matrix representation of S with respect to the decomposition $\mathcal{H} = (\mathcal{H} \ominus H_z) \oplus H_z$. Since $P_z = T_{\chi\{z\}} \in \mathcal{T}_a(T)$, it follows that $SP_z - P_zS \in \mathcal{K}(\mathcal{H})$, or equivalently, that $S_{12}$and $S_{21}$are compact. Using Lemma (4.3.7) and passing to the equivalence classes in the Calkin algebra, we find that

$$[F_{(\chi\{z\})}] = [P_z(S_{11} \oplus S_{22}) - \Phi_T(S)P_z] = [P_z(0 \oplus S_{22}) - \Phi_T(S_{11} \oplus S_{22})P_z].$$

For each μ-inner function $\theta \in I_\mu$, we have

$$(T_\theta^*(S_{11} \oplus S_{22})T_\theta|_{H_z} = (T_\theta^* S_{22} T_\theta)|_{H_z} = S_{22}.$$

Hence the definition of Φ_T implies that $\Phi_T(S_{11} \oplus S_{22})|_{H_z} = S_{22}$. But then $P_z(0 \oplus S_{22}) - \Phi_T(S_{11} \oplus S_{22})P_z = 0$and therefore $F_{(\chi\{z\})}$is compact.

Let us suppose in addition that $T$is essentially normal. Then it follows from Lemma 3.9 (c) in [134] that all operators in the image of the map

$$F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H}),\ f \mapsto T_f S - \mathcal{P}_{\mathcal{H}}(Y_S\Psi_U(f))\big|_{\mathcal{H}}$$

belong to the essential commutant $(T)^{ec}$of T. Hence we can apply the following consequence of the Allan-Douglas localization principle to every operator inran (F).

Proposition (4.3.9)[128]: Suppose that the regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$is essentially normal. Then for every operator $X \in (T)^{ec}$, we have

$$\|X\|_e = \sup_{z \in \partial_A} \inf\left\{\|T_f X\|_e ; f \in C(\partial_A) \ \ with\, f(z) = 1\right\}.$$

Proof. By Lemma3.9 (c) and Lemma 4.1 in [134], the essential normality of T yields that $\mathcal{D} = (T)^{ec}$is a C^*-algebra containing $\mathcal{T}(T) \cup \mathcal{K}(H)$, that the C^*-algebra

$$\mathcal{A} = \left(C^*(\{T_f : f \in C(\partial_A)\})\right) + \mathcal{K}(H)/\mathcal{K}(H)$$

is contained in the center of the C^*-algebra $\mathcal{T} = \mathcal{D}/\mathcal{K}(H)$ and that the mapping$\tau: C(\partial_A) \to \mathcal{A}, f \mapsto [T_f]$,is a surjective C^*-algebra homo-morphism. Hence, for each functional $\lambda \in \Delta_{\mathcal{A}}$in the character space of $\mathcal{A}$, there is a unique point $z(\lambda) \in \partial_A$with

$$\lambda([T_f]) = f(z(\lambda))(f \in C(\partial_A)).$$

For $\lambda \in \Delta_{\mathcal{A}}$ and $z \in \partial_A$, let $I_\lambda \subset \mathcal{T}$be the closed ideal generated by all elements $[T_f]$ where $f \in C(\partial_A)$ and $\lambda([T_f]) = 0$, and let $I_\lambda \subset \mathcal{T}$be the closed ideal generated by all elements $[T_f]$

such that $f \in C(\partial_A)$ satisfies $f(z) = 0$. Then $I_\lambda = I_{z(\lambda)}$for all $\lambda \in \Delta_{\mathcal{A}}$, and the Allan-Douglas localization principle (Theorem 7.47 in [5]) implies that

$$\|X\|_e = \sup_{\lambda \in \Delta_{\mathcal{A}}} \|[X] + I_\lambda\|_{\mathcal{T}/I_\lambda} \leq \sup_{z \in \partial_A} \|[X] + I_z\|_{\mathcal{T}/I_z}$$

for every $X \in (T)^{ec}$. But for $X \in (T)^{ec}$and $f \in C(\partial_A)$ with $f(z) = 1$, the estimate

$$\|[X] + I_z\|_{\mathcal{T}/I_z} = \left\|\left[T_f X\right] + I_z\right\|_{\mathcal{T}/I_z} \leq \left\|T_f X\right\|_{\mathrm{e}}$$

holds. This observation completes the proof.

An application of the dominated convergence theorem (Lemma 3.4 in [134]) shows that the mapping

$$F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H})\,, f \mapsto T_f S - \mathcal{P}_{\mathcal{H}}\left(Y_S \Psi_U(f)\right)\big|_{\mathcal{H}}$$

is pointwise boundedly SOT-continuous, that is, for every bounded sequence $(f_k)_k$in $L^\infty(\mu)$ converging pointwise μ-almost everywhere to some function $f \in L^\infty(\mu)$, it follows that $F(f) = \lim_{k\to\infty} F(f_k)$ in the strong operator topology.

Corollary (4.3.10)[128]: Suppose that the regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$is essentially normal. For a given operator $S \in \mathcal{T}_a(T)^{ec}$, let $F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H})$ be defined as above. If $F\left(L^\infty(\mu)\right) \subsetneq \mathcal{K}(\mathcal{H})$, then there is a sequence $(f_k)_k$of continuous functions $f_k \in C(\partial_A, [0,1])$with pairwise disjoint supports such that

$$\inf_{k\geq 1} \|F(f_k)\| > 0.$$

Proof. Suppose that $F\left(L^\infty(\mu)\right) \nsubseteq \mathcal{K}(\mathcal{H})$. Since every bounded measurable function can be approximated uniformly by linear combinations of characteristic functions of Borel sets, we can choose a characteristic function $\chi$of some Borel set in ∂_A such that $\varrho = \frac{\|F(\chi)\|_e}{2} > 0$. By Proposition (4.3.9) there is a point $z \in \partial_A$with

$$\|F(f\chi)\|_e = \left\|T_f F(\chi)\right\|_e > \varrho$$

for all $f \in C(\partial_A)$ with $f(z) = 1$. Here the first equality follows from Lemma 3.9 (c) in [134]. Let $k \geq 0$ be an integer. Suppose that $g_1, \ldots, g_k \in C(\partial_A, [0,1])$are functions with pairwise disjoint supports such that $\|F(g_i\chi)\|_e > \varrho$ and $z \notin \operatorname{supp}(g_i)$ for $j = 0, \ldots, k$. Choose a function $f \in C(\partial_A, [0,1])$with $f(z) = 1$ and $\operatorname{supp}(f) \cap \operatorname{supp}(g_i) = \emptyset$ for all $j = 0, \ldots, k$. Let $\left(\theta_j\right)_j$be a sequence of functions in $C(\partial_A, [0,1])$ with $z \notin \operatorname{supp}\left(\theta_j\right)$for all $j$such that $\theta_j(w) \to 1$ as $j \to \infty$for every point $w \in \partial_A \backslash \{z\}$. Since $F$is pointwise boundedly SOT-continuous, it follows that

$$F\left(\chi_{\{z\}^c} f\chi\right) = \mathrm{SOT} - \lim_{j\to\infty} F\left(\theta_j f\chi\right).$$

As an application of Lemma (4.3.8), we obtain that

$$\left\|F\left(\chi_{\{z\}^c} f\chi\right)\right\| \geq \|F(f\chi)\|_e > \varrho.$$

Hence there is an integer $j \geq 1$such that $\left\|F\left(\theta_j f\chi\right)\right\| > \varrho$.

Inductively one obtains a sequence of functions $g_k \in C(\partial_A, [0,1])$ with pairwise disjoint supports and $\|F(g_k\chi)\| > \varrho$ for all j. In the inductive step, one can define $g_{k+1} = \theta_j f$ with $f$and $\theta_j$as above. A standard application of Lusin's theorem (Theorem 7.4.3 and Proposition 3.1.2 in [130]) shows that there is a sequence $\left(h_j\right)_j$ in $C(\partial_A, [0,1])$ such that $h_j \to \chi$ for $j \to \infty$ μ-almost everywhere. Since

$$F(g_k\chi) = \mathrm{SOT} - \lim_{j\to\infty} F\left(g_k h_j\right)\,(k \geq 1),$$

for each $k \geq 1$, there is an index $j_k \geq 1$ with $\|F(g_k h_{j_k})\| > \varrho$. But then the resulting functions $f_k = g_k h_{j_k}$have all the required properties.

We are able to show the main result.

Theorem (4.3.11)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$be an essentially normal regular A-isometry. Then for every operator $S \in \mathcal{T}_a(T)^{ec}$, we have

$$S - \Phi_T(S) \in \mathcal{K}(\mathcal{H}).$$

Proof. Let $F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H})\, , f \mapsto T_f S - \mathcal{P}_\mathcal{H}\left(Y_S \Psi_U(f)\right)\big|_{\mathcal{H}}$, be the map considered above. Since $S - \Phi_\mathrm{T}(S) = F(1)$, it suffices to show that $F\left(L^\infty(\mu)\right) \subset \mathcal{K}(\mathcal{H})$. Let us assume that this inclusion does not hold. Then by Corollary (4.3.10) there are a positive real number $\varrho > 0$ and a sequence of functions $f_k \in C(\partial_A, [0,1])$ with pairwise disjoint supports $A_k = supp(f_k)$ such that $\|F(f_k)\| > \varrho$ for all $k \geq 1$.

Exactly as in [134] , one can use the regularity of $T$to replace $(f_k)_k$ by a sequence $(g_k)_k$of functions in A such that

$$\|g_k\|_{\infty,\partial_A} \leq 2\, , \|g_k\|_{\infty,\partial_A \setminus A_k} < 2^{-k} \|F(g_k)\| > \frac{\varrho}{4}$$

for all $k \geq 1$. Recall that $F(g_j) = T_{g_j} S - \Phi_T(S) T_{g_j}$is the weak*limit of a net consisting of operators of the form

$$T_{g_j} S - \frac{1}{k^k} \sum_{i \in \{1,\dots,k\}^k} T^*_{\theta(i)} S T_{\theta(i)} T_{g_j} = \frac{1}{k^k} \sum_{i \in \{1,\dots,k\}^k} T^*_{\theta(i)} \left(T_{g_j \theta(i)} S - S T_{g_j \theta(i)}\right)$$

with suitable μ-inner functions $\theta(i) \in I_\mu$. Hence, for each $j \geq 1$, there is a function $\theta_j : \partial_A \to \mathbb{C}$ with $|\theta_j| = 1$ on ∂_Asuch that $\theta_j \in I_\mu$and such that the function $h_j = g_j \theta_j \in H^\infty_A(\mu)$ satisfies

$$\|h_j\|_{\infty,\partial_A} \leq 2, \|h_j\|_{\infty,\partial_A \setminus A_j} < 2^{-J}, \left\|T_{h_j} S - S T_{h_j}\right\| > \frac{\varrho}{4}.$$

By hypothesis the commutators $K_j = \left[T_{h_j}, S\right]$ are compact. By passing to a subsequence, one can achieve that the limit

$$c = \lim_{j \to \infty} \|K_j\| \in \left[\frac{\varrho}{4}, \infty\right]$$

exists. Since the sequence $(h_j)_j$is uniformly bounded on $\partial_A$and converges to zero point-wise on ∂_A, it follows that the sequences $(K_j)_j$and $(K_j^*)_j$converge to zero strongly. Aresult due to Muhly and Xia (Lemma 2.1 in [137]) shows that, by passing to a subsequence again, one can achieve that the series

$$K = \sum_{j=1}^{\infty} K_j$$

converges in the strong operator topology and satisfies $\|K\|_e = c > 0$. Since each point of ∂_Abelongs to at most one of the sets A_j, the partial sums of the series $\sum_{j=1}^{\infty} h_j$are uniformly bounded on $\partial_A$and converge pointwise to a function $h: \partial_A \to \mathbb{C}$. By the dominated convergence theorem it follows that $h \in H^\infty_A A(\mu)$. Again using Lemma 3.4 from [134] , one obtains that

$$T_h = \mathrm{SOT} - \sum_{j=1}^{\infty} T_{h_j}\ , [T_h S] = \mathrm{SOT} - \sum_{j=1}^{\infty} \left[T_{h_j}, S\right] = K.$$

But then $T_h \in \mathcal{T}_a(T)$ would be an operator with non-compact commutator $[T_h, S] = K$. This contradiction completes the proof.

Let $T \in \mathcal{B}(\mathcal{H})^n$ be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\mathcal{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. Suppose that $W^*(U) \subset \mathcal{B}(\widehat{\mathcal{H}})$ is a maximal abelian von Neumann algebra, that is, $W^*(U) = (U)'$.
Then Corollary (4.3.3) implies that $\mathcal{T}(T) = \{T_f; f \in L^\infty(\mu)\}$. As a consequence, we obtain a complete characterization of the essential commutant $\mathcal{T}_a(T)^{ec}$ of the analytic Toeplitz operators in this case.

Corollary (4.3.12)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$ be an essentially normal regular A-isometry with minimal normal extension $U \in \mathcal{B}(\mathcal{H})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$. If $W^*(U) \subset \mathcal{B}(\widehat{\mathcal{H}})$ is a maximal abelian von Neumann algebra and $S \in \mathcal{B}(\mathcal{H})$, then equivalent are:

(i) $S \in \mathcal{T}_a(T)^{ec}$.

(ii) $S = T_f + K$ with a compact operator $K \in \mathcal{K}(\mathcal{H})$ and a symbol $f \in L^\infty(\mu)$ with the property that the associated Hankel operator H_f is compact.

Proof. First, suppose that $S \in \mathcal{T}_a(T)^{ec}$. Then $\Phi_T(S) = T_f$ with a suitable function $f \in L^\infty(\mu)$. The proof of Theorem (4.3.11) shows that the image of the bounded linear map

$$F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H}),\, g \mapsto T_g S - \mathcal{P}_{\mathcal{H}}\left(\Psi_U(gf)\right)\big|_{\mathcal{H}}$$

is contained in $\mathcal{K}(\mathcal{H})$. It follows that $K = F(1) = S - T_f$ is compact and the identity

$$\begin{aligned} F(\bar{f}) = T_{\bar{f}}S - T_{|f|^2} &= T_{\bar{f}}T_f - T_{|f|^2} + T_{\bar{f}}K \\ &= \mathcal{P}_{\mathcal{H}}\Psi_U(\bar{f})\mathcal{P}_{\mathcal{H}}\Psi_U(f)\big|_{\mathcal{H}} - \mathcal{P}_{\mathcal{H}}\Psi_U(\bar{f})\Psi_U(f)\big|_{\mathcal{H}} + T_{\bar{f}}K \\ &= -\mathcal{P}_{\mathcal{H}}\Psi_U(\bar{f})\mathcal{P}_{\mathcal{H}^\perp}\Psi_U(f)\big|_{\mathcal{H}} + T_{\bar{f}}K = -H_{\bar{f}}^*H_f + +T_{\bar{f}}K \end{aligned}$$

shows that also the operator H_f is compact.

In order to prove the remaining implication, it suffices to verify that all Toeplitz operators T_f such that the corresponding Hankel operators H_f are compact essentially commute with $\mathcal{T}_a(T)$. But this follows from the formula

$$T_f T_g - T_g T_f = T_{gf} - T_g T_f = \mathcal{P}_{\mathcal{H}}\Psi_U(g)H_f$$

which holds for all $f \in L^\infty(\mu)$ and $g \in H_A^\infty(\mu)$.

By considering Hankel operators $H_Y = (1 - \mathcal{P}_{\mathcal{H}})Y|_{\mathcal{H}} \in B(\mathcal{H}, \mathcal{H}^\perp)$ with symbol $Y \in (U)'$, we obtain a similar characterization of the essential commutant of the analytic Toeplitz operators in the general case.

Corollary (4.3.13)[128]: If $T \in B(\mathcal{H})^n$ is an essentially normal regular A-isometry with minimal normal extension $U \in B(\widehat{\mathcal{H}})^n$ and scalar spectral measure $\mu \in M^+(\partial_A)$, then the following statements are equivalent:

(i) $S \in \mathcal{T}_a(T)^{ec}$.

(ii) $S = T_Y + K$ with a compact operator $K \in \mathcal{K}(\mathcal{H})$ and a symbol $Y \in (U)'$ such that the associated Hankel operator H_Y has the property that $H_{\bar{f}}^*H_Y$ is compact for every $f \in L^\infty(\mu)$.

(iii) $S = T_Y + K$ with a compact operator $K \in \mathcal{K}(\mathcal{H})$ and a symbol $Y \in (U)'$ such that the associated Hankel operator H_Y has the property that $H_{\bar{f}}^*H_Y$ is compact for every $f \in H_A^\infty(\mu)$.

Proof. For arbitrary symbols $f \in L^\infty(\mu)$ and $Y \in (U)'$, an elementary calculation shows that

$$-H_{\bar{f}}^*H_Y = T_f T_Y - \mathcal{P}_{\mathcal{H}}\Psi_U(f)|_{\mathcal{H}}.$$

Suppose that $S \in \mathcal{T}_a(T)^{ec}$. Then Theorem (4.3.11) implies that $S = T_Y + K$ is a sum of the Toeplitz operator $T_Y = \Phi_T(S) \in \mathcal{B}(\mathcal{H})$with symbol $Y \in (U)'$ and the compact operator $K = S - \Phi_T(S) \in \mathcal{K}(\mathcal{H})$. By the proof of Theorem (4.3.11) the range of the mapping

$$F: L^\infty(\mu) \to \mathcal{B}(\mathcal{H}), f \mapsto T_f S - \mathcal{P}_\mathcal{H}\Psi_U(f)Y|_\mathcal{H}$$

is contained in $\mathcal{K}(\mathcal{H})$. Consequently, $H_{\bar{f}}^* H_Y = T_f K - F(f)$ is compact for every symbol $f \in L^\infty(\mu)$. To complete the proof note that the identity

$$H_{\bar{f}}^* H_Y = \mathcal{P}_\mathcal{H}\Psi_U(f)Y|_\mathcal{H} - T_f T_Y = T_Y T_f - T_f T_Y$$

holds for $f \in H_A^\infty(\mu)$ and $Y \in (U)'$.

Johnson and Parrott characterized the essential commutant $\mathfrak{U}^{ec}$of an abelian von Neumann algebra $\mathfrak{U} \subset \mathcal{B}(\mathcal{H})$ as the sum $\mathfrak{U}' + \mathcal{K}(\mathcal{H})$ of its commutant and the compact operators [8]. This result has been generalized in [138] to the non-abelian case. We present an alternative proof of Johnson and Parrott's result for finitely generated abelian von Neumann algebras. To this end, let us observe that, for every compact subset $K \subset \mathbb{C}^n$, the Shilov boundary of $C(K)$ is equal to Kitself and the triple $(C(K), K, \mu)$ is regular [129] for every choice of $\mu \in M^+(K)$. Consequently, every commuting tuple $N = (N_1, \dots, N_n) \in \mathcal{B}(\mathcal{H})^n$of normal operators is a regular $C(\sigma(N))$-isometry.

Corollary (4.3.14)[128]: (Johnson–Parrott). The essential commutant of a finitely generated abelian von Neumann algebra $\mathfrak{U} \subset \mathcal{B}(\mathcal{H})$is given by

$$\mathfrak{U}^{ec} = \mathfrak{U}' + \mathcal{K}(\mathcal{H}).$$

Proof. Since $\mathfrak{U}$is abelian, its generators $N_1, \dots, N_n \in \mathcal{B}(\mathcal{H})^n$ form a commuting tuple of normal operators and hence a normal regular $C(\sigma(N))$-isometry $N \in \mathcal{B}(\mathcal{H})^n$. By Theorem (4.3.11), the inclusion $\mathcal{T}_a(N)^{ec} \subset \mathcal{T}(N) + \mathcal{K}(\mathcal{H})$ holds. Hence it suffices to check that the analytic Toeplitz operators associated with Ncoincide with $\mathfrak{U} = W^*(N)$ and that the abstract N-Toeplitz operators are precisely those operators that commute with $\mathfrak{U}$. Let $\mu \in M^+(\sigma(N))$ denote the scalar spectral measure associated with N. Then $C(\sigma(N))$ is weak$*$-dense in $L^\infty(\mu)$, which implies that $H^\infty_{C(\sigma(N))}(\mu) = L^\infty(\mu)$ and hence

$$W^*(N) = \Psi_N(L^\infty(\mu)) = \Psi_N\left(H^\infty_{C(\sigma(N))}(\mu)\right) = \mathcal{T}_a(N).$$

To conclude the proof, we combine the fact that $(N)' = W^*(N)' = \mathfrak{U}'$with Corollary (4.3.4) to obtain the remaining identity $\mathcal{T}(N) = \mathfrak{U}'$.

By [132], the preceding result applies in particular to every abelian von Neumann algebra on a separable Hilbert space.

As another application of Theorem (4.3.11) we characterize those regular A-isometries for which the associated Toeplitz projection $\Phi_T$vanishes on the compact operators. By following the lines of the proof of [133] and adapting it to the setting of regular A-isometries, we observe that a regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$has empty point spectrum if and only if

$$\mathcal{T}(T) \cap \mathcal{K}(\mathcal{H}) = \{0\}.$$

Corollary (4.3.15)[128]: The Toeplitz projection Φ_T associated with a regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$ vanishes on $\mathcal{K}(\mathcal{H})$ if and only if $\sigma_p(T) = \emptyset$.

Proof. Recall that the Toeplitz projection acts as the identity on the Toeplitz operators. Thus, if T has an eigenvalue, we can choose a compact Toeplitz operator $X \neq 0$ satisfying $\Phi_T(X) = X \neq 0$. On the other hand, the minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$ of T is a normal regular A-isometry. Moreover, the mapping $\Phi_U$is the corresponding Toeplitz projection. A look at Theorem (4.3.11) reveals that

$$S - \Phi_U(S) \in \mathcal{K}(\widehat{\mathcal{H}})$$
for every element $S \in \mathcal{T}_a(U)^{ec}$. Now assume that $K \in \mathcal{K}(\mathcal{H})$ is a compact operator. Then $\widehat{K} = K \oplus 0 \in \mathcal{K}(\widehat{\mathcal{H}})$ is compact and thus belongs to $\mathcal{T}_a(U)^{ec}$. Hence the above calculation implies that $\Phi_U(\widehat{K}) \in \mathcal{K}(\widehat{\mathcal{H}}) \cap \mathcal{T}(U)$ is a compact U-Toeplitz operator. Assuming that $\sigma_p(T) = \emptyset$, we infer that $\Phi_T(K) = \mathcal{P}_{\mathcal{H}} \Phi_U(\widehat{K})\big|_{\mathcal{H}} = 0$.

Using Corollary (4.3.15) we prove an essential spectral inclusion theorem for Toeplitz operators.

Theorem (4.3.16)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$. Then Thas empty point spectrum if and only if the spectral inclusion $\sigma(Y) \subset \sigma_e(T_Y)$holds for every operator $Y \in (U)'$.

Proof. Suppose that $\sigma_p(T) = \emptyset$ and fix an operator $Y \in (U)'$. We first show that the left spectrum of $Y$is contained in the left essential spectrum of T_Y. To prove this inclusion it suffices to verify that $Y$is left invertible in $\mathcal{B}(\widehat{\mathcal{H}})$ whenever $T_Y$is left invertible in the Calkin algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let us suppose that $X \in \mathcal{B}(\mathcal{H})$ is an operator with $XT_Y - 1_{\mathcal{H}} \in \mathcal{K}(\mathcal{H})$. Using Corollary (4.3.4) and the proof of Theorem (4.3.5), we find that
$$\hat{\pi}(XY) = \hat{\pi}(X)\hat{\pi}(T_Y) = \hat{\pi}(XT_Y) = 1_{\widehat{\mathcal{H}}} + \hat{\pi}(XT_Y - 1_{\mathcal{H}}).$$
Since $\sigma_p(U) = \sigma_p(T) = \emptyset$, it follows from Corollary (4.3.15) applied to U that Φ_U annihilates the compact operators. But then, using the definition of $\hat{\pi}$, we find that
$$\hat{\pi}(X)Y = 1_{\mathcal{H}} + \Phi_U\big((XT_Y - 1_{\mathcal{H}}) \oplus 0\big) = 1_{\mathcal{H}}.$$
Thus we have shown that the left spectrum of $Y$is contained in the left essential spectrum of T_Y. Applying the same argument to $Y^* \in (U)'$, we obtain that the left spectrum of Y^* is contained in the left essential spectrum of $T_Y^* = T_{Y^*}$. By standard duality results this means precisely that the right spectrum of Y is contained in the right essential spectrum of T_Y. In total we have shown that $\sigma(Y) \subset \sigma_e(T_Y)$for every operator $Y \in (U)'$under the hypothesis that the point spectrum of $T$is empty. If $x$is a joint eigenvector for T, then the orthogonal projection $Y$of $\widehat{\mathcal{H}}$ onto the one-dimensional subspace spanned by $x$belongs to the commutant $(U)'$ and $T_Y$is the corresponding rank-one projection on $\mathcal{H}$. Then $1 \in \sigma(Y)$ while $\sigma_e(T_Y) \subset \{0\}$. Hence the essential spectral inclusion result does not hold.

For regular A-isometries with empty point spectrum, we even proved spectral inclusion theorems for the left (essential) spectra and right (essential) spectra separately. If $\sigma_p(T) = \emptyset$and μdenotes the scalar spectral measure of the minimal normal extension $U \in \mathcal{B}(\widehat{\mathcal{H}})^n$, then we obtain in particular that
$$\operatorname{essran}(f) = \sigma_{L^\infty(\mu)}(f) = \sigma\big(\Psi_U(f)\big) \subset \sigma_e\big(T_f\big)$$
for every function $f \in L^\infty(\mu)$. For the particular case of Toeplitz operators on the Hardy space of the unit disc or the unit ball, this result is contained in [5] and [12] .

For a given subalgebra $B \subset (U)'$, we denote by
$$\mathcal{T}_{\mathcal{B}} = \overline{\operatorname{alg}}(\{T_X;\ X \in \mathcal{B}\}) \subset \mathcal{B}(\mathcal{H})$$
the smallest norm-closed subalgebra containing all operators $T_X$with $X \in \mathcal{B}$. The semi-commutator ideal $\mathcal{SC}(\mathcal{T}_{\mathcal{B}})$ of $\mathcal{T}_{\mathcal{B}}$ is defined as the norm-closed ideal in $\mathcal{T}_{\mathcal{B}}$generated by all operators $T_X T_Y - T_{XY}$ with $X, Y \in \mathcal{B}$. Since $\mathcal{T}_{\mathcal{B}}$is the norm-closure of the set of all finite sums of finite products of operators of the form $T_X$with $X \in B$, a straightforward argument using

part (iii) of Corollary (4.3.6) shows that $\mathcal{T}_{\mathcal{B}}$is invariant under the Toeplitz projection $\Phi_T$and that

$$\mathcal{SC}(\mathcal{T}_{\mathcal{B}}) = \ker\left(\Phi_T|_{\mathcal{T}_{\mathcal{B}}}\right) = \ker\left(\pi|_{\mathcal{T}_{\mathcal{B}}}\right).$$

The last equality follows from Theorem (4.3.5) together with Corollary (4.3.4).

Corollary (4.3.17)[128]: Let $T \in \mathcal{B}(\mathcal{H})^n$be a regular A-isometry with minimal normal extension $U \in \mathcal{B}(\mathcal{H})^n$. For each subalgebra $\mathcal{B} \subset (U)'$, there is a short exact sequence

$$0 \to \mathcal{SC}(\mathcal{T}_{\mathcal{B}}) \hookrightarrow \mathcal{T}_{\mathcal{B}} \xrightarrow{\pi} \mathcal{B} \longrightarrow 0$$

Of Banach algebras with $\pi(T_X) = X$ for all $X \in \mathcal{B}$. If $\sigma_p(T) = \emptyset$, then

$$\mathcal{T}_{\mathcal{B}} \cap \mathcal{K}(\mathcal{H}) \subset \mathcal{SC}(\mathcal{T}_{\mathcal{B}}).$$

Proof. The existence of the short exact sequence follows from the remarks preceding the corollary. The last assertion is a consequence of Corollary (4.3.15).

Using part (ii) and part(iii) of Corollary (4.3.6) one obtains that, for every regular A-isometry $T \in \mathcal{B}(\mathcal{H})^n$and each subalgebra $\mathcal{B} \subset (U)'$, the direct sum decomposition

$$\mathcal{T}_{\mathcal{B}} = \mathcal{SC}(\mathcal{T}_{\mathcal{B}}) \oplus \{T_X;\ X \in \overline{\mathcal{B}}\}$$

holds with $\mathcal{SC}(\mathcal{T}_{\mathcal{B}}) = \ker\left(\Phi_T|_{\mathcal{T}_{\mathcal{B}}}\right)$ and $\{T_X;\ X \in \overline{\mathcal{B}}\} = \Phi_T(\mathcal{T}_{\mathcal{B}})$. If in addition the subalgebra $\mathcal{B} \subset (U)'$is self-adjoint in the sense that $X^* \in B$whenever $X \in B$, then the sequence described in Corollary (4.3.17) is a short exact sequence of C^*-algebras.

Chapter 5
Toeplitz Corona Theorems

We extend and refine some work of Agler-McCarthy and Amar concerning the Corona problem for the polydisk and the unit ball in $\mathbb{C}^n$. For free functions a and b on a free domain $\mathcal{K}$ defined free polynomial inequalities, a sufficient condition on the difference $aa^* - bb^*$ to imply the existence a free function c taking contractive values on $\mathcal{K}$ such that $a = bc$ is established. The connection to recent work of Agler and McCarthy and their free Toeplitz Corona Theorem is exposited.

Section (5.1): The Polydisk and the Unit Ball

Theorem (5.1.1)[139]: (Corona) Let $\{f_j\}_{j=1}^m \subseteq H^\infty(D)$. Assume that

$$0 < \epsilon^2 \le \sum_{j=1}^m |f_j(z)|^2 \le 1 \quad for\ all\ z \in D.$$

There exists a constant $C(\epsilon, m) < \infty$ and $\{g_j\}_{j=1}^m \subseteq H^\infty(D)$. so that

$$\sup_{z\in D} \sum_{j=1}^m |g_j(z)|^2 \le C(\epsilon, m)^2 \ \ and \ \sum_{j=1}^m f_j(z) g_j(z) = 1 \ \ for\ all\ z \in D.$$

Theorem (5.1.1) and especially the techniques utilized in its proof have been very influential. See, for example, Garnett [42]. Among many questions raised by this theorem, we wish to consider the analogous Corona problem for the polydisk and the unit ball in $\mathbb{C}^n$.

For the case of the bidisk, D^2, Agler and McCarthy proved the following:

Theorem (5.1.2)[139]: (Agler and McCarthy) Let $\{f_j\}_{j=1}^m \subseteq H^\infty(D)$. Then there exist $\{g_j\}_{j=1}^m \subseteq H^\infty(D^2)$. with

$$\sum_{j=1}^m f_j g_j \equiv 1 \ \ and \ \sup_{z\in D^2} \sum_{j=1}^m |g_j(z)|^2 \le \frac{1}{\delta^2}$$

if and only if

$$T_F^\mu (T_F^\mu)^* \ge \delta^2 I_\mu$$

for all probability measures μ on T^2.

Although the Theorem (5.1.2) and its proof seemed to be restricted to $n = 2$ by the classical and beautiful counterexample of Parrot [143], Amar managed to extend it to D^n (and to B^n).

Theorem (5.1.3)[139]:(Amar) Let $\{f_j\}_{j=1}^m \subseteq H^\infty(\Omega)$. Then there exist $\{g_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ with

$$\sum_{j=1}^m f_j g_j \equiv 1 \ \ and \ \sup_{z\in \Omega} \sum_{j=1}^m |g_j(z)|^2 \le \frac{1}{\delta^2}$$

if and only if

$$T_F^\mu (T_F^\mu)^* \ge \delta^2 I_\mu$$

for all probability measures μ on $\partial\Omega$.

In other words, Amar shows that for $\{f_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ and $\delta > 0$ the following are equivalent:

(i) There exist $\{g_j\}_{j=1}^m \subseteq H^\infty(\Omega)$ with

$$\sum_{j=1}^m f_j g_j = 1 \quad on\, \Omega\, and\ \sup_{z\in\Omega} \sum_{j=1}^m |g_j(z)|^2 \le \frac{1}{\delta^2}\,.$$

(ii) For all probability measures μ on $\partial\Omega$ and all $h \in P^2(\mu)$ there exists $\{k_j\}_{j=1}^m \subseteq H^\infty(\Omega)$

$$\sum_{j=1}^m f_j k_j = h \quad and \ \sum_{j=1}^m \|k_j\|_\mu^2 \le \frac{1}{\delta^2}\|h\|_\mu^2$$

By results of Andersson-Carlsson [141] for the unit ball and Varopoulos [148], Li [61], Lin [74], Trent [147], and Treil-Wick [70] for the polydisk case, we know that if the input functions are bounded away from 0 on Ω, we have an $H^p(\Omega)$ Theorem (5.1.1) for $1 \le p < \infty$. That is, if

$$\{f_j\}_{j=1}^m \subseteq H^\infty(\Omega) \ \text{and}\ 0 < \epsilon^2 \le \inf_{z\in\Omega} \sum_{j=1}^m \|f_j(z)\|^2 \le 1,$$

then for $1 \le p < \infty$ there exists a $\delta_p > 0$ so that

$$T_F T_F^* \ge \delta_p^2 I_{H^p(\Omega)},$$

where $F = (f_1, f_2, \dots)$. Unfortunately, the best of these estimateshave $\delta_p \downarrow 0$ as $p \uparrow \infty$.

Thus Theorem (5.1.3) tells us that a solution to the Corona problem for $H^p(\Omega)$ follows from the following statement:

$$T_F T_F^* \ge \delta_p^2 I\,, \delta > 0 \overset{?}{\Rightarrow} \exists \epsilon > 0 \qquad \text{such that}\ T_p^\mu (T_p^\mu)^* \ge \epsilon^2 I_\mu$$

for all probability measures μ on $\partial\Omega$.

Necessity in Theorem (5.1.3) is trivial; so we will concentrate on weakening the sufficient conditions to get the same Corona output.

We will extend Theorem (5.1.3) to an infinite number of input functions and refine his theorem, so that we need only consider probability measures, μ, of the form $|H|^2 d\sigma$, where $H \in \mathcal{H}$. In addition, we weaken the hypotheses to just have our operators dominate a certain rank one operator. We begin with a series of lemmas.

Lemma (5.1.4)[139]: Let $\mathcal{F}(z) = [f_{ij}(z)]_{i,j=1}^\infty$, $f_{ij} \in H^\infty(\Omega)$. Then

$$\|T_\mathcal{F}\|_{B\left(\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)\right)} = \sup_{z\in\Omega} \|\mathcal{F}(z)\|_{B(l^2)}\,.$$

Proof. Let $\underline{h} \in \overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)$. Then

$$\|T_\mathcal{F}\underline{h}\|^2_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)} = \sup_{0\le r<1} \left(\int_{\delta\Omega} \|\mathcal{F}(re^{it})\underline{h}(re^{it})\|_{l^2}^2 \, d\sigma \right)$$

$$\le \sup_{z\in\Omega} \int_{\delta\Omega} \|\mathcal{F}(z)\|^2_{B(l^2)} \sup_{0\le r<1} \left(\int_{\delta\Omega} \|\underline{h}(re^{it})\|_{l^2}^2 \, d\sigma \right) \le \sup_{z\in\Omega} \|\mathcal{F}(z)\|^2_{B(l^2)} \|\underline{h}\|^2_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}.$$

For $\underline{x} \in \text{Ball}_1(l^2)$ and $z \in \Omega$

$$\left\| T_{\mathcal{F}}^{*}\left(\underline{x}\frac{k_z}{\|k_z\|_{H^2(\Omega)}}\right)\right\|^2_{\bigoplus_1^\infty H^2(\Omega)} = \left\|\mathcal{F}(z)^*\underline{x}\frac{k_z}{\|k_z\|_{H^2(\Omega)}}\right\|^2_{\bigoplus_1^\infty H^2(\Omega)} = \|\mathcal{F}(z)^*\underline{x}\|^2_{l^2}.$$

Thus,

$$\|T_{\mathcal{F}}\| = \|\mathcal{F}\| \geq \sup_{z\in\Omega}\sup_{\underline{x}\in \mathrm{Ball}_1(l^2)}\|\mathcal{F}(z)^*\underline{x}\|^2_{l^2} \geq \sup_{z\in\Omega}\|\mathcal{F}(z)^*\|_{B(l^2)} = \sup_{z\in\Omega}\|\mathcal{F}(z)\|.$$

For a Hilbert space, K, and vectors $x, y, h \in K$, we let $x \otimes y$denote the rank one operator defined on $K$by

$$(x \otimes y)(h) = \langle h, y\rangle x.$$

The next lemma will be used repeatedly with $A = T_F^H$ and $k = H$, for $H \in \mathcal{H}$.

Lemma (5.1.5)[139]: Assume that for $A \in B(K)$ and $k \in K$with $\|k\|_K = 1,\ AA^* \geq \delta^2 k \otimes k$. Then there exists $u_k \in (\operatorname{Ker} A)^{\perp}$, so that $Au_k = k$ and $\|u_k\|_K \leq \frac{1}{\delta}$.

Proof. By the Douglas Range Inclusion Theorem, see [142], there exists a $C \in B(H, \operatorname{Ker} A^{\perp})$ such that $AC = k \otimes k$and $\|C\| \leq \frac{1}{\delta}$. Let $u_k = Ck$.

Lemma (5.1.6)[139]: For f a positive, bounded, lower semi-continuous function on $\partial\Omega$, there exists a nonvanishing $H \in H^\infty(\Omega)$, so that

$$f = |H|^2\sigma - a.e. on\ \partial\Omega.$$

Proof. For $\Omega = D^n$ and $\partial\Omega = T^n$, this is a result of Rudin [145]. For $\Omega = B^n$ and $\partial\Omega = \partial B^n$, this is a theorem of Alexandrov (see Rudin[146], p. 32).

Recall that

$$\mathcal{H} \triangleq \left\{H \in H^\infty(\Omega): H\text{nonvanishing in } \Omega, \quad \frac{1}{H} \in L^\infty(\delta\Omega, d\sigma), \text{and } \|H\|_2 = 1\right\}.$$

For $\{a_j\}_{j=1}^\infty$ a fixed countable dense set in Ω with $a_1 = 0$, define for each $N = 1, 2, \ldots$

$$\mathcal{C}_N \triangleq co\left\{\frac{\left|k_{a_j}\right|^2}{\left|k_{a_j}\right|_2^2} : j = 1, \ldots, N\right\}$$

Here $k_a(\cdot)$ is the reproducing kernel for $H^2(\Omega)$. It is clear that $\mathcal{C}_N$ is compact and convex in $L^1(\delta\Omega, d\sigma)$.

Calculating, we see that for $\Omega = D^n$ and $g \in \mathcal{C}_N$, we have

$$0 < \left(\frac{1 - \|a\|}{1 + \|a\|}\right)^n \leq g(z) \leq \left(\frac{1 + \|a\|}{1 - \|a\|}\right)^n < \infty \ \ for\ all\ z \in \Omega,$$

where $\|a\| = \max\{\|a_j\|: j = 1, \ldots, N\}$.

For $\Omega = B^n$ and $g \in \mathcal{C}_N$, we have

$$0 < \left(\frac{1 - \|a\|}{1 + \|a\|}\right)^n \leq g(z) \leq \left(\frac{1 + \|a\|}{1 - \|a\|}\right)^n < \infty \ \ for\ all\ z \in \Omega,$$

where $\|a\| = \left(\sum_{j=1}^N \|a_j\|_2^2\right)^{\frac{1}{2}}$.

Note that for $g \in \mathcal{C}_N$, the above calculation shows that, as sets, $P^2(g\ d\sigma)$equals $H^2(\Omega)$.

Assume that $T_F T_F^* \geq \delta^2 1 \otimes 1$ and choose $\underline{x} \in \bigoplus_1^\infty H^2(\Omega)$so that $T_F\underline{x} = 1$ and $\|\underline{x}\|_2 \leq \frac{1}{\delta}$.

For $N = 1, 2, \ldots$ define

$$\mathcal{F}_N : \mathcal{C}_N \times \bigoplus_1^\infty H^2(\Omega) \to [0, \infty)$$

By

$$\mathcal{F}_N(g, \underline{a}) \triangleq \int_{\delta\Omega} \left\|\underline{x} - P_{\mathrm{Ker}(T_F)}\underline{a}\right\|_{l^2}^2 g d\sigma,$$

for $g \in \mathcal{C}_N$ and $\underline{a} \in \bigoplus_1^\infty H^2(\Omega)$.

Since $g \in \mathcal{C}_N$,

$$\underline{x} - P_{\mathrm{Ker}(T_F)}\underline{a} \in \bigoplus_1^\infty H^2(\Omega)$$

and $\mathcal{F}_N(g, \underline{a})$ is finite and positive.

For fixed $\underline{a} \in \bigoplus_1^\infty L^2(d\sigma)$, $g \longmapsto \mathcal{F}_N(g, \underline{a})$ is linear and thus concave on the compact convex set $\mathcal{C}_N$. For fixed $g \in \mathcal{C}_N$, $\underline{a} \longmapsto \mathcal{F}_N(g, \underline{a})$ is convex and continuous on $\bigoplus_1^\infty H^2(\Omega)$.

Lemma (5.1.7)[139]: Assume that $T_F T_F^* \geq \delta^2 1 \otimes 1$. For each $N = 1, 2, \ldots$,

$$\inf_{\underline{a} \in \bigoplus_1^\infty H^2(\Omega)} \sup_{g \in \mathcal{C}_N} \mathcal{F}_N(g, \underline{a}) = \sup_{g \in \mathcal{C}_N} \inf_{\underline{a} \in \bigoplus_1^\infty H^2(\Omega)} \mathcal{F}_N(g, \underline{a}).$$

Proof. By our remarks above, we may apply von Neumann's minimax theorem. See, for example, Gamelin [119].

We are now ready to present our extension of Theorem (5.1.3).

Theorem (5.1.8)[139]: Assume that for some $\delta > 0$, $T_F T_F^* \geq \delta^2 H \otimes H$ for all $H \in \mathcal{H}$. Then there exists a $G \in \bigoplus_1^\infty H^2(\Omega)$ with

$$FG \equiv 1 \text{ in } \Omega \text{ and} \sup_{z \in \Omega} \|G(z)\|_{l^2} \leq \frac{1}{\delta}.$$

That is,

$$T_F T_{G^T} \equiv I \; in \; H^2(\Omega).$$

Proof. Since $T_F T_F^* \geq \delta^2 1 \otimes 1$, we may choose $\underline{x} \in \bigoplus_1^\infty H^2(\Omega)$ so that $T_F \underline{x} = 1$ and $\left\|\underline{x}_0\right\|_2 \leq \frac{1}{\delta}$.

Fix any positive integer, N, and any $g \in \mathcal{C}_N$. By Lemma (5.1.6), we may find an $H \in \mathcal{H}$, so that $|H|^2 = g$ σ-a.e. on $\partial\Omega$.

By our assumption

$$T_F^H (T_F^H)^* \geq \delta^2 H \otimes H,$$

so there exists an $\underline{x}_H \in \bigoplus_1^\infty H^2(\Omega)$ with

$$T_F^H(\underline{x}_H) = 1 \quad and \ \|\underline{x}_H\|_{2,g\,d\sigma} \le \frac{1}{\delta}. \qquad (1)$$

Since $\underline{x} - \underline{x}_H \in \mathrm{Ker}(T_F)$, we have $\underline{x}_H - \underline{x} = P_{\mathrm{Ker}(T_F)}\underline{\alpha}$for $\underline{\alpha} = \underline{x} - \underline{x}_H$.
Thus (1) says that

$$\int_{\partial\Omega} \|\underline{x} - P_{\mathrm{Ker}(T_F)}\underline{\alpha}\|^2 g\,d\sigma = \mathcal{F}_N(g,\underline{\alpha}) \le \frac{1}{\delta^2}.$$

Since this is true for every $g \in \mathcal{C}_N$, we may apply the minimax theorem, Lemma (5.1.7), and deduce that

$$\inf_{\underline{\alpha}\in\bigoplus_1^\infty H^2(\Omega)} \sup_{g\in\mathcal{C}_N} \mathcal{F}_N(g,\underline{\alpha}) \le \frac{1}{\delta^2}. \qquad (2)$$

Then using (2), choose $\underline{\alpha} \in \bigoplus_1^\infty H^2(\Omega)$ so that

$$\int_{\partial\Omega} \|\underline{x} - P_{\mathrm{Ker}(T_F)}\underline{\alpha}\|_{l^2}^2 g\,d\sigma \le \left(\frac{1}{\delta^2} + \frac{1}{N}\right) \ for\ all\ g \in \mathcal{C}_N. \ (3)$$

Since

$$\frac{|k_{a_j}|^2}{\|k_{a_j}\|_2^2} \in \mathcal{C}_N \quad for\ j = 1,2,\dots,N,$$

we see that if

$$G^N \triangleq \underline{x} - P_{\mathrm{Ker}(T_F)}\underline{\alpha},$$

Then

(i) $\|G^{(N)}(a_j)\|_{l^2}^2 \le \int_{\partial\Omega} \|G^{(N)}\|_{l^2}^2 \frac{|k_{a_j}|^2}{\|k_{a_j}\|_2^2} d\sigma \le \frac{1}{\delta^2} + \frac{1}{N}$,

for $j = 1,2,\dots,N$

(ii) $\|G^{(N)}\|_2^2 \le \frac{1}{\delta^2} + \frac{1}{N}$, and

(iii) $FG^{(N)} \equiv 1\ in\ \Omega$.

Repeating this argument for each $N = 1,2,\dots$, we get a sequence of elements, $G^{(N)} \in \bigoplus_1^\infty H^2(\Omega)$, satisfying (i), (ii), and (iii).

By relabeling the sequence of elements, $\{G^{(N)}\}$, if necessary, let G be a weak limit of $\{G^{(N)}\}_{N=1}^\infty$ in $\bigoplus_1^\infty H^2(\Omega)$. Fix any $a_p \in \{a_j\}_{j=1}^\infty$. Then

$$\|G(a_p)\|_{l^2} = \lim_{N\to\infty} \|G^{(N)}(a_j)\|_{l^2} \le \frac{1}{\delta}\ by \text{(b)}.$$

Since $G$is continuous in Ω and $\{a_j\}_{j=1}^\infty$ is dense in Ω, we have shown that

$$\sup_{z\in\Omega} \|G(z)\|_{l^2} \le \frac{1}{\delta}.$$

By (iii),

$$I = \text{weak}\lim_{N\to\infty} T_F(G^{(N)}) = T_F(G).$$

Thus, by Lemma (5.1.4), $T_F T_G = 1$. This completes the proof of Theorem (5.1.8).

We need the fact that $\operatorname{Ker} T_F = \operatorname{Ran} T_{\mathcal{F}}$, for an appropriate analytic $\mathcal{F}$. For $\Omega = B^n$, the unit ball in $\mathbb{C}^n$, the fact that $\operatorname{Ker} T_F = \operatorname{Ran} T_{\mathcal{F}}$follows from results of Andersson and Carlsson [141]. For $\Omega = D^2$, $\operatorname{Ker} T_F = \operatorname{Ran} T_{\mathcal{F}}$follows from Taylor spectrum results of Putinar [144]. That $\operatorname{Ker} T_F = \operatorname{Ran} T_{\mathcal{F}}$in the general case, $\Omega = D^n$, follows from an extension of the techniques of Trent [147] and will appear in a forthcoming concerning the Taylor spectrum of T_F.

The following shows that the Theorem (5.1.1) for the polydisk or unit ball, reduces to an estimation of a lower bound for $T_F^H (T_F^H)^*$ where $H \in \mathcal{H}$, but $H$is not cyclic for $H^2(\Omega)$. (Note that we always have $\frac{1}{H} \in L^\infty(\partial\Omega, d\sigma)$.)

Theorem (5.1.9)[139]: For $H \in \mathcal{H}$ and Hcyclic in $H^2(\Omega)$, then

$$T_F T_F^* (T_F^H)^* \geq \delta^2 1 \otimes 1 \Longrightarrow T_F^H (T_F^H)^* \geq \delta^2 H \otimes H.$$

Proof. To show that, when $H$is cyclic, $T_F^H (T_F^H)^* \geq \delta^2 H \otimes H$, it suffices to find $\underline{u}_H \in \overset{\infty}{\underset{1}{\bigoplus}} H^2(\Omega)$, satisfying

$$F\underline{u}_H = 1 \left(so\ F\left(H\underline{u}_H\right) = H\right) \text{and} \left\|H\underline{u}_H\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)} \leq \frac{1}{\delta} \|H\|_{H^2(\Omega)} = \frac{1}{\delta}. \ (4)$$

Let $\underline{x} = T_F (T_F T_F^*)^{-1} 1$. Then such a $\underline{u}_H$ must have the form $\underline{u}_H = \underline{x} - P_{\operatorname{Ker}(T_F)} \underline{\alpha}$for some $\underline{\alpha} \in \overset{\infty}{\underset{1}{\bigoplus}} H^2(\Omega)$. To see that such an $\underline{\alpha}$ exists, satisfying (4), we compute

$$\inf_{\underline{\alpha} \in \overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)} \int_{\partial\Omega} \left\|\underline{x} - P_{\operatorname{Ker}(T_F)} \underline{\alpha}\right\|_{l^2}^2 |H|^2 \, d\sigma$$

$$= \inf_{\underline{\alpha} \in \overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)} \int_{\partial\Omega} \left\|\underline{x}H - T_{\mathcal{F}}\left(H\underline{\alpha}\right)\right\|_{l^2}^2 d\sigma \left(\text{since} \operatorname{Ker}(T_F) = \operatorname{Ran}(T_{\mathcal{F}})\right)$$

$$= \inf_{\underline{\beta} \in \overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)} \int_{\partial\Omega} \left\|\underline{x}H - T_{\mathcal{F}}\left(\underline{\beta}\right)\right\|_{l^2}^2 d\sigma \ (\text{since } H \text{ is cyclic}) = \left\|P_{\operatorname{ran}(T_{\mathcal{F}})}^{\perp}\left(\underline{x}H\right)\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}^2$$

$$= \left\|P_{\operatorname{Ker}(T_F)}^{\perp}\left(\underline{x}H\right)\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}^2 (\text{since } \operatorname{Ran}(T_{\mathcal{F}}) = \operatorname{Ker}(T_F)) = \left\|P_{\operatorname{ran}(T_F^*)}\left(H\underline{x}\right)\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}^2$$

$$= \left\|T_F^* (T_F T_F^*)^{-1} T_F H T_F^* (T_F T_F^*)^{-1} \underline{1}\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}^2 = \left\|T_F^* (T_F T_F^*)^{-1} H\right\|_{\overset{\infty}{\underset{1}{\oplus}} H^2(\Omega)}^2$$

$$\leq \frac{1}{\delta^2} \|H\|_{H^2(\Omega)}^2 = \frac{1}{\delta^2}.$$

In the case that $n = 1$, we may choose H in Lemma (5.1.6) to be outer and thus cyclic for $H^2(\mathrm{D})$. So Carleson's Theorem (5.1.1) for $H^\infty(\mathrm{D})$ follows from Theorems (5.1.8) and (5.1.9).

A very natural and interesting question arises from our work. Thanks to Treil's remarkable example [67], we know that for an analytic $\mathcal{F} = \left[f_{ij}\right]_{ij=1}^{\infty}$ with

$$\epsilon^2 I_{l^2} \leq \mathcal{F}(z)\mathcal{F}(z)^* \leq I_{l^2} \;\; for\ all\ z \in \Omega,$$

there does not necessarily exist an analytic $\mathcal{G} = [g_{ij}]_{ij=1}^{\infty}$

$$\text{with } \mathcal{F}(z)\mathcal{G}(z) = I_{l^2} \;\; for\ all\ z \in \Omega$$
$$and \; \sup_{z\in\Omega} \|\mathcal{G}(z)\|_{B(l^2)} < \infty.$$

How do we know when such a $\mathcal{G}$must exist? For the case of the unit disk, D, it is necessary and sufficient that there exist a $\delta > 0$ with

$$\delta^2 I \leq T_{\mathcal{F}} T_{\mathcal{F}}^*.$$

For the polydisk and ball in $\mathbb{C}^n$, a natural question is: Does $T_{\mathcal{F}}^H T_{\mathcal{F}}^{H*} \geq \delta^2 I_H$ for some $\delta > 0$ and for all $H \in \mathcal{H}$ imply the existence of a bounded analytic Toeplitz operator $T_{\mathcal{G}}$with

$$T_{\mathcal{F}} T_{\mathcal{G}} = I_{l^2}?$$

For $\mathcal{T}(z)$, a$q \times \infty$matrix with $q < \infty$, a modification of our techniques works, but we only get an estimate

$$\|T_{\mathcal{G}}\| \leq \frac{q}{\delta}.$$

Section (5.2): Toeplitz Corona Problem

While isomorphic Banach algebras of continuous complex-valued functions with the supremum norm can be defined on distinct topological spaces, the results of Gelfand (cf. [5]) showed that for an algebra $A \subseteq C(X)$, there is a canonical choice of domain, the maximal space of the algebra. If the algebra Acontains the function 1, then its maximal ideal space, M_A, is compact. Determining $M_A$for a concrete algebra is not always straightforward. New points can appear, even when the original space $X$is compact, as the disk algebra, defined on the unit circle T, demonstrates. If Aseparates the points of X, then one can identify Xas a subset of $M_A$with a point $x_0$in Xcorresponding to the maximal ideal of all functions in Avanishing at x_0. When $X$is not compact, new points must be present but there is still the question of whether the closure of Xin $M_A$is all of $M_A$or does there exist a "corona" $M_A \backslash X \neq \emptyset$.

The celebrated theorem of Carleson states that the algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions on the unit disk $\mathbb{D}$has no corona. There is a corona problem for $H^\infty(\Omega)$ for every domain Ω in $\mathbb{C}^m$but a positive solution exists only for the case $m = 1$ with Ω a finitely connected domain in $\mathbb{C}$.

One can show with little difficulty that the absence of a corona for an algebra A means that for $\{\varphi_i\}_{i=1}^n$in A, the statement that

$$\sum_{i=1}^{n} |\varphi_i(x)|^2 \geq \varepsilon^2 > 0 \;\text{ for all } x \text{ in } X \qquad (5)$$

is equivalent to the existence of functions$\{\varphi_i\}_{i=1}^n$in$A$ such that

$$\sum_{i=1}^{n} \varphi_i(x)\psi_i(x) = 1 \;\text{ for all } x \text{ in } X \qquad (6)$$

The original proof of Carleson [40] for $H^\infty(\mathbb{D})$ has been simplified over the years but the original ideas remain vital and important. One attempt at an alternate approach, pioneered by Arveson [39] and Schubert [159], and extended by Agler –McCarthy [56], Amar [140], and finally Trent – Wick [139] for the ball and polydisk, involves an analogous question about Toeplitz operators.

