Chapter 1

Weak Solutions and Sobolev Spaces

Section (1.1): Introduction

The finite element method is a computational technique for obtaining approximate solutions to the partial differential equations that arise in scientific and engineering applications. Rather than approximating the partial differential equation directly as with, e.g., finite difference methods, the finite element method utilizes a variational problem that involves an integral of the differential equation over the problem domain. This domain is divided into a number of subdomains called finite elements and the solution of the partial differential equation is approximated by a simpler polynomial function on each element. These polynomials have to be pieced together so that the approximate solution has an appropriate degree of smoothness over the entire domain. Once this has been done, the variational integral is evaluated as a sum of contributions from each finite element .The result is an algebraic system for the approximate solution having a finite size rather than the original infinite-dimensional differential equation. Thus, like finite difference methods, the finite element process has discretized the partial differential equation but, unlike finite difference method, the approximate solution is known throughout the domain as a piecewise polynomial function and not just at a set of points. Logan [9] attributes the discovery of the finite element method to Hrennik of [10] and McHenry [8] who decomposed a two-dimensional problem domain into an assembly of one-dimensional bars and beams.Courant [3] used a variational formulation to describe a partial differential equation with a piecewise linear polynomial approximation of the solution relative to a decomposition of the problem domain into triangular elements to solve equilibrium and vibration problems.

Section (1.2): Weak Solution and Sobolev Spaces

As a first simple example let us consider the two-point boundary value problem

$$-u''(x) + b(x)\,u'(x) + c(x)(u(x) = f(x) \;\; in \quad \Omega := (0,1)\,; \tag{1.1}$$

$$u(0) = u(1) = 0 \tag{1.2}$$

Let b, c and f be given continuous functions. Assume that a classical solution exists, i.e., a twice continuously differentiable function u that satisfies (1.1) and (1.2). Then for an arbitrary continuous function v we have

$$\int_{\Omega} (-u" + bu' + cu)v\, dx = \int_{\Omega} fv\, dx \tag{1.3}$$

The reverse implication is also valid: if a function $u \in C^2(\bar{\Omega})$ satisfies equation (1.3) for all $v \in C(\bar{\Omega})$, then u is a classical solution of the differential equation (1.1).

If $v \in C^1(\bar{\Omega})$, then we can integrate by parts (1.3) and obtain

$$-u'v \Big|_{x=0}^{1} + \int_{\Omega} u'v'\, dx + \int_{\Omega} (bu' + cu)v\, dx = \int_{\Omega} fv\, dx$$

Under the additional condition $v(0) = v(1) = 0$ this is equivalent to

$$\int_{\Omega} u'v' dx + \int_{\Omega} (bu' + cu)v\, dx = \int_{\Omega} fv\, dx \tag{1.4}$$

Unlike (1.1) or (1.3), equation (1.4) still makes sense if we know only that It turns out that Soblev spaces, which generalize L_p spaces to spaces of functions whose generalized derivatives also lie in L_p, are the correct setting in which to examine weak formulations of differential equations. In Section 2.2 we shall give some basic properties of Sobolev spaces that will allow us to analyse discretization methods at least in standard situations. Variational problems often appear when modelling problems applied in the natural and technical sciences because in many situations nature follows minimum or maximum laws such as the principle of minimum energy. Next we consider a simple elliptic model problem in two dimensions.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set with a (piecewise) smooth boundary Γ. Let $f: \bar{\Omega} \to \mathbb{R}$ be a given function. We seek a twice differentiable function u that satisfies

$$-\Delta u(\xi, \eta) = f\,(\xi, \eta) \quad in \quad \Omega \tag{1.5}$$

the application of an integral theorem (the two dimensional analogue of integration by parts see the next section for details) yields

$$\int_{\Omega} \left(\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \xi} + \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \eta}\right) dx = \int_{\Omega} f v \, dx \tag{1.7}$$

So far we have said nothing about the existence and uniqueness of solutions I of the variational formulation of boundary value problems, because to deal adequately with these topics it is necessary to work in the framework of Sobolev spaces. For more information about the existence and uniqueness of solution, see [1,2,5,11,12].

Chapter 2

Function Spaces for the Variational Formulation of Boundary Value Problems

Section (2.1): Introduction

In the classical treatment of differential equations, the solution and certain of its derivatives are required to be continuous functions. One therefore works in the spaces $C^k(\bar{\Omega})$ that contain functions with continuous derivatives up to order k on the given domain Ω, or in spaces where these derivatives are Holder continuous. When the strong form (e.g. (1.7)) of a differential equation is replaced by a variational formulation, then instead of point wise differentiability we need only ensure the existence of some integrals that contain the unknown function as certain derivatives. Thus it makes sense to use function spaces that are especially suited to this situation.

Section (2.2): Function Spaces

We start with some basic facts from functional analysis.

Let U be a linear (vector) space. A mapping $\|.\|: U \to \mathbb{R}$ is called a norm if it has the following properties:

i) $\|u\| \geq 0 \; for\; all\; u \in U, \; \|u\| = 0 \Leftrightarrow u = 0,$

ii) $\|\lambda u\| = |\lambda| \, \|u\| \; for\; all\; u \in U, \lambda \in \mathbb{R},$

iii) $\|u + v\| \leq \|u\| + \|v\| \; for\; all\; u, v \in U.$

A linear space U endowed with a norm is called a normed space. A sequence $\{u^k\}$ in a normed space is a Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N(\varepsilon)$ such that

$$\|u^k - u^l\| \leq \varepsilon \quad for\; all \quad k, l \geq N(\varepsilon) .$$

The next property is of fundamental importance both in existence theorems for solutions of variational problems and in proofs of convergence of numerical methods. A normed space is called complete if every Cauchy sequence $\{u^k\} \subset U$ converges in U, i.e., there exists $u \in U$ with

$$lim_{k\to\infty}\|u^k - u\| = 0$$

Equivalently,

$$u = \lim_{k\to\infty} u^k$$

Complete normed spaces are often called Banach spaces.

Let U, V be two normed spaces with norms $\|.\|_U$ and $\|.\|_V$ respectively.

A mapping $P : U \to V$ is continuous at $u \in U$ if for any sequence

$\{u^k\} \subset U$ converging to u one has

$$\lim_{k\to\infty} Pu^k = Pu$$

$$\lim_{k\to\infty}\left\|u^k - u\right\|_U = 0 \;\Rightarrow\; \lim_{k\to\infty}\left\|Pu^k - Pu\right\|_V = 0$$

A mapping is continuous if it is continuous at every point $u \in U$.

A mapping $P : U \to V$ is called linear if

$$P(\lambda u + \mu v) = \lambda Pu + \mu Pv \; for\; all\; u, v \in U, \lambda, \mu \in \mathbb{R}$$

A linear mapping is continuous if there exists a constant $M \geq 0$ such that

$$\|Pu\| \leq M\|u\| \;\; for\; all\; u \in U.$$

A mapping $f : U \to \mathbb{R}$ is usually called a functional. Consider the set of all continuous linear functional $f : U \to \mathbb{R}$. These form a normed space with norm defined by

$$\|f\|_* = \begin{matrix} sup \\ v \neq 0 \end{matrix} \; \frac{|f(v)|}{\|v\|}$$

This space is in fact a Banach space. It is the dual space $U^*\; of\; U$. When

$f \in U^*$ and $u \in U$ we shall sometimes write $\langle f, u\rangle$ instead of $f(u)$.

Occasionally it is useful to replace convergence in the normed space by convergence in a weaker sense:

if

$$\lim_{k\to\infty} \langle f, u^k \rangle = \langle f, u \rangle \ \ for\ all\ f \in U^*$$

For a sequence $\{u^k\} \subset U$ and $u \in U$, then we say that the sequence $\{u^k\}$ converges weakly to u. It is standard to use the notation

$$u^k \rightharpoonup u\ for\ k \to \infty$$

To denote weak convergence. If $u = \lim_{k\to\infty} u^k \ \ then\ u^k \rightharpoonup u$, i.e. convergence implies weak convergence, but the converse is false: a weakly convergent sequences not necessarily convergent.

It is particularly convenient to work in linear spaces that are endowed with a scalar product. A mapping $(\cdot,\cdot): U \times U \to \mathbb{R}$ is called a (real-valued) scalar product if U has the following properties:

- **i)** $(u, u) \geq 0\ for\ all\ u \in U, \quad (u, u) = 0 \Leftrightarrow u = 0,$
- **ii)** $(\lambda u, v) = \lambda(u, v)\ for\ all\ u, v \in U, \quad \lambda \in R,$
- **iii)** $(u, v) = (v, u)\ for\ all\ u, v \in U,$
- **iv)** $(u + v, w) = (u, w) + (v, w)\ for\ all\ u, v, w \in U.$

Given a scalar product, one can define an induced norm by $\|u\| := \sqrt{(u, u)}$.

But not all norms are induced by related scalar products.

A Banach space in which the norm is induced by a scalar product is called a (real) Hilbert space. From the properties of the scalar product one can deduce the useful Cauchy-Schwarz inequality:

$$|u, v| \leq \|u\|\|v\| \ \ for\ all\ \ u, v \in U.$$

Continuous linear functional on Hilbert spaces have a relatively simple structure that is important in many applications. It is stated in the next result.

Theorem (2.2.1): (Riesz). Let $f : V \to \mathbb{R}$ be a continuous linear functional on a Hilbert space V . Then there exists a unique w $\in$ V such that

$$(w, v) = f(v) \text{ for all } v \in V.$$

Moreover, one has

$$\|f\|_* = \|\omega\|$$

The Lebesgue spaces of integrable functions are the starting point for the construction of the Sobolev spaces. Let $\Omega \subset$ Rn (for n = 1, 2, 3) be a bounded domain (i.e., open and connected) with boundary $\Gamma := \partial\Omega.\ Let\, p \in [1, +\infty)$.

The class of all functions whose $p\ ^{th}$power is integrable on Ω is denoted by

$$L_p(\Omega) := \left\{ v : \int_\Omega |v(x)|^p \, dx < +\infty. \right\}$$

Furthermore

$$\|v\|_{L_p(\Omega)} := \left[\int_\Omega |v(x)|^P dx \right]^{\frac{1}{p}}$$

Is a norm on L_p. It is important to remember that we work with Lebesgue integrals (see, e.g., [12], so all functions that differ only on a set of measure zero are identified. It is in this sense that $\|v\| = 0$ implies $v = 0$. Moreover, the space $Lp(\Omega)$ is complete, i.e., is a Banach space.

In the case p = 2 the integral

$$(u, v) := \int_\Omega u(x)\, v(x)\, dx$$

Defines a scalar product, so $L_2(\Omega)$ is a Hilbert space.

The definition of these spaces can be extended to the case $p = \infty$ with

$$L_\infty(\Omega) := \left\{ v : \operatorname*{ess\,sup}_{x \in \Omega} |v(x)| < +\infty \right\}$$

And associated norm

$$L_\infty(\Omega) \coloneqq \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|$$

Here ess sup denotes the essential supremum, i.e., the lowest upper bound over Ω excluding subsets of Ω of Lebesgue measure zero.

To treat differential equations, the next step is to introduce derivatives into the definitions of suitable spaces. In extending the Lebesgue spaces to Sobolev spaces one needs generalized derivatives, which we now describe.

Denote by $cl_v A$ the closure of a subset $A \subset V$ with respect to the topology of the space . For $v \in C(\bar{\Omega})$ the support of v is then defined by

$$supp\ v := cl_{\mathbb{R}n} | \{x \in \Omega : v(x) \neq 0\}.$$

For our bounded domain Ω, set

$$C_0^\infty(\Omega) := \{v \in C^\infty(\Omega) : supp\ v \subset \Omega\}.$$

In our further considerations the role of integration by parts in several dimensions is very important. For instance, for arbitrary $u \in C^1(\bar{\Omega})\ and\ v \in C_0^\infty(\Omega)$ one has

$$\int_\Omega \frac{\partial u}{\partial x_i} v\, dx = \int_\Gamma uv \cos(n, e^i)\, ds - \int_\Omega u \frac{\partial v}{\partial x_i} dx$$

Where e^i is the unit vector in the i^{th} coordinate direction and n is the outward-pointing unit vector normal to Γ. Taking into account that $v|_\Gamma = 0$ we get

$$\int_\Omega u \frac{\partial v}{\partial x_i} dx = - \int_\Omega \frac{\partial u}{\partial x_i} v\, dx. \tag{2.1}$$

This identity is the starting point for the generalization of standard derivatives on Lebesgue spaces.

First we need more notation. To describe partial derivatives one uses a multi-index $\alpha := (\alpha_1, \ldots, \alpha_n)$ where each α_i is a non-negative integer. Set

$|\alpha| = \sum_i \alpha_i$. We introduce

$$D_u^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots x_n^{\alpha_n}} u$$

For the derivative of order $|\alpha|$ with respect to the multi-index α.

Now, recalling (2.1), we say that an integrable function u is in a generalized sense differentiable with respect to the multi-index α if there exists an integrable function w with

$$\int_{\Omega} uD^{\alpha}v \, dx = (-1)^{|\alpha|} \int_{\Omega} wv \, dx \; for \; all \; v \in C_0^{\infty}(\Omega) . \tag{2.2}$$

The function $D^{\alpha}u := w$ is called the generalized derivative of u with respect to the multi-index α.

Applying this definition to each first-order coordinate derivative, we obtain a generalized gradient ∇u. Furthermore, if for a componentwise integrable vector-valued function u there exists an integrable function z with

$$\int_{\Omega} \underline{u} \nabla v \, dx = -\int_{\Omega} zv \, dx \quad for \; all \; v \in C_0^{\infty}(\Omega),$$

Then we call z the generalized divergence of u and we write div $\underline{u} := z$.

