Chapter 1

Spectral Properties and Invariant Subspaces

We show that if an invariant subspace J contains a function that is bounded away from 0 on some neighborhood of a point λ on the unit circle $\mathbb{T}$, then the spectrum of $z[J]$, multiplication by z, when regarded as operating on the quotient space $L_a^2(\mathbb{D})/J$, does not contain the point λ. A consequence of this result is that the spectrum associated with the invariant subspace of all functions vanishing on a prescribed Bergman space zero sequence coincides with the closer of the sequence.

Section (1.1): Invariant Subspaces in the Bergman Space

Let $L_a^2(\mathbb{D})$ denote the standard Bergman space of all holomorphic functions on the open unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$ that satisfy the integrability condition

$$\|f\|_{L^2} = \left(\int_{\mathbb{D}} |f(z)|^2 dS(z)\right)^{1/2} < \infty \tag{1}$$

Here, dS denotes area measure in $\mathbb{C}$, normalized by a constant factor

$$dS(z) = {dxdy}/{\pi}, \quad z = x + iy$$

A closed subspace J of $L_a^2(\mathbb{D})$ is said to be z-invariant, or just invariant, proved the product zf belongs to J whenever $f \in J$. Here, we use the standard notation z for the coordinate function

$$z(\lambda) = \lambda, \ \ \lambda \in \mathbb{D}$$

The structure of the lattice of invariant subspace in $L_a^2(\mathbb{D})$ has attracted a lot of attention from operator theorists as well as function theorists , but most results have been disappointing, in the sense that one realizes that no simple characterization such as is known for the Hardy space $H^2(\mathbb{D})$ is possible for the Bergman space. The famous theorem on the invariant subspace of $H^2(\mathbb{D})$ is due to Arne Beurling [2], and it asserts that every z-invariant subspace J of $H^2(\mathbb{D})$, analogously defined as for the Bergman space, is either trivial, that is, $J = \{0\}$,or has the form $J = uH^2(\mathbb{D})$,where u is an inner function, that is, a bounded analytic function on $\mathbb{D}$ with non-tangential boundary values having modulus 1 almost everywhere .

Given an invariant subspace J of Bergman space $L_a^2(\mathbb{D})$, consider the operator $z[J]: L_a^2(\mathbb{D})\,/\,J \to L_a^2(\mathbb{D})\,/\,J$ defined by the relation

$$z[J](f + J) = zf + J, \ \ f \in L_a^2(\mathbb{D}) \tag{2}$$

We write $\sigma(z[J])$ for the spectrum of the operator $z[J]$, which consists of those $\lambda \in \mathbb{C}$ for which the operator $\lambda - z[J]$, acting on $L_a^2(\mathbb{D})\,/\,J$,is not invertible. It is well known that the spectrum $\sigma(z[J])$ is a compact subset of the closed unit disc $\overline{\mathbb{D}}$. Because the lattice of invariant subspace of the Bergman space is very rich, it is appropriate to also consider another spectral notation (we may call it the weak spectrum): $\sigma'(z[J])$ denote the collection of all $\lambda \in$

$\mathbb{C}$ for which the operator $\lambda - z[J]: L_a^2(\mathbb{D}) / J \to L_a^2(\mathbb{D}) / J$ is not onto. What can go wrong is that $\lambda - z[J]$ need not be one-to-one even if it is onto; this occurs precisely (for $\lambda \in \mathbb{D}$) when the invariant subspace fails to have what Richter[3] calls the codimension 1 property. It is not difficult to see that the set $\sigma'(z[J])$ is also a compact subset of $\overline{\mathbb{D}}$, and clearly we have the inclusion $\sigma'(z[J]) \subset \sigma(z[J])$. There are cases when $\sigma(z[J]) = \overline{\mathbb{D}}$ and $\sigma'(z[J]) = \mathbb{T}$; for an example [4].

The weak spectrum $\acute{\sigma}(z[J])$ is as follows: if λ is a complex number, we have $\lambda \in \mathbb{C} \setminus \sigma'(z[J])$ if and only if

$$(\lambda - z)L_a^2(\mathbb{D}) + J = L_a^2(\mathbb{D}) \quad (3)$$

The work on spectra associated with invariant subspaces in the Bergman space can be found in [3,5,6].

Theorem (1.1.1)(Richter):[1] Let J be an invariant subspace of $L_a^2(\mathbb{D})$, other than the trivial subspace $\{0\}$, if J has the codimension 1 property $\sigma'(z[J]) = \sigma(z[J])$. If, on the other hand, J does not have the codimension 1 property, then $\sigma(z[J]) = \overline{\mathbb{D}}$, and $\sigma'(z[J]) \supset \mathbb{T}$.

Lemma (1.1.2):[1] Every invariant subspace of J of the Bergman space $L_a^2(\mathbb{D})$, other than $\{0\}$, contains a non-identically vanishing function G_J ,which extends to a holomorphic function on the region

$$\{z \in \mathbb{C}: 1 / \bar{z} \notin \sigma(z[J])\}$$

and has $|G_J(z)| \geq 1$ on the union of arcs $\mathbb{T} \setminus \sigma(z[J])$.

Proof: The assertion is void if $\sigma(z[J]) = \overline{\mathbb{D}}$, so we may as well assume that J has the codimension 1 property, by Theorem (1.1.1). then the subspace zJ also has the codimension 1 property [3], so by Theorem (1.1.1) $\sigma(z[zJ]) = \sigma'(z[zJ])$. We show that

$$\sigma(z[zJ]) = \sigma(z[J]) \cup \{0\} \quad (4)$$

It is sufficient to prove this equality with the $\sigma(.)$'s replaced by $\sigma'(.)$'s. By definition, if I is an invariant subspace, $\lambda \in \mathbb{C} \setminus \sigma(z[I])$ if and only if

$$(\lambda - z)L_a^2(\mathbb{D}) + I = L_a^2(\mathbb{D})$$

Clearly, the weak spectrum has the monotonicity property that $\sigma'(z[I']) \supset \sigma'(z[I])$ if I' is another invariant subspace with $I' \subset I$. From this we see that $\sigma'(z[zJ]) \supset \sigma'(z[J])$, and it is not difficult to see that $0 \in \sigma'(z[zJ])$ directly from the definition. For these reasons, to verify (4) we just need to show $\sigma'(z[zJ]) \subset \sigma'(z[J]) \cup \{0\}$. To this end, let us take a $\lambda \in \mathbb{C} \setminus \sigma'(z[J]) \setminus \{0\}$, and try to show that $\lambda \in \mathbb{C} \setminus \sigma'(z[zJ])$. By the definition of the spectrum, we have that

$$(\lambda - z)L_a^2(\mathbb{D}) + J = L_a^2(\mathbb{D}),$$

So by multiplying both sides by z, we have in particular

$$(\lambda - z)L_a^2(\mathbb{D}) + zJ \supset zL_a^2(\mathbb{D})$$

There are functions in $(\lambda - z)L_a^2(\mathbb{D})$ that do not vanish at 0, for instance the function $\lambda - z$ itself, so that since $zL_a^2(\mathbb{D})$ has codimension 1 in $L_a^2(\mathbb{D})$, we must in fact have

$$(\lambda - z)L_a^2(\mathbb{D}) + zJ = L_a^2(\mathbb{D}).$$

This show that $\lambda \in \mathbb{C} \setminus \sigma'(z[zJ])$, as asserted.

We prove the assertion of the Lemma. Let $G_J \in J \ominus zJ$ have norm 1 then the kernel representation formula

$$G_J(\lambda) = \langle G_J, (1 - \bar{\lambda}z)^{-2} \rangle_{L^2}, \quad \lambda \in \mathbb{D},$$

generalizes to

$$G_J(\lambda) = \langle G_J + zJ, \left(1 - \bar{\lambda}z[zJ]\right)^{-2}(1 + zJ) \rangle_{L_\alpha^2/zJ},$$

where $1 + zJ$ denotes the coset containing the constant function 1 in the quotient space $L_\alpha^2(\mathbb{D}) \,/\, zJ$, and we see that the expression on the right-hand side is a well-defined holomorphic function in the variable λ on the set

$$\left\{z \in \mathbb{C}: {}^{1}\!/_{\bar{z}} \notin \sigma(z[zJ])\right\},$$

which coincides with

$$\left\{z \in \mathbb{C}: {}^{1}\!/_{\bar{z}} \notin \sigma(z[J])\right\},$$

because the additional point 0 in (4) now corresponds to the point at infinity. The functions G_J were studied in [7], for instance, it is clear that G_J has modulus ≥ 1 at every boundary point to which it extends continuously.

Lemma (1.1.3): [1] Let $f \in L_a^2(\mathbb{D})$ be such that on an open disk $D(z_0, \rho)$, centered at $z_0 \in \mathbb{T}$, with radius $\rho > 0$, we have

$$|f(z)| > \varepsilon, \qquad z \in \mathbb{D} \cap D(z_0, \rho),$$

for some constant $\varepsilon > 0$. Then the exists a bounded analytic function g on $\mathbb{D}$ such that

$$\frac{1}{2} < |f(z)g(z)| < 2, \qquad z \in \mathbb{D} \cap D(z_0, \rho'),$$

for some smaller radius ρ', $0 < \rho' < \rho$.

Proof: Consider the function $1/f$, which is homomorphic, zero-free, and bounded on $\mathbb{D} \cap D(z_0, \rho)$, we are now in the whole unit disk $\mathbb{D}$. On the region $\mathbb{D} \cap D(z_0, \rho)$, we are now in a situation where we may apply the standard Nevanlinna theory, to show that the harmonic function $\log|\varepsilon/f|$ has boundary values in the sense of distribution theory on $\mathbb{D} \cap D(z_0, \rho)$, and these boundary values form a negative Borel measure μ. We may then pick a slightly smaller radius ρ'', $0 < \rho'' < \rho$, and let φ be the Poisson extension to the whole disk $\mathbb{D}$ corresponding to the part of the measure μ that falls upon the arc $\mathbb{T} \cap D(z_0, \rho'')$. The negative measure μ is finite on that arc, because we can map $\mathbb{D} \cap D(z_0, \rho)$ conformally onto

$\mathbb{D}$, and on $\mathbb{D}$, and the mapped measure on $\mathbb{T}$ corresponding to μ must be bounded; the rest is an exercise in conformal mapping. we now find a bounded holomorphic function g on $\mathbb{D}$ having $|g| = \varepsilon^{-1} \exp(\varphi)$ on $\mathbb{D}$,and by construction and the Schwarz reflection principle , fg extends holomorphically across the arc $\mathbb{T} \cap D(z_0, \rho'')$, and has modulus 1 on it. The function fg clearly meets the assertion, for some small radius ρ'.

Theorem (1.1.4):[1] Let J be an invariant subspace of $L_a^2(\mathbb{D})$. Then $\sigma'(z[J]) = Z_*(J)$.