In particular, for $\{\varphi_i\}_{i=1}^n$ in $H^\infty(\Omega)$ for $\Omega = \mathbb{B}^m$or $\mathbb{D}^m$, one considers the Toeplitz operator $T_\Phi: H^2(\Omega)^2 \to H^2(\Omega)$ defined $T_\Phi \boldsymbol{f} = \sum_{i=1}^n \varphi_i f_i$ for $\boldsymbol{f}$ in $H^2(\Omega)$, where $\boldsymbol{f} = f_1 \oplus \dots \oplus f_n$ and $\mathcal{X}^n = \mathcal{X} \oplus \dots \oplus \mathcal{X}$ for any space $\mathcal{X}$. One considers the relation between the operator inequality

$$T_\Phi T_\Phi^* \geq \varepsilon^2 I \qquad \text{for some } \varepsilon > 0 \tag{7}$$

and statement (5). One can readily show that (7) implies that one can solve (6) where the functions $\{\psi_i\}_{i=1}^n$are in $H^2(\Omega)$. We will call the existence of such functions, statement (8). The original hope was that one would be able to modify the method or the functions obtained to achieve $\{\psi_i\}_{i=1}^n$in $H^\infty(\Omega)$. That (5) implies (7) follows from earlier work of Andersson – Carlsson [141] for the unit ball and of Varopoulos [148], Li [158], Lin [74], Trent [147] and Treil –Wick [70] for the polydisk.

In the Trent –Wick [139] this goal was at least partially accomplished with the use of (7) to obtain a solution to (8) for the case $m = 1$ and for the case $m > 1$ if one assumes (7) for a family of weighted Hardy spaces. Their method was based on that of Amar [140].

We provide a modest generalization of the result of Trent – Wick in which weighted Hardy spaces are replaced by cyclic submodules or cyclic invariant subspaces of the Hardy space and reinterpretations are given in the language of Hilbert modules for some of their other results. It is believed that this reformulation clarifies the situation and raises several interesting questions about the corona problem and Hilbert modules. Moreover, it shows various ways the Corona Theorem could be established for the ball and polydisk algebras. However, most of our effort is directed at analyzing the proof in [139] and identifying key hypotheses.

A Hilbert module over the algebra $A(\Omega)$, for Ω a bounded domain in $\mathbb{C}^m$, is a Hilbert space $\mathcal{H}$which is a unital module over $A(\Omega)$ for which there exists $C \geq 1$ so that $\|\varphi \cdot f\|_{\mathcal{H}} \leq C\|\varphi\|_{A(\Omega)}\|f\|_{\mathcal{H}}$for φin $A(\Omega)$ and $f$in $\mathcal{H}$. Here $A(\Omega)$ is the closure in the supremum norm over Ω of all functions holomorphic in a neighborhood of the closure of Ω.

We consider Hilbert modules with more structure which better imitate the classical examples of the Hardy and Bergman spaces.

The Hilbert module $\mathcal{R}$over $A(\Omega)$ is said to be quasi-free of multiplicity one if it has a canonical identification as a Hilbert space closure of $A(\Omega)$ such that:

(i) Evaluation at a point $z$in Ω has a continuous extension to $\mathcal{R}$for which the norm is locally uniformly bounded.

(ii) Multiplication by a φin $A(\Omega)$ extends to a bounded operator $T_\varphi$in $\mathcal{L}(\mathcal{R})$.

(iii) For a sequence $\{\varphi_k\}$ in $A(\Omega)$ which is Cauchy in $\mathcal{R}, \varphi_k(z) \to 0$ for all $z$in Ω if and only if $\|\varphi_k\|_{\mathcal{R}} \to 0$.

We normalize the norm on $\mathcal{R}$ so that $\|1\|_{\mathcal{R}} = 1$.

We are interested in establishing a connection between the corona problem for $\mathcal{M}(\mathcal{R})$ and the Toeplitz corona problem on $\mathcal{R}$. Here $\mathcal{M}(\mathcal{R})$ denotes the multiplier algebra for $\mathcal{R}$; that is, $\mathcal{M}\ \big(= \mathcal{M}(\mathcal{R})\big)$ consists of the functions ψ on Ω for which $\psi\mathcal{R} \subset \mathcal{R}$. Since 1 is in $\mathcal{R}$, we see that $\mathcal{M}$ is a subspace of $\mathcal{R}$ and hence consists of holomorphic functions on Ω. Moreover, a standard argument shows that $\psi$is bounded (cf. [154]) and hence $\mathcal{M} \subset H^\infty(\Omega\,)$. In general, $\mathcal{M} \neq H^\infty(\Omega)$.

For ψin $\mathcal{M}$we let $T_\psi$denote the analytic Toeplitz operator in $\mathcal{L}(\mathcal{R})$ defined by module multiplication by ψ. Given functions $\{\varphi_i\}_{i=1}^n$in $\mathcal{M}$, the set is said to

(i) satisfy the corona condition if $\sum_{i=1}^n |\varphi_i(z)|^2 \geq \varepsilon^2$for some $\varepsilon > 0$ and all z in Ω;

(ii) have a corona solution if there exist $\{\psi_i\}_{i=1}^n$ in $\mathcal{M}$ such that $\sum_{i=1}^n \varphi_i(z)\psi_i(z) = 1$ for z in Ω;

(iii) satisfy the Toeplitz corona condition if $\sum_{i=1}^n T_{\varphi_i}T_{\varphi_i}^* \geq \varepsilon^2 I_{\mathcal{R}}$ for some $\varepsilon > 0$;

and

(iv) satisfy the $\mathcal{R}$-corona problem if there exist $\{f_i\}_{i=1}^n$ in $\mathcal{R}$ such that $\sum_{i=1}^n T_{\varphi_i} f_i = 1$ or $\sum_{i=1}^n \varphi_i(z) f(z_i) = 1$ for z in Ω with $\sum_{i=1}^n \|f_i\|^2 \leq 1/\varepsilon^2$.

It is easy to show that (ii) ⇒ (i), (iv) ⇒ (iii) and (ii) ⇒ (iv). As mentioned in the introduction, it has been shown that (i) ⇒ (iii) in case Ω is the unit ball $\mathbb{B}^m$ or the polydisk $\mathbb{D}^m$ and (i) ⇒ (iii) for $\Omega = \mathbb{D}$ is Carleson's Theorem. For a class of reproducing kernel Hilbert spaces with complete Nevanlinna –Pick kernels one knows that (ii) and (iii) are equivalent [152] (cf. [151], [157]). These results are closely related to generalizations of the commutant lifting Theorem [48]. Finally, (iii) ⇒ (iv) results from the range inclusion theorem see (cf. [142]).

Lemma (5.2.1)[149]: If $\{\varphi_i\}_{i=1}^n$ in $\mathcal{M}$ satisfy $\sum_{i=1}^n T_{\varphi_i}T_{\varphi_i}^* \geq \varepsilon^2 I_{\mathcal{R}}$ for some $\varepsilon > 0$, then there exist $\{f_i\}_{i=1}^n$ in $\mathcal{R}$ such that $\sum_{i=1}^n \varphi_i(z) f_i(z) = 1$ for z in Ω and $\sum_{i=1}^n \|f_i\|_{\mathcal{R}}^2 \leq 1/\varepsilon^2$.

Proof. The assumption that $\sum_{i=1}^n T_{\varphi_i}T_{\varphi_i}^* \geq \varepsilon^2 I$ implies that the operator $X: \mathcal{R}^n \to \mathcal{R}$ defined by $X\boldsymbol{f} = \sum_{i=1}^n T_{\varphi_i} f_i$ satisfies $XX^* = \sum_{i=1}^n T_{\varphi_i}T_{\varphi_i}^* \geq \varepsilon^2 I_{\mathcal{R}}$ and hence by [142] there exists $Y: \mathcal{R} \to \mathcal{R}^n$ such that $XY = I_{\mathcal{R}}$ with $\|Y\| \leq \frac{1}{\varepsilon}$. Therefore, with $Y_1 = f_1 \oplus \dots \oplus f_n$, we have $\sum_{i=1}^n \varphi_i(z) f_i(z) = \sum_{i=1}^n T_{\varphi_i} f_i = XY_1 = 1$ and $\sum_{i=1}^n \|f_i\|_{\mathcal{R}}^2 = \|Y\|^2 \|1\|_{\mathcal{R}}^2 \leq 1/\varepsilon^2$. Thus the result is proved.

To compare our results to those in [139], we need the following observations.

Lemma (5.2.2)[149]: Let $\mathcal{R}$ be the Hilbert module $L_a^2(\mu)$ over $A(\Omega)$ defined to be the closure of $A(\Omega)$ in $L^2(\mu)$ for some probability measure μ on clos Ω. For f in $L_a^2(\mu)$, the Hilbert modules $L_a^2(|f|^2 d\mu)$ and $[f]$, the cyclic submodule of $\mathcal{R}$ generated by f, are isomorphic such that $1 \to f$.

Proof. Note that $\|\varphi \cdot 1\|_{L^2(|f|^2 d\mu)} = \|\varphi f\|_{L^2(\mu)}$ for φ in $A(\Omega)$ and the closure of this map sets up the desired isomorphism.

Lemma (5.2.3)[149]: If $\{f_i\}_{i=1}^n$ are functions in $L_a^2(\mu)$ and $g(z) = \sum_{i=1}^n \|f_i(z)\|^2$, then $L_a^2(g\, d\mu)$ is isomorphic to the cyclic submodule $[f_1 \oplus \dots \oplus f_n]$ of $L_a^2(\mu)^n$ with $1 \to f_1 \oplus \dots \oplus f_n$.

In [139], Trent –Wick prove this result and use it to replace the L_a^2 spaces used by Amar [140] by weighted Hardy spaces. However, before proceeding we want to explore the meaning of this result from the Hilbert module point of view.

Lemma (5.2.4)[149]: For $\mathcal{R} = H^2(\mathbb{B}^m)$ (or $H^2(\mathbb{D}^m)$) the cyclic submodule of $\mathcal{R}^n$ generated by $\varphi_1 \oplus \dots \oplus \varphi_n$ with $\{\varphi_i\}_{i=1}^n$ in $A(\mathbb{B}^m)$ (or $A(\mathbb{D}^m)$) is isomorphic to a cyclic submodule of $H^2(\mathbb{B}^m)$ (or $H^2(\mathbb{D}^m)$).

Proof. Combining Lemma 3 in [139] with the observations made in Lemmas (5.2.2) and (5.2.3) above yields the result.

There are several remarks and questions that arise at this point. First, does this result hold for arbitrary cyclic submodules in $H^2(\mathbb{B}^m)^n$ or $H^2(\mathbb{D}^m)^n$, which would require an extension of Lemma 3 in [139] to arbitrary $\boldsymbol{f}$ in $H^2(\mathbb{B}^m)^n$ or $H^2(\mathbb{D}^m)^n$? (This equivalence follows from the fact that a converse to Lemma (5.2.2) is valid.) It is easy to see that the lemma can be extended to an n-tuple of the form $f_1 h \oplus \dots \oplus f_n h$, where the $\{f_i\}_{i=1}^n$ are in

$A(\Omega)$ and $h$is in $\mathcal{R}$. Thus one need only assume that the quotients $\{f_i/f_j\}_{i,j=1}^{n}$are in $A(\Omega)$ or even only equal a.e. to some continuous functions on $\partial\Omega$.

Second, the argument works for cyclic submodules in $H^2(\mathbb{B}^m)\otimes l^2$ or $H^2(\mathbb{D}^m)\otimes l^2$as long as the generating vectors are in $A(\Omega)$ since Lemma 3 in [139] holds in this case also.

Since every cyclic submodule of $H^2(\mathbb{D}^m)\otimes l^2$is isomorphic to $H^2(\mathbb{D}^m)$, the classical Hardy space has the property that all cyclic submodules for the case of infinite multiplicity already occur, up to isomorphism, in the multiplicity one case. Although less trivial to verify, the same is true for the bundle shift Hardy spaces of multiplicity one over a finitely connected domain in $\mathbb{C}$[150].

Third, one can ask if there are other Hilbert modules $\mathcal{R}$that possess the property that every cyclic submodule of $\mathcal{R}\otimes\mathbb{C}^n$or $\mathcal{R}\otimes l^2$is isomorphic to a submodule of $\mathcal{R}$? The Bergman module $L_a^2(\mathbb{D})$ does not have this property since the cyclic submodule of $L_a^2(\mathbb{D})\oplus L_a^2(\mathbb{D})$ generated by $1\oplus z$ is not isomorphic to a submodule of $L_a^2(\mathbb{D})$. If it were, we could write the function $1+|z|^2=|f(z)|^2$ for some $f$in $L_a^2(\mathbb{D})$ which a simple calculation using a Fourier expansion in terms of $\{z^n\bar{z}^m\}$ shows is not possible.

We now abstract some other properties of the Hardy modules over the ball and polydisk.

We say that the Hilbert module $\mathcal{R}$over $A(\Omega)$ has the modulus approximation property (MAP) if for vectors $\{f_i\}_{i=1}^N$in $\mathcal{M}\subseteq\mathcal{R}$, there is a vector $k$in $\mathcal{R}$such that $\|\theta k\|_{\mathcal{R}}^2=\sum_{j=1}^N\|\theta f_i\|^2$for θin $\mathcal{M}$. The map $\theta k\to\theta f_i\oplus\ldots\oplus\theta f_N$thus extends to a module isomorphism of $[k]\subset\mathcal{R}$and $[f_i\oplus\ldots\oplus f_N]\subset\mathcal{R}^N$.

For $z_0$in Ω, let I_{z_0}denote the maximal ideal in $A(\Omega)$ of all functions that vanish at z_0. The quasi-free Hilbert module $\mathcal{R}$over $A(\Omega)$ of multiplicity one is said to satisfy the weak modulus approximation property (WMAP) if

(i) A nonzero vector k_{z_0}in $\mathcal{R}\ominus I_{z_0}.\mathcal{R}$ can be written in the form $k_{z_0}\cdot 1$, where $k_{z_0}$is in $\mathcal{M}$, and $T_{k_{z_0}}$has closed range acting on $\mathcal{R}$. In this case $\mathcal{R}$ is said to have a good kernel function.

(ii)Property MAP holds for $f_i=\lambda_i k_{z_i}, i=1,\ldots,N$with $0\le\lambda_i\le 1$ and $\sum_{i=1}^N\lambda_i^2=1$.

Our main result relating properties (ii) and (iii) is the following one which generalizes Theorem 1 of [139].

Theorem (5.2.5)[149]: Let $\mathcal{R}$be a WMAP quasi-free Hilbert module over $A(\Omega)$of multiplicity one and $\{\varphi_i\}_{i=1}^n$ be functions in $\mathcal{M}$. Then the following are equivalent:

(i) There exist functions $\{\psi_i\}_{i=1}^n$ in $H^\infty(\Omega)$such that $\sum_{i=1}^n\varphi_i(z)\psi_i(z)=1$ and $\sum|\psi_i(z)|\le 1/\varepsilon^2$ for some $\varepsilon>0$ and all z in Ω, and

(ii) there exists $\varepsilon>0$such that for every cyclic submodule $\mathcal{S}$of $\mathcal{R}$, $\sum_{i=1}^n T_{\varphi_i}^{\mathcal{S}}T_{\varphi_i}^{\mathcal{S}^*}\ge\varepsilon^2 I_{\mathcal{S}}$, where $T_\varphi^{\mathcal{S}}=T_\varphi\big|_{\mathcal{S}}$for φin $\mathcal{M}$.

Proof. We follow the proof in [139] making a few changes. Fix a dense set $\{z_i\}_{i=2}^\infty$ of Ω.

First, we define for each positive integer N, the set $\mathcal{C}_N$to be the functions $\left\{\frac{|k_{z_i}|^2}{\|k_{z_i}\|^2}\right\}_{i=1}^2$,

and the function 1 for $i=1$ with abuse of notation. Since $\mathcal{R}$ being WMAP implies that it has a good kernel function, $\mathcal{C}_N$consists of nonnegative continuous functions on Ω. For a function g in the convex hull of the set $\left\{\frac{|k_{z_i}|^2}{\|k_{z_i}\|^2}\right\}_{i=1}^{N}$, the vector $\frac{\lambda_1 k_{z_1}}{\|k_{z_1}\|^2} \oplus \dots \oplus \frac{\lambda_N k_{z_N}}{\|k_{z_N}\|^2}$ is in $\mathcal{R}^N$. By definition there exists G in $\mathcal{R}$ such that $[G] \cong \left[\frac{\lambda_1 k_{z_1}}{\|k_{z_1}\|} \oplus \dots \oplus \frac{\lambda_N k_{z_N}}{\|k_{z_N}\|}\right]$ by extending the map $\theta G \to \frac{\lambda_1 \theta k_{z_1}}{\|k_{z_1}\|} \oplus \dots \oplus \frac{\lambda_N \theta k_{z_N}}{\|k_{z_N}\|}$ for θ in $\mathcal{M}$.

Second, let $\{\varphi_1, \dots, \varphi_n\}$ be in $\mathcal{M}$ and let T_Φ denote the column operator defined from $\mathcal{R}^n$ to $\mathcal{R}$ by $T_\Phi(f_1 \oplus \dots \oplus f_n) = \sum_{i=1}^n T_{\varphi_i} f_i$ for $\boldsymbol{f} = (f_1 \oplus \dots \oplus f_n)$ in $\mathcal{R}^n$ and set $\mathcal{K} = \ker T_\Phi \subset \mathcal{R}^n$. Fix $\boldsymbol{f}$ in $\mathcal{R}^n$. Define the function

$$\mathcal{F}_N: \mathcal{C}_N \times \mathcal{K} \to [0, \infty)$$

by

$$\mathcal{F}_N(g, \boldsymbol{h}) = \sum_{i=1}^{N} \lambda_i^2 \left\| \frac{k_{z_i}}{\|k_{z_i}\|} (\boldsymbol{f} - \boldsymbol{h}) \right\|^2 \text{ for } \boldsymbol{h} = h_1 \oplus \dots \oplus h_n \ in\ \mathcal{R}^n,$$

where $g = \sum_{i=1}^n \lambda_i^2 |k_{z_i}|^2 / \|k_{z_i}\|^2$ and $\sum_{i=1}^n \lambda_i^2 = 1$. We are using the fact that the k_{z_i} are in $\mathcal{M}$ to realize $k_{z_i}(\boldsymbol{f} - \boldsymbol{h})$ in $\mathcal{R}^n$.

Except for the fact we are restricting the domain of $\mathcal{F}_N$ to $\mathcal{C}_N \times \mathcal{K}$ instead of $\mathcal{C}_N \times \mathcal{R}^n$, this definition agrees with that of [139]. Again, as in [139], this function is linear in g for fixed $\boldsymbol{h}$ and convex in $\boldsymbol{h}$ for fixed g. (Here one uses the triangular inequality and the fact that the square function is convex.)

Third, we want to identify $\mathcal{F}_N(g, \boldsymbol{h})$ in terms of the product of Toeplitz operators $\left(T_\Phi^{\mathcal{S}_g}\right)\left(T_\Phi^{\mathcal{S}_g}\right)^*$, where $\mathcal{S}_g$ is the cyclic submodule of $\mathcal{R}$ generated by a vector P in $\mathcal{R}$ as given in Lemma (5.2.3) such that the map $P \to \left(\frac{\lambda_1 k_{z_1}}{\|k_{z_1}\|} \oplus \dots \oplus \frac{\lambda_N k_{z_N}}{\|k_{z_N}\|}\right)$ extends to a module isomorphism with $g = \sum_{i=1}^N \frac{\lambda_i^2 |k_{z_i}|^2}{\|k_{z_i}\|^2}$, $0 \le \lambda_j^2 \le 1$, and $\sum_{i=1}^N \lambda_j^2 = 1$.

Note for $\boldsymbol{f}$ in $\mathcal{R}^n$, $\inf_{\boldsymbol{h} \in \mathcal{K}} \mathcal{F}_N(g, \boldsymbol{h}) \le 1/\varepsilon^2 \|T_\Phi \boldsymbol{f}\|^2$ if $T_\Phi^{\mathcal{S}_g}\left(T_\Phi^{\mathcal{S}_g}\right)^* \ge \varepsilon^2 I_{\mathcal{S}_g}$. Thus, if $T_\Phi^{\mathcal{S}}\left(T_\Phi^{\mathcal{S}}\right)^* \ge \varepsilon^2 I_{\mathcal{S}_g}$ for every cyclic submodule of $\mathcal{R}$, we have $\inf_{\boldsymbol{h} \in \mathcal{K}} \mathcal{F}_N(g, \boldsymbol{h}) \le 1/\varepsilon^2 \|T_\Phi \boldsymbol{f}\|^2$. Thus from the von Neumann min-max theorem we obtain

$$\inf_{\boldsymbol{h} \in \mathcal{K}} \sup_{g \in \mathcal{C}_N} \mathcal{F}_N(g, \boldsymbol{h}) = \sup_{g \in \mathcal{C}_N} \inf_{\boldsymbol{h} \in \mathcal{K}} \mathcal{F}_N(g, \boldsymbol{h}) \le 1/\varepsilon^2 \|T_\Phi \boldsymbol{f}\|^2.$$

From the inequality $T_\Phi T_\Phi^* \ge \varepsilon^2 I_{\mathcal{R}}$, we know that there exists f_0 in $\mathcal{R}^n$ such that $\|f_0\| \le 1/\varepsilon \|1\| = 1/\varepsilon$ and $T_\Phi f_0 = 1$. Moreover, we can find $\boldsymbol{h}_N$ in $\mathcal{K}$ such that $\mathcal{F}_N(g, \boldsymbol{h}_N) \le (1/\varepsilon^2 + 1/N)\|T_\Phi f_0\|^2 = 1/\varepsilon^2 + 1/N$ for all g in $\mathcal{C}_N$. In particular, for $g_i = \frac{|k_{z_i}|^2}{\|k_{z_i}\|^2}$, we have $T_\Phi^{\mathcal{S}_{g_i}}\left(T_\Phi^{\mathcal{S}_{g_i}}\right)^* \ge \varepsilon^2 I_{\mathcal{S}_{g_i}}$, where $\left\|k_{z_i}/\|k_{z_i}\|(f_0 - \boldsymbol{h}_N)\right\|^2 < \frac{1}{\varepsilon^2} + 1/N$.

There is one subtle point here in that 1 may not be in the range of $T_\Phi^{\mathcal{S}}$. However, if P is a vector generating the cyclic module $\mathcal{S}_g$, then P is in $\mathcal{M}$ and T_P has closed range. To see this recall that the map

$$\theta P \to \lambda_1 \frac{\theta k_{z_i}}{\|k_{z_i}\|} \oplus \dots \oplus \lambda_N \frac{\theta k_{z_N}}{\|k_{z_N}\|}$$

for θ in $\mathcal{M}$ is an isometry. Since the functions $\{k_{z_i}/\|k_{z_i}\|\}_{i=1}^{N}$ are in $\mathcal{M}$ by assumption, it follows that the operator $\mathcal{M}_P$is bounded on $\mathcal{M} \subseteq \mathcal{R}$ and has closed range on $\mathcal{R}$ since the operators $M_{k_{z_i}/\|k_{z_i}\|}$ have closed range, again by assumption. Therefore, find a vector $\boldsymbol{f}$ in $\mathcal{S}_g^n$so that $T_\Phi \boldsymbol{f} = P$. But if $\boldsymbol{f} = f_1 \oplus ... \oplus f_n$, then f_i is in $[P]$ and hence has the form $f_i = P\tilde{f}_i$ for $\tilde{f}_i$ in $\mathcal{R}$. Therefore, $T_\Phi T_P \tilde{\boldsymbol{f}} = P$or $T_\Phi \tilde{\boldsymbol{f}} = 1$ which is what is needed since in the proof $f_0 - \tilde{\boldsymbol{f}}$ is in $\mathcal{K}$.

To continue the proof we need the following lemma.

Lemma (5.2.6)[149]: If z_0 is a point in $\Omega$and $\boldsymbol{h}$ is a vector in $\mathcal{R}^n$, then $\|\boldsymbol{h}(z_0)\|_{\mathbb{C}^n}^2 \leq \|k_{z_0}/\|k_{z_0}\|\boldsymbol{h}\|^2$.

Proof. Suppose $\boldsymbol{h} = h_1 \oplus ... \oplus h_n$with $\{h_i\}_{i=1}^n$in $A(\Omega)$. Then $T_{h_i}^* k_{z_0} = \overline{h_\iota(z_0)} k_{z_0}$and hence

$$\overline{h_\iota(z_0)} \|k_{z_0}\|^2 = \langle T_{h_i}^* k_{z_0}, k_{z_0} \rangle = \langle k_{z_0}, T_{h_i} k_{z_0} \rangle$$

since $T_{k_{z_0}} h_i = T_{h_i} k_{z_0}$.(We are using the fact the $k_{z_0} h_i = k_{z_0} h_i \cdot 1 = h_i k_{z_0} = h_i k_{z_0} \cdot$) Therefore,

$$\overline{h_\iota(z_0)} \|k_{z_0}\|^2 = \left|\langle k_{z_0}, T_{k_{z_0}} h_i \rangle\right| \leq \|k_{z_0}\|^2 \left\|T_{k_{z_0}/\|k_{z_0}\|} h_i\right\|,$$

or,

$$\overline{h_\iota(z_0)} \leq \left\|T_{k_{z_0}/\|k_{z_0}\|} h_i\right\|.$$

Finally,

$$\|\boldsymbol{h}(z_0)\|_{\mathbb{C}^n}^2 = \sum_{i=1}^{n} |h_\iota(z_0)|^2 \leq \left\|T_{k_{z_0}/\|k_{z_0}\|} \boldsymbol{h}\right\|^2,$$

and since both terms of this inequality are continuous in the $\mathcal{R}$-norm, we can eliminate the assumption that $\boldsymbol{h}$ is in $A(\Omega)^n$.

Returning to the proof of the theorem, we can apply the lemma to conclude that $\|(\boldsymbol{f}_0 - \boldsymbol{h}_0)(z)\|_{\mathbb{C}^n}^2 \leq \|k_{z_i}/\|k_{z_i}\|(\boldsymbol{f}_0 - \boldsymbol{h}_0)\|^2 \leq 1/\varepsilon^2 + 1/N$. Therefore, we see that the vector $\boldsymbol{f}_N = \boldsymbol{f}_0 - \boldsymbol{h}_N$ in $\mathcal{R}^n$satisfies

(i) $T_\Phi(\boldsymbol{f}_N - \boldsymbol{h}_N) = 1$,

(ii) $\|\boldsymbol{f}_N - \boldsymbol{h}_N\|_{\mathcal{R}}^2 \leq 1/\varepsilon^2 + 1/N \; and$

(iii) $\|(\boldsymbol{f}_N - \boldsymbol{h}_N)(z_i)\|_{\mathbb{C}^n}^2 \leq 1/\varepsilon^2 + 1/N \;\; for \; i = 1, \dots, N$.

Since the sequence $\{\boldsymbol{f}_N\}_{N=1}^{\infty}$ in $\mathcal{R}^n$ is uniformly bounded in norm, there exists a subsequence converging in the weak$*$-topology to a vector ψin $\mathcal{R}^n$. Since weak$*$-convergence implies pointwise convergence, we see that $\sum_{j=1}^{n} \varphi_j \psi_j = 1$ and $\|\psi_j(z_i)\|_{\mathbb{C}^n}^2 \leq \frac{1}{\varepsilon^2}$for all$z_i$. Since $\psi$is continuous on Ω and the set $\{z_i\}$ is dense in Ω, it follows that ψ is in $H_{\mathbb{C}^n}^{\infty}(\Omega)$ and $\|\psi\| \leq 1/\varepsilon^2$ which concludes the proof.

Note that we conclude that ψ is in $H^\infty(\Omega)$ and not in $\mathcal{M}$which would be the hoped for result.

One can note that the argument involving the min-max theorem enables one to show that there are vectors $\boldsymbol{h}$in $\mathcal{K}$which satisfy

$$\|k_{z_i}(\boldsymbol{f} - \boldsymbol{h})\|^2 \leq \frac{1}{\varepsilon^2} + \frac{1}{N}.$$

Moreover, this shows that there are vectors $\tilde{\boldsymbol{f}}$ so that $T_\Phi \tilde{\boldsymbol{f}} = 1, \|\tilde{\boldsymbol{f}}\|^2 \leq 1/\varepsilon^2 + 1/N$, and $\|\tilde{\boldsymbol{f}}(z_i)\|^2 \leq 1/\varepsilon^2 + 1/N$for $i = 1, \dots, N$. An easy compactness argument completes the proof since the sets of vectors for each $N$are convex, compact and nested and hence have a point in common.

With the definitions given, the question arises of which Hilbert modules are (MAP) or which quasi-free ones are WMAP. Lemma (5.2.4) combined with observations in [139] show that both $H^2(\mathbb{B}^m)$ and $H^2(\mathbb{D}^m)$ are WMAP. Indeed any L_a^2 space for a measure supported on $\partial\mathbb{B}^m$or the distinguished boundary of $\mathbb{D}^m$ has these properties. One could also ask for which quasi-free Hilbert modules $\mathcal{R}$the kernel functions $\{k_z\}_{z\in\Omega}$are in $\mathcal{M}$and whether the Toeplitz operators T_{k_z} are invertible operators as they are in the cases of $H^2(\mathbb{B}^m)$ and $H^2(\mathbb{D}^m)$. It seems possible that the kernel functions for all quasi-free Hilbert modules might have these properties when Ω is strongly pseudo-convex, with smooth boundary. In many concrete cases, the $k_{z_0}$are actually holomorphic on a neighborhood of the closure of Ω for $z_0$in Ω, where the neighborhood, of course, depends on z_0.

Note that the formulation of the criteria in terms of a cyclic submodule $\mathcal{S}$ of the quasi-free Hilbert modules makes it obvious that the condition

$$T_\Phi^{\mathcal{S}}\left(T_\Phi^{\mathcal{S}}\right)^* \geq \varepsilon^2 I_{\mathcal{S}}$$

is equivalent to

$$T_\Phi T_\Phi^* \geq \varepsilon^2 I_{\mathcal{R}}$$

if the generating vector for $\mathcal{S}$is a cyclic vector. This is Theorem 2 of [139]. Also it is easy to see that the assumption on the Toeplitz operators for all cyclic submodules is equivalent to assuming it for all submodules. That is because

$$\|(P_{\mathcal{S}} \otimes I_{\mathbb{C}^n})T_\Phi^* f\| \geq \left\|\left(P_{|\boldsymbol{f}|} \otimes I_{\mathbb{C}^n}\right)T_\Phi^* f\right\|$$

for $\boldsymbol{f}$in the submodule $\mathcal{S}$.

If for the ball or polydisk we knew that the function "representing" the modulus of a vector-valued function could be taken to be continuous on $\operatorname{clos}(\Omega)$ or cyclic, the corona problem would be solved for those cases. No such result is known, however and it seems likely that such a result is false.

Finally, one would also like to reach the conclusion that the function $\psi$is in the multiplier algebra even if it is smaller than $H^\infty(\Omega)$. In [153] Costea, Sawyer and Wick this goal is achieved for a family of spaces which includes the Drury –Arveson space. It seems possible that one might be able to modify the line of proof discussed here to involve derivatives of the $\{\varphi_i\}_{i=1}^n$ to accomplish this goal in this case, but that would clearly be more difficult.

Section (5.3): Douglas Property for Free Functions

Free functions can be traced back to the work of Taylor [177], [178] and generalize formal power series which appear in the study of finite automata [176]. They have been of interest for their connections with free probablity and engineering systems theory, [181], [180], [179], [BGT], [169], [170], [161], [171], [172], [173], [174], [175], [162], [163], [164], [165].

We provide a conceptually different proof of a result in [162] of a sufficient condition for the existence of a factorization $b = ac$, for free functions $a, b$and a free contractive-valued function $c$on a free domin determined by free polynomials. As a consquence, the Toeplitz Corona Theorem of [162] is obtained. For more on the Corona and the Toeplitz-Corona problems, see [162], [40], [153], [158], [74], [159], [70], [139], [149].

All Hilbert spaces considered here are Complex and separable. Let $\mathbb{M}(\mathbb{C}^d)$ denote graded set $\left(\mathbb{M}_n(\mathbb{C}^d)\right)_n$, where $\mathbb{M}_n(\mathbb{C}^d)$ is the set of d-tuples $X = (X_1, \dots, X_d)$ of $n \times n$ matrices. Observe that the graded set $\mathbb{M}_n(\mathbb{C}^d)$ is closed with respect to direct sums and unitary conjugations. More generally,

A non-commutative set $\mathcal{L} = \left(\mathcal{L}(n)\right)_n$ is a graded set where $\mathcal{L}(n) \subset \mathbb{M}_n(\mathbb{C}^d)$ such that for $X \in \mathcal{L}(m), Y \in \mathcal{L}(n)$ and a unitary matrix $U \in M_m(\mathbb{C})$,

(i) $X \oplus Y = (X_1 \oplus Y_1, \dots, X_d \oplus Y_d) \in \mathcal{K}(m+n);\ and$

(ii) $U^*XU = (U^*XU_1, \dots, U^*X_dU) \in \mathcal{K}(m)$.

A $B(\mathcal{H}, \mathcal{E})$-valued non-commutative function defined on the non- commutative set $\mathcal{L}$ is a function such that for $X \in \mathcal{L}(m), Y \in \mathcal{L}(n)$,

(i) $f(X) \in B(\mathcal{H} \otimes \mathbb{C}^m, \mathcal{E} \otimes \mathbb{C}^m)$.

(ii) $f(X \oplus Y) = f(X) \oplus f(Y)$

(iii) $f(S^{-1}XS) = (I_{\mathcal{E}} \otimes S^{-1})f(X)(I_{\mathcal{H}} \otimes S)$ whenever $S \in M_m(\mathbb{C})$ is invertible and $S^{-1}XS \in \mathcal{L}(m)$.

We will say that such a function is bounded if $\sup_{n \in \mathbb{N}} E_n < \infty$, where $E_n = \sup_{x \in \mathcal{L}(n)} \|f(X)\|$. Henceforth we will use the abbreviation ”nc” for ”non-commutative”.

A typical example of an nc function is a free polynomial in the d non- commuting variables $x_1, \dots, x_d$, which is defined as follows.

Let $\mathcal{F}_d$ be the semigroup of words formed using the d-symbols $x_1, \dots, x_d$ and the empty word $\emptyset$ denote the identity element of $\mathcal{F}_d$. A $B(\mathbb{C}^k)$-valued free polynomial in the non-commuting variables $x_1, \dots, x_d$ is a finite formal sum of the form $\sum_{w \in \mathcal{F}_d} p_w w$, where $p_w \in B(\mathbb{C}^k)$. For $w = x_{j_1}x_{j_2} \dots x_{j_m}$, the evaluation of p at $X \in \mathbb{M}_n(\mathbb{C}^d)$, is given by $p(X) = \sum_{w \in \mathcal{F}_d} p_w \otimes X^w \in B(C^k \otimes \mathbb{C}^n)$, where $X^w = X_{j_1}X_{j_2} \dots X_{j_m}$. For $0 \in \mathbb{M}_n(\mathbb{C}^d)$, $p(0) := p_\emptyset \otimes I_n$. It is easy to see that p is a $B(\mathbb{C}^k)$-valued nc function defined on the nc set $\mathbb{M}(\mathbb{C}^d)$.

Let ϵ and δ be $B(\mathbb{C}^k)$-valued free polynomials in $x_1, \dots, x_d$ and let $\mathcal{K}$ denote the graded set $\left(\mathcal{K}(n)\right)_n$, where

$$\mathcal{K}(n) = \{X \in \mathbb{M}_n(\mathbb{C}^d) : \exists\, c > 0 \text{ such that } \epsilon(X)\epsilon(X)^* - \delta(X)\delta(X)^* \succ c(I_k \otimes I_n)\}. \quad (8)$$

Observe that the graded set $\mathcal{K} = \left(\mathcal{K}(n)\right)_n$ is an nc set. We will consider this nc set with the additional assumption that $0 \in \mathcal{K}(1)$. Our main result is the following.

A key ingredient in the proof is the existence of a left-invariant Haar probablity measure on the compact group of unitary matrices in $M_n(\mathbb{C})$.

Observe that if $\epsilon = I_k\emptyset$, where $\emptyset \in \mathcal{F}_d$ is the empty word, then $\mathcal{K}$ is the domain $G_\delta = \left(G_\delta(n)\right)$ considered in [162], where

$$G_\delta(n) = \{X = (X_1, \dots, X_d) : \|\delta(X)\| < 1\} \subset \mathbb{M}_n(\mathbb{C}^d), \quad (9)$$

with the additional assumption that $0 \in G_\delta(1)$. The following theorem for the domain G_δ has been proved in [162].

Lemma (5.3.1)[160]: Let X, Y be separable Hilbert spaces and $W \in B(\mathcal{X} \otimes \mathbb{C}^n, Y \otimes \mathbb{C}^n)$. If $W = (I_Y \otimes V)W(I_{\mathcal{X}} \otimes V^*)$ for all unitaries $V \in M_n(\mathbb{C})$, then there exists an operator $\mathcal{W} \in B(\mathcal{X}, Y)$ such that $W = \mathcal{W} \otimes I_n$In.

Proof. The result is an embodiment of the fact that the only $n \times n$ matrices which commute with all $n \times n$ matrices are multiples of the identity. Since $(I_Y \otimes V)W = W(I_{\mathcal{X}} \otimes V)$ for every unitary $V \in M_n(\mathbb{C})$, it follows that

$$(I_Y \otimes X)W = W(I_{\mathcal{X}} \otimes X) \quad (10)$$

for every $X \in M_n(\mathbb{C})$. Let $\{e_1, \dots, e_n\}$ denote an orthonormal basis for $\mathbb{C}^n$and let $E_{j,k} = e_j e_k^*$denote the resulting matrix units. Write $W = \sum W_{j,k} \otimes E_{j,k}$ for operators $W_{j,k}: \mathcal{X} \to Y$. Choosing, for $1 \leq \alpha, \beta \leq n$, the matrix $X = e_\alpha e_\beta^*$, from equation (10) it follows that

$$\sum_k W_{\beta,k} \otimes e_\alpha e_k^* = \sum_j W_{j,\alpha} \otimes e_\alpha e_\beta^* .$$

Hence, $W_{\beta,k} = 0$ for $k \neq \beta$, $W_{j,\alpha} = 0$ for $j \neq \alpha$and $W_{\alpha\alpha} = W_{\beta,\beta}$and the result follows by taking $\mathcal{W} = W_{\alpha\alpha}$.

Lemma (5.3.2)[160]: Let $\mathcal{H}$be a Hilbert space and suppose $A, B \in B(\mathcal{H})$. If $AA^* - BB^* \succ cI$for some $c > 0$, then there exists a unique $E \in B(\mathcal{H})$such that $B^* = E^* A^*$and $\|E^*\| \leq 1$. Moreover, if $\mathcal{H}$is finite dimensional, then E is unique and $\|E^*\| < 1$.

Proof. The Douglas lemma ([142]) implies the existence of a contraction E such that $B = AE$assuming only that $AA^* - BB^* \succcurlyeq 0$. Since the hypotheses imply that $AA^* \succcurlyeq cI$is invertible, in the case that $\mathcal{H}$is finite dimensional, it follows that $A$is invertible and $E = A^{-1}B$is uniquely determined. Moreover, since $A(I - EE^*)A^* \succcurlyeq cI$ and A is invertible, E is a strict contraction.

Let $G^{(n)} = \{U \in M_n(\mathbb{C}): U^*U = I\}$. It is well known that $G^{(n)}$ is a compact group with respect to multiplication. Hence there exists a unique left-invariant Haar measure $h^{(n)}$on $G^{(n)}$such that $h^{(n)}(G) = 1$ and

$$\int_{G^{(n)}} f(U) dh^{(n)}(U) = \int_{G^{(n)}} f(VU) dh^{(n)}(U), \tag{11}$$

for all $f \in C\left(G^{(n)}\right), U, V \in G^{(n)}$. For more details see [40].

Recall the nc set $\mathcal{K}$defined in (8) and the assumption that $0 \in \mathcal{K}(1)$.

Proposition (5.3.3)[160]: Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be Hilbert spaces and suppose that $a$and $b$are bounded $B(\mathcal{E}_2, \mathcal{E}_3)$and $B(\mathcal{E}_1, \mathcal{E}_3)$valued nc-functions on $\mathcal{K}$. There exists a $B(\mathcal{E}_1, \mathcal{E}_2)$valued nc-function $f$such that, for all $n$and $X \in \mathcal{K}(n)$,

(i) $\|f(X)\| \leq 1;$ *and*

(ii) $a(X)\, f(X) = b(X),$

if there exists a $B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_3)$-valued nc function $h$defined on $\mathcal{K}$such that

$$a(T)a(\mathcal{R})^* - b(T)b(\mathcal{R})^* = h(T)[I_{\ell^2} \otimes (\epsilon(T)\epsilon(\mathcal{R})^* - \delta(T)\delta(\mathcal{R})^*)]h(\mathcal{R})^* \\ \text{for all } n \in \mathbb{N} \textit{ and } \mathcal{R}, T \in \mathcal{K}(n). \tag{12}$$

Proof. Fix $n \in \mathbb{N}$. For all $R, T \in \mathcal{K}(n)$, rearranging (12) yields,

$$a(T)a(\mathcal{R})^* + h(T)[I_{\ell^2} \otimes \delta(T)\delta(\mathcal{R})^*]h(\mathcal{R})^* \\ = h(T)[I_{\ell^2} \otimes \epsilon(T)\epsilon(\mathcal{R})^*]h(\mathcal{R})^* + b(T)b(\mathcal{R})^*. \tag{13}$$

Consider the closed subspaces:

$$\mathcal{D}^{(n)} = \overline{span}\left\{\begin{bmatrix} (I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* \\ a(\mathcal{R})^* \end{bmatrix} x : x \in \mathcal{E}_3 \otimes \mathbb{C}^n, \mathcal{R} \in \mathcal{K}(n)\right\},$$

$$\mathcal{R}^{(n)} = \overline{span}\left\{\begin{bmatrix} (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* \\ b(\mathcal{R})^* \end{bmatrix} x : x \in \mathcal{E}_3 \otimes \mathbb{C}^n, \mathcal{R} \in \mathcal{K}(n)\right\},$$

of $(\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_2 \otimes \mathbb{C}^n)$ and $(\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_1 \otimes \mathbb{C}^n)$ respectively.

Let $W(n): \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$be the linear map obtained by extending the map

$$\begin{bmatrix} (I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* \\ a(\mathcal{R})^* \end{bmatrix} x \to \begin{bmatrix} (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* \\ b(\mathcal{R})^* \end{bmatrix} x$$

linearly to all of $\mathcal{D}^{(n)}$. It follows from equation (13) that $W_n: \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$ is an isometry (and hence the map is indeed well defined). Since the codimensions of $\mathcal{D}^{(n)}$and $\mathcal{R}^{(n)}$agree, it follows that $W^n: \mathcal{D}^{(n)} \to \mathcal{R}^{(n)}$can be extended to a unitary $V^{(n)}$. Thus

$$V^{(n)} := \begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} : (\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_2 \otimes \mathbb{C}^n) \to (\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n) \oplus (\mathcal{E}_1 \otimes \mathbb{C}^n)$$

and satisfies

$$\begin{pmatrix} A^{(n)} & B^{(n)} \\ C^{(n)} & D^{(n)} \end{pmatrix} \begin{pmatrix} (I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^* \\ a(\mathcal{R})^* \end{pmatrix} = \begin{pmatrix} (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*) h(\mathcal{R})^* \\ b(\mathcal{R})^* \end{pmatrix}$$

i.e.

$$\sum_{\ell=1}^{k} A^{(n)} (I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^* + B^{(n)} a(\mathcal{R})^* = (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*) h(\mathcal{R})^*, \qquad (14)$$

$$C^{(n)} (I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^* + D^{(n)} a(\mathcal{R})^* + b(\mathcal{R})^*. \qquad (15)$$

Let $U \in G^{(n)}$. Observe that $U^* R U \in \mathcal{K}(n)$. Replacing $\mathcal{R}$in equations (14) and (15) by $U^* R U$yields,

$$A^{(n)} (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(\mathcal{R})^*)\left(I_{\mathcal{E}_3} \otimes U\right) + B^{(n)} \left(I_{\mathcal{E}_2} \otimes U^*\right) a(\mathcal{R})^* \left(I_{\mathcal{E}_3} \otimes U\right)$$
$$= (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \epsilon(\mathcal{R})^*) h(\mathcal{R})^* \left(I_{\mathcal{E}_3} \otimes U\right), \qquad (16)$$

and

$$C^{(n)} (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^* \left(I_{\mathcal{E}_3} \otimes U\right)$$
$$+ D^{(n)} \left(I_{\mathcal{E}_3} \otimes U^*\right) a(\mathcal{R})^* \left(I_{\mathcal{E}_3} \otimes U\right)$$
$$= \left(I_{\mathcal{E}_1} \otimes U^*\right) b(\mathcal{R})^* \left(I_{\mathcal{E}_3} \otimes U\right). \qquad (17)$$

Multiplying equation (16) on the left by $(I_{\ell^2} \otimes I_k \otimes U)$ and on the right by $\left(I_{\mathcal{E}_3} \otimes U^*\right)$ and equation (17) on the left by $\left(I_{\mathcal{E}_1} \otimes U\right)$ and on the left by $\left(I_{\mathcal{E}_3} \otimes U^*\right)$ yields,

$$(I_{\ell^2} \otimes I_k \otimes U) A^{(n)} (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^*$$
$$+ (I_{\ell^2} \times I_k \otimes U) B^{(n)} \left(I_{\mathcal{E}_3} \otimes U^*\right) a(\mathcal{R})^*$$
$$= (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*) h(\mathcal{R})^*, \qquad (18)$$

and

$$\left(I_{\mathcal{E}_1} \otimes U\right) C^{(n)} (I_{\ell^2} \otimes I_k \otimes U^*)(I_{\ell^2} \otimes \delta(\mathcal{R})^*) h(\mathcal{R})^*$$
$$+ \left(I_{\mathcal{E}_1} \otimes U\right) D^{(n)} \left(I_{\mathcal{E}_2} \otimes U^*\right) a(\mathcal{R})^* = b(\mathcal{R})^*. \qquad (19)$$

Let $\tilde{A}^{(n)}, \tilde{B}^{(n)}, \tilde{C}^{(n)}$and $\widetilde{D}^{(n)}$denote the bounded (in fact, contractive) operators that satisfy

$$\langle \tilde{A}^{(n)}x, y\rangle = \int_{G^{(n)}} \langle A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)x, (I_{\ell^2} \otimes I_k \otimes U^*)y\rangle dh^{(n)}(U)$$

$$\langle \tilde{B}^{(n)}a, b\rangle = \int_{G^{(n)}} \langle B^{(n)}\left(I_{\mathcal{E}_2} \otimes U^*\right)a, (I_{\ell^2} \otimes I_k \otimes U^*)b\rangle dh^{(n)}(U)$$

$$\langle \tilde{C}^{(n)}z, w\rangle = \int_{G^{(n)}} \langle C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)z, \left(I_{\mathcal{E}_1} \otimes U^*\right)w\rangle dh^{(n)}(U) \tag{20}$$

$$\langle \widetilde{D}^{(n)}g, h\rangle = \int_{G^{(n)}} \langle D^{(n)}\left(I_{\mathcal{E}_2} \otimes U^*\right)g, \left(I_{\mathcal{E}_1} \otimes U^*\right)h\rangle dh^{(n)}(U)$$

for all $x, y, b, z \in \ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n$; $a, g \in \mathcal{E}_2 \otimes \mathbb{C}^n$; $w, h \in \mathcal{E}_1 \otimes \mathbb{C}^n$. Moreover, For $x \in \mathcal{E}_3 \otimes \mathbb{C}^n$and $y \in \ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n, u \in \mathcal{E}_3 \otimes \mathbb{C}^n$and $v \in \mathcal{E}_1 \otimes \mathbb{C}^n$, it follows from equations (20), (18) and (19) that (21)

$$\langle \left[\tilde{A}^{(n)}(I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* + \tilde{B}^{(n)}a(\mathcal{R})^*\right]x, y\rangle$$

$$= \int_{G^{(n)}} \langle \left[(I_{\ell^2} \otimes I_k \otimes U)A^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)\left((I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^*\right) + (I_{\ell^2} \otimes I_k \otimes U)B^{(n)}\left(I_{\mathcal{E}_2} \otimes U^*\right)a(\mathcal{R})^*\right]x, y\rangle dh^{(n)}(U)$$

$$= \int_{G^{(n)}} \langle (I_{\ell^2} \otimes \varepsilon(\mathcal{R})^*)h(\mathcal{R})^*x, y\rangle dh^{(n)}(U)$$

$$= \langle (I_{\ell^2} \otimes \varepsilon(\mathcal{R})^*)h(\mathcal{R})^*x, y\rangle \tag{21}$$

as well as

$$\langle \left[\tilde{C}^{(n)}(I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* + \widetilde{D}^{(n)}a(\mathcal{R})^*\right]u, v\rangle$$

$$= \int_{G^{(n)}} \langle \left[\left(I_{\mathcal{E}_1} \otimes U\right)C^{(n)}(I_{\ell^2} \otimes I_k \otimes U^*)\left((I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^*\right) + \left(I_{\mathcal{E}_1} \otimes U\right)D^{(n)}\left(I_{\mathcal{E}_2} \otimes U^*\right)a(\mathcal{R})^*\right]u, v\rangle dh^{(n)}(U)$$

$$= \int_{G^{(n)}} \langle b(\mathcal{R})^*u, v\rangle dh^{(n)}(U) = \langle b(\mathcal{R})^*u, v\rangle \tag{22}$$

Equations (21) and (22) together imply that

$$\begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \widetilde{D}^{(n)} \end{pmatrix} \begin{pmatrix} (I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* \\ a(\mathcal{R})^* \end{pmatrix} = \begin{pmatrix} (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* \\ b(\mathcal{R})^* \end{pmatrix}$$

Also, observe that $\begin{pmatrix} \tilde{A}^{(n)} & \tilde{B}^{(n)} \\ \tilde{C}^{(n)} & \widetilde{D}^{(n)} \end{pmatrix}$is a contraction. Lastly, for $V \in G^{(n)}$, the left invariance property of the Haar measure h implies that $\tilde{A}^{(n)}, \tilde{B}^{(n)}, \tilde{C}^{(n)}$ and$\widetilde{D}^{(n)}$are invariant under conjugation by $I \otimes V$and hence

$$\tilde{A}^{(n)} = (I_{\ell^2} \otimes I_k \otimes V)\tilde{A}^{(n)}(I_{\ell^2} \otimes I_k \otimes V^*)$$

$$\tilde{B}^{(n)} = (I_{\ell^2} \otimes I_k \otimes V)\tilde{B}^{(n)}\left(I_{\mathcal{E}_3} \otimes V^*\right)$$
$$\tilde{C}^{(n)} = \left(I_{\mathcal{E}_1} \otimes V\right)\tilde{C}^{(n)}(I_{\ell^2} \otimes I_k \otimes V^*)$$
$$\widetilde{D}^{(n)} = \left(I_{\mathcal{E}_1} \otimes V\right)\widetilde{D}^{(n)}\left(I_{\mathcal{E}_3} \otimes V^*\right).$$

It follows from Lemma (5.3.1) that there exists bounded operators $\mathcal{A}^{(n)}, \mathcal{B}^{(n)}, \mathcal{C}^{(n)}$, and $\mathcal{D}^{(n)}$ such that $\tilde{A}^{(n)} = \mathcal{A}^{(n)} \otimes I_n, \tilde{B}^{(n)} = \mathcal{B}^{(n)} \otimes I_n, \tilde{C}^{(n)} = \mathcal{C}^{(n)} \otimes I_n$ and $\widetilde{D}^{(n)} = \mathcal{D}^{(n)} \otimes I_n$, where $\mathcal{A}^{(n)} \in B(\ell^2 \otimes \mathbb{C}^k), \mathcal{B}^{(n)} \in B(\mathcal{E}_2, \ell^2 \otimes \mathbb{C}^k), \mathcal{C}^{(n)} \in B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_1)$ and $\mathcal{D}^{(n)} \in B(\mathcal{E}_2, \mathcal{E}_1)$. Moreover,

$$\begin{pmatrix} \mathcal{A}^{(n)} & \mathcal{B}^{(n)} \\ \mathcal{C}^{(n)} & \mathcal{D}^{(n)} \end{pmatrix} : (\ell^2 \otimes \mathbb{C}^k) \otimes \mathcal{E}_2 \to (\ell^2 \otimes \mathbb{C}^k) \otimes \mathcal{E}_1$$

is a contraction.

Let $\mathcal{H} = (\ell^2 \otimes \mathbb{C}^k) \otimes \mathcal{E}_2$ and $\mathcal{E} = (\ell^2 \otimes \mathbb{C}^k) \otimes \mathcal{E}_1$. Observe that $\mathcal{H} \oplus \mathcal{E}$ is separable. At this point, it has been proved that there exists an operator $\mathcal{V} \in B(\mathcal{H}, \mathcal{E})$ such that $\|\mathcal{V}\| \leq 1$ and

$$\mathcal{V} \otimes I_n \begin{pmatrix} (I \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* \\ a(\mathcal{R})^* \end{pmatrix} = \begin{pmatrix} (I \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* \\ b(\mathcal{R})^* \end{pmatrix} \tag{23}$$

Let

$$L_n = \left\{\begin{pmatrix} 0 & 0 \\ \mathcal{V} & 0 \end{pmatrix} : \|\mathcal{V}\| \leq 1 \text{ and} (\mathcal{V} \otimes I_n) \text{solves} (16)\right\} \subset B(\mathcal{H} \oplus \mathcal{E}).$$

The argument above implies that $L_n \neq \emptyset$ for each $n \in \mathbb{N}$. It is also the case that L_n is a WOT-closed subset of the WOT-compact unit ball of $B(\mathcal{H} \oplus \mathcal{E})$. Thus L_n is WOT-compact for each $n \in \mathbb{N}$. Moreover since $0 \in \mathcal{K}(1)$, it follows that $L_n \supset L_{n+1}$. By the nested intersection property of compact sets, $\bigcap_{n\in\mathbb{N}} L_n$ is non-empty. Say $\begin{pmatrix} 0 & 0 \\ \mathcal{V} & 0 \end{pmatrix} \in \bigcap_{n\in\mathbb{N}} L_n$, where $V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in B\ (\ell^2 \otimes \mathbb{C}^k),\ B \in B\ (\mathcal{E}_2, \ell^2 \otimes \mathbb{C}^k),\ C \in B\ (\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_1)$ and $D \in B(\mathcal{E}_2, \mathcal{E}_1)$.