Now to define the Sobolev spaces, let l be a non-negative integer. Let $p \in [2, \infty)$. Consider the subspace of all functions from $L_p(\Omega)$ whose generalized derivatives up to order l exist and belong to $L_p(\Omega)$. This subspace is called the Sobolev space $W_p^l(\Omega)$ (Sobolev, 1938). The norm in $W_p^l(\Omega)$ is chosen to be

$$\|u\|_{W_p^l(\Omega)} = \left[\int_{\Omega} \sum_{|\alpha| \leq l} |\alpha| \leq |[D_{\alpha}u](x)|^p \, dx \right]^{\frac{1}{p}} . \tag{2.3}$$

Starting from $L_{\infty}(\Omega)$, the Sobolev space $W_{\infty}^l(\Omega)$ is defined analogously.

Today it is known that Sobolev spaces can be defined in several other equivalent ways. For instance, Meyers and Serrin (1964) proved the following (see [6]):

For $1 \leq p < \infty$ the space $C^{\infty}(\Omega) \cap W_p^l(\Omega)$ is dense in $W_p^l(\Omega)$. (2.4)

That is, for these values of p the space $W_p^l(\Omega)$ can be generated by completing the space $C^\infty(\Omega)$ with respect to the norm defined by (2.3). In other words,

$$W_p^l(\Omega) = cl_{W_p^l(\Omega)} C^\infty(\Omega).$$

This result makes clear that one can approximate functions in Sobolev spaces by functions that are differentiable in the classical sense. Hence, various desirable properties of Sobolev spaces can be proved by first verifying them for classical functions and then using the above density identity to extend them to Sobolev spaces. When $p = 2$ the spaces $W_p^l(\Omega)$ are Hilbert spaces with scalar product

$$(u, v) = \int_\Omega \left(\sum_{|\alpha| \leq l} D_\alpha u D^\alpha v \right) dx. \tag{2.5}$$

It is standard to use the notation $H^l(\Omega)$ in this case, i.e., $H^l(\Omega) = W_2^l(\Omega)$. In the treatment of second-order elliptic boundary value problems the Sobolev spaces $H^1(\Omega)$ play a fundamental role, while for fourth-order elliptic problems one uses the spaces $H^2(\Omega)$.If additional boundary conditions come into the game, then additional information concerning certain subspaces of these Sobolev spaces is required. For more information see [1,3,5,11,12].

Chapter 3

Numerical Computations and Results

Section (3.1): A Simple Boundary Value Problem

Let us consider the two- point boundary value problem

$$-pu'' + qu = f(x) \qquad 0 < x < 1$$

$$u(0) = 0 \;, \quad u(1) = 0$$

with constant coefficients p, q where $p > 0$ and q≥ 0

As described before ,we construct a variational or (weak formulation) using Galerkin's method.

The weak form is

$$\int_0^1 (-pu'' + qu)v dx = \int_0^1 f v dx$$

$$\Rightarrow \int_0^1 (pu'v' + quv)dx - pu'v \Big|_0^1 = \int_0^1 f v dx$$

$$\Rightarrow \int_0^1 (pu'v' + quv)dx - pu'(1)v(1) + pu'(0)v(0) = \int_0^1 f v dx$$

$$\Rightarrow \int_0^1 (pu'v' + quv)dx = \int_0^1 f v dx$$

$$\Rightarrow a(u, v) = (f, v)$$

The weak form of the problem is to find $v \in H_0^1$ such that

$$a(u, v) = (f, v) \qquad \forall\, v \in H_0^1 \tag{3.1}$$

Now we obtain an approximate solution of (3.1) and we write

$$\breve{u} = \sum_{i=1}^{m-1} C_i\psi_i \Rightarrow a(\tilde{u}, v) = (f, v)$$

$$a\left(\sum_{i=1}^{m-1} C_i\psi_i , v\right) = (f, v) \qquad (3.2)$$

Then we choose the weight function v by Galerkin's method $v_i = \psi_i \quad i = 1, \dots M-1$

$$\Rightarrow a\left(\sum_{i=1}^{m-1} C_i\psi_i , \psi_i\right) = (f, \psi_i)$$

$$\Rightarrow \sum_{i=1}^{m-1} a(\psi_i, \psi_j)C_j = (f, \psi_i)$$

$$i = 1,2, \dots \dots M-1$$

$$a(\psi_i, \psi_j) = a_{ij} = \int_0^1 \left(p\psi'_i(x)\psi'_j(x) + q\psi_i(x)\psi_j(x)\right)dx$$

$$j = 1,2, \dots , M \qquad (3.3)$$

$$b_i = \int_0^1 \psi_i(x)f(x)dx$$

Now let us choose the simplest continuous piecewise polynomial approximations of u and v , this would be a piecewise linear polynomial with respect to a mesh

$$0 = x_0 < x_1 < x_2 < \cdots x_m = 1 \quad on\ [0,1]$$

Each subinterval $(x_{i-1}, x_i) \quad i = 1, \dots \dots M-1$ is called finite element

basis $\psi_i(x)$ is created from the hat function $\psi_1(x), \psi_2(x), \dots . \psi_m(x)$ are

$$\psi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & if \quad x_{i-1} \leq x < x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & if \quad x_i \leq x < x_{i+1} \\ 0 & otherwise \end{cases} \tag{3.4}$$

and

$$\psi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases},$$

also

$$\psi_{i+1}(x) = \begin{cases} \dfrac{x - x_i}{x_{i+1} - x_i} & if \quad x_i \leq x < x_{i+1} \\ \dfrac{x_{i+2} - x}{x_{i+2} - x_{i+1}} & if \quad x_{i+1} \leq x < x_{i+2} \\ 0 & otherwise \end{cases} \tag{3.5}$$

$$\psi_{i-1}(x) = \begin{cases} \dfrac{x - x_{i-2}}{x_{i-1} - x_{i-2}} & if \quad x_{i-2} \leq x < x_{i-1} \\ \dfrac{x_i - x}{x_i - x_{i-1}} & if \quad x_{i-1} \leq x < x_i \\ 0 & otherwise \end{cases} \tag{3.6}$$

we can write $h_i = x_{i+1} - x_i$

Also $\psi'_i(x)$ form

$$\psi'_i(x) = \begin{cases} \dfrac{1}{h_{i-1}} & if \quad x_{i-1} \leq x < x_i \\ \dfrac{-1}{h_i} & if \quad x_i \leq x < x_{i+1} \\ 0 & otherwise \end{cases} \tag{3.7}$$

simply by making the mesh uniform

$h_i = h = \frac{1}{M} \quad i = 1,2, \dots M\text{-}1$

$$\psi'_i(x) = \begin{cases} \dfrac{1}{h} & if \quad x_{i-1} \le x < x_i \\ \dfrac{-1}{h} & if \quad x_i \le x < x_{i+1} \\ 0 & otherwise \end{cases} \tag{3.8}$$

For example we take ($p = 1$, $q = 0$) $\Rightarrow -u'' = f(x)$

$$a_{i,j} = \int_0^1 \psi'_i \psi'_j \, dx = \int_{x_{i-1}}^{x_i} \psi'_i \psi'_j dx + \int_{x_i}^{x_{i+1}} \psi'_i \psi'_j dx$$

$$a_{i,i} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx = \left(\frac{1}{h^2}\right) \int_{x_{i-1}}^{x_i} dx + \left(\frac{1}{h^2}\right) \int_{x_i}^{x_{i+1}} dx$$

$$= \left(\frac{1}{h^2}\right)\left(x \Big|_{x_{i-1}}^{x_i} + x \Big|_{x_i}^{x_{i+1}}\right) = \left(\frac{1}{h^2}\right)[(x_i - x_{i-1}) + (x_{i+1} - x_i)] = \left(\frac{1}{h^2}\right)(h + h) = \frac{2}{h}$$

$$a_{i,i+1} = \int_{x_{i-1}}^{x_i} \psi'_i \psi'_{i+1} dx = \int_{x_i}^{x_{i+1}} \psi'_i \psi'_{i+1} dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)(0) + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx$$

$$= 0 + \left(-\frac{1}{h^2}\right) \int_{x_i}^{x_{i+1}} dx = x \Big|_{x_i}^{x_{i+1}} = \left(-\frac{1}{h^2}\right)(x_{i+1} - x_i) = -\frac{1}{h}$$

$$a_{i,i-1} = \int_{x_{i-1}}^{x_i} \psi'_i \psi_{i-1} dx + \int_{x_i}^{x_{i+1}} \psi'_i \psi'_{i-1} dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)(0) dx$$

$$\left(-\frac{1}{h^2}\right) \int_{x_{i-1}}^{x_i} dx + 0 = x \Big|_{x_{i-1}}^{x_i} = \left(-\frac{1}{h^2}\right)(x_i - x_{i-1}) = -\frac{1}{h}$$

$$a_{i,i-2} = \int_{x_{i-1}}^{x_i} \psi'_i \psi_{i-2} dx + \int_{x_i}^{x_{i+1}} \psi'_i \psi'_{i-2} dx = 0$$

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)dx \qquad (3.9)$$

Then we approximate the integrals in eq.(3.9) using the following trapezoidal integration rule :

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)],$$

$$b_i \approx \frac{x_i - x_{i-1}}{2}[f(x_i)\psi_i(x_i) + f(x_{i-1})\psi_i(x_{i-1})] + \frac{x_{i+1} - x_i}{2}[f(x_i)\psi_i(x_i) + f(x_{i+1})\psi_i(x_{i+1})]$$

$$\frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) \approx hf(x_i) \qquad i = 1, \ldots\ldots M-1.$$

Now we approximate the above integration using the following Simpson's integration rule :

$$\int_0^h f(x)dx \approx \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

$$b_i \approx \frac{h}{6}\left[f(x_i)\psi_i(x_i) + 4f\left(x_{i-\frac{1}{2}}\right)\psi_i\left(x_{i-\frac{1}{2}}\right) + f(x_{i-1})\psi_i(x_{i-1})\right] + \frac{h}{6}\left[f(x_{i+1})\psi_i(x_{i+1}) + 4f\left(x_{i+\frac{1}{2}}\right)\psi_i\left(x_{i+\frac{1}{2}}\right) + f(x_i)\psi_i(x_i)\right]$$

$$\approx \frac{h}{6}\left[f(x_i) + \frac{4}{2}f\left(x_{i-\frac{1}{2}}\right) + \frac{4}{2}f\left(x_{i+\frac{1}{2}}\right) + f(x_i)\right] = \frac{h}{6}\left[2f(x_i) + 2f\left(x_{i-\frac{1}{2}}\right) + 2f\left(x_{i+\frac{1}{2}}\right)\right]$$

$$\approx \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \dfrac{2}{h} & if \;\; i = j \\ \\ -\dfrac{1}{h} & if \;\; |i - j| = 1 \\ \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.3) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i;i+1}C_{i+1} = b_i \qquad i = 1{,}2, \dots . M - 1$$

$$\Rightarrow -\frac{1}{h}C_{i-1} + \frac{2}{h}C_i - \frac{1}{h}C_{i+1} = b_i \qquad i = 1, \dots . . M - 1$$

when $i = 1 \Rightarrow -\frac{1}{h}C_0 + \frac{2}{h}C_1 - \frac{1}{h}C_2 = b_1$

when $i = 2 \Rightarrow -\frac{1}{h}C_1 + \frac{2}{h}C_2 - \frac{1}{h}C_3 = b_2$

when $i = 3 \Rightarrow -\frac{1}{h}C_2 + \frac{2}{h}C_3 - \frac{1}{h}C_4 = b_3$

$\vdots$

when $i = M - 2 \Rightarrow -\frac{1}{h}C_{m-3} + \frac{2}{h}C_{m-2} - \frac{1}{h}C_{m-1} = b_{m-2}$

when $i = M - 1 \Rightarrow -\frac{1}{h}C_{m-2} + \frac{2}{h}C_{m-1} - \frac{1}{h}C_m = b_{m-1}$

also

$C_0 = C_m = 0$ from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

$$\breve{u}(x) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} \frac{2}{h} & & -\frac{1}{h} & & 0 \\ \frac{-1}{h} & \ddots & \ddots & \ddots & \frac{-1}{h} \\ 0 & & \frac{-1}{h} & & \frac{2}{h} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = h \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ f_{m-1} \end{bmatrix}$$

To find the approximate solution , we substitute C_i , $i = 1, \dots . M - 1$,

in the approximate solution

$$\check{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

we get the solution as a vector

$$\begin{bmatrix} C_0 & \sum_{i=1}^{m-1} C_i \psi_i(x) & C_m \end{bmatrix}.$$

Example (3.1.1):

Let us study the two- point boundary value problem

$-u'' = (3x + x^2)e^x \qquad in\ (0,1)$ (3.10)

$u(0) = 0$, $u(1) = 0$

where the exact solution is given by $u = x(1 - x)e^x$

Using Galerkin's method, the weak form is

$$\int_0^1 -u''v dx = \int_0^1 fv dx$$

$$\Rightarrow \int_1^1 u'v'dx - u'v \Big|_0^1 = \int_0^1 (3x + x^2)e^x v dx$$

$$\Rightarrow \int_0^1 u'v'dx - u'(1)v(1) + u'(0)v(0) = \int_0^1 (3x + x^2)e^x v dx$$

$$\Rightarrow \int_0^1 u'v'dx = \int_0^1 (3x + x^2)e^x v dx$$

$$\Rightarrow a(u, v) = (f, v)$$

The weak form of the problem is to find $v \in H_0^1$ such that

$$a(u, v) = (f, v) \qquad \forall\, v \in H_0^1 \tag{3.11}$$

Now we obtain an approximate solution of (3.11) and we write

$$\check{u} = \sum_{i=1}^{m-1} C_i \psi_i$$

$$\Rightarrow a(\tilde{u}, v) = (f, v)$$

$$a\left(\sum_{i=1}^{m-1} C_i \psi_i\, , v\right) = (f, v) \tag{3.12}$$

Then we choose the weight function v by Galerkin's method $v_i = \psi_i$

$i = 1, \dots M - 1$

$$\Rightarrow a\left(\sum_{i=1}^{m-1} C_i \psi_i\, , \psi_i\right) = (f, \psi_i) \qquad i = 1, \dots . M - 1$$

$$\Rightarrow \sum_{i=1}^{m-1} a(\psi_i, \psi_j) C_j = (f, \psi_i) \qquad j = 1,2, \dots . . M - 1$$

$$a(\psi_i, \psi_j) = a_{ij} = \int_0^1 \psi'_i(x) \psi'_j(x) dx \qquad j = 1,2, \dots, M - 1 \tag{3.13}$$

$$b_i = \int_0^1 \psi_i(x)3x + x^2)e^x dx$$

Now we choose the hat function $\psi_i(x)$ which satisfies equation (3.4), (3.5), (3.6) and (3.8), therefore

$$a_{i,j} = \int_0^1 \psi_i' \psi_j' \, dx = \int_{x_{i-1}}^{x_i} \psi_i' \psi_j' dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_j' dx$$

when j=i

$$a_{i,i} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx = \frac{1}{h} + \frac{1}{h} = \frac{2}{h}$$

when j=i+1

$$a_{i,i+1} = \int_{x_{i-1}}^{x_i} \psi_i' \psi_{i+1}' dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_{i+1}' dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)(0) + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h}$$

when $j = i - 1$

$$a_{i,i-1} = \int_{x_{i-1}}^{x_i} \psi_i' \psi'_{i-1} dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_{i-1}' dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)(0) dx = -\frac{1}{h}$$

$$a_{i,i-2} = \int_{x_{i-1}}^{x_i} \psi_i' \psi'_{i-2} dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_{i-2}' dx = 0$$

$$b_i = \int_0^1 f(x)\psi_i(x) \, dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x) dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x) dx \qquad (3.14)$$