Proof: Richter [3] has shown that

$$\sigma'(z[J]) \cap \mathbb{D} = Z_*(J) \cap \mathbb{D}$$

By Lemma (1.1.2) $Z_*(J) \cap \mathbb{T}$ is contained within $\sigma(z[J]) \cap \mathbb{T}$. this entails that $Z_*(J) \cap \mathbb{T} \subset \sigma'(z[J]) \cap \mathbb{T}$, for the following reasons. if J fails to have the codimension 1 property, then by Theorem (1.1.1) $\sigma'(z[J]) \supset \mathbb{T}$, which makes the assertion trivial. If, on the other hand, J dose have the codimension 1 property, then $\sigma(z[J]) = \sigma'(z[J])$, and all is well.

The rest of the proof is devoted to obtaining the reverse inclusion

$$Z_*(J) \cap \mathbb{T} \supset \sigma'(z[J]) \cap \mathbb{T}.$$

Let f be a function in J, and suppose there exists a point $\lambda \in \mathbb{T}$ such that for some disk centered at λ with radius $R > 0$,

$$D(\lambda, R) = \{z \in \mathbb{C}: |z - \lambda| < R\},$$

we have

$$1/2 < |f(z)| < 2, \qquad z \in D(\lambda, R) \cap \mathbb{D},$$

Such a function f exists in J if and only if $\lambda \in \mathbb{T} \setminus Z_*(J)$, by Lemma(1.1.3). we need to show that $\lambda \notin \sigma'(z[J])$; this amounts to proving that

$$(\lambda - z)L_a^2(\mathbb{D}) + J = L_a^2(\mathbb{D}).$$

In other words, we need to show that for every $g \in L_a^2(\mathbb{D})$ and $h \in L_a^2(\mathbb{D})$ can be found such that

$$(\lambda - z)h - g \in J.$$

Fix three real parameters r_1, r_2, r_3 with $0 < r_1 < r_2 < r_3 < R$, and let

$$D(\lambda, r_j) = \{z \in \mathbb{C}: |z - \lambda| < r_j\}, \qquad j = 1,2,3,$$

Be the disk around λ with radius r_j. Let χ_λ be an infinitely differentiable compactly supported function on $\mathbb{C}$ with values between 0 and 1, which vanishes off the disk $D(\lambda, r_2)$ and has value 1 on the smaller disk $D(\lambda, r_1)$. Let the function q_λ solve the $\bar{\partial}$-problem

$$\bar{\partial} q_\lambda(z) = \frac{g(z)\bar{\partial}\varkappa_\lambda(z)}{(\lambda - z)f(z)}, \qquad z \in \mathbb{D}; \tag{5}$$

just put

$$q_\lambda(z) = \int_{\mathbb{D}} \frac{g(\zeta)\bar{\partial}\varkappa_\lambda(\zeta)}{(\lambda-\zeta)(z-\zeta)f(\zeta)} dS(\zeta), \quad z \in \mathbb{C}. \tag{6}$$

Note that since the right-hand side of (5) is in $L^2(\mathbb{D}, dS)$, and since we are in fact considering the convolution of that $L^2(\mathbb{D}, dS)$ function with the $\bar{\partial} - \ker(\pi z)^{-1}$, which locally belongs to L^q for every $q < 2$, we see that q_λ, as defined by (6), belongs to $L^p(\mathbb{D}, dS)$ for all $p < \infty$. One more thing that is immediate is that q_λ is holomorphic off the closure of $D(\lambda, r_2) \cap \mathbb{D}$, and in particular bounded on $\mathbb{C} \setminus D(\lambda, r_3)$ ($q_\lambda(z)$ tends to 0 as $|z| \to \infty$). We consider the function

$$P_\lambda(z) = -g(z)\varkappa_\lambda(z)/f(z) + (\lambda - z)q_\lambda(z), \quad z \in \mathbb{D},$$

which belongs to $L^2(\mathbb{D}, dS)$, because f is bounded away from 0 on the support of $\varkappa_\lambda$. Moreover, P_λ is holomorphic on $\mathbb{D}$, since

$$\bar{\partial}P_\lambda(z) = -g(z)\bar{\partial}\varkappa_\lambda(z)/f(z) + (\lambda - z)\bar{\partial}q_\lambda(z) = 0, \quad z \in \mathbb{D},$$

Let us assume that we know that fP_λ belongs to J. We then put

$$h(z) = g(z)\frac{1-\varkappa_\lambda(z)}{\lambda - z} + f(z)q_\lambda(z), \quad z \in \mathbb{D}, \tag{7}$$

and note that f is bounded on $D(\lambda, R) \cap \mathbb{D}$, and q_λ is bounded on $\mathbb{C} \setminus D(\lambda, r_3)$ and belongs to $L^2(dS)$ on $\mathbb{D}$, so that the product fq_λ clearly is in $L^2(\mathbb{D}, dS)$. The function h thus belongs to $L^2(\mathbb{D}, dS)$, and since

$$\bar{\partial}h(z) = -g(z)\bar{\partial}\varkappa_\lambda(z)/(\lambda - z) + f(z)\bar{\partial}q_\lambda(z) = 0, \quad z \in \mathbb{D},$$

h belongs to $L^2_a(\mathbb{D})$. To check that the function h dose what we set out for it to do, observe that

$$(\lambda - z)h(z) - g(z) = -\varkappa_\lambda(z)g(z) + (\lambda - z)f(z)q_\lambda(z) = f(z)P_\lambda(z),$$

so that the assertion is immediate once we know that fP_λ is in J. The way P_λ is constructed, this function is bounded on $\mathbb{D} \setminus D(\lambda, r_3)$, and $L^2(dS)$ on $\mathbb{D} \cap D(\lambda, r_3)$. The properties of function f complement those of P_λ: f is bounded on $D(\lambda, R)$, and $L^2(dS)$ on $\mathbb{D} \setminus D(\lambda, R)$. Using this information, it is not difficult to show that

$$f(z)P_\lambda(\rho z) \to f(z)P_\lambda(z), \quad as\ 1 > \rho \to 1;$$

in the norm of $L^2_a(\mathbb{D})$. Since the function $f(z)P_\lambda(\rho z)$ belong to J, for all ρ with $0 < \rho < 1$, we see that $fP_\lambda \in J$.

Corollary (1.1.5):[1] Let A be a zero sequence in $\mathbb{D}$ for a function in $L^2_a(\mathbb{D})$, and consider the associated invariant subspace

$$\mathcal{I}(A) = \{f \in L^2_a(\mathbb{D}): f = 0 \text{ on } A\};$$

counting multiplicities when necessary. Then $\sigma(z[\mathcal{J}(A)]) = \bar{A}$, the closure of A in $\bar{\mathbb{D}}$.

Proof: Invariant subspace of the type $\mathcal{J}(A)$ always have the codimension 1 property [3], and consequently $\sigma(z[\mathcal{J}(A)]) = \sigma'(z[\mathcal{J}(A)])$, by Theorem(1.1.1) so, by Theorem (1.1.4) all we need to do is show that $Z_*(\mathcal{J}(A)) = \bar{A}$. Clearly, $Z_*(\mathcal{J}(A)) \supset \bar{A}$; in [7], there exists a function G_A which vanishes precisely on A in $\mathbb{D}$, extends holomorphically across the set $\mathbb{T} \setminus \bar{A}$, and has modulus ≥ 1 there.

Section (1.2): Bergman Space Having the Codimension Two Property

Following Kristion Seip [7,8,10],we say that a sequence $A = \{a_j\}_j$ is a sampling sequence for $L^2_a(\mathbb{D})$ provided we can find positive constants K_1, K_2 such that

$$K_1 \int_{\mathbb{D}} |f(z)|^2 dS(z) \leqq \sum_j \left(1 - |a_j|^2\right)^2 |f(a_j)|^2 \leqq K_2 \int_{\mathbb{D}} |f(z)|^2 dS(z)$$

holds for all $f \in L^2_a(\mathbb{D})$, the sequence A is said to be an interpolating sequence for $L^2_a(\mathbb{D})$, provided that to every l^2 sequence $\{w_j\}_j$,there exists a function $f \in L^2_a(\mathbb{D})$ having

$$\left(1 - |a_j|^2\right) f(a_j) = w_j \qquad \text{for all } j.$$

If f is a holomorphic function on $\mathbb{D}$, we write $Z(f)$ for the sequence of zero f, counting multiplicities, provided f does not vanish identically. If f vanishes identically, we write $Z(f) = \mathbb{D}$. A sequence of points in $\mathbb{D}$ is called a Bergman space zero sequence provided it coincides with $Z(f)$ for some nonidentically vanishing function $f \in L^2_a(\mathbb{D})$. Every interpolating sequence A for $L^2_a(\mathbb{D})$ is also a Bergman space zero sequence: just take an interpolant for the sequence $w_1 = 1, w_j = 0$ for all other j, and multiply this function by $z - a_1$ to get a non identically vanishing function that vanishes on the sequence A. This actually only shows that A must be a subsequence of a Bergman space zero sequence, but it is well known, and not too hard to show, that every subsequence of Bergman space zero sequence is itself a zero sequence [7,8,9]. However, the union of two zero sequence need not be a zero sequence [9]; in fact, it may be so far away from being a zero sequence as to be a sampling sequence, as we shall see in Theorem(1.2.1).

Theorem (1.2.1):[4] There exists a sampling sequence for $L^2_a(\mathbb{D})$ which is the union of two disjoint zero sequences.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are tow Hilbert spaces, it is standard to denote by $\mathcal{H}_1 \oplus \mathcal{H}_2$ their direct sum, that is, the linear space of all pairs (x_1, x_2), with $x_1 \in \mathcal{H}_1$ and $x_2 \in \mathcal{H}_2$, supplied with the norm

$$\|(x_1, x_2)\|_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \left(\|x_1\|^2_{\mathcal{H}_1} + \|x_2\|^2_{\mathcal{H}_2}\right)^{\frac{1}{2}},$$

which makes $\mathcal{H}_1 \oplus \mathcal{H}_2$ a Hilbert space. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are closed subspaces of a bigger Hilbert space $\mathcal{H}$, one can consider their sum $\mathcal{H}_1 + \mathcal{H}_2$, and in case $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal subspace, one then replaces the plus sign(+) with a direct plus ($\oplus$) sign. We shall take the liberty to write $\mathcal{H}_1 \oplus \mathcal{H}_2$ provided the closed subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ have $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$, and the direct sum norm on $\mathcal{H}_1 \oplus \mathcal{H}_2$ is equivalent to the restriction of the $\mathcal{H}$-norm to $\mathcal{H}_1 + \mathcal{H}_2$,given that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$, this last property is equivalent to requiring the sum $\mathcal{H}_1 + \mathcal{H}_2$ to be closed in $\mathcal{H}$, by the closed graph theorem.

Proposition (1.2.2):[4] Let A and B be two disjoint zero sequences whose union is a sampling sequence. Then $\mathcal{I}(A) \cap \mathcal{I}(B) = \{0\}$, and the subspace $\mathcal{I}(A) + \mathcal{I}(B)$ is closed in $L^2_a(\mathbb{D})$, allowing us to write $\mathcal{I}(A) \oplus \mathcal{I}(B)$ instead of $\mathcal{I}(A) + \mathcal{I}(B)$.the subspace $\mathcal{I}(A) \oplus \mathcal{I}(B)$ is z-invariant, and it has the codimension 2 property.