For all $n \in \mathbb{N}$ and $R \in \mathcal{K}(n)$, we have,

$$(A \otimes I_n)(I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* + (B \otimes I_n)a(\mathcal{R})^* = (I_{\ell^2} \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* \tag{24}$$

$$(C \otimes I_n)(I_{\ell^2} \otimes \delta(\mathcal{R})^*)h(\mathcal{R})^* + (D \otimes I_n)a(\mathcal{R})^* = b(\mathcal{R})^* \tag{25}$$

By Lemma (5.3.2), for each $n \in \mathbb{N}$ and $R \in \mathcal{K}(n)$ there exists a uniquely determined strict contraction $\gamma(R) \in B(\mathbb{C}^k \otimes \mathbb{C}^n)$ such that

$$\delta(R)^* = \gamma(R)^*\epsilon(\mathcal{R})^*. \tag{26}$$

Since $\|A \otimes I_n\| \leq 1$ and $\|\gamma(\mathcal{R})^*\| < 1$, rearranging equation (24) and using (26) yields,

$$(I_{\ell^2} \otimes \epsilon(\mathcal{R})^*)h(\mathcal{R})^* = \{I_{\ell^2} \otimes I_k \otimes I_n - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n)a(R)^*. \tag{27}$$

Using (27) and (26) in (25) yields,

$$[(C \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\{I_{\ell^2} \otimes I_k \otimes I_n - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n) + (D \otimes I_n)]a(R)^* = b(R)^*. \tag{28}$$

For $n \in \mathbb{N}, R \in \mathcal{K}(n)$, define the function f on $\mathcal{K}$ by

$$f(R) = [(C \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\{I_{\ell^2} \otimes I_k \otimes I_n - (A \otimes I_n)(I_{\ell^2} \otimes \gamma(R)^*)\}^{-1}(B \otimes I_n) + (D \otimes I_n)]^* \tag{29}$$

Thus f is a $B(\mathcal{E}_1, \mathcal{E}_2)$-valued graded function which satisfies $a(R)f(R) = b(R)$. It is also easy to see that f preserves direct sums.

Finally, to show that f is an nc function, suppose $R \in \mathcal{K}(n)$ and S is an invertible $n \times n$ matrix such that $S^{-1}RS \in \mathcal{K}(n)$. We need to show that $f(S^{-1}RS) = (I_{\mathcal{E}_2} \otimes S^{-1}) f(R) (I_{\mathcal{E}_1} \otimes S)$. Observe that $\gamma(R)^*$ is uniquely determined by (26), since $\epsilon(\mathcal{R})^*$ is invertible. From the form of f, it is enough to show $\gamma(S^{-1}RS) = (I_k \otimes S^{-1})\gamma(R)(I_k \otimes S)$. To this end, observe that,

$$(I_k \otimes S^*)\delta(R)^*(I_k \otimes (S^*)^{-1}) = \delta(S^{-1}RS)^* = \gamma(S^{-1}RS)^*\epsilon(S^{-1}RS)^*$$
$$= \gamma(S^{-1}RS)^*(I_k \otimes S^*)\epsilon(R)^*(I_k \otimes (S^*)^{-1}). \quad (30)$$

Thus

$$(I_k \otimes S^*)\gamma(R)^*\epsilon(R)^*(I_k \otimes (S^*)^{-1}) = \gamma(S^{-1}RS)^*(I_k \otimes S^*)\epsilon(R)^*(I_k \otimes (S^*)^{-1}).$$

Since $\epsilon(R)^*(I_k \otimes (S^*)^{-1})$ is invertible, taking adjoints, it follows that

$$(I_k \otimes S^{-1})\gamma(R)(I_k \otimes S) = \gamma(S^{-1}RS).$$

The proof is complete if we show that $\|f(R)\| \leq 1$ for every $n \in \mathbb{N}$ and $R \in K(n)$.

Recall that for all $n \in \mathbb{N}$, $V \otimes I_n = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} = \begin{pmatrix} A \otimes I_n & B \otimes I_n \\ C \otimes I_n & D \otimes I_n \end{pmatrix}$ is a contraction. Thus there exists bounded operators $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\begin{pmatrix} \mathcal{P}^*\mathcal{P} & \mathcal{P}^*\mathcal{Q} \\ \mathcal{Q}^*\mathcal{P} & \mathcal{Q}^*\mathcal{Q} \end{pmatrix} = \begin{pmatrix} I_{\ell^2 \otimes \mathbb{C}^k \otimes \mathbb{C}^n} & 0 \\ 0 & I_{\mathcal{E}_2 \otimes \mathbb{C}^n} \end{pmatrix} - \begin{pmatrix} \mathcal{A}^* & \mathcal{B}^* \\ \mathcal{C}^* & \mathcal{D}^* \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \succcurlyeq 0.$$

For notational convenience, let $\Gamma(R) := (I_{\ell^2} \otimes \gamma(R)^*)$, $\Delta(R) := (I_{\ell^2} \otimes I_k \otimes I_n - A\Gamma(R))$ and $\Phi(R) := \Delta(R)^{-1}$. We have $f(R)^* = \mathcal{D} + \mathcal{C}\Gamma(R)^*(R)\mathcal{B}$.

Using equation (9), for $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$, we have

$$\begin{aligned}
&(I_{\mathcal{E}_2} \otimes I_n) - f(R)f(R)^* \\
&= (I_{\mathcal{E}_2} \otimes I_n) - \mathcal{D}^*\mathcal{D} - \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{C}^*\mathcal{D} - \mathcal{D}^*\mathcal{C}\Gamma(R)\Phi(R)\mathcal{B} \\
&- \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{C}^*\mathcal{C}\Gamma(R)\Phi(R)\mathcal{B} \\
&= \mathcal{Q}^*\mathcal{Q} + \mathcal{B}^*\mathcal{B} + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*(\mathcal{A}^*\mathcal{B} + \mathcal{P}^*\mathcal{Q}) + (\mathcal{B}^*\mathcal{A} + \mathcal{Q}^*\mathcal{P})\Gamma(R)\Phi(R)\mathcal{B} \\
&- \mathcal{B}^*\Phi(R)^*\Gamma(R)^*(I - \mathcal{A}^*\mathcal{A} + \mathcal{P}^*\mathcal{P})\Gamma(R)\Phi(R)\mathcal{B} \\
&= \mathcal{B}^*\Phi(R)^*[\Delta(R)^*\Delta(R) + \Gamma(R)^*\mathcal{A}^*\Delta(R) + \Delta(R)^*\mathcal{A}\Gamma(R) \\
&- \Gamma(R)^*(I - \mathcal{A}^*\mathcal{A})\Gamma(R)]\Phi(R)\mathcal{B} + \mathcal{Q}^*\mathcal{Q} + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{P}^*\mathcal{Q} \\
&+ \mathcal{Q}^*\mathcal{P}\Gamma(R)\Phi(R)\mathcal{B} + \mathcal{B}^*\Phi(R)^*\Gamma(R)^*\mathcal{P}^*\mathcal{P}\Gamma(R)\Phi(R)\mathcal{B} \\
&= \mathcal{B}^*\Phi(R)^*[I - \Gamma(R)^*\Gamma(R)]\Phi(R)\mathcal{B} \\
&+ (\mathcal{Q} + \mathcal{P}\Gamma(R)\Phi(R)\mathcal{B})^*(\mathcal{Q} + \mathcal{P}\Gamma(R)\Phi(R)\mathcal{B}) \succcurlyeq 0.
\end{aligned}$$

Theorem (5.3.4)[160]: Let $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ be finite-dimensional Hilbert spaces and suppose that a and b are bounded $B(\mathcal{E}_2, \mathcal{E}_3)$ and $B(\mathcal{E}_1, \mathcal{E}_3)$ valued nc-functions on $\mathcal{K} = G_\delta$. The following are equivalent.

(i) There exists a $B(\ell^2 \otimes \mathbb{C}^k, \mathcal{E}_3)$ valued nc-function h defined on $\mathcal{K}$ such that $a(T)a(\mathcal{R})^* - b(T)b(\mathcal{R})^* = h(T)[I_{\ell^2} \otimes (I_k \otimes I_n) - \delta(T)\delta(\mathcal{R})^*]h(\mathcal{R})^*$ for all $n \in \mathbb{N}$ *and* $\mathcal{R}, T \in \mathcal{K}(n)$.

(ii) There exists a bounded $B(\mathcal{E}_1, \mathcal{E}_2)$ valued nc-function f such that $\|f(X)\| \leq 1$ and $a(X)f(X) = b(X)$, for all $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$.

(iii) $a(X)a(X)^* - b(X)b(X)^* \succcurlyeq 0$ *for all* $n \in \mathbb{N}$ *and* $X \in \mathcal{K}(n)$.

It is immediate that a proof of the implication (i) ⇒ (ii) of Theorem (5.3.4), follows from Proposition (5.3.3) by taking $\epsilon = I_k\emptyset$. Thus the proof given here of Proposition (5.3.3), exploiting the Haar measure, provides an alternate and conceptually different proof of (i) ⇒ (ii) than the one given in [162].

Proof. (i) implies (ii): Follows from Proposition (5.3.3), by letting $\epsilon = I_k\emptyset$.

(ii) implies (iii): Observe that for each $n \in \mathbb{N}$ and $X \in \mathcal{K}(n)$,

$$a(X)a(X)^* - b(X)b(X)^* = a(X)a(X)^* - a(X)f(X)f(X)^*a(X)^*$$
$$= a(X)\big(I_{\mathcal{E}_2} \otimes I_n - f(X)f(X)^*\big)a(X)^* \succcurlyeq 0. \quad (31)$$

(iii) implies (i): This is the content of Theorem 7.10 in [162].

Recall the non-commutative set $G_\delta = \big(G_\delta(n)\big)_n$ from (9). The following is the Toeplitz-Corona theorem of [162] for the non-commutative domain $G_\delta = \big(G_\delta(n)\big)$ with the assumption that $0 \in G_\delta(1)$. Observe that certain well-known non-commutative domains, for example, the non-commutative polydisc, can be realized as such G_δ, for suitable δ.

Theorem (5.3.5)[160]: Let $a_1, \dots, a_\ell$ be bounded $\mathbb{C}$-valued nc–functions defined on G_δ and $\mu > 0$. If for all $n \in \mathbb{N}$ and $R \in G_\delta(n)$, $\sum_{i=1}^{\ell} a_i(R)a_i(R)^* \succcurlyeq \mu^2 I_n$, then there exists $\mathbb{C}$-valued nc functions $g_1, \dots, g_\ell$ defined on G_δ such that $\sum_{i=1}^{\ell} a_i(R)g_i(R) = I_n$ for each $n \in \mathbb{N}$ and $R \in G_\delta(n)$. Moreover the $B\big(\mathbb{C}, \mathbb{C}^\ell\big)$valued nc function g satisfies $\|g(R)\| \le \frac{1}{\mu}$ for all $n \in \mathbb{N}$ and $R \in G_\delta(n)$, where $g(R) = e_1 \otimes g_1(R) + \cdots e_\ell \otimes g_j(R)$ and $e_1, e_2, \dots, e_\ell$ are the standard unit (column) vectors in $\mathbb{C}^\ell$.

Proof. Letting $\mathcal{E}_1 = \mathcal{E}_3 = \mathbb{C}$ and $\mathcal{E}_2 = \mathbb{C}^\ell, a(R) = e_1^* \otimes a_1(R) + \cdots + e_\ell^* \otimes a_\ell(R)$ and $b(R) = \mu I_n$ for $R \in G_\delta(n)$ in Theorem (5.3.4), the hypothesis becomes $a(R)a(R)^* - b(R)b(R)^* \succcurlyeq 0$. Theorem (5.3.4) now implies that there exists a $B\big(\mathbb{C}, \mathbb{C}^\ell\big)$ valued nc function f such that $\|f(R)\| \le 1$ and
$$\big[e_1^* \otimes a_1(R) + \cdots + e_\ell^* \otimes a_j(R)\big]f(R) = \mu I_n. \quad (32)$$
Choose $\mathbb{C}$ -valued nc functions $f_1, \dots, f_\ell$ such that $f(R) = e_1 \otimes f_1(R) + \cdots\ e_\ell \otimes f_\ell(R)$. Using this in equation (32) yields,
$$\sum_{i=1}^{\ell} a_i(R)f_i(R) = \mu I_n.$$
Taking $g_i = \frac{1}{\mu} f_i; i = 1, 2, \dots, \ell$, completes the proof.

Let Λ denote a linear $r \times r$ matrix-valued nc polynomial,
$$\Lambda(x) = \sum_{j=1}^{g} A_j x_j,$$
where the A_j are $r \times r$ matrices. The corresponding linear pencil is the expression
$$L(x) = I - \Lambda(x) - \Lambda^*(x),$$
where Λ^* is the formal adjoint of Λ determined by,
$$\Lambda^*(X) = \Lambda(X)^*$$
for tuples $X = (X_1, \dots, X_d)$ of $n \times n$ matrices. In this case the graded set $\mathcal{K} = \big(\mathcal{K}(n)\big)_n$ is known as a free (non-commutative) spectrahedron (See [168]). A bit of algebra shows
$$L(x) = (I - \Lambda)(x)(I - \Lambda)(x)^* - \Lambda(x)\Lambda(x)^*.$$

Chapter 6
Toeplitz Algebra on the Bergman Space of the Unit Ball in $\mathbb{C}^n$ and Localization the Commutator Ideal

We can show that for $n > 1$, the closed bilateral ideal generated by operators of the above form, where f , g can be required to be continuous on the open unit ball or supported in a nowhere dense set, is also all of $\mathfrak{T}$. The main results extend results of Xia and Zheng to the case of the Bergman space. In the case of the Bargmann-Fock space, our results provide new, more general conditions, that imply the work of Xia and Zheng via a more familiar approach. We show that the norm closure of $\{T_f : f \in L^1(B; dv)\}$ actually coincides with the Toeplitz algebra $\mathfrak{T}$, i.e., the C^*-algebra generated by $\{T_f : f \in L^1(B; dv)\}$. A key ingredient in the proof is the class of weakly localized operators recently introduced by Isralowitz, Mitkovski and Wick. The approach simultaneously gives us the somewhat surprising result that $\mathfrak{T}$ also coincides with the C^*-algebra generated by the class of weakly localized operators.

Section (6.1): Unit Ball in $\mathbb{C}^n$

For $n > 1$, let $\mathbb{C}^n$ denote the cartesian product of n copies of $\mathbb{C}$. For any two points $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we use the notations $\langle z, w\rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$ and $|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$ for the inner product and the associated Euclidean norm. Let B_n denote the open unit ball which consists of points $z \in \mathbb{C}^n$ with $|z| < 1$. Let dvdenote the Lebesgue measure on B_n so normalized that $v(B_n) = 1$. Let $d\mu(z) = (1 - |z|^2)^{-n-1} dv(z)$. Then $d\mu$ is invariant under the action of the group of automorphisms $\mathrm{Aut}(B_n)$ of B_n. Even though $d\mu$ is an infinite measure on B_n, it will be very useful for us later.

Let $L^2 = L^2(B_n, dv)$ and $L^\infty = L^\infty(B_n, dv)$. The Bergman space L_a^2 is the subspace of L^2 which consists of all holomorphic functions. The orthogonal projection from L^2 onto L_a^2 is given by

$$Pf(z) = \int_{B_n} \frac{f(w)}{(1 - \langle z, w\rangle)^{n+1}} dv\,(w) \ \ , f \in L^2 , z \in B_n.$$

The normalized reproducing kernels for L_a^2 are of the form

$$k_z(w) = (1 - |z|^2)^{(n+1)/2}(1 - \langle w, z\rangle)^{-n-1}, \qquad |z|, |w| < 1.$$

We have $\|k_z\| = 1$ and $\langle g, k_z\rangle = (1 - |z|^2)^{(n+1)/2} g(z)$ for all $g \in L_a^2$.

Let $\mathfrak{B}(L_a^2)$ be the C^*-algebra of all bounded linear operators on L_a^2. Let $\mathcal{K}$ denote the ideal of compact operators on L_a^2.

For any $\eta \in L^\infty$ let $M_\eta: L^2 \to L^2$ be the operator of multiplication by η and $P_\eta = PM_\eta$. Then $\|P_\eta\| \le \|\eta\|_\infty$. The Toeplitz operator $T_\eta: L_a^2 \to L_a^2$ is the restriction of $P_\eta$to L_a^2. For any subset G of L^∞, let $\mathfrak{T}(G)$ denote the C^*-subalgebra of $\mathfrak{B}(L_a^2)$generated by $\{T_\eta : \eta \in G\}$. The commutator ideal of this algebra is denoted by $\mathfrak{CT}(G)$. It is well-know that $\mathfrak{CT}(C(\bar{B}_n))$ is the same as $\mathcal{K}$, see [13].

The algebra $\mathfrak{T}(L^\infty)$ which is generated by all Toeplitz operators with bounded symbols is called the full Toeplitz algebra. Its commutator ideal is $\mathfrak{CT}(L^\infty)$.

There have been many results on commutator ideals and abelianizations of Toeplitz algebras acting on Hardy spaces. In contrast with this, there are only few results for Toeplitz algebras on Bergman spaces. Suárez showed in [81] that the Toeplitz algebra $\mathfrak{T}(L^\infty)$ on the Bergman space of the unit disk coincides with its commutator ideal $\mathfrak{CT}(L^\infty)$. Suárez used some explicit computations and identities which are readily available on the unit disk to construct a function $\eta \in L^\infty$ with the property that $\eta > c > 0$ on the disk and T_η is in the commutator ideal $\mathfrak{CT}(L^\infty)$. In higher dimensions, the computations become more complicated and some of the identities which were used by Suárez are not available. We could not find a way to get around these difficulties to construct a function similar to that of Suárez so we tried a different approach. It turns out that our new approach gives more general results about commutator ideals of the Toeplitz algebras. Indeed, we do not need G to be all the functions in L^∞ to get $\mathfrak{CT}(G) = \mathfrak{T}(G)$. We can take G to be $L^\infty \cap C(B_n)$, the set of all bounded continuous functions on the open unit ball, or we can take G to be all the functions in L^∞ which are supported in a set E where E can be a nowhere dense set with $\nu(E)$ as small as we please.

We next describe a metric on the unit ball which we will mainly use here. For any $z \in B_n$, let φ_z denote the Mobius automorphism of B_n that interchanges 0 and z. For any $z, w \in B_n$, let $\rho(z,w) = |\varphi_z(w)|$. Then ρ is a metric which is invariant under the action of the group of automorphisms $\mathrm{Aut}(B_n)$ of B_n. These properties of ρcan be proved by using identities in [79]. Further discussion of this metric will appear later.

A collection $\mathcal{W} = \{w_j : j \in J\}$ of points in B_n is said to be separated if $r = \inf\{\rho(w_j, w_k) : j \neq k\} > 0$. It is a consequence of Lemma (6.1.2) that in this case the index set J is necessarily at most countable. The number r is called the degree of separation of $\mathcal{W}$.

For $z \in B_n$ and $0 < r < 1$, let

$$E(z,r) = \{w \in B_n : \rho(w,z) \leq r\}$$

denote the closed r-ball centered at z in the ρmetric.

Theorem (6.1.1)[182]: Let $\{w_j : j \in \mathbb{N}\}$ be a separated sequence of points in B_n so that $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R)$ for some $0 < R < 1$. Let η be a measurable function defined on $[0,\infty)$ with $\eta \geq 0, \eta(t) = 0$ if $t \leq 1$ and $\|\eta\|_\infty = 1$. For each $0 < \varepsilon < 1$ put $\eta_\varepsilon(z) = z_1\eta(|z|/\varepsilon)$. Let G_ε be the set of all functions of the form $\sum_{j\in F} \eta_\varepsilon \circ \varphi_{w_j}$ or $\sum_{j\in F} \bar{\eta}_\varepsilon \circ \varphi_{w_j}$ where F is a subset of $\mathbb{N}$. Then the operator

$$A_\varepsilon = \sum_{j\in\mathbb{N}} \left[T_{\eta_\varepsilon \circ \varphi_{w_j}}, T_{\bar{\eta}_\varepsilon \circ \varphi_{w_j}}\right]^2$$

belongs to the commutator ideal $\mathfrak{CT}(G_\varepsilon)$. Furthermore, for all but countably many ε, the operator $A_\varepsilon$is invertible.

Put $E_\varepsilon = \bigcup_{j\in\mathbb{N}} \varphi_{w_j}(\mathrm{supp}(\eta_\varepsilon))$. Then $G_\varepsilon$is contained in the sub-space $\{\zeta \in L^\infty : \zeta \text{ is supported on} E_\varepsilon\}$. If $\eta$is supported in a nowhere dense subset of $[0,1]$then $\eta_\varepsilon$is supported in a nowhere dense subset of B_n, hence E_ε, being the union of a locally finite collection of nowhere dense sets, is a nowhere dense subset of B_n, too. Furthermore, we will show that for $\varepsilon > 0$, the Lebesgue measure of E_ε is $O(\varepsilon^{2n})$. We will also show that if η is a continuous function then $G_\varepsilon$is a subspace of $C(B_n)$ for all $0 < \varepsilon < 1$.

The fact that A_ε belongs to the ideal $\mathfrak{CT}(G_\varepsilon)$ is proved exactly as in Suárez's. The reason is that all the properties of the metric ρ and the kernel functions which were crucial for Suárez's proof hold true in higher dimensions.

The invertibility of A_ε follows from a general fact about operators which are diagonalizable with respect to the standard orthonormal basis of L_a^2. In fact, sums of a “large enough” number of operators which are unitarily equivalent to operators of the above type are invertible. This is the content of Theorem (6.1.10) which follows.

For any $z \in B_n$, the formula

$$\mathcal{U}_z(f) = (f \circ \varphi_z)k_z, \qquad f \in L^2$$

defines a bounded operator on L^2. It is well-known that $\mathcal{U}_z$ is a unitary self-adjoint operator and $\mathcal{U}_z T_\eta \mathcal{U}_z^* = T_{\eta\circ\varphi_z}$ for all $z \in B_n$ and all $\eta \in L^\infty$, see, for example, Lemma 7 and 8 in [183].

Also a simple computation reveals that for all $z, w \in B_n$,

$$\mathcal{U}_z(k_w) = \left(\frac{|1-\langle z,w\rangle|}{1-\langle z,w\rangle}\right)^{n+1} k_{\varphi_z}(w).$$

This implies

$$\mathcal{U}_z(k_w \otimes k_w)\mathcal{U}_z^* = k_{\varphi_z(w)} \otimes k_{\varphi_z(w)}.$$

Now for any multi-index $\alpha = (\alpha_1, \alpha_2 \dots, \alpha_n)$, let $|\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1!, \dots, \alpha_n!$ and $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$. Put

$$e_\alpha = \left(\frac{(n+|\alpha|)!}{n!\,\alpha!}\right)^{1/2} z^\alpha.$$

Then $\{e_\alpha : \alpha \in \mathbb{N}^n\}$is the standard orthonormal basis for L_a^2, see Proposition 1.4.9 in [79].

Recall that for any two nonzero elements f and g in $L_a^2, f \otimes g$ denotes the rank one operator $(f \otimes g)u = \langle u, g\rangle f$, for all $u \in L_a^2$.

The following inequalities illustrate the fact that the metric ρ in higher dimensions also possesses all the properties used in Suárez’s. These results are well-known but since we are not aware of an appropriate reference, we sketch here a proof.

Lemma (6.1.2)[182]: For any z, w in B_n, the followings hold:

$$\left|\frac{|z|-|w|}{1-|z||w|}\right| \le \rho(z,w) \le \frac{|z-w|}{1-\langle z,w\rangle}.$$

Proof. Using $|\langle z,w\rangle| \le |z||w|$, we get the inequalities

$$1-\frac{|z-w|^2}{|1-\langle z,w\rangle|^2} \le \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \le \frac{(1-|z|^2)(1-|w|^2)}{(1-|z||w|)^2}.$$

Combining the above inequalities with the identity

$$1-|\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \text{(see Theorem 2.2.2 in [79])}$$

we obtain

$$1-\frac{|z-w|^2}{|1-\langle z,w\rangle|^2} \le 1-|\varphi_z(w)|^2 \le 1-\frac{(|z|-|w|)^2}{(1-|z||w|)^2},$$

from which the stated inequalities follow.

From Lemma (6.1.2) and the invariance of ρ under the action of $\mathrm{Aut}(B_n)$, we have for any $z, w, u \in B_n$,

$$\rho(z,w)=\rho\big(\varphi_u(z),\varphi_u(w)\big)>\left|\frac{|\varphi_u(z)|-|\varphi_u(w)|}{1-|\varphi_u(z)||\varphi_u(w)|}\right|$$
$$=\frac{|\rho(z,u)-\rho(u,w)|}{1-\rho(z,u)\rho(u,w)}. \qquad (1)$$

From the second inequality in Lemma (6.1.2), we see that if $|z|,|w|\le R<1$ then
$$\rho(z,w)\le\frac{|z-w|}{|1-\langle z,w\rangle|}\le\frac{|z-w|}{1-R^2}. \qquad (2)$$

For all $0<r<1$ and all $0<R<1$, from the compactness of $E(0,R)$ in the Euclidean metric, there is an M which depends only on n,r and R so that if $\{w_1,\dots,w_m\}$ is a subset of $E(0,R)$ and $|w_j-w_k|\ge(1-R^2)r$ for all $j\ne k$ then $m\le M$. Then (2) implies that if $\{w_1,\dots,w_m\}$ is a subset of $E(0,R)$ so that $\rho(w_j,w_k)\ge r$ for all $j\ne k$ then $m\le M$.

The above properties of ρallow us to prove the following characteristic of a separated collection of points in B_n.

Lemma (6.1.3)[182]: Let $\{w_j: j\in J\}$ be a collection of points in B_n so that $\rho(w_j,w_k)>r$ for all $j\ne k$, where $0<r<1$. Let $0<R_1,R_2<1$ be given. Then there is an N depending only on n,r,R_1 and R_2 so that for any $u\in B_n$ the set $\{j\in J: E(u,R_1)\cap E(w_j,R_2)\ne\emptyset\}$ has at most N elements.

Proof. By applying the Möbius automorphism that interchanges 0 and u if necessary, we can assume without loss of generality that $u=0$. Let $\tilde R=(R_1+R_2)/(1+R_1R_2)$. Suppose $z,w\in B_n$ with $|w|\le R_1$ and $|z|>\tilde R$. Then from Lemma (6.1.2),
$$\rho(z,w)\ge\frac{|z|-|w|}{1-|w||z|}>\frac{\tilde R-R_1}{1-\tilde R R_1}=R_2.$$

So $E(0,R_1)\cap E(z,R_2)\ne\emptyset$ implies that $|z|\le\tilde R$. Hence, $\{j\in J: E(0,R_1)\cap E(z_j,R_2)\ne\emptyset\}$ is a subset of the set $\{j\in J: |w_j|\le\tilde R\}$. From the remark preceding the lemma, the second set has at most N elements, where N depends only on n,r,R_1 and R_2. The conclusion of the lemma follows from here.

The following lemma is similar to [81] but somewhat stronger even though the proof is almost identical. We state here the lemma and give the proof, too.

Lemma (6.1.4)[182]: Let $\mathcal{W}=\{w_j: j\in J\}$ be a separated collection of points in B_n and $0<\sigma<1$. Then there is a finite decomposition $\mathcal{W}=\mathcal{W}_1\cup\dots\cup\mathcal{W}_N$ such that for every $1\le i\le N$, $E(z,\sigma)\cap E(w,\sigma)=\emptyset$ for all $z\ne w$ in $\mathcal{W}_i$.

Proof. Let $\mathcal{W}_1\subset\mathcal{W}$be a maximal subset so that $E(z,\sigma)\cap E(w,\sigma)=\emptyset$ for all $z\ne w$ in $\mathcal{W}_i$. If $\mathcal{W}_1=\mathcal{W}$ we are done. Otherwise suppose that $m\ge 2$ and $\mathcal{W}_1,\dots,\mathcal{W}_{m-1}$ are chosen so that $E(z,\sigma)\cap E(w,\sigma)=\emptyset$ for all $z\ne w$ in $\mathcal{W}_i$, all $1\le i\le m-1$ and $\mathcal{W}\backslash(\mathcal{W}_1\cup\dots\cup\mathcal{W}_{m-1})\ne\emptyset$. Let $\mathcal{W}_m\subset\mathcal{W}\backslash(\mathcal{W}_1\cup\dots\cup\mathcal{W}_{m-1})$ be a maximal subset so that $E(z,\sigma)\cap E(w,\sigma)=\emptyset$ for all $z\ne w$ in $\mathcal{W}_m$. By the maximality at each of the previous steps, if $u\in\mathcal{W}_m$ then for every $1\le i\le m-1$, there is a $u_i\in\mathcal{W}_i$ so that $E(u_i,\sigma)\cap E(u,\sigma)\ne\emptyset$. Therefore $\{u,u_1,\dots,u_{m-1}\}\subset\{j\in J: E(u,\sigma)\cap E(w_j,\sigma)\ne\emptyset\}$. From Lemma (6.1.2), there is an N depending on $n,\sigma$and the degree of separation of $\mathcal{W}$ so that $m\le N$.

From now, fix an $r\in(0,1)$ and a sequence of points $\mathcal{W}=\{w_j: j\in\mathbb{N}\}$ in B_n so that $E(w_j,r)\cap E(w_k,r)=\emptyset$ for all $j\ne k$ in $\mathbb{N}$.

Now we state some lemmas which are in Suárez's for the case $n = 1$ and for L_a^p with $1 < p < \infty$, see [81]. Here we are interested in the case $n \geq 2$ and $p = 2$. The conclusions of those lemmas in our case still hold true with no major changes in the proofs.

Lemma (6.1.5)[182]: Let $0 < \beta < 1$ and $r < R < 1$ and let

$$\Phi(z,w) = \sum_{j\in\mathbb{N}} \chi_{E(w_j,r)}(z)\chi_{B_n\backslash E(w_j,R)}(W)|1 - \langle z,w\rangle|^{n-1} .$$

Then we have the following, where $c_1(\beta) > 0$:

$$\int_{B_n} \Phi(z,w)(1 - |z|^2)^{-\beta} dv(z) \leq c_1(\beta)(1 - |w|^2)^{-\beta} .$$

Lemma (6.1.6)[182]: Let $0 < \beta < 1$ and $r < R < 1$ and $\Phi(z,w)$ as in Lemma (6.1.5). Then

$$\int_{B_n} \Phi(z,w)(1 - |w|^2)^{-\beta} dv(w) \leq c_2(\beta,R)(1 - |z|^2)^{-\beta} ,$$

where $c_2(\beta,R) \to 0$ when $R \to 1$.

Lemma (6.1.7)[182]: Suppose that $R \in (r,1)$ and $a_j, A_j \in L^\infty$ are functions of norm ≤ 1 such that

$$\operatorname{supp} a_j \subset E(w_j,r) \quad \text{and} \quad \operatorname{supp} A_j \subset B_n\backslash E(w_j,R).$$

Then the operator $\sum_{j\in\mathbb{N}} M_{a_j}PM_{A_j}$ is bounded on L^2, with norm bounded by some constant $k(R) \to 0$ when $R \to 1$.

The following proposition is the case $n \geq 1$ and $p = 2$ of [81]. Since we have all the needed properties of the metric ρ and all the necessary lemmas, the proof is identical to that of Suárez.

Proposition (6.1.8)[182]: For each $j \in \mathbb{N}$, let $c_j^1, \dots, c_j^l, a_j, b_j, d_j^1, \dots, d_j^m \in L^\infty$ be functions of norm ≤ 1 supported on $E(w_j,r)$. Then the following belongs to the commutator ideal $\mathfrak{CT}(L^\infty)$ of the full Toeplitz algebra:

$$\sum_{j\in\mathbb{N}} T_{c_j^1} \dots T_{c_j^1} \left(T_{a_j}T_{b_j} - T_{b_j}T_{a_j}\right) T_{d_j^1} \dots T_{d_j^m}.$$

In the proof of Proposition (6.1.8), we are dealing only with Toeplitz operators with symbols in the subset G of L^∞ which consists of functions of the form $\sum_{j\in F} f_i$, where F is a subset of $\mathbb{N}$ and f is one of the symbols $c^1, \dots, c^l, a, b, d^1, \dots, d^m$. So in the above conclusion, we can replace $\mathfrak{CT}(L^\infty)$ by the smaller ideal $\mathfrak{CT}(G)$.

Fix a bounded set $\{S_\alpha : \alpha \in \mathbb{N}^n\}$ of strictly positive real numbers.

Lemma (6.1.9)[182]: Fix $0 < R < 1$ and $\varepsilon > 0$ so that $(1 + \varepsilon)R < 1$. Let $\delta > 0$ be given. Then there is a constant $C(\delta) > 0$ so that for all $|z| \leq R$,

$$k_z \otimes k_z C(\delta) \sum_{\alpha\in\mathbb{N}^n} S_\alpha e_\alpha \otimes e_\alpha + \delta \int_{|w|<(1+\varepsilon)R} k_w \otimes k_w \quad d\mu(w). \tag{3}$$

Proof. Let f be in $L_a^2$and $|z| \leq R$. Let J be a finite subset of $\mathbb{N}^n$. Put

$$g_J = \sum_{\alpha \in J} \langle f, e_\alpha \rangle e_\alpha \text{ and } h_J = \sum_{\alpha \in \mathbb{N}^n \setminus J} \langle f, e_\alpha \rangle e_\alpha .$$

Then

$$\begin{aligned} \langle (k_z \otimes k_z) f, f \rangle = |\langle f, k_z \rangle|^2 &= \left| \langle g_J, k_z \rangle + \langle h_J, k_z \rangle \right|^2 \\ &\le 2 \left(\left| \langle g_J, k_z \rangle \right|^2 + \left| \langle h_J, k_z \rangle \right|^2 \right). \end{aligned} \tag{4}$$

Now,

$$\begin{aligned} \left| \langle h_J, k_z \rangle \right|^2 &= \left| \sum_{\alpha \in \mathbb{N}^n \setminus J} \langle f, e_\alpha \rangle \langle e_\alpha, k_z \rangle \right|^2 = (1 - |z|^2)^{n+1} \left| \sum_{\alpha \in \mathbb{N}^n \setminus J} \langle f, e_\alpha \rangle e_\alpha(z) \right|^2 \\ &\le \left| \sum_{\alpha \in \mathbb{N}^n \setminus J} \langle f, e_\alpha \rangle e_\alpha(z) \right|^2 \le \left(\sum_{\alpha \in \mathbb{N}^n \setminus J} |\langle f, e_\alpha \rangle| \left| \left(\frac{(n + |\alpha|!)}{n!\,\alpha!} \right)^{\frac{1}{2}} |z^\alpha| \right| \right)^2 \\ &\le \left(\sum_{\alpha \in \mathbb{N}^n \setminus J} |\langle f, e_\alpha \rangle|^2 \left((1 + \varepsilon) R \right)^{2|\alpha|} \right) \\ &\times \left(\sum_{\alpha \in \mathbb{N}^n \setminus J} \frac{(n + |\alpha|!)}{n!\,\alpha!} |z^\alpha|^2 \left((1 + \varepsilon) R \right)^{-2|\alpha|} \right). \end{aligned} \tag{5}$$

On the other hand, the homogeneity of the e_α's shows that

$$f\big((1 + \varepsilon) R \zeta\big) = \sum_{\alpha \in \mathbb{N}^n} \langle f, e_\alpha \rangle \left((1 + \varepsilon) R \right)^{|\alpha|} e_\alpha(\zeta)$$

so that the change-of-variable $w = (1 + \varepsilon) R \zeta$ gives

$$\begin{aligned} \int_{|w| < (1+\varepsilon)R} \langle (k_w \otimes k_w) f, f \rangle d\mu(w) &= \int_{|w| < (1+\varepsilon)R} |f(w)|^2 dv(w) \\ &= \left((1 + \varepsilon) R \right)^{2n} \int_{B_n} |f(1 + \varepsilon) R \zeta|^2 dv(\zeta) \\ &= \left((1 + \varepsilon) R \right)^{2n} \sum_{\alpha \in \mathbb{N}^n} |\langle f, e_\alpha \rangle|^2 \left((1 + \varepsilon) R \right)^{2|\alpha|} \\ &\ge \left((1 + \varepsilon) R \right)^{2n} \sum_{\alpha \in \mathbb{N}^n \setminus J} |\langle f, e_\alpha \rangle|^2 \left((1 + \varepsilon) R \right)^{2|\alpha|}. \end{aligned} \tag{6}$$

This implies

$$\sum_{\alpha\in\mathbb{N}^n\setminus J} |\langle f, e_\alpha\rangle|^2 \left((1+\varepsilon)R\right)^{2|\alpha|}$$

$$\leq \left((1+\varepsilon)R\right)^{-2n} \int\limits_{|w|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w). \tag{7}$$

Inequalities (5) and (7) imply

$$\left|\langle h_J, k_z\rangle\right|^2 \leq \left(\sum_{\alpha\in\mathbb{N}^n\setminus J} \frac{(n+|\alpha|!)}{n!\,\alpha!} |z^\alpha|^2 \left((1+\varepsilon)R\right)^{-2|\alpha|}\right)\left((1+\varepsilon)R\right)^{-2n}$$

$$\times \int\limits_{|z|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w).$$

Now from the identity

$$K_w(\zeta) = \sum_{\alpha\in\mathbb{N}^n} \overline{e_\alpha(w)} e_\alpha(\zeta),$$

for $w, z \in B_n$, where $K_w(\zeta)$ is the Bergman reproducing kernel, we have

$$\sum_{\alpha\in\mathbb{N}^n} |e_\alpha(w)|^2 = K_w(w) = \frac{1}{(1-|w|^2)^{n+1}}.$$

If we take $w = z/(1+\varepsilon)R$, where $|z| \leq R$, we obtain

$$\sum_{\alpha\in\mathbb{N}^n} \frac{(n+|\alpha|!)}{n!\,\alpha!} |z^\alpha|^2 \left((1+\varepsilon)R\right)^{-2|\alpha|} = \sum_{\alpha\in\mathbb{N}^n} \left|e_\alpha\left(\frac{z}{(1+\varepsilon)R}\right)\right|^2$$

$$\leq \frac{1}{\left(1-\frac{1}{(1+\varepsilon)^2}\right)^{n+1}}. \tag{8}$$

So there is a finite subset J of $\mathbb{N}^n$ which is independent of z so that

$$\sum_{\alpha\in\mathbb{N}^n\setminus J} \frac{(n+|\alpha|!)}{n!\,\alpha!} |z^\alpha|^2 \left((1+\varepsilon)R\right)^{-2|\alpha|} \leq \frac{\delta}{2}\left((1+\varepsilon)R\right)^{2n}.$$

Hence for this J,

$$\left|\langle h_J, k_z\rangle\right|^2 \leq \frac{\delta}{2} \int\limits_{|w|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w). \tag{9}$$

Also,

$$\left|\langle g_J, k_z\rangle\right|^2 \leq \sum_{\alpha\in J} |\langle f, e_\alpha\rangle|^2. \tag{10}$$

From inequalities (4), (9) and (10), we conclude that

$$\langle (k_z \otimes k_z)f, f\rangle \leq 2\sum_{\alpha\in J} \langle (e_\alpha \otimes e_\alpha)f, f\rangle + \delta \int\limits_{|w|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w).$$

Since $S_\alpha > 0$ for all $\alpha \in J$ and J is finite, there is a constant $C(\delta) > 0$ so that $C(\delta)S_\alpha \geq 2$ for all $\alpha \in J$. Then for any $f \in L_a^2$, and any $|z| \leq R$,

$$\langle (k_z \otimes k_z)f, f\rangle \leq C(\delta)\sum_{\alpha\in J} S_\alpha\langle (e_\alpha \otimes e_\alpha)f, f\rangle + \delta \int_{|w|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w)$$

$$\leq C(\delta)\sum_{\alpha\in \mathbb{N}^n} S_\alpha\langle (e_\alpha \otimes e_\alpha)f, f\rangle + \delta \int_{|w|<(1+\varepsilon)R} \langle (k_w \otimes k_w)f, f\rangle d\mu(w).$$

In other words, for any $|z| \leq R$,

$$k_z \otimes k_z \leq C(\delta)\sum_{\alpha\in \mathbb{N}^n} S_\alpha e_\alpha \otimes e_\alpha + \delta \int_{|w|<(1+\varepsilon)R} k_w \otimes k_w d\mu(w).$$

Theorem (6.1.10)[182]: Let $\{e_\alpha : \alpha \in \mathbb{N}^n\}$be a bounded set of strictly positive real numbers. Let

$$S = \sum_{\alpha\in \mathbb{N}^n} S_\alpha e_\alpha \otimes e_\alpha .$$

Let $\{w_j : j \in \mathbb{N}\}$ be a separated sequence of points in B_n so that $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R)$ for some $0 < R < 1$. Then there is a positive constant c so that

$$\sum_{j\in \mathbb{N}^n} \mathcal{U}_{w_j} S \mathcal{U}_{w_j}^* \geq c \geq 0.$$

Proof. Let $S = \sum_{\alpha\in\mathbb{N}^n} S_\alpha e_\alpha \otimes e_\alpha$ and $\mathcal{W} = \{w_j : j \in \mathbb{N}\}$ be as in the hypothesis of Theorem (6.1.10). Choose an $\varepsilon > 0$ so that $(1+\varepsilon)R < 1$.

For each $a \in B_n$, apply $\mathcal{U}_a$ to the left and $\mathcal{U}_a^*$ to the right of both sides of inequality (3) in Lemma (6.1.9), we get, for $|z| \leq R$:

$$\mathcal{U}_a(k_z \otimes k_z)\mathcal{U}_a^* \leq C(\delta)\mathcal{U}_a S \mathcal{U}_a^* + \delta \int_{|w|<(1+\varepsilon)R} \mathcal{U}_a(k_w \otimes k_w)\mathcal{U}_a^* d\mu(w)$$

$$= C(\delta)\mathcal{U}_a S \mathcal{U}_a^* + \delta \int_{|w|<(1+\varepsilon)R} k_{\varphi_a} \otimes k_{\varphi_a}\, d\mu(w)$$

$$= C(\delta)\mathcal{U}_a S \mathcal{U}_a^*$$

$$+ \delta \int_{|\varphi_a(\zeta)|<(1+\varepsilon)R} k_\zeta \otimes k_\zeta \, d\mu(\zeta) \left(\text{by the change} - \text{of} - \text{variable } w\right.$$

$$\left. = \varphi_a(\zeta)\right) = C(\delta)\mathcal{U}_a S \mathcal{U}_a^* + \delta \int_{E(a,(1+\varepsilon)R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta).$$

Since $\mathcal{U}_a(k_z \otimes k_z)\mathcal{U}_a^* = k_{\varphi_a(z)} \otimes k_{\varphi_a(z)}$, the above implies

$$k_{\varphi_a(z)} \otimes k_{\varphi_a(z)} \leq C(\delta)\mathcal{U}_a S \mathcal{U}_a^* + \delta \int_{E(a,(1+\varepsilon)R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta). \tag{11}$$

For each $|z| \leq R$, let

$$T(z) = \sum_{j\in\mathbb{N}} k_{\varphi_{w_j}(z)} \otimes k_{\varphi_{w_j}(z)}.$$

Then (11) gives, for $|z| \leq R$:

$$T(z) \leq C(\delta) \sum_{j\in\mathbb{N}} \mathcal{U}_{w_j} S \mathcal{U}_{w_j}^* + \delta \sum_{j\in\mathbb{N}} \int_{E(w_j,(1+\varepsilon)R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta). \tag{12}$$

Decompose $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_N$ as in Lemma (6.1.4), where N depends only on $n, (1+\varepsilon)R$ and the degree of separation of $\mathcal{W}$. Then

$$\sum_{j\in\mathbb{N}} \int_{E(w_j,(1+\varepsilon)R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta) \leq \sum_{i=1}^{N} \sum_{w\in\mathcal{W}_i} \int_{E(w,(1+\varepsilon)R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta)$$

$$\leq \sum_{i=1}^{N} \int_{B_n} k_\zeta \otimes k_\zeta \, d\mu(\zeta) = N.$$

Hence, for $|z| \leq R$,

$$T(z) \leq C(\delta) \sum_{j\in\mathbb{N}} \mathcal{U}_{w_j} S \mathcal{U}_{w_j}^* + \delta N. \tag{13}$$

By integrating $T(z)$with respect to $dv(z)$over the ball $|z| < R$, we get

$$\int_{|z|<R} T(z) dv(z) \geq (1-R^2)^{n+1} \int_{|z|<R} T(z)(1-|z|^2)^{-(n+1)} dv(z)$$

$$= (1-R^2)^{n+1} \int_{|z|<R} T(z) d\mu(z)$$

$$= (1-R^2)^{n+1} \sum_{j\in\mathbb{N}} \int_{|z|<R} k_{\varphi_{w_j}(z)} \otimes k_{\varphi_{w_j}(z)} \, d\mu(z)$$

$$= (1-R^2)^{n+1} \sum_{j\in\mathbb{N}} \int_{E(w_j,R)} k_\zeta \otimes k_\zeta \, d\mu(\zeta)$$

$$\geq (1-R^2)^{n+1} \int_{B_n} k_\zeta \otimes k_\zeta \, d\mu(\zeta) \left(\text{since} B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R) \right)$$

$$= (1-R^2)^{n+1}. \tag{14}$$

Inequalities (13) and (14) together imply

$$C(\delta) \sum_{j\in\mathbb{N}} \mathcal{U}_{w_j} S \mathcal{U}_{w_j}^* + \delta N \geq (1-R^2)^{n+1} R^{-2n}.$$

Now choose δ so small that

$$\delta N \leq 2^{-1}(1-R^2)^{n+1} R^{-2n}.$$

Then we have

$$C(\delta)\sum_{j\in\mathbb{N}}\mathcal{U}_{w_j}S\mathcal{U}_{w_j}^{*} \geq 2^{-1}(1-R^2)^{n+1}R^{-2n} > 0. \tag{15}$$

Suppose $\eta_\varepsilon(z) = z_1\eta(|z|/\varepsilon)$ for all $z=(z_1,\dots,z_n)\in B_n$ as in the hypothesis of Theorem (6.1.1). We will compute directly $\left[T_{\eta_\varepsilon}, T_{\overline{\eta}_\varepsilon}\right]$ to see that it is a diagonal operator with respect to the standard orthonormal basis.

For any multi-indices α and β in $\mathbb{N}^n$, we have

$$\langle T_{\eta_\varepsilon}e_\alpha, e_\beta\rangle = \int_{B_n}\eta_\varepsilon(z)e_\alpha(z)\bar{e}_\beta(z)\ dv(z) = \int_{|z|<\varepsilon}\eta(|z|/\varepsilon)z_1e_\alpha(z)\bar{e}_\beta(z)\ \ dv(z).$$

Now,

$$z_1e_\alpha(z) = \left(\frac{(n+|\alpha|!)}{n!\,\alpha!}\right)^{1/2} z_1z^\alpha = \left(\frac{(n+|\alpha|)!}{n!\,\alpha!}\frac{n!\left(\alpha+(1,0,\dots,0)\right)!}{(n+|\alpha|+1)!}\right)^{1/2} e_{\alpha+(1,0,\dots,0)}(z)$$
$$= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} e_{\alpha+(1,0,\dots,0)}(z).$$

So

$$\langle T_{\eta_\varepsilon}e_\alpha, e_\beta\rangle = \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2}\int_{|z|<\varepsilon}\eta(|z|/\varepsilon)e_{\alpha+(1,0,\dots,0)}(z)\bar{e}_\beta(z)\ \ dv(z)$$
$$= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2}\int_0^\varepsilon(2n)r^{2n-1}\eta\left(\frac{r}{\varepsilon}\right)\int_{S^n}e_{\alpha+(1,0,\dots,0)}(r\zeta)\bar{e}_\beta(r\zeta)\ \ d\sigma(\zeta)\,dr$$
$$= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2}$$
$$\times\begin{Bmatrix} 0 & if\ \beta\neq\alpha+(1,0,\dots,0), \\ \int_0^\varepsilon(2n)r^{2n-1}\eta\left(\frac{r}{\varepsilon}\right)\left(\frac{n+|\alpha|+1}{n}\right)r^{2|\alpha|+2}dr & if\ \beta=\alpha+(1,0,\dots,0), \end{Bmatrix}$$

(see [79]).

We have

$$\int_0^\varepsilon(2n)r^{2n-1}\eta\left(\frac{r}{\varepsilon}\right)\left(\frac{n+|\alpha|+1}{n}\right)r^{2|\alpha|+2}dr = \int_0^\varepsilon 2(n+|\alpha|+1)r^{2n+2|\alpha|+1}\eta\left(\frac{r}{\varepsilon}\right)dr$$
$$= \varepsilon^{2n+2|\alpha|+2}\int_0^1(n+|\alpha|+1)t^{n+|\alpha|}\eta\left(t^{\frac{1}{2}}\right)dt$$

$\left(\text{by the change} - \text{of} - \text{variable}\ r\ =\ \varepsilon t^{\frac{1}{2}}\right)$.

For $m\geq 0$, put $\gamma_m = \int_0^1(m+1)t^m\eta\left(t^{1/2}\right)\ \ dt > 0$. Note that γ_m depends only on m and the function η. We then have

$$T_{\eta_\varepsilon}e_\alpha = \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2}\varepsilon^{2n+2|\alpha|}\gamma_{n+|\alpha|}e_{\alpha+(1,0,\dots,0)}$$

From this we see that for any multi-index α,

$$T_{\overline{\eta}_\varepsilon} e_\alpha = \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \varepsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1} e_{\alpha-(1,0,\dots,0)}$$

if $\alpha_1 \geq 1$ and $T_{\overline{\eta}_\varepsilon} e_\alpha = 0$ if $\alpha_1 = 1$.

Now for multi-indices α with $\alpha_1 \geq 1$,

$$\begin{aligned} T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon} e_\alpha &= T_{\eta_\varepsilon}\left(\left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \varepsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1} e_{\alpha-(1,0,\dots,0)}\right) \\ &= \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \varepsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1} \left(\frac{\alpha_1}{n+|\alpha|}\right)^{1/2} \varepsilon^{2n+2|\alpha|} \gamma_{n+|\alpha|-1} e_\alpha \\ &= \frac{\alpha_1}{n+|\alpha|} \varepsilon^{4(n+|\alpha|)} \gamma_{n+|\alpha|-1}^2 e_\alpha, \end{aligned}$$

and

$$\begin{aligned} T_{\overline{\eta}_\varepsilon} T_{\eta_\varepsilon} e_\alpha &= T_{\overline{\eta}_\varepsilon}\left(\left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \varepsilon^{2n+2|\alpha|+2} \gamma_{n+|\alpha|} e_{\alpha+(1,0,\dots,0)}\right) \\ &= \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \varepsilon^{2n+2|\alpha|+2} \left(\frac{\alpha_1+1}{n+|\alpha|+1}\right)^{1/2} \varepsilon^{2n+2|\alpha|+2} \gamma_{n+|\alpha|} e_\alpha \\ &= \frac{\alpha_1+1}{n+|\alpha|+1} \varepsilon^{4(n+|\alpha|+1)} \gamma_{n+|\alpha|}^2 e_\alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} \left[T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon}\right] e_\alpha &= \left(\frac{\alpha_1}{n+|\alpha|} \varepsilon^{4(n+|\alpha|)} \gamma_{n+|\alpha|-1}^2 - \frac{\alpha_1+1}{n+|\alpha|+1} \varepsilon^{4(n+|\alpha|+1)} \gamma_{n+|\alpha|}^2\right) e_\alpha \\ &= \left(\frac{\alpha_1}{\alpha_1+1} \frac{n+|\alpha|+1}{n+|\alpha|} \frac{\gamma_{n+|\alpha|-1}^2}{\gamma_{n+|\alpha|}^2} - \varepsilon^4\right) \times \frac{\alpha_1+1}{n+|\alpha|+1} \varepsilon^{4(n+|\alpha|)} \gamma_{n+|\alpha|}^2 e_\alpha. \end{aligned}$$

This formula also holds for multi-indices α with $\alpha_1 = 0$.

For all $0 < \varepsilon < 1$ so that

$$\varepsilon^4 \notin \left\{\frac{\alpha_1}{\alpha_1+1} \frac{n+|\alpha|+1}{n+|\alpha|} \frac{\gamma_{n+|\alpha|-1}^2}{\gamma_{n+|\alpha|}^2} : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\right\},$$

the operator $T = \left[T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon}\right]^2$ can be written as

$$T = \sum_{\alpha \in \mathbb{N}^n} s_\alpha e_\alpha \otimes e_\alpha,$$

where $s_\alpha > 0$ for all α.

Since $\{w_j : j \in \mathbb{N}\}$ is separated and $B_n = \bigcup_{j\in\mathbb{N}} E(w_j, R)$ for some $0 < R < 1$, Theorem (6.1.10) implies that there is a positive number c so

$$A_\varepsilon = \sum_{j\in\mathbb{N}} \mathcal{U}_{w_j} T \mathcal{U}_{w_j}^* = \sum_{j\in\mathbb{N}} \mathcal{U}_{w_j} \left[T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon}\right]^2 \mathcal{U}_{w_j}^* \geq c \geq 0.$$

Now for each $j \in \mathbb{N}$,

$$\begin{aligned} \mathcal{U}_{w_j}\left[T_{\eta_\varepsilon}, T_{\overline{\eta}_\varepsilon}\right] \mathcal{U}_{w_j}^* &= \mathcal{U}_{w_j}\left[T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon} - T_{\overline{\eta}_\varepsilon} T_{\eta_\varepsilon}\right] \mathcal{U}_{w_j}^* = \mathcal{U}_{w_j} T_{\eta_\varepsilon} T_{\overline{\eta}_\varepsilon} \mathcal{U}_{w_j}^* - \mathcal{U}_{w_j} T_{\overline{\eta}_\varepsilon} T_{\eta_\varepsilon} \mathcal{U}_{w_j}^* \\ &= \left(\mathcal{U}_{w_j} T_{\eta_\varepsilon} \mathcal{U}_{w_j}^*\right)\left(\mathcal{U}_{w_j} T_{\overline{\eta}_\varepsilon} \mathcal{U}_{w_j}^*\right) - \left(\mathcal{U}_{w_j} T_{\overline{\eta}_\varepsilon} \mathcal{U}_{w_j}^*\right)\left(\mathcal{U}_{w_j} T_{\eta_\varepsilon} \mathcal{U}_{w_j}^*\right) \\ &= T_{\eta_\varepsilon \circ \varphi_{w_j}} T_{\overline{\eta}_\varepsilon \circ \varphi_{w_j}} - T_{\overline{\eta}_\varepsilon \circ \varphi_{w_j}} T_{\eta_\varepsilon \circ \varphi_{w_j}} = \left[T_{\eta_\varepsilon \circ \varphi_{w_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{w_j}}\right]. \end{aligned}$$

Hence $A_\varepsilon = \sum_{j\in\mathbb{N}} \left[T_{\eta_\varepsilon \circ \varphi_{w_j}}, T_{\overline{\eta}_\varepsilon \circ \varphi_{w_j}}\right]^2$.