Using trapezoidal rule

$$\int_a^b f(x) dx \approx \frac{b-a}{2} [f(a) + f(b)],$$

$$b_i \approx \frac{x_i - x_{i-1}}{2}[f(x_i)\psi_i(x_i) + f(x_{i-1})\psi_i(x_{i-1})] + \frac{x_{i+1} - x_i}{2}[f(x_i)\psi_i(x_i) + f(x_{i+1})\psi_i(x_{i+1})]$$

$$\frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) \approx h\,f(x_i) \qquad i = 1, \dots.. M-1$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \frac{2}{h} & if \;\; i = j \\ -\frac{1}{h} & if \;\; |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.13) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = h\,f(x_i) \qquad i = 1,2, \dots. M-1$$

$$\Rightarrow -\frac{1}{h}C_{i-1} + \frac{2}{h}C_i - \frac{1}{h}C_{i+1} = h\,f(x_i) \qquad i = 1, \dots.. M-1$$

when $i = 1 \Rightarrow -\frac{1}{h}C_0 + \frac{2}{h}C_1 - \frac{1}{h}C_2 = h\,f(x_1)$
when $i = 2 \Rightarrow -\frac{1}{h}C_1 + \frac{2}{h}C_2 - \frac{1}{h}C_3 = h\,f(x_2)$
when $i = 3 \Rightarrow -\frac{1}{h}C_2 + \frac{2}{h}C_3 - \frac{1}{h}C_4 = h\,f(x_3)$
$\vdots$
when $i = M-1 \Rightarrow -\frac{1}{h}C_{m-2} + \frac{2}{h}C_{m-1} - \frac{1}{h}C_m = h\,f(x_{m-1})$
also

$C_0 = C_m = 0$ from the boundary condition

$$\check{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

$$\check{u}(x) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{m-1} \end{bmatrix} = h^2 \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_2) \end{bmatrix}$$

To find the approximate solution, we substitute $C_i \quad i = 1, \dots . M - 1$, in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

We get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i \psi_i(x) \quad\quad C_m \right].$$

Then we solve this system using MATLAB code(1) and show the results in figure(3.1).

Now we use Simpson's integration rule for approximating the integration in equation(3.14):

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)dx$$

$$\int_0^h f(x)dx \approx \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

$$\begin{aligned} b_i &\approx \frac{h}{6}\left[f(x_i)\psi_i(x_i) + 4f\left(x_{i-\frac{1}{2}}\right)\psi_i\left(x_{i-\frac{1}{2}}\right) + f(x_{i-1})\psi_i(x_{i-1})\right] \\ &+ \frac{h}{6}\left[f(x_{i+1})\psi_i(x_{i+1}) + 4f\left(x_{i+\frac{1}{2}}\right)\psi_i\left(x_{i+\frac{1}{2}}\right) + f(x_i)\psi_i(x_i)\right] \\ &\approx \frac{h}{6}\left[f(x_i) + \frac{4}{2}f\left(x_{i-\frac{1}{2}}\right) + \frac{4}{2}f\left(x_{i+\frac{1}{2}}\right) + f(x_i)\right] \\ &\approx \frac{h}{6}\left[2f(x_i) + 2f\left(x_{i-\frac{1}{2}}\right) + 2f\left(x_{i+\frac{1}{2}}\right)\right] \approx \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right] \end{aligned}$$

We can write the following

$$a(\psi_i,\psi_j) = \begin{cases} \dfrac{2}{h} & if\ \ i = j \\ -\dfrac{1}{h} & if\ \ |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.13) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right],$$

$i = 1,2, \dots . M-1$

$\Rightarrow -\frac{1}{h}C_{i-1} + \frac{2}{h}C_i - \frac{1}{h}C_{i+1} = \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$, $i = 1, \dots . . M-1$

when $i = 1 \Rightarrow -\frac{1}{h}C_0 + \frac{2}{h}C_1 - \frac{1}{h}C_2 = \frac{h}{3}\left[f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right)\right],$

when $i = 2 \Rightarrow -\frac{1}{h}C_1 + \frac{2}{h}C_2 - \frac{1}{h}C_3 = \frac{h}{3}\left[f\left(x_{\frac{3}{2}}\right) + f(x_2) + f\left(x_{\frac{5}{2}}\right)\right],$

when $i = 3 \Rightarrow -\frac{1}{h}C_2 + \frac{2}{h}C_3 - \frac{1}{h}C_4 = \frac{h}{3}\left[f\left(x_{\frac{5}{2}}\right) + f(x_3) + f\left(x_{\frac{7}{2}}\right)\right],$

$\vdots$

when $i = M-2 \Rightarrow$

$$-\frac{1}{h}C_{m-3} + \frac{2}{h}C_{m-2} - \frac{1}{h}C_{m-1} = \frac{h}{3}\left[f\left(x_{m-\frac{5}{2}}\right) + f(x_{m-2}) + f\left(x_{m-\frac{3}{2}}\right)\right],$$

when $i = M-1 \Rightarrow -\frac{1}{h}C_{m-2} + \frac{2}{h}C_{m-1} - \frac{1}{h}C_m = \frac{h}{3}\left[f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + f\left(x_{m-\frac{1}{2}}\right)\right],$

also

$C_0 = C_m = 0$ from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

$$\breve{u}(x) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = \frac{h^2}{3} \begin{bmatrix} f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right) \\ \vdots \\ f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + f\left(x_{m-\frac{1}{2}}\right) \end{bmatrix}$$

to find the approximate solution, we substitute C_i, $i = 1, \dots . M-1$,in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

we get the solution as a vector

$$\begin{bmatrix} C_0 & \sum_{i=1}^{m-1} C_i\psi_i(x) & C_m \end{bmatrix}.$$

Then we solve this system using MATLAB code(2) and show the results in figure(3.2).

Example (3.1.2):

Let us study the two- point boundary value problem

$$u'' - u = -sin2\pi x \qquad \text{in} \quad (0,1) \tag{3.15}$$
$$u(0) = 0 \quad , \quad u(1) = 0$$

where the exact solution is given by

$$u = \frac{sin2\pi x}{1 + 4\pi^2}$$

Using Galerkin's method, the weak form is

$$\int_0^1 (u'' - u)vdx = -\int_0^1 (sin2\pi x)\ vdx$$

$$\Rightarrow \int_0^1 (-u'v' - uv)dx + u'v\Big|_0^1 = -\int_0^1 (sin2\pi x)vdx$$

$$\Rightarrow \int_0^1 (-u'v' - uv)dx + u'(1)v(1) + u'(0)v(0) = -\int_0^1 (sin2\pi x)vdx$$

$\forall v \in H_0$

$$\Rightarrow \int_0^1 (-u'v' - uv)dx = -\int_0^1 (sin2\pi x)vdx$$

$$\Rightarrow a(u,v) = (f,v) \qquad \forall v \in H_0^1$$

The weak form of the problem is to find $v \in H_0^1$ such that

$$a(u,v) = (f,v) \qquad \forall\ v \in H_0^1 \tag{3.16}$$

Now we obtain an approximate solution of (3.16) and we write

$$\breve{u} = \sum_{i=1}^{m-1} C_i \psi_i$$

$$\Rightarrow a(\tilde{u}, v) = (f, v)$$

$$a\left(\sum_{i=1}^{m-1} C_i \psi_i \, , v\right) = (f, v) \tag{3.17}$$

Then we choose the weight function v by Galerkin's method $v_i = \psi_i$, $i = 1, \dots M - 1$

$$\Rightarrow a\left(\sum_{i=1}^{m-1} C_i \psi_i \, , \psi_i\right) = (f, \psi_i) \qquad i = 1, \dots, M - 1$$

$$\Rightarrow \sum_{i=1}^{m-1} a(\psi_i, \psi_j) C_j = (f, \psi_i) \qquad j = 1,2, \dots, M - 1$$

$$a(\psi_i, \psi_j) = a_{ij} = \int_0^1 (-\psi'_i(x)\psi'_j(x) - \psi_i \, \psi_j) dx$$

$$j = 1,2, \dots \dots M - 1 \tag{3.18}$$

$$b_i = -\int_0^1 \psi_i(x) sin 2\pi x dx$$

Now we choose the het function $\psi_i(x)$ which satisfies equation (3.4), (3.5), (3.6) and (3.8), therefore

$$a_{i,j} = -\int_0^1 (\psi'_i \psi'_j + \psi_i \psi_j) dx$$

$$= -\int_{x_{i-1}}^{x_i} \psi'_i \psi'_j dx - \int_{x_i}^{x_{i+1}} \psi'_i \psi'_j dx - \int_{x_{i-1}}^{x_i} \psi_i \psi_j dx - \int_{x_i}^{x_{i+1}} \psi_i \psi_j dx$$

when $j = i$

$$a_{i,i} = -\int_{x_{i-1}}^{x_i}\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)dx - \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx - \int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)dx - $$
$$-\int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)dx$$

$$a_{i,i} = -\int_{x_{i-1}}^{x_i}\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)dx - \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx - \int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2 dx$$
$$-\int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^2 dx = -\left(\frac{2}{h}+\frac{2h}{3}\right)$$

when $j = i + 1$

$$a_{i,i+1} = -\int_0^1(\psi_i'\psi_{i+1}' + \psi_i\psi_{i+1})dx$$
$$= -\int_{x_{i-1}}^{x_i}\psi_i'\psi_{i+1}'dx - \int_{x_i}^{x_{i+1}}\psi_i'\psi_{i+1}'dx - \int_{x_{i-1}}^{x_i}\psi_i\psi_{i+1}dx - \int_{x_i}^{x_{i+1}}\psi_i\psi_{i+1}dx$$
$$= -\int_{x_{i-1}}^{x_i}\left(\frac{1}{h}\right)(0) - \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx - \int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)(0)dx - $$
$$-\int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)\left(\frac{x-x_i}{x_{i+1}-x_i}\right)dx = \left(\frac{1}{h}-\frac{h}{6}\right)$$

when $j = i - 1$

$$a_{i,i-1} = -\int_0^1(\psi_i'\psi_{i-1}' + \psi_i\psi_{i-1})dx$$
$$= -\int_{x_{i-1}}^{x_i}\psi_i'\psi_{i-1}'dx - \int_{x_i}^{x_{i+1}}\psi_i'\psi_{i-1}'dx - \int_{x_{i-1}}^{x_i}\psi_i\psi_{i-1}dx - \int_{x_i}^{x_{i+1}}\psi_i\psi_{i-1}dx$$

$$= -\int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) - \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)(0)dx - \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x_i - x}{x_i - x_{i-1}}\right)dx -$$

$$- \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)(0\,)dx \quad = \left(\frac{1}{h} - \frac{h}{6}\right)$$

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)dx \qquad (3.19)$$

Using trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)],$$

$$b_i \approx \frac{x_i - x_{i-1}}{2}[f(x_i)\psi_i(x_i) + f(x_{i-1})\psi_i(x_{i-1})]$$
$$+ \frac{x_{i+1} - x_i}{2}[f(x_i)\psi_i(x_i) + f(x_{i+1})\psi_i(x_{i+1})]$$

$$\approx \frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) \approx -h\,f(x_i) \qquad i = 1, \ldots\ldots M-1,$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} -\left(\frac{2}{h} + \frac{2h}{3}\right) & if \quad i = j \\ \left(\frac{1}{h} - \frac{h}{6}\right) & if \;\; |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.18) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = -h\,f(x_i) \qquad i = 1,2, \ldots . M-1$$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right)C_{i-1} - \left(\frac{2}{h} + \frac{2h}{3}\right)C_i + \left(\frac{1}{h} - \frac{h}{6}\right)C_{i+1} = -h\,f(x_i), \quad i = 1, \ldots\ldots M-1$$

when $i = 1 \Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right)\frac{1}{h} C_0 - \left(\frac{2}{h} + \frac{2h}{3}\right) C_1 + \left(\frac{1}{h} - \frac{h}{6}\right) C_2 = -h\, f(x_1)$,
when $i = 2 \Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_1 - \left(\frac{2}{h} + \frac{2h}{3}\right) C_2 + \left(\frac{1}{h} - \frac{h}{6}\right) C_3 = -h\, f(x_2)$,
when $i = 3 \Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_2 - \left(\frac{2}{h} + \frac{2h}{3}\right) C_3 + \left(\frac{1}{h} - \frac{h}{6}\right) C_4 = -h\, f(x_3)$,
when $i = M - 1 \Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_{m-2} - \left(\frac{2}{h} + \frac{2h}{3}\right) C_{m-1} + \left(\frac{1}{h} - \frac{h}{6}\right) C_m = -h\, f(x_{m-1})$,
also
$C_0 = C_m = 0$, from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

$$\breve{u}(x) = C_0 \psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} -\frac{2}{h} - \frac{2h}{3} & & \frac{1}{h} - \frac{h}{6} & & \\ \frac{1}{h} - \frac{h}{6} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \frac{1}{h} - \frac{h}{6} \\ & & \frac{1}{h} - \frac{h}{6} & & -\frac{2}{h} - \frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = -h \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix}$$

To find the approximate solution, we substitute C_i , $i = 1, \dots . M - 1$, in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

we get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i \psi_i(x) \quad C_m \right]$$

Then we solve this system using MATLAB code(3) and show the results in figure(3.3).