Proof: Write $C = A \cup B$, with $C = \{c_j\}_j$, and let $f \in \mathcal{I}(A)$ and $g \in \mathcal{I}(B)$ be arbitrary. Clearly, a sampling sequence cannot be a zero sequence, and hence $\mathcal{I}(A) \cap \mathcal{I}(B) = \{0\}$. our next job is to check that the subspace $\mathcal{I}(A) + \mathcal{I}(B)$ is closed in $L^2_a(\mathbb{D})$. Denote by K_1, K_2 the positive constants associated with the sampling property of the sequence C.

$$K_1 \int_{\mathbb{D}} |f(z)|^2 dS(z) \leqq \sum_j \left(1 - |c_j|^2\right)^2 |f(c_j)|^2 \leqq K_2 \int_{\mathbb{D}} |f(z)|^2 dS(z).$$

Let $f \in \mathcal{I}(A)$ and $g \in \mathcal{I}(B)$ be arbitrary. Then for every point c_j in the sequence C, we have

$$|f(c_j) + g(c_j)|^2 = |f(c_j)|^2 + |g(c_j)|^2$$

so that if we use the sampling property of sequence C, we get

$$\|f + g\|^2_{L^2} = \int_{\mathbb{D}} |f(z) + g(z)|^2 dS(z) \geqq K_2^{-1} \sum_j \left(1 - |c_j|^2\right)^2 |f(c_j) + g(c_j)|^2$$
$$= K_2^{-1} \sum_j \left(1 - |c_j|^2\right)^2 \left(|f(c_j)|^2 + |g(c_j)|^2\right) \geqq K_1 \, K_2^{-1}\left(\|f\|^2_{L^2} + \|g\|^2_{L^2}\right).$$

The property we shall focus on is that with $\varepsilon = K_1 \, K_2^{-1} > 0$

$$\|f + g\|^2_{L^2} \geqq \varepsilon\left(\|f\|^2_{L^2} + \|g\|^2_{L^2}\right), \quad f \in \mathcal{I}(A), \; g \in \mathcal{I}(B).$$

It implies the assertion that $\mathcal{I}(A) + \mathcal{I}(B)$ is a closed subspace of $L^2_\alpha(\mathbb{D})$, justifying the change of notation to $\mathcal{I}(A) \oplus \mathcal{I}(B)$. Since the subspaces $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are z-invariant, their(direct) sum $\mathcal{I}(A) \oplus \mathcal{I}(B)$ is z-invariant as well. What remains to be done is to demonstrate that the codimension of $z(\mathcal{I}(A) \oplus \mathcal{I}(B))$ in $f \in \mathcal{I}(A)$ and $g \in \mathcal{I}(B)$ is 2. Note that if $f \in \mathcal{I}(A)$ and $g \in \mathcal{I}(B)$, then

$$z(f + g) = zf + zg \in z\mathcal{I}(A) \oplus z\mathcal{I}(B).$$

So that

$$z\left(\mathcal{I}(A) \oplus \mathcal{I}(B)\right) = z\mathcal{I}(A) \oplus z\mathcal{I}(B).$$

The direct sum sign is justified because $z\mathcal{I}(A)$ and $z\mathcal{I}(B)$ are closed subspace of $\mathcal{I}(A)$ and $\mathcal{I}(B)$, respectively. The subspaces $z\mathcal{I}(A)$ and $z\mathcal{I}(B)$ have codimension 1 in the spaces $\mathcal{I}(A)$

and $\mathcal{I}(B)$ [3], respectively and consequently, their direct sum $z\mathcal{I}(A) \oplus z\mathcal{I}(B)$ must have codimension 2 in $\mathcal{I}(A) \oplus \mathcal{I}(B)$.

Corollary (1.2.3):[4] There exist a z-invariant subspace of $L_a^2(\mathbb{D})$ which has the codimension 2 property. Moreover, this z-invariant subspace can be of the form $J = \mathcal{I}(A) + \mathcal{I}(B)$, where A and B are tow disjoint Bergman space zero sequences.

Theorem (1.2.4):[4] Let A and B be two disjoint Bergman spaces zero sequences. Then the smallest z-invariant subspace of $L_a^2(\mathbb{D})$ containing both $\mathcal{I}(A)$ and $\mathcal{I}(B)$ having the codimension 1 property is $L_a^2(\mathbb{D})$ itself.

Proof: Let us for convenience denote by J the smallest invariant subspace of $L_a^2(\mathbb{D})$ containing $\mathcal{I}(A)$ and $\mathcal{I}(B)$ with the codimension 1 property; the assertion we wish to prove is that $J = L_a^2(\mathbb{D})$.

Only one of sequences A and B can contain the point 0, since they are disjoint. Let $A = \{a_j\}_{j=1}^{\infty}$ be the one that does not contain 0. We may then consider the extremal function G_A for the problem

$$\sup\{ Re\, f(0) : f = 0 \text{ on } A, \|f\|_{L^2} \leqq 1\}$$

which has the property of vanishing only on the sequence A, among other things, according to [7]. Let A_N be the finite subsequence $A_N = \{a_j\}_{j=1}^{N}$, and let G_{A_N} be the extremal function associated with zero sequence A_N. We know from [7] that G_{A_N} is a rational function whose poles are located at the reflected points $\{1/\bar{a}_j\}_{j=1}^{N}$; it vanishes precisely at A_N in the unit disk $\mathbb{D}$, has $|G_{A_N}(z)| \geqq 1$ on the circle $\mathbb{T}$, has $\|G_{A_N}\|_{L^2} = 1$, and moreover, it has the expansive property

$$\|f\|_{L^2} \leqq \|G_{A_N} f\|_{L^2} \qquad f \in L_a^2(\mathbb{D}) \tag{8}$$

Denote by $Z(J)$ the common zero set in $\mathbb{D}$ of the functions in J. Since the sequences A and B are disjoint, we have $Z(J) = \phi$. Richter [3] has shown that if a z-invariant subspace I has the codimension 1 property, then if $\lambda \in \mathbb{D}$ is any point which does not belong to the common zero set $Z(I)$ of I, and $f \in I$ has $f(\lambda) = 0$, the function $f(z)/(z-\lambda)$ also belongs to I. Let us apply this argument repeatedly to the invariant subspace J, and the function G_A. We then obtain as a conclusion that G_A / G_{A_N} also belongs to J, for every positive integer N. But as $N \to \infty, G_A / G_{A_N} \to 1$ pointwise in $\mathbb{D}$, and by (8), with $f = G_A / G_{A_N}$, we have

$$\|G_A / G_{A_N}\|_{L^2} \leqq \|G_A\|_{L^2} = 1$$

So that, by [7], $G_A / G_{A_N} \to 1$ in norm as $N \to \infty$. Hence the constant function 1 has to belong to J as well. But the constant function 1 generates, as an invariant subspace, to whole Bergman space $L_a^2(\mathbb{D})$, and we arrive at the function $J = L_a^2(\mathbb{D})$.

Corollary (1.2.5):[4] Let A and B be two disjoint Bergman spaces zero sequences. If one of the sequences A and B does not accumulate at every point of the unit circle $\mathbb{T}$, then $\mathcal{I}(A) + \mathcal{I}(B)$ is dense in $L_a^2(\mathbb{D})$.

Proof: Let J be the norm closure of sum $\mathcal{I}(A)+\mathcal{I}(B)$, which is an invariant subspace in $L_a^2(\mathbb{D})$. In [1,3], the concept of the weak spectrum $\sigma'(z[J])$ associated with an invariant subspace I as introduced, and by [1], we see that for our particular choice $I=J$, we have

$$\sigma'(z[J]) \subset \bar{A} \cap \bar{B}.$$

Note that by assumption, the set $\bar{A} \cap \bar{B}$ is a proper closed subset of the unit circle $\mathbb{T}$. But Richter [1,3] has shown that weak spectra associated with invariant subspace not having the codimension 1 property must contain the whole unit circle ,and therefore, the invariant subspace J must necessarily have the codimension 1 property.

The pseudohyperbolic metric on $\mathbb{D}$ given by the expression

$$\varrho(z,\zeta) = \left|\frac{\zeta - z}{1-\bar{\zeta}z}\right|, \quad z,\zeta \in \mathbb{D}.$$

A sequence (or set) of points $A=\{\lambda_j\}_j$ in $\mathbb{D}$, finite or infinite, is said to be uniformly discrete provided that

$$\inf\{\varrho(a_j,a_k) : j \neq k\} > 0.$$

Clearly, a uniformly discrete sequence has to consist of distinct points, and for finite sequence, this is the only restriction.

If A is a sequence of points in the unit disk $\mathbb{D}$, and $\zeta \in \mathbb{D}$ an arbitrary point, let A_ζ denote the image of A under the conformal automorphism of the unit disk

$$\varphi_\zeta(z) = \frac{\zeta - z}{1-\bar{\zeta}z}, \quad z \in \mathbb{D}.$$

Associate with the sequence A_ζ the function $n(r,A_\zeta)$, which counts the number of points of A_ζ contained with the disk

$$\{z \in \mathbb{D} : |z| < r\}.$$

Moreover, we shall need the definite integral

$$N(r,A_\zeta) = \int_0^r n(r,A_\zeta)dt, 0<r<1.$$

If $A(r)$ now stands for the function

$$A(r) = \log\frac{1+r}{1-r}, \quad 0<r<1,$$

Seip defines his upper density of A as

$$D^+(A) = \limsup_{1>r\to 1} \sup_{\zeta\in D} \frac{N(r,A_\zeta)}{A(r)},$$

and this lower density of A as

$$D^-(A) = \liminf_{1>r\to 1} \inf_{\zeta\in D} \frac{N(r,A_\zeta)}{A(r)}.$$

For the special case of the un weighted Bergman space.

Theorem (1.2.6)(Siep):[4] A sequence A of distinct points in $\mathbb{D}$ is sampling for $L_a^2(\mathbb{D})$ if and only if it can be expressed as a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence A' for which $D^-(A') > 1/2$.

Theorem (1.2.7)(Siep):[4] A sequence A of distinct points in $\mathbb{D}$ is interpolating for $L_a^2(\mathbb{D})$ if and only if it is uniformly discrete and $D^+(A) < 1/2$.

Theorem (1.2.8):[4] The exists a sampling sequence C for $L_a^2(\mathbb{D})$ which is the union of two disjoint interpolating sequences A and B for $L_a^2(\mathbb{D})$.

Proof: The upper half plane

$$\mathbb{U} = \{z \in \mathbb{C}: \operatorname{Im} z > 0\}$$

is mapped conformally onto the unit disk $\mathbb{D}$ by the Moebius mapping

$$\varphi(z) = (z-i)/(z+i), \qquad z \in \mathbb{C}\setminus\{-i\}.$$

We will construct three sequences $A', B',$ and C' in $\mathbb{U}$, and then define $A = \varphi(A')$, $B = \varphi(B')$, and $C = \varphi(C')$. Fix two real-valued parameters $\beta, \beta > 1$, and $\gamma, \gamma > 0$, with the property that

$$2\pi < \gamma \log \beta < 4\pi.$$

The sequence C' will consist of all points in the upper half plane $\mathbb{U}$ of the from

$$c_{j,k} = \beta^j(k\gamma + i),$$

where j, k range over the integers, and i, as always, is the square root of -1. The subsequence A' will consist of all points

$$a_{j,k} = \beta^j(2k\gamma + i),$$

with j, k ranging over the integers, and $\acute{B}$ will be the sequence of all points

$$b_{j,k} = \beta^j\big((2k+1)\gamma + i\big),$$

again with j, k ranging over the integers, the sequences A and C are very regular, and Siep[10] has already computed their densities:

$$D^+(C) = D^-(C) = \frac{2\pi}{\gamma \log \beta} > 1/2,$$

and

$$D^+(A) = D^-(A) = \frac{\pi}{\gamma \log \beta} < 1/2.$$

The sequence B is also very regular, although it does not fit the class of regular sequences for which Seip has calculated the densities, and it is in fact possible verify that

$$D^+(B) = D^-(B) = \frac{\pi}{\gamma \log \beta} < 1/2$$

By Theorem (1.2.6) and (1.2.7) A and B are interpolating sequences, and C is a sampling sequence.