Note that for each j, the function $\eta_\varepsilon \circ \varphi_{w_j}$ is supported in the set

$$\left\{z \in B_n: \left|\varphi_{w_j}(z)\right| \le \varepsilon\right\} = \left\{z \in B_n: \rho\left(z, w_j\right) \le \varepsilon\right\} = E\left(w_j, \varepsilon\right).$$

We now decompose $\mathcal{W} = \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_N$ such that $E(z,\varepsilon) \cap E(w,\varepsilon) = \emptyset$ for all $z \neq w$ in $\mathcal{W}_j$, for all $1 \le j \le N$ as in Lemma (6.1.4). Hence

$$A_\varepsilon = \sum_{i=1}^{N} \sum_{w \in \mathcal{W}_i} \left[T_{\eta_\varepsilon \circ \varphi_w}, T_{\bar{\eta}_\varepsilon \circ \varphi_w}\right]^2,$$

where, by Proposition (6.1.8) and the remark following it, each of the summands is in $\mathfrak{CT}(G_\varepsilon)$. Here we remind that G_ε is the subset of L^∞ consisting of all functions of the form $\sum_{j\in F} \eta_\varepsilon \circ \varphi_{w_j}$ or $\sum_{j\in F} \bar{\eta}_\varepsilon \circ \varphi_{w_j}$, where F is a subset of $\mathbb{N}$.

It then follows that A_ε itself belongs to $\mathfrak{CT}(G_\varepsilon)$.

We are discussing some remarks about Theorem (6.1.1). Our first remark is the existence of a separated sequence as in the hypothesis of Theorem (6.1.1). This is actually a consequence of Zorn's lemma. In fact, let $0 < r < 1$ and Ω_r be the collection of all sets of points $\{w_j: j \in J\}$in B_n so that $\rho\left(w_j, w_k\right) > r$ for all $j \neq k$. The sets in Ω_r are ordered by inclusion. Apply Zorn's lemma, we get a maximal set in Ω_r. Denote this set by $\{w_j: j \in J\}$. Since J must be infinite and countable, we can assume that J is $\mathbb{N}$. Then for any $z \in B_n$, by maximality there is a $j \in \mathbb{N}$ so that $\rho\left(z, w_j\right) \le r$. Hence $B_n = \bigcup_{j\in\mathbb{N}} E\left(w_j, r\right)$.

Note that all functions in G_ε vanish on $B_n \backslash E_\varepsilon$, where E_ε is a subset of $V_\varepsilon = \bigcup_{j\in\mathbb{N}} E\left(w_j, \varepsilon\right)$. The following lemma gives an upper estimate for the Lebesgue measure of V_ε for small $\varepsilon > 0$.

Lemma (6.1.11)[182]: Suppose $0 < \varepsilon_0 < 1$ so that $E\left(w_j, \varepsilon_0\right) \cap E(w_l, \varepsilon_0) = \emptyset$ for all $j \neq l$. Then for any $\varepsilon < \varepsilon_0$,

$$\upsilon(V_\varepsilon) \le \left(\frac{\varepsilon}{\varepsilon_0}\right)^{2n} \upsilon\left(V_{\varepsilon_0}\right).$$

Proof. For any $0 < \delta < 1$ and $z \in B_n$, we have

$$\begin{aligned}
\upsilon\left(E(z,\delta)\right) &= \int_{E(z,\delta)} d\nu\,(w) \\
&= \int_{E(0,\delta)} \frac{(1-|z|^2)^{n+1}}{|1-\langle\zeta,z\rangle|^{2(n+1)}}\, d\nu\,(\zeta) \left(\text{by the change} - \text{of} - \text{variable } w = \varphi_z(\zeta)\right) \\
&= (1-|z|^2)^{n+1} \int_{E(0,1)} \frac{\delta^{2n}\, d\nu(\zeta)}{|1-\langle\zeta,z\rangle|^{2(n+1)}} \\
&= (1-|z|^2)^{n+1}\delta^{2n} \int_{E(0,1)} (1-\delta|z|^2)^{-n-1} \left\|k_{\delta_z}(\zeta)\right\|^2 d\nu(\zeta) \\
&= (1-|z|^2)^{n+1}\delta^{2n}(1-\delta|z|^2)^{-n-1} \left\|k_{\delta_z}(\zeta)\right\|^2 \\
&= (1-|z|^2)^{n+1}\delta^{2n}(1-\delta|z|^2)^{-n-1}.
\end{aligned}$$

Now for $0 < \varepsilon < \varepsilon_0 < 1$ as in the hypothesis,

$$\begin{aligned}
\upsilon(V_\varepsilon) &= \sum_{j\in\mathbb{N}} \upsilon\left(E\left(w_j,\varepsilon\right)\right) = \sum_{j\in\mathbb{N}} \left(1-\left|w_j\right|^2\right)^{n+1} \varepsilon^{2n}\left(1-\left|w_j\right|^2\right)^{-n-1} \\
&= \sum_{j\in\mathbb{N}} \left(1-\left|w_j\right|^2\right)^{n+1} \varepsilon_0^{2n} \times \left(1-\left(\varepsilon_0\left|w_j\right|\right)^2\right)^{-n-1} \left(\frac{\varepsilon}{\varepsilon_0}\right)^{2n} \left(\frac{1-\left(\varepsilon_0\left|w_j\right|\right)^2}{1-\left(\varepsilon\left|w_j\right|\right)^2}\right)^{n+1} \\
&\le \left(\frac{\varepsilon}{\varepsilon_0}\right)^{2n} \sum_{j\in\mathbb{N}} \upsilon\left(E\left(w_j,\varepsilon_0\right)\right) \left(\text{because } \frac{1-\left(\varepsilon_0\left|w_j\right|\right)^2}{1-\left(\varepsilon\left|w_j\right|\right)^2} \le 1 \; for \; \varepsilon < \varepsilon_0\right) \\
&= \left(\frac{\varepsilon}{\varepsilon_0}\right)^{2n} \upsilon\left(V_{\varepsilon_0}\right).
\end{aligned}$$

This lemma implies that if the separated set is fixed then the Lebesgue measure $\upsilon(V_\varepsilon)$ can be made as small as we please provided that ε is small.

We will show that if $\eta$is a continuous function on $[0,1]$ then G_ε is contained in $C(B_n)$, the space of continuous functions on the open unit ball B_n. This remark together with Theorem (6.1.1) implies that $\mathfrak{C}\mathfrak{T}(C(B_n)\cap L^\infty)$ coincides with the full Toeplitz algebra $\mathfrak{T}(L^\infty)$. The reader should compare this with the fact that $\mathfrak{C}\mathfrak{T}\left(C(\bar{B}_n)\right)$ is the same as the ideal $\mathcal{K}$ of compact operators.

Suppose $\eta$is continuous on $[0,1]$, then for each $j\in\mathbb{N}$ the function $\eta_\varepsilon\circ\varphi_{w_j}$ is continuous and supported in the ball $E\left(w_j,\varepsilon\right)$. Suppose F is a subset of $\mathbb{N}$. Let $f=\sum_{j\in F}\eta_\varepsilon\circ\varphi_{w_j}$. Let $0<R<1$ be given. By Lemma (6.1.3), all but a finite number of functions in the series vanish on $E(0,R)$. Thus f, being a finite sum of continuous functions on $E(0,R)$, is continuous on $E(0,R)$for all $0<R<1$. So f is continuous on the open unit ball B_n. Similarly, functions of the form $\sum_{j\in F}\bar{\eta}_\varepsilon\circ\varphi_{w_j}$ are also continuous on B_n.

Section (6.2): Compactness in Bergman and Fock Spaces

The Bargmann-Fock space $\mathcal{F}^2 \coloneqq \mathcal{F}^2(\mathbb{C}^n)$ is the collection of entire functions on $\mathbb{C}^n$ such that

$$\int\limits_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\, d\upsilon(z) < \infty.$$

It is well known that $\mathcal{F}^2$ is a reproducing kernel Hilbert space with reproducing kernel given by $K_z(w)=e^{\bar{z}w}$. As usual, we denote by k_z the normalized reproducing kernel at z. For a bounded operator Ton $\mathcal{F}^2$, the Berezin transform of $T$is the function defined by

$$\tilde{T}(z)=\langle Tk_z,k_z\rangle_{\mathcal{F}^2}.$$

It was proved recently by Bauer and Isralowitz that the vanishing Berezin transform is sufficient for compactness whenever the operator is in the Toeplitz algebra [185]. However, it is generally very difficult to check when a given operator T belongs in the Toeplitz algebra, unless T is itself a Toeplitz operator or a combination of a few Toeplitz operators, and as such one would like a “simpler” sufficient condition to guarantee this.

In [190] Xia and Zheng introduced a class of “sufficiently localized” operators on the Fock space which include the algebraic closure of the Toeplitz operators. These are the operators T acting on the Fock space such that there exist constants $2n<\beta<\infty$ and $0<C<\infty$ with

$$|\langle Tk_z, k_w\rangle_{\mathcal{F}^2}| \leq \frac{C}{(1+|z-w|)^\beta} \tag{16}$$

It was proved by Xia and Zheng that every bounded operator T from the C^*algebra generated by the sufficiently localized operators whose Berezin transform vanishes at infinity, i.e.,

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{\mathcal{F}^2} = 0 \tag{17}$$

is compact on $\mathcal{F}^2$. One of their main innovations is providing an easily checkable condition (16) which is general enough to imply compactness from the seemingly much weaker condition (17).

We wish to extend the Xia-Zheng notion of sufficiently localized to a larger class of operators. Note that (16) easily implies

$$\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|\, dv(w) < \infty;$$

and consequently one should look at generalizations of sufficiently localized that allow for weaker integral conditions. Next, we want to provide a simpler, more direct, proof of the main result in [190] which follows a more traditional route and can be extended to other spaces. Finally, we show that an analog of our more general theorem, remains true in the case of the Bergman space.

We provide the following extension of the main result from [190]. For operators T acting on the Fock space we impose the following conditions. We first assume that

$$\sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|\, dv(w) < \infty\,, \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle T^*k_z, k_w\rangle_{\mathcal{F}^2}|\, dv(w) < \infty \tag{18}$$

which is enough to conclude that the operator T initially defined on the linear span of the reproducing kernels extends to a bounded operator on $\mathcal{F}^2$. To show that the operator is compact, we impose the following additional assumptions on T:

$$\lim_{r\to\infty} \sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|\, dv(w) = 0\,, \lim_{r\to\infty} \sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle T^*k_z, k_w\rangle_{\mathcal{F}^2}|\, dv(w) = 0. \tag{19}$$

Definition (6.2.1)[184]: We will say that a linear operator T on $\mathcal{F}^2$ is weakly localized if it satisfies the conditions (18) and (19).

Notice that every sufficiently localized operator in the sense of Xia and Zheng obviously satisfies (18) and (19) and is weakly localized in our sense too. We prove the following result.

Theorem (6.2.2)[184]: Let T be an operator on $\mathcal{F}^2$ which belongs to the C^*algebra generated by weakly localized operators satisfying Definition (6.2.1). If

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{\mathcal{F}^2} = 0,$$

then T is compact.

We can easily extend the result to the weighted Fock space $\mathcal{F}^2_\alpha$.

Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$ and let the space $A^2 := A^2(\mathbb{B}_n)$ denote the classical Bergman space, i.e., the collection of all holomorphic functions on $\mathbb{B}_n$ such that

$$\|f\|^2_{A^2} := \int_{\mathbb{B}_n} |f(z)|^2\, dv(z) < \infty.$$

The function $K_z(w) \coloneqq (1-\bar{z}w)^{-(n+1)}$ is the reproducing kernel for A^2 and
$$K_z(w) \coloneqq \frac{(1-|z|^2)^{\frac{n+1}{2}}}{(1-\bar{z}w)^{(n+1)}}$$
is the normalized reproducing kernel at the point z. We also will let $d\lambda$ denote the invariant measure on $\mathbb{B}_n$, i.e.,
$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}.$$

We are interested in operators T acting on the Bergman space that satisfy the following conditions. First, we assume that there exists $\frac{n-1}{n+1} < a < \frac{n}{n+1}$ such that
$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \infty\,,$$
$$\sup_{z\in\mathbb{B}_n} \int_{\mathbb{B}_n} |\langle T^*k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \infty \quad (20)$$
These are enough to conclude that the operator Tinitially defined on the linear span of the reproducing kernels extends to a bounded operator on A^2. To treat compactness we make the following additional assumptions on T: There exists $\frac{n-1}{n+1} < a < \frac{n}{n+1}$ such that
$$\lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$
$$= 0\ ,\ \lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} |\langle T^*k_z, k_w\rangle| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) = 0. \quad (21)$$
Definition (6.2.3)[184]: We say that a linear operator T on A^2 is weakly localized if it satisfies the conditions (20) and (21).

We prove the following result.

Theorem (6.2.4)[184]: Let T be an operator on A^2 which belongs to the C^*algebra generated by weakly localized operators satisfying Definition (6.2.3). If
$$\lim_{|z|\to 1} \|Tk_z\|_{A^2} = 0,$$
then T is compact.

For weakly localized operators we can deduce compactness under even weaker assumption. We currently cannot prove this result for the C^*algebra of weakly localized operators on the Bergman space.

Theorem (6.2.5)[184]: Let T be a weakly localized operator on A^2. If
$$\lim_{|z|\to 1} \langle Tk_z, k_z\rangle_{A^2} = 0,$$
then T is compact.

It will be clear that the method of proof also will work in the weighted Bergman space A_α^2.

We wish to emphasize that when the operator T belongs to the Toeplitz algebra generated by L^∞symbols, this result is known through deep work of Suarez, [189] in the case of A^2. See also [188] for the case of weighted Bergman spaces. We will prove below that the Toeplitz algebra generated by L^∞symbols is a subalgebra of the C^*algebra generated by the weakly localized operators.

Let φ_z be the Möbius map of $\mathbb{B}_n$ that interchanges 0 and z. Then, we have the following well known fact that

$$1-|\varphi_z(w)|^2=\frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^2}\ .$$

As a consequence,

$$|\langle k_z,k_w\rangle_{A^2}|=\frac{1}{\left\|k_{\varphi_z(w)}\right\|_{A^2}}. \tag{22}$$

Using the automorphism φ_z, the pseudohyperbolic and hyperbolic metrics on $\mathbb{B}_n$ are defined by

$$\rho(z,w):=|\varphi_z(w)| \text{ and } \beta(z,w):=\frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

Recall that these metrics are connected by $\rho=\frac{e^{2\beta}-1}{e^{2\beta}+1}=\tanh\beta$ and it is well-known that these metrics are invariant under the automorphism group of $\mathbb{B}_n$. We let

$$D(z,r):=\{w\in\mathbb{B}_n:\beta(z,w)\le r\}=\{w\in\mathbb{B}_n:\rho(z,w)\le s=\tanh r\},$$

denote the hyperbolic disc centered at $z$of radius r. For $z\in\mathbb{B}_n$, define

$$U_zf(w):=f\big(\varphi_z(w)\big)k_z(w),$$

which via a simple change of variables argument is clearly a unitary operator on A^2. Then for an operator Ton the Bergman space A^2, for $z\in\mathbb{B}_n$ we set

$$T_z:=U_zTU_z^*.$$

Recall also that the orthogonal (Bergman) projection of $L^2(\mathbb{B}_n,dv)$ onto A^2 is given by the integral operator

$$P(f)(z):=\int_{\mathbb{B}_n}\langle K_w,K_z\rangle_{A^2}\,f(w)dv(w).$$

Therefore, for all $f\in A^2$ we have $f(z)=\int_{\mathbb{B}_n}\langle f,k_w\rangle_{A^2}k_w(z)d\lambda(w)$. Moreover,

$$\|f\|_{A^2}^2=\int_{\mathbb{B}_n}|\langle f,k_w\rangle_{A^2}|^2\,d\lambda(w).$$

As usual an important ingredient in our treatment will be the Rudin-Forelli estimates, see [191] or [187]. Recall the standard Rudin-Forelli estimates:

$$\int_{\mathbb{B}_n}\frac{|\langle K_w,K_z\rangle_{A^2}|^{\frac{r+s}{2}}}{\|K_z\|_{A^2}^s\|K_w\|_{A^2}^r}d\lambda(w)\le C=C(r,s)<\infty,\quad \forall z\in\mathbb{B}_n \tag{23}$$

for all $r>\kappa>s>0$, where $\kappa=\kappa_n:=\frac{2n}{n+1}$. We will use these in the following form: There exists $\frac{n-1}{n+1}<a<\frac{n}{n+1}$ such that

$$\int_{\mathbb{B}_n}\langle k_z,k_w\rangle_{A^2}\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}d\lambda(w)\le C=C(a)<\infty,\quad \forall z\in\mathbb{B}_n \tag{24}$$

To see that this is true in the classical Bergman space setting, for a given $\frac{n-1}{n+1} < a < \frac{n}{n+1}$ set $r = 1 + a$ and $s = 1 - a$. Then $r + s = 2$, and since $a > \frac{n-1}{n+1}$ we have that $r = 1 + a > \frac{2n}{n+1}$ (equivalently, $a > \frac{2n}{n+1} - 1 = \frac{n-1}{n+1}$). Similarly, we have that $s = 1 - a < \kappa$ if and only if $a > -\frac{n-1}{n+1}$. By plugging these in (23) we obtain (24).

We will also need the following uniform version of the Rudin-Forelli estimates.

Lemma (6.2.6)[184]: Let $\frac{n-1}{n+1} < a < \frac{n}{n+1}$. Then

$$\lim_{R\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,R)^c} |\langle k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) = 0. \tag{25}$$

Proof. Notice first that

$$\int_{D(z,R)^c} |\langle k_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) = \int_{D(0,R)^c} \left|\langle k_z, k_{\varphi_z(w)}\rangle_{A^2}\right| \frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a} d\lambda(w)$$

$$= \int_{D(0,R)^c} |\langle k_z, k_w\rangle_{A^2}|^a \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}} d\lambda(w) = \int_{D(0,R)^c} \frac{|\langle k_z, k_w\rangle_{A^2}|^a}{\|K_w\|_{A^2}^{1+a}} d\lambda(w)$$

$$= \int_{D(0,R)^c} \frac{dv(w)}{|1-\overline{w}z|^{(n+1)a}(1-|w|^2)^{\frac{n+1}{2}(1-a)}}$$

$$= \int_{R'}^{1} \int_{\mathbb{S}_n} \frac{r^{2n-1} d\zeta dr}{|1-zr\bar{\zeta}|^{(n+1)a}(1-r^2)^{\frac{n+1}{2}(1-a)}}.$$

In the last integral $R := \log\frac{1+R'}{1-R'}$. Notice that $R' \to 1$ when $R \to \infty$. Now the last integral can be written as

$$\int_{R'}^{1} I_{(n+1)a-n}(rz) \frac{r^{2n-1} dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}},$$

where

$$I_c(z) := \int_{\mathbb{S}_n} \frac{d\zeta}{|1-zr\bar{\zeta}|^{c+n}}.$$

It is well known, and very simple to check in the n-dimensional case, see [191], that

$$I_{(n+1)a-n}(rz) = \sum_{k=0}^{\infty} \left| \frac{\Gamma\left(k + \frac{n+1}{2}a\right)}{k!\,\Gamma\left(\frac{n+1}{2}a\right)} \right|^2 |rz|^{2k} \simeq \sum_{k=0}^{\infty} \frac{1}{k^{(n+1)(1-a)}} |rz|^{2k}.$$

The last relation follows from the Stirling formula. Thus,

$$\int_{R'}^{1} I_{(n+1)a-n}(rz)\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}} \simeq \int_{R'}^{1}\sum_{k=0}^{\infty}\frac{1}{k^{(n+1)(1-a)}}|rz|^{2k}\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}}$$
$$\leq \int_{R'}^{1}\sum_{k=0}^{\infty}\frac{1}{k^{(n+1)(1-a)}}\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}} = \sum_{k=0}^{\infty}\frac{1}{k^{(n+1)(1-a)}}\int_{R'}^{1}\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}}$$
$$= \int_{0}^{1-R'^2}\frac{(1-x)^{n-1}}{x^{\frac{n+1}{2}(1-a)}}dx\sum_{k=0}^{\infty}\frac{1}{k^{(n+1)(1-a)}} \leq \int_{0}^{1-R'^2}\frac{1}{x^{\frac{n+1}{2}(1-a)}}dx\sum_{k=0}^{\infty}\frac{1}{k^{(n+1)(1-a)}}.$$

Our condition $\frac{n-1}{n+1} < a < \frac{n}{n+1}$ implies that the series above is convergent (here we simply need that $a < \frac{n}{n+1}$. It also implies that $x^{\frac{n+1}{2}(1-a)}$ is integrable on $(0,1)$ (here we require that $a > \frac{n-1}{n+1}$. Thus, we have that

$$\int_{R'}^{1} I_{(n+1)a-n}(rz)\frac{r^{2n-1}dr}{(1-r^2)^{\frac{n+1}{2}(1-a)}} \lesssim (1-(R')^2)^{1-\frac{n+1}{2}(1-a)}.$$

Therefore, taking the limit as $R \to \infty$ (which is the same as $R' \to 1$) we obtain the desired conclusion.

First, we want to make sure that the class of weakly localized operators is large enough to contain some interesting operators. This is indeed true since every Toeplitz operator with a bounded symbol belongs to this class.

Proposition (6.2.7)[184]: Each Toeplitz operator T_u on A^2 with a bounded symbol $u(z)$ is weakly localized.

Proof. By definition

$$T_u k_z(w) = P(uk_z)(w) = \int_{\mathbb{B}_n}\langle K_z, K_x\rangle_{A^2} u(x)k_z(x)\ dv(x).$$

Therefore,

$$|\langle T_u k_z, k_w\rangle_{A^2}| \leq \int_{\mathbb{B}_n}|\langle K_z, K_x\rangle_{A^2}||u(x)||\langle k_z, k_x\rangle_{A^2}|\,d\lambda(x)$$

$$\leq \|u\|_\infty \int_{\mathbb{B}_n}\langle k_w, k_x\rangle_{A^2}|\langle k_x, k_z\rangle_{A^2}|\,d\lambda(x).$$

To check (21) we proceed as follows. For $z, x \in \mathbb{B}_n$, set

$$I_{z(x)} := \int_{D(z,R)^c}\langle k_w, k_x\rangle_{A^2}\frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}}\,d\lambda(w)|\langle k_x, k_z\rangle_{A^2}|$$

First note that

$$\int_{D(z,R)^c} \langle T_u k_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$\leq \|u\|_\infty \int_{D(z,R)^c} \int_{\mathbb{B}_n} \langle k_w, k_x\rangle_{A^2} |\langle k_x, k_z\rangle_{A^2}| \, d\lambda(x) \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$= \|u\|_\infty \int_{\mathbb{B}_n} I_z(x)\, d\lambda(x) = \|u\|_\infty \left(\int_{D\left(z,\frac{r}{2}\right)} + \int_{D\left(z,\frac{r}{2}\right)^c} \right) I_z(x) d\lambda(x).$$

To estimate the first integral notice that for $x \in D\left(z, \frac{r}{2}\right)$ we have $D(z,r)^c \subset D\left(x, \frac{r}{2}\right)^c$. Therefore, the first integral is no greater than

$$\int_{D\left(z,\frac{r}{2}\right)} \int_{D\left(z,\frac{r}{2}\right)^c} \langle k_w, k_x\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) |\langle k_x, k_z\rangle_{A^2}| \, d\lambda(x).$$

It is easy to see that the last expression is no greater than $C(a)\|u\|_\infty A\left(\frac{r}{2}\right)$, where

$$A(r) = \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \langle k_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w),$$

and $C(a)$ is just the bound from the standard Rudin-Forelli estimates (24).

Estimating the second integral is simpler. The second integral is clearly no greater than

$$\int_{D\left(z,\frac{r}{2}\right)} \int_{\mathbb{B}_n} \langle k_w, k_x\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) |\langle k_x, k_z\rangle_{A^2}| \, d\lambda(x).$$

By the standard Rudin-Forelli estimates (24) the inner integral is no greater than

$$C(a) \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a},$$

where the constant $C(a)$ is independent of $z$and x. So, the whole integral is bounded by $C(a)A\left(\frac{r}{2}\right)$. Therefore

$$\sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \langle T_u k_z, k_w\rangle_{A^2} \frac{\|K_z\|^a}{\|K_w\|^a} d\lambda(w) \leq \|u\|_\infty \left(C(a)A\left(\frac{r}{2}\right) + C(a)A\left(\frac{r}{2}\right) \right).$$

Applying the uniform Rudin-Forelli estimates (25) in Lemma (6.2.6) we prove the Proposition since $2C(a)\|u\|_\infty A(\frac{r}{2}) \to 0$ as $r \to \infty$.

We next show that the class of weakly localized operators forms a $*$-algebra.

Proposition (6.2.8)[184]: The collection of all weakly localized operators on A^2 forms a $*$-algebra.

Proof. It is trivial that $T \in \mathcal{A}$ implies $T^* \in \mathcal{A}$. It is also easy to see that any linear combination of two operators in $\mathcal{A}$ must be also in $\mathcal{A}$. It remains to prove that if $T, S \in \mathcal{A}$, then $TS \in \mathcal{A}$.

$$\int_{D(z,r)^c} \langle TSk_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) = \int_{D(z,r)^c} \langle Sk_z, T^*k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$= \int_{D(z,r)^c} \left| \int_{\mathbb{B}_n} \langle Sk_z, k_x\rangle_{A^2} \langle k_x, T^*k_w\rangle_{A^2} d\lambda(x) \right| \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w)$$

$$\leq \int_{\mathbb{B}_n} \int_{D(z,r)^c} |\langle k_x, T^*k_w\rangle_{A^2}| \frac{d\lambda(w)}{\|K_w\|_{A^2}^a} |\langle Sk_z, k_w\rangle_{A^2}| \|K_z\|_{A^2}^a \, d\lambda(x).$$

Proceeding exactly as in the proof of the previous Proposition and using the conditions following from $T, S \in \mathcal{A}$ in the place of the local Rudin-Forelli estimates (25) we obtain that

$$\lim_{r\to\infty} \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \langle TSk_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) = 0.$$

The corresponding version for $(TS)^*$ is proved in exactly the same way.

We next show that every weakly localized operator can be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [187].

Proposition (6.2.9)[184]: There exists an integer $N > 0$ such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{f_i\}$ of $\mathbb{B}_n$ by disjoint Borel sets satisfying

(i) every point of $\mathbb{B}_n$ belongs to at most $N$of the sets $G_j := \{z \in \Omega: d(z, F_j) \leq r\}$,

(ii) $\text{diam}_d F_j \leq 2r$ for every j.

We use this to prove the following proposition, which is similar to what appears in [187], but exploits condition (21).

Proposition (6.2.10)[184]: Let $T: A^2 \to A^2$ be a weakly localized operator. Then for every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{f_j\}$ (associated to r) from Proposition (6.2.9)

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^2 \to L^2(\mathbb{B}_n\, dv)} < \epsilon.$$

Proof. Define

$$S = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}$$

It suffices to show that given $\epsilon > 0$ we can find an $r > 0$ so that

$$\|Sf\|_{A^2} \lesssim \epsilon \|f\|_{A^2}.$$

Given ϵ choose r large enough so that

$$\sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \langle Tk_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \epsilon$$

$$\text{and } \sup_{z\in\mathbb{B}_n} \int_{D(z,r)^c} \langle T^*k_z, k_w\rangle_{A^2} \frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a} d\lambda(w) < \epsilon$$

Note that for any $z \in \mathbb{B}_n$ we have that

$$|Sf(z)| \leq \int_{\mathbb{B}_n} \sum_j 1_{F_j(z)} 1_{G_j^c}(w) \langle T^* K_z, K_w \rangle_{A^2} |f(w)| dv(w) = \int_{G_j^c} |\langle T^* K_z, K_w \rangle_{A^2}| |f(w)| dv(w)$$

$$\leq \int_{D(z,r)^c} |\langle T^* K_z, K_w \rangle_{A^2}| |f(w)| dv(w) .$$

Now

$$\|Sf\|_{A^2}^2 = \int_{\mathbb{B}_n} |Sf(z)|^2 dv(w) \leq \int_{\mathbb{B}_n} \left(\int_{D(z,r)^c} |\langle T^* K_z, K_w \rangle_{A^2}| |f(w)| d\sigma(w) \right)^2 dv(w)$$

$$\leq \int_{\mathbb{B}_n} \left(\int_{D(z,r)^c} |\langle T^* K_z, K_w \rangle_{A^2}| \|K_w\|_{A^2}^a |f(w)|^2 dv(w) \right) \left(\int_{D(z,r)^c} \frac{|\langle T^* K_z, K_w \rangle_{A^2}|}{\|K_w\|_{A^2}^a} \, dv(w) \right) dv(z)$$

$$= \int_{\mathbb{B}_n} \left(\int_{D(z,r)^c} |\langle T^* K_z, K_w \rangle_{A^2}| \|K_w\|_{A^2}^a |\langle f, k_w \rangle_{A^2}|^2 d\lambda(w) \right) \times \left(\int_{D(z,r)^c} \frac{|\langle T^* k_z, k_w \rangle_{A^2}|}{\|K_w\|_{A^2}^a} d\lambda(w) \right) d\lambda(z)$$

$$\leq \epsilon \int_{\mathbb{B}_n} \left(\int_{D(z,r)^c} |\langle T^* k_z, k_w \rangle_{A^2}| \|K_w\|_{A^2}^a |\langle f, k_w \rangle_{A^2}|^2 d\lambda(w) \right) \frac{1}{\|K_z\|_{A^2}^a} d\lambda(z)$$

$$= \epsilon \int_{\mathbb{B}_n} |\langle f, k_w \rangle_{A^2}|^2 \left(\int_{D(w,r)^c} |\langle k_z, T k_w \rangle_{A^2}| \frac{\|K_w\|_{A^2}^a}{\|K_z\|_{A^2}^a} d\lambda(z) \right) d\lambda(w) \leq \epsilon \int_{\mathbb{B}_n} |\langle f, k_w \rangle_{A^2}|^2 \, d\lambda(w)$$

$$= \epsilon^2 \|f\|_{A^2}^2.$$

We can now prove one of our main results. The proof is very similar to [187]. For completeness we give the details.

Theorem (6.2.11)[184]: Let $T: A^2 \to A^2$ be a linear operator in the C^*-algebra generated by the weakly localized operators on A^2. If $\lim_{|z| \to 1} \|T k_z\|_{A^2} = 0$, then T must be compact.

Proof. Let $\epsilon > 0$. By Proposition 6.2.10 there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ associated to r (from Proposition (6.2.18))

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^2 \to L^2(\mathbb{B}_n \, dv)} < \epsilon.$$

Since $\sum_{j \leq m} M_{1_{F_j}} TPM_{1_{G_j}}$ is compact for every $m \in \mathbb{N}$ we have that the essential norm of TP as an operator from $L^2(\mathbb{B}_n \, dv)$ to A^2 can be estimated in the following way.

$$\|TP\|_e \leq \left\| TP - \sum_{j \leq m} M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^2 \to L^2(\mathbb{B}_n \, dv)}$$

$$\leq \left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^2 \to L^2(\mathbb{B}_n \, dv)} + \|T_m\| \lesssim \epsilon,$$

where

$$T_m = \sum_{j \leq m} M_{1_{F_j}} TPM_{1_{G_j}} .$$

If we can show that

$$\limsup_{m\to\infty} \|T_m\|_{A^2\to L^2(\mathbb{B}_n\, dv)} \lesssim \epsilon,$$

then since $\epsilon > 0$ is arbitrary we will have that TP is compact and hence T is compact on A^2.

Let $f \in A^2$ be arbitrary of norm no greater than 1. Then,

$$\|T_m f\|_{A^2}^2 = \sum_{j\geq m} \left\|M_{1_{F_j}} TPM_{1_{G_j}} f\right\|_{A^2}^2 = \sum_{j\geq m} \frac{\left\|M_{1_{F_j}} TPM_{1_{G_j}} f\right\|_{A^2}^2}{\left\|M_{1_{G_j}} f\right\|_{A^2}^2} \left\|M_{1_{G_j}} f\right\|_{A^2}^2$$

$$\leq N \sup_{j\geq m} \left\|M_{1_{F_j}} T l_j\right\|_{A^2}^2 \lesssim \sup_{j\geq m} \left\|T l_j\right\|_{A^2}^2,$$

where

$$l_j := P\frac{M_{1_{G_j}} f}{\left\|M_{1_{G_j}}\right\|_{A^2}}.$$

We now have

$$\|T_m\|_{A^2\to L^2(\mathbb{B}_n\, dv)} \lesssim \sup_{j\geq m} \sup_{\|f\|\leq 1} \left\{ \left\|T l_j\right\|_{A^2} : l_j = \frac{PM_{1_{G_j}} f}{\left\|M_{1_{G_j}}\right\|_{A^2}} \right\}$$

and hence

$$\limsup_{m\to\infty} \|T_m\|_{A^2\to L^2(\mathbb{B}_n\, dv)} \lesssim \limsup_{j\to\infty} \sup_{\|f\|_{A^2}\leq 1} \left\{ \left\|T l_j\right\|_{A^2} : l_j = \frac{PM_{1_{G_j}} f}{\left\|M_{1_{G_j}}\right\|_{A^2}} \right\}$$

There exists a sequence $\{f_j\}$ in A^2 with $\left\|f_j\right\|_{A^2} \leq 1$ such that

$$\limsup_{j\to\infty} \sup_{\|f\|\leq 1} \left\{ \left\|T_g\right\|_{A^2} : g = \frac{PM_{1_{G_j}} f}{\left\|M_{1_{G_j}}\right\|_{A^2}} \right\} - \epsilon \leq \limsup_{j\to\infty} \|T g_i\|_{A^2},$$

where

$$g_j := P\frac{M_{1_{G_j}} f_i}{\left\|M_{1_{G_j}}\right\|_{A^2}} = \frac{\int_{G_j} \langle f_i, k_w\rangle_{A^2} k_w \, d\lambda(w)}{\left(\int_{G_j} |\langle f_i, k_w\rangle_{A^2}|^2 k_w \, d\lambda(w)\right)^{\frac{1}{2}}}.$$

For each $j \geq m$ pick $z_j \in G_j$. There exists $\rho > 0$ such that $G_j \subset D(z_j, \rho)$. Doing a simple change of variables we obtain

$$g_j = \int_{\varphi_{z_j}(G_j)} a_j\left(\varphi_{z_j}(w)\right) U_{z_j}^* k_w \, d\lambda\left(\varphi_{z_j}(w)\right),$$

where $a_j(w)$ is defined to be

$$\frac{\langle f_i, k_w\rangle_{A^2}}{\left(\int_{G_j} |\langle f_i, k_w\rangle_{A^2}|^2 \, d\lambda(w)\right)^{\frac{1}{2}}}$$

on G_j, and zero otherwise.

We claim that $g_j = U_{z_j}^* h_j$, where

$$h_j(z) := \int_{\varphi_{z_j}(G_j)} a_j\left(\varphi_{z_j}(w)\right) k_w(z) d\lambda\left(\varphi_{z_j}(w)\right).$$

First, using the generalized Minkowski Inequality it is easy to see that $h_j \in L^2(\mathbb{B}_n; dv)$ and consequently in A^2. To prove that the claim is correct we only need to show that for each $g \in L^2(\mathbb{B}_n;\ dv)$ we have that $\langle g_j, g\rangle = \langle h_j, U_{z_j} g\rangle$. This can be readily done using Fubini's Theorem. The total variation of each member of the sequence of measures $a_j\left(\varphi_{z_j}(w)\right) d\lambda\left(\varphi_{z_j}(w)\right)$ (as measures on the compact set $\overline{D(0,\rho)}$) satisfies $\left\|a_j\left(\varphi_{z_j}(w)\right) d\lambda\left(\varphi_{z_j}(w)\right)\right\|_{A^2} \lesssim \lambda\left(\varphi_{z_j}(G_j)\right) \leq \lambda(D(0,\rho))$.

Therefore, there exists a weak-$*$ convergent subsequence which approaches some measure ν. Abusing notation slightly we keep indexing this subsequence by j. Let

$$h(z) := \int_{D(0,\rho)} k_w(z) d\,\nu(w).$$

The mentioned weak-$*$ convergence implies that h_j converges to h pointwise. Using the Lebesgue Dominated Convergence Theorem we obtain that $h_j \to h$ in $L^2(\mathbb{B}_n;\ dv)$. This implies that $h \in A^2$. In addition, $1 \geq \|g_j\|_{A^2} = \left\|U_{z_j}^* h_j\right\|_{A^2} = \|h_j\|_{A^2}$.Thus, $\|h\|_{A^2} \lesssim 1$. So, we finally have

$$\limsup_{m\to\infty} \|T_m\|_{A^2\to L^2(\mathbb{B}_n, dv)} \lesssim \limsup_{j\to\infty} \|T g_j\|_{A^2} + \epsilon = \limsup_{j\to\infty} \left\|T U_{z_j}^* h\right\|_{A^2} + \epsilon$$
$$\lesssim \limsup_{j\to\infty} \left\|T U_{z_j}^* h\right\|_{A^2} + \epsilon.$$

Choose h' in the linear span of normalized reproducing kernels such that $\|h - h'\|_{A^2} < \epsilon/\|T\|$. The assumption $\|T k_z\|_{A^2} \to 0$ as $|z| \to 1$ implies that $\left\|T U_{z_j}^* h'\right\|_{A^2} \to 0$ as $j \to \infty$. Therefore we finally obtain

$$\limsup_{m\to\infty} \|T_m\|_{A^2\to L^2(\mathbb{B}_n, dv)} \lesssim \limsup_{j\to\infty} \left\|T U_{z_j}^* h\right\|_{A^2} \leq \limsup_{j\to\infty} \left\|T U_{z_j}^* h'\right\|_{A^2} + \epsilon = \epsilon.$$

Our next goal is to weaken the assumption $\|T k_z\|_{A^2} \to 0$ to $\langle T k_z, k_z\rangle_{A^2} \to 0$ as $|z| \to 1$.

We can do this only for weaklylocalized operators and not for the entire C^*algebra. We will make use of the following result due to Axler and Zheng in the case of the disc, and Raimondo (with the same proof) in the case of the unit ball. See also the work of Engliŝ for the case of bounded symmetric domains [186]. In all cases the proof appearing in [76], [186], [75] applies to all bounded linear operators.

Theorem (6.2.12)[184]: (Axler and Zheng, [76]; Raimondo [75]). For any bounded operator T on the Bergman space A^2 if

$$\lim_{|z|\to 1} \langle T k_z, k_z\rangle_{A^2} = 0,$$

then $T_z k_0 \to 0$ weakly. Consequently, in this case $T_z k_0$ converges to 0 uniformly on compact subsets of the ball $\mathbb{B}_n$.

We can show the following Lemma.

Lemma (6.2.13)[184]: If T is weakly localized operator on A^2, then

$$\lim_{|z|\to 1}\int_{\mathbb{B}_n}|\langle Tk_z,k_z\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\,\mathrm{d}\lambda(\mathrm{w})=0.$$

Proof. First, observe that:

$$\int_{\mathbb{B}_n}|\langle Tk_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\,\mathrm{d}\lambda(w)=\left(\int_{D(z,r)}+\int_{D(z,r)^c}\right)|\langle Tk_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\,d\lambda(w).$$

Let $\epsilon>0$. Our assumption on T implies that we can choose $r>0$ large enough so that the second integral is less than ϵ. We need to show that the first integral can be made smaller than (a constant times) ϵ. Fix this $r>0$ and then consider the first integral,

$$\int_{D(z,r)}|\langle Tk_z,k_w\rangle_{A^2}|\frac{\|K_z\|^a}{\|K_w\|^a}\,d\lambda(w)=\int_{D(z,r)^c}\left|\langle T_zk_0,k_{\varphi_z(w)}\rangle_{A^2}\right|\frac{\|K_z\|_{A^2}^a}{\|K_w\|_{A^2}^a}\,d\lambda(w)$$

$$=\int_{D(0,r)}|\langle T_zk_0,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a}\,d\lambda(w)$$

$$=\int_{D(0,r)}|T_zk_0(w)|\frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a}\frac{d\lambda(w)}{\|K_w\|_{A^2}}.$$

We can choose z with $|z|$ large enough so that $|T_zk_0(w)|<\epsilon$ for all $w\in\overline{D(0,r)}$ (we have uniform convergence to 0 on the compact set $\overline{D(0,r)}$) . Therefore the second integral is no greater than

$$\epsilon\int_{D(0,r)}\frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a}\frac{d\lambda(w)}{\|K_w\|_{A^2}}=\epsilon\int_{D(0,r)}|\langle k_z,k_w\rangle_{A^2}|\frac{\|K_z\|_{A^2}^a}{\left\|K_{\varphi_z(w)}\right\|_{A^2}^a}\,d\lambda(w)\le C(a)\epsilon.$$

Here we have used (22). Since $\epsilon>0$ was arbitrary we are done with the proof.

We can now show the following result.

Theorem (6.2.14)[184]: Let T be a weakly localized operator on A^2. If

$$\lim_{|z|\to 1}\langle Tk_z,k_z\rangle_{A^2}=0,$$

then T is compact.

Proof. Notice first that

$$\langle Tf,k_w\rangle_{A^2}=\langle f,T^*k_w\rangle_{A^2}=\int_{\mathbb{B}_n}\langle f,k_z\rangle_{A^2}\langle k_z,T^*k_w\rangle_{A^2}\,d\lambda(z)$$

$$=\int_{\mathbb{B}_n}\langle Tk_z,k_w\rangle_{A^2}\langle f,k_z\rangle_{A^2}\,d\lambda(z).$$

For each $r>0$ define an operator T_r by

$$\langle T_rf,k_w\rangle_{A^2}:=\int_{D(0,r)}\langle Tk_z,k_w\rangle_{A^2}\langle f,k_z\rangle_{A^2}d\lambda(z).$$

These are basically the operators $S_{[r]}$ that Axler and Zheng use in [76]. It is easy to see that these are all Hilbert-Schmidt operators by testing the square integrability of the kernel. Therefore, it suffices to show that $T_r\to T$ in the operator norm.

Let $\epsilon>0$ and let $f\in A^2$ be an arbitrary element of norm 1. We have

$$\|(T-T_r)f\|_{A^2}^2 = \int_{\mathbb{B}_n} |\langle (T-T_r)f, k_w\rangle_{A^2}|^2 \, d\lambda(w).$$

We first examine the integrand.

$$|\langle (T-T_r)f, k_w\rangle_{A^2}| \le \int_{D(0,r)^c} |\langle Tk_z, k_w\rangle_{A^2} \langle f, k_z\rangle_{A^2}| d\lambda(z)$$

$$= \int_{D(0,r)} |\langle Tk_z, k_w\rangle_{A^2}|^{1/2} \frac{\|K_w\|_{A^2}^{a/2}}{\|K_z\|_{A^2}^{a/2}} |\langle Tk_z, k_w\rangle_{A^2}|^{1/2} |\langle f, k_z\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a/2}}{\|K_w\|_{A^2}^{a/2}} d\lambda(z)$$

$$\le \left(\int_{D(0,r)^c} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_w\|_{A^2}^{a}}{\|K_z\|_{A^2}^{a}} d\lambda(z) \right)^{\frac{1}{2}}$$

$$\times \left(\int_{D(0,r)^c} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}} |\langle f, k_z\rangle_{A^2}|^2 d\lambda(z) \right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{D(0,r)^c} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}} |\langle f, k_z\rangle_{A^2}|^2 d\lambda(z) \right)^{\frac{1}{2}}.$$

Here, the implied constant depends on (20). Thus, after an application of Fubini, we obtain

$$\|(T-T_r)f\|_{A^2}^2 \lesssim \int_{D(0,r)^c} \left(\int_{\mathbb{B}_n} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}} d\lambda(w) \right) |\langle f, k_z\rangle_{A^2}|^2 d\lambda(z).$$

Notice that the variable z in the inner integral is from $D(0,r)^c$. Therefore, by Lemma (6.2.13), we can choose $r > 0$ large enough so that for $z \in \mathbb{B}_n$ with $|z|$ sufficiently close to 1

$$\int_{\mathbb{B}_n} |\langle Tk_z, k_w\rangle_{A^2}| \frac{\|K_z\|_{A^2}^{a}}{\|K_w\|_{A^2}^{a}} d\lambda(w) < \epsilon.$$

Therefore, for such large r we have

$$\|(T-T_r)f\|_{A^2}^2 \lesssim \int_{D(0,r)^c} \epsilon |\langle f, k_z\rangle_{A^2}|^2 d\lambda(z) \lesssim \epsilon \|f\|_{A^2}^2 = \epsilon.$$

This proves that $T_r \to T$ in the operator norm, and so we are done.

In Bargmann-Fock space case, we prove somewhat more general statement. Some parts of the proof are essentially identical to the case of the Bergman space. We only outline the necessary modifications.

Let

$$D(z,r) := \{w \in \mathbb{C}^n : |w-z| < r\}$$

denote the standard Euclidean disc centered at the point z of radius $r > 0$. For $z \in \mathbb{C}^n$, we define

$$U_z f(w) := f(z-w) k_z(w),$$

which via a simple change of variables argument is clearly a unitary operator on $\mathcal{F}^2$. Then for an operator T on the Bargmann-Fock space $\mathcal{F}^2$, for $z \in \mathbb{C}^n$ we set

$$T_z := U_z T U_z^*.$$

We denote

$$d\sigma(z) := e^{-|z|^2} dv(z) = \frac{dv(z)}{\|K_z\|_{\mathcal{F}^2}^2}.$$

Recall also that the orthogonal projection of $L^2(\mathbb{C}^n, d\sigma(z))$ onto $\mathcal{F}^2$ is given by the integral operator

$$P(f)(z) := \int_{\mathbb{C}^n} \langle K_w, K_z \rangle_{\mathcal{F}^2} f(w) d\sigma(w) \, .$$

Therefore, for all $f \in \mathcal{F}^2$ we have $f(z) = \int_{\mathbb{C}^n} \langle f, k_w \rangle_{\mathcal{F}^2} dv(w)$. Moreover,

$$\|f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}^n} |\langle f, k_w \rangle_{\mathcal{F}^2}|^2 dv(w) \, .$$

The following analog of Lemma (6.2.6) is simpler to prove in this case.

Lemma (6.2.15)[184]:

$$\lim_{R\to\infty} \sup_{z\in\mathbb{C}^n} \int_{D(z,R)^c} \langle k_z, k_w \rangle_{\mathcal{F}^2} dv(w) = 0.$$

The proof of this is immediate since

$$\int_{D(z,R)^c} \langle k_z, k_w \rangle_{\mathcal{F}^2} dv(w) = \int_{D(0,R)^c} e^{-|w|^2} dv(w)$$

which clearly goes to zero as $R \to \infty$.

As in the Bergman case the class of weakly localized operators contains all Toeplitz operators with bounded symbols. In addition this class forms a $*$-algebra. The proof of these two facts is basically the same as in the Bergman space details.

Proposition (6.2.16)[184]: Each Toeplitz operator T_u on A^2 with a bounded symbol $u(z)$ is weakly localized.

Note that T_u is sufficiently localized even in the sense of Xia and Zheng by [190].

Proposition (6.2.17)[184]: The collection of all weakly localized operators on $\mathcal{F}^2$ forms a $*$-algebra.

We next prove that operators from the $*$-algebra of weakly localized operators can also be approximated by infinite sums of well localized pieces. To state this property we need to recall the following proposition proved in [187].

Proposition (6.2.18)[184]: There exists an integer $N > 0$ such that for any $r > 0$ there is a covering $\mathcal{F}_r = \{F_j\}$ of $\mathbb{C}^n$ by disjoint Borel sets satisfying

(i) every point of $\mathbb{C}^n$ belongs to at most N of the sets $G_j := \{z \in \Omega : d(z, F_j) \leq r\}$,

(ii) $\operatorname{diam}_d F_j \leq 2r$ for every j.

We use this to prove the following proposition, which is similar to what appears in [187], but exploits condition (21).

Proposition (6.2.19)[184]: Let $T: \mathcal{F}^2 \to \mathcal{F}^2$ be a linear operator satisfying (21). Then for every $\epsilon > 0$ there exists $r > 0$ such that for the covering $\mathcal{F}_r = \{F_j\}$ (associated to r) from Proposition (6.2.18).

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{\mathcal{F}^2 \to L^2\left(\mathbb{C}^n;\, e^{-|z|^2} dv(z)\right)} < \epsilon.$$

Proof. Define

$$S = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}$$

It suffices to show that given $\epsilon > 0$ we can find an $r > 0$ so that

$$\|Sf\|_{\mathcal{F}^2} \lesssim \epsilon \|f\|_{\mathcal{F}^2}.$$

Given ϵ choose r large enough so that

$$\sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| dv(w) < \epsilon \quad \text{and} \quad \sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle T k_z, k_w\rangle_{\mathcal{F}^2}| dv(w) < \epsilon.$$

Note that for any $z \in \mathbb{C}^n$ we have that

$$|Sf(z)| \le \int_{\mathbb{C}^n} \sum_j 1_{F_j(z)} 1_{G_j^c}(w) |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| |f(w)| d\sigma(w)$$

$$= \int_{G_j^c} |\langle T^* K_z, K_w\rangle_{\mathcal{F}^2}| |f(w)| d\sigma(w).$$

Now

$$\|Sf\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}^n} |Sf(z)|^2 d\sigma(z) \le \int_{\mathbb{C}^n} \left(\int_{D(z,r)^2} |\langle T^* K_z, K_w\rangle_{\mathcal{F}^2}| |f(w)| d\sigma(w) \right)^2 d\sigma(z)$$

$$\le \int_{\mathbb{C}^n} \left(\int_{D(z,r)^2} |\langle T^* K_z, K_w\rangle_{\mathcal{F}^2}| |f(w)|^2 d\sigma(w) \right) \left(\int_{D(z,r)^2} |\langle T^* K_z, K_w\rangle_{\mathcal{F}^2}| d\sigma(w) \right) d\sigma(z)$$

$$= \int_{\mathbb{C}^n} \left(\int_{D(z,r)^2} |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| |\langle f, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) \right) \left(\int_{D(z,r)^2} |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| dv(w) \right) dv(z)$$

$$\le \epsilon \int_{\mathbb{C}^n} \left(\int_{D(z,r)^2} |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| |\langle f, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) \right) dv(z)$$

$$= \epsilon \int_{\mathbb{C}^n} |\langle f, k_w\rangle_{\mathcal{F}^2}|^2 \left(\int_{D(z,r)^2} |\langle T^* k_z, k_w\rangle_{\mathcal{F}^2}| dv(w) \right) dv(w)$$

$$\le \epsilon^2 \int_{\mathbb{C}^n} |\langle f, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) = \epsilon^2 \|f\|_{\mathcal{F}^2}^2.$$

The proof of the next result is basically the same as the proof of Theorem (6.2.10) and therefore we skip it.

Theorem (6.2.20)[184]: Let $T: \mathcal{F}^2 \to \mathcal{F}^2$ be a bounded linear operator in the C^* algebra generated by weakly localized operators. If

$$\lim_{|z|\to\infty} \|Tk_z\|_{\mathcal{F}^2} = 0,$$

then T is compact.

By combining this result with the following Proposition we obtain the desired proof of Theorem (6.2.2).