Now we use Simpson's integration rule for approximating the integration in equation (3.19):

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)dx$$

$$\int_0^h f(x)dx \approx \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

$$b_i \approx \frac{h}{6}\left[f(x_i)\psi_i(x_i) + 4f\left(x_{i-\frac{1}{2}}\right)\psi_i\left(x_{i-\frac{1}{2}}\right) + f(x_{i-1})\psi_i(x_{i-1})\right] +$$
$$+\frac{h}{6}\left[f(x_{i+1})\psi_i(x_{i+1}) + 4f\left(x_{i+\frac{1}{2}}\right)\psi_i\left(x_{i+\frac{1}{2}}\right) + f(x_i)\psi_i(x_i)\right]$$
$$\approx \frac{h}{6}\left[f(x_i) + \frac{4}{2}f\left(x_{i-\frac{1}{2}}\right) + \frac{4}{2}f\left(x_{i+\frac{1}{2}}\right) + f(x_i)\right]$$
$$\approx \frac{h}{6}\left[2f(x_i) + 2f\left(x_{i-\frac{1}{2}}\right) + 2f\left(x_{i+\frac{1}{2}}\right)\right]$$
$$\approx \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} -\left(\frac{2}{h} + \frac{2h}{3}\right) & if\ \ i = j \\ \left(\frac{1}{h} - \frac{h}{6}\right) & if\ \ |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin’s equation (3.18) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = -\frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$$
$$i = 1,2, \dots .M-1$$
$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right)C_{i-1} - \left(\frac{2}{h} + \frac{2h}{3}\right)C_i + \left(\frac{1}{h} - \frac{h}{6}\right)C_{i+1}$$
$$= -\frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right] \qquad i = 1, \dots ..M-1,$$

when $i = 1$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right)\frac{1}{h}C_0 - \left(\frac{2}{h} + \frac{2h}{3}\right)C_1 + \left(\frac{1}{h} - \frac{h}{6}\right)C_2 = -\frac{h}{3}\left[f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right)\right],$$

when $i = 2$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_1 - \left(\frac{2}{h} + \frac{2h}{3}\right) C_2 + \left(\frac{1}{h} - \frac{h}{6}\right) C_3 = -\frac{h}{3}\left[f\left(x_{\frac{3}{2}}\right) + f(x_2) + f\left(x_{\frac{5}{2}}\right)\right],$$

when $i = 3$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_2 - \left(\frac{2}{h} + \frac{2h}{3}\right) C_3 + \left(\frac{1}{h} - \frac{h}{6}\right) C_4 = -\frac{h}{3}\left[f\left(x_{\frac{5}{2}}\right) + f(x_3) + f\left(x_{\frac{7}{2}}\right)\right],$$

$\vdots$

when $i = M - 2$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_{m-3} - \left(\frac{2}{h} + \frac{2h}{3}\right) C_{m-2} + \left(\frac{1}{h} - \frac{h}{6}\right) C_{m-1}$$
$$= -\frac{h}{3}\left[f\left(x_{m-\frac{5}{2}}\right) + f(x_{m-2}) + f\left(x_{m-\frac{3}{2}}\right)\right],$$

when $i = M - 1$

$$\Rightarrow \left(\frac{1}{h} - \frac{h}{6}\right) C_{m-2} - \left(\frac{2}{h} + \frac{2h}{3}\right) C_{m-1} + \left(\frac{1}{h} - \frac{h}{6}\right) C_m$$
$$= -\frac{h}{3}\left[f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + f\left(x_{m-\frac{1}{2}}\right)\right],$$

also

$C_0 = C_m = 0$, from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

$$\breve{u}(x) = C_0 \psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} -\frac{2}{h} - \frac{2h}{3} & \frac{1}{h} - \frac{h}{6} & & \\ \frac{1}{h} - \frac{h}{6} & \ddots & \ddots & \frac{1}{h} - \frac{h}{6} \\ & \ddots & \ddots & \\ & \frac{1}{h} - \frac{h}{6} & & -\frac{2}{h} - \frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} =$$

$$= -\frac{h}{3} \begin{bmatrix} f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right) \\ f\left(x_{\frac{3}{2}}\right) + f(x_2) + f\left(x_{\frac{5}{2}}\right) \\ \vdots \\ f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + f\left(x_{m-\frac{1}{2}}\right) \end{bmatrix}$$

To find the approximate solution, we substitute C_i, $i = 1, \ldots . M - 1$, in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

We get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i \psi_i(x) \quad C_m \right].$$

Then we solve this system using MATLAB code(4) and show the results in figure(3.4).

Example (3.1.3):

Let us study the two- point boundary value problem

$$-u'' + u = x \qquad \text{in } (0,1) \tag{3.20}$$
$$u(0) = 0 \ , \quad u(1) = 0$$

where the exact solution is given by

$$u = x - \frac{\sinh x}{\sinh 1}.$$

Using Galerkin's method, the weak form is

$$\int_0^1 (-u'' + u) v dx = \int_0^1 x \; v dx$$

$$\Rightarrow \int_0^1 (u'v' + uv) dx - u'v \Big|_0^1 = \int_0^1 xv dx$$

$$\Rightarrow \int_0^1 (u'v' + uv) dx + u'(1)v(1) + u'(0)v(0) = \int_0^1 xv dx \qquad \forall v \in H_0^1$$

$$\Rightarrow \int_0^1 (u'v' + uv) dx = \int_0^1 xv dx$$

$$\Rightarrow a(u,v) = (f,v) \qquad \forall v \in H_0^1$$

The weak form of the problem is to find $v \in H_0^1$ such that

$$a(u,v) = (f,v) \qquad \forall \, v \in H_0^1 \tag{3.21}$$

Now we obtain an approximate solution of (3.21) and we write

$$\breve{u} = \sum_{i=1}^{m-1} C_i \psi_i$$

$$\Rightarrow a(\tilde{u}, v) = (f, v)$$

$$a\left(\sum_{i=1}^{m-1} C_i \psi_i \, , v\right) = (f, v) \tag{3.22}$$

Then we choose the weight function v by Galerkin's method

$v_i = \psi_i \qquad\qquad i = 1, \dots M-1$

$$\Rightarrow a\left(\sum_{i=1}^{m-1} C_i \psi_i \, , \psi_i\right) = (f, \psi_i) \qquad i = 1, \dots . M-1$$

$$\Rightarrow \sum_{i=0}^{m-1} a(\psi_i, \psi_j) C_j = (f, \psi_i) \qquad i = 1,2, \dots . . M-1$$

$$a(\psi_i, \psi_j) = a_{ij} = \int_0^1 \left(\psi'_i(x)\psi'_j(x) + \psi_i \, \psi_j\right) dx$$

$$j = 1,2, \dots . . M-1 \tag{3.23}$$

$$b_i = \int_0^1 \psi_i(x) x dx$$

Now we choose the het function $\psi_i(x)$ which satisfies equation (3.4), (3.5), (3.6) and (3.8), therefore

$$a_{i,j} = \int_0^1 (\psi'_i \psi'_j + \psi_i \psi_j) dx = \int_{x_{i-1}}^{x_i} \psi'_i \psi'_j dx + \int_{x_i}^{x_{i+1}} \psi'_i \psi'_j dx + \int_{x_{i-1}}^{x_i} \psi_i \psi_j dx + \int_{x_i}^{x_{i+1}} \psi_i \psi_j dx \, ,$$

when j=i

$$a_{i,i} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) dx +$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right) dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right) dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 dx$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)^2 dx = \left(\frac{2}{h} + \frac{2h}{3}\right)$$

when j=i+1

$$a_{i,i+1} = \int_0^1 (\psi_i' \psi_{i+1}' + \psi_i \psi_{i+1}) dx$$

$$= \int_{x_{i-1}}^{x_i} \psi_i' \psi_{i+1}' dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_{i+1}' dx + \int_{x_{i-1}}^{x_i} \psi_i \psi_{i+1} dx + \int_{x_i}^{x_{i+1}} \psi_i \psi_{i+1} dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)(0) + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)(0) dx$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{x - x_i}{x_{i+1} - x_i}\right) dx = \left(-\frac{1}{h} + \frac{h}{6}\right)$$

when $j = i - 1$

$$a_{i,i-1} = \int_0^1 (\psi_i' \psi_{i-1}' + \psi_i \psi_{i-1}) dx$$

$$= \int_{x_{i-1}}^{x_i} \psi_i' \psi_{i-1}' dx + \int_{x_i}^{x_{i+1}} \psi_i' \psi_{i-1}' dx + \int_{x_{i-1}}^{x_i} \psi_i \psi_{i-1} dx + \int_{x_i}^{x_{i+1}} \psi_i \psi_{i-1} dx$$

$$= \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)(0) dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x_i - x}{x_i - x_{i-1}}\right) dx$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)(0) dx = \left(-\frac{1}{h} + \frac{h}{6}\right)$$

$$b_i = \int_0^1 f(x)\psi_i(x)\, dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x) dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x) dx \qquad (3.24)$$

Using trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)],$$

$$b_i \approx \frac{x_i - x_{i-1}}{2}[f(x_i)\psi_i(x_i) + f(x_{i-1})\psi_i(x_{i-1})] + \frac{x_{i+1} - x_i}{2}[f(x_i)\psi_i(x_i) + f(x_{i+1})\psi_i(x_{i+1})]$$

$$\frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) \approx hf(x_i) \qquad i = 1, \dots .. M-1$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \left(\frac{2}{h} + \frac{2h}{3}\right) & if \;\; i = j \\ \left(-\frac{1}{h} + \frac{h}{6}\right) & if \;\; |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.23) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = hf(x_i), \;\; i = 1,2, \dots . M-1$$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)\frac{1}{h}C_{i-1} + \left(\frac{2}{h} + \frac{2h}{3}\right)C_i + \left(-\frac{1}{h} + \frac{h}{6}\right)C_{i+1} = hf(x_i)$$

when $i = 1 \Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_0 + \left(\frac{2}{h} + \frac{2h}{3}\right)C_1 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_2 = hf(x_1),$

when $i = 2 \Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_1 + \left(\frac{2}{h} + \frac{2h}{3}\right)C_2 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_3 = hf(x_2),$

when $i = 3 \Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_2 + \left(\frac{2}{h} + \frac{2h}{3}\right)C_3 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_4 = hf(x_3),$

$\vdots$

When$i = M - 1 \Rightarrow$

$$\left(-\frac{1}{h} + \frac{h}{6}\right)\frac{1}{h}C_{m-2} + \left(\frac{2}{h} + \frac{2h}{3}\right)C_{m-1} + \left(-\frac{1}{h} + \frac{h}{6}\right)C_m = hf(x_{m-1}),$$

also

$C_0 = C_m = 0$ from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

$$\breve{u}(x) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} \frac{2}{h}+\frac{2h}{3} & -\frac{1}{h}+\frac{h}{6} & & \\ -\frac{1}{h}+\frac{h}{6} & \ddots & \ddots & -\frac{1}{h}+\frac{h}{6} \\ & -\frac{1}{h}+\frac{h}{6} & & \frac{2}{h}+\frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = h \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix}.$$

To find the approximation solution, we substitute $C_i, i = 1, \dots. M-1$,in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

We get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i \psi_i(x) \quad C_m \right].$$

Then we solve this system using MATLAB code(5) and show the results in figure(3.5).

Now we use Simpson's integration rule for approximating the integration in equation (3.24):

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)d$$

$$\int_0^h f(x)dx \approx \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

$$b_i \approx \frac{h}{6}\left[f(x_i)\psi_i(x_i) + 4f\left(x_{i-\frac{1}{2}}\right)\psi_i\left(x_{i-\frac{1}{2}}\right) + f(x_{i-1})\psi_i(x_{i-1})\right] +$$
$$+\frac{h}{6}\left[f(x_{i+1})\psi_i(x_{i+1}) + 4f\left(x_{i+\frac{1}{2}}\right)\psi_i\left(x_{i+\frac{1}{2}}\right) + f(x_i)\psi_i(x_i)\right]$$

$$\approx \frac{h}{6}\left[f(x_i) + \frac{4}{2}f\left(x_{i-\frac{1}{2}}\right) + \frac{4}{2}f\left(x_{i+\frac{1}{2}}\right) + f(x_i)\right]$$
$$\approx \frac{h}{6}\left[2f(x_i) + 2f\left(x_{i-\frac{1}{2}}\right) + 2f\left(x_{i+\frac{1}{2}}\right)\right]$$
$$\approx \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \left(\frac{2}{h} + \frac{2h}{3}\right) & if\ \ i = j \\ \\ \left(-\frac{1}{h} + \frac{h}{6}\right) & if\ \ |i-j| = 1 \\ \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.23) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right],$$

$i = 1, 2, \dots . M-1$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_{i-1} + \left(\frac{2}{h} + \frac{2h}{3}\right)C_i + \left(-\frac{1}{h} + \frac{h}{6}\right)C_{i+1}$$

$$= \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right],$$

when $i = 1$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)\frac{1}{h}C_0 + \left(\frac{2}{h} + \frac{2h}{3}\right)C_1 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_2 = \frac{h}{3}\left[f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right)\right],$$

when $i = 2$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_1 - \left(\frac{2}{h} + \frac{2h}{3}\right)C_2 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_3 = \frac{h}{3}\left[f\left(x_{\frac{3}{2}}\right) + f(x_2) + f\left(x_{\frac{5}{2}}\right)\right],$$

when $i = 3$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_2 - \left(\frac{2}{h} + \frac{2h}{3}\right)C_3 + \left(-\frac{1}{h} + \frac{h}{6}\right)C_4 = \frac{h}{3}\left[f\left(x_{\frac{5}{2}}\right) + f(x_3) + f\left(x_{\frac{7}{2}}\right)\right],$$