To carry out a construction of an invariant subspace I of $L_a^2(\mathbb{D})$ having the codimension n property, that is, zI should have codimension n in I, along the lines, we need to find invariant subspaces $I_1, I_2, \dots, I_n$, all having the codimension 1 property,

$$\|f_1 + \cdots + f_n\|_{L^2} \geqq \varepsilon(\|f_1\|_{L^2} + \ldots + \|f_n\|_{L^2}) \tag{9}$$

holds for some $\varepsilon > 0$ and for all $f_1 \in I_1, f_2 \in I_2, \dots, f_n \in I_n$. Then the (direct) sum

$$I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$$

has the codimension n property. There are several ways to get such a collection of invariant subspace; the one outlined here was suggested to me by Boris Korenbllum [11].

Let the sequence C' be the regular sequence in the upper half plane $\mathbb{U}$ appearing in the proof of Theorem (1.2.8). only this time the parameters $\beta, \beta > 1$, and $\gamma, \gamma > 0$, must be chosen such that

$$4\pi(n-1)/n < \gamma \log \beta < 4\pi.$$

Let $B_1', \dots, B_n'$ be subsequences of C', the sequence $B_m (m = 1, \dots, n)$ consisting of all the points

$$b_{j,k}^m = \beta^j\big((nk + m)\gamma + i\big),$$

with j, k ranging over the integers. Let $A'_m = C' \setminus B'_m \, for \, m = 1, \dots, n$ and put $A_m = \varphi(A'_m)$, where φ is the Moebius mapping

$$\varphi(z) = (z - i)/(z + i), \quad z \in \mathbb{C} \setminus \{-i\},$$

which sends $\mathbb{U}$ onto $\mathbb{D}$. Note that by the regular nature of the sets A_m, it can in fact be shown that

$$D^+(A_m) = D^-(A_m) = \frac{2(n-1)\pi}{n\gamma \log \beta} < 1/2,$$

so that by Seip's Theorem (1.2.6) A_m is interpolating for $L_a^2(\mathbb{D})$, for each m. Since the sequence $C = \varphi(\acute{C})$ is sampling, an argument analogous to the one used in the proof of Proposition (1.2.2) now shows that the invariant subspaces

$$I_m = \mathcal{I}(A_m), \quad m = 1, \ldots, n,$$

meet condition (9) for some constant $\varepsilon > 0$.

Chapter 2

Spectra and Index of Invariant Subspaces

We consider M_z−invariant subspaces $\mathcal{M} \subseteq H_d^2(\mathcal{D})$. The fiber dimension of $\mathcal{M}$ is defined to be $\sup_{\lambda\in\mathbb{B}_d} \dim\{f(\lambda): f \in \mathcal{M}\}$. We show that if $\mathcal{M}$ has finite positive fiber dimension m, then the essential Taylor spectrum of $M_z|\mathcal{M}, \sigma_e(M_z|\mathcal{M})$, equals $\partial\mathbb{B}_d$ plus possibly a subset of the zero set of a nonzero bounded analytic function on $\mathbb{B}_d$ and $\operatorname{ind} M_z - \lambda|\mathcal{M} = (-1)^d m$ for every $\lambda \in \mathbb{B}_d\backslash\sigma_e(M_z|\mathcal{M})$. As a Corollary we show that if $T = (T_1, .., T_d)$ is a pure d-contraction of finite rank, then $\sigma_e(T) \cap \mathbb{B}_d$ is contained in the zero set of a nonzero bounded analytic function and $(-1)^d$ in $\mathrm{d}(T-\lambda) = k(T)$ for all $\lambda \in \mathbb{B}_d\backslash\sigma_e(T)$. Here $k(T)$ denotes Arveson's curvature invariant. We also show that for $d > 1$ there are such d-contractions with $\sigma_e(T) \cap \mathbb{B}_d \neq \emptyset$. These results answer a question of Arveson, [19]. We also show related results for the Hardy and Bergman spaces of the unit ball and unit poly disc of $\mathbb{C}^d$.

Section (2.1): Translation Invariant Spaces

Let $l_2(-\infty,\infty)$ be the classical Hilbert space of complex valued functions defined on the discrete group of integers, and let $L_2(-\infty,\infty)$ be the Hilbert space of complex valued square-integrable functions defined on the continuous group of real numbers. We use the symbol $\widetilde{H}_d$ to denote the subspace of all functions in the discrete space $l_2(-\infty,\infty)$ which vanish for negative values of their argument, and the symbol $\widetilde{H}_c$ to denote the subspace of all functions in the continuous space $L_2(-\infty,\infty)$ which vanish for negative values of their argument. Let $\tilde{L}$ be any subspace of $\widetilde{H}$ which is invariant with respect to left translation, i.e., $\tilde{h}(x)$ in $\tilde{L}$ implies that the projection of $\tilde{h}(x+\tau)$ on $\widetilde{H}$ belongs to $\tilde{L}$ for all positive τ. Let τ denote the restriction of the left unit shift operator to the nonnegative integers or real numbers, i.e.,

$$(T\tilde{h})(x) = \begin{cases} \tilde{h}(x+\tau) & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and let $\sigma(T)$ denote the spectrum of T acting on $\tilde{L}$. $\sigma(T)$ is defined to be the set of all complex numbers λ for which $(T-\lambda)^{-1}$ does not exist as a bounded linear operator, and the point spectrum of T, denoted by $\sigma_p(T)$ is taken to be the set of complex numbers λ for which $(T-\lambda)$ is not one-to-one.[2]

Theorem(2.1.1):[12] Every nontrivial closed R space is of the form GH, where $G(z)$ is an analytic function in the interior of the unit disc and $|G(z)| \leq 1$ there. For z restricted to the unit circle, $|G(z)| = 1$ almost everywhere. The function $G(z)$ is uniquely determined by the space R, except for multiplication by a complex constant of modulus 1.

Let $\widetilde{H}$ denote the Hilbert space of one sided sequences $h = \{h_n\}_0^\infty$ with inner product

$$(h,g) = \sum_{n=0}^{\infty} h_n g_n^* < \infty,$$

where the asterisk which appears is used to denote complex conjugation.

The Fourier transform of h, denoted by $F(h)$, is taken to be the function

$$h(z) = \sum_{n=0}^{\infty} h_n z^n = F(h),$$

which is holomorphic in the interior of the unit disc and satisfies the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \left|hre^{i\theta}\right|^2 d\theta \leq \|h\|^2 \quad 0 \leq r < 1,$$

According to well-known properties, the radial limit

$$h\left(e^{i\theta}\right) = \lim_{r\to 1-0} h(re^{i\theta})$$

exists almost everywhere and

$$\|h\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|h(e^{i\theta})\right|^2 d\theta.$$

Let H denote the Fourier transform of the space $\widetilde{H}$ equipped with inner product

$$(h, g) = \frac{1}{2\pi}\int_0^{2\pi} h\left(e^{i\theta}\right) g^*(e^{i\theta}) d\theta.$$

Under these conditions the Fourier transform is a unitary mapping of $\widetilde{H}$ onto H. It is easy to see that the orthogonal complement $H^{\perp}$ of H with respect to the space of square integrable functions on the unit circle is given by

$$H^{\perp} = z^* H^*,$$

and every function in $H^{\perp}$ can be analytically continued into the complement of the closed unit disc.

Let $\tilde{R}$ be any subspace of $\widetilde{H}$ which is invariant under right translation, i.e. whenever the sequence $(h_0, h_1, \ldots)$ belongs to $\tilde{R}$ we require that the sequence $(0,0, \ldots, h_0, h_1, \ldots)$ with n initial zeros should also belong to $\tilde{R}$ for all positive n. Let $\tilde{L}$ denote the orthogonal complement of $\tilde{R}$ with respect to $\widetilde{H}$ and let $T: \tilde{L} \to \tilde{L}$ be the unit left translation operator defined by

$$T(h_n) = h_{n+1} \quad n \geq 0.$$

If we set $\tilde{T} = FTF^{-1}$ an easy computation shows that $\tilde{T}$ is expressed by the equation

$$\left(\tilde{T}h\right)(z) = \frac{h(z) - h(0)}{z}$$

Since F is unitary, the spectrum of T acting on $\tilde{L}$ may be determined directly from the spectrum $\tilde{T}$ acting on L.

Lemma (2.1.2):[12] Let $f(z)$ be analytic in the interior of the unit disc. Then $f(z)$ belong to L if and only if it is square integrable on $|z| = 1$ and the function $G(z)z^*f^*(z)$ can be continued as an analytic function throughout the interior of the unit disc.

Proof: Notice that $f(z)$ belongs to L if and only if it is square integrable on the unit circle and satisfies the condition $(G^*f, H) = 0$. This means that $f(z)$ belongs to L if and only if there is a function $k(z)$ in H with the property that

$$G^*(z)f(z) = z^*k^*(z) \tag{1}$$

when $|z| = 1$. Taking the complex conjugate of (1), we see that $f(z)$ belongs to L if and only if the function $G(z)z^*f^*(z)$ is square integrable on $|z| = 1$ and can be analytically continued as a function k in H, i.e.throughout the interior of the unit disc.

Theorem (2.1.3):[12] Let λ be any complex number whose modulus is less than 1. Then $\lambda \notin \sigma(T)$ if and only if $G(\lambda^*) \neq 0$.

Proof: We first show that the condition $G(\lambda^*) \neq 0$ implies that the equation

$$\frac{f(z) - f(0)}{z} - \lambda f(z) = l(z) \tag{2}$$

has a unique solution $f(z)$ in L for every $l(z)$ in L. It will then follow from the interior mapping principle [14] that $\lambda \notin \sigma(T)$.

Solving (2) formally, we find

$$f(z) = \frac{zl(z) - f(0)}{1 - \lambda z} \tag{3}$$

which is valid when $|z| = 1$. Since we assumed that $l(z)$ belongs to L, it follows from Lemma (2.1.2) that the function

$$m(z) = G(z)z^*l^*(z) \tag{4}$$

defined for $|z| = 1$ can be analytically continued throughout the interior of the unit disc as a function in H. Taking the complex conjugate of (3), multiplying both sides by $z^*G(z)$, and then substituting (4) into the resulting expression, we finally obtain the relation

$$G(z)z^*f^*(z) = \frac{m(z) + G(z)f^*(0)}{z - \lambda^*} \tag{5}$$

which is valid a.e. whenever $|z| = 1$.