Proposition (6.2.21)[184]: Let $T:\mathcal{F}^2 \to \mathcal{F}^2$ be a bounded linear operator in the C^* algebra generated by weakly localized operators satisfying (19). If

$$\lim_{|z|\to\infty} \langle Tk_z, k_z\rangle_{\mathcal{F}^2} = 0,$$

then

$$\lim_{|z|\to\infty} \|Tk_z\|_{\mathcal{F}^2} = 0.$$

Proof. Let $\epsilon > 0$ be given. Note that we have

$$\|Tk_z\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}^n} |\langle Tk_z, k_z\rangle_{\mathcal{F}^2}|^2 dv(w) = \left(\int_{D(z,r)} + \int_{D(z,r)}\right) |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) = I + II.$$

For term II we use a standard approximation result. Since T is in the C^* algebra generated by (19), given $\epsilon > 0$ we can find T' that is weakly localized so that

$$\|T - T'\|_{\mathcal{F}^2\to\mathcal{F}^2} < \epsilon.$$

Now choose r sufficiently large so that

$$\sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle T'k_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) < \epsilon$$

which is possible since T' satisfies (19). We use this to estimate II.

$$II = \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w)$$

$$\lesssim \int_{D(z,r)^c} |\langle T'k_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) + \int_{D(z,r)^c} |\langle (T - T')k_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w).$$

The first term is controlled by the choice of r above. We also have that for this choice of r (actually all r) that

$$\sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle (T - T')k_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) \le \sup_{z\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle (T - T')k_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w)$$

$$= \sup_{z\in\mathbb{C}^n} \|(T - T')k_z\|_{\mathcal{F}^2}^2 \le \|(T - T')k_z\|_{\mathcal{F}^2\to\mathcal{F}^2} < \epsilon.$$

Then then yields that

$$\sup_{z\in\mathbb{C}^n} \int_{D(z,r)^c} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) \lesssim \epsilon$$

Now for term I, note that we have

$$\int_{D(z,r)} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) = \int_{D(z,r)} \left|\langle T_z k_0, k_{\varphi_z(w)}\rangle_{\mathcal{F}^2}\right|^2 dv(w)$$

$$= \int_{D(0,r)} \left|\langle T_z k_0, k_{\varphi_z(w)}\rangle_{\mathcal{F}^2}\right|^2 dv(w) = \int_{D(0,r)} |T_z k_0(w)|^2 \frac{dv(w)}{\|K_w\|_{\mathcal{F}^2}^2}$$

$$= \int_{D(0,r)} |T_z k_0(w)|^2 d\sigma(w).$$

By Englis [186] we can choose z with $|z|$ large enough so that $|T_z k_0(w)| < \epsilon$ for all $w \in \overline{D(0,r)}$ (we have uniform convergence to 0 on the compact set $\overline{D(0,r)}$). Furthermore, we have that $\sigma\big(D(0,r)\big) \le \sigma(\mathbb{C}^n) < \infty$, and so

$$\int_{\overline{D(z,r)}} |\langle Tk_z, k_w\rangle_{\mathcal{F}^2}|^2 dv(w) = \int_{\overline{D(z,r)}} \left|\langle T_z k_0, k_{\varphi_z(w)}\rangle_{\mathcal{F}^2}\right|^2 dv(w) < \epsilon.$$

Thus, we have

$$\lim_{|z|\to\infty} \|Tk_z\|_{\mathcal{F}^2}^2 = 0.$$

Corollary (6.2.22)[200]: Let $\frac{2n}{n+1} < a^2 < \frac{2n+1}{n+1}$. Then

$$\lim_{R\to\infty} \sup_{(z^2-1)\in\mathbb{B}_n} \int_{D(z^2-1,R)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1) = 0. \quad (26)$$

Proof. Notice first that

$$\int_{D(z^2-1,R)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{D(z^2-1,R)^c} \left|\langle k_{z^2-1}, k_{\varphi_{z^2-1}(w^2-1)}\rangle_{A^2}\right| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{D(0,R)^c} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|^{a^2-1} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{D(0,R)^c} \frac{|\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|^2}{\|K_{w^2-1}\|_{A^2}^{a^2}} d\lambda(w^2-1)$$

$$= \int_{D(0,R)^c} \frac{dv(w^2-1)}{\left|1-\overline{(w^2-1)}(z^2-1)\right|^{(a^2-1)(n+1)}(1-|w^2-1|^2)^{-a^2\left(\frac{n+1}{2}\right)}}$$

$$= \int_{R'}^{1}\int_{\mathbb{S}_n} \frac{(1+3\epsilon)^{2n-1} d\zeta\, d(1+3\epsilon)}{|1-(z^2-1)(1+3\epsilon)\bar{\zeta}|^{(n+1)(a^2-1)}\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}.$$

In the last integral $R := \log\frac{1+R'}{1-R'}$. Notice that $R' \to 1$ when $R \to \infty$. Now the last integral can be written as

$$\int_{R'}^{1} I_{(n+1)a^2-1-n}((1+3\epsilon)(z^2-1)) \frac{(1+3\epsilon)^{2n-1} d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}},$$

where

$$I_c(z^2-1) := \int_{\mathbb{S}_n} \frac{d\zeta}{|1-(z^2-1)(1+3\epsilon)\bar{\zeta}|^{c+n}}.$$

It is well known, and very simple to check in the n-dimensional case, see [191], that

$$I_{(a^2-1)(n+1)-n}((1+3\epsilon)(z^2-1))\sum_{k=0}^{\infty}\left|\frac{\Gamma\left(k+\frac{n+1}{2}(a^2-1)\right)}{k!\,\Gamma\left(\frac{n+1}{2}(a^2-1)\right)}\right|^2|(1+3\epsilon)(z^2-1)|^{2k}$$

$$\simeq\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}|(1+3\epsilon)(z^2-1)|^{2k}.$$

The last relation follows from the Stirling formula. Thus,

$$\int_{R'}^{1}I_{(a^2-1)(n+1)-n}\big((1+3\epsilon)(z^2-1)\big)\frac{(1+3\epsilon)^{2n-1}d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}$$

$$\simeq\int_{R'}^{1}\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}|(1+3\epsilon)(z^2-1)|^{2k}\frac{(1+3\epsilon)^{2n-1}d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}$$

$$\leq\int_{R'}^{1}\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}\frac{(1+3\epsilon)^{2n-1}d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}$$

$$=\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}\int_{R'}^{1}\frac{(1+3\epsilon)^{2n-1}d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}$$

$$=\int_{0}^{1-{R'}^2}\frac{(1-x)^{n-1}}{x^{-a^2\left(\frac{n+1}{2}\right)}}dx\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}\leq\int_{0}^{1-{R'}^2}\frac{1}{x^{-a^2\left(\frac{n+1}{2}\right)}}dx\sum_{k=0}^{\infty}\frac{1}{k^{-a^2(n+1)}}.$$

Our condition $\frac{2n}{n+1}<a^2<\frac{2n+1}{n+1}$ implies that the series above is convergent (here we simply need that $a^2<\frac{2n+1}{n+1}$. It also implies that $x^{-a^2\left(\frac{n+1}{2}\right)}$ is integrable on $(0,1)$ (here we require that $a^2\lesssim\frac{2n}{n+1}$. Thus, we have that

$$\int_{R'}^{1}I_{(n+1)(a^2-1)-n}((1+3\epsilon)(z^2-1))\frac{(1+3\epsilon)^{2n-1}d(1+3\epsilon)}{\big(3\epsilon(2+3\epsilon)\big)^{-a^2\left(\frac{n+1}{2}\right)}}\simeq(1-(R')^2)^{1+a^2\left(\frac{n+1}{2}\right)}.$$

Therefore, taking the limit as $R\to\infty$ (which is the same as $R'\to 1$) we obtain the desired conclusion.

Corollary (6.2.23)[200]: Each Toeplitz operator T_u on A^2 with a bounded symbol $u(z^2-1)$ is weakly localized.

Proof. By definition

$$T_u k_{z^2-1}(w^2-1)=P(uk_{z^2-1})(w^2-1)$$

$$=\int_{\mathbb{B}_n}\langle K_{z^2-1},K_{x^2-1}\rangle_{A^2}u(x^2-1)k_{z^2-1}(x^2-1)\quad dv(x^2-1).$$

Therefore,

$$|\langle T_u k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \leq \int_{\mathbb{B}_n} |\langle K_{z^2-1}, K_{x^2-1}\rangle_{A^2}||u(x^2-1)||\langle k_{z^2-1}, k_{x^2-1}\rangle_{A^2}|\, d\lambda(x^2-1)$$
$$\leq \|u\|_\infty \int_{\mathbb{B}_n} \langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}|\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1).$$

To check (21) we proceed as follows. For $(z^2-1), (x^2-1) \in \mathbb{B}_n$, set

$$I_{(z^2-1)(x^2-1)} := \int_{D(z^2-1,R)^c} \langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)|\langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}|$$

First note that

$$\int_{D(z^2-1,R)^c} \langle T_u k_{z^2-1}, k_{w^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$
$$\leq \|u\|_\infty \int_{D(z^2-1,R)^c} \int_{\mathbb{B}_n} \langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2}|\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2$$
$$-1) \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1) = \|u\|_\infty \int_{\mathbb{B}_n} I_{z^2-1}(x^2-1)\, d\lambda(x^2-1)$$
$$= \|u\|_\infty \left(\int_{D\left(z^2-1,\frac{1+3\epsilon}{2}\right)} + \int_{D\left(z^2-1,\frac{1+3\epsilon}{2}\right)^c} \right) I_{z^2-1}(x^2-1) d\lambda(x^2-1).$$

To estimate the first integral notice that for $(x^2-1) \in D\left(z^2-1, \frac{1+3\epsilon}{2}\right)$ we have $D(z^2-1, 1+3\epsilon)^c \subset D\left(x^2-1, \frac{1+3\epsilon}{2}\right)^c$.

Therefore, the first integral is no greater than

$$\int_{D\left(z^2-1,\frac{1+3\epsilon}{2}\right)} \int_{D\left(z^2-1,\frac{1+3\epsilon}{2}\right)^c} \langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2$$
$$-1)|\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1).$$

It is easy to see that the last expression is no greater than $C(a^2-1)\|u\|_\infty A\left(\frac{1+3\epsilon}{2}\right)$, where

$$A(1+3\epsilon) = \sup_{(z^2-1)\in\mathbb{B}_n} \int_{D(z^2-1,1+3\epsilon)^c} \langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)\,,$$

and $C(a^2-1)$ is just the bound from the standard Rudin-Forelli estimates (24).

Estimating the second integral is simpler. The second integral is clearly no greater than

$$\int_{D\left(z^2-1,\frac{1+3\epsilon}{2}\right)} \int_{\mathbb{B}_n} \langle k_{w^2-1}, k_{x^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)|\langle k_{x^2-1}, k_{z^2-1}\rangle_{A^2}|\, d\lambda(x^2-1).$$

By the standard Rudin-Forelli estimates (24) the inner integral is no greater than

$$C(a^2-1)\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}},$$

where the constant $C(a^2-1)$ is independent of (z^2-1) and (x^2-1). So, the whole integral is bounded by $C(a^2-1)A\left(\frac{1+3\epsilon}{2}\right)$. Therefore

$$\sup_{(z^2-1)\in\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c}\langle T_u k_{z^2-1},k_{w^2-1}\rangle_{A^2}\frac{\|K_{z^2-1}\|^{a^2-1}}{\|K_{w^2-1}\|^{a^2-1}}d\lambda(w^2-1)$$
$$\leq 2\|u\|_\infty\left(C(a^2-1)A\left(\frac{1+3\epsilon}{2}\right)\right).$$

Applying the uniform Rudin-Forelli estimates (26) in Corollary (6.2.22) we prove the Proposition since $2C(a^2-1)\|u\|_\infty A\left(\frac{1+3\epsilon}{2}\right)\to 0$ as $\epsilon\to\infty$.

Corollary (6.2.24)[200]: The collection of all weakly localized operators on A^2 forms a $*$-algebra.

Proof. It is trivial that $T\in\mathcal{A}$ implies $T^*\in\mathcal{A}$. It is also easy to see that any linear combination of two operators in $\mathcal{A}$ must be also in $\mathcal{A}$. It remains to prove that if $T,(T+\epsilon)\in\mathcal{A}$, then $T(T+\epsilon)\in\mathcal{A}$.

$$\int_{D(z^2-1,1+3\epsilon)^c}\langle T(T+\epsilon)k_{z^2-1},k_{w^2-1}\rangle_{A^2}\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}d\lambda(w^2-1)$$
$$=\int_{D(z^2-1,1+3\epsilon)^c}\langle (T+\epsilon)k_{z^2-1},T^*k_{w^2-1}\rangle_{A^2}\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}d\lambda(w^2-1)$$
$$=\int_{D(z^2-1,1+3\epsilon)^c}\left|\int_{\mathbb{B}_n}\langle (T+\epsilon)k_{z^2-1},k_{x^2-1}\rangle_{A^2}\langle k_{x^2-1},T^*k_{w^2-1}\rangle_{A^2}d\lambda(x^2-1)\right|$$
$$\times\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}d\lambda(w^2-1)$$
$$\leq\int_{\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c}|\langle k_{x^2-1},T^*k_{w^2-1}\rangle_{A^2}|\frac{d\lambda(w^2-1)}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}$$
$$\times|\langle (T+\epsilon)k_{z^2-1},k_{w^2-1}\rangle_{A^2}|\|K_{z^2-1}\|_{A^2}^{a^2-1}d\lambda(x^2-1).$$

Proceeding exactly as in the proof of the previous Proposition and using the conditions following from $T,(T+\epsilon)\in\mathcal{A}$ in the place of the local Rudin-Forelli estimates (25) we obtain that

$$\lim_{\epsilon\to\infty}\sup_{(z^2-1)\in\mathbb{B}_n}\int_{D(z^2-1,1+3\epsilon)^c}\langle T(T+\epsilon)k_{z^2-1},k_{w^2-1}\rangle_{A^2}\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}d\lambda(w^2-1)=0.$$

The corresponding version for $(T(T+\epsilon))^*$ is proved in exactly the same way.

Corollary (6.2.25)[200]: Let $T: A^2\to A^2$ be a weakly localized operator. Then there exists $\epsilon\geq 0$ such that for the covering $\mathcal{F}_{1+\epsilon}=\{f_j\}$ (associated to $(1+\epsilon)$) from Proposition (6.2.9)

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{A^2 \to L^2(\mathbb{B}_n\, dv)} < \epsilon.$$

Proof. Define

$$T + \epsilon = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}$$

It suffices to show that given $\epsilon > 0$, so that

$$\|(T+\epsilon)f\|_{A^2} \lesssim \epsilon \|f\|_{A^2}.$$

Given ϵ choose $(1+\epsilon)$ large enough so that

$$\sup_{(z^2-1)\in\mathbb{B}_n} \int_{D(z^2-1,1+\epsilon)^c} \langle Tk_{z^2-1}, k_{w^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1) < \epsilon$$

$$\text{and} \quad \sup_{(z^2-1)\in\mathbb{B}_n} \int_{D(z^2-1,1+\epsilon)^c} \langle T^*k_{z^2-1}, k_{w^2-1}\rangle_{A^2} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1) < \epsilon$$

Note that for any $(z^2-1) \in \mathbb{B}_n$ we have that

$$\begin{aligned}
&|(T+\epsilon)f(z^2-1)| \\
&\quad \leq \int_{\mathbb{B}_n} \sum_j 1_{F_j(z^2-1)} 1_{G_j^c}(w^2-1) \langle T^*K_{z^2-1}, K_{w^2-1}\rangle_{A^2} |f(w^2-1)| dv(w^2-1) \\
&\quad = \int_{G_j^c} |\langle T^*K_{z^2-1}, K_{w^2-1}\rangle_{A^2}||f(w^2-1)| dv(w^2-1) \\
&\quad \leq \int_{D(z^2-1,1+\epsilon)^c} |\langle T^*K_{z^2-1}, K_{w^2-1}\rangle_{A^2}||f(w^2-1)| dv(w^2-1).
\end{aligned}$$

Now

$$\begin{aligned}
\|(T+\epsilon)f\|_{A^2}^2 &= \int_{\mathbb{B}_n} |(T+\epsilon)f(z^2-1)|^2 dv(w^2-1) \\
&\leq \int_{\mathbb{B}_n} \left(\int_{D(z^2-1,1+\epsilon)^c} |\langle T^*K_{z^2-1}, K_{w^2-1}\rangle_{A^2}||f(w^2-1)| d\sigma(w^2-1) \right)^2 dv(w^2 \\
&\quad - 1)
\end{aligned}$$

$$\times\left(\int_{D(z^2-1,1+\epsilon)^c}\frac{|\langle T^*k_{z^2-1},k_{w^2-1}\rangle_{A^2}|}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}d\lambda(w^2-1)\right)d\lambda(z^2-1)$$

$$\leq\epsilon\int_{\mathbb{B}_n}\left(\int_{D(z^2-1,1+\epsilon)^c}|\langle T^*k_{z^2-1},k_{w^2-1}\rangle_{A^2}|\|K_{w^2-1}\|_{A^2}^{a^2-1}|\langle f,k_{w^2-1}\rangle_{A^2}|^2d\lambda(w^2-1)\right)\times\frac{1}{\|K_{z^2-1}\|_{A^2}^{a^2-1}}d\lambda(z^2-1)$$

$$=\epsilon\int_{\mathbb{B}_n}|\langle f,k_{w^2-1}\rangle_{A^2}|^2\left(\int_{D(w^2-1,1+\epsilon)^c}|\langle k_{z^2-1},Tk_{w^2-1}\rangle_{A^2}|\frac{\|K_{w^2-1}\|_{A^2}^{a^2-1}}{\|K_{z^2-1}\|_{A^2}^{a^2-1}}d\lambda(z^2-1)\right)d\lambda(w^2-1)\leq\epsilon^2\int_{\mathbb{B}_n}|\langle f,k_{w^2-1}\rangle_{A^2}|^2\,d\lambda(w^2-1)=\epsilon^2\|f\|_{A^2}^2.$$

Corollary (6.2.26)[200]: Let $T: A^2\to A^2$ be a linear operator in the C^*-algebra generated by the weakly localized operators on A^2. If $\lim_{|z^2-1|\to1}\|Tk_{z^2-1}\|_{A^2}=0$, then T must be compact.

Proof. Let by Corollary (6.2.25) there exists $\epsilon\geq0$ such that for the covering $\mathcal{F}_{1+\epsilon}=\{F_j\}$ associated to $(1+\epsilon)$ (from Proposition (6.2.18))

$$\left\|TP-\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{A^2\to L^2(\mathbb{B}_n\,dv)}<\epsilon.$$

Since $\sum_{j+\epsilon}M_{1_{F_j}}TPM_{1_{G_j}}$is compact for every $(j+\epsilon)\in\mathbb{N}$ we have that the essential norm of TP as an operator from $L^2(\mathbb{B}_n\,dv)$ to A^2 can be estimated in the following way.

$$\|TP\|_e\leq\left\|TP-\sum_{j-\epsilon}M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{A^2\to L^2(\mathbb{B}_n\,dv)}$$

$$\leq\left\|TP-\sum_j M_{1_{F_j}}TPM_{1_{G_j}}\right\|_{A^2\to L^2(\mathbb{B}_n\,dv)}+\|T_{j-\epsilon}\|\lesssim\epsilon,$$

Where $(\epsilon\geq0)$

$$T_{j-\epsilon}=\sum_{j-\epsilon}M_{1_{F_j}}TPM_{1_{G_j}}.$$

If we can show that

$$\limsup_{j-\epsilon\to\infty}\|T_{j-\epsilon}\|_{A^2\to L^2(\mathbb{B}_n\,dv)}\lesssim\epsilon,$$

then since $\epsilon>0$ is arbitrary we will have that TP is compact and hence T is compact on A^2.

Let $f\in A^2$ be arbitrary of norm no greater than 1. Then,

$$\left\|T_{j-\epsilon}f\right\|_{A^2}^2 = \sum_{j-\epsilon}\left\|M_{1_{F_j}}TPM_{1_{G_j}}f\right\|_{A^2}^2 = \sum_{j-\epsilon}\frac{\left\|M_{1_{F_j}}TPM_{1_{G_j}}f\right\|_{A^2}^2}{\left\|M_{1_{G_j}}f\right\|_{A^2}^2}\left\|M_{1_{G_j}}f\right\|_{A^2}^2$$

$$\leq N\sup_{j-\epsilon}\left\|M_{1_{F_j}}Tl_j\right\|_{A^2}^2 \lesssim \sup_{j-\epsilon}\left\|Tl_j\right\|_{A^2}^2,$$

where

$$l_j := P\frac{M_{1_{G_j}}f}{\left\|M_{1_{G_j}}\right\|_{A^2}}.$$

We now have

$$\left\|T_{j-\epsilon}\right\|_{A^2\to L^2(\mathbb{B}_n\,dv)} \lesssim \sup_{j-\epsilon}\sup_{\|f\|\leq 1}\left\{\left\|Tl_j\right\|_{A^2}: l_j = \frac{PM_{1_{G_j}}f}{\left\|M_{1_{G_j}}\right\|_{A^2}}\right\}$$

and hence

$$\limsup_{j-\epsilon\to\infty}\left\|T_{j-\epsilon}\right\|_{A^2\to L^2(\mathbb{B}_n\,dv)} \lesssim \limsup_{j\to\infty}\sup_{\|f\|_{A^2}\leq 1}\left\{\left\|Tl_j\right\|_{A^2}: l_j = \frac{PM_{1_{G_j}}f}{\left\|M_{1_{G_j}}\right\|_{A^2}}\right\}$$

There exists a sequence $\{f_j\}$ in A^2 with $\left\|f_j\right\|_{A^2} \leq 1$ such that

$$\limsup_{j\to\infty}\sup_{\|f\|\leq 1}\left\{\left\|T_g\right\|_{A^2}: g = \frac{PM_{1_{G_j}}f}{\left\|M_{1_{G_j}}\right\|_{A^2}}\right\} - \epsilon \leq \limsup_{j\to\infty}\left\|Tg_j\right\|_{A^2},$$

where

$$g_j := P\frac{M_{1_{G_j}}f_j}{\left\|M_{1_{G_j}}\right\|_{A^2}} = \frac{\int_{G_j}\langle f_j, k_{w^2-1}\rangle_{A^2}k_{w^2-1}\,d\lambda(w^2-1)}{\left(\int_{G_j}\left|\langle f_j, k_{w^2-1}\rangle_{A^2}\right|^2 k_{w^2-1}\,d\lambda(w^2-1)\right)^{\frac{1}{2}}}.$$

For each $\epsilon \geq 0$ pick $(z^2-1)_j \in G_j$. There exists $\epsilon \geq 0$ such that $G_j \subset D\big((z^2-1)_j, 1+\epsilon\big)$. Doing a simple change of variables we obtain

$$g_j = \int_{\varphi_{(z^2-1)_j}(G_j)}(a^2-1)_j\left(\varphi_{(z^2-1)_j}(w^2-1)\right)U^*_{(z^2-1)_j}k_{w^2-1}\,d\lambda\left(\varphi_{(z^2-1)_j}(w^2-1)\right),$$

where $(a^2-1)_j(w^2-1)$ is defined to be

$$\frac{\langle f_j, k_{w^2-1}\rangle_{A^2}}{\left(\int_{G_j}\left|\langle f_j, k_{w^2-1}\rangle_{A^2}\right|^2 d\lambda(w^2-1)\right)^{\frac{1}{2}}}$$

on G_j, and zero otherwise.

We claim that $g_j = U^*_{(z^2-1)_j}h_j$, where

$$h_j(z^2-1):=\int\limits_{\varphi_{(z^2-1)_j}(G_j)}(a^2-1)_j\left(\varphi_{(z^2-1)_j}(w^2-1)\right)k_{w^2-1}((z^2-1))d\lambda\left(\varphi_{(z^2-1)_j}(w^2-1)\right).$$

First, using the generalized Minkowski Inequality it is easy to see that $h_j \in L^2(\mathbb{B}_n; dv)$ and consequently in A^2. To prove that the claim is correct we only need to show that for each $g \in L^2(\mathbb{B}_n;\, dv)$ we have that $\langle g_j, g\rangle = \langle h_j, U_{(z^2-1)_j} g\rangle$. This can be readily done using Fubini's Theorem. The total variation of each member of the sequence of measures $(a^2-1)_j\left(\varphi_{(z^2-1)_j}(w^2-1)\right)d\lambda\left(\varphi_{(z^2-1)_j}(w^2-1)\right)$ (as measures on the compact set $\overline{D(0,1+\epsilon)}$) satisfies $\left\|(a^2-1)_j\left(\varphi_{(z^2-1)_j}(w^2-1)\right)d\lambda\left(\varphi_{(z^2-1)_j}(w^2-1)\right)\right\|_{A^2} \lesssim \lambda\left(\varphi_{(z^2-1)_j}(G_j)\right) \le \lambda(D(0,1+\epsilon))$.

Therefore, there exists a weak-$*$ convergent subsequence which approaches some measure ν. Abusing notation slightly we keep indexing this subsequence by j. Let

$$h(z^2-1):=\int\limits_{D(0,1+\epsilon)} k_{w^2-1}(z^2-1)d\,\nu(w^2-1).$$

The mentioned weak-$*$ convergence implies that h_j converges to h pointwise. Using the Lebesgue Dominated Convergence Theorem we obtain that $h_j \to h$ in $L^2(\mathbb{B}_n;\, dv)$. This implies that $h \in A^2$. In addition, $1 \ge \|g_j\|_{A^2} = \left\|U^*_{(z^2-1)_j} h_j\right\|_{A^2} = \|h_j\|_{A^2}$.Thus, $\|h\|_{A^2} \lesssim 1$. So, we finally have

$$\limsup_{j-\epsilon\to\infty}\|T_{j-\epsilon}\|_{A^2\to L^2(\mathbb{B}_n,dv)} \lesssim \limsup_{j\to\infty}\|Tg_j\|_{A^2}+\epsilon = \limsup_{j\to\infty}\left\|TU^*_{(z^2-1)_j}h\right\|_{A^2}+\epsilon$$
$$\lesssim \limsup_{j\to\infty}\left\|TU^*_{(z^2-1)_j}h\right\|_{A^2}+\epsilon.$$

Choose h' in the linear span of normalized reproducing kernels such that $\|h-h'\|_{A^2} < \epsilon/\|T\|$. The assumption $\|Tk_{z^2-1}\|_{A^2} \to 0$ as $|z^2-1| \to 1$ implies that $\left\|TU^*_{(z^2-1)_j}h'\right\|_{A^2} \to 0$ as $j \to \infty$. Therefore we finally obtain

$$\limsup_{j-\epsilon\to\infty}\|T_{j-\epsilon}\|_{A^2\to L^2(\mathbb{B}_n,dv)} \lesssim \limsup_{j\to\infty}\left\|TU^*_{(z^2-1)_j}h\right\|_{A^2} \le \limsup_{j\to\infty}\left\|TU^*_{(z^2-1)_j}h'\right\|_{A^2}+\epsilon$$
$$=\epsilon.$$

Corollary (6.2.27)[200]: If T is weakly localized operator on A^2, then

$$\lim_{|z^2-1|\to1}\int\limits_{\mathbb{B}_n}|\langle Tk_{z^2-1},k_{z^2-1}\rangle_{A^2}|\frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}}\mathrm{d}\lambda(w^2-1)=0.$$

Proof. First, observe that:

$$\int_{\mathbb{B}_n} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} \mathrm{d}\lambda(w^2-1)$$

$$= \left(\int_{\overline{D(z^2-1,1+\epsilon)}} + \int_{\overline{D(z^2-1,1+\epsilon)^c}} \right) |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1).$$

Let $\epsilon > 0$. Our assumption on T implies that we can choose $\epsilon \geq 0$ large enough so that the second integral is less than ϵ. We need to show that the first integral can be made smaller than (a constant times) ϵ. Fix this $\epsilon \geq 0$ and then consider the first integral,

$$\int_{\overline{D(z^2-1,1+\epsilon)}} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|^{a^2-1}}{\|K_{w^2-1}\|^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{\overline{D(z^2-1,1+\epsilon)^c}} \left|\langle T_{z^2-1}k_0, k_{\varphi_{(z^2-1)}(w^2-1)}\rangle_{A^2}\right| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{\overline{D(0,1+\epsilon)}} \left|\langle T_{z^2-1}k_0, k_{\varphi_{(z^2-1)}(w^2-1)}\rangle_{A^2}\right| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\left\|K_{\varphi_{(z^2-1)}(w^2-1)}\right\|_{A^2}^{a^2-1}} d\lambda(w^2-1)$$

$$= \int_{\overline{D(0,1+\epsilon)}} |T_{z^2-1}k_0(w^2-1)| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\left\|K_{\varphi_{(z^2-1)}(w^2-1)}\right\|_{A^2}^{a^2-1}}.$$

We can choose (z^2-1) with $|z^2-1|$ large enough so that $|T_{z^2-1}k_0(w^2-1)| < \epsilon$ for all $(w^2-1) \in \overline{D(0,1+\epsilon)}$ (we have uniform convergence to 0 on the compact set $\overline{D(0,1+\epsilon)}$) . Therefore the second integral is no greater than

$$\epsilon \int_{\overline{D(0,1+\epsilon)}} \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\left\|K_{\varphi_{(z^2-1)}(w^2-1)}\right\|_{A^2}^{a^2-1}} \frac{d\lambda(w^2-1)}{\|K_{w^2-1}\|_{A^2}}$$

$$= \epsilon \int_{\overline{D(0,1+\epsilon)}} |\langle k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\left\|K_{\varphi_{(z^2-1)}(w^2-1)}\right\|_{A^2}^{a^2-1}} \frac{d\lambda(w^2-1)}{\left\|K_{(w^2-1)}\right\|_{A^2}}$$

$$\leq C(a^2-1)\epsilon.$$

Here we have used (22). Since $\epsilon > 0$ was arbitrary we are done with the proof.

Corollary (6.2.28)[200]: Let T be a weakly localized operator on A^2. If

$$\lim_{|z^2-1|\to 1} \langle Tk_{z^2-1}, k_{z^2-1}\rangle_{A^2} = 0,$$

then T is compact.

Proof. Notice first that

$$\langle Tf, k_{w^2-1}\rangle_{A^2} = \langle f, T^* k_{w^2-1}\rangle_{A^2} = \int_{\mathbb{B}_n} \langle f, k_{z^2-1}\rangle_{A^2} \langle k_{z^2-1}, T^* k_{w^2-1}\rangle_{A^2}\, d\lambda(z^2-1)$$

$$= \int_{\mathbb{B}_n} \langle Tk_{z^2-1}, k_{z^2-1}\rangle_{A^2} \langle f, k_{z^2-1}\rangle_{A^2}\, d\lambda(z^2-1).$$

For each $\epsilon \geq 0$ define an operator $T_{1+\epsilon}$ by

$$\langle T_{1+\epsilon} f, k_{w^2-1}\rangle_{A^2} := \int\limits_{D(0,1+\epsilon)} \langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2} \langle f, k_{z^2-1}\rangle_{A^2} d\lambda(z^2-1).$$

These are basically the operators $(T+\epsilon)_{[1+\epsilon]}$ that Axler and Zheng use in [76]. It is easy to see that these are all Hilbert-Schmidt operators by testing the square integrability of the kernel. Therefore, it suffices to show that $T_{1+\epsilon} \to T$ in the operator norm.

Let $\epsilon > 0$ and let $f \in A^2$ be an arbitrary element of norm 1. We have

$$\|(T - T_{1+\epsilon})f\|_{A^2}^2 = \int\limits_{\mathbb{B}_n} |\langle (T - T_{1+\epsilon})f, k_{w^2-1}\rangle_{A^2}|^2 \, d\lambda(w^2-1).$$

We first examine the integrand.

$$|\langle (T - T_{1+\epsilon})f, k_{w^2-1}\rangle_{A^2}| \le \int\limits_{D(0,1+\epsilon)^c} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2} \langle f, k_{z^2-1}\rangle_{A^2}| d\lambda(z^2-1)$$

$$= \int\limits_{D(0,1+\epsilon)} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|^{1/2} \frac{\|K_{w^2-1}\|_{A^2}^{\frac{a^2-1}{2}}}{\|K_{z^2-1}\|_{A^2}^{\frac{a^2-1}{2}}} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}|^{1/2} |\langle f, k_{z^2-1}\rangle_{A^2}|$$

$$\times \frac{\|K_{z^2-1}\|_{A^2}^{\frac{a^2-1}{2}}}{\|K_{w^2-1}\|_{A^2}^{\frac{a^2-1}{2}}} d\lambda(z^2-1)$$

$$\le \left(\int\limits_{D(0,1+\epsilon)^c} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{w^2-1}\|_{A^2}^{a^2-1}}{\|K_{z^2-1}\|_{A^2}^{a^2-1}} d\lambda(z^2-1) \right)^{\frac{1}{2}}$$

$$\times \left(\int\limits_{D(0,1+\epsilon)^c} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} |\langle f, k_{z^2-1}\rangle_{A^2}|^2 d\lambda(z^2-1) \right)^{\frac{1}{2}}$$

$$\lesssim \left(\int\limits_{D(0,1+\epsilon)^c} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} |\langle f, k_{z^2-1}\rangle_{A^2}|^2 d\lambda(z^2-1) \right)^{\frac{1}{2}}.$$

Here, the implied constant depends on (1.5). Thus, after an application of Fubini, we obtain

$$\|(T - T_{1+\epsilon})f\|_{A^2}^2$$

$$\lesssim \int\limits_{D(0,1+\epsilon)^c} \left(\int\limits_{\mathbb{B}_n} |\langle T k_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2 - 1) \right) |\langle f, k_{z^2-1}\rangle_{A^2}|^2 d\lambda(z^2-1).$$

Notice that the variable (z^2-1) in the inner integral is from $D(0,1+\epsilon)^c$. Therefore, by Corollary (6.2.27), we can choose $\epsilon \ge 0$ large enough so that for $(z^2-1) \in \mathbb{B}_n$ with $|z^2-1|$ sufficiently close to 1

$$\int_{\mathbb{B}_n} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{A^2}| \frac{\|K_{z^2-1}\|_{A^2}^{a^2-1}}{\|K_{w^2-1}\|_{A^2}^{a^2-1}} d\lambda(w^2-1) < \epsilon.$$

Therefore, for such large $(1+\epsilon)$ we have

$$\|(T-T_{1+\epsilon})f\|_{A^2}^2 \lesssim \int_{D(0,1+\epsilon)^c} \epsilon|\langle f, k_{z^2-1}\rangle_{A^2}|^2 d\lambda(z^2-1) \lesssim \epsilon\|f\|_{A^2}^2 = \epsilon.$$

This proves that $T_{1+\epsilon} \to T$ in the operator norm, and so we are done.

Corollary (6.2.29)[200]: Let $T: \mathcal{F}^2 \to \mathcal{F}^2$ be a linear operator satisfying (21). Then there exists $\epsilon \geq 0$ such that for the covering $\mathcal{F}_{1+\epsilon} = \{F_j\}$ (associated to $(1+\epsilon)$) from Proposition (6.2.18).

$$\left\| TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}} \right\|_{\mathcal{F}^2 \to L^2\left(\mathbb{C}^n;\, e^{-|z^2-1|^2}\, dv(z^2-1)\right)} < \epsilon.$$

Proof. Define

$$T+\epsilon = TP - \sum_j M_{1_{F_j}} TPM_{1_{G_j}}$$

It suffices to show that we can find an $\epsilon \geq 0$ so that

$$\|(T+\epsilon)f\|_{\mathcal{F}^2} \lesssim \epsilon\|f\|_{\mathcal{F}^2}.$$

Given ϵ choose $(1+\epsilon)$ large enough so that

$$\sup_{(z^2-1)\in\mathbb{C}^n} \int_{D(z^2-1,1+\epsilon)^c} |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}| dv(w^2-1) < \epsilon$$

and

$$\sup_{(z^2-1)\in\mathbb{C}^n} \int_{D(z^2-1,1+\epsilon)^c} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}| dv(w^2-1) < \epsilon.$$

Note that for any $(z^2-1) \in \mathbb{C}^n$ we have that

$$\begin{aligned} &|(T+\epsilon)f(z^2-1)| \\ &\leq \int_{\mathbb{C}^n} \sum_j 1_{F_j(z^2-1)} 1_{G_j^c}(w^2-1) |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}| |f(w^2-1)| d\sigma(w^2-1) \\ &= \int_{G_j^c} |\langle T^* K_{z^2-1}, K_{w^2-1}\rangle_{\mathcal{F}^2}| |f(w^2-1)| d\sigma(w^2-1). \end{aligned}$$

Now

$$\|(T+\epsilon)f\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}^n} |(T+\epsilon)f(z^2-1)|^2 d\sigma(z^2-1)$$

$$\leq \int_{\mathbb{C}^n} \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* K_{z^2-1}, K_{w^2-1}\rangle_{\mathcal{F}^2}||f(w^2-1)| d\sigma(w^2-1) \right)^2 d\sigma(z^2-1)$$

$$\leq \int_{\mathbb{C}^n} \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* K_{z^2-1}, K_{w^2-1}\rangle_{\mathcal{F}^2}||f(w^2-1)|^2 d\sigma(w^2-1) \right)$$

$$\times \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* K_{z^2-1}, K_{w^2-1}\rangle_{\mathcal{F}^2}| d\sigma(w^2-1) \right) d\sigma(z^2-1)$$

$$= \int_{\mathbb{C}^n} \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}||\langle f, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) \right)$$

$$\times \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}| dv(w^2-1) \right) dv(z^2-1)$$

$$\leq \epsilon \int_{\mathbb{C}^n} \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}||\langle f, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) \right) dv(z^2-1)$$

$$= \epsilon \int_{\mathbb{C}^n} |\langle f, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 \left(\int_{D(z^2-1,1+\epsilon)^2} |\langle T^* k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}| dv(w^2-1) \right) dv(w^2-1)$$

$$\leq \epsilon \int_{\mathbb{C}^n} |\langle f, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) = \epsilon^2 \|f\|_{\mathcal{F}^2}^2.$$

Corollary (6.2.30)[200]: Let $T: \mathcal{F}^2 \to \mathcal{F}^2$ be a bounded linear operator in the C^*-algebra generated by weakly localized operators satisfying (19). If

$$\lim_{|z^2-1|\to\infty} \langle T k_{z^2-1}, k_{z^2-1}\rangle_{\mathcal{F}^2} = 0,$$

then

$$\lim_{|z^2-1|\to\infty} \|T k_{z^2-1}\|_{\mathcal{F}^2} = 0.$$

Proof. Let $\epsilon > 0$ be given. Note that we have

$$\|T k_{z^2-1}\|_{\mathcal{F}^2}^2 = \int_{\mathbb{C}^n} |\langle T k_{z^2-1}, k_{z^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$= \left(\int_{\overline{D(z^2-1,1+\epsilon)}} + \int_{\overline{D(z^2-1,1+\epsilon)}} \right) |\langle T k_{z^2-1}, k_{z^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) = I + II.$$

For term II we use a standard approximation result. Since T is in the C^*-algebra generated by (1.6), given $\epsilon > 0$ we can find T' that is weakly localized so that

$$\|T - T'\|_{\mathcal{F}^2 \to \mathcal{F}^2} < \epsilon.$$

Now choose $(1+\epsilon)$ sufficiently large so that

$$\sup_{(z^2-1)\in\mathbb{C}^n} \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle T' k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) < \epsilon$$

which is possible since T' satisfies (26). We use this to estimate II.

$$II = \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$\lesssim \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle T'k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$+ \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle (T-T')k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1).$$

The first term is controlled by the choice of $(1+\epsilon)$ above. We also have that for this choice of $(1+\epsilon)$ (actually all $(1+\epsilon)$) that

$$\sup_{(z^2-1)\in\mathbb{C}^n} \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle (T-T')k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$\leq \sup_{(z^2-1)\in\mathbb{C}^n} \int_{\mathbb{C}^n} |\langle (T-T')k_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$= \sup_{(z^2-1)\in\mathbb{C}^n} \|(T-T')k_{z^2-1}\|_{\mathcal{F}^2}^2 \leq \|(T-T')k_{z^2-1}\|_{\mathcal{F}^2\to\mathcal{F}^2} < \epsilon.$$

Then then yields that

$$\sup_{(z^2-1)\in\mathbb{C}^n} \int_{\overline{D(z^2-1,1+\epsilon)^c}} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1) \lesssim \epsilon$$

Now for term I, note that we have

$$\int_{\overline{D(z^2-1,1+\epsilon)}} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$

$$= \int_{\overline{D(z^2-1,1+\epsilon)}} \left|\langle T_{z^2-1}k_0, k_{\varphi_{(z^2-1)}(w^2-1)}\rangle_{\mathcal{F}^2}\right|^2 dv(w^2-1)$$

$$= \int_{\overline{D(0,1+\epsilon)}} \left|\langle T_{z^2-1}k_0, k_{\varphi_{(z^2-1)}(w^2-1)}\rangle_{\mathcal{F}^2}\right|^2 dv(w^2-1)$$

$$= \int_{\overline{D(0,1+\epsilon)}} |T_{z^2-1}k_0(w^2-1)|^2 \frac{dv(w^2-1)}{\|K_{w^2-1}\|_{\mathcal{F}^2}^2}$$

$$= \int_{\overline{D(0,1+\epsilon)}} |T_{z^2-1}k_0(w^2-1)|^2 d\sigma(w^2-1).$$

We can choose (z^2-1) with $|z^2-1|$ large enough so that $|T_{z^2-1}k_0(w^2-1)| < \epsilon$ for all $(w^2-1) \in \overline{D(0,1+\epsilon)}$ (we have uniform convergence to 0 on the compact set $\overline{D(0,1+\epsilon)}$). Furthermore, we have that $\sigma\big(D(0,1+\epsilon)\big) \leq \sigma(\mathbb{C}^n) < \infty$, and so

$$\int_{D(z^2-1,1+\epsilon)} |\langle Tk_{z^2-1}, k_{w^2-1}\rangle_{\mathcal{F}^2}|^2 dv(w^2-1)$$
$$= \int_{D(z^2-1,1+\epsilon)} \left|\langle T_{z^2-1}k_0, k_{\varphi_{(z^2-1)}(w^2-1)}\rangle_{\mathcal{F}^2}\right|^2 dv(w^2-1) < \epsilon.$$

Thus, we have

$$\lim_{|z^2-1|\to\infty} \|Tk_{z^2-1}\|^2_{\mathcal{F}^2} = 0.$$

Section (6.3): Toeplitz Algebra on the Bergman Space

We begin with a discussion of localized operators. Let **B** denote the open unit ball $\{z \in \mathbb{C}^n : |z| < 1\}$ in $\mathbf{C}^n$. The Bergman metric on **B** is given by the formula

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, z, w \in \mathbf{B},$$

where φ_z is the Möbius transform of the ball given in [79]. For each $z \in \mathbf{B}$ and each $r > 0$, the corresponding β-ball will be denoted by $D(z,r)$. That is,

$$D(z,r) = \{w \in \mathbf{B}: \beta(z,w) < r\}$$

Let dv be the volume measure on **B** with the normalization $v(\mathbf{B}) = 1$. Then the formula

$$d\lambda(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$$

gives us the standard Möbius-invariant measure on **B**.

Recall that the Bergman space $L_a^2(\boldsymbol{B}, dv)$ is the subspace

$$\{h \in L^2(\boldsymbol{B}, dv): h \text{ is analytic on } \mathbf{B}\}$$

of $L^2(\boldsymbol{B}, dv)$. It is well known that the normalized reproducing kernel for the Bergman space is given by the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{(n+1)/2}}{(1-\langle\zeta,z\rangle)^{n+1}}, z, \zeta \in \mathbf{B}. \tag{27}$$

It was first discovered in [190] that localization is a powerful tool for analyzing operators on reproducing-kernel Hilbert spaces. This idea was further explored in [184]. More specially, in [184] Isralowitz, Mitkovski and Wick introduced the notion of weakly localized operators on the Bergman space. Below we give a slightly more refined version of their definition. Our refinement lies in the realization that we can define a class of localized operators for each given localization parameter s.

Definition (6.3.1)[192]: Let a positive number $(n-1)/(n+1) < s < 1$ be given.

(a) A bounded operator B on the Bergman space $L_a^2(\boldsymbol{B}, dv)$ is said to be s-weakly localized if it satisfies the conditions

$$\sup_{z\in\mathbf{B}} \int |\langle Bk_z, k_w\rangle| \left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\sup_{z\in\mathbf{B}} \int |\langle B^*k_z, k_w\rangle| \left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2} d\lambda(w) < \infty,$$

$$\lim_{r\to\infty}\sup_{z\in\mathbf{B}} \int_{\mathbf{B}\backslash D(z,r)} |\langle Bk_z, k_w\rangle| \left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2} d\lambda(w) = 0$$

and

$$\lim_{r\to\infty} \sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle B^* k_z, k_w\rangle| \left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

(b) Let $\mathcal{A}_S$ denote the collection of s-weakly localized operators defined as above.

(c) Let $C^*(\mathcal{A}_S)$ denote the C^*-algebra generated by $\mathcal{A}_S$.

For each $(n-1)/(n+1) < s < 1$, the simplest examples of s-weakly localized operators are the Toeplitz operators, which, as we recall, are defined as follows. Let $P: L^2(\boldsymbol{B}, dv) \to L_a^2(\boldsymbol{B}, dv)$ be the orthogonal projection. Then for $f \in L^\infty(\boldsymbol{B}, dv)$, the formula

$$T_f h = P(fh), \quad h \in L_a^2(\boldsymbol{B}, dv),$$

defines the Toeplitz operator T_f. Also recall that the Toeplitz algebra $\mathcal{T}$ on $L_a^2(\boldsymbol{B}, dv)$ is the C^*-algebra generated by the collection of Toeplitz operators

$$\{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}.$$

It was shown in [184] that $\mathcal{A}_s \supset \{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}$, hence $C^*(\mathcal{A}_s) \supset \mathcal{T}$.

In [189], Suárez showed that for $A \in \mathcal{T}$, the condition

$$\lim_{|z|\uparrow 1} \langle A k_z, k_z\rangle = 0 \quad (28)$$

implies that A is compact. In [184], Isralowitz, Mitkovski and Wick showed that for $A \in C^*(\mathcal{A}_s)$, condition (28) also implies that A is compact. The introduction of the notion of weakly localized operators in [184] has the added virtue that it significantly simplifies the work necessary to obtain the above result. Indeed the approach in [184] explains why such results should hold true.

The results in [184], [189] certainly inspire further examinations of the inclusion relation

$$\mathcal{T} \subset C^*(\mathcal{A}_s). \quad (29)$$

Given what we know about Toeplitz operators (see, e.g., [193]-[86],[196],[197],[198],[199]), the C^*-algebra $\mathcal{T}$ is certainly much better understood than $C^*(\mathcal{A}_s)$. It is known, for example, that $\mathcal{T}$ coincides with its commutator ideal [81], [182]. Thus an obvious question is, is the C^*-algebra $C^*(\mathcal{A}_s)$ structurally different from $\mathcal{T}$?

Question (6.3.2)[192]: Is the inclusion in (29) proper for any $(n-1)/(n+1) < s < 1$? Is there any difference between $C^*(\mathcal{A}_s)$ and $C^*(\mathcal{A}_t)$ for $s \neq t$ in the interval $((n-1)/(n+1), 1)$?

The answer, as it turns out, is somewhat surprising:

Theorem (6.3.3)[192]: For every $(n-1)/(n+1) < s < 1$ we have $C^*(\mathcal{A}_s) = \mathcal{T}$.

An immediate consequence of Theorem (6.3.3) is, of course, that $C^*(\mathcal{A}_s) = C^*(\mathcal{A}_t)$ for all $s, t \in ((n-1)/(n+1), 1)$. We emphasize that this equality at the level of C^*-algebras is obtained without knowing whether there is any kind of inclusion relation between the classes $\mathcal{A}_s$ and $\mathcal{A}_t$ in the case $s \neq t$.

Although Question (6.3.2) was the original motivation, our approach to this problem naturally leads us to a stronger result, a result that simultaneously settles a much older question. Let us introduce

Definition (6.3.4)[192]: Let $\mathcal{T}^1$ denote the closure of $\{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}$ with respect to the operator norm.

Below is our main result, which not only answers Question (6.3.2), but also tells us something significant about the Toeplitz algebra $\mathcal{T}$ itself.

The documented history of interest in $\mathcal{T}^{(1)}$ can be traced at least back to [83], [195], where Engliš showed that it contains all the compact operators on $L_a^2(\boldsymbol{B}, dv)$.

Later in [198], Suárez took another look at $\mathcal{T}^{(1)}$. There he introduced a sequence of higher Berezin transforms $B_1, \dots, B_k, \dots$,, which are generalizations of the original Berezin transform B_0. Suárez expressed his belief that every operator S in $\mathcal{T}$ is the limit in operator norm of the sequence of Toeplitz operators $\{T_{B_k(S)}\}$. If this is true, then it certainly implies that $\mathcal{T}^{(1)} = \mathcal{T}$. One can only speculate that, perhaps, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ was what Suárez had in mind all along, and the higher Berezin transforms were his tools to try to prove it. While we still do not know if it is true that

$$\lim_{k\to\infty} \left\|T_{B_k(S)} - S\right\| = 0$$

for every $S \in \mathcal{T}$, the equality $\mathcal{T}^{(1)} = \mathcal{T}$ is now proven using completely different ideas. From the proof of Theorem (6.3.13), we see that the approximation of a general $S \in \mathcal{T}$ by Toeplitz operators is quite complicated: it takes several stages.

Let us give an outline for the proof of Theorem (6.3.13). Since each $\mathcal{A}_s$ is known to be a *-algebra that contains $\{T_f : f \in L^\infty(\boldsymbol{B}, dv)\}$ [184], it suffices to show that $\mathcal{A}_s \subset \mathcal{T}^{(1)}$. An elementary C^*-algebraic argument further reduces this to the proof of the inclusion

$$T_\Phi \mathcal{A}_s T_\Phi \subset \mathcal{T}^{(1)}$$

for a suitably chosen Toeplitz operator T_Φ that is both positive and invertible. We can pick the function Φ in such a way that for every $B \in \mathcal{A}_s$, the operator $T_\Phi B T_\Phi$ is "resolved" in the form

$$T_\Phi B T_\Phi = \int\int_{D(0,2)\times D(0,2)} E_w B E_z \, d\lambda(w) d\lambda(z),$$

where each E_z is a sum of rank-one operators over a lattice:

$$E_z = \sum_{u\in\mathcal{L}} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}\,.$$

A crucial ingredient in the proof is the norm estimate in Lemma (6.3.10) below. This estimate has a number of implications, and one of the implications is that the map $(w, z) \mapsto E_w B E_z$ is continuous with respect to the operator norm. This norm continuity immediately implies that $T_\Phi B T_\Phi$ is contained in the norm closure of the linear span of

$$\{E_w B E_z : w, z \in \mathbf{B}\}.$$

Thus we can complete the proof by showing that $E_w B E_z \in \mathcal{T}^{(1)}$ for all $z, w \in \mathbf{B}$. One can think of $E_w B E_z$ as an infinite matrix. The localization condition for B ensures that the terms in $E_w B E_z$ that are "far from the diagonal" form an operator of small norm. The rest of the terms in $E_w B E_z$ are a linear combination of operators in a special class $\mathcal{D}_0$ (see Definition (6.3.14)). In other words, $E_w B E_z$ can be approximated in norm by operators in the linear span of $\mathcal{D}_0$. Then, with several applications of the estimate in Lemma (6.3.10), we are able to show that $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$, accomplishing our goal.

We discuss the analogue of Theorem (6.3.13) on the Fock space.

Definition (6.3.5)[192]: A subset Γ of $\mathbf{B}$ is said to be separated if there is a $\delta = \delta(\Gamma) > 0$ such that the inequality $\beta(u, v) \geq \delta$ holds for all $u \neq v$ in Γ.

Recall that for each $z \in \mathbf{B}\backslash\{0\}$, the Möbius transform φ_z is given by the formula

$$\varphi_z(\zeta) = \frac{1}{1 - \langle\zeta, z\rangle}\left\{z - \frac{\langle\zeta, z\rangle}{|z|^2} z - (1 - |z|^2)^{1/2}\left(\zeta - \frac{\langle\zeta, z\rangle}{|z|^2} z\right)\right\}$$

Also, we define $\varphi_0(\zeta) = -\zeta$. Recall that each φ_z is an involution, i.e., $\varphi_z \circ \varphi_z = \mathrm{id}$ [79]. Let us list some of the elementary properties of separated sets that will be used repeatedly in the sequel.

Lemma (6.3.6)[192]: Let Γ be a separated set in **B**.

(a) For each $0 < R < \infty$, there is a natural number $N = N(\Gamma; R)$ such that $\mathrm{card}\,\{v \in \Gamma: \beta(u, v) \le R\} \le N$ for every $u \in \Gamma$.

(b) For every pair of $z \in \mathbf{B}$ and $\rho > 0$, there is a finite partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$ such that for every $i \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta\big(\varphi_u(z), \varphi_v(z)\big) > \rho$.

Proof. By definition, there is a $\delta > 0$ such that $\beta(u, v) \ge \delta$ for all $u \ne v$ in Γ. Thus

$$D(u, \delta/2) \cap D(v, \delta/2) = \emptyset \quad \text{for all} \quad u \ne v \quad \text{in } \Gamma.$$

Let $R > 0$ be given. Then for every pair of $u, v \in \Gamma$, the condition $\beta(u, v) \le R$ implies $D(v, \delta/2) \subset D\big(u, R + (\delta/2)\big)$. By the Möbius invariance of the Bergman metric β and the measure $d\lambda$, we have

$$\lambda\big(D(\delta/2)\big) = \lambda\big(\varphi_v(D(0, \delta/2)\big) = \lambda\big(D(0, \delta/2)\big).$$

Therefore if we write $N(u)$ for the cardinality of the set $\{v \in \Gamma: \beta(u, v) \le R\}$, then

$$N(u)\lambda\big(D(0, \delta/2)\big) = \sum_{\substack{v \in \Gamma \\ \beta(u,v) \le R}} \lambda\big(D(v, \delta/2)\big) \le \lambda(D\big(u, R + (\delta/2)\big) = \lambda\left(D\big(0, R + (\delta/2)\big)\right).$$

That is, $N(u) \le \lambda(D\big(0, R + (\delta/2)\big)/\lambda\big(D(0, \delta/2)\big)$, which proves (i).

To prove (ii), let $z \in \mathbf{B}$ and $\rho > 0$ be given, and set $r = \rho + 2\beta(z, 0)$. By (i), there is an $m \in N$ such that $\mathrm{card}\,\{v \in \Gamma: \beta(u, v) \le r\} \le m$ for every $u \in \Gamma$. By a standard maximality argument, there is a partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$ such that for every $i \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \ne v$ imply $\beta(u, v) > r$. But if u, v satisfy the condition $\beta(u, v) > r$, then by the Möbius invariance of β we have

$$\begin{aligned}\beta\big(\varphi_u(z), \varphi_v(z)\big) &\ge \beta(u, v) - \beta(\varphi_u(z), u) - \beta\big(v, \varphi_v(z)\big) \\ &= \beta(u, v) - \beta\big(\varphi_u(z), \varphi_u(0)\big) - \beta\big(\varphi_v(0), \varphi_v(z)\big) \\ &= \beta(u, v) - \beta(z, 0) - \beta(0, z) > r - 2\beta(z, 0) = \rho.\end{aligned}$$

This completes the proof.

Lemma (6.3.7)[192]: For all $u, v, x, y \in \mathbf{B}$ we have

$$\frac{(1 - |\varphi_u(x)|^2)^{1/2}(1 - |\varphi_v(y)|^2)^{1/2}}{|1 - \langle \varphi_u(x), \varphi_v(y)\rangle|} \le 2e^{\beta(x,0)+\beta(y,0)} \frac{(1 - |u|^2)^{1/2}(1 - |v|^2)^{1/2}}{|1 - \langle u, v\rangle|}.$$

Proof. For $a, b \in \mathbf{B}$, we have $1 - |\varphi_a(b)|^2 = (1 - |a|^2)(1 - |b|^2)/|1 - \langle a, b\rangle^2|$ [79]. Thus if we write

$$\alpha = \frac{(1 - |a|^2)(1 - |b|^2)^{1/2}}{|1 - \langle a, b\rangle^2|},$$

then

$$\log\frac{1}{\alpha} \le \frac{1}{2}\log\frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|} \le \log\frac{2}{\alpha}.$$

Consequently

$$e^{-\beta(a,b)} \le \frac{(1 - |a|^2)^{\frac{1}{2}}(1 - |b|^2)^{\frac{1}{2}}}{|1 - \langle a, b\rangle|} \le 2e^{-\beta(a,b)}. \tag{30}$$

For $u, v, x, y \in \mathbf{B}$, by the Möbius invariance of the Bergman metric, we have
$\beta(\varphi_u(z), \varphi_v(y)) \geq \beta(u,v) - \beta(\varphi_u(x), u) - \beta(\varphi_v(y), v) = \beta(u,v) - \beta(x,0) - \beta(y,z)$.
Combining (30) with this inequality, we find that

$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_v(y)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_v(y)\rangle|} \leq 2e^{-\beta(\varphi_u(x),\varphi_v(y))} \leq 2e^{\beta(x,0)+\beta(y,0)}e^{-\beta(u,v)}$$
$$\leq 2e^{\beta(x,0)+\beta(y,0)}\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}.$$

This proves the lemma.

Lemma (6.3.8)[192]: Let Γ be a separated set in $\mathbf{B}$. Then there is a $0 < C(\Gamma) < \infty$ such that

$$\sum_{v\in\Gamma}\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+1}(1-|v|^2)^{(4n+1)/8} \leq C(\Gamma)(1-|\xi|^2)^{(4n+1)/8}$$

for every $\xi \in \mathbf{B}$.