$\vdots$

when $i = M-2$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_{m-3} - \left(\frac{2}{h} + \frac{2h}{3}\right)C_{m-2} + \left(-\frac{1}{h} + \frac{h}{6}\right)C_{m-1} = \frac{h}{3}\left[f\left(x_{m-\frac{5}{2}}\right) + \right.$$
$$\left. f(x_{m-2}) + +f\left(x_{m-\frac{3}{2}}\right)\right],$$

when $i = M - 1$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right) C_{m-2} + \left(\frac{2}{h} + \frac{2h}{3}\right) C_{m-1} + \left(-\frac{1}{h} + \frac{h}{6}\right) C_m$$
$$= \frac{h}{3}\left[f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + +f\left(x_{m-\frac{1}{2}}\right)\right],$$

also

$C_0 = C_m = 0$ from the boundary condition

$$\check{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

$$\check{u}(x) = C_0 \psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} \frac{2}{h} + \frac{2h}{3} & \ddots & -\frac{1}{h} + \frac{h}{6} & \ddots & \\ -\frac{1}{h} + \frac{h}{6} & \ddots & \ddots & \ddots & -\frac{1}{h} + \frac{h}{6} \\ & \ddots & \ddots & & \\ & & -\frac{1}{h} + \frac{h}{6} & & \frac{2}{h} + \frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} =$$

$$= \frac{h}{3} \begin{bmatrix} f\left(x_{\frac{1}{2}}\right) + f(x_1) + f\left(x_{\frac{3}{2}}\right) \\ f\left(x_{\frac{3}{2}}\right) + f(x_2) + f\left(x_{\frac{5}{2}}\right) \\ \vdots \\ f\left(x_{m-\frac{3}{2}}\right) + f(x_{m-1}) + f\left(x_{m-\frac{1}{2}}\right) \end{bmatrix}.$$

To find the approximate solution, we substitute C_i, $i = 1, \dots . M - 1$in the approximate solution

$$\check{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

we get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i \psi_i(x) \quad C_m\right].$$

Then we solve this system using MATLAB code(6) and show the results in figure(3.6).

Example (3.1.4) [4]:

Let us study the two- point boundary value problem

$$-\varepsilon u'' + u = -(cos^2\pi x + 2\,\varepsilon\pi^2 cos2\pi x) \qquad \text{in } (0,1) \qquad (3.25)$$

$$u(0) = 0 \quad , \quad u(1) = 0$$

where the exact solution is given by

$$y(x) = \frac{\exp\left[-\frac{(1-x)}{\sqrt{\varepsilon}}\right] + \exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{1 + \exp\left[-\frac{1}{\sqrt{\varepsilon}}\right]} - cos^2\pi x$$

Using Galerkin's method, the weak form is

$$\int_0^1 (-\varepsilon u'' + u)v dx = \int_0^1 x \; v dx$$

$$\Rightarrow \int_0^1 (\varepsilon u'v' + uv)dx - \varepsilon u'v \Big|_0^1 = \int_0^1 xvdx$$

$$\Rightarrow \int_0^1 (u'v' + uv)dx + \varepsilon u'(1)v(1) + \varepsilon u'(0)v(0) = \int_0^1 xvdx \qquad \forall v \in H_0^1$$

$$\Rightarrow \int_0^1 (\varepsilon u'v' + uv)dx = \int_0^1 xvdx$$

$$\Rightarrow a(u,v) = (f,v) \qquad \forall v \in H_0^1$$

the weak form of the problem is to find $v \in H_0^1$ such that

$$a(u,v) = (f,v) \qquad \forall\, v \in H_0^1 \qquad (3.26)$$

Now we obtain an approximate solution of (3.26) and we write

$$\breve{u} = \sum_{i=1}^{m-1} C_i\psi_i$$

$$\Rightarrow a(\tilde{u},v) = (f,v)$$

$$a\left(\sum_{i=1}^{m-1} C_i\psi_i\,, v\right) = (f,v) \qquad (3.27)$$

Then we choose the weight function v by Galerkin's method

$$v_i = \psi_i \qquad i = 1, \dots M-1$$

$$\Rightarrow a\left(\sum_{i=1}^{m-1} C_i\psi_i , \psi_i\right) = (f, \psi_i) \qquad i = 1, \dots . M-1$$

$$\Rightarrow \sum_{i=0}^{m-1} a(\psi_i, \psi_j)C_j = (f, \psi_i) \qquad i = 1,2, \dots . . M-1$$

$$a(\psi_i, \psi_j) = a_{ij} = \int_0^1 \varepsilon(\psi'_i(x)\psi'_j(x) + \psi_i\,\psi_j)dx$$

$$j = 1,2, \dots . . M-1 \tag{3.28}$$

$$b_i = \int_0^1 \psi_i(x)xdx$$

Now we choose the het function $\psi_i(x)$ which satisfies equation (3.4), (3.5), (3.6) and (3.8), therefore

$$a_{i,j} = \int_0^1 (\varepsilon\psi_i'\psi_j' + \psi_i\psi_j)dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\psi_i'\psi_j'dx + \int_{x_i}^{x_{i+1}} \varepsilon\psi_i'\psi_j'dx + \int_{x_{i-1}}^{x_i} \psi_i\psi_j dx + \int_{x_i}^{x_{i+1}} \psi_i\psi_j dx$$

when $j = i$

$$a_{i,i} = \int_{x_{i-1}}^{x_i} \varepsilon\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)dx + \int_{x_i}^{x_{i+1}} \varepsilon\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)dx +$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)dx + \int_{x_i}^{x_{i+1}} \varepsilon\left(-\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2 dx$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)^2 dx = \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)$$

when $j = i + 1$

$$a_{i,i+1} = \int_0^1 (\varepsilon\psi_i'\psi_{i+1}' + \psi_i\psi_{i+1})dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\psi_i'\psi_{i+1}'dx + \int_{x_i}^{x_{i+1}} \varepsilon\psi_i'\psi_{i+1}'dx + \int_{x_{i-1}}^{x_i} \psi_i\psi_{i+1}dx + \int_{x_i}^{x_{i+1}} \psi_i\psi_{i+1}dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\left(\frac{1}{h}\right)(0) + \int_{x_i}^{x_{i+1}} \varepsilon\left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)(0)dx +$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)\left(\frac{x - x_i}{x_{i+1} - x_i}\right)dx = \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)$$

when $j = i - 1$

$$a_{i,i-1} = \int_0^1 (\varepsilon\psi_i'\psi_{i-1}' + \psi_i\psi_{i-1})dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\psi_i'\psi_{i-1}'dx + \int_{x_i}^{x_{i+1}} \varepsilon\psi_i'\psi_{i-1}'dx + \int_{x_{i-1}}^{x_i} \psi_i\psi_{i-1}dx + \int_{x_i}^{x_{i+1}} \psi_i\psi_{i-1}dx$$

$$= \int_{x_{i-1}}^{x_i} \varepsilon\left(\frac{1}{h}\right)\left(-\frac{1}{h}\right) + \int_{x_i}^{x_{i+1}} \varepsilon\left(-\frac{1}{h}\right)(0)dx + \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x_i - x}{x_i - x_{i-1}}\right)dx +$$

$$+ \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right)(0)dx = \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)$$

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)dx \qquad (3.29)$$

Using trapezoidal rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2}[f(a) + f(b)],$$

$$b_i \approx \frac{x_i - x_{i-1}}{2}[f(x_i)\psi_i(x_i) + f(x_{i-1})\psi_i(x_{i-1})] + \frac{x_{i+1} - x_i}{2}[f(x_i)\psi_i(x_i) + f(x_{i+1})\psi_i(x_{i+1})]$$

$$\frac{h}{2}f(x_i) + \frac{h}{2}f(x_i) \approx hf(x_i) \qquad i = 1, \dots..M-1$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right) & if \;\; i = j \\ \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right) & if \;\; |i-j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation(3.28) yield in this case the linear tridiagonal system

$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = hf(x_i), \;\; i = 1,2, \dots .M-1$

$$\Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)\frac{1}{h}C_{i-1} + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_i + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_{i+1} = hf(x_i),$$
$$i = 1, \dots, M-1$$

when $i = 1 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_0 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_1 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_2 = hf(x_1),$

when $i = 2 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_1 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_2 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_3 = hf(x_2),$

when $i = 3 \;\Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_2 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_3 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_4 = hf(x_3),$

$\vdots$

when $i = M-1 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)\frac{1}{h}C_{m-2} + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_{m-1} + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\quad\right)C_m = hf(x_{m-1}),$

Also

$C_0 = C_m = 0$ from the boundary condition

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

$$\breve{u}(u) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} \frac{2\varepsilon}{h} + \frac{2h}{3} & -\frac{\varepsilon}{h} + \frac{h}{6} & & \\ -\frac{\varepsilon}{h} + \frac{h}{6} & \ddots & \ddots & -\frac{\varepsilon}{h} + \frac{h}{6} \\ & \ddots & \ddots & \\ & -\frac{\varepsilon}{h} + \frac{h}{6} & & \frac{2\varepsilon}{h} + \frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = h \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ \vdots \\ f(x_{m-1}) \end{bmatrix}.$$

To find the approximation solution ,we substitute $C_i,\ i = 1, \dots . M-1$, in the approximate solution

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

we get the solution as a vector

$$\left[C_0 \quad \sum_{i=1}^{m-1} C_i\psi_i(x) \qquad C_m\right].$$

Then we solve this system using MATLAB code(7) and show the results in figure(3.7).

Now we use Simpson's integration rule for approximating the integration in equation(3.29):

$$b_i = \int_0^1 f(x)\psi_i(x)\,dx = \int_{x_{i-1}}^{x_i} f(x)\psi_i(x)dx + \int_{x_i}^{x_{i+1}} f(x)\psi_i(x)d$$

$$\int_0^h f(x)dx \approx \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

$$\begin{aligned}
b_i &\approx \frac{h}{6}\left[f(x_i)\psi_i(x_i) + 4f\left(x_{i-\frac{1}{2}}\right)\psi_i\left(x_{i-\frac{1}{2}}\right) + f(x_{i-1})\psi_i(x_{i-1})\right] + \\
&+ \frac{h}{6}\left[f(x_{i+1})\psi_i(x_{i+1}) + 4f\left(x_{i+\frac{1}{2}}\right)\psi_i\left(x_{i+\frac{1}{2}}\right) + f(x_i)\psi_i(x_i)\right] \\
&\approx \frac{h}{6}\left[f(x_i) + \frac{4}{2}f\left(x_{i-\frac{1}{2}}\right) + \frac{4}{2}f\left(x_{i+\frac{1}{2}}\right) + f(x_i)\right] \\
&\approx \frac{h}{6}\left[2f(x_i) + 2f\left(x_{i-\frac{1}{2}}\right) + 2f\left(x_{i+\frac{1}{2}}\right)\right] \\
&\approx \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right].
\end{aligned}$$

We can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \left(\frac{2}{h} + \frac{2h}{3}\right) & if\ \ i = j \\ \left(-\frac{1}{h} + \frac{h}{6}\ \right) & if\ \ |i-j| = 1 \\ 0 & otherwise \end{cases}.$$

The Galerkin’s equation (3.28) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = \frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right]$$

$$i = 1,2, \dots . M - 1$$

$$\Rightarrow \left(-\frac{1}{h} + \frac{h}{6}\right)C_{i-1} + \left(\frac{2}{h} + \frac{2h}{3}\right)C_i + \left(-\frac{1}{h} + \frac{h}{6}\right)C_{i+1}$$

$$= -\frac{h}{3}\left[f\left(x_{i-\frac{1}{2}}\right) + f(x_i) + f\left(x_{i+\frac{1}{2}}\right)\right],$$

we can write the following

$$a(\psi_i, \psi_j) = \begin{cases} \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right) & if \;\; i = j \\ \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right) & if \;\; |i - j| = 1 \\ 0 & otherwise \end{cases}$$

The Galerkin's equation (3.28) yield in this case the linear tridiagonal system

$$a_{i,i-1}C_{i-1} + a_{i,i}C_i + a_{i,i+1}C_{i+1} = hf(x_i), \;\; i = 1,2, \dots . M - 1$$

$$\Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)\frac{1}{h}C_{i-1} + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_i + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_{i+1} = hf(x_i),$$

$$i = 1, \dots , M - 1$$

when $i = 1 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_0 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_1 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_2 = hf(x_1),$

when $i = 2 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_1 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_2 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_3 = hf(x_2),$

when $i = 3 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_2 + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_3 + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_4 = hf(x_3),$

$\vdots$

when $i = M - 1 \Rightarrow \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)\frac{1}{h}C_{m-2} + \left(\frac{2\varepsilon}{h} + \frac{2h}{3}\right)C_{m-1} + \left(-\frac{\varepsilon}{h} + \frac{h}{6}\right)C_m = hf(x_{m-1}),$

Also

$C_0 = C_m = 0$ from the boundary condition

$$\check{u}(x) = \sum_{i=1}^{m-1} C_i\psi_i(x)$$

$$\check{u}(x) = C_0\psi_0(x_0) = 0 \Rightarrow C_0 = 0$$

$$\begin{bmatrix} \frac{2\varepsilon}{h}+\frac{2h}{3} & -\frac{\varepsilon}{h}+\frac{h}{6} & & \\ -\frac{\varepsilon}{h}+\frac{h}{6} & \ddots & \ddots & -\frac{\varepsilon}{h}+\frac{h}{6} \\ & \ddots & \ddots & \\ & & -\frac{\varepsilon}{h}+\frac{h}{6} & \frac{2\varepsilon}{h}+\frac{2h}{3} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ \vdots \\ \vdots \\ C_{m-1} \end{bmatrix} = -\frac{h}{3} \begin{bmatrix} f\left(x_{\frac{1}{2}}\right)+f(x_1)+f\left(x_{\frac{3}{2}}\right) \\ f\left(x_{\frac{3}{2}}\right)+f(x_2)+f\left(x_{\frac{5}{2}}\right) \\ \vdots \\ f\left(x_{m-\frac{3}{2}}\right)+f(x_{m-1})+f\left(x_{m-\frac{1}{2}}\right) \end{bmatrix}.$$