The right hand side of (5) is an analytic function throughout the interior of the unit disc whose L_2 norm is bounded on the unit circle if and only if

$$f^*(0) = -\frac{m(\lambda^*)}{G(\lambda^*)} \tag{6}$$

Using the fact that $G(\lambda^*) \neq 0$, we may define $f^*(0)$ by (6) and substitute this value into the right hand side of (5) to obtain a function in H. Since the left hand side of (5) agrees with this function whenever $|z| = 1$, it follows that the left hand side of (5) can be extended as a function in H as well. We conclude that $\lambda \neq \sigma(T)$.

Conversely, if $G(\lambda^*) = 0$, we may express $G(z)$ as

$$G(z) = \frac{z - \lambda^*}{1 - \lambda z} G_1(z),$$

where $|G_1(z)| = 1$ a.e. $|z| = 1$. From this factorization it follows that

$$\frac{z^*}{(1 - \lambda z)^*} G(z) = \frac{G_1(z)}{1 - \lambda z}$$

whenever $|z| = 1$. Consequently, we deduce from Lemma (2.1.2) that L already contains the eigenfunction $(1 - \lambda z)^{-1}$. This means that $\lambda \in \sigma_\rho(T)$.

Lemma (2.1.4):[12] Let $u(z)$ belong to the space H and let $v(z)$ be analytic in any domain, $\mathcal{D}$, contained in the complement of the unit disc whose boundary contains the circular arc$\Gamma = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}$. Assume that no inner point of Γ is the limit point of boundary points of $\mathcal{D}$ which do not belong to Γ. If

$$\lim_{r \to 1-0} u(re^{i\theta}) - v(r^{-1}e^{i\theta}) = 0 \quad for\ \theta_1 < \theta < \theta_2$$

and

$$\int_{\theta_1}^{\theta_2} \left|u(re^{i\theta}) - v(r^{-1}e^{i\theta})\right|^2 d\theta \leq M \text{ for } 0 < r_0 < r < 1$$

then $u(x)$ and $v(z)$ are analytic continuations of each other across Γ.

Proof: Let $C(r_2)$ be the boundary of the circular sector whose vertices are $\{r_k e^{i\gamma_j}\}_{k,j-1,2}$ and let $\tilde{C}(r_2)$ be the boundary of the sector whose vertices are $\{r_k^{-1} e^{i\gamma_j}\}_{k,j-1,2}$ where $\theta_1 < \gamma_1 < \gamma_2 < \theta_2$ and $r_0 < r_1 < r_2 < 1$. Since $u(z)$ belongs to H, we may assume without loss of generality that it has a finite radial limit at the points $e^{i\gamma_1}$ and $e^{i\gamma_2}$.If z is any point inside $C(r_2)$ we may write

$$u(z) = \frac{1}{2\pi i} \oint_{C(r_2)} \frac{u(\xi)d\xi}{z - \xi} + \frac{1}{2\pi i} \oint_{\tilde{C}(r_2)} \frac{v(\xi)d\xi}{z - \xi}.$$

Now pass to the limit as r_2 goes to 1. By virtue of the conditions stated in the hypothesis, we may apply Schwarz's inequality and the Lebesgue dominated convergence theorem to obtain the relation

$$\lim_{r_2 \to 1-0} \int_{\gamma_1}^{\gamma_2} \frac{u\left(r_2 e^{i\theta}\right) - v\left({r_2}^{-1} e^{i\theta}\right)}{r_2 e^{i\theta} - z} d\theta = 0.$$

This implies that $u(z)$ can be expressed in the form

$$u(z) = \frac{1}{2\pi i} \oint_{C(r_1)} \frac{w(\xi) d\xi}{z - \xi} \tag{7}$$

where $C(r_1)$ is the sector whose vertices are $\{r_1 e^{i\gamma_j}\}_{j=1,2}$ and $\{{r_1}^{-1} e^{i\gamma_j}\}_{j=1,2}$ while

$$\omega(\xi) = \begin{cases} u(\xi) & \text{if } |\xi| \leq 1 \\ v(\xi) & \text{if } |\xi| > 1. \end{cases}$$

A similar argument shows that $v(z)$ may also be represented by (7). Thus $u(z)$ and $v(z)$ are analytic continuations of each other across Γ.

Theorem (2.1.5):[12] Let λ be any complex number such that $|\lambda| = 1$. If there exists a $\delta > 0$ such that $|G(z)| > \delta$ in the intersection of some neighborhood of λ^* with the interior of the unit disc, then $\lambda \notin \sigma(T)$.

Proof: Let $\bar{G}(z)$ and $\bar{m}(z)$ be analytic continuations of the functions $1/G^*(z)$ and $z^* l^*(z)/G^*(z)$ defined for $|z| = 1$, into the complement of the closed unit disc. Since $|G(z)| > \delta$ in the intersection of some neighborhood of λ with the unit disc, it follows from Lemma (2.1.4) that the functions $\bar{G}(z)$ and $\tilde{m}(z)$ are analytic continuations of $G(z)$ and $m(z)$ defined in Theorem (2.1.3) throughout some circular neighborhood of λ^*.As in the proof of Theorem (2.1.3), if we set

$$f^*(0) = -\frac{m(\lambda^*)}{G(\lambda^*)}$$

and substitute this value into (5), it is evident that the right hand side of (5) is a function in H. Using our previous reasoning, this means that the left hand side of (5) can also be extended as a function in H.

Theorem (2.1.6):[12] Let λ be any complex number of modulus 1 with the property that

$$\lim_{z \to \lambda^*} \inf |G(z)| = 0$$

when $|z| = 1$. Then $\lambda \in \sigma(T)$.

Proof: Let $\lambda_n \to \lambda$ be any sequence of complex numbers having modulus less than 1 for which $\lim_{n\to\infty} |G(\lambda_n^*)| = 0$. It remains to be shown that $\lambda \in \sigma(T)$.

We first consider the identity

$$\frac{1 - |\lambda_n|^2}{1 - \lambda_n z} = (1 - |\lambda_n|^2) \frac{1 - G^*(\lambda_n^*) G(z)}{1 - \lambda_n z} + (1 - |\lambda_n|^2) \frac{G^*(\lambda_n^*) G(z)}{1 - \lambda_n z}. \tag{8}$$

According to the Theorem(2.1.1) , the second function on the right hand side of (8) belongs to R; the first function, on the other hand, belongs to L by virtue of Lemma (2.1.2). Call these two functions $r_n(z)$ and $l_n(z)$ respectively, so that (8) may be written in the form

$$\frac{1-|\lambda_n|^2}{1-\lambda_n z} = l_n(z) + r_n(z), \tag{9}$$

where $l_n(z) \in L$ and $r_n(z) \in R$.

Using the fact that $|G(z)| = 1$ when $|z| = 1$, we may compute norms to obtain the relation

$$\|l_n(z)\|^2 = 1 - |G(\lambda_n^*)|^2 , \tag{10}$$

because $r_n(z)$ and $l_n(z)$ are mutually orthogonal.

Applying the operator $T - \lambda$ to both sides of (9) and using the fact that

$$\left\|(T-\lambda)\frac{1-|\lambda_n|^2}{1-\lambda_n z}\right\| = |\lambda - \lambda_n|, \tag{11}$$

we obtain the inequality

$$\begin{aligned} \|(T-\lambda)l_n(z)\| &\le |\lambda-\lambda_n| + |G(\lambda_n^*)|\|T-\lambda\| \\ &\le |\lambda-\lambda_n| + (1+|\lambda|)|G(\lambda_n^*)|. \end{aligned} \tag{12}$$

Passing to the limit as $n \to \infty$, we conclude from (12) that

$$\lim_{n\to\infty} \|(T-\lambda)l_n(z)\| = 0. \tag{13}$$

But (10) implies

$$\lim_{n\to\infty} \|l_n(z)\| = 1, \tag{14}$$

We therefore deduce from (13) and (14) that $T - \lambda$ cannot have abounded inverse.

Theorem (2.1.7) Paley-Wiener:[12] Every function h in H can be extended as a regular analytic function into the upper half-plane in such a way that

$$\int_{-\infty}^{\infty} h^*(s+it)h(s+it)ds \le \text{constant}$$

for all positive values of t. Conversely, the restriction to the real axis of any such function belongs to H [15].

For fixed t, $h(s+it)$ is the Fourier transform of $e^{-xt}\tilde{h}(x)$; since $\tilde{h}(x)$ vanishes for negative values of x, the L_2, norm of $e^{-xt}\tilde{h}(x)$ decreases with increasing t. Hence, by Parseval's theorem, we have the following.

Corollary (2.1.8):[12] If $h(\sigma)$ belongs to H, its L_2, norm along the line $\text{Im}\sigma = t, t \geq 0$, decreases with increasing t.

The orthogonal complement of $\widetilde{H}$ with respect to the space of square integrable functions on the entire real axis is the space of square integrable functions which vanish for x positive. The Fourier transforms of these functions form the orthogonal complement $\widetilde{H}^{\perp}$ of $\widetilde{H}$. Functions in $\widetilde{H}^{\perp}$ can be continued analytically into the lower half plane. It is easy to check that $\widetilde{H}^{\perp}$ is the conjugate of $\widetilde{H}$, i.e.

$$\widetilde{H}^{\perp} = \widetilde{H}^{*}.$$

Let $\tilde{R}$ be any right translation invariant subspace of $\widetilde{H}$ and let R be its Fourier transform. Such an R-space can be characterized intrinsically by the property that $e^{ia_a}R$ is contained in R for all positive a.

Lemma (2.1.9):[12] The function

$$g(s) = (s - i)(s + i)^{-1}$$

can be uniformly approximated on every compact subset of the real line by a sequence of trigonometric polynomials of the form

$$t_n(s) = \sum_{k=0}^{N_n} c_k(n) e^{ib_k(n)s}$$

where $b_k(n) \geq 0$ and $|t_n(s)| \leq M$.

Proof: Define

$$g_n(s) = 1 - 2\int_0^n e^{-(1-is)x}dx$$

and observe that the g_n's converge uniformly to g on the real axis. Replacing the integral by its Riemann sum, we may write

$$g_n(s) = 1 - \lim_{k\to\infty} g_{nk}(s)$$

where

$$g_{nk}(s) = \frac{2}{\mathcal{K}}\sum_{j=0}^{k} \exp[-\frac{jn}{\mathcal{K}}(1 - is)]$$

Using the summation formula for a geometric series, it is easy to verify that the g_{nk}'s are uniformly bounded and converge uniformly to g_n, on every compact subset of the real line.

Theorem (2.1.10):[12] Every nonempty closed R-space is of the form GH, where $G(\sigma)$ is a regular analytic function in the upper half plane and $|G(\sigma)| \leq 1$ there. For σ restricted to the real axis, $|G(\sigma)| = 1$ almost everywhere. The function $G(\sigma)$ is uniquely determined by the space R, except for multiplication by a complex constant of modulus 1.

Proof: Let $\widetilde{H}_d$ be the discrete space of sequences, and let H_d be the Fourier transform of $\widetilde{H}_d$.