Proof. If Γ is a separated set in $\mathbf{B}$, then there is a $\delta > 0$ such that $\beta(u,v) \geq \delta$ for all $u \neq v$ in Γ. Thus $D(u,\delta/2) \cap D(v,\delta/2) = \emptyset$ for all $u \neq v$ in Γ. If $w \in D(v,\delta/2)$, then $v \in D(w,\delta/2) = \varphi_w(D(0,\delta/2))$. Thus If $w \in D(v,\delta/2)$, then there is a $v' \in D(v,\delta/2)$ such that $v = \varphi_w(v')$. Let $\xi \in \mathbf{B}$. Since $\xi = \varphi_\xi(0)$, we can apply Lemma (6.3.8) to obtain

$$\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|} \leq 2e^{\frac{\delta}{2}}\frac{(1-|\xi|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|1-\langle\xi,w\rangle|} \quad (31)$$

for every $w \in D(v,\delta/2)$. Also, since $v = \varphi_w(v')$ and $v' \in D(0,\delta/2)$, we have

$$1-|v|^2 = 1-|\varphi_w(v')|^2 = \frac{(1-|v'|^2)(1-|w|^2)}{|1-\langle v',w\rangle|^2} \leq \frac{4}{1-|v'|^2}(1-|w|^2)$$
$$\leq 4e^{2\beta(v',0)}(1-|w|^2) \leq 4e^{\delta}(1-|w|^2). \quad (32)$$

Set $C_1 = (2e^{\delta/2})^{n+1}(4e^{\delta})^{(4n+1)/8}$. Then it follows from (30) and (31) that

$$\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+1}(1-|v|^2)^{(4n+1)/8}$$
$$\leq C_1\left(\frac{(1-|\xi|^2)^{1/2}(1-|w|^2)^{1/2}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{(4n+1)/8}$$

for every $w \in D(v,\delta/2)$. Hence for each $\zeta \in \mathbf{B}$ we have

$$\sum_{v\in\Gamma}\left(\frac{(1-|\xi|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle\xi,v\rangle|}\right)^{n+1}(1-|v|^2)^{(4n+1)/8}$$
$$\leq \sum_{v\in\Gamma}\frac{C_1}{\lambda\left(D\left(v,\frac{\delta}{2}\right)\right)}\int_{D\left(v,\frac{\delta}{2}\right)}\left(\frac{(1-|\xi|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w)$$
$$\leq \frac{C_1}{\lambda\left(D\left(v,\frac{\delta}{2}\right)\right)}\int\left(\frac{(1-|\xi|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w). \quad (33)$$

To estimate the last integral, note that

$$\frac{(1-|\xi|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|1-\langle\xi,\varphi_\xi(\zeta)\rangle|}=(1-|\zeta|^2)^{1/2}.$$

Thus, making the substitution $w=\varphi_\xi(\zeta)$ and using the Möbius invariance of $d\lambda$, we have

$$\int\left(\frac{(1-|\xi|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|1-\langle\xi,w\rangle|}\right)^{n+1}(1-|w|^2)^{\frac{4n+1}{8}}d\lambda(w)$$

$$=\int(1-|\zeta|^2)^{\frac{(n+1)}{2}}\left(1-\left|\varphi_\xi(\zeta)\right|^2\right)^{\frac{(4n+1)}{8}}d\lambda(\zeta)$$

$$=\int(1-|\zeta|^2)^{\frac{(n+1)}{2}}\left(\frac{(1-|\xi|^2)(1-|\zeta|^2)}{(1-\langle\xi,\zeta\rangle^2)}\right)^{\frac{(4n+1)}{8}}d\lambda(\zeta)$$

$$=(1-|\xi|^2)^{(4n+1)/8}\int\frac{dv(\zeta)}{|1-\langle\xi,\zeta\rangle|^{n+\left(\frac{1}{4}\right)}(1-|\zeta|^2)^{3/8}}=(*).$$

To further estimate $(*)$, let $d\sigma$ be the standard spherical measure on the unit sphere $\{x\in\mathbf{C}^n:|x|=1\}$. There is a constant C_2 such that

$$\int\frac{d\sigma(x)}{|1-\langle z,x\rangle|^{n+\left(\frac{1}{4}\right)}}\le\frac{C_2}{(1-|z|^2)^{1/4}}$$

for every $z\in\mathbf{B}$ [79]. Combining this with the radial-spherical decomposition $dv=2nr^{2n-1}drd\sigma$ of the volume measure, we have

$$\int\frac{dv(\zeta)}{|1-\langle\xi,\zeta\rangle|^{n+\left(\frac{1}{4}\right)}(1-|\zeta|^2)^{3/8}}\le\int_0^1\frac{C_2 2nr^{2n-1}dr}{(1-r^2)^{(1/4)+(3/8)}}\le nC_2\int_0^1\frac{dt}{(1-t)^{5/8}}=\frac{8}{3}nC_2.$$

Therefore

$$(*)\le 3nC_2(1-|\xi|^2)^{(4n+1)/8}.$$

Substituting this in (33), we conclude that the desired inequality holds for the constant

$$C(\Gamma)=\frac{3nC_1C_2}{\lambda\big(D(0,\delta/2)\big)}.$$

This completes the proof.

Recall that each Toeplitz operator has an "integral representation" in terms of the normalized reproducing kernel $\{k_w: w\in\mathbf{B}\}$. Indeed for each $f\in L^\infty(\mathbf{B},dv)$, we have

$$T_f=\int f(w)k_w\otimes k_w d\lambda(w). \tag{34}$$

This formula is obtained through direct verification.

Let $\mathcal{L}$ be a subset of $\mathbf{B}$ which is maximal with respect to the property that

$$D(u,1)\cap D(v,1)=\emptyset\qquad\text{for all } u\ne v \text{ in } \mathcal{L}. \tag{35}$$

This $\mathcal{L}$ will be fixed for the rest. The maximality of $\mathcal{L}$ implies that

$$\bigcup_{u\in\mathcal{L}}D(u,2)=\mathbf{B}. \tag{36}$$

Now, for each $z\in\mathbf{B}$, define

$$E_z=\sum_{u\in\mathcal{L}}k_{\varphi_u(u)}\otimes k_{\varphi_u(u)}. \tag{37}$$

Define the function

$$\Phi = \sum_{u\in\mathcal{L}} \chi_{D(u,2)} \tag{38}$$

on **B**. By (35) and Lemma (6.3.6) (i), there is a natural number $\mathcal{N} \in N$ such that

$$\text{card}\,\{v \in \mathcal{L}: D(u,2) \cap D(v,2) \neq \emptyset\} \le \mathcal{N}$$

for every $u \in \mathcal{L}$. This and (35) together tell us that the inequality

$$1 \le \Phi \le \mathcal{N} \tag{39}$$

holds on the unit ball **B**. By (34) and the Möbius invariance of β and $d\lambda$, we have

$$T_\Phi = \int \Phi(w) k_w \otimes k_w d\lambda(w) = \sum_{u\in\mathcal{L}} \int_{D(u,2)} k_w \otimes k_w d\lambda(w)$$

$$= \sum_{u\in\mathcal{L}} \int_{D(0,2)} k_{\varphi_u(u)} \otimes k_{\varphi_u(u)} d\lambda(z)\,.$$

That is, we have

$$T_\Phi = \int_{D(0,2)} E_z d\lambda(z). \tag{40}$$

Lemma (6.3.9)[192]: There is a constant $0 < C_{2.5} < 1$ such that $\|E_z\| \le C_{2.5}$ for every $z \in D(0,2)$.

Proof. By Lemma (6.3.7), for $u, v, z \in$ **B** we have

$$\left|\langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle\right| = \left(\frac{(1-|\varphi_v(z)|^2)^{1/2}(1-|\varphi_u(z)|^2)^{1/2}}{|1-\langle\varphi_u(z),\varphi_v(z)\rangle|}\right)^{n+1}$$

$$\le \left(2e^{2\beta(z,0)}\right)^{n+1}\left(\frac{(1-|u|^2)^{\frac{1}{2}}(1-|v|^2)^{\frac{1}{2}}}{|1-\langle u,v\rangle|}\right)^{n+1}. \tag{41}$$

Let $\{\epsilon_u: u \in \mathcal{L}\}$ be an orthonormal set. For each $z \in$ **B**, define the operator

$$F_z = \sum_{u\in\mathcal{L}} \epsilon_u \otimes k_{\varphi_u(z)}. \tag{42}$$

Since $E_z = F_z^* F_z$ and $\|F_z^* F_z\| = \|F_z F_z^*\|$, it suffices to estimate the later. We have

$$F_z F_z^* = \sum_{u,v\in\mathcal{L}} \langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle \epsilon_u \otimes \epsilon_v.$$

Now suppose that $z \in D(0,2)$ and write $C_1 = (2e^4)^{n+1}$. By (41), for every vector $x = \sum_{u,\in\mathcal{L}} x_u \epsilon_u$ we have

$$\langle F_z F_z^* x, x\rangle \le \sum_{u,v\in\mathcal{L}} \left|\langle k_{\varphi_v(z)}, k_{\varphi_u(z)}\rangle\right| |x_u||x_v|$$

$$\le C_1 \sum_{u,v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+1} |x_u||x_v| = C_1 \sum_{u\in\mathcal{L}} |x_u| y_u, \tag{43}$$

where

$$y_u = \sum_{v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+1} |x_v|$$

for each $u \in \mathcal{L}$. Next we apply the Schur test. Indeed by the Cauchy-Schwarz inequality and Lemma (6.3.8), we have

$$y_u^2 \le C(\mathcal{L})(1-|u|^2)^{\frac{4n+1}{8}} \times \sum_{v\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}}.$$

Applying Lemma (6.3.8) again, we have

$$\sum_{u\in\mathcal{L}} y_u^2 \le C(\mathcal{L}) \sum_{v\in\mathcal{L}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} \sum_{u\in\mathcal{L}} (1-|u|^2)^{\frac{4n+1}{8}} \times \left(\frac{(1-|u|^2)^{1/2}(1-|v|^2)^{1/2}}{|1-\langle u,v\rangle|}\right)^{n+1}$$
$$\le C^2(\mathcal{L}) \sum_{v\in\mathcal{L}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} (1-|v|^2)^{\frac{4n+1}{8}} = C^2(\mathcal{L}) \sum_{v\in\mathcal{L}} |x_v|^2.$$

Combining this with (43), we find that

$$\langle F_z F_z^* x, x\rangle \le C_1 C(\mathcal{L}) \sum_{u\in\mathcal{L}} |x_v|^2 = C_1 C(\mathcal{L}) \|x\|^2.$$

Since the vector x is arbitrary, we conclude that $\|E_z\| = \|F_z F_z^*\| \le C_1 C(\mathcal{L})$ for every $z \in D(0,2)$. This completes the proof.

Recall that for each $z \in \mathbf{B}$, the formula

$$(U_Z h)(\zeta) = k_z(\zeta) h(\varphi_z(\zeta)), \quad \zeta \in \mathbf{B} \quad \text{and} \quad h \in L_a^2(\mathbf{B}, dv), \qquad (44)$$

defines a unitary operator. These unitary operators will play an essential role.

As usual, we write $H^\infty(\mathbf{B})$ for the collection of bounded analytic functions on $\mathbf{B}$. Also, we write $\|h\|_\infty = \sup_{\zeta\in\mathbf{B}} |h\zeta|$ for $h \in H^\infty(\mathbf{B})$. Naturally, we consider $H^\infty(\mathbf{B})$ as a subset of the Bergman space $L_a^2(\mathbf{B}, dv)$.

Lemma (6.3.10)[192]: Given any separated set Γ in $\mathbf{B}$, there exists a constant $0 < B(\Gamma) < \infty$ such that the following estimate holds: Let $\{h_u : u \in \Gamma\}$ be functions in $H^\infty(\mathbf{B})$ such that $\sup_{u\in\Gamma} \|h_u\|_\infty < \infty$, and let $\{e_u : u \in \Gamma\}$ be any orthonormal set. Then

$$\left\|\sum_{u\in\Gamma} (U_u h_u) \otimes e_u\right\| \le B(\Gamma) \sup_{u\in\Gamma} \|h_u\|_\infty.$$

Proof. Given $\Gamma, \{h_u : u \in \Gamma\}$ and $\{e_u : u \in \Gamma\}$ as in the statement, let us write

$$A = \sum_{u\in\Gamma} (U_u h_u) \otimes e_u$$

for convenience. By (39), the self-adjoint Toeplitz operator T_Φ is invertible with $\|T_\Phi^{-1}\| < 1$. Therefore $\|A\| = \|T_\Phi^{-1} T_\Phi A\| \le \|T_\Phi A\|$. Combining this with (40), we see that

$$\|A\| \le \lambda(D(0,2)) \sup_{z\in D(0,2)} \|E_z A\|. \qquad (45)$$

Thus it suffices to estimate $\|E_z A\|$ for $z \in D(0,2)$. Let F_z be the operator defined by (42). Then Lemma (6.3.9) implies that $\|F_z^*\| \le C_{2.5}^{1/2}$ for $z \in D(0,2)$. Hence

$$\|E_z A\| \le C_{2.5}^{\frac{1}{2}} \|F_z A\| \qquad z \in D(0,2). \qquad (46)$$

Consequently, we only need to estimate $\|F_z A\|$.

To estimate $\|F_z A\|$, let us denote

$$H = \sup_{u\in\Gamma} \|h_u\|_\infty.$$

Let $z \in D(0,2)$. Then note that

$$F_z A = \sum_{u\in\mathcal{L}} \sum_{u\in\Gamma} \langle U_v h_v, k_{\varphi_u(z)}\rangle \epsilon_u \otimes e_v . \tag{47}$$

Since $U_v h_v = k_v \cdot h_v \circ \varphi_v$, the reproducing property of $k_{\varphi_u(z)}$ gives us

$$\langle U_v h_v, k_{\varphi_u(z)}\rangle = h_v\left(\varphi_v\big(\varphi_u(z)\big)\right)\langle k_v, k_{\varphi_u(z)}\rangle,$$

which is one of the key facts on which depends. Thus

$$\left|\langle U_v h_v, k_{\varphi_u(z)}\rangle\right| \le H\left|\langle k_v, k_{\varphi_u(z)}\rangle\right| = H\left(\frac{(1-|v|^2)^{1/2}(1-|\varphi_u(z)|^2)^{1/2}}{|1-\langle \varphi_u(z), v\rangle|}\right)^{n+1}.$$

Since $v = \varphi_v(0)$ and $z \in D(0,2)$, an application of Lemma (6.3.7) gives us

$$\left|\langle U_v h_v, k_{\varphi_u(z)}\rangle\right| \le C_1 H\left(\frac{(1-|v|^2)^{\frac{1}{2}}(1-|u|^2)^{\frac{1}{2}}}{|1-\langle u, v\rangle|}\right)^{n+1}, \tag{48}$$

where $C_1 = (2e^2)^{n+1}$. Now consider vectors

$$x = \sum_{u\in\Gamma} x_v e_v \qquad \text{and} \qquad y = \sum_{u\in\mathcal{L}} y_e \epsilon_u .$$

It follows from (47) and (48) that

$$|\langle F_z A x, y\rangle| \le C_1 H \sum_{u\in\mathcal{L}} \sum_{u\in\Gamma}\left(\frac{(1-|v|^2)^{\frac{1}{2}}(1-|u|^2)^{\frac{1}{2}}}{|1-\langle u, v\rangle|}\right)^{n+1} |x_v||y_u|$$

$$= C_1 H \sum_{u\in\mathcal{L}} b_u |y_u|, \tag{49}$$

where

$$b_u = \sum_{u\in\Gamma}\left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} |x_v|,$$

$u \in \mathcal{L}$. We apply the Schur test as we did in the proof of Lemma (6.3.9). By the Cauchy-Schwarz inequality and the bound given in Lemma (6.3.8), we have

$$b_u^2 \le C(\Gamma)(1-|u|)^{\frac{4n+1}{8}} \times \sum_{v\in\Gamma}\left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}},$$

$u \in \mathcal{L}$. Applying Lemma (6.3.8) again, we obtain

$$\sum_{u\in\mathcal{L}} b_u^2 \le C(\Gamma) \sum_{v\in\Gamma}\sum_{u\in\mathcal{L}} (1-|u|)^{\frac{4n+1}{8}} \times \left(\frac{(1-|v|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, v\rangle|}\right)^{n+1} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}}$$

$$= C(\Gamma)C(\mathcal{L}) \sum_{v\in\Gamma} (1-|v|^2)^{\frac{4n+1}{8}} \frac{|x_v|^2}{(1-|v|^2)^{\frac{4n+1}{8}}} = C(\Gamma)C(\mathcal{L})\|x\|^2.$$

Combining this with (49), we obtain

$$|\langle F_z A x, y\rangle| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H\|x\|\|y\|.$$

Since the vectors x and y are arbitrary, this means

$$\|F_z A\| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H$$

for $z \in D(0,2)$. Recalling (45) and (46), we see that the lemma holds for the constant

$$B(\Gamma) = \lambda\big(D(0,2)\big) C_{2.5}^{1/2} C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2}.$$

This completes the proof.

Proposition (6.3.11)[192]: Suppose that Γ is a separated set in $\mathbf{B}$. Furthermore, suppose that $\{c_u: u \in \Gamma\}$ are complex numbers satisfying the condition

$$\sup_{u\in\Gamma}|c_u| < \infty. \tag{50}$$

Then for each $z \in \mathbf{B}$, the operator

$$Y_z = \sum_{u\in\Gamma} c_u k_{\varphi_u(z)} \otimes k_{\varphi_u(u)} \tag{51}$$

is bounded on the Bergman space. Moreover, the map $z \longmapsto Y_z$ from $\mathbf{B}$ into $\mathcal{B}\left(L_a^2(\mathbf{B}, dv)\right)$ is continuous with respect to the operator norm.

Proof. For $u, z \in \mathbf{B}$, simple computation shows that

$$U_u k_z = \left(\frac{|1-\langle u,v\rangle|}{1-\langle u,v\rangle}\right)^{n+1} k_{\varphi_u(z)}. \tag{52}$$

Therefore

$$k_{\varphi_u(u)} \otimes k_{\varphi_u(u)} = (U_u k_z) \otimes (U_u k_z).$$

Let $\{e_u: u \in \Gamma\}$ be an orthonormal set. Then for every $z \in \mathbf{B}$ we have the factorization

$$Y_z = A_z B_z^*,$$

where

$$A_z = \sum_{u\in\Gamma} c_u (U_u k_z) \otimes e_u \qquad \text{and} \qquad B_z = \sum_{u\in\Gamma} (U_u k_z) \otimes e_u.$$

Applying Lemma (6.3.10) to the case $h_u = c_u k_z, u \in \Gamma$, we see that each A_z is a bounded operator. Similarly, each B_z is also bounded. Hence $Y_z = A_z B_z^*$ is bounded.

To show that the map $z \longmapsto Y_z$ is continuous with respect to the operator norm, it suffices to show that the maps $z \longmapsto A_z$ and $z \longmapsto B_z$ are continuous with respect to the operator norm. Since B_z is just a special case of A_z, it suffices to consider the map $z \longmapsto A_z$.

For any $z, w \in \mathbf{B}$, we have

$$A_z - A_w = \sum_{u\in\Gamma} c_u\big(U_u(k_z - k_w)\big) \otimes e_u.$$

Applying Lemma (6.3.10) to the case where $h_u = c_u(k_z - k_w), u \in \Gamma$, we find that

$$\|A_z - A_w\| \le B(\Gamma) C \|k_z - k_w\|_\infty,$$

where $C = \sup_{u\in\Gamma}|c_u|$. For each $z \in \mathbf{B}$, it is elementary that

$$\lim_{w\to z}\|k_z - k_w\|_\infty = 0.$$

Hence the map $z \longmapsto A_z$ is continuous with respect to operator norm. This completes the proof.

Let us recall two known facts about $\mathcal{A}_S$. First, for each given $(n-1)/(n+1) < s < 1$, we have $\mathcal{A}_s \supset \{T_f: f \in L^\infty(\boldsymbol{B}, dv)\}$ [184]. Indeed by (34), this is a consequence of the fact

$$\lim_{r\to\infty}\sup_{z\in\mathbf{B}} \int_{\mathbf{B}\setminus D(z,r)} |\langle k_z, k_w\rangle||\langle k_x, k_w\rangle|\, d\lambda(x) \left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2} d\lambda(w) = 0.$$

To prove this limit, the idea in [184] is to split the inner x-integral above as the sum of the part on $D(z, r/2)$ and the part on $\mathbf{B}\setminus D(z, r/2)$. With such split, this limit follows from the Rudin-Forelli estimate [184].

Second, each $\mathcal{A}_s$ is a $*$-algebra [184]. In this case, the gist of the matter is the limit

$$\limsup_{r\to\infty}\sup_{z\in\mathbf{B}}\int_{\mathbf{B}\backslash D(z,r)}\int|\langle Tk_z,k_x\rangle||\langle k_x,S^*k_w\rangle|\,d\lambda(x)\times\left(\frac{(1-|w|^2)}{(1-|z|)^2}\right)^{s(n+1)/2}d\lambda(w)=0 \quad (53)$$

for $S,T\in\mathcal{A}_s$. To prove this, [184] splits the inner x-integral in the same way as above. Then it is easy to see that (53) follows from the localization condition for S and T.

Next comes the most crucial step in the proof of Theorem (6.3.13):

Proposition (6.3.12)[192]: Let $(n-1)/(n+1)<s<1$. If $B\in\mathcal{A}_s$, then $E_wBE_z\in\mathcal{T}^{(1)}$ for all $z,w\in\mathbf{B}$.

Assuming Proposition (6.3.12), we have

Theorem (6.3.13)[192]: For every $(n-1)/(n+1)<s<1$ we have $\mathcal{T}^{(1)}=C^*(\mathcal{A}_s)$. Consequently, $\mathcal{T}^{(1)}=\mathcal{T}=C^*(\mathcal{A}_s)$.

Proof. Let $(n-1)/(n+1)<s<1$ be given. By the fact that $\mathcal{A}_s$ is a $*$-algebra mentioned above, $C^*(\mathcal{A}_s)$ is just the norm closure of $\mathcal{A}_s$. Since we also know that $\mathcal{A}_s\supset\{T_f:f\in L^\infty(\boldsymbol{B},dv)\}$, Theorem (6.3.13) will follow if we can show that $\mathcal{A}_s\subset\mathcal{T}^{(1)}$. We prove this inclusion into two steps.

(i) Let $B\in\mathcal{A}_s$ be given. As the first step, let us show that $T_\Phi BT_\Phi\in\mathcal{T}^{(1)}$. Indeed it follows from (40) that

$$T_\Phi BT_\Phi=\int\int_{D(0,2)\times D(0,2)}E_wBE_z\,d\lambda(w)d\lambda(z). \quad (54)$$

Consider the map

$$(w,z)\mapsto E_wBE_z \quad (55)$$

from $\mathbf{B}\times\mathbf{B}$ into $\mathcal{B}(L_a^2(\mathbf{B},dv))$. Proposition (6.3.12) tells us that the range of map (55) is contained in $\mathcal{T}^{(1)}$. Hence every Riemann sum corresponding to the integral in (54) belongs to $\mathcal{T}^{(1)}$. On the other hand, by Proposition (6.3.11), the map $z\mapsto E_z$ is continuous with respect to the operator norm. Hence map (55) is also continuous with respect to the operator norm. Since the closure of $D(0,2)\times D(0,2)$ is a compact subset of $\mathbf{B}\times\mathbf{B}$, the norm continuity of (55) means that the integral in (54) is the limit with respect to the operator norm of a sequence of Riemann sums $s_1,s_2,\dots,s_k,\dots$. Since each s_k belongs to $\mathcal{T}^{(1)}$, so does $T_\Phi BT_\Phi$.

(ii) Given $B\in\mathcal{A}_s$, we will now show that $B\in\mathcal{T}^{(1)}$. Since $T_\Phi\in\mathcal{A}_s$ and since $\mathcal{A}_s$ is an algebra, we have $T_\Phi^jBT_\Phi^k\in\mathcal{A}_s$ for all $j,k\in\mathbf{Z}_+$. Thus it follows from (i) that

$$T_\Phi^{j+1}BT_\Phi^{k+1}\in\mathcal{T}^{(1)}\qquad\text{for all integers } j\ge 0 \text{ and } k\ge 0. \quad (56)$$

Let $C^*(T_\Phi)$ be the unital C^*-algebra generated by T_Φ. Since T_Φ is self-adjoint, (56) implies that

$$T_\Phi XBT_\Phi X\in\mathcal{T}^{(1)}\qquad\text{for every } X\in C^*(T_\Phi).$$

We again use the invertibility of T_Φ, which is guaranteed by (39). It is elementary that the inverse T_Φ^{-1}, once it exists, must belong to the C^*-algebra $C^*(T_\Phi)$. Thus, letting $X=T_\Phi^{-1}$ in the above, we obtain $B\in\mathcal{T}^{(1)}$. This completes the proof of Theorem (6.3.13).

Our goal is to prove Proposition (6.3.12).

Definition (6.3.14)[192]: (a) Let $\mathcal{D}_0$ denote the collection of operators of the form

$$\sum_{u\in\Gamma}c_uk_u\otimes k_{\gamma(u)},$$

where Γ is any separated set in $\mathbf{B}, \{c_u : u \in \Gamma\}$ is any bounded set of complex coefficients, and $\gamma: \Gamma \to \mathbf{B}$ is any map for which there is a $0 < C < \infty$ such that

$$\beta(u, \gamma(u)) \le C \tag{57}$$

for every $u \in \Gamma$.

(b) Let $\mathcal{D}$ denote the operator-norm closure of the linear span of $\mathcal{D}_0$.

With $\mathcal{D}_0$ and $\mathcal{D}$ we can divide the proof of Proposition (6.3.12) into two independent parts:

We will see that the proofs of these two propositions are based on different ideas. More specifically, the proof of Proposition (6.3.20) relies on the estimate provided by Lemma (6.3.10), whereas the proof of Proposition (6.3.16) takes advantage of the localization condition of the operators in As. The proof of Proposition (6.3.10) begins with

Lemma (6.3.15)[192]: Let $(n-1)/(n+1) < s < 1$ be given. If $B \in \mathcal{A}_s$, then for every separated set Γ in $\mathbf{B}$ and every pair of $z, w \in \mathbf{B}$ we have

$$\lim_{R\to\infty} \sup_{u\in\Gamma} \sum_{\substack{u\in\Gamma \\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} = 0 \tag{57}$$

and

$$\lim_{R\to\infty} \sup_{u\in\Gamma} \sum_{\substack{u\in\Gamma \\ \beta(u,v)>R}} \left|\langle k_{\varphi_u(z)}, Bk_{\varphi_v(w)} \rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2} = 0. \tag{59}$$

Proof. Given such s and $B \in \mathcal{A}_s$, by Definition (6.3.1) we have

$$\lim_{r\to\infty} \sup_{x\in\mathbf{B}} \int_{\mathbf{B}\setminus D(x,r)} \left|\langle Bk_x, k_\zeta\rangle\right| \left(\frac{(1-|\zeta|^2)}{(1-|x|)^2}\right)^{s(n+1)/2} d\lambda(\zeta) = 0 \tag{60}$$

and

$$\lim_{r\to\infty} \sup_{x\in\mathbf{B}} \int_{\mathbf{B}\setminus D(x,r)} \left|\langle B^* k_x, k_\zeta\rangle\right| \left(\frac{(1-|\zeta|^2)}{(1-|x|)^2}\right)^{s(n+1)/2} d\lambda(\zeta) = 0. \tag{61}$$

Let Γ, z and w also be given as in the lemma. Denote $G = D(0,1)$ and $G_w = \varphi_w(G)$. Then it is easy to see that $G_w \subset D\big(0,1+\beta(w,0)\big)$. For $h \in L_a^2(\mathbf{B}, dv)$ and $v \in \Gamma$, we have

$$h\big(\varphi_w(w)\big) = (h\circ\varphi_v\circ\varphi_w)(0) = \frac{1}{\lambda(G)}\int_G h\circ\varphi_v\circ\varphi_w d\lambda = \frac{1}{\lambda(G)}\int_{(\varphi_v\circ\varphi_w)(G)} hd\lambda$$

$$= \frac{1}{\lambda(G)}\int_{\varphi_v(G_w)} hd\lambda = \frac{1}{\lambda(G)}\int_{\varphi_v(G_w)} \frac{\langle h, k_\zeta\rangle}{(1-|\zeta|^2)^{(n+1)/2}} d\lambda(\zeta).$$

Thus

$$\langle h, k_{\varphi_v(w)}\rangle = \frac{1}{\lambda(G)}\int_{\varphi_v(G_w)} \langle h, k_\zeta\rangle \left(\frac{1-|\varphi_v(w)|^2}{1-|\zeta|^2}\right)^{(n+1)/2} d\lambda(\zeta).$$

If $\zeta \in \varphi_v(G_w)$, then $\zeta = \varphi_v(\xi)$ for some $\xi \in G_w \subset D\big(0,1+\beta(w,0)\big)$, which means

$$1-|\zeta|^2 = 1-|\varphi_v(\xi)|^2 = \frac{(1-|v|^2)(1-|\zeta|^2)}{|1-\langle\xi, v\rangle|^2} \ge \frac{1}{4}(1-|\zeta|^2)(1-|v|^2).$$

On the other hand,

$$1 - |\varphi_v(w)|^2 = \frac{(1-|v|^2)(1-|w|^2)}{|1-\langle w,v\rangle|^2} \le \frac{2}{1-|w|}(1-|v|^2).$$

Hence there is a $0 < C_1 < \infty$ which depends only on n and w such that

$$\left|\langle h, k_{\varphi_v(w)}\rangle\right|(1-|v|^2)^{s(n+1)/2} \le \frac{C_1}{\lambda(G)} \int_{k_{\varphi_v(G_w)}} \langle h, k_\zeta\rangle (1-|\zeta|^2)^{s(n+1)/2} d\lambda(\zeta)$$

for all $h \in L_a^2(\mathbf{B}, dv)$ and $v \in \Gamma$. Applying this inequality to the case where $h = Bk_{\varphi_u(z)}, u \in \Gamma$, we have

$$\left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2}$$
$$\le \frac{C_1}{\lambda(G)} \int_{\varphi_v(G_w)} \langle Bk_{\varphi_u(z)}, k_\zeta\rangle \left(\frac{1-|\zeta|^2}{1-|u|^2}\right)^{s(n+1)/2} d\lambda(\zeta),$$

$v \in \Gamma$. Since

$$1 - |\varphi_u(z)|^2 = \frac{(1-|u|^2)(1-|z|^2)}{|1-\langle z,u\rangle|^2} \le \frac{2}{1-|z|}(1-|u|^2),$$

there is a $0 < C_2 < \infty$ which depends only on n and z such that

$$\left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2}$$
$$\le \frac{C_1 C_2}{\lambda(G)} \int_{\varphi_v(G_w)} \langle Bk_{\varphi_u(z)}, k_\zeta\rangle \left(\frac{1-|\zeta|^2}{1-|\varphi_u(z)|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(\zeta), \tag{62}$$

$u, v \in \Gamma$. Set $L = 1 + \beta(w,0) + \beta(z,0)$ and consider any $R > L$. If u, $v \in \Gamma$ are such that $\beta(u,v) > R$, then for every $\zeta \in \varphi_v(G_w) \subset \varphi_v\left(D\big(0,1+\beta(w,0)\big)\right)$ we have

$$\beta(\varphi_u(z),\zeta) \ge \beta(u,v) - \beta\big(u,\varphi_u(z)\big) > R - 1 - \beta(w,0) - \beta(z,0) = R - L. \tag{63}$$

Thus the combination of (62) and (63) gives us

$$\sum_{\substack{v\in\Gamma \\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, Bk_{\varphi_v(w)}\rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2}$$
$$\le \frac{C_1 C_2}{\lambda(G)} \int_{\beta(\varphi_u(z),\zeta)>R-L} \sum_{v\in\Gamma} \chi_{\varphi_v(G_w)}(\zeta) \left|\langle Bk_{\varphi_u(z)}, k_\zeta\rangle\right|$$
$$\times \left(\frac{1-|\zeta|^2}{1-|\varphi_u(z)|^2}\right)^{\frac{s(n+1)}{2}} d\lambda(\zeta), \tag{64}$$

$u \in \Gamma$. By the Möbius invariance of β and the fact that $G_w \subset D\big(0,1+\beta(w,0)\big)$, we have $\varphi_v(G_w) \subset D\big(v, 1+\beta(w,0)\big)$. Since Γ is separated, it follows from Lemma (6.3.6) (i) that there is an $N \in \mathbf{N}$ which depends only on Γ and w such that the inequality

$$\sum_{v\in\Gamma} \chi_{\varphi_v(G_w)} \le N$$

holds on $\mathbf{B}$. Substituting this in (64), we conclude that

$$\sum_{\substack{v\in\Gamma\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|v|^2}{1-|u|^2}\right)^{s(n+1)/2}$$
$$\le \frac{C_1C_2N}{\lambda(G)} \int_{\beta(\varphi_u(z),\zeta)>R-L} \langle Bk_{\varphi_u(z)}, k_\zeta\rangle \left(\frac{1-|\zeta|^2}{1-|\varphi_u(z)|^2}\right)^{s(n+1)/2} d\lambda(\zeta)$$
for every $u \in \Gamma$. By this inequality, (57) follows from (59). Since
$$\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle = \langle B^* k_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle,$$
(58) follows from (60) by the same argument. This completes the proof.

Proposition (6.3.16)[192]: Let $(n-1)/(n+1) < s < 1$. If $B \in \mathcal{A}_s$, then for every pair of $z, w \in \mathbf{B}$ we have $E_w B E_z \in \mathcal{D}$.

Proof. Let $(n-1)/(n+1) < s < 1$. For $B \in \mathcal{A}_s$ and $z, w \in \mathbf{B}$, we have
$$E_w B E_z = \sum_{u,v\in\mathcal{L}} k_{\varphi_v(w)} \otimes k_{\varphi_v(w)} \cdot B \cdot k_{\varphi_u(z)} \otimes k_{\varphi_u(z)}$$
$$= \sum_{u,v\in\mathcal{L}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)}.$$
Thus for any $R > 0$, we can write $E_w B E_z = V_R + W_R$, where
$$V_R = \sum_{\substack{u,v\in\mathcal{L}\\ \beta(u,v)\le R}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)}$$
and
$$W_R = \sum_{\substack{u,v\in\mathcal{L}\\ \beta(u,v)>R}} \langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)}.$$
Obviously, the proposition will follow if we can prove the following two statements:

(i) $\lim_{R\to\infty} \|W_R\| = 0$.

(ii) $V_R \in \text{span}(\mathcal{D}_0)$ for every $R > 0$.

To prove (i), note that by (52) and Lemma (6.3.10), there are constants C_1, C_2 such that
$$\sum_{u\in\mathcal{L}} \left|\langle h, k_{\varphi_u(z)}\rangle\right|^2 \le C_1\|h\|^2 \quad \text{and} \quad \sum_{v\in\mathcal{L}} \left|\langle h, k_{\varphi_v(w)}\rangle\right|^2 \le C_2\|h\|^2 \tag{65}$$
for every $h \in L_a^2(\mathbf{B}, dv)$. Given $h, g \in L_a^2(\mathbf{B}, dv)$, we have
$$|\langle W_R h, g\rangle| \le \sum_{\substack{u,v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \mathscr{s}_u \mathscr{t}_v, \tag{66}$$
where
$$\mathscr{s}_u = \left|\langle h, k_{\varphi_u(z)}\rangle\right| \qquad \text{and} \qquad \mathscr{t}_v = \left|\langle k_{\varphi_v(w)}, g\rangle\right|.$$
We apply the Schur test one more time. Indeed for each $u \in \mathcal{L}$, let us write
$$\mathscr{y}_u = \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \mathscr{t}_v \tag{67}$$
Then for each $u \in \mathcal{L}$, the Cauchy-Schwarz inequality gives us

$$\mathcal{y}_u^2 \leq \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right|(1 - |v|^2)^{\frac{s(n+1)}{2}} \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \frac{\mathcal{t}_v^2}{(1-|v|^2)^{\frac{s(n+1)}{2}}}$$

$$\leq H(R) \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}} \mathcal{t}_v^2,$$

where

$$H(R) = \sup_{\xi\in\mathcal{L}} \sum_{\substack{v\in\mathcal{L}\\ \beta(\xi,v)>R}} \left|\langle Bk_{\varphi_\xi(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|u|^2}{1-|\xi|^2}\right)^{\frac{s(n+1)}{2}}.$$

Therefore

$$\sum_{u\in\mathcal{L}} \mathcal{y}_u^2 \leq H(R) \sum_{u\in\mathcal{L}} \sum_{\substack{v\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}} \mathcal{t}_v^2$$

$$= H(R) \sum_{v\in\mathcal{L}} \mathcal{t}_v^2 \sum_{\substack{u\in\mathcal{L}\\ \beta(u,v)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_v(w)}\rangle\right| \left(\frac{1-|u|^2}{1-|v|^2}\right)^{\frac{s(n+1)}{2}}$$

$$\leq H(R)G(R) \sum_{v\in\mathcal{L}} \mathcal{t}_v^2,$$

where

$$G(R) = \sup_{\xi\in\mathcal{L}} \sum_{\substack{u\in\mathcal{L}\\ \beta(u,\xi)>R}} \left|\langle Bk_{\varphi_u(z)}, k_{\varphi_{\varphi_\xi}(w)}\rangle\right| \left(\frac{1-|u|^2}{1-|\xi|^2}\right)^{\frac{s(n+1)}{2}}.$$

By (66) and (67), we now have

$$|\langle W_R h, g\rangle| \leq \sum_{u\in\mathcal{L}} \mathcal{s}_u \mathcal{y}_u \leq \left(\sum_{u\in\mathcal{L}} \mathcal{s}_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} \mathcal{y}_u^2\right)^{1/2}$$

$$\leq \{H(R)G(R)\}^{1/2} \left(\sum_{u\in\mathcal{L}} \mathcal{s}_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} \mathcal{t}_u^2\right)^{1/2}.$$

Combining this with (66), we find that

$$|\langle W_R h, g\rangle| \leq \{C_1 C_2 H(R) G(R)\}^{1/2} \|h\| \|g\|.$$

Since $h, g \in L_a^2(\mathbf{B}, dv)$ are arbitrary, this means

$$W_R \leq \{C_1 C_2 H(R) G(R)\}^{1/2}.$$

Applying Lemma (6.3.15), we have $\lim_{R\to\infty} H(R) = 0$ and $\lim_{R\to\infty} G(R) = 0$. Therefore $\lim_{R\to\infty} \|W_R\| = 0$ as promised.

We now turn to the proof of (ii). First of all, given an $R > 0$, for each $v \in \mathcal{L}$ we define

$$F_v = \{u \in \mathcal{L}: \beta(u,v) \le R\}.$$
By Lemma (6.3.6) (i), there is an $N \in \mathbf{N}$ such that
$$\text{card}\,(F_v) \le N$$
for every $v \in \mathcal{L}$. Also, by Lemma (6.3.6) (ii), for the given $w \in \mathbf{B}$, there is a partition
$$\mathcal{L} = L_1 \cup \ldots \cup L_m$$
such that for each $i \in \{1, \ldots, m\}$, if $v, v' \in L_i$ and $v \ne v'$, then $\beta\big(\varphi_v(w), \varphi_{v'}(w)\big) \ge 1$. That is, for each $i \in \{1, \ldots, m\}$, the set
$$K_i = \{\varphi_v(w): v \in L_i\}$$
is separated. We have $V_R = X_1 + \cdots + X_m$, where
$$X_i = \sum_{\varphi_v(w) \in K_i} \sum_{u \in F_v} \langle B k_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)},$$
$i \in \{1, \ldots, m\}$. To prove (ii), it suffices to show that $X_i \in \text{span}(\mathcal{D}_0)$ of every $i \in \{1, \ldots, m\}$. For this purpose we further decompose each K_i. Indeed for each pair of $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, N\}$, we define
$$L_{i,j} = \{v \in L_i: \text{card}\,(F_v) = j\} \quad \textit{and} \quad K_{i,j} = \{\varphi_v(w): v \in L_{i,j}\}.$$
Then $X_i = X_{i,1} + \cdots + X_{i,N}$, where
$$X_{i,j} = \sum_{\varphi_v(w) \in K_{i,j}} \sum_{u \in F_v} \langle B k_{\varphi_u(z)}, k_{\varphi_v(w)} \rangle k_{\varphi_v(w)} \otimes k_{\varphi_u(z)},$$
$i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, N\}$. Thus it suffices to show that $X_{i,j} \in \text{span}(\mathcal{D}_0)$ for every such pair of i, j. But it is obvious that given a pair of such i, j, we can define maps
$$\gamma_{i,j}^{(1)}, \ldots, \gamma_{i,j}^{(j)}: K_{i,j} \to \mathbf{B}$$
such that
$$\{\varphi_u(z): u \in F_v\} = \left\{\gamma_{i,j}^{(1)}\big(\varphi_v(w)\big), \ldots, \gamma_{i,j}^{(j)}\big(\varphi_v(w)\big)\right\}$$
for every $v \in L_{i,j}$. Thus $X_{i,j} = X_{i,j}^{(1)} + \cdots + X_{i,j}^{(j)}$, where for each $v \in \{1, \ldots, j\}$ we have
$$X_{i,j}^{(v)} = \sum_{u \in F_v} \langle B k_{\gamma_{i,j}^{(v)}(\xi)}, k_\xi \rangle\, k_\xi \otimes k_{\gamma_{i,j}^{(v)}(\xi)}.$$
Hence the proof will be complete if we can show that $\gamma_{i,j}^{(v)} \in (\mathcal{D}_0)$ for every triple of indices $i \in \{1, \ldots, m\}, j \in \{1, \ldots, N\}$ and $v \in \{1, \ldots, j\}$.

By the above definitions, for every such triple of i, j, v, if $\zeta \in K_{i,j}$, then there exist $v \in L_{i,j}$ and $u \in F_v$ such that $\xi = \varphi_v(w)$ and $\gamma_{i,j}^{(v)}(\xi) = \varphi_u(z)$. Therefore
$$\begin{aligned}\beta\left(\xi, \gamma_{i,j}^{(v)}(\xi)\right) = \beta\big(\varphi_v(w), \varphi_u(z)\big) &\le \beta(\varphi_v(w), v) + \beta(v, u) + \beta\big(u, \varphi_u(z)\big) \\ &\le \beta(w, 0) + R + \beta(0, z).\end{aligned}$$
This shows that the map $\gamma_{i,j}^{(v)}: K_{i,j} \to \mathbf{B}$ satisfies condition (57). By Definition (6.3.14) (a), we have $X_{i,j}^{(v)} \in (\mathcal{D}_0)$. This completes the proof of Proposition (6.3.16).

Next we turn to the proof of Proposition (6.3.20), which involves a few steps.

Proposition (6.3.17)[192]: Suppose that Γ is a separated set in $\mathbf{B}$. Furthermore, suppose that $\{c_u: u \in \Gamma\}$ are complex numbers for which (50) holds. Then for each $z \in \mathbf{B}$, the operator Y_z defined by (51) belongs to $\mathcal{T}^{(1)}$.

Proof. (i) Let us first show that $Y_0 \in \mathcal{T}^{(1)}$. Since Γ is separated, there is $\delta > 0$ such that $\beta(u,v) \geq \delta$ for all $u \neq v$ in Γ. That is, if $u, v \in \Gamma$ and $u \neq v$, then $D(u,\delta/2) \cap D(u,\delta/2) = \emptyset$. For each $0 < \epsilon < \delta/2$, define the operator

$$A_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \int_{D(0,\epsilon)} Y_z d\lambda(z).$$

By the norm continuity of the map $z \longmapsto Y_z$ provided by Proposition (6.3.11), we have

$$\lim_{\epsilon\downarrow 0} \|Y_0 - Y_\epsilon\| = 0.$$

Thus to prove the membership $Y_0 \in \mathcal{T}^{(1)}$, it suffices to show that each A_ϵ is a Toeplitz operator with a bounded symbol. Indeed by the Möbius invariance of β and $d\lambda$, we have

$$A_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \int_{D(0,\epsilon)} k_{\varphi_u(z)} \otimes k_{\varphi_u(z)} d\lambda(z)$$

$$= \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \int_{D(u,\epsilon)} k_w \otimes k_w d\lambda(w) = \int f_\epsilon(w) k_w \otimes k_w d\lambda(w),$$

where

$$f_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \chi_{D(u,\epsilon)}.$$

Since, $0 < \epsilon < \delta/2$, we have $D(u,\epsilon) \cap D(v,\epsilon) = \emptyset$ for $u \neq v$ in Γ. Hence $f_\epsilon \in L^\infty(\mathbf{B}, dv)$. By (34), we have $A_\epsilon = T_{f_\epsilon}$. This proves the membership $Y_0 \in \mathcal{T}^{(1)}$.

(ii) Now consider an arbitrary $z \in \mathbf{B}$. By Lemma (6.3.6) (ii), there is a partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_{\mathrm{m}}$ such that for every $i \in \{1, \ldots, m\}$, the conditions $u, v \in \Gamma_i$ and $u \neq v$ imply $\beta\big(\varphi_u(z), \varphi_v(z)\big) \geq 1$. That is, for each $i \in \{1, \ldots, m\}$, the set

$$G_i = \{\varphi_u(z) : u \in \Gamma_i\}$$

is separated. Obviusly, $Y_z = Y_{z,1} + \cdots + Y_{z,m}$, where

$$Y_{z,i} = \sum_{\varphi_u(z)\in G_i} c_u k_{\varphi_u(z)} \otimes k_{\varphi_u(z)},$$

$i = 1, \ldots, m$. By (1) we have $Y_{z,i} \in \mathcal{T}^{(1)}$ for every $i \in \{1, \ldots, m\}$. Hence $Y_z \in \mathcal{T}^{(1)}$.

In addition to the normalized reproducing kernel k_z given by (27), it will be convenient for our next step to use the unnormalized reproducing kernel

$$K_z(\zeta) = \frac{1}{(1 - \langle \zeta, z\rangle)^{n+1}}, z, \zeta \in \mathbf{B},$$

and other kernel-like functions. This involves monomials in the complex variables $\zeta_1, \ldots, \zeta_n$ and the standard multi-index convention (see [79]). For each pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we define

$$K_{z;\alpha}(\zeta) = \frac{\zeta^\alpha}{(1 - \langle \zeta, z\rangle)^{n+1+|\alpha|}}, \tag{68}$$

$\zeta \in \mathbf{B}$. Note that $K_z = K_{z;0}$ for every $z \in \mathbf{B}$.

Proposition (6.3.18)[192]: Let Γ be a separated set in $\mathbf{B}$ and suppose that $\{c_u : u \in \Gamma\}$ is a bounded set of complex coefficients. Then for every pair of $\alpha \in \mathbf{Z}_+^n$ and $z \in \mathbf{B}$, we have

$$\sum_{u\in\Gamma} c_u (U_u K_z) \otimes \big(U_u K_{z;\alpha}\big) \in \mathcal{T}^{(1)}.$$

Proof. We prove the proposition by an induction on $|\alpha|$. If $|\alpha| = 0$, i.e. $\alpha = 0$, then

$$(U_uK_z)\otimes(U_uK_{z;0})=(U_uK_z)\otimes(U_uK_z)=\frac{1}{(1-|Z|^2)^{n+1}}\,k_{\varphi_u(z)}\otimes k_{\varphi_u(z)}.$$

Hence the case where $|\alpha|=0$ follows from Proposition (6.3.17). Suppose that $k\in\mathbf{Z}_+$ and that the proposition holds true for every $\alpha\in\mathbf{Z}_+^n$ satisfying the condition $|\alpha|\le k$. Now consider the case where $\alpha\in\mathbf{Z}_+^n$ is such that $|\alpha|=k+1$. Then we can decompose α in the form

$$\alpha=a+b\,,$$

where $|a|=k$ and $|b|=1$. That is, there is some $\upsilon\in\{1,\dots,n\}$ such that the υ –th component of b is 1 and the other components of b are all 0. We will also consider b as a vector in $\mathbf{C}^n$. By the induction hypothesis, we have

$$\sum_{u\in\Gamma}c_u(U_uK_z)\otimes(U_uK_{z;\alpha})\in\mathcal{T}^{(1)}\qquad\text{for every}\quad z\in\mathbf{B}. \tag{69}$$

Let $z\in\mathbf{B}$ be given. Then there is an $\epsilon=\epsilon(z)>0$ such that $z+c\in\mathbf{B}$ for every $c\in\mathbf{C}^n$ satisfying the condition $|c|\le\epsilon$. For each $t\in[0,\epsilon]$, define the operators

$$A_t=\sum_{u\in\Gamma}c_u(U_uK_{z+tb})\otimes(U_uK_{z+tb;a})\quad\text{and}\quad B_t=\sum_{u\in\Gamma}c_u(U_uK_{z+itb})\otimes(U_uK_{z+itb;a}).$$

Also, we define

$$X=\sum_{u\in\Gamma}c_u\{(n+1+k)(U_uK_z)\otimes(U_uK_{z;\alpha})+(n+1)(U_uK_{z;b})\otimes(U_uK_{z;a})\}$$

and

$$Y=\sum_{u\in\Gamma}c_u\{(n+1+k)(U_uK_z)\otimes(U_uK_{z;\alpha})-(n+1)(U_uK_{z;b})\otimes(U_uK_{z;a})\}.$$

We will show that

$$\lim_{t\downarrow 0}\left\|\frac{1}{t}(A_t-A_0)-X\right\|=0 \tag{70}$$

and

$$\lim_{t\downarrow 0}\left\|\frac{1}{it}(B_t-B_0)-Y\right\|=0. \tag{71}$$

Before getting to their proofs, let us first see the consequence of these limits. By (69) we have $A_t\in\mathcal{T}^{(1)}$ and $B_t\in\mathcal{T}^{(1)}$ for all $t\in[0,\epsilon]$. Hence it follows from (70) and (71) that $X,Y\in\mathcal{T}^{(1)}$. Thus

$$\sum_{u\in\Gamma}c_u(U_uK_z)\otimes(U_uK_{z;\alpha})=\frac{1}{2(n+1+k)}(X+Y)\in\mathcal{T}^{(1)}\ ,$$

completing the induction on $|\alpha|$.

Let us now turn to the proof of (70). Note that $t^{-1}(A_t-A_0)=G_t+H_t$, where

$$H_t=\frac{1}{t}\sum_{u\in\Gamma}c_u(U_uK_{z+tb})\otimes\{U_u(K_{z+tb;a}-K_{z;a})\}$$

and

$$G_t=\frac{1}{t}\sum_{u\in\Gamma}c_u\{U_u(K_{z+tb}-K_z)\}\otimes(U_uK_{z;a}).$$

Similarly, we write $X = V + W$, where
$$V = \sum_{u\in\Gamma} c_u(n+1+k)(U_uK_z) \otimes (U_uU_{z;\alpha})$$
and
$$W = \sum_{u\in\Gamma} c_u(n+1)(U_uK_{z;b}) \otimes (U_uU_{z;a}).$$
Since $\|t^{-1}(A_t - A_0) - X\| \leq \|H_t - V\| + \|G_t - W\|$, (70) will follow if we can show
$$\lim_{t\downarrow 0}\|H_t - V\| = 0 \tag{72}$$
and
$$\lim_{t\downarrow 0}\|G_t - W\| = 0. \tag{73}$$
To prove (72), for $0 < t \leq \epsilon$ we write $H_t - V = S_t + T_t$, where
$$S_t = \sum_{u\in\Gamma} c_u(U_uK_{z+tb}) \otimes \{U_u(t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)k_{z;\alpha})\}$$
and
$$T_t = (n+1+k)\sum_{u\in\Gamma} c_u\{U_u(K_{z+tb} - K_z)\} \otimes (U_uK_{z;\alpha}).$$
Thus the proof of (72) is reduced to the proof of the fact that $\|S_t\| \to 0$ and $\|T_t \to 0\|$ as t descends to 0. To prove this, we pick an orthonormal set $\{e_u : u \in \Gamma\}$ and factor S_t in the form $S_t = S_t^{(1)}S_t^{(2)*}$, where
$$S_t^{(1)} = \sum_{u\in\Gamma} c_u(U_uK_{z+tb}) \otimes e_u$$
and
$$S_t^{(2)} = \sum_{u\in\Gamma} \{U_u(t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)k_{z;\alpha})\} \otimes e_u.$$
Set $C = \sup_{u\in\Gamma}|c_u|$. Then it follows from Lemma (6.3.10) that
$$\left\|S_t^{(1)}\right\| \leq CB(\Gamma)\|K_{z+tb}\|_\infty$$
and
$$\left\|S_t^{(2)}\right\| \leq B(\Gamma)\left\|t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)k_{z;\alpha}\right\|_\infty.$$
Since $a + b = \alpha$ and $k = |a|$, by (68) and elementary algebra, we have
$$\lim_{t\downarrow 0}\left\|t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)k_{z;\alpha}\right\|_\infty = 0.$$
Also, it is trivial that $\|k_{z+tb}\|_\infty$ remains bounded as t descends to 0. Hence
$$\|S_t\| \leq \left\|S_t^{(1)}\right\|\left\|S_t^{(2)}\right\| \leq C(B(\Gamma))^2\|k_{z+tb}\|_\infty\left\|t^{-1}(K_{z+tb;a} - K_{z;a}) - (n+1+k)k_{z;\alpha}\right\|_\infty \to 0$$
as t descends to 0. For T_t, we have the factorization $T_t = T_t^{(1)}T^{(2)*}$, where
$$T_t^{(1)} = (n+1+k)\sum_{u\in\Gamma} c_u\{U_u(K_{z+tb} - K_z)\} \otimes e_u$$
and

$$T^{(2)} = \sum_{u\in\Gamma} \left(U_u K_{z;\alpha}\right) \otimes e_u.$$

By Lemma (6.3.10), $\left\|T_t^{(1)}\right\| \le (n+1+k)CB(\Gamma)\|k_{z+tb} - K_z\|_\infty$, and $T^{(2)}$ is a bounded operator. It is obvious that

$$\lim_{t\downarrow 0} \|k_{z+tb} - K_z\|_\infty = 0.$$

Hence $\|T_t\| \le \left\|T_t^{(1)}\right\| \left\|T^{(2)}\right\| \to 0$ as t descends to 0. This completes the proof of (72).

To prove (73), note that

$$G_t - W = \sum_{u\in\Gamma} c_u\{U_u\left(t^{-1}(K_{z+tb} - K_z) - (n+1)k_{z;b}\right)\} \otimes \left(U_u K_{z;a}\right) = Z_t T^{(2)*},$$

where

$$Z_t = \sum_{u\in\Gamma} c_u\{U_u\left(t^{-1}(K_{z+tb} - K_z) - (n+1)k_{z;b}\right)\} \otimes e_u\,.$$

Applying Lemma (6.3.10) again, we have

$$\|Z_t\| \le CB(\Gamma)\left\|t^{-1}(K_{z+tb} - K_z) - (n+1)k_{z;b}\right\|_\infty.$$

Another easy exercise shows that

$$\lim_{t\downarrow 0} \left\|t^{-1}(K_{z+tb} - K_z) - (n+1)k_{z;b}\right\|_\infty = 0.$$

Hence $\|G_t - W\| \le \|Z_t\|\left\|T^{(2)}\right\| \to 0$ as t descends to 0, proving (73). Thus we have completed the proof of (70).