To find the approximation solution ,we substitute $C_i, i = 1, \dots . M-1$,in the approximate solution,

$$\breve{u}(x) = \sum_{i=1}^{m-1} C_i \psi_i(x)$$

we get the solution as a vector

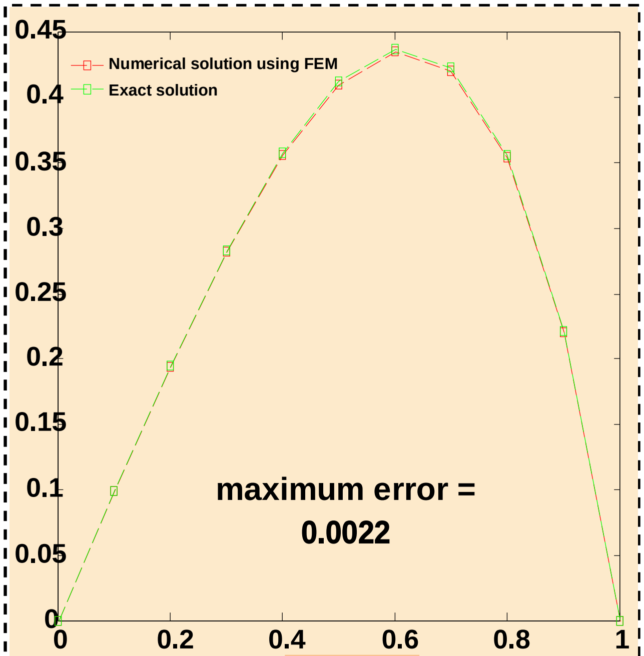
$$\begin{bmatrix} C_0 & \sum_{i=1}^{m-1} C_i \psi_i(x) & C_m \end{bmatrix}.$$

Then we solve this system using MATLAB code(8) and show the results in figure(3.8)

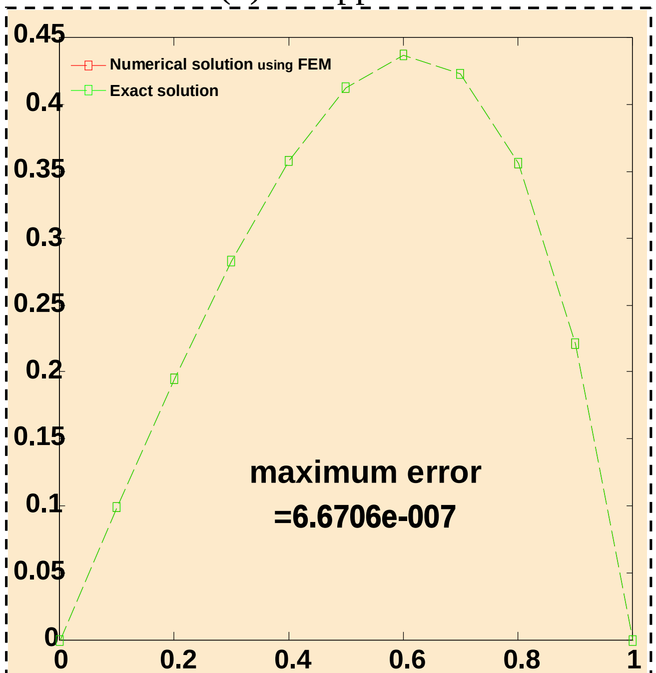
Section (3.2): Numerical results using FEM:

Figure (3.1) illustrates the numerical solution using FEM and exact solution for example (3.1) using Trapezoidal integral rule. We use MATLAB code(1) in Appendix.

Figure (3.2) illustrates the numerical solution using FEM and exact solution for example (3.1) using Simpson's integral rule . we use MATLAB code(2) in Appendix.



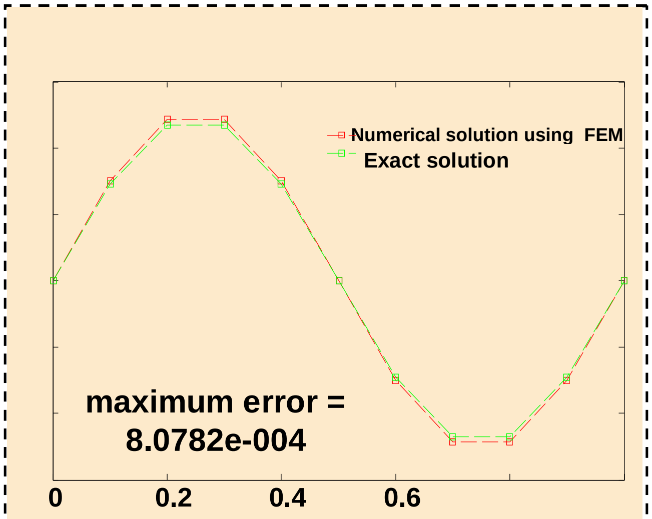
figure(3.1)



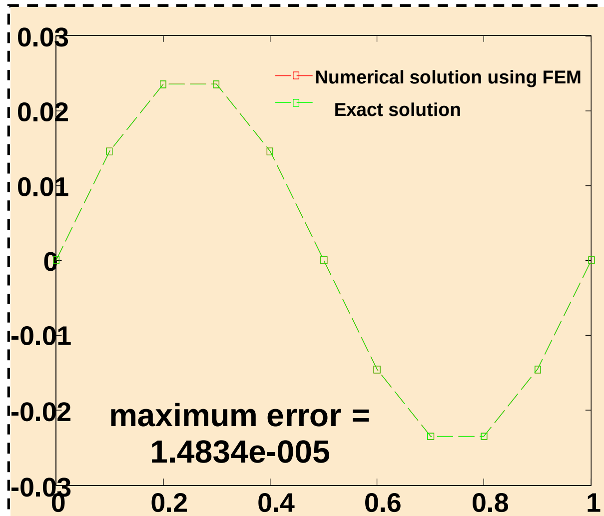
figure(3.2)

Figure (3.3) illustrates the numerical solution using FEM and exact solution for example (3.2) using Trapezoidal integral rule . we use MATLAB code(3) in Appendix .

Figure (3.4) illustrates the numerical solution using FEM and exact solution for example (3.2) using Simpson's integral rule . we use MATLAB code(4) in Appendix .



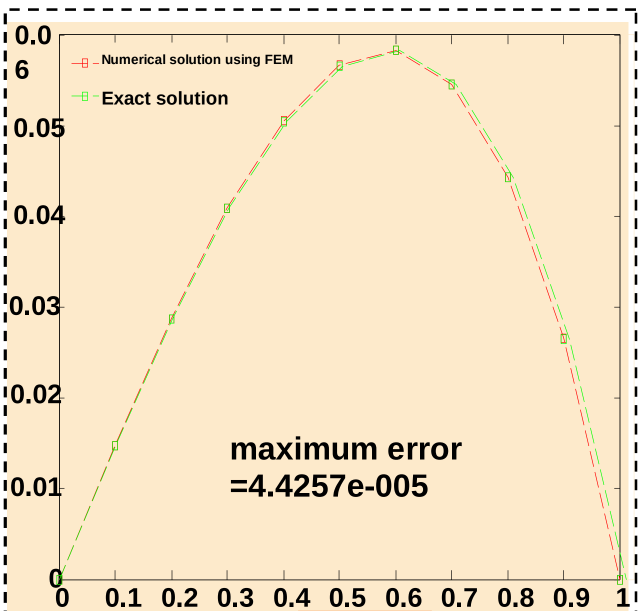
figure(3.3)



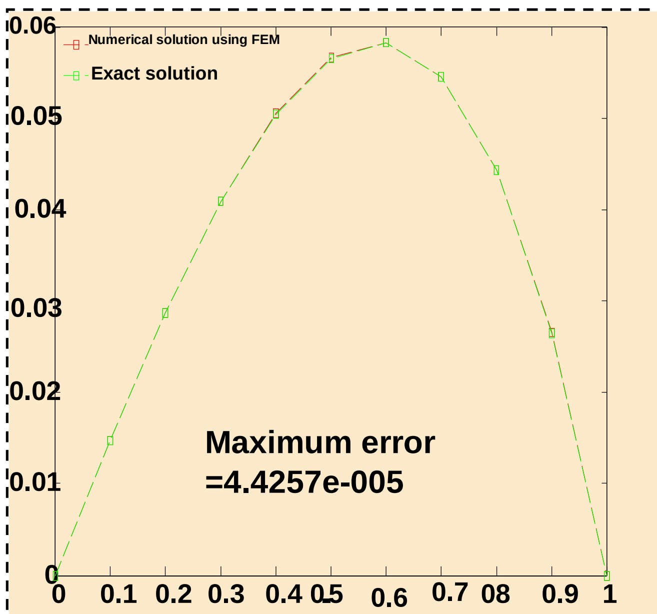
figure(3.4)

Figure (3.5) illustrates the numerical solution using FEM and exact solution for example (3.3) using Trapezoidal integral rule . we use MATLAB code(5) in Appendix .

Figure (3.6) illustrates the numerical solution using FEM and exact solution for example (3.3) using Simpson's integral rule . we use MATLAB code(6) in Appendix.



figure(3.5)



figure(3.6)

Figure (3.7) illustrates Maximum error between numerical using FEM and exact solution for example (3.4) using Trapezoidal integral rule, with $n = 64$ and different values of ε . we use MATLAB code(7) in Appendix.

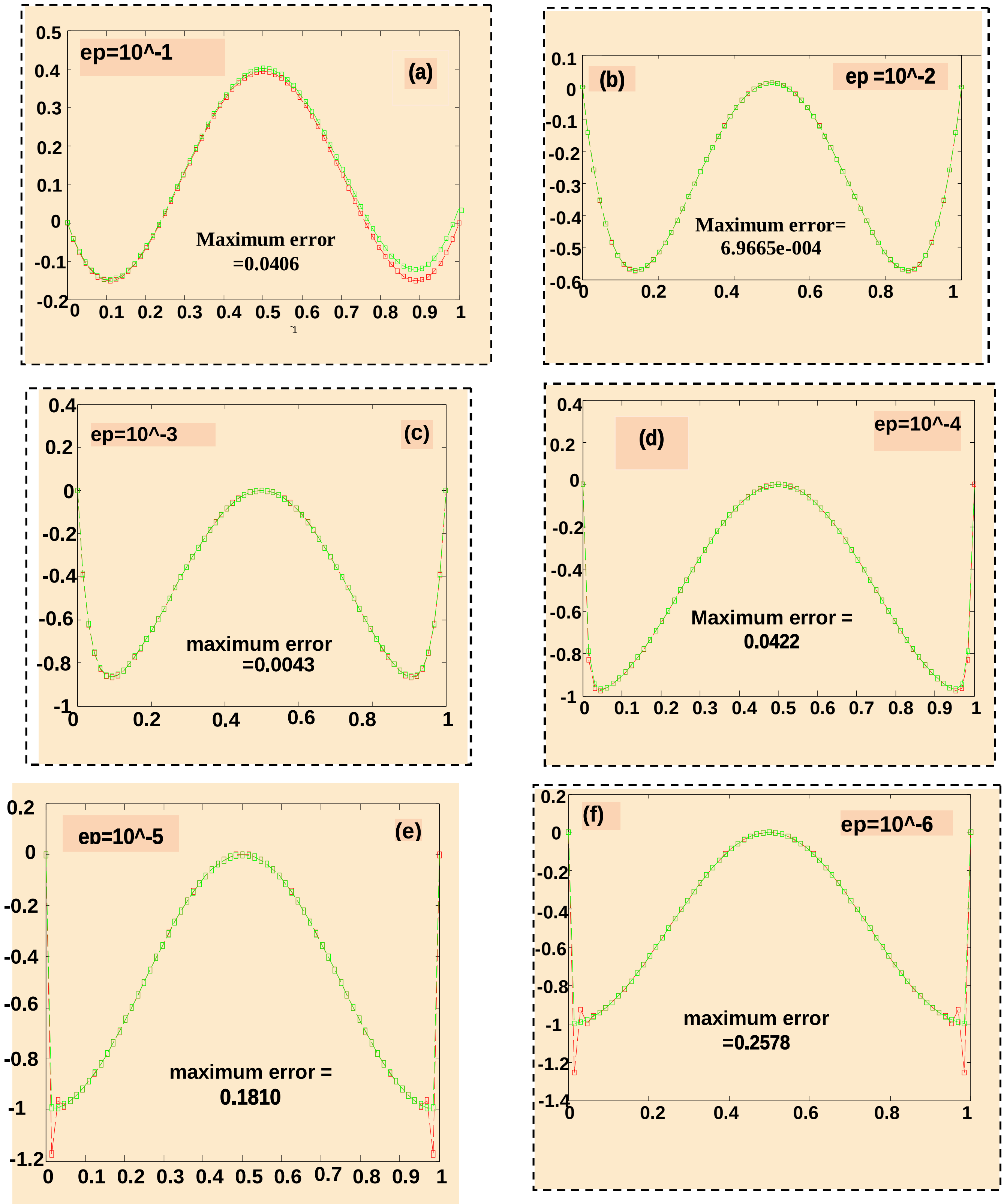
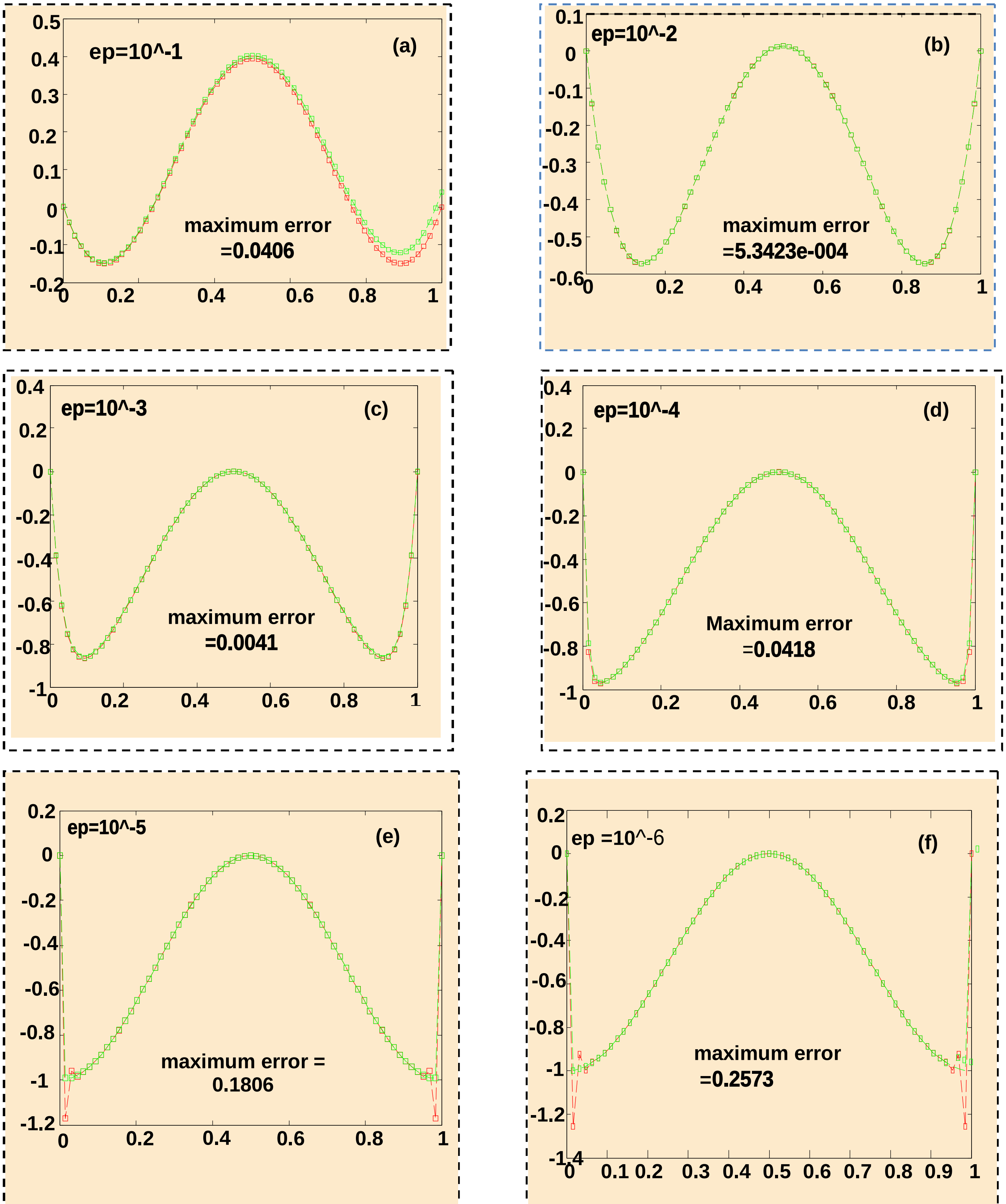