The mapping $U : H_d \to H$ defined by

$$(Uf)(\sigma) = \frac{1}{\sqrt{2}(\sigma+i)} f\left(\frac{\sigma-i}{\sigma+i}\right) = \hat{f}(\sigma) \tag{15}$$

has an inverse which may be expressed in the form

$$\left(U^{-1}\hat{f}\right)(z) = \frac{2\sqrt{2}}{i(z-1)} \hat{f}\left(\frac{z+1}{i(z-1)}\right) = f(z) \tag{16}$$

and satisfies the relation

$$\int_0^{2\pi} \left|f(e^{i\theta})\right|^2 d\theta = \int_{-\infty}^{\infty} \left|\hat{f}(s)\right|^2 ds.$$

We therefore conclude that U is an isometry from H_d onto H.

Let R be any closed R-space of H. Then $e^{ias}R \subset R$ for all positive a, and this implies that multiplication by any trigonometric polynomial with positive exponents maps R into itself. Let $V_n: R \to R$ be defined by

$$\left(V_n\hat{f}\right)(\sigma) = t_n(\sigma)\hat{f}(\sigma).$$

According to Lemma (2.1.9), the operators V_n converge strongly to the operator V defined by

$$\left(V\hat{f}\right)(\sigma) = \left(\frac{\sigma-i}{\sigma+i}\right)\hat{f}(\sigma)$$

Since R is closed, $V(R) \subset R$. If we set $U^{-1}VU = \hat{V}$, a computation using (15) and (16) shows that $\tilde{V}$ is expressed by

$$\left(\hat{V}f\right)(z) = zf(z).$$

Therefore,

$$\hat{V}U^{-1}(R) = zU^{-1}(R) = U^{-1}V(R) \subset U^{-1}(R),$$

and we conclude that U^{-1} maps R spaces onto R_d spaces because multiplication by z maps $U^{-1}(R)$ into itself.

According to Beurling's Theorem, we may write

$$U^{-1}(R) = G(z)H_d.$$

Applying U to both sides of this identity yields the relation

$$R = G\left(\frac{\sigma - i}{\sigma + i}\right)H$$

which proves the theorem.

Lemma (2.1.11):[12] The function $f(\sigma)$ belong to the space L if and only if it is analytic in the upper half plane, square integrable on the real axis, and the function $G(\sigma)f^*(\sigma)$ can be continued into the upper half plane as an analytic function in such a way that its L_2, norm is uniformly bounded on every line parallel to the real axis.

Proof: Notice that $f(\sigma)$ belongs to L if and only if it is square integrable on the real axis, can be continued into the upper half plane as an analytic function in such a way that its norm is uniformly bounded on every line parallel to the real axis, and further satisfies the condition that $(G^*f, H) = 0$.This implies that $f(\sigma)$ belongs to L if and only if there is a function $k(\sigma)$ in H with the property that

$$G^*(\sigma)f(\sigma) = k^*(\sigma) \tag{17}$$

when σ is real. Taking the complex conjugate of (16), we see that $f(\sigma)$ belongs to L if and only if the function $G(\sigma)\, f^*(\sigma)$ can be analytically continued into the upper half plane as a function in H, i.e. in such a way that its L_2, norm is uniformly bounded on every line parallel to the real axis.

Corollary (2.1.12):[12] Let λ be any complex number in the upper half plane. Denote by d_λ the distance of the normalized exponential function

$$\tilde{e}_\lambda(x) = \begin{cases} \sqrt{2\ \mathrm{Im}\ \lambda e^{i\lambda x}} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

from the space $\tilde{L}$. Then it follows that

$$d_\lambda = |G(-\lambda^*)| \tag{18}$$

Proof: Consider the identity

$$\frac{i\sqrt{2\mathrm{Im}\lambda}}{\sigma + \lambda} = i\sqrt{2\mathrm{Im}\lambda}\frac{G^*(-\lambda^*)G(\sigma)}{\sigma + \lambda} + i\sqrt{2\mathrm{Im}\lambda}\frac{1 - G^*(-\lambda^*)G(\sigma)}{\sigma + \lambda}. \tag{19}$$

The function appearing on the left hand side of (19) is the Fourier transform of $\sqrt{2\pi}\tilde{e}_\lambda(x)$. According to the Theorem (2.1.1), the first function on the right hand side of (19) belongs to R. Using Lemma (2.1.9) and the fact that $G(\sigma)$ has modulus equal to 1 along the real axis, it is easy to verify that the second function on the right belongs to L. Formula (18) may be derived immediately by computing norms.

Lemma (2.1.13):[12] Let μ be any complex number in the upper half plane $\mathrm{Im}(\sigma) > 0$ and let $m(\sigma)$ be an arbitrary function in H. Then the sequence $\{m(\mu + 2k\pi)\}_{-\infty}^{\infty}$ belongs to $l_2(-\infty, \infty)$.

Proof: Using the analyticity of $m(\sigma)$ we may write

$$m(\sigma) = \frac{1}{2\pi}\int_{0}^{2\pi} m(\sigma + \rho e^{i\phi})\, d\phi,$$

from which it follows that

$$m(\sigma)\frac{\delta^2}{2} = \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{\delta} m\big(\sigma + \rho e^{i\phi}\big)\rho\, d\rho d\phi$$

for some sufficiently small $\delta > 0$.

An application of Schwarz's inequality to this last expression yields

$$|m(\sigma)|^2 \le \frac{1}{\pi\delta^2}\int_{0}^{\delta}\int_{0}^{2\pi} \big|m\big(\sigma + \rho e^{i\phi}\big)\big|^2 \rho\, d\rho\, d\phi.$$

Changing from polar to rectangular coordinates and replacing the domain of integration by a square centered at σ, we obtain the relation

$$|m(s + it)|^2 \le \frac{1}{\pi\delta^2}\int_{t-\delta}^{t+\delta}\int_{s-\delta}^{s+\delta} |m(u + iv)|^2\, du\, dv \tag{20}$$

Since

$$\int_{-\infty}^{\infty} |m(u + iv)|^2\, du \le M < \infty$$

for all $v > 0$, (20) implies that

$$\sum_{n=-\infty}^{\infty} |m(\mu + 2n\pi)|^2 \le \frac{1}{\pi\delta^2}\int_{\mathrm{Im}\,\mu-\delta}^{\mathrm{Im}\,\mu+\delta}\int_{-\infty}^{\infty} |m(u + iv)|^2\, du\, dv$$

We thus have

$$\sum_{k=-\infty}^{\infty} |m(\mu + 2k\pi)|^2 \le \frac{2M}{\pi\delta}$$

which means that the sequence $\{m(\mu + 2k\pi)\}_{-\infty}^{\infty}$ belongs to $L_2(-\infty, \infty)$.

Theorem (2.1.14):[12] Let λ be any complex number such that $0 < |\lambda| < 1$.Then $\lambda \in \sigma(T)$ if and only if there exists a $\delta > 0$ such that

$$|G(2\pi n - i ln\, \lambda^*)| > \delta$$

for all $n = 0, \pm 1, ...$

Proof: We will first demonstrate that the condition $|G(2\pi n - iln\,\lambda^*)| > \delta$ for all $n = 0, \pm 1, ...$ implies that the functional equation

$$\tilde{f}(x+1) - \lambda\tilde{f}(x) = \tilde{l}(x) \tag{21}$$

has a unique solution $\tilde{f}(x)$ in $\tilde{L}$ for every $\tilde{l}(x)$ in $\tilde{L}$. It will then follow from the interior mapping principle that $\lambda \notin \sigma(T)$.

We begin by taking the Fourier transform of both sides of (21) to get

$$e^{-i\sigma}[f(\sigma) - f_1(\sigma)] - \lambda f(\sigma) = l(\sigma) \tag{22}$$

where $f(x)$ is the Fourier transform of $\tilde{f}(x)$restricted to the interval (0,l).

Solving,‘) for $f(\sigma)$, we obtain

$$f(\sigma) = \frac{l(\sigma) + e^{-i\sigma}f_1(\sigma)}{e^{-i\sigma} - \lambda} \tag{23}$$

which is yalid for all real σ.

Since $l(\sigma)$ belongs to L, it follows from Lemma(2.1.10) that the function

$$m(\sigma) = G(\sigma)l^*(\sigma) \qquad (\sigma\, real) \tag{24}$$

can be analytically continued into the upper half plane as a function in H.Taking complex conjugates, multiplying both sides of (23) by $G(\sigma)$ and applying (24), we get

$$G(\sigma)f^*(\sigma) = \frac{m(\sigma) + e^{iv}G(\sigma)f_1^*(\sigma)}{e^{i\sigma} - \lambda^*} \tag{25}$$

for σ real.

Both the numerator and the denominator of the right hand side of (25) are analytic functions in the upper half plane; consequently, the left hand side has an analytic continuation into the upper half plane if and only if

$$f_1(2n\pi + i\ln\lambda) = -\frac{m^*(2n\pi - i\ln\lambda^*)}{\lambda G^*(2n\pi - i\ln\lambda^*)}. \tag{26}$$

Since $m(\sigma)$ belongs to H and since the sequence$\{G(2n\pi - i\ln\lambda^*)\}_{-\infty}^{\infty}$ is bounded away from zero by δ, we may invoke Lemma (2.1.13) to conclude that the right hand side of (26) belongs to $L_2(-\infty,\infty)$.Thus, according to the Riesz-Fisher Theorem, there exists a unique square integrable function, $\tilde{k}(x)$, whose support is contained in the interval (0,l) having the property that

$$-\frac{m^*(2n\pi - i\ln\lambda^*)}{\lambda G^*(2n\pi - i\ln\lambda^*)} = \frac{1}{\sqrt{2\pi}}\int_0^1 \tilde{k}(x)e^{2\pi inx}\,dx \qquad (27)$$

Define

$$f_1(\sigma) = \frac{1}{\sqrt{2\pi}}\int_0^1 [\tilde{k}(x)e^{\ln\lambda x}]e^{i\sigma x}\,dx \qquad (28)$$

and observe that $f_1(\sigma)$ is the Fourier transform of a square integrable function whose support is also contained in (0,l). Furthermore, $f_1(\sigma)$ agrees with the right hand side of (26) on the points $\sigma = 2n\pi + i\ln\lambda$. If we substitute (28) back into (25), we see that the right hand side of (25) is now an analytic function in the upper half plane whose L_2, norm is uniformly bounded on every line parallel to the real axis. Applying the Paley-Wiener Theorem, this means that the left hand side of (25) can be analytically continued into the upper half plane as a function in H - which is exactly what is wanted to prove.

Lemma (2.1.15):[12] Let $u(\sigma)$ belong to H and let $v(\sigma)$ be analytic in any domain, $\mathcal{D}$ contained in the lower half plane whose boundary contains the interval $\Gamma = (s_1, s_2)$. Assume that no inner point of Γ is the limit point of boundary points of $\mathcal{D}$ which do not belong to Γ. If

$$\lim_{t\to 0^+} u(s+it) - v(s-it) = 0 \quad for\ s_1 < s < s_2,$$

and

$$\int_{s_1}^{s_2} |u(s+it) - v(s-it)|^2 \le M \qquad for\ 0 < t < t_0,$$

then $u(\sigma)$ and $v(\sigma)$ are analytic continuations of each other across Γ.