The proof of (71) uses essentially the same argument as above, and the only additional care that needs to be taken is the following: The rank-one operator $f \otimes g$ is linear with respect to f and conjugate linear with respect to g. Moreover, the inner product $\langle\zeta, z\rangle$ on $\mathbf{C}^n$ is conjugate linear with respect to z. These are the properties that determine the + and $-$ signs in each term $c_u\{\dots\}$ in the sum that defines the operator Y. This completes the proof of the proposition.

Proposition (6.3.19)[192]: Let Γ be a separated set in $\mathbf{B}$ and let $\{c_u : u \in \Gamma\}$ be a bounded set of complex coefficients. Then for every $w \in \mathbf{B}$ we have

$$\sum_{u\in\Gamma} c_u k_u \otimes k_{\varphi_u(w)} \in \mathcal{T}^{(1)}. \tag{74}$$

Proof. For each $\alpha \in \mathbf{Z}_+^n$, define the monomial function

$$p_\alpha(\zeta) = \zeta^\alpha$$

on $\mathbf{B}$. Given a $w \in \mathbf{B}$, let us define

$$d_u(w) = c_u \left(\frac{1 - \langle w, u\rangle}{|1 - \langle w, u\rangle|}\right)^{n+1},$$

$u \in \Gamma$. Note that $K_{0;\alpha} = p_\alpha$ for every $\alpha \in \mathbf{Z}_+^n$. Also, $U_u K_0 = U_1 = k_u$ for every $u \in \Gamma$. Thus, applying Proposition (6.3.18) to the case where $z = 0$, we have

$$\sum_{u\in\Gamma} d_u(w) k_u \otimes (U_u p_\alpha) \in \mathcal{T}^{(1)} \tag{75}$$

for every $\alpha \in \mathbf{Z}_+^n$. Define the function

$$g_w(\zeta) = \langle\zeta, w\rangle\,, \zeta \in \mathbf{B}.$$

For each $j \in \mathbf{Z}_+$, define the operator

$$A_j = \sum_{u\in\Gamma} d_u(w)k_u \otimes \left(U_u g_w^j\right).$$

Since each g_w^j is in the linear span of $\{p_\alpha : \alpha \in \mathbf{Z}_+^n\}$, (75) implies that $A_j \in \mathcal{T}^{(1)}$ for every $j \in \mathbf{Z}_+$. Let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then we have the factorization $A_j = TB_j^*$ for each $j \in \mathbf{Z}_+$, where

$$T = \sum_{u\in\Gamma} d_u(w)k_u \otimes e_u \qquad \text{and} \qquad B_j = \sum_{u\in\Gamma} \left(U_u g_w^j\right) \otimes e_u .$$

Lemma (6.3.10) tells us that T is a bounded operator. Define

$$G = \sum_{u\in\Gamma} (U_u K_w) \otimes e_u .$$

It also follows from Lemma (6.3.10) that

$$\left\| G - \sum_{j=0}^{k} \frac{(n+j)!}{n!\,j!} B_j \right\| \le B(\Gamma) \left\| K_w - \sum_{j=0}^{k} \frac{(n+j)!}{n!\,j!} g_w^j \right\|_\infty \tag{76}$$

for every $k \in \mathbf{Z}_+$. By the expansion formula

$$\frac{1}{(1-c)^{n+1}} = \sum_{j=0}^{\infty} \frac{(n+j)!}{n!\,j!} c^j \,, |c| < 1,$$

and the fact that $|w| < 1$, we have

$$\lim_{k\to\infty} \left\| K_w - \sum_{j=0}^{k} \frac{(n+j)!}{n!\,j!} g_w^j \right\|_\infty = 0.$$

Combining this with (76), we obtain

$$\lim_{k\to\infty} \left\| TG^* - \sum_{j=0}^{k} \frac{(n+j)!}{n!\,j!} A_j \right\| = \lim_{k\to\infty} \left\| TG^* - T \sum_{j=0}^{k} \frac{(n+j)!}{n!\,j!} B_j^* \right\| = 0.$$

Since each A_j belongs to $\mathcal{T}^{(1)}$, we conclude that

$$\sum_{u\in\Gamma} d_u(w)k_u \otimes (U_u K_w) = TG^* \in \mathcal{T}^{(1)}.$$

Since $k_w = (1-|w|^2)^{(n+1)/2} K_w$, this implies

$$\sum_{u\in\Gamma} d_u(w)k_u \otimes (U_u K_w) \in \mathcal{T}^{(1)}. \tag{77}$$

Recalling the definition of $d_u(w)$ and (52), we see that (77) implies (74).

Proposition (6.3.20)[192]: We have $\mathcal{D}_0 \subset \mathcal{T}^{(1)}$.

Since $\mathcal{T}^{(1)}$ is a norm closed linear subspace of $\mathcal{B}(L_a^2(\mathbf{B}, dv))$, Proposition (6.3.12) follows immediately from Propositions (6.3.16) and (6.3.20).

Proof. Let Γ be a separated set in $\mathbf{B}$, let $\{c_u : u \in \Gamma\}$ be a bounded set of coefficients, and let $\gamma : \Gamma \to \mathbf{B}$ be a map satisfying (58). Let $K = \{w \in \mathbf{B} : \beta(0, w) \le C\}$, where C is the constant that appears in (58). We want to show that the operator

$$T = \sum_{u\in\Gamma} c_u k_u \otimes k_{\gamma(u)}$$

belongs to $\mathcal{T}^{(1)}$. For this purpose, define

$$\psi(u) = \varphi_u(\gamma(u)),\, u \in \Gamma.$$

Since $\beta(u,\gamma(u)) \le C$, by the Möbius invariance of β and the fact $\varphi_u(u) = 0$, we have $\beta(0,\psi(u)) = \beta(u,\gamma(u)) \le C$ for every $u \in \Gamma$. That is, $\psi(u) \in K$ for every $u \in \Gamma$. Since $\varphi_u(\psi(u)) = \gamma(u), u \in \Gamma$, by (52) we have

$$T = \sum_{u\in\Gamma} d_u k_u \otimes (U_u k_{\psi(u)}),$$

where $|d_u| = |c_u|$ for every $u \in \Gamma$. Let $\{e_u : u \in \Gamma\}$ be an orthonormal set. Then we have the factorization $T = AB^*$, where

$$A = \sum_{u\in\Gamma} d_u k_u \otimes e_u \qquad \text{and} \qquad B = \sum_{u\in\Gamma} (U_u k_{\psi(u)}) \otimes e_u\,.$$

We again use the fact that the map $z \mapsto k_z$ is $\|\cdot\|_\infty$-continuous. That is,

$$\lim_{w\to z} \|k_z - k_w\|_\infty = 0 \qquad \text{for every} \qquad z \in \mathbf{B}.$$

Let $\epsilon > 0$ be given. Since K is compact, there are non-empty open sets $\Omega_1, \dots, \Omega_m$ in $\mathbf{B}$ and $z_i \in \Omega_i, i = 1, \dots, m$, such that

$$\Omega_1 \cup \dots \cup \Omega_m \supset K \tag{78}$$

and

$$\|k_{z_i} - k_w\|_\infty < \epsilon \qquad \text{wherever} \quad w \in \Omega_i,$$

$i = 1, \dots, m$. From the open cover (78) we obtain a partition

$$K = E_1 \cup \dots \cup E_m$$

such that $E_i \subset \Omega_i$ for every $i \in \{1, \dots, m\}$. We now define

$$\Gamma_i = \{u \in \Gamma : \psi(u) \in E_i\},$$

$i = 1, \dots, m$. Then $\|k_{z_i} - k_{\psi(u)}\|_\infty < \epsilon$ if $u \in \Gamma_i$. For every $i \in \{1, \dots, m\}$, we also define

$$B_i = \sum_{u\in\Gamma_i} (U_u k_{z_i}) \otimes e_u\,.$$

For each $i \in \{1, \dots, m\}$ we have

$$AB_i^* = \sum_{u\in\Gamma_i} d_u k_u \otimes (U_u k_{z_i}) = \sum_{u\in\Gamma_i} d_{u,i} k_u \otimes k_{\varphi_u(z_i)},$$

where $|d_{u,i}| = |d_u|$ for $u \in \Gamma_i$. Thus it follows from Proposition (6.3.19) that

$$\{AB_1^*, \dots, AB_m^*\} \subset \mathcal{T}^{(1)}. \tag{79}$$

On the other hand, we have

$$B - (B_1 + \cdots + B_m) = \sum_{i=1}^{m} \sum_{u\in\Gamma_i} \{U_u(k_{\psi(u)} - k_{z_i})\} \otimes e_u\,.$$

Since the sets $\Gamma_1, \dots, \Gamma_m$ form a partition of Γ, i.e., $\Gamma_i \cap \Gamma_j = \emptyset$ whenever $i \ne j$, Lemma (6.3.10) tells us that

$$\|B - (B_1 + \cdots + B_m)\| \le B(\Gamma) \max_{1\le i\le m} \sup_{u\in\Gamma_i} \|k_{\psi(u)} - k_{z_i}\|_\infty \le B(\Gamma)\epsilon.$$

Lemma (6.3.10) also tells us that A is a bounded operator. Hence

$$\begin{aligned}\|T - (AB_1^* + \cdots + AB_m^*)\| &= \|AB^* - (AB_1^* + \cdots + AB_m^*)\| \le \|A\|\|B^* - (B_1^* + \cdots + B_m^*)\| \\ &= \|A\|\|B - (B_1 + \cdots + B_m)\| \le \|A\|B(\Gamma)\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, combining this inequality with (79), we conclude that $T \in \mathcal{T}^{(1)}$.

This completes the proof of Proposition (6.3.20).

The analogue of Theorem (6.3.13) also holds in the setting of the Fock space. To discuss the details, let us first recall the necessary definitions.

Let $d\mu$ be the Gaussian measure on $\mathbf{C}^n$. It is well known that, in terms of the standard volume measure dV on $\mathbf{C}^n$, we have

$$d\mu(z) = \pi^{-n} e^{-|z|^2} dV(z).$$

Recall that the Fock space $H^2(\mathbf{C}^n, d\mu)$ is defined to be the subspace $h \in L^2(\mathbf{C}^n, d\mu)$: h is analytic on $\mathbf{C}^n\}$ of $L^2(\mathbf{C}^n, d\mu)$. The symbol k_z will denote the normalized reproducing kernel for $H^2(\mathbf{C}^n, d\mu)$. That is,

$$k_z(\zeta) = e^{\langle \zeta, z\rangle} e^{-|z|^2/2}, z, \zeta \in \mathbf{C}^n$$

In [190], the notion of sufficiently localized operators was introduced:

Definition (6.3.21)[192]: A bounded operator B on $H^2(\mathbf{C}^n, d\mu)$ is said to be sufficiently localized if there exist constants $2n < \beta < \infty$ and $0 < C < \infty$ such that

$$|\langle Bk_z, k_w\rangle| \leq \frac{C}{(1 + |z - w|)^\beta}$$

for all $z, w \in \mathbf{C}^n$.

Let $C^*(\mathcal{SL})$ be the C^*-algebra generated by the collection of sufficiently localized operators on $H^2(\mathbf{C}^n, d\mu)$. Combining localization properties with a new approach, it was shown in [190] that for $A \in C^*(\mathcal{SL})$,

$$\text{the condition } \lim_{|z|\to\infty} \langle Ak_z, k_z\rangle = 0 \text{ implies that } A \text{ is compact:} \qquad (80)$$

This was the result that motivated Isralowitz, Mitkovski and Wick to introduce the notion of weakly localized operators in [184]. On the Fock space, weakly localized operators are defined as follows.

Definition (6.3.22)[192]: [184] A bounded operator T on $H^2(\mathbf{C}^n, d\mu)$ is said to be weakly localized if it satisfies the conditions

$$\sup_{z\in\mathbf{C}^n} \int |\langle Tk_z, k_w\rangle|\, dV(w) < \infty\ , \quad \sup_{z\in\mathbf{C}^n} \int |\langle T^*k_z, k_w\rangle|\, dV(w) < \infty\,,$$

and

$$\lim_{r\to\infty} \sup_{z\in\mathbf{C}^n} \int\limits_{|z-w|\geq r} |\langle Tk_z, k_w\rangle| dV(w) = 0, \qquad \lim_{r\to\infty} \sup_{z\in\mathbf{C}^n} \int\limits_{|z-w|\geq r} |\langle T^*k_z, k_w\rangle| dV(w) = 0.$$

It is easy to see that any sufficiently localized operator is weakly localized. Moreover, it was shown in [184] that (80) also holds true if A is in the C^*-algebra generated by the weakly localized operators on $H^2(\mathbf{C}^n, d\mu)$.

Replacing the class $\mathcal{A}_s$ by the class of operators defined in Definition (6.3.22), one can prove the analogue of Theorem (6.3.13) on the Fock space $H^2(\mathbf{C}^n, d\mu)$. The proof is in fact easier in the Fock space case. This is because, compared with the Bergman space, the structure of the Fock space is much simpler, and one generally gets much better "decaying rate" in estimates.

For example, instead of general separated sets, in the Fock space setting we only need to be concerned with the standard lattice

$$\mathbf{Z}^{2n} = \{(j_1 + ik_1, \dots, j_n + ik_n): j_1, \dots, j_n, k_1, \dots, k_n \in \mathbf{Z}\}$$

and its subsets. What replaces $D(0,2)$ is the fundamental cube

$$S = \{(x_1 + iy_1, \dots, x_n + iy_n): x_1, \dots, x_n, y_1, \dots, y_n \in \{0,1)\}$$

in $\mathbf{C}^n$. With $\mathbf{Z}^{2n}$ and S we have

$$\bigcup_{u\in\mathbf{Z}^{2n}}\{u-S\}=\mathbf{C}^n\,, \tag{81}$$
which is a tiling of the space, meaning that there is no overlap between $u-S$ and $v-S$ for $u\neq v$ in $\mathbf{Z}^{2n}$. Compared with the covering scheme (33), the tiling scheme (81) offers considerable advantages. For example, the Toeplitz operator T_Φ used in the proof of Theorem (6.3.13) can simply be replaced by the identity operator 1 in the case of Fock space.

There is, however, one technical issue in the Fock space case that warrants mentioning. This stems from the fact that there are no bounded analytic functions on $\mathbf{C}^n$ other than constants. Thus the straightforward analogue of Lemma (6.3.10) on $H^2(\mathbf{C}^n,d\mu)$, while true, is not very useful. In the Fock-space setting, the supremum norm $\|\cdot\|_\infty$ must be replaced by something else.

Definition (6.3.23)[192]: For an analytic function h on $\mathbf{C}^n$, we write
$$\|h\|_*=\left(\int|h(\zeta)|^2e^{-(1/2)|\zeta|^2}dV(\zeta)\right)^{1/2}.$$
Let $\mathcal{H}_*$ be the collection of analytic functions h on $\mathbf{C}^n$ satisfying the condition $\|h\|_*<\infty$.

For each $z\in\mathbf{C}^n$, let U_z be the unitary operator defined by the formula
$$(U_zf)(\zeta)=f(z-\zeta)k_z(\zeta)\ ,\qquad \zeta\in\mathbf{C}^n, \tag{82}$$
$f\in H^2(\mathbf{C}^n,d\mu)$. The following is what replaces Lemma (6.3.10) in the Fock-space setting:

Lemma (6.3.24)[192]: There is a constant $0<C_{4.4}<\infty$ such that the following estimate holds: Let $e_u: u\in\mathbf{Z}^{2n}$ be any orthonormal set and let $h_u\in\mathcal{H}_*\,, u\in\mathbf{Z}^{2n}$, be functions satisfying the condition $\sup_{u\in\mathbf{Z}^{2n}}\|h\|_*<\infty$. Then
$$\left\|\sum_{u\in\mathbf{Z}^{2n}}(U_uh_u)\ \otimes e_u\right\|\leq C_{4.4}\sup_{u\in\mathbf{Z}^{2n}}\|h\|_*.$$
Proof. Let us first estimate $|\langle U_uh_u,U_vh_v\rangle|$. By (82), for $u\in\mathbf{Z}^{2n}$ we have
$$|\langle U_uh_u,U_vh_v\rangle|=\int h_u(u-\zeta)\overline{h_v(v-\zeta)}k_u(\zeta)\overline{k_v(\zeta)}e^{-|\zeta|^2}\,dV(\zeta). \tag{83}$$
Moreover,
$$\left|k_u(\zeta)\overline{k_v(\zeta)}\right|e^{-|\zeta|^2}=e^{-(1/2)\,(|u-\zeta|^2+|v-\zeta|^2)}, \tag{84}$$
$\zeta\in\mathbf{C}^n$. Observe that
$$(|u-\zeta|^2+|v-\zeta|^2\geq\frac{1}{2}(|u-\zeta|^2+|v-\zeta|^2\geq\frac{1}{2}|u-v|^2.$$
Thus, splitting the $1/2$ in (84) as $(1/4)+(1/4)$, we find that
$$\left|k_u(\zeta)\overline{k_v(\zeta)}\right|e^{-|\zeta|^2}\leq e^{-(1/8)|u-v|^2}\,e^{-(1/4)|u-\zeta|^2}\,e^{-(1/4)|v-\zeta|^2}.$$
Combining this with (83) and applying the Cauchy-Schwarz inequality, we obtain
$$|\langle U_uh_u,U_vh_v\rangle|\leq e^{-\left(\frac{1}{8}\right)|u-v|^2}\,\|h_u\|_*\|h_v\|_*\leq e^{-\left(\frac{1}{8}\right)|u-v|^2}\,H_*^2, \tag{85}$$
where
$$H_*=\sup_{u\in\mathbf{Z}^{2n}}\|h_u\|_*.$$
Write
$$A=\sum_{u\in\mathbf{Z}^{2n}}(U_uh_u)\otimes e_u$$
and consider any vector $x=\sum_{u\in\mathbf{Z}^{2n}}x_ue_u$. By (85), we have

$$\|Ax\|^2 \le \sum_{u,v\in\mathbf{Z}^{2n}} |\langle U_v h_v , U_u h_u\rangle||x_u||x_v| \le H_*^2 \sum_{u,v\in\mathbf{Z}^{2n}} e^{-(1/8)|u-v|^2} |x_u||x_v|.$$

Applying the Schur test to the right-hand side, we find that

$$\|Ax\|^2 \le CH_*^2 \sum_{u\in\mathbf{Z}^{2n}} |x_u|^2 = CH_*^2\|x\|^2,$$

where $C = \sum_{z\in\mathbf{Z}^{2n}} e^{-(1/8)|v|^2}$, which is finite. Since the vector x is arbitrary, we conclude that $\|A\| \le C^{1/2}H_*$. Thus the lemma holds for the constant $C_{4.4} = C^{1/2}$.

In the proof of the Fock-space analogue of Theorem (6.3.13), the $\|\cdot\|_\infty$-continuities of the previous are replaced by the corresponding $\|\cdot\|_*$-continuities. For example, for the normalized reproducing kernel of the Fock space one easily verifies that

$$\lim_{w\to z}\|k_z - k_w\|_* = 0$$

for every $z \in \mathbf{C}^n$. Thus, using Lemma (6.3.24) in place of Lemma (6.3.10), the analogue of Theorem (6.3.13) on the Fock space can be obtained by following the argument in the previously.

Corollary (6.3.25)[200]: Let Γ be a separated set in **B**.

(c) For each $0 < \epsilon < \infty$, there is a natural number $N = N(\Gamma; 1+\epsilon)$such that $\text{card}\{(u+\epsilon) \in \Gamma: \beta(u, u+\epsilon) \le 1+\epsilon\} \le N$ for every $u \in \Gamma$.

(d) For every pair of $(z^2-1) \in \mathbf{B}$ and $\epsilon > 0$, there is a finite partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$such that for every $i \in \{1, \ldots, m\}$, the conditions $u, (u+\epsilon) \in \Gamma_i$and $\epsilon > 0$imply $\beta\big(\varphi_u(z^2-1), \varphi_{u+\epsilon}(z^2-1)\big) > 1+\epsilon$.

Proof. By definition, there is a $\delta > 0$ such that $\beta(u, u+\epsilon) \ge \delta$ for all $\epsilon > 0$in Γ. Thus

$$D(u, \delta/2) \cap D\left(u+\epsilon, \frac{\delta}{2}\right) = \emptyset \quad \text{for all} \epsilon > 0 \ \text{ in } \Gamma.$$

Let $\epsilon > 0$ be given. Then for every pair of $u, (u+\epsilon) \in \Gamma$, the condition $\beta(u, u+\epsilon) \le 1+\epsilon$implies $D(u+\epsilon, \delta/2) \subset D\big(u, 1+\epsilon+(\delta/2)\big)$. By the Möbius invariance of the Bergman metric $\beta$and the measure $d\lambda$, we have

$$\lambda\big(D(\delta/2)\big) = \lambda\big(\varphi_{u+\epsilon}(D(0,\delta/2)\big) = \lambda\big(D(0,\delta/2)\big).$$

Therefore if we write $N(u)$ for the cardinality of the set $\{(u+\epsilon) \in \Gamma: \beta(u, u+\epsilon) \le 1+\epsilon\}$, then

$$N(u)\lambda\big(D(0,\delta/2)\big)$$

$$= \sum_{\substack{(u+\epsilon)\in\Gamma \\ \beta(u,u+\epsilon)\le 1+\epsilon}} \lambda\big(D(u+\epsilon, \delta/2)\big) \le \lambda(D\big(u, 1+\epsilon+(\delta/2)\big)$$

$$= \lambda\Big(D\big(0{,}1+\epsilon+(\delta/2)\big)\Big).$$

That is, $N(u) \le \lambda(D\big(0{,}1+\epsilon+(\delta/2)\big)/\lambda\big(D(0,\delta/2)\big)$, which proves (a).

To prove (b), let $(z^2-1) \in \mathbf{B}$ and $\epsilon > 0$ be given, and set $2\beta(z^2-1{,}0) = 0$. By (a), there is an $m \in N$ such that $\text{card}\{(u+\epsilon) \in \Gamma: \beta(u, u+\epsilon) \le 1+\epsilon\} \le m$ for every $u \in \Gamma$. By a standard maximality argument, there is a partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$such that for every $i \in \{1, \ldots, m\}$, the conditions $u, (u+\epsilon) \in \Gamma_i$and $\epsilon > 0$imply $\beta(u, u+\epsilon) > 1+\epsilon$. But if $u, (u+\epsilon)$satisfy the condition $\beta(u, u+\epsilon) > 1+\epsilon$, then by the Möbius invariance of βwe have

$$\beta\big(\varphi_u(z^2-1),\varphi_{u+\epsilon}(z^2-1)\big)$$
$$\geq \beta(u,u+\epsilon)-\beta(\varphi_u(z^2-1),u)-\beta\big(u+\epsilon,\varphi_{u+\epsilon}(z^2-1)\big)$$
$$=\beta(u,u+\epsilon)-\beta\big(\varphi_u(z^2-1),\varphi_u(0)\big)-\beta\big(\varphi_{u+\epsilon}(0),\varphi_{u+\epsilon}(z^2-1)\big)$$
$$=\beta(u,u+\epsilon)-\beta(z^2-1,0)-\beta(0,z^2-1)>1+\epsilon-2\beta(z^2-1,0)$$
$$=1+\epsilon.$$
This completes the proof.

Corollary (6.3.26)[200]: For all $u,(u+\epsilon),x,(x+\epsilon)\in\mathbf{B}$ we have
$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_{u+\epsilon}(x+\epsilon)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_{u+\epsilon}(x+\epsilon)\rangle|}$$
$$\leq 2e^{\beta(x,0)+\beta(x+\epsilon,0)}\frac{(1-|u|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle u,u+\epsilon\rangle|}.$$
Proof. For $a,a+\epsilon\in\mathbf{B}$, we have $1-|\varphi_a(a+\epsilon)|^2=(1-|a|^2)(1-|a+\epsilon|^2)/|1-\langle a,a+\epsilon\rangle^2|$[79]. Thus if we write
$$\alpha^2-1=\frac{(1-|a|^2)(1-|a+\epsilon|^2)^{1/2}}{|1-\langle a,a+\epsilon\rangle^2|},$$
then
$$\log\frac{1}{\alpha^2-1}\leq\frac{1}{2}\log\frac{1+|\varphi_a(a+\epsilon)|}{1-|\varphi_a(a+\epsilon)|}\leq\log\frac{2}{\alpha^2-1}.$$
Consequently
$$e^{-\beta(a,a+\epsilon)}\leq\frac{(1-|a|^2)^{\frac{1}{2}}(1-|a+\epsilon|^2)^{\frac{1}{2}}}{|1-\langle a,a+\epsilon\rangle|}\leq 2e^{-\beta(a,a+\epsilon)}. \tag{86}$$
For $u,(u+\epsilon),x,(x+\epsilon)\in\mathbf{B}$, by the Möbius invariance of the Bergman metric, we have
$$\beta\big(\varphi_u(z^2-1),\varphi_{u+\epsilon}(x+\epsilon)\big)\geq\beta(u,u+\epsilon)-\beta(\varphi_u(x),u)-\beta(\varphi_{u+\epsilon}(x+\epsilon),u+\epsilon)$$
$$=\beta(u,u+\epsilon)-\beta(x,0)-\beta(x+\epsilon,z^2-1).$$
Combining (86) with this inequality, we find that
$$\frac{(1-|\varphi_u(x)|^2)^{1/2}(1-|\varphi_{u+\epsilon}(x+\epsilon)|^2)^{1/2}}{|1-\langle\varphi_u(x),\varphi_{u+\epsilon}(x+\epsilon)\rangle|}\leq 2e^{-\beta(\varphi_u(x),\varphi_{u+\epsilon}(x+\epsilon))}$$
$$\leq 2e^{\beta(x,0)+\beta(x+\epsilon,0)}e^{-\beta(u,u+\epsilon)}$$
$$\leq 2e^{\beta(x,0)+\beta(x+\epsilon,0)}\frac{(1-|u|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle u,u+\epsilon\rangle|}.$$
This proves the lemma.

Corollary (6.3.27)[200]: Let Γbe a separated set in $\mathbf{B}$. Then there is a $0<C(\Gamma)<\infty$such that
$$\sum_{u+\epsilon\in\Gamma}\left(\frac{(1-|\xi^2-1|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle\xi^2-1,u+\epsilon\rangle|}\right)^{n+1}(1-|u+\epsilon|^2)^{(4n+1)/8}$$
$$\leq C(\Gamma)(1-|\xi^2-1|^2)^{(4n+1)/8}$$
for every $(\xi^2-1)\in\mathbf{B}$.

Proof. If Γ is a separated set in $\mathbf{B}$, then there is a $\delta>0$ such that $\beta(u,u+\epsilon)\geq\delta$for all $\epsilon>0$in Γ. Thus $D(u,\delta/2)\cap D(u+\epsilon,\delta/2)=\emptyset$for all $\epsilon>0$in Γ. If $(w^2-1)\in D(u+\epsilon,\delta/2)$, then $(u+\epsilon)\in D(w^2-1,\delta/2)=\varphi_{w^2-1}\big(D(0,\delta/2)\big)$. Thus If $(w^2-1)\in D(u+\epsilon,\delta/2)$, then there is a $u'+\epsilon\in D(u+\epsilon,\delta/2)$ such that $u+\epsilon=\varphi_{w^2-1}(u'+\epsilon)$. Let $(\xi^2-1)\in\mathbf{B}$. Since $\xi^2-1=\varphi_{\xi^2-1}(0)$, we can apply Corollary (6.3.27) to obtain

$$\frac{(1-|\xi^2-1|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle \xi^2-1, u+\epsilon\rangle|} \le 2e^{\frac{\delta}{2}} \frac{(1-|\xi^2-1|^2)^{\frac{1}{2}}(1-|w^2-1|^2)^{\frac{1}{2}}}{|1-\langle \xi^2-1, w^2-1\rangle|} \quad (87)$$

for every $(w^2-1) \in D(u+\epsilon, \delta/2)$. Also, since $u+\epsilon = \varphi_{w^2-1}(u'+\epsilon)$ and $u'+\epsilon \in D(0, \delta/2)$, we have

$$\begin{aligned} 1-|u+\epsilon|^2 &= 1-|\varphi_{w^2-1}(u'+\epsilon)|^2 \\ &= \frac{(1-|u'+\epsilon|^2)(1-|w^2-1|^2)}{|1-\langle u'+\epsilon, w^2-1\rangle|^2} \\ &\le \frac{4}{1-|u'+\epsilon|^2}(1-|w^2-1|^2) \\ &\le 4e^{2\beta(u'+\epsilon,0)}(1-|w^2-1|^2) \\ &\le 4e^{\delta}(1-|w^2-1|^2). \end{aligned} \quad (88)$$

Set $C_1 = \left(2e^{\delta/2}\right)^{n+1}\left(4e^{\delta}\right)^{(4n+1)/8}$. Then it follows from (87) and (88) that

$$\begin{aligned} &\left(\frac{(1-|\xi^2-1|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle \xi^2-1, u+\epsilon\rangle|}\right)^{n+1} (1-|u+\epsilon|^2)^{(4n+1)/8} \\ &\qquad \le C_1 \left(\frac{(1-|\xi^2-1|^2)^{1/2}(1-|w^2-1|^2)^{1/2}}{|1-\langle \xi^2-1, w^2-1\rangle|}\right)^{n+1} (1-|w^2-1|^2)^{(4n+1)/8} \end{aligned}$$

for every $(w^2-1) \in D(u+\epsilon, \delta/2)$. Hence for each $(\xi^2-1) \in$ **B**we have

$$\begin{aligned} &\sum_{(u+\epsilon)\in\Gamma} \left(\frac{(1-|\xi^2-1|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle \xi^2-1, u+\epsilon\rangle|}\right)^{n+1} (1-|u+\epsilon|^2)^{(4n+1)/8} \\ &\le \sum_{(u+\epsilon)\in\Gamma} \frac{C_1}{\lambda\left(D\left(u+\epsilon, \frac{\delta}{2}\right)\right)} \int_{D\left(u+\epsilon,\frac{\delta}{2}\right)} \left(\frac{(1-|\xi^2-1|^2)^{\frac{1}{2}}(1-|w^2-1|^2)^{\frac{1}{2}}}{|1-\langle \xi^2-1, w^2-1\rangle|}\right)^{n+1} \\ &\times (1-|w^2-1|^2)^{\frac{4n+1}{8}} d\lambda(w^2-1) \\ &\le \frac{C_1}{\lambda\left(D\left(u+\epsilon, \frac{\delta}{2}\right)\right)} \int \left(\frac{(1-|\xi^2-1|^2)^{\frac{1}{2}}(1-|w^2-1|^2)^{\frac{1}{2}}}{|1-\langle \xi^2-1, w^2-1\rangle|}\right)^{n+1} \\ &\times (1-|w^2-1|^2)^{\frac{4n+1}{8}} d\lambda(w^2-1). \end{aligned} \quad (89)$$

To estimate the last integral, note that

$$\frac{(1-|\xi^2-1|^2)^{\frac{1}{2}}\left(1-\left|\varphi_{\xi^2-1}(\zeta^2-1)\right|^2\right)^{\frac{1}{2}}}{\left|1-\langle \xi^2-1, \varphi_{\xi^2-1}(\zeta^2-1)\rangle\right|} = (1-|\zeta^2-1|^2)^{1/2}.$$

Thus, making the substitution $w^2-1 = \varphi_{\xi^2-1}(\zeta^2-1)$ and using the Möbius invariance of $d\lambda$, we have

$$= \int (1-|\zeta^2-1|^2)^{\frac{(n+1)}{2}} \left(1-\left|\varphi_{\xi^2-1}(\zeta^2-1)\right|^2\right)^{\frac{(4n+1)}{8}} d\lambda(\zeta^2-1)$$

$$= \int (1-|\zeta^2-1|^2)^{\frac{(n+1)}{2}} \left(\frac{(1-|\xi^2-1|^2)(1-|\zeta^2-1|^2)}{(1-\langle \xi^2-1, \zeta^2-1\rangle^2)}\right)^{\frac{(4n+1)}{8}} d\lambda(\zeta^2 - 1)$$

$$= (1-|\xi^2-1|^2)^{(4n+1)/8} \int \frac{d(u+\epsilon)(\zeta^2-1)}{|1-\langle \xi^2-1, \zeta^2-1\rangle|^{n+\left(\frac{1}{4}\right)}(1-|\zeta^2-1|^2)^{3/8}}$$

$$= (*).$$

To further estimate $(*)$, let $d\sigma$be the standard spherical measure on the unit sphere $\{x \in \mathbf{C}^n : |x| = 1\}$. There is a constant C_2such that

$$\int \frac{d\sigma(x)}{|1-\langle z^2-1, x\rangle|^{n+\left(\frac{1}{4}\right)}} \leq \frac{C_2}{(1-|z^2-1|^2)^{1/4}}$$

for every $(z^2-1) \in \mathbf{B}$ [79]. Combining this with the radial-spherical decomposition $d(u+\epsilon) = 2n(1+\epsilon)^{2n-1} d(1+\epsilon) d\sigma$ of the volume measure, we have

$$\int \frac{d(u+\epsilon)(\zeta^2-1)}{|1-\langle \xi^2-1, \zeta^2-1\rangle|^{n+\left(\frac{1}{4}\right)}(1-|\zeta^2-1|^2)^{3/8}} \leq \int_0^1 \frac{C_2 2n(1+\epsilon)^{2n-1} d(1+\epsilon)}{(1-(1+\epsilon)^2)^{(1/4)+(3/8)}}$$

$$\leq nC_2 \int_0^1 \frac{d(1-2\epsilon)}{(2\epsilon)^{5/8}} = \frac{8}{3} nC_2.$$

Therefore

$$(*) \leq 3nC_2(1-|\xi^2-1|^2)^{(4n+1)/8}.$$

Substituting this in (89), we conclude that the desired inequality holds for the constant

$$C(\Gamma) = \frac{3nC_1C_2}{\lambda\big(D(0,\delta/2)\big)}.$$

This completes the proof.

Corollary (6.3.28)[200]: There is a constant $0 < C_{2.5} < 1$ such that $\|E_{z^2-1}\| \leq C_{2.5}$ for every $(z^2-1) \in D(0,2)$.

Proof. By Corollary (6.3.26), for $u, (u+\epsilon), (z^2-1) \in \mathbf{B}$we have

$$\left|\langle k_{\varphi_{u+\epsilon}(z^2-1)}, k_{\varphi_u(z^2-1)}\rangle\right| = \left(\frac{(1-|\varphi_{u+\epsilon}(z^2-1)|^2)^{1/2}(1-|\varphi_u(z^2-1)|^2)^{1/2}}{|1-\langle \varphi_u(z^2-1), \varphi_{u+\epsilon}(z^2-1)\rangle|}\right)^{n+1}$$

$$\leq \left(2e^{2\beta(z^2-1,0)}\right)^{n+1} \left(\frac{(1-|u|^2)^{\frac{1}{2}}(1-|u+\epsilon|^2)^{\frac{1}{2}}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1}. \quad (90)$$

Let $\{\epsilon_u : u \in \mathcal{L}\}$be an orthonormal set. For each $(z^2-1) \in \mathbf{B}$, define the operator

$$F_{z^2-1} = \sum_{u\in\mathcal{L}} \epsilon_u \otimes k_{\varphi_u(z^2-1)}. \quad (91)$$

Since $E_{z^2-1} = F^*_{z^2-1} F_{z^2-1}$ and $\left\|F^*_{z^2-1} F_{z^2-1}\right\| = \left\|F_{z^2-1} F^*_{z^2-1}\right\|$, it suffices to estimate the later. We have

$$F_{z^2-1}F^*_{z^2-1} = \sum_{u,(u+\epsilon)\in\mathcal{L}} \langle k_{\varphi_{u+\epsilon}(z^2-1)}, k_{\varphi_u(z^2-1)}\rangle \epsilon_u \otimes \epsilon_{u+\epsilon}.$$

Now suppose that $(z^2-1)\in D(0,2)$ and write $C_1 = (2e^4)^{n+1}$. By (40), for every vector $x = \sum_{u,\in\mathcal{L}} x_u \epsilon_u$we have

$$\langle F_{z^2-1}F^*_{z^2-1}x, x\rangle \leq \sum_{u,(u+\epsilon)\in\mathcal{L}} \left|\langle k_{\varphi_{u+\epsilon}(z^2-1)}, k_{\varphi_u(z^2-1)}\rangle\right| |x_u||x_{u+\epsilon}|$$

$$\leq C_1 \sum_{u,(u+\epsilon)\in\mathcal{L}} \left(\frac{(1-|u|^2)^{\frac{1}{2}}(1-|u+\epsilon|^2)^{\frac{1}{2}}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} |x_u||x_{u+\epsilon}|$$

$$= C_1 \sum_{u\in\mathcal{L}} |x_u|(x+\epsilon)_u, \tag{92}$$

where

$$(x+\epsilon)_u = \sum_{(u+\epsilon)\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} |x_{u+\epsilon}|$$

for each $u\in\mathcal{L}$. Next we apply the Schur test. Indeed by the Cauchy-Schwarz inequality and Corollary (6.3.27), we have

$$(x+\epsilon)_u^2 \leq C(\mathcal{L})(1-|u|^2)^{\frac{4n+1}{8}} \times \sum_{(u+\epsilon)\in\mathcal{L}} \left(\frac{(1-|u|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}}.$$

Applying Corollary (6.3.27) again, we have

$$\sum_{u\in\mathcal{L}} (x+\epsilon)_u^2 \leq C(\mathcal{L}) \sum_{(u+\epsilon)\in\mathcal{L}} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}} \sum_{u\in\mathcal{L}} (1-|u|^2)^{\frac{4n+1}{8}} \times \left(\frac{(1-|u|^2)^{1/2}(1-|u+\epsilon|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1}$$

$$\leq C^2(\mathcal{L}) \sum_{(u+\epsilon)\in\mathcal{L}} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}} (1-|u+\epsilon|^2)^{\frac{4n+1}{8}}$$

$$= C^2(\mathcal{L}) \sum_{(u+\epsilon)\in\mathcal{L}} |x_{u+\epsilon}|^2.$$

Combining this with (92), we find that

$$\langle F_{z^2-1}F^*_{z^2-1}x, x\rangle \leq C_1 C(\mathcal{L}) \sum_{u\in\mathcal{L}} |x_{u+\epsilon}|^2 = C_1 C(\mathcal{L})\|x\|^2.$$

Since the vector $x$is arbitrary, we conclude that $\|E_{z^2-1}\| = \left\|F_{z^2-1}F^*_{z^2-1}\right\| \leq C_1 C(\mathcal{L})$ for every $(z^2-1)\in D(0,2)$. This completes the proof.

Corollary (6.3.29)[200]: Given any separated set Γ in $\mathbf{B}$, there exists a constant $0 < (A + \epsilon)(\Gamma) < \infty$ such that the following estimate holds: Let $\{h_u : u \in \Gamma\}$be functions in $H^\infty(\mathbf{B})$such that $\sup_{u\in\Gamma}\|h_u\|_\infty < \infty$, and let $\{e_u : u \in \Gamma\}$be any orthonormal set. Then

$$\left\|\sum_{u\in\Gamma}(U_u h_u) \otimes e_u\right\| \leq (A+\epsilon)(\Gamma) \sup_{u\in\Gamma}\|h_u\|_\infty .$$

Proof. Given $\Gamma, \{h_u : u \in \Gamma\}$and $\{e_u : u \in \Gamma\}$as in the statement, let us write

$$A = \sum_{u\in\Gamma}(U_u h_u) \otimes e_u$$

for convenience. By (38), the self-adjoint Toeplitz operator $T_{\sum_{\widetilde{\widetilde{m}}}\Phi_{\widetilde{m}}}$is invertible with $\left\|T^{-1}_{\sum_{\widetilde{\widetilde{m}}}\Phi_{\widetilde{m}}}\right\| < 1$. Therefore $\|A\| = \left\|T^{-1}_{\sum_{\widetilde{\widetilde{m}}}\Phi_{\widetilde{m}}} T_{\sum_{\widetilde{\widetilde{m}}}\Phi_{\widetilde{m}}} A\right\| \leq \left\|T_{\sum_{\widetilde{\widetilde{m}}}\Phi_{\widetilde{m}}} A\right\|$. Combining this with (39), we see that

$$\|A\| \leq \lambda\big(D(0,2)\big) \sup_{(z^2-1)\in D(0,2)} \|E_{z^2-1}A\|. \tag{93}$$

Thus it suffices to estimate $\|E_{z^2-1}A\|$for $(z^2 - 1) \in D(0,2)$. Let F_{z^2-1}be the operator defined by (91). Then Corollary (6.3.28) implies that $\left\|F^*_{z^2-1}\right\| \leq C_{2.5}^{1/2}$for $(z^2 - 1) \in D(0,2)$. Hence

$$\|E_{z^2-1}A\| \leq C_{2.5}^{\frac{1}{2}}\|F_{z^2-1}A\|(z^2-1) \in D(0,2). \tag{94}$$

Consequently, we only need to estimate $\|F_{z^2-1}A\|$.

To estimate $\|F_{z^2-1}A\|$, let us denote

$$H = \sup_{u\in\Gamma}\|h_u\|_\infty .$$

Let $(z^2 - 1) \in D(0,2)$. Then note that

$$F_{z^2-1}A = \sum_{u\in\mathcal{L}}\sum_{u\in\Gamma}\langle U_{u+\epsilon}h_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle \epsilon_u \otimes e_{u+\epsilon} . \tag{95}$$

Since $U_{u+\epsilon}h_{u+\epsilon} = k_{u+\epsilon} \cdot h_{u+\epsilon} \circ \varphi_{u+\epsilon}$, the reproducing property of $k_{\varphi_u(z^2-1)}$gives us

$$\langle U_{u+\epsilon}h_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle = h_{u+\epsilon}\left(\varphi_{u+\epsilon}\big(\varphi_u(z^2-1)\big)\right)\langle k_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle,$$

which is one of the key facts on which depends. Thus

$$\left|\langle U_{u+\epsilon}h_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle\right| \leq H\left|\langle k_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle\right|$$
$$= H\left(\frac{(1-|u+\epsilon|^2)^{1/2}(1-|\varphi_u(z^2-1)|^2)^{1/2}}{|1-\langle\varphi_u(z^2-1), u+\epsilon\rangle|}\right)^{n+1}.$$

Since $u + \epsilon = \varphi_{u+\epsilon}(0)$ and $(z^2 - 1) \in D(0,2)$, an application of Corollary (6.3.26) gives us

$$\left|\langle U_{u+\epsilon}h_{u+\epsilon}, k_{\varphi_u(z^2-1)}\rangle\right| \leq C_1 H\left(\frac{(1-|u+\epsilon|^2)^{\frac{1}{2}}(1-|u|^2)^{\frac{1}{2}}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1}, \tag{96}$$

where $C_1 = (2e^2)^{n+1}$. Now consider vectors

$$x = \sum_{u\in\Gamma} x_{u+\epsilon}e_{u+\epsilon} \text{ and} x+\epsilon = \sum_{u\in\mathcal{L}}(x+\epsilon)_e\epsilon_u .$$

It follows from (95) and (96) that

$$|\langle F_{z^2-1}Ax, x+\epsilon\rangle| \le C_1 H \sum_{u\in\mathcal{L}}\sum_{u\in\Gamma}\left(\frac{(1-|u+\epsilon|^2)^{\frac{1}{2}}(1-|u|^2)^{\frac{1}{2}}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} |x_{u+\epsilon}||(x+\epsilon)_u| = C_1 H \sum_{u\in\mathcal{L}} (a+\epsilon)_u |(x+\epsilon)_u|, \quad (97)$$

where

$$(a+\epsilon)_u = \sum_{u\in\Gamma}\left(\frac{(1-|u+\epsilon|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} |x_{u+\epsilon}|,$$

$u \in \mathcal{L}$. We apply the Schur test as we did in the proof of Corollary (6.3.28). By the Cauchy-Schwarz inequality and the bound given in Corollary (6.3.27), we have

$$(a+\epsilon)_u^2 \le C(\Gamma)(1-|u|)^{\frac{4n+1}{8}} \times \sum_{(u+\epsilon)\in\Gamma}\left(\frac{(1-|u+\epsilon|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}},$$

$u \in \mathcal{L}$. Applying Corollary (6.3.27) again, we obtain

$$\begin{aligned}\sum_{u\in\mathcal{L}}(a+\epsilon)_u^2 &\le C(\Gamma)\sum_{(u+\epsilon)\in\Gamma}\sum_{u\in\mathcal{L}}(1-|u|)^{\frac{4n+1}{8}} \\ &\quad\times\left(\frac{(1-|u+\epsilon|^2)^{1/2}(1-|u|^2)^{1/2}}{|1-\langle u, u+\epsilon\rangle|}\right)^{n+1} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}} \\ &= C(\Gamma)C(\mathcal{L})\sum_{(u+\epsilon)\in\Gamma}(1-|u+\epsilon|^2)^{\frac{4n+1}{8}} \frac{|x_{u+\epsilon}|^2}{(1-|u+\epsilon|^2)^{\frac{4n+1}{8}}} \\ &= C(\Gamma)C(\mathcal{L})\|x\|^2.\end{aligned}$$

Combining this with (97), we obtain

$$|\langle F_{z^2-1}Ax, x+\epsilon\rangle| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H\|x\|\|x+\epsilon\|.$$

Since the vectors $x$and $(x+\epsilon)$are arbitrary, this means

$$\|F_{z^2-1}A\| \le C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2} H$$

for $(z^2-1) \in D(0,2)$. Recalling (93) and (94), we see that the lemma holds for the constant

$$(A+\epsilon)(\Gamma) = \lambda\big(D(0,2)\big)C_{2.5}^{1/2}C_1\{C(\Gamma)C(\mathcal{L})\}^{1/2}.$$

This completes the proof.

Corollary (6.3.30)[200]: Suppose that $\Gamma$is a separated set in **B**. Furthermore, suppose that $\{c_u : u \in \Gamma\}$ are complex numbers satisfying the condition

$$\sup_{u\in\Gamma}|c_u| < \infty. \quad (98)$$

Then for each $(z^2-1) \in \mathbf{B}$, the operator

$$Y_{z^2-1} = \sum_{u\in\Gamma} c_u k_{\varphi_u(z^2-1)} \otimes k_{\varphi_u(u)} \quad (99)$$

is bounded on the Bergman space. Moreover, the map $z^2-1 \longmapsto Y_{z^2-1}$from **B** into $\mathcal{B}\left(L_a^2(\mathbf{B}, d(u+\epsilon))\right)$is continuous with respect to the operator norm.

Proof. For $u,(z^2-1) \in \mathbf{B}$, simple computation shows that

$$U_u k_{z^2-1} = \left(\frac{|1-\langle u,u+\epsilon\rangle|}{1-\langle u,u+\epsilon\rangle}\right)^{n+1} k_{\varphi_u(z^2-1)}. \tag{100}$$

Therefore

$$k_{\varphi_u(u)} \otimes k_{\varphi_u(u)} = (U_u k_{z^2-1}) \otimes (U_u k_{z^2-1}).$$

Let $\{e_u: u \in \Gamma\}$be an orthonormal set. Then for every $(z^2-1) \in \mathbf{B}$we have the factorization

$$Y_{z^2-1} = A_{z^2-1}(A_{z^2-1}^* + \epsilon),$$

where

$$A_{z^2-1} = \sum_{u\in\Gamma} c_u (U_u k_{z^2-1}) \otimes e_u \text{ and } \qquad A_{z^2-1}+\epsilon = \sum_{u\in\Gamma} (U_u k_{z^2-1}) \otimes e_u\,.$$

Applying Corollary (6.3.29) to the case $h_u = c_u k_{z^2-1}, u \in \Gamma$, we see that each $A_{z^2-1}$is a bounded operator. Similarly, each $(A_{z^2-1}+\epsilon)$is also bounded. Hence $Y_{z^2-1} = A_{z^2-1}(A_{z^2-1}^*+\epsilon)$is bounded.

To show that the map $z^2-1 \longmapsto Y_{z^2-1}$is continuous with respect to the operator norm, it suffices to show that the maps $z^2-1 \longmapsto A_{z^2-1}$and $z^2-1 \longmapsto A_{z^2-1}+\epsilon$are continuous with respect to the operator norm. Since $(A_{z^2-1}+\epsilon)$is just a special case of A_{z^2-1}, it suffices to consider the map $z^2-1 \longmapsto A_{z^2-1}$.

For any $(z^2-1),(w^2-1) \in \mathbf{B}$, we have

$$A_{z^2-1} - A_{w^2-1} = \sum_{u\in\Gamma} c_u\left(U_u(k_{z^2-1}-k_{w^2-1})\right) \otimes e_u.$$

Applying Corollary (6.3.29) to the case where $h_u = c_u(k_{z^2-1}-k_{w^2-1}), u \in \Gamma$, we find that

$$\|A_{z^2-1}-A_{w^2-1}\| \le (A+\epsilon)(\Gamma)C\|k_{z^2-1}-k_{w^2-1}\|_\infty\,,$$

where $C = \sup_{u\in\Gamma}|c_u|$. For each $(z^2-1) \in \mathbf{B}$, it is elementary that

$$\lim_{w^2-1\to z^2-1}\|k_{z^2-1}-k_{w^2-1}\|_\infty = 0.$$

Hence the map $z^2-1 \longmapsto A_{z^2-1}$is continuous with respect to operator norm. This completes the proof.

Corollary (6.3.31)[200]: For every $(n-1)/(n+1) < 1-\epsilon,\ \epsilon < 1$we have $\mathcal{T}^{(1)} = C^*(\mathcal{A}_{1-\epsilon})$. Consequently, $\mathcal{T}^{(1)} = \mathcal{T} = C^*(\mathcal{A}_{1-\epsilon})$.

Proof. Let $(n-1)/(n+1) < 1-\epsilon$, $\epsilon < 1$ be given. By the fact that $\mathcal{A}_{1-\epsilon}$is a $*$-algebra mentioned above, $C^*(\mathcal{A}_{1-\epsilon})$ is just the norm closure of $\mathcal{A}_{1-\epsilon}$. Since we also know that $\mathcal{A}_{1-\epsilon} \supset \left\{T_{\sum_{\widetilde{m}} f_{\widetilde{m}}}: \sum_{\widetilde{m}} f_{\widetilde{m}} \in L^\infty(\boldsymbol{B}, d(u+\epsilon))\right\}$, Corollary (6.3.31) will follow if we can show that $\mathcal{A}_{1-\epsilon} \subset \mathcal{T}^{(1)}$. We prove this inclusion into two steps.

(i) Let $(A+\epsilon) \in \mathcal{A}_{1-\epsilon}$be given. As the first step, let us show that $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}(A+\epsilon)T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}} \in \mathcal{T}^{(1)}$. Indeed it follows from (40) that

$$T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}(A+\epsilon)T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}} = \int \int_{D(0,2)\times D(0,2)} E_{w^2-1}(A+\epsilon)E_{z^2-1}d\lambda(w^2-1)d\lambda(z^2-1)\,. \tag{101}$$

Consider the map

$$(w^2-1,z^2-1)\mapsto E_{w^2-1}(A+\epsilon)E_{z^2-1} \tag{102}$$

from $\mathbf{B}\times\mathbf{B}$into $\mathcal{B}(L_a^2(\mathbf{B},d(u+\epsilon)))$. Proposition (6.3.12) tells us that the range of map (102) is contained in $\mathcal{T}^{(1)}$. Hence every Riemann sum corresponding to the integral in (101) belongs to $\mathcal{T}^{(1)}$. On the other hand, by Proposition (102), the map $z^2-1\mapsto E_{z^2-1}$is continuous with respect to the operator norm. Hence map (54) is also continuous with respect to the operator norm. Since the closure of $D(0,2)\times D(0,2)$ is a compact subset of $\mathbf{B}\times\mathbf{B}$,the norm continuity of (54) means that the integral in (101) is the limit with respect to the operator norm of a sequence of Riemann sums $1-\epsilon_1,1-\epsilon_2,\dots,1-\epsilon_k,\dots$. Since each $(1-\epsilon_k)$ belongs to $\mathcal{T}^{(1)}$, so does $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}(A+\epsilon)T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$.

(ii) Given $(A+\epsilon)\in\mathcal{A}_{1-\epsilon}$, we will now show that $(A+\epsilon)\in\mathcal{T}^{(1)}$. Since $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\in\mathcal{A}_{1-\epsilon}$and since $\mathcal{A}_{1-\epsilon}$is an algebra, we have $T^j_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}(A+\epsilon)T^k_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\in\mathcal{A}_{1-\epsilon}$for all $j,k\in\mathbf{Z}_+$. Thus it follows from (i) that

$$T^{j+1}_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}(A+\epsilon)T^{k+1}_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\in\mathcal{T}^{(1)}\text{for all integers } j\geq 0 \text{ and } k\geq 0. \tag{103}$$

Let $C^*\left(T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\right)$ be the unital C^*-algebra generated by $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$. Since $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$is self-adjoint, (103) implies that

$$T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}X(A+\epsilon)T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}X\in\mathcal{T}^{(1)}\text{for every } X\in C^*\left(T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\right).$$

We again use the invertibility of $T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$, which is guaranteed by (39). It is elementary that the inverse $T^{-1}_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$, once it exists, must belong to the C^*-algebra $C^*\left(T_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}\right)$. Thus, letting $X=T^{-1}_{\sum_{\widetilde{m}}\Phi_{\widetilde{m}}}$in the above, we obtain $(A+\epsilon)\in\mathcal{T}^{(1)}$. This completes the proof of Corollary (6.3.31).