Figure (3.8) illustrates Maximum error between numerical solution using FEM and the exact solution for example (3.4) ,using Simpson's integral rule with $n = 64$ and different values of ε. we use MATLAB code(8) in Appendix .



Section (3.3): Comparison of the Finite Element Method (FEM) with the Finite Difference Method (FDM)

In this section we use the finite difference method (FDM) for solving the problems of section 3.1 which are solved previously by the finite element method for Comparison reasons.

Example (3.3.1):

Let us study the two- point boundary value problem (3.10)

$-u'' = (3x + x^2)e^x \qquad in(0,1)$

$u(0) = 0 \;\; , \;\; u(1) = 0$

where the exact solution is given by

$$u = x(1 - x)e^x$$

first we divide the domain (the interval [0,1]) in to sub intervals

$$x_j = x_0 + jh \quad , \qquad j = 1,2, \dots . n \qquad , \quad h = \frac{1}{n}$$

we have

$$u''(x) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \qquad j = 1,2 \dots \dots . n - 1.$$

Evaluate the equation at $x = x_j$

$$-\frac{d^2u(x_j)}{dx^2} = (3x_j + x^2{}_j)e^{x_j} \qquad j \le n - 1$$

$\Rightarrow -u_{j-1} + 2u_j - u_j = h^2 f(x_j) \qquad j = 1,2 \dots \dots . n - 1$

when j=$1 \Rightarrow -u_0 + 2u_1 - u_2 = h^2(3x_1 + x^2{}_1)e^{x_1}$

when j=$2 \Rightarrow -u_1 + 2u_2 - u_3 = h^2(3x_2 + x^2{}_2)e^{x_2}$

when j=$3 \Rightarrow -u_2 + 2u_3 - u_4 = h^2(3x_3 + x^2{}_3)e^{x_3}$

$\vdots$

when j=n-2$\Rightarrow -u_{n-3} + 2u_{n-2} - u_{n-1} = h^2(3x_{n-2} + {x^2}_{n-2})e^{x_{n-2}}$

when j=n-1$\Rightarrow -u_{n-2} + 2u_{n-1} - u_n = h^2(3x_{n-1} + {x^2}_{n-1})e^{x_{n-1}}$

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = h^2\begin{bmatrix} (3x_1 + {x^2}_1)e^{x_1} \\ (3x_2 + {x^2}_2)e^{x_2} \\ \vdots \\ \vdots \\ (3x_{n-1} + {x^2}_{n-1})e^{x_{n-1}} \end{bmatrix}.$$

Then we solve this system using MATLAB code(1) and show the results in figure(3.9).

Example (3.3.2):

Let us study the two- point boundary value problem (3.15)

$u'' - u = -sin2\pi x \qquad in \quad (0,1)$

$u(0) = 0 \ , \ u(1) = 0$

where the exact solution is given by

$$u = \frac{sin2\pi}{1 + 4\pi^2}$$

first we divide the domain (the interval [0,1]) in to sub intervals

$$x_j = x_0 + jh \qquad , \qquad j = 1,2, \dots . n \qquad , \qquad h = \frac{1}{n}$$

$$u''(x) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \qquad j = 1,2 \dots \dots . n - 1$$

Evalute the equation at $x = x_j$

$$\frac{d^2u(x_j)}{dx^2} - u(x_j) = -sin(2\pi x_j) \qquad j \le n - 1$$

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} - u_j = -sin(2\pi x_j)$$

$\Longrightarrow u_{j-1} - (2 + h^2)\ u_j + u_{j+1} = -h^2 sin(2\pi x_j) \qquad j = 1,2 \dots \dots . n - 1$

when $j=1 \Rightarrow u_0-(2+h^2)u_1+u_2=-h^2 sin(2\pi x_1)$

when $j=2 \Rightarrow u_1-(2+h^2)u_2+u_3=-h^2 sin(2\pi x_2)$

when $j=3 \Rightarrow u_2-(2+h^2)u_3+u_4=-h^2 sin(2\pi x_3)$

$\vdots$

when $j=n-2 \Rightarrow u_{n-3}-(2+h^2)u_{n-2}+u_{n-1}=-h^2 sin(2\pi x_{n-1})$

when $j=n-1 \Rightarrow u_{n-2}-(2+h^2)u_{n-1}+u_n=-h^2 \sin(2\pi x_{n-1})$

$$\begin{bmatrix} -(2+h^2) & 1 & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -(2+h^2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{bmatrix} = -h^2 \begin{bmatrix} sin(2\pi x_1) \\ sin(2\pi x_2) \\ \vdots \\ \vdots \\ sin(2\pi x_{n-1}) \end{bmatrix}.$$

Then we solve this system using MATLAB code(9) and show the results in figure(3.10).

Example (3.3.3):

Let us study the two- point boundary value problem (3.20)

$-u'' + u = x \qquad\qquad in \quad (0,1)$

$u(0)=0 \quad , \quad u(1)=0$

where the exact solution is given by

$$u = x - \frac{sinhx}{sinh1}$$

first we divide the domain (the interval [0,1]) in to sub intervals

$$x_j = x_0 + jh \qquad , \qquad j = 1,2,\dots.n \qquad , \qquad h=\frac{1}{n}$$

$$u''(x) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \qquad j = 1,2 \ldots \ldots . n-1$$

Evalute the equation at $x = x_j$

$$-\frac{d^2u(x_j)}{dx^2} + u(x_j) = x_j \qquad j \leq n-1$$

$$\frac{-u_{j-1} + 2u_j - u_{j+1}}{h^2} + u_j = x_j$$

$$\Longrightarrow -u_{j-1} + (2+h^2)\ u_j - u_{j+1} = h^2 x_j \qquad j = 1,2 \ldots \ldots . n-1$$

when $j = 1 \Rightarrow -u_0 + (2+h^2)u_1 - u_2 = h^2 x_1$

when $j = 2 \Rightarrow -u_1 + (2+h^2)u_2 - u_3 = h^2 x_2$

when $j = 3 \Rightarrow -u_2 + (2+h^2)u_3 - u_4 = h^2 x_3$

$\vdots$

when $j = n-2 \Rightarrow -u_{n-3} + (2+h^2)u_{n-2} - u_{n-1} = h^2 x_{n-1}$

when $j = n-1 \Rightarrow -u_{n-2} + (2+h^2)u_{n-1} - u_n = h^2 x_{n-1}$

$$\begin{bmatrix} (2+h^2) & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & (2+h^2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{n-1} \end{bmatrix} = h^2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Then we solve this system using MATLAB code (10) and show the results in figure (3.11).

Example(3.3.4) :[4].

Let us study the two- point boundary value problem (3.25)

$$-\varepsilon u'' + u = -(cos^2\pi x + 2\ \varepsilon\pi^2 cos2\pi x) \qquad \text{in} \quad (0,1)$$

$u(0) = 0 \quad , \quad u(1) = 0$

where the exact solution is given by

$$u(x) = \frac{\exp\left[-\frac{(1-x)}{\sqrt{\varepsilon}}\right] + \exp\left[-\frac{x}{\sqrt{\varepsilon}}\right]}{1 + \exp\left[-\frac{1}{\sqrt{\varepsilon}}\right]} - cos^2\pi x$$

first we divide the domain (the interval [0,1]) in to sub intervals

$$x_j = x_0 + jh \qquad , \qquad j = 1,2,\ldots.n \qquad , \qquad h = \frac{1}{n}$$

$$u''(x) \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \qquad j = 1,2\ldots\ldots.n-1$$

Evalute the equation at $x = x_j$

$$-\frac{\varepsilon d^2u(x_j)}{dx^2} + u(x_j) = x_j \qquad j \le n-1$$

$$\frac{-\varepsilon u_{j-1} + 2\varepsilon u_j - \varepsilon u_{j+1}}{h^2} + u_j = f(x_j)$$

$$\Longrightarrow -\varepsilon u_{j-1} + (2\varepsilon + h^2)\ u_j - \varepsilon u_{j+1} = h^2 f(x_j\ ) \qquad j = 1,2\ldots\ldots.n-1$$

when $j = 1 \Rightarrow -\varepsilon u_0 + (2\varepsilon + h^2)u_1 - \varepsilon u_2 = h^2 f(x_1)$

when $j = 2 \Rightarrow -\varepsilon u_1 + (2\varepsilon + h^2)u_2 - \varepsilon u_3 = h^2 f(x_2)$

when $j = 3 \Rightarrow -\varepsilon u_2 + (2\varepsilon + h^2)u_3 - \varepsilon u_4 = h^2 f(x_3)$

$\vdots$

when $j = n-2 \Rightarrow -\varepsilon u_{n-3} + (2\varepsilon + h^2)u_{n-2} - \varepsilon u_{n-1} = h^2 f(x_{n-1})$

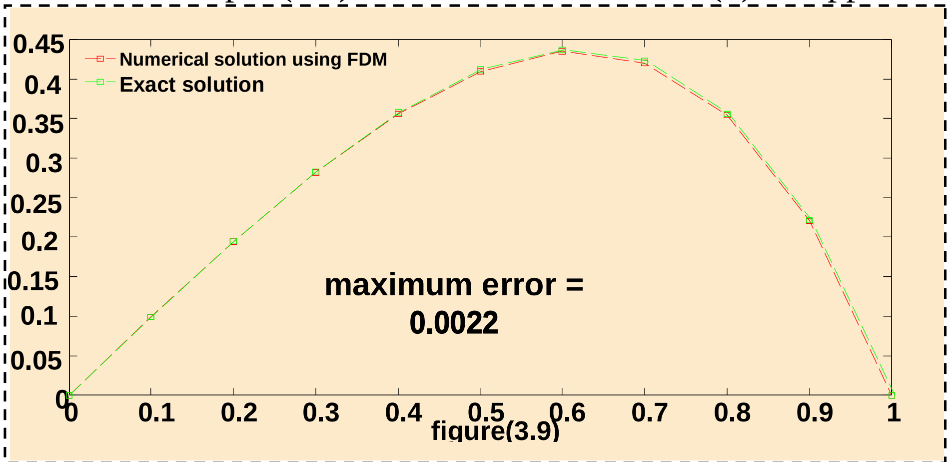
when $j = n-1 \Rightarrow -\varepsilon u_{n-2} + (2\varepsilon + h^2)u_{n-1} - \varepsilon u_n = h^2 f(x_{n-1})$

$$\begin{bmatrix} (2\varepsilon + h^2) & -\varepsilon & & & \\ -\varepsilon & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\varepsilon \\ & & & -\varepsilon & (2\varepsilon + h^2) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{n-1} \end{bmatrix} = h^2 \begin{bmatrix} f(x_1\ ) \\ f(x_2) \\ \vdots \\ \vdots \\ f(x_{n-1}) \end{bmatrix}$$

Then we solve this system using MATLAB code(11) and show the results in figure(12).