Theorem (2.1.16):[12] Let λ be any complex number such that $|\lambda| = 1$. If there exists a $\delta > 0$ such that $|G(\sigma)| > \delta$ in an ϵ-neighborhood of every point $\sigma_n = 2n\pi - i\ln\lambda^*$ then $\lambda \notin \sigma(T)$.

Proof: If $|G(\sigma)| > \delta$ in an ϵ-neighborhood of every point $\sigma_n = 2n\pi - i\ln\lambda^*$, it follows from Lemma (2.1.15) that the functions $1/G^*(\sigma^*)$ and $l^*(\sigma^*)/G^*(\sigma^*)$ are analytic continuations of the functions $G(\sigma)$ and $m(\sigma)$ defined in Theorem (2.1.14) throughout an ϵ-neighborhood of every point $2n\pi - i\ln\lambda^*$. By Lemma (2.1.13), we further note that the sequence $\{G(2n\pi - i\ln\lambda^*)\}_{-\infty}^{\infty}$ belongs to $L_2(-\infty,\infty)$.

Thus, as in the proof of Theorem (2.1.13), there exists a unique square integrable function, $\tilde{k}(x)$, whose support is contained in the unit interval and which has the property that

$$-\frac{m^*(2n\pi - i\ln\lambda^*)}{\lambda G^*(2n\pi - i\ln\lambda^*)} = \frac{1}{\sqrt{2\pi}}\int_0^1 \tilde{k}(x)e^{2\pi inx}\,dx$$

If we again set

$$f_1(\sigma)=\frac{1}{\sqrt{2\pi}}\int_0^1[\tilde{k}(x)e^{\ln\lambda x}]e^{i\sigma x}\,dx$$

and substitute this value into (25), the resulting expression is still an analytic function whose L_2, norm is uniformly bounded on every line parallel to the real axis. This means that the left hand side of (25) belongs to H.

Theorem (2.1.17):[12] If λ is any complex number of modulus 1 which satisfies the condition

$$\inf_{-\infty<n<\infty}\left\{\lim_{\sigma\to 2n\pi-i\ln\lambda^*}\inf|G(\sigma)|\right\}=0,\ \ \operatorname{Im}\sigma>0$$

then $\lambda\in\sigma(T)$.

Proof: Suppose the condition stated in the theorem is satisfied. Then there exists a sequence of integers $\{n_j\}_1^\infty$ and a sequence of complex numbers $\{\sigma_j\}_1^\infty$ in the upper half plane with the property that

$$\lim_{j\to\infty}\sigma_j-2n_j\pi+i\ln\lambda^*=0\ \text{ and }\ \lim_{j\to\infty}|G(\sigma_j)|=0.$$

Consider the normalized exponential function

$$\tilde{e}_j(x)=\begin{cases}\sqrt{2\operatorname{Im}\sigma_j}e^{-i\sigma_j x} & if\ x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

and let $\tilde{r}_j(x)$ and $\tilde{l}_j(x)$ be its projections onto the spaces $\tilde{R}$ and $\tilde{L}$ respectively.

We then have

$$\tilde{e}_j(x)=\tilde{r}_j(x)+\tilde{l}_j(x) \tag{29}$$

where

$$\|\tilde{r}_j(x)\|=|G(\sigma_j)| \tag{30}$$

by (18) and

$$\|\tilde{l}_j(x)\|=\sqrt{1-\|\tilde{r}_j(x)\|^2} \tag{31}$$

by virtue of the orthogonality between $\tilde{r}_j(x)$ and $\tilde{l}_j(x)$. Using the fact that

$$(T-\lambda)\tilde{e}_j(x)=(e^{i\sigma_j}-\lambda)\tilde{e}_j(x)$$

and applying the operator $T-\lambda$ to both sides of (33), we obtain the inequality

$$\left\|(T-\lambda)\tilde{l}_j(x)\right\| \leq \left|e^{-i\sigma_j}-\lambda\right| + (1+|\lambda|)\left\|\tilde{r}_j(x)\right\|$$
$$\leq \left|e^{-i\sigma_j}-\lambda\right| + (1+|\lambda|)\left|G(\sigma_j)\right|$$

By hypothesis, this implies

$$\lim_{j\to\infty}\left\|(T-\lambda)\tilde{l}_j(x)\right\| = 0$$

However, it follows from (34) and (35) that

$$\lim_{j\to\infty}\left\|\tilde{l}_j(x)\right\| = 1$$

We therefore conclude that $(T-\lambda)$ cannot have a bounded inverse.

Theorem (2.1.18):[12] Let λ be any complex number such that $0<|\lambda|\leq 1$. Then $\lambda \in \sigma_p(T)$ if and only if

$$G(2\pi n_0 - i\ln\lambda^*) = 0$$

for some integer n_0.

Proof: Clearly, $\tilde{f}_\lambda(x)$ is an eigenfunction corresponding to the eigenvalue λ if and only if

$$\tilde{f}_\lambda(x) = \lambda^{[x]}\tilde{f}(x) \quad (32)$$

where $[x]$ denotes the largest integer in x and $\tilde{f}(x)$ is the periodic continuation of an L_2, function whose support is contained in the interval $(0,1)$. Equation (32) implies that there are no points in $\sigma_p(T)$ which have modulus $=1$ because the associated eigenfunctions would not be square integrable. Let $0<|\lambda|<1$, and assume that there is a non-zero eigenfunctions $\tilde{f}_\lambda(x)$ in the space $\tilde{L}$. We will show that $G(2\pi n - i\ln\lambda^*)$ vanishes for some value of n.

Rewrite (32) in the form

$$\tilde{f}_\lambda(x) = (\lambda)^x\hat{f}(x) \quad (33)$$

where

$$\tilde{f}(x) = \lambda^{[x]-x}f(x)$$

Since $\hat{f}(x)$ is square integrable over the unit interval and has period 1, we may represent it by its Fourier series:

$$\hat{f}(x) \sim \sum_{n=-\infty}^{\infty}\hat{f}_n e^{2\pi i n x}$$

where

$$\hat{f}_n = \int_0^1 \hat{f}(x) e^{-2\pi i n x}\, dx.$$

Observe from (33) that $\hat{f}(x)$ vanishes if and only if $\hat{f}(x)$ also vanishes. Consequently, for some n_0, we have $f_n, \hat{f}_n \neq 0$. It is well known from classical theory that the closed subspace spanned by the left translates of $\acute{f}(x)$ in the $L_2(0,1)$ topology contains the exponential $e^{2\pi i n_0 x}$. This means that, for any $\epsilon > 0$, there exist constants $c_1, c_2, \dots, c_k$ (all distinct from zero) and positive numbers $t_1, t_2, \dots, t_k$ such that

$$\int_0^1 \left| e^{2\pi i n_0 x} - \sum_{n=1}^{K} c_n \hat{f}(x + t_n) \right|^2 dx \leq \epsilon^2.$$

Define new constants $d_n = c_n \lambda^{-t_n}$ and notice that

$$\int_0^\infty \left| \lambda^x e^{2\pi i n_0 x} - \sum_{n=1}^{K} d_n \lambda^{x+t_n} \hat{f}(x + t_n) \right|^2 dx$$

$$\leq \int_0^\infty |\lambda|^{2x} \left| e^{2\pi i n_0 x} - \sum_{n=1}^{K} c_n \hat{f}(x + t_n) \right|^2 dx \leq \frac{\epsilon^2}{1 - |\lambda|^2}$$

This last inequality implies that the exponential $exp\,[(2\pi i n_0 + \ln \lambda)\, x]$ belongs to the space $\tilde{L}$. Thus, from formula (18), it follows that

$$G(2\pi n_0 - i \ln \lambda^*) = 0$$

Conversely, if

$$G(2\pi n_0 - i \ln \lambda^*) = 0$$

it follows from the same formula that an eigenfunctions of the form

$$\tilde{f}_\lambda(x) = \exp(2\pi n_0 + \ln \lambda)\, x$$

is already in $\tilde{L}$.

Theorem (2.1.19):[12] The origin belongs to $\sigma_p(T)$ if and only if there exists an $a > 0$ such that $\left| e^{-ia\sigma} G(\sigma) \right| \leq 1$in the upper half plane.

Proof: Assume that the origin belongs to $\sigma_p(T)$. Then there exists a nonzero function $\tilde{l}(x) \in \tilde{L}$ whose support is contained in the interval $(0,1)$. Let a be the smallest number such that the support of $\tilde{l}(x)$ is contained in $(0,a)$. According to the well-known Titchmarsh convolution theorem [16], $\tilde{L}$ contains the space of all square integrable functions whose supports are contained in $(0,a)$. This can occur if and only if every function in $\tilde{R}$ vanishes for $x < a$. Let $\tilde{R}_1$ be the right translation invariant space obtained from a by translating every function in $\tilde{R}$ a units to the left. From the Theorem(2.1.1) it follows that

$$e^{-ia\sigma}G(\sigma)H = R_1 \subset H.$$

Since multiplication by $e^{-ia\sigma}G(\sigma)$ maps H into H, multiplication by any power of $e^{-ia\sigma}G(\sigma)$ also maps H into H. It is clear that the L_2 norm on the real axis is preserved in this multiplication, so it follows from the Paley-Wiener theorem that it cannot be increased on any line parallel to the real axis. This is the case if and only if $\left|e^{-ia\sigma}G(\sigma)\right| \leq 1$ in the upper half plane.

Conversely, if $\left|e^{-ia\sigma}G(\sigma)\right| \leq 1$ in the upper half plane, it follows that

$$R = e^{ia\sigma}R_1.$$

Taking inverse Fourier transforms, this implies that every function in $\tilde{R}$ vanishes for $x < a$. Consequently, $\tilde{L}$ contains all square integrable functions whose supports are contained in $(0, a)$. This means that the origin must belong to $\sigma_p(T)$.

Using the information at hand, we will now construct an $\tilde{L}$ space for which $\sigma(T)$ is the closed unit disc, but which nevertheless has a void point spectrum.

To this end let $\tilde{L}$ be the space whose characteristic function is the convergent infinite product

$$G(\sigma) = \prod_{n=1}^{\infty} e^{-in/\sigma - 2\pi n^3}$$

If a is any real number greater than zero, we then have

$$\lim_{t\to\infty}|e^{at}G(it)| = \lim_{t\to\infty} \exp\left[\left(a - \sum_{n=1}^{\infty}\frac{n}{4\pi^2 n^6 + t^2}\right)t\right] = \infty.$$

According to Theorem (2.1.19), this implies that $\{0\} \notin \sigma_p(T)$; moreover, since $G(\sigma)$ never vanishes in the upper half plane, we may invoke Theorem (2.1.18) to justify the assertion that $\sigma_p(T) = \phi$.

let λ be any complex number such that $0 < |\lambda| < 1$ and let $k_j = j^3$ for $j = 1, 2, \dots$. Then

$$\log\left|G\left(2\pi k_j - i\ln\lambda^*\right)\right| = \sum_{n=1}^{\infty}\frac{n\ln\lambda}{|2\pi(j^3 - n^3) - i\ln\lambda^*|^2},$$

and, by setting $j = n$, we obtain the inequality

$$\log|G(2\pi j^3 - i\ln\lambda^*)| < j\frac{\ln|\lambda|}{|\ln\lambda^*|^2}.$$

A passage to the limit yields the relation

$$\lim_{j\to\infty} G(2\pi j^3 - i\ln\lambda^*) = 0,$$

from whence it follows that $\lambda \in \sigma(T)$ by Theorem (2.1.13).[14].

Section (2.2): Spaces of Analytic Functions of Several Complex Variables

Let $d \geq 1$, let Ω be a region in $\mathbb{C}^d$ with $0 \in \Omega$, and let $\mathcal{H}$ be a Hilbert space of analytic functions on Ω. We will be particularly interested in the cases where $\mathcal{H}$ equals one of the usual Hardy or Bergman spaces of the ball, $\mathbb{B}_d = \{z \in \mathbb{C}^d: |z| < 1\}$, or the polydisc, $\mathbb{D}_d = \{z \in \mathbb{C}^d: |z_i| < 1 \text{ for } i = 1,\dots,\mathrm{d}\}$, or where $\mathcal{H} = H_d^2$ where H_d^2 is the Hilbert space of analytic functions on $\mathbb{B}_d$ determined by the reproducing kernel $k_\omega(z) = \frac{1}{1-\langle z,\omega\rangle}$, where $\langle z,\omega\rangle = \sum_{i=1}^{d} z_i\overline{\omega}_i$.

Associated with each such space $\mathcal{H}$ we have a multiplier algebra $M(\mathcal{H})$ consisting of all analytic functions φ on Ω such that $\varphi f \in \mathcal{H}$ for each $f \in \mathcal{H}$. It is easy to see that each multiplier gives rise to a bounded operator $M_\varphi: \mathcal{H} \to \mathcal{H}, f \to \varphi f$, and the multiplier norm $\|\varphi\|_M$ is defined to be the operator norm of M_φ. One always has $M(\mathcal{H}) \subseteq H^\infty(\Omega)$, the algebra of bounded analytic functions on Ω, and it is well-known that for the Hardy and Bergman spaces we have $M(\mathcal{H}) = H^\infty(\Omega)$. It is also known that $M(H_d^2) \subsetneq H^\infty(\Omega)$ ([18]). We will assume that $M(\mathcal{H}) \subseteq \mathcal{H}$, and that for each $i = 1,\dots,d$ the i-th coordinate function z_i is a multiplier of $\mathcal{H}$. We will write M_z for the d-tuple $(M_{z_1},\dots,M_{z_d})$ of commuting operators on $\mathcal{H}$.

A subspace $\mathcal{M}$ of $\mathcal{H}$ is called multiplier invariant if $\varphi f \in \mathcal{M}$ for each $f \in \mathcal{M}$ and $\varphi \in M(\mathcal{H})$. We will investigate the Fredholm spectrum and Fredholm index of $M_z|\mathcal{M}$ for nonzero multiplier invariant subspaces of $\mathcal{H}$.[19].

Assume $d = 1$. Then the index of an invariant subspace $\mathcal{M}$ of $\mathcal{H}$ is defined to be the dimension of $\mathcal{M}/z\mathcal{M}$. If we assume that for each $\lambda \in \Omega$ that $M_z - \lambda$ is a Fredholm operator on $\mathcal{H}$, then it follows that $(M_z - \lambda)|\mathcal{M}$ is bounded below for each multiplier invariant subspace $\mathcal{M}$ of $\mathcal{H}$. Thus, $(M_z - \lambda)|\mathcal{M}$ is a semi-Fredholm operator and the continuity properties of the Fredholm index imply the following stability of the index of an invariant subspace: $\operatorname{ind}\mathcal{M} = \dim\mathcal{M}/z\mathcal{M} = -\operatorname{ind} M_z|\mathcal{M} = -\operatorname{ind}(M_z - \lambda)|\mathcal{M} = \dim\mathcal{M}/(z-\lambda)\mathcal{M}$ for all $\lambda \in \Omega$.[4,20,21,22,23].

Let $d \geq 1$ again, and let $\mathcal{M}$ be an invariant subspace of $\mathcal{H}$. By analogy with the situation in $d = 1$ one would like to consider

$$\dim\mathcal{M}/((z_1-\lambda_1)\mathcal{M} + \dots + (z_d-\lambda_d)\mathcal{M})$$

for $\lambda = (\lambda_1,\dots,\lambda_d) \in \Omega$.

However two problems arise. The first is if we assume that $(z_1-\lambda_1)\mathcal{H} + \dots + (z_d-\lambda_d)\mathcal{H}$ is closed in $\mathcal{H}$ it is not clear that the same is true for an arbitrary invariant subspace. The second is for very simple examples $\dim\mathcal{M}/((z_1-\lambda_1)\mathcal{M} + \dots + (z_d-\lambda_d)\mathcal{M})$ will depend on the choice of the point λ. Indeed, if $\mathcal{H}$ is the Hardy or Bergman space of the ball or poly disc, or $\mathcal{H} = H_d^2$, and if $\mathcal{M} = \{f \in \mathcal{H}: f(0) = 0\}$, then one easily checks that

$\dim \mathcal{M}/(z_1\mathcal{M}+\cdots+z_d\mathcal{M}) = d$ while $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\cdots+(z_d-\lambda_d)\mathcal{M}) = 1$ for all $\lambda \in \mathbb{B}_d\backslash\{0\}$. The second problem in this case is connected to the fact that all functions in $\mathcal{M}$ are zero at $\lambda = 0$. It is well-known that one obtains a more stable definition of index if one uses the Fredholm index of the tuple $(M_z - \lambda)|\mathcal{M}$, where $\lambda \in \mathbb{B}_d\backslash\sigma_e(M_z|\mathcal{M})$, to define the index of an invariant subspace. Here we set $M_z - \lambda = (M_{z_1} - \lambda_1 I, \ldots, M_{z_d} - \lambda_d I)$ and we use $\sigma_e(T)$ to denote the essential Taylor spectrum of the operator tuple T; the Fredholm index of T is defined to be the alternating sum of the Betti numbers of the Koszul complex associated with T - we shall give the full definitions. It will still turn out that for $d > 1$ and for all the spaces $\mathcal{H}$ mentioned above there are invariant subspaces $\mathcal{M}$ of $\mathcal{H}$ such that $\sigma_e(M_z|\mathcal{M}) \cap \mathbb{B}_d$ is nonempty. Thus in general the Fredholmness of $(M_z - \lambda)|\mathcal{M}$ may depend on the base point $\lambda \in \Omega$.

Theorem (2.2.1):[17] Let $\mathcal{H}$ be the Hardy or Bergman space of the ball or polydisc of $\mathbb{C}^d$, or let $\mathcal{H} = H_d^2$.

If an invariant subspace $\mathcal{M}$ of $\mathcal{H}$ contains a nonzero multiplier φ, then

$$\sigma_e(M_z|\mathcal{M}) \cap \Omega \subseteq \mathrm{Z}(\varphi)$$

and for every $\lambda \in \Omega\backslash\sigma_e(M_z|\mathcal{M})$ the tuple $(M_z - \lambda)|\mathcal{M}$ has Fredholm index $(-1)^d$. In fact, for all $\lambda \in \Omega\backslash\mathrm{Z}(\varphi)$ we have

$$\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\cdots+(z_d-\lambda_d)\mathcal{M}) = 1.$$

Proof: For $d = 1$ a similar argument was given in [3,27]. Recall that if $\mathcal{B}$ is a Banach space of analytic functions on Ω, then one says that one can solve Gleason's problem for $\mathcal{B}$ if whenever $g \in \mathcal{B}$ and $\lambda \in \Omega$ then there are functions $g_1,\ldots,g_d \in \mathcal{B}$ such that $g - g(\lambda) = \sum_{i=1}^{d}(z_i - \lambda_i)g_i$ [29]. We may assume that the multiplier $\varphi \in \mathcal{M}$ satisfies $\varphi(\lambda) = 1$. Let $f \in \mathcal{M}$. Then

$$f = f(\lambda)\varphi + \varphi\big(f - f(\lambda)\big) - (\varphi - 1)f.$$

Now, if we assume that one can solve Gleason's problem for both the space $\mathcal{H}$ and the multiplier algebra $\mathcal{M}(\mathcal{H})$, then there are functions $f_1,\ldots,f_d \in \mathcal{H}$ and multipliers $\varphi_1,\ldots,\varphi_d \in M(\mathcal{H})$ such that

$$f = f(\lambda)\varphi + \sum_{i=1}^{d}(z_i - \lambda_i)(\varphi f_i - \varphi_i f).$$

It is clear that for each i the function $\varphi_i f$ is in the multiplier invariant subspace $\mathcal{M}$, and the same is true for φf_i, if one assumes for example that the multipliers are dense in $\mathcal{H}$. Thus $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\cdots+(z_d-\lambda_d)\mathcal{M}) = 1$ provided all the assumptions are satisfied. We will show that one can solve Gleason's problem for the multiplier algebra of H_d^2. All the other assumptions are already known to be true for the spaces mentioned in the theorem.

If $\mathcal{D}$ is a separable complex Hilbert space, then we denote by $\mathcal{H}_\mathcal{D}$ the space of $\mathcal{D}$-valued $\mathcal{H}$-functions. It is the set of all analytic functions $f: \Omega \to \mathcal{D}$ such that for each $x \in \mathcal{D}$ the function $f_x(\lambda) = \langle f(\lambda), x\rangle_\mathcal{D}$ defines a function in $\mathcal{H}$ and such that

$$\|f\|^2 = \sum_{n=1}^{\infty} \|f_{e_n}\|^2 < \infty$$

for some orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{D}$. One shows that the above expression is independent of the choice of orthonormal basis. In particular, one has for $f \in \mathcal{H}, x \in \mathcal{D}$ that the function $fx: \lambda \to f(\lambda)x$ is in $\mathcal{H}_{\mathcal{D}}$ and $\|fx\| = \|f\|\|x\|_{\mathcal{D}}$. If $f \in \mathcal{H}_{\mathcal{D}}, x \in \mathcal{D}$, and $\lambda \in \mathbb{B}_d$ we have $\langle f(\lambda), x\rangle_{\mathcal{D}} = \langle f, k_\lambda x\rangle$, where we have used $k_\lambda \in \mathcal{H}$ to denote the reproducing kernel for $\mathcal{H}$ at λ. There is an obvious identification of the tensor product $\mathcal{H} \otimes \mathcal{D}$ with $\mathcal{H}_{\mathcal{D}}$, where one identifies the elementary tensors $f \otimes x$ with the functions fx. Considering the definition of the norm in $\mathcal{H}_{\mathcal{D}}$, one may also think of $\mathcal{H}_{\mathcal{D}}$ as a direct sum of $\dim \mathcal{D}$ copies of the scalar valued space $\mathcal{H}$.

Each (scalar valued) multiplier $\varphi \in M(\mathcal{H})$ defines an operator on $\mathcal{H}_{\mathcal{D}}$ of the same norm, and we shall also denote this operator by M_φ. We shall say that a subspace $\mathcal{M}$ of $\mathcal{H}_{\mathcal{D}}$ is scalar multiplier invariant if $M_\varphi \mathcal{M} \subseteq \mathcal