Corollary (6.3.32)[200]: Let $(n-1)/(n+1)<1-\epsilon$, $\epsilon<1$ be given. If $(A+\epsilon)\in\mathcal{A}_{1-\epsilon}$, then for every separated set Γin $\mathbf{B}$ and every pair of $(z^2-1),(w^2-1)\in\mathbf{B}$ we have

$$\limsup_{\epsilon\to\infty}\sup_{u\in\Gamma}\sum_{\substack{u\in\Gamma\\ \beta(u,u+\epsilon)>1+\epsilon}}\left|\langle(A+\epsilon)k_{\varphi_u(z^2-1)},k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right|\left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} = 0 \tag{104}$$

and

$$\limsup_{\epsilon\to\infty}\sup_{u\in\Gamma}\sum_{\substack{u\in\Gamma\\ \beta(u,u+\epsilon)>1+\epsilon}}\left|\langle k_{\varphi_u(z^2-1)},(A+\epsilon)k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right|\left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} = 0. \tag{105}$$

Proof. Given such $(1-\epsilon)$and $(A+\epsilon)\in\mathcal{A}_{1-\epsilon}$, by Definition (1.1) we have

$$\limsup_{\epsilon\to\infty}\sup_{x\in\mathbf{B}}\int_{\mathbf{B}\setminus D(x,1+\epsilon)}\left|\langle(A+\epsilon)k_x,k_{\zeta^2-1}\rangle\right|\left(\frac{(1-|\zeta^2-1|^2)}{(1-|x|)^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} d\lambda(\zeta^2-1) = 0 \tag{106}$$

and

$$\limsup_{\epsilon\to\infty}\sup_{x\in\mathbf{B}}\int_{\mathbf{B}\setminus D(x,1+\epsilon)}\left|\langle (A^*+\epsilon)k_x,k_{\zeta^2-1}\rangle\right|\left(\frac{(1-|\zeta^2-1|^2)}{(1-|x|)^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}d\lambda(\zeta^2-1)$$
$$=0. \qquad (107)$$

Let $\Gamma,(z^2-1)$and (w^2-1)also be given as in the lemma. Denote $G=D(0,1)$ and $G_{w^2-1}=\varphi_{w^2-1}(G)$. Then it is easy to see that $G_{w^2-1}\subset D\big(0,1+\beta(w^2-1,0)\big)$. For $h\in L_a^2(\mathbf{B},d(u+\epsilon))$and $(u+\epsilon)\in\Gamma$, we have

$$h\big(\varphi_{w^2-1}(w^2-1)\big)=(h\circ\varphi_{u+\epsilon}\circ\varphi_{w^2-1})(0)=\frac{1}{\lambda(G)}\int_G h\circ\varphi_{u+\epsilon}\circ\varphi_{w^2-1}d\lambda$$

$$=\frac{1}{\lambda(G)}\int_{(\varphi_{u+\epsilon}\circ\varphi_{w^2-1})(G)}hd\lambda=\frac{1}{\lambda(G)}\int_{\varphi_{u+\epsilon}(G_{w^2-1})}hd\lambda$$

$$=\frac{1}{\lambda(G)}\int_{\varphi_{u+\epsilon}(G_{w^2-1})}\frac{\langle h,k_{\zeta^2-1}\rangle}{(1-|\zeta|^2)^{(n+1)/2}}d\lambda(\zeta^2-1)\,.$$

Thus

$$\langle h,k_{\varphi_{u+\epsilon}(w^2-1)}\rangle$$
$$=\frac{1}{\lambda(G)}\int_{\varphi_{u+\epsilon}(G_{w^2-1})}\langle h,k_{\zeta^2-1}\rangle\left(\frac{1-|\varphi_{u+\epsilon}(w^2-1)|^2}{1-|\zeta^2-1|^2}\right)^{(n+1)/2}d\lambda(\zeta^2-1)\,.$$

If $(\zeta^2-1)\in\varphi_{u+\epsilon}(G_{w^2-1})$, then $\zeta^2-1=\varphi_{u+\epsilon}(\xi^2-1)$ for some $(\xi^2-1)\in G_{w^2-1}\subset D\big(0,1+\beta(w^2-1,0)\big)$, which means

$$1-|\zeta^2-1|^2=1-|\varphi_{u+\epsilon}(\xi^2-1)|^2=\frac{(1-|u+\epsilon|^2)(1-|\zeta^2-1|^2)}{|1-\langle\xi^2-1,u+\epsilon\rangle|^2}$$
$$\geq\frac{1}{4}(1-|\zeta^2-1|^2)(1-|u+\epsilon|^2).$$

On the other hand,

$$1-|\varphi_{u+\epsilon}(w^2-1)|^2=\frac{(1-|u+\epsilon|^2)(1-|w^2-1|^2)}{|1-\langle w^2-1,u+\epsilon\rangle|^2}\leq\frac{2}{1-|w^2-1|}(1-|u+\epsilon|^2).$$

Hence there is a $0\leq\epsilon_1<\infty$which depends only on $n$and (w^2-1)such that

$$\left|\langle h,k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right|(1-|u+\epsilon|^2)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\leq\frac{1+\epsilon_1}{\lambda(G)}\int_{k_{\varphi_{u+\epsilon}(G_{w^2-1})}}\langle h,k_{\zeta^2-1}\rangle(1-|\zeta^2-1|^2)^{\frac{(1-\epsilon)(n+1)}{2}}d\lambda(\zeta^2-1)$$

for all $h\in L_a^2(\mathbf{B},d(u+\epsilon))$and $(u+\epsilon)\in\Gamma$. Applying this inequality to the case where $h=(A+\epsilon)k_{\varphi_u(z^2-1)},u\in\Gamma$, we have

$$\left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\leq \frac{1+\epsilon_1}{\lambda(G)} \int_{\varphi_{u+\epsilon}\left(G_{w^2-1}\right)} \langle (A$$

$$+\epsilon)k_{\varphi_u(z^2-1)}, k_{\zeta^2-1}\rangle \left(\frac{1-|\zeta^2-1|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} d\lambda(\zeta^2-1)\,,$$

$(u+\epsilon) \in \Gamma$. Since

$$1-|\varphi_u(z^2-1)|^2 = \frac{(1-|u|^2)(1-|z^2-1|^2)}{|1-\langle z^2-1, u\rangle|^2} \leq \frac{2}{1-|z^2-1|}(1-|u|^2),$$

there is a $0 \leq \epsilon_2 < \infty$which depends only on $n$and z^2-1such that

$$\left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\leq \frac{(1+\epsilon_1)(1+\epsilon_2)}{\lambda(G)} \int_{\varphi_{u+\epsilon}\left(G_{w^2-1}\right)} \langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\zeta^2-1}\rangle$$

$$\times \left(\frac{1-|\zeta^2-1|^2}{1-|\varphi_u(z^2-1)|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} d\lambda(\zeta^2-1), \tag{108}$$

$u, (u+\epsilon) \in \Gamma$. Set $L = 1+\beta(w^2-1,0)+\beta(z^2-1,0)$ and consider any $\epsilon > 0$. If $u, (u+\epsilon) \in \Gamma$ are such that $\beta(u, u+\epsilon) > L+\epsilon$, then for every $(\zeta^2-1) \in \varphi_{u+\epsilon}(G_w) \subset \varphi_{u+\epsilon}\left(D\left(0,1+\beta(w^2-1,0)\right)\right)$ we have

$$\beta(\varphi_u(z^2-1), \zeta^2-1) \geq \beta(u, u+\epsilon) - \beta\left(u, \varphi_u(z^2-1)\right)$$

$$> L+\epsilon-1-\beta(w^2-1,0)-\beta(z^2-1,0) = \epsilon. \tag{109}$$

Thus the combination of (108) and (109) gives us

$$\sum_{\substack{(u+\epsilon)\in\Gamma \\ \beta(u,u+\epsilon)>L+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, (A+\epsilon)k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\leq \frac{(1+\epsilon_1)(1+\epsilon_2)}{\lambda(G)} \int_{\beta(\varphi_u(z^2-1),\zeta^2-1)>\epsilon} \sum_{(u+\epsilon)\in\Gamma} \chi_{\varphi_{u+\epsilon}\left(G_{w^2-1}\right)}(\zeta^2-1)$$

$$\times \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\zeta^2-1}\rangle\right|$$

$$\times \left(\frac{1-|\zeta^2-1|^2}{1-|\varphi_u(z^2-1)|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} d\lambda(\zeta^2-1), \tag{110}$$

$u \in \Gamma$. By the Möbius invariance of $\beta$and the fact that $G_{w^2-1} \subset D\left(0,1+\beta(w^2-1,0)\right)$, we have $\varphi_{u+\epsilon}(G_{w^2-1}) \subset D\left(u+\epsilon, 1+\beta(w^2-1,0)\right)$. Since Γ is separated, it follows from Corollary (6.3.25) (a) that there is an $N \in \mathbf{N}$which depends only on Γ and (w^2-1)such that the inequality

$$\sum_{(u+\epsilon)\in\Gamma} \chi_{\varphi_{u+\epsilon}(G_{w^2-1})} \leq N$$

holds on **B**. Substituting this in (110), we conclude that

$$\sum_{\substack{(u+\epsilon)\in\Gamma \\ \beta(u,u+\epsilon)>L+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u+\epsilon|^2}{1-|u|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\leq \frac{(1+\epsilon_1)(1+\epsilon_2)N}{\lambda(G)} \int_{\beta(\varphi_u(z^2-1),\zeta^2-1)>\epsilon} \langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\zeta^2-1}\rangle$$

$$\times \left(\frac{1-|\zeta^2-1|^2}{1-|\varphi_u(z^2-1)|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} d\lambda(\zeta^2-1)$$

for every $u \in \Gamma$. By this inequality, (104) follows from (106). Since

$$\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle = \langle (A^*+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle,$$

(105) follows from (107) by the same argument. This completes the proof.

Corollary (6.3.33)[200]: Let $(n-1)/(n+1) < 1-\epsilon, \epsilon < 1$. If $(A+\epsilon) \in \mathcal{A}_{1-\epsilon}$, then for every pair of $(z^2-1), (w^2-1) \in$ **B** we have $E_{w^2-1}(A+\epsilon)E_{z^2-1} \in \mathcal{D}$.

Proof. Let $(n-1)/(n+1) < 1-\epsilon, \epsilon < 1$. For $(A+\epsilon) \in \mathcal{A}_{1-\epsilon}$ and $(z^2-1), (w^2-1) \in$ **B**, we have

$$E_{w^2-1}(A+\epsilon)E_{z^2-1}$$

$$= \sum_{u,(u+\epsilon)\in\mathcal{L}} k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_{u+\epsilon}(w^2-1)} \cdot (A+\epsilon) \cdot k_{\varphi_u(z^2-1)} \otimes k_{\varphi_u(z^2-1)}$$

$$= \sum_{u,(u+\epsilon)\in\mathcal{L}} \langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_u(z^2-1)}.$$

Thus for any $\epsilon > 0$, we can write $E_{w^2-1}(A+\epsilon)E_{z^2-1} = V_{1+\epsilon} + W_{1+\epsilon}$, where

$$V_{1+\epsilon} = \sum_{\substack{u,(u+\epsilon)\in\mathcal{L} \\ \beta(u,u+\epsilon)\leq 1+\epsilon}} \langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_u(z^2-1)}$$

and

$$W_{1+\epsilon} = \sum_{\substack{u,(u+\epsilon)\in\mathcal{L} \\ \beta(u,u+\epsilon)>1+\epsilon}} \langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_u(z^2-1)}.$$

Obviously, the proposition will follow if we can prove the following two statements:

(i) $\lim_{\epsilon\to\infty} \|W_{1+\epsilon}\| = 0$.

(ii) $V_{1+\epsilon} \in \text{span}(\mathcal{D}_0)$ for every $\epsilon > 0$.

To prove (i), note that by (100) and Corollary (6.3.29), there are constants $(1+\epsilon_1), (1+\epsilon_2)$ such that

$$\sum_{u\in\mathcal{L}}\left|\langle h,k_{\varphi_u(z^2-1)}\rangle\right|^2 \le (1+\epsilon_1)\|h\|^2 \text{and} \sum_{(u+\epsilon)\in\mathcal{L}}\left|\langle h,k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right|^2 \le (1+\epsilon_2)\|h\|^2 \quad (111)$$

for every $h \in L_a^2(\mathbf{B}, d(u+\epsilon))$. Given $h, (h+\epsilon) \in L_a^2(\mathbf{B}, d(u+\epsilon))$, we have

$$|\langle W_{1+\epsilon}h, h+\epsilon\rangle| \le \sum_{\substack{u,(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \mathscr{s}_u \mathscr{t}_{u+\epsilon}, \quad (112)$$

where

$$\mathscr{s}_u = \left|\langle h, k_{\varphi_u(z^2-1)}\rangle\right| \text{and} \mathscr{t}_{u+\epsilon} = \left|\langle k_{\varphi_{u+\epsilon}(w^2-1)}, h+\epsilon\rangle\right|.$$

We apply the Schur test one more time. Indeed for each $u \in \mathcal{L}$, let us write

$$\mathscr{y}_u = \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \mathscr{t}_{u+\epsilon} \quad (113)$$

Then for each $u \in \mathcal{L}$, the Cauchy-Schwarz inequality gives us

$$\mathscr{y}_u^2 \le \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| (1-|u+\epsilon|^2)^{\frac{(1-\epsilon)(n+1)}{2}}$$

$$\times \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \frac{\mathscr{t}_{u+\epsilon}^2}{(1-|u+\epsilon|^2)^{\frac{(1-\epsilon)(n+1)}{2}}}$$

$$\le H(1+\epsilon) \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u|^2}{1-|u+\epsilon|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} \mathscr{t}_{u+\epsilon}^2,$$

where

$$H(1+\epsilon) = \sup_{(\xi^2-1)\in\mathcal{L}} \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(\xi^2-1,u+\epsilon)>1+\epsilon}} \left|\langle (A+\epsilon)k_{\varphi_{\xi^2-1}(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\right| \left(\frac{1-|u|^2}{1-|\xi^2-1|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}.$$

Therefore

$$\sum_{u\in\mathcal{L}} \mathscr{y}_u^2 \le H(1$$

$$+\epsilon)\sum_{u\in\mathcal{L}} \sum_{\substack{(u+\epsilon)\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \Big|\langle (A$$

$$+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\Big| \left(\frac{1-|u|^2}{1-|u+\epsilon|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} \mathscr{t}_{u+\epsilon}^2$$

$$= H(1+\epsilon)\sum_{(u+\epsilon)\in\mathcal{L}} \mathscr{t}_{u+\epsilon}^2 \sum_{\substack{u\in\mathcal{L}\\ \beta(u,u+\epsilon)>1+\epsilon}} \Big|\langle (A+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)}\rangle\Big|$$

$$\times\left(\frac{1-|u|^2}{1-|u+\epsilon|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}} \le H(1+\epsilon)G(1+\epsilon)\sum_{(u+\epsilon)\in\mathcal{L}} \mathscr{t}_{u+\epsilon}^2,$$

where

$$G(1+\epsilon) = \sup_{(\xi^2-1)\in\mathcal{L}} \sum_{\substack{u\in\mathcal{L}\\ \beta(u,\xi^2-1)>1+\epsilon}} \Big|\langle (A$$

$$+\epsilon)k_{\varphi_u(z^2-1)}, k_{\varphi_{\varphi_{\xi^2-1}}(w^2-1)}\rangle\Big| \left(\frac{1-|u|^2}{1-|\xi^2-1|^2}\right)^{\frac{(1-\epsilon)(n+1)}{2}}.$$

By (112) and (113), we now have

$$|\langle W_{1+\epsilon}h, h+\epsilon\rangle| \le \sum_{u\in\mathcal{L}} \mathscr{s}_u \mathscr{y}_u \le \left(\sum_{u\in\mathcal{L}} \mathscr{s}_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} \mathscr{y}_u^2\right)^{1/2}$$

$$\le \{H(1+\epsilon)G(1+\epsilon)\}^{1/2} \left(\sum_{u\in\mathcal{L}} \mathscr{s}_u^2\right)^{1/2} \left(\sum_{u\in\mathcal{L}} \mathscr{t}_u^2\right)^{1/2}.$$

Combining this with (111), we find that

$$|\langle W_{1+\epsilon}h, h+\epsilon\rangle| \le \{(1+\epsilon_1)(1+\epsilon_2)H(1+\epsilon)G(1+\epsilon)\}^{1/2}\|h\|\|h+\epsilon\|.$$

Since $h,(h+\epsilon)\in L_a^2(\mathbf{B},d(u+\epsilon))$are arbitrary, this means

$$W_{1+\epsilon} \le \{(1+\epsilon_1)(1+\epsilon_2)H(1+\epsilon)G(1+\epsilon)\}^{1/2}.$$

Applying Corollary (6.3.32), we have $\lim_{\epsilon\to\infty} H(1+\epsilon)=0$ and $\lim_{\epsilon\to\infty} G(1+\epsilon)=0$. Therefore $\lim_{\epsilon\to\infty}\|W_{1+\epsilon}\|=0$ as promised.

We now turn to the proof of (ii). First of all, given an $\epsilon>0$, for each $(u+\epsilon)\in\mathcal{L}$we define

$$F_{u+\epsilon} = \{u\in\mathcal{L}: \beta(u,u+\epsilon)\le 1+\epsilon\}.$$

By Corollary (6.3.25) (a), there is an $N\in\mathbf{N}$such that

$$\mathrm{card}(F_{u+\epsilon}) \le N$$

for every $(u+\epsilon)\in\mathcal{L}$. Also, by Corollary (6.3.25) (b), for the given $(w^2-1)\in\mathbf{B}$, there is a partition

$$\mathcal{L} = L_1 \cup \ldots \cup L_m$$
such that for each $i \in \{1, \ldots, m\}$, if $u+\epsilon, u+\epsilon' \in L_i$and $u+\epsilon \neq u+\epsilon'$, then $\beta\big(\varphi_{u+\epsilon}(w^2-1), \varphi_{u+\epsilon'}(w^2-1)\big) \geq 1$. That is, for each $i \in \{1, \ldots, m\}$, the set
$$K_i = \{\varphi_{u+\epsilon}(w^2-1) : (u+\epsilon) \in L_i\}$$
is separated. We have $V_{1+\epsilon} = X_1 + \cdots + X_m$, where

$$X_i = \sum_{\varphi_{u+\epsilon}(w^2-1) \in K_i} \sum_{u \in F_{u+\epsilon}} \langle (A+\epsilon) k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)} \rangle k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_u(z^2-1)},$$
$i \in \{1, \ldots, m\}$. To prove (ii), it suffices to show that $X_i \in \operatorname{span}(\mathcal{D}_0)$ of every $i \in \{1, \ldots, m\}$. For this purpose we further decompose each K_i. Indeed for each pair of $i \in \{1, \ldots, m\}$and $j \in \{1, \ldots, N\}$, we define
$L_{i,j} = \{(u+\epsilon) \in L_i : \operatorname{card}(F_{u+\epsilon}) = j\}$ *and* $K_{i,j} = \{\varphi_{u+\epsilon}(w^2-1) : (u+\epsilon) \in L_{i,j}\}$.
Then $X_i = X_{i,1} + \cdots + X_{i,N}$, where

$$X_{i,j} = \sum_{\varphi_{u+\epsilon}(w^2-1) \in K_{i,j}} \sum_{u \in F_{u+\epsilon}} \langle (A+\epsilon) k_{\varphi_u(z^2-1)}, k_{\varphi_{u+\epsilon}(w^2-1)} \rangle k_{\varphi_{u+\epsilon}(w^2-1)} \otimes k_{\varphi_u(z^2-1)},$$
$i \in \{1, \ldots, m\}$and $j \in \{1, \ldots, N\}$. Thus it suffices to show that $X_{i,j} \in \operatorname{span}(\mathcal{D}_0)$ for every such pair of i, j. But it is obvious that given a pair of such i, j, we can define maps
$$\gamma_{i,j}^{(1)}, \ldots, \gamma_{i,j}^{(j)} : K_{i,j} \to \mathbf{B}$$
such that
$$\{\varphi_u(z^2-1) : u \in F_{u+\epsilon}\} = \left\{\gamma_{i,j}^{(1)}\big(\varphi_{u+\epsilon}(w^2-1)\big), \ldots, \gamma_{i,j}^{(j)}\big(\varphi_{u+\epsilon}(w^2-1)\big)\right\}$$
for every $(u+\epsilon) \in L_{i,j}$. Thus $X_{i,j} = X_{i,j}^{(1)} + \cdots + X_{i,j}^{(j)}$, where for each $(u+\epsilon) \in \{1, \ldots, j\}$we have

$$X_{i,j}^{(u+\epsilon)} = \sum_{u \in F_{u+\epsilon}} \langle (A+\epsilon) k_{\gamma_{i,j}^{(u+\epsilon)}(\xi^2-1)}, k_{\xi^2-1} \rangle\, k_{\xi^2-1} \otimes k_{\gamma_{i,j}^{(u+\epsilon)}(\xi^2-1)}.$$
Hence the proof will be complete if we can show that $\gamma_{i,j}^{(u+\epsilon)} \in (\mathcal{D}_0)$for every triple of indices $i \in \{1, \ldots, m\}, j \in \{1, \ldots, N\}$and $(u+\epsilon) \in \{1, \ldots, j\}$.

By the above definitions, for every such triple of $i, j, u+\epsilon$, if $(\zeta^2-1) \in K_{i,j}$, then there exist $(u+\epsilon) \in L_{i,j}$and $u \in F_{u+\epsilon}$such that $\xi^2 - 1 = \varphi_{u+\epsilon}(w^2-1)$ and $\gamma_{i,j}^{(u+\epsilon)}(\xi^2-1) = \varphi_u(z^2-1)$. Therefore
$$\begin{aligned}\beta\left(\xi^2-1, \gamma_{i,j}^{(u+\epsilon)}(\xi^2-1)\right) &= \beta\big(\varphi_{u+\epsilon}(w^2-1), \varphi_u(z^2-1)\big) \\ &\leq \beta(\varphi_{u+\epsilon}(w^2-1), u+\epsilon) + \beta(u+\epsilon, u) + \beta\big(u, \varphi_u(z^2-1)\big) \\ &\leq \beta(w^2-1, 0) + 1 + \epsilon + \beta(0, z^2-1).\end{aligned}$$
This shows that the map $\gamma_{i,j}^{(u+\epsilon)} : K_{i,j} \to \mathbf{B}$satisfies condition (57). By Definition (6.3.14) (a), we have $X_{i,j}^{(u+\epsilon)} \in (\mathcal{D}_0)$. This completes the proof of Corollary (6.3.33).

Corollary (6.3.34)[200]: Suppose that $\Gamma$is a separated set in $\mathbf{B}$. Furthermore, suppose that $\{c_u : u \in \Gamma\}$are complex numbers for which (98) holds. Then for each $(z^2-1) \in \mathbf{B}$, the operator Y_{z^2-1}defined by (99) belongs to $\mathcal{T}^{(1)}$.

Proof. (1) Let us first show that $Y_0 \in \mathcal{T}^{(1)}$. Since Γ is separated, there is $\delta > 0$ such that $\beta(u, u+\epsilon) \geq \delta$for all $\epsilon \neq 0$in Γ. That is, if $u, (u+\epsilon) \in \Gamma$ and $\epsilon \neq 0$, then $D(u, \delta/2) \cap D(u, \delta/2) = \emptyset$. For each $0 < \epsilon < \delta/2$, define the operator

$$A_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \int\limits_{D(0,\epsilon)} Y_{z^2-1} d\lambda(z^2-1).$$

By the norm continuity of the map $z^2 - 1 \mapsto Y_{z^2-1}$provided by Corollary (6.3.30), we have

$$\lim_{\epsilon \downarrow 0} \|Y_0 - Y_\epsilon\| = 0.$$

Thus to prove the membership $Y_0 \in \mathcal{T}^{(1)}$, it suffices to show that each $A_\epsilon$is a Toeplitz operator with a bounded symbol. Indeed by the Möbius invariance of $\beta$and $d\lambda$, we have

$$A_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \int\limits_{D(0,\epsilon)} k_{\varphi_u(z^2-1)} \otimes k_{\varphi_u(z^2-1)} d\lambda(z^2-1)$$

$$= \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \int\limits_{D(u,\epsilon)} k_{w^2-1} \otimes k_{w^2-1} d\lambda(w^2-1)$$

$$= \int f_\epsilon(w^2-1) k_{w^2-1} \otimes k_{w^2-1} d\lambda(w^2-1),$$

where

$$f_\epsilon = \frac{1}{\lambda\big(D(0,\epsilon)\big)} \sum_{u\in\Gamma} c_u \chi_{D(u,\epsilon)}.$$

Since, $0 < \epsilon < \delta/2$, we have $D(u,\epsilon) \cap D(u+\epsilon, \epsilon) = \emptyset$for $\epsilon \neq 0$in Γ. Hence $f_\epsilon \in L^\infty(\mathbf{B}, d(u+\epsilon))$. By (34), we have $A_\epsilon = T_{f_\epsilon}$. This proves the membership $Y_0 \in \mathcal{T}^{(1)}$.

(ii) Now consider an arbitrary $(z^2-1) \in \mathbf{B}$. By Corollary (6.3.25) (b), there is a partition $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_{\mathrm{m}}$such that for every $i \in \{1, \ldots, m\}$, the conditions $u, (u+\epsilon) \in \Gamma_i$and $\epsilon \neq 0$imply $\beta\big(\varphi_u(z^2-1), \varphi_{u+\epsilon}(z^2-1)\big) \geq 1$. That is, for each $i \in \{1, \ldots, m\}$, the set

$$G_i = \{\varphi_u(z^2-1) : u \in \Gamma_i\}$$

is separated. Obviusly, $Y_{z^2-1} = Y_{(z^2-1),1} + \cdots + Y_{(z^2-1),m}$, where

$$Y_{(z^2-1),i} = \sum_{\varphi_u(z^2-1)\in G_i} c_u k_{\varphi_u(z^2-1)} \otimes k_{\varphi_u(z^2-1)},$$

$i = 1, \ldots, m$. By (1) we have $Y_{(z^2-1),i} \in \mathcal{T}^{(1)}$for every $i \in \{1, \ldots, m\}$. Hence $Y_{z^2-1} \in \mathcal{T}^{(1)}$.

Corollary (6.3.35)[200]: Let Γ be a separated set in $\mathbf{B}$ and suppose that $\{c_u : u \in \Gamma\}$is a bounded set of complex coefficients. Then for every pair of $(\alpha^2-1) \in \mathbf{Z}_+^n$ and $(z^2-1) \in \mathbf{B}$, we have

$$\sum_{u\in\Gamma} c_u (U_u K_{z^2-1}) \otimes \big(U_u K_{z^2-1;\alpha^2-1}\big) \in \mathcal{T}^{(1)}.$$

Proof. We prove the proposition by an induction on $|\alpha^2-1|$. If $|\alpha^2-1| = 0$, i.e. $\alpha^2 = 1$, then

$$(U_uK_{z^2-1})\otimes\left(U_uK_{z^2-1;0}\right)=(U_uK_{z^2-1})\otimes(U_uK_{z^2-1})$$
$$=\frac{1}{(1-|z^2-1|^2)^{n+1}}k_{\varphi_u(z^2-1)}\otimes k_{\varphi_u(z^2-1)}.$$

Hence the case where $|\alpha^2-1|=0$ follows from Corollary (6.3.34). Suppose that $k\in\mathbf{Z}_+$and that the proposition holds true for every $(\alpha^2-1)\in\mathbf{Z}_+^n$satisfying the condition $|\alpha^2-1|\le k$. Now consider the case where $(\alpha^2-1)\in\mathbf{Z}_+^n$is such that $|\alpha^2-1|=k+1$. Then we can decompose (α^2-1)in the form

$$\alpha^2-1=2a+\epsilon\,,$$

where $|a|=k$and $|a+\epsilon|=1$. That is, there is some $(u+\epsilon)\in\{1,\dots,n\}$such that the $(u+\epsilon)$ –th component of $(a+\epsilon)$is 1 and the other components of $(a+\epsilon)$are all 0. We will also consider $(a+\epsilon)$as a vector in $\mathbf{C}^n$. By the induction hypothesis, we have

$$\sum_{u\in\Gamma}c_u(U_uK_{z^2-1})\otimes\left(U_uK_{(z^2-1);(\alpha^2-1)}\right)\in\mathcal{T}^{(1)}\text{for every}\quad(z^2-1)\in\mathbf{B}.\quad(114)$$

Let $(z^2-1)\in\mathbf{B}$be given. Then there is an $\epsilon=\epsilon(z^2-1)>0$ such that $z^2-1+c\in\mathbf{B}$for every $c\in\mathbf{C}^n$satisfying the condition $|c|\le\epsilon$. For each $(1-2\epsilon)\in[0,\epsilon]$, define the operators

$$A_{1-2\epsilon}=\sum_{u\in\Gamma}c_u\left(U_uK_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right)\otimes\left(U_uK_{z^2-1+(1-2\epsilon)(a+\epsilon);a}\right)$$

and

$$A_{1-2\epsilon}+\epsilon=\sum_{u\in\Gamma}c_u\left(U_uK_{z^2-1+i(1-2\epsilon)(a+\epsilon)}\right)\otimes\left(U_uK_{z^2-1+i(1-2\epsilon)(a+\epsilon);a}\right).$$

Also, we define

$$X=\sum_{u\in\Gamma}c_u\{(n+1+k)(U_uK_{z^2-1})\otimes\left(U_uK_{(z^2-1);(\alpha^2-1)}\right)+(n+1)\left(U_uK_{(z^2-1);(a+\epsilon)}\right)\otimes\left(U_uK_{(z^2-1);a}\right)\}$$

and

$$Y=\sum_{u\in\Gamma}c_u\{(n+1+k)(U_uK_{z^2-1})\otimes\left(U_uK_{(z^2-1);(\alpha^2-1)}\right)-(n+1)\left(U_uK_{(z^2-1);(a+\epsilon)}\right)\otimes\left(U_uK_{(z^2-1);a}\right)\}.$$

We will show that

$$\lim_{1-2\epsilon\downarrow 0}\left\|\frac{1}{1-2\epsilon}(A_{1-2\epsilon}-A_0)-X\right\|=0\qquad(115)$$

and

$$\lim_{1-2\epsilon\downarrow 0}\left\|\frac{1}{i(1-2\epsilon)}(A_{1-2\epsilon}-A_0)-Y\right\|=0.\qquad(116)$$

Before getting to their proofs, let us first see the consequence of these limits. By (114) we have $A_{1-2\epsilon}\in\mathcal{T}^{(1)}$and $(A_{1-2\epsilon}+\epsilon)\in\mathcal{T}^{(1)}$for all $(1-2\epsilon)\in[0,\epsilon]$. Hence it follows from (115) and (116) that $X,Y\in\mathcal{T}^{(1)}$. Thus

$$\sum_{u\in\Gamma} c_u(U_uK_{z^2-1})\otimes\left(U_uK_{(z^2-1);(\alpha^2-1)}\right)=\frac{1}{2(n+1+k)}(X+Y)\in\mathcal{T}^{(1)}\ ,$$

completing the induction on $|\alpha^2-1|$.

Let us now turn to the proof of (69). Note that $(1-2\epsilon)^{-1}(A_{1-2\epsilon}-A_0)=G_{1-2\epsilon}+H_{1-2\epsilon}$, where

$$H_{1-2\epsilon}=\frac{1}{1-2\epsilon}\sum_{u\in\Gamma} c_u\left(U_uK_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right)\otimes\left\{U_u\left(K_{z^2-1+(1-2\epsilon)(a+\epsilon);a}-K_{(z^2-1);a}\right)\right\}$$

and

$$G_{1-2\epsilon}=\frac{1}{1-2\epsilon}\sum_{u\in\Gamma} c_u\left\{U_u\left(K_{z^2-1+(1-2\epsilon)(a+\epsilon)}-K_{z^2-1}\right)\right\}\otimes\left(U_uK_{(z^2-1);a}\right).$$

Similarly, we write $X = V + W$, where

$$V=\sum_{u\in\Gamma} c_u(n+1+k)(U_uK_{z^2-1})\otimes\left(U_uU_{(z^2-1);(\alpha^2-1)}\right)$$

and

$$W=\sum_{u\in\Gamma} c_u(n+1)\left(U_uK_{(z^2-1);(a+\epsilon)}\right)\otimes\left(U_uU_{(z^2-1);a}\right).$$

Since $\|(1-2\epsilon)^{-1}(A_{1-2\epsilon}-A_0)-X\|\le\|H_{1-2\epsilon}-V\|+\|G_{1-2\epsilon}-W\|$, (115) will follow if we can show

$$\lim_{1-2\epsilon\downarrow 0}\|H_{1-2\epsilon}-V\|=0 \qquad (117)$$

and

$$\lim_{1-2\epsilon\downarrow 0}\|G_{1-2\epsilon}-W\|=0. \qquad (118)$$

To prove (117), for $0<1-2\epsilon\le\epsilon$ we write $H_{1-2\epsilon}-V=A_{1-2\epsilon}+\epsilon+T_{1-2\epsilon}$, where

$$A_{1-2\epsilon}+\epsilon=\sum_{u\in\Gamma} c_u\left(U_uK_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right)\otimes\left\{U_u\left((1-2\epsilon)^{-1}\left(K_{z^2-1+(1-2\epsilon)(a+\epsilon);a}-K_{(z^2-1);a}\right)-(n+1+k)k_{(z^2-1);(\alpha^2-1)}\right)\right\}$$

and

$$T_{1-2\epsilon}=(n+1+k)\sum_{u\in\Gamma} c_u\left\{U_u\left(K_{z^2-1+(1-2\epsilon)(a+\epsilon)}-K_{z^2-1}\right)\right\}\otimes\left(U_uK_{(z^2-1);(\alpha^2-1)}\right).$$

Thus the proof of (117) is reduced to the proof of the fact that $\|A_{1-2\epsilon}+\epsilon\|\to 0$ and $\|T_{1-2\epsilon}\to 0\|$as $(1-2\epsilon)$descends to 0. To prove this, we pick an orthonormal set $\{e_u: u\in\Gamma\}$and factor $(A_{1-2\epsilon}+\epsilon)$ in the form $A_{1-2\epsilon}+\epsilon=(A^{(1)}_{1-2\epsilon}+2\epsilon)(A^{(2)*}_{1-2\epsilon}+2\epsilon)$, where

$$A^{(1)}_{1-2\epsilon}+2\epsilon=\sum_{u\in\Gamma} c_u\left(U_uK_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right)\otimes e_u$$

and

$$A_{1-2\epsilon}^{(2)} + 2\epsilon = \sum_{u\in\Gamma}\{U_u((1-2\epsilon)^{-1}(K_{z^2-2\epsilon(a+\epsilon);a} - K_{(z^2-1);a}) - (n+1 + k)k_{(z^2-1);(\alpha^2-1)})\} \otimes e_u.$$

Set $1+\epsilon = \sup_{u\in\Gamma}|c_u|$. Then it follows from Corollary (6.3.29) that

$$\left\|A_{1-2\epsilon}^{(1)} + 2\epsilon\right\| \leq (1+\epsilon)(A+\epsilon)(\Gamma)\left\|K_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right\|_\infty$$

and

$$\left\|A_{1-2\epsilon}^{(2)} + 2\epsilon\right\| \leq (A+\epsilon)(\Gamma) \times \left\|(1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon);a} - K_{(z^2-1);a}) - (n+1 + k)k_{(z^2-1);(\alpha^2-1)}\right\|_\infty.$$

Since $2a+\epsilon = \alpha^2 - 1$and $k = |a|$, by (68) and elementary algebra, we have

$$\lim_{1-2\epsilon\downarrow 0}\left\|(1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon);a} - K_{(z^2-1);a}) - (n+1+k)k_{(z^2-1);(\alpha^2-1)}\right\|_\infty = 0.$$

Also, it is trivial that $\left\|k_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right\|_\infty$remains bounded as $(1-2\epsilon)$descends to 0. Hence

$$\begin{aligned}\|A_{1-2\epsilon}+\epsilon\| &\leq \left\|A_{1-2\epsilon}^{(1)} + 2\epsilon\right\| \left\|A_{1-2\epsilon}^{(2)} + 2\epsilon\right\| \\ &\leq (1+\epsilon)((A+\epsilon)(\Gamma))^2\left\|k_{z^2-1+(1-2\epsilon)(a+\epsilon)}\right\|_\infty \\ &\times \left\|(1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon);a} - K_{(z^2-1);a}) - (n+1 + k)k_{(z^2-1);(\alpha^2-1)}\right\|_\infty \to 0\end{aligned}$$

as $(1-2\epsilon)$descends to 0. For $T_{1-2\epsilon}$, we have the factorization $T_{1-2\epsilon} = T_{1-2\epsilon}^{(1)}T^{(2)*}$, where

$$T_{1-2\epsilon}^{(1)} = (n+1+k)\sum_{u\in\Gamma} c_u\{U_u(K_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1})\} \otimes e_u$$

and

$$T^{(2)} = \sum_{u\in\Gamma}(U_u K_{(z^2-1);(\alpha^2-1)}) \otimes e_u.$$

By Corollary (6.3.29), $\left\|T_{1-2\epsilon}^{(1)}\right\| \leq (n+1+k)(1+\epsilon)(A+\epsilon)(\Gamma)\left\|k_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}\right\|_\infty$, and $T^{(2)}$is a bounded operator. It is obvious that

$$\lim_{1-2\epsilon\downarrow 0}\left\|k_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}\right\|_\infty = 0.$$

Hence $\|T_{1-2\epsilon}\| \leq \left\|T_{1-2\epsilon}^{(1)}\right\| \left\|T^{(2)}\right\| \to 0$ as $(1-2\epsilon)$descends to 0. This completes the proof of (117).

To prove (118), note that

$$G_{1-2\epsilon} - W = \sum_{u\in\Gamma} c_u\{U_u((1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}) - (n + 1)k_{(z^2-1);(a+\epsilon)})\} \otimes (U_u K_{(z^2-1);a}) = Z_{1-2\epsilon}T^{(2)*},$$

where

$$Z_{1-2\epsilon} = \sum_{u\in\Gamma} c_u\{U_u((1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}) - (n+1)k_{(z^2-1);(a+\epsilon)})\} \otimes e_u .$$

Applying Corollary (6.3.29) again, we have

$$\|Z_{1-2\epsilon}\| \le (1+\epsilon)(A+\epsilon)(\Gamma)\left\|(1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}) - (n+1)k_{(z^2-1);(a+\epsilon)}\right\|_\infty.$$

Another easy exercise shows that

$$\lim_{1-2\epsilon\downarrow 0}\left\|(1-2\epsilon)^{-1}(K_{z^2-1+(1-2\epsilon)(a+\epsilon)} - K_{z^2-1}) - (n+1)k_{(z^2-1);(a+\epsilon)}\right\|_\infty = 0.$$

Hence $\|G_{1-2\epsilon} - W\| \le \|Z_{1-2\epsilon}\|\|T^{(2)}\| \to 0$ as $(1-2\epsilon)$descends to 0, proving (118). Thus we have completed the proof of (115).

The proof of (116) uses essentially the same argument as above, and the only additional care that needs to be taken is the following: The rank-one operator $\sum_{\widetilde{m}} f_{\widetilde{m}} \otimes h + \epsilon$is linear with respect to the series $\sum_{\widetilde{m}} f_{\widetilde{m}}$and conjugate linear with respect to $(h+\epsilon)$. Moreover, the inner product $\langle \zeta^2 - 1, z^2 - 1\rangle$on $\mathbf{C}^n$is conjugate linear with respect to $(z^2 - 1)$. These are the properties that determine the + and −signs in each term $c_u\{\dots\}$in the sum that defines the operator Y. This completes the proof of the proposition.

Corollary (6.3.36)[200]: Let Γbe a separated set in $\mathbf{B}$ and let $\{c_u : u\in\Gamma\}$be a bounded set of complex coefficients. Then for every $(w^2-1)\in\mathbf{B}$ we have

$$\sum_{u\in\Gamma} c_u k_u \otimes k_{\varphi_u(w^2-1)} \in \mathcal{T}^{(1)}. \tag{119}$$

Proof. For each $(\alpha^2-1)\in\mathbf{Z}_+^n$, define the monomial function

$$p_{\alpha^2-1}(\zeta^2-1) = (\zeta^2-1)^{\alpha^2-1}$$

on $\mathbf{B}$. Given a $(w^2-1)\in\mathbf{B}$, let us define

$$d_u(w^2-1) = c_u\left(\frac{1-\langle w^2-1, u\rangle}{|1-\langle w^2-1,u\rangle|}\right)^{n+1},$$

$u\in\Gamma$. Note that $K_{0;\alpha^2-1} = p_{\alpha^2-1}$for every $(\alpha^2-1)\in\mathbf{Z}_+^n$. Also, $U_uK_0 = U_1 = k_u$for every $u\in\Gamma$. Thus, applying Corollary (6.3.35) to the case where $z^2 = 1$, we have

$$\sum_{u\in\Gamma} d_u(w^2-1)k_u \otimes (U_u p_{\alpha^2-1}) \in \mathcal{T}^{(1)} \tag{120}$$

for every $(\alpha^2-1)\in\mathbf{Z}_+^n$. Define the function

$$(h+\epsilon)_{w^2-1}(\zeta^2-1) = \langle \zeta^2-1, w^2-1\rangle, \qquad (\zeta^2-1)\in\mathbf{B}.$$

For each $j\in\mathbf{Z}_+$, define the operator

$$A_j = \sum_{u\in\Gamma} d_u(w^2-1)k_u \otimes \left(U_u(h+\epsilon)^j_{w^2-1}\right).$$

Since each $(h+\epsilon)^j_{w^2-1}$is in the linear span of $\{p_{\alpha^2-1} : (\alpha^2-1)\in\mathbf{Z}_+^n\}$, (120) implies that $A_j\in\mathcal{T}^{(1)}$for every $j\in\mathbf{Z}_+$. Let $\{e_u : u\in\Gamma\}$be an orthonormal set. Then we have the factorization $A_j = T(A_j^* + \epsilon)$for each $j\in\mathbf{Z}_+$, where

$$T=\sum_{u\in\Gamma} d_u(w^2-1)k_u \otimes e_u \text{ and} A_j+\epsilon=\sum_{u\in\Gamma}\left(U_u(h+\epsilon)^j_{w^2-1}\right)\otimes e_u .$$

Corollary (6.3.29) tells us that $T$is a bounded operator. Define

$$G=\sum_{u\in\Gamma}(U_uK_{w^2-1})\otimes e_u .$$

It also follows from Corollary (6.3.29) that

$$\left\| G-\sum_{j=0}^{k}\frac{(n+j)!}{n!\,j!}(A_j+\epsilon)\right\| \le (A+\epsilon)(\Gamma)\left\| K_{w^2-1}-\sum_{j=0}^{k}\frac{(n+j)!}{n!\,j!}(h+\epsilon)^j_{w^2-1}\right\|_\infty \quad (121)$$

for every $k\in\mathbf{Z}_+$. By the expansion formula

$$\frac{1}{(1-c)^{n+1}}=\sum_{j=0}^{\infty}\frac{(n+j)!}{n!\,j!}c^j, |c|<1,$$

and the fact that $|w^2-1|<1$, we have

$$\lim_{k\to\infty}\left\| K_{w^2-1}-\sum_{j=0}^{k}\frac{(n+j)!}{n!\,j!}(h+\epsilon)^j_{w^2-1}\right\|_\infty=0.$$

Combining this with (121), we obtain

$$\lim_{k\to\infty}\left\| TG^*-\sum_{j=0}^{k}\frac{(n+j)!}{n!\,j!}A_j\right\|=\lim_{k\to\infty}\left\| TG^*-T\sum_{j=0}^{k}\frac{(n+j)!}{n!\,j!}(A_j^*+\epsilon)\right\|=0.$$

Since each $A_j$belongs to $\mathcal{T}^{(1)}$, we conclude that

$$\sum_{u\in\Gamma} d_u(w^2-1)k_u\otimes(U_uK_{w^2-1})=TG^*\in\mathcal{T}^{(1)}.$$

Since $k_{w^2-1}=(1-|w^2-1|^2)^{(n+1)/2}K_{w^2-1}$, this implies

$$\sum_{u\in\Gamma} d_u(w^2-1)k_u\otimes(U_uK_{w^2-1})\in\mathcal{T}^{(1)}. \quad (122)$$

Recalling the definition of $d_u(w^2-1)$ and (100), we see that (122) implies (119).

Corollary (6.3.37)[200]: We have $\mathcal{D}_0\subset\mathcal{T}^{(1)}$.

Since $\mathcal{T}^{(1)}$is a norm closed linear subspace of $\mathcal{B}(L^2_a(\mathbf{B},d(u+\epsilon)))$, Proposition (6.3.12) follows immediately from Propositions (6.3.16) and (6.3.20).

Proof. Let Γ be a separated set in $\mathbf{B}$, let $\{c_u: u\in\Gamma\}$be a bounded set of coefficients, and let $\gamma:\Gamma\to\mathbf{B}$ be a map satisfying (58). Let $K=\{w^2-1\in\mathbf{B}: \beta(0,w^2-1)\le 1+\epsilon\}$, where $(1+\epsilon)$is the constant that appears in (58). We want to show that the operator

$$T=\sum_{u\in\Gamma} c_uk_u\otimes k_{\gamma(u)}$$

belongs to $\mathcal{T}^{(1)}$. For this purpose, define

$$\psi(u)=\varphi_u(\gamma(u)), u\in\Gamma.$$

Since $\beta(u,\gamma(u)) \le 1+\epsilon$, by the Möbius invariance of $\beta$and the fact $\varphi_u(u) = 0$, we have $\beta(0,\psi(u)) = \beta(u,\gamma(u)) \le 1+\epsilon$for every $u \in \Gamma$. That is, $\psi(u) \in K$for every $u \in \Gamma$. Since $\varphi_u(\psi(u)) = \gamma(u), u \in \Gamma$, by (3.1) we have

$$T = \sum_{u\in\Gamma} d_u k_u \otimes \left(U_u k_{\psi(u)}\right),$$

where $|d_u| = |c_u|$for every $u \in \Gamma$. Let $\{e_u : u \in \Gamma\}$be an orthonormal set. Then we have the factorization $T = A(A^* + \epsilon)$, where

$$A = \sum_{u\in\Gamma} d_u k_u \otimes e_u \text{ and } \qquad A+\epsilon = \sum_{u\in\Gamma} \left(U_u k_{\psi(u)}\right) \otimes e_u .$$

We again use the fact that the map $z^2 - 1 \longmapsto k_{z^2-1}$is $\|\cdot\|_\infty$-continuous. That is,

$$\lim_{w^2-1 \to z^2-1} \|k_{z^2-1} - k_{w^2-1}\|_\infty = 0 \qquad \text{for every} \qquad (z^2-1) \in \mathbf{B}.$$

Let $\epsilon > 0$ be given. Since $K$is compact, there are non-empty open sets $\Omega_1, \dots, \Omega_m$in **B** and $(z_i^2 - 1) \in \Omega_i, i = 1, \dots, m$, such that

$$\Omega_1 \cup \dots \cup \Omega_m \supset K \tag{123}$$

and

$$\left\|k_{z_i^2-1} - k_{w^2-1}\right\|_\infty < \epsilon \qquad \text{wherever} \quad (w^2-1) \in \Omega_i,$$

$i = 1, \dots, m$. From the open cover (123) we obtain a partition

$$K = E_1 \cup \dots \cup E_m$$

such that $E_i \subset \Omega_i$for every $i \in \{1, \dots, m\}$. We now define

$$\Gamma_i = \{u \in \Gamma : \psi(u) \in E_i\},$$

$i = 1, \dots, m$. Then $\left\|k_{z_i^2-1} - k_{\psi(u)}\right\|_\infty < \epsilon$ if $u \in \Gamma_i$. For every $i \in \{1, \dots, m\}$, we also define

$$A_i + \epsilon = \sum_{u\in\Gamma_i} \left(U_u k_{z_i^2-1}\right) \otimes e_u .$$

For each $i \in \{1, \dots, m\}$we have

$$A(A_i^* + \epsilon) = \sum_{u\in\Gamma_i} d_u k_u \otimes \left(U_u k_{z_i^2-1}\right) = \sum_{u\in\Gamma_i} d_{u,i} k_u \otimes k_{\varphi_u(z_i^2-1)},$$

where $|d_{u,i}| = |d_u|$for $u \in \Gamma_i$. Thus it follows from Corollary (6.3.36) that

$$\{A(A_1^* + \epsilon), \dots, A(A_m^* + \epsilon)\} \subset \mathcal{T}^{(1)}. \tag{124}$$

On the other hand, we have

$$A + \epsilon - ((A_1+\epsilon) + \cdots + (A_m+\epsilon)) = \sum_{i=1}^{m} \sum_{u\in\Gamma_i} \left\{U_u \left(k_{\psi(u)} - k_{z_i^2-1}\right)\right\} \otimes e_u .$$

Since the sets $\Gamma_1, \dots, \Gamma_m$form a partition of Γ, i.e., $\Gamma_i \cap \Gamma_j = \emptyset$whenever $i \ne j$, Corollary (6.3.29) tells us that

$$\|A + \epsilon - ((A_1+\epsilon) + \cdots + (A_m+\epsilon))\| \le (A+\epsilon)(\Gamma) \max_{1\le i\le m} \sup_{u\in\Gamma_i} \left\|k_{\psi(u)} - k_{z_i^2-1}\right\|_\infty$$
$$\le (A+\epsilon)(\Gamma)\epsilon.$$

Corollary (6.3.29) also tells us that $A$is a bounded operator. Hence

$$\|T-(A(A_1^*+\epsilon)+\cdots+A(A_m^*+\epsilon))\| = \|A(A^*+\epsilon)-(A(A_1^*+\epsilon)+\cdots+A(A_m^*+\epsilon))\|$$
$$\le \|A\|\left\|(A^*+\epsilon)-\left((A_1^*+\epsilon)+\cdots+(A_m^*+\epsilon)\right)\right\|$$
$$= \|A\|\left\|(A+\epsilon)-\left((A_1+\epsilon)+\cdots+(A_m+\epsilon)\right)\right\| \le \|A\|(A+\epsilon)(\Gamma)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, combining this inequality with (2.6), we conclude that $T \in \mathcal{T}^{(1)}$. This completes the proof of Corollary (6.3.37).

Corollary (6.3.38)[200]: There is a constant $0 \le \epsilon_{4.4} < \infty$ such that the following estimate holds: Let $e_u : u \in \mathbf{Z}^{2n}$ be any orthonormal set and let $h_u \in \mathcal{H}_*, u \in \mathbf{Z}^{2n}$, be functions satisfying the condition $\sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_* < \infty$. Then

$$\left\|\sum_{u\in\mathbf{Z}^{2n}} (U_u h_u) \otimes e_u\right\| \le (1+\epsilon_{4.4}) \sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_* .$$

Proof. Let us first estimate $|\langle U_u h_u, U_{u+\epsilon} h_{u+\epsilon}\rangle|$. By (81), for $u \in \mathbf{Z}^{2n}$ we have

$$|\langle U_u h_u, U_{u+\epsilon} h_{u+\epsilon}\rangle| = \int h_u(u-\zeta^2+1)\overline{h_{u+\epsilon}(u+\epsilon-\zeta^2+1)}k_u(\zeta^2-1)\overline{k_{u+\epsilon}(\zeta^2-1)}e^{-|\zeta^2-1|^2}\, dV(\zeta^2-1). \tag{125}$$

Moreover,

$$\left|k_u(\zeta^2-1)\overline{k_{u+\epsilon}(\zeta^2-1)}\right|e^{-|\zeta^2-1|^2} = e^{-(1/2)\left(|u-\zeta^2-1|^2+|u+\epsilon-\zeta^2-1|^2\right)}, \tag{126}$$

$(\zeta^2-1) \in \mathbf{C}^n$. Observe that

$$(|u-\zeta^2-1|^2+|u+\epsilon-\zeta^2+1|^2 \ge \frac{1}{2}(|u-\zeta^2-1|^2+|u+\epsilon-\zeta^2+1|^2 \ge \frac{1}{2}|\epsilon|^2.$$

Thus, splitting the $1/2$ in (126) as $(1/4)+(1/4)$, we find that

$$\left|k_u(\zeta^2-1)\overline{k_{u+\epsilon}(\zeta^2-1)}\right|e^{-|\zeta^2-1|^2} \le e^{-(1/8)|\epsilon|^2}e^{-(1/4)|u-\zeta^2-1|^2}e^{-(1/4)|u+\epsilon-\zeta^2-1|^2}.$$

Combining this with (125) and applying the Cauchy-Schwarz inequality, we obtain

$$|\langle U_u h_u, U_{u+\epsilon} h_{u+\epsilon}\rangle| \le e^{-\left(\frac{1}{8}\right)|\epsilon|^2}\|h_u\|_*\|h_{u+\epsilon}\|_* \le e^{-\left(\frac{1}{8}\right)|\epsilon|^2}H_*^2, \tag{127}$$

where

$$H_* = \sup_{u\in\mathbf{Z}^{2n}} \|h_u\|_* .$$

Write

$$A = \sum_{u\in\mathbf{Z}^{2n}} (U_u h_u) \otimes e_u$$

and consider any vector $x = \sum_{u\in\mathbf{Z}^{2n}} x_u e_u$. By (127), we have

$$\|Ax\|^2 \le \sum_{u,(u+\epsilon)\in\mathbf{Z}^{2n}} |\langle U_{u+\epsilon} h_{u+\epsilon}, U_u h_u\rangle||x_u||x_{u+\epsilon}| \le H_*^2 \sum_{u,(u+\epsilon)\in\mathbf{Z}^{2n}} e^{-(1/8)|\epsilon|^2}|x_u||x_{u+\epsilon}|.$$

Applying the Schur test to the right-hand side, we find that

$$\|Ax\|^2 \le (1+\epsilon)H_*^2 \sum_{u\in\mathbf{Z}^{2n}} |x_u|^2 = (1+\epsilon)H_*^2\|x\|^2,$$

where $1+\epsilon = \sum_{(z^2-1)\in\mathbf{Z}^{2n}} e^{-(1/8)|z^2-1|^2}$, which is finite. Since the vector x is arbitrary, we conclude that $\|A\| \le (1+\epsilon)^{1/2}H_*$. Thus the lemma holds for the constant $1+\epsilon_{4.4} = (1+\epsilon)^{\frac{1}{2}}$.

Corollary (6.3.39)[200]: Let Γ in $\mathbf{B} \subset \boldsymbol{Z}^{2z}$, $e_u : u \in \boldsymbol{Z}^{2n}$ be any orthonormalset and h_u be bounded function in $\mathcal{H}_*$ satisfying $\sup_{u \in \boldsymbol{Z}^{2n}} \|h_u\|_* < \infty$. Then

(i) $(A+\epsilon)(\Gamma) = 1 + \epsilon_{4.4}$ for a constant$0 \leq \epsilon_{4.4} \leq \infty$

(ii)$\|A+\epsilon\| \leq (1+\epsilon)^{\frac{1}{2}}$ for $\|u\| = 1$ where $u \in \Gamma$.

Proof: (i) From the two inequalities of Lemmas (6.3.29) and (6.3.38) we have that

$$(A+\epsilon)(\Gamma) = 1 + \epsilon_{4.4}.$$

(ii) Taking the supremum over all values of Γ s.t. $\|u\| = 1$, we have, by Corollary (6.3.38), that $\|A+\epsilon\| \leq (1+\epsilon)^{\frac{1}{2}}$.

In the proof of the Fock-space analogue of Corollary (6.3.31), the $\|\cdot\|_\infty$-continuities of the previous are replaced by the corresponding $\|\cdot\|_*$-continuities. For example, for the normalized reproducing kernel of the Fock space one easily verifies that

$$\lim_{w^2 \to z^2} \|k_{z^2-1} - k_{w^2-1}\|_* = 0$$

for every $(z^2-1) \in \mathbf{C}^n$. Thus, using Corollary (6.3.38) in place of Corollary (6.3.29), the analogue of Corollary (6.3.31) on the Fock space can be obtained by following the argument in the previously.

List of Symbols

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