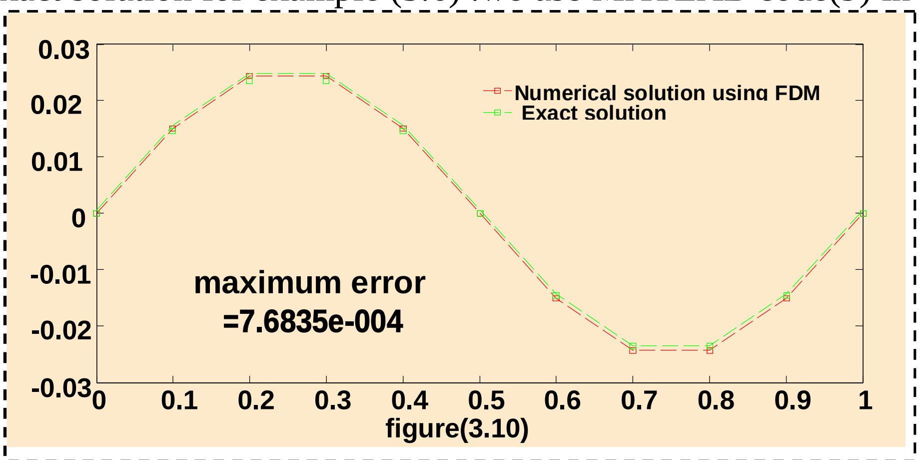
Section (3.4): Numerical results using FDM:

Figure (3.9) illustrates the numerical solution using the finite difference method (FDM). and exact solution for example (3.5) we use MATLAB code(1) in Appendix



figure(3.9)

Figure (3.10) illustrates the numerical solution using the finite difference method (FDM). and exact solution for example (3.6) .we use MATLAB code(9) in Appendix.



figure(3.10)

Figure (3.11) illustrates the numerical solution using the finite difference method (FDM). and exact solution for example (3.7). we use MATLAB code(10) in Appendix.

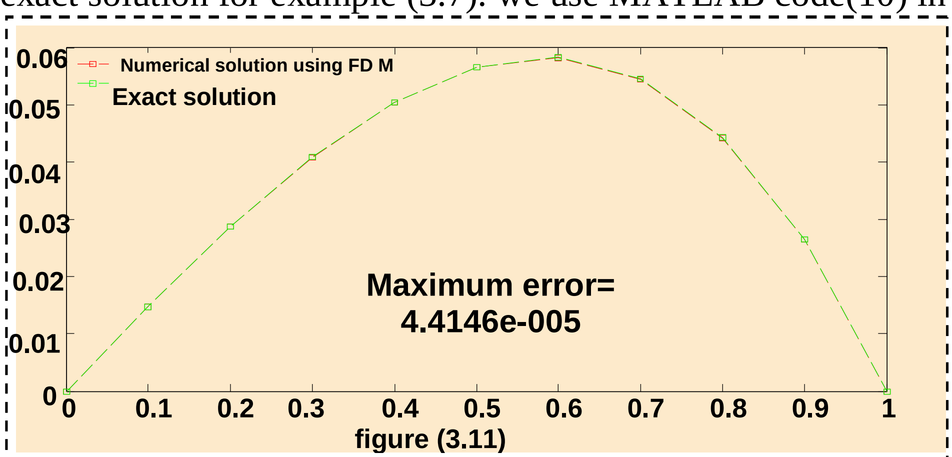
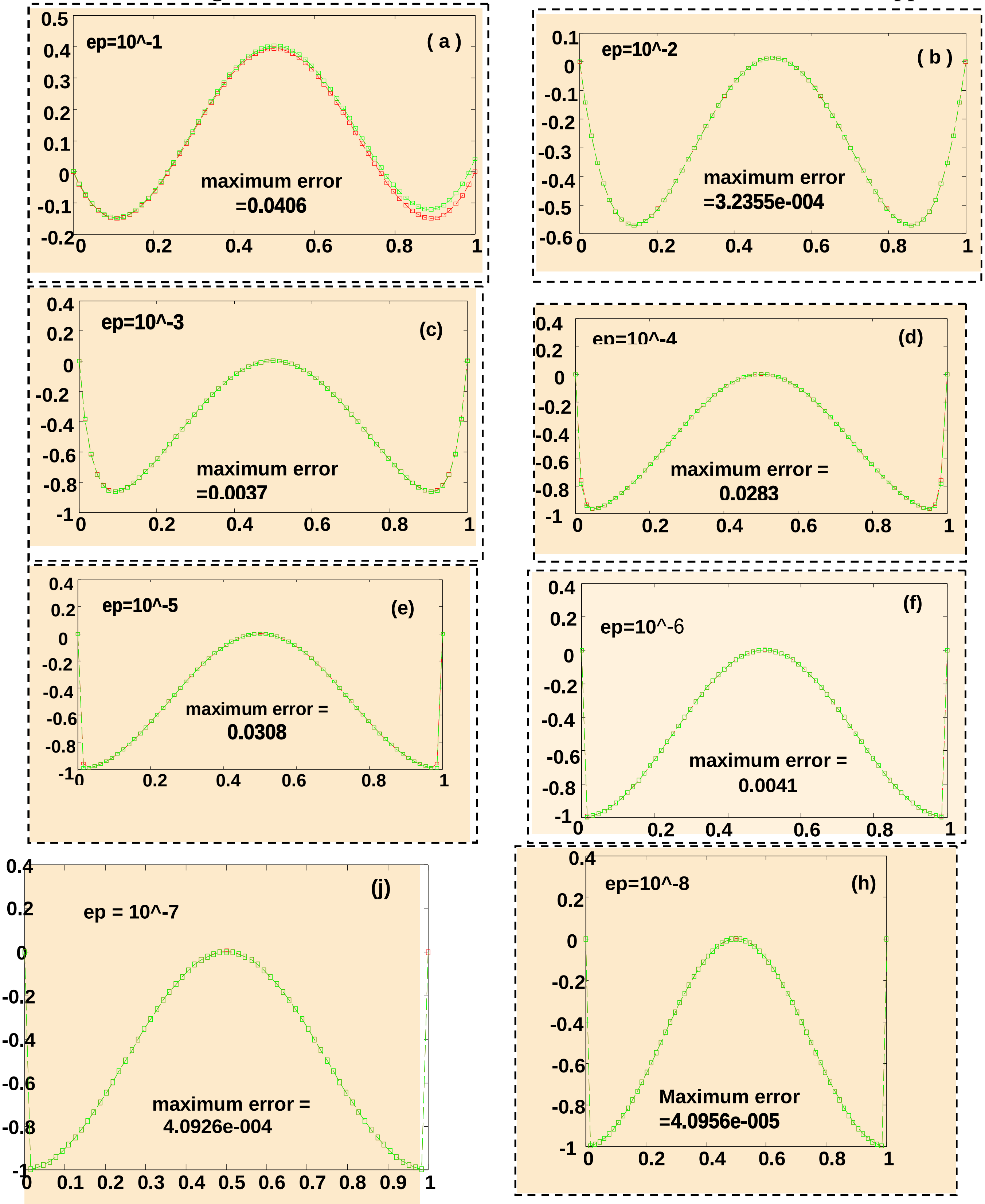


figure (3.11)

Figure (3.12) illustrates the numerical solution Maximum error between numerical solution using FDM and the exact solution for example (3.8) with $n = 64$ and different values of ε using the finite difference method ,we use MATLAB code(11) in Appendix.



Chapter 4

Conclusion

In this thesis, we have applied the Finite Element Method (FEM) to solve som boundary value problem. We approximate the solution by using a piecewise linear polynomial, then we approximate the resulting integrals by using Trapezoidal and Simpson's rules. We studied a two-point boundary value problem with regular and singular coefficients. Also, we used the Finite Difference Method (FDM) for solving some of these problems that mentioned before comparison reasons. Results showed that FEM gives acceptable solution for the two-point boundary value problem with a regular coefficients. Also, when we approximate the resulting integrals by using Simpson's rule, we got a much butter approximation than the use of Trapezoidal rule. For comparison reasons, we used the Finite Difference Method (FDM) results showed that the solution by using FDM for equation with a singular coefficient, is stable but fails to be stable for problem with a singular coefficient where this singular parameter because very small. Also, the maximum error when using FEM with Simpson's rule is less than the maximum error when using FDM. We may expect when using a piecewise polynomial of a higher order as a basis functions, that the finite element method to be better than the finite difference method for problem with regular and singular coefficients.

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Appendix

code(1)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
y =(3*x+x.^2).*exp(x);
%f= x.*(1-x).*exp(x);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i) = 2 ;
end
for i = 1:M-3
A(i,i+1) = -1;
end
for i = 2:M-2
A(i,i-1) =-1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = h^2*(3*x+x.^2).*exp(x);
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code (2)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
xg = -0.05:h:0.95;
%f =(3*x+x.^2).*exp(x);
y = x.*(1-x).*exp(x);
ana_u = y;
plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i) = 2 ;
end
for i = 1:M-3
A(i,i+1) = -1;
end
for i = 2:M-2
A(i,i-1) =-1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = (h^2/3)*(3*x+x.^2).*exp(x);
bg = (h^2/3)*(3*xg+xg.^2).*exp(xg);
B = bg(2:M-1) + b(2:M-1) + bg(3:M);
B = B';
c = A\B;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code (3)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
%f =- sin(2*Pi*x);
y = sin(2*pi*x)/(1 + 4*pi^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i)=(-2/h - 2*h/3);
end
for i = 1:M-3
A(i,i+1)=(1/h+h/6);
end
for i = 2:M-2
A(i,i-1)=(1/h+h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = -h*sin(2*pi*x);
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs
```

code (4)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
xg = -0.05:h:0.95;
%f =- sin(2*PI*x);
y = sin(2*pi*x)/(1 + 4*pi^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i)=-(2/h+2*h/3);
end
for i = 1:M-3
A(i,i+1)=(1/h-h/6);
end
for i = 2:M-2
A(i,i-1)=(1/h-h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b =-(h/3)*sin(2*pi*x);
bg = -(h/3)*sin(2*pi*xg);
B = bg(2:M-1) + b(2:M-1) + bg(3:M);
B = B';
c = A\B;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code (5)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
%f=x;
y =x- (sinh(x)/sinh(1));
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i) = (2/h + 2*h/3);
end
for i = 1:M-3
A(i,i+1) = -(1/h+h/6);
end
for i = 2:M-2
A(i,i-1) =-(1/h+h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = h*x;
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code (6)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
xg = -0.05:h:0.95;
%f =x;
y =x- (sinh(x)/sinh(1));
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i) = (2/h + 2*h/3);
end
for i = 1:M-3
A(i,i+1) = (-1/h+h/6);
end
for i = 2:M-2
A(i,i-1) =(-1/h+h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = (h/3)*x;
bg=(h/3)*xg;
B = bg(2:M-1) + b(2:M-1) + bg(3:M);
B = B';
c = A\B;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

Code (7)

```
clc
clear
n = 64;
h = 1/n;
x0 = 0; xf = 1;
x = x0:h:xf;
%f =-(cos(pi*x).^2+2*ep*pi.^2*cos(2*pi*x));
ep=10^-4;
y =(exp(-(1-x)/sqrt(ep))+exp(-x/sqrt(ep))/(1+exp(-1/sqrt(ep))))-(cos(pi*x).^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
    A(i,i) = (2*ep/h + 2*h/3);
end
for i = 1:M-3
    A(i,i+1) = (-ep/h+h/6);
end
for i = 2:M-2
    A(i,i-1) =(-ep/h+h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b =-h*(cos(pi*x).^2+(2*ep*pi.^2)*cos(2*pi*x));
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

Code (8)

```
clc
clear
n = 64;
h = 1/n;
x0 = 0; xf = 1;
x = x0:h:xf;
xg = -0.0078:h:0.9922;
%f =-(cos(pi*x).^2+2*ep*pi.^2*cos(2*pi*x));
ep=10^-5;
y =(exp(-(1-x)/sqrt(ep))+exp(-x/sqrt(ep))/(1+exp(-1/sqrt(ep))))-(cos(pi*x).^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
    A(i,i)=(2*ep/h + 2*h/3);
end
for i = 1:M-3
    A(i,i+1)=(-ep/h+h/6);
end
for i = 2:M-2
    A(i,i-1)=(-ep/h+h/6);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b =-(h/3)*(cos(pi*x).^2+(2*ep*pi.^2)*cos(2*pi*x));
bg =-(h/3)*(cos(pi*xg).^2+(2*ep*pi.^2)*cos(2*pi*xg));
B = bg(2:M-1) + b(2:M-1) + bg(3:M);
B = B';
c = A\B;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

Code(9)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
%f=- sin(2*Pi*x);
y = sin(2*pi*x)/(1 + 4*pi^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
A(i,i)=-(2+h^2 );
end
for i = 1:M-3
A(i,i+1)=1;
end
for i = 2:M-2
A(i,i-1)=1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = -h^2*sin(2*pi*x);
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code(10)

```
clc
clear
h = 0.1;
x0 = 0; xf = 1;
x = x0:h:xf;
%f =x;
y =x- (sinh(x)/sinh(1));
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2),
for i = 1:M-2
A(i,i) = (2+ h^2);
end
for i = 1:M-3
A(i,i+1) = -1;
end
for i = 2:M-2
A(i,i-1) =-1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = h^2*x;
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
plot(x,num_u,'--rs',x,ana_u,'--gs')
```

code(11)

```
clc
clear
n = 64;
h = 1/n;
x0 = 0; xf = 1;
x = x0:h:xf;
%y =-(cos(pi*x).^2+2*ep*pi.^2*cos(2*pi*x));
ep=10^-5;
y =(exp(-(1-x)/sqrt(ep))+exp(-x/sqrt(ep))/(1+exp(-1/sqrt(ep))))-(cos(pi*x).^2);
ana_u = y;
%plot(x,y)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
M  = length(x);
A = zeros(M-2,M-2);
for i = 1:M-2
    A(i,i) = (2*ep+ h^2);
end
for i = 1:M-3
    A(i,i+1) = -ep;
end
for i = 2:M-2
    A(i,i-1) =-ep;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
b = -h^2*(cos(pi*x).^2+(2*ep*pi.^2*cos(2*pi*x));
bb =b([2:M-1])';
c = A\bb;
cc = [0 c' 0];
num_u = cc';
%plot(x, num_u)
maximum_error = max(abs(ana_u' - num_u))
plot(x,num_u,'--rs',x,ana_u,'--gs')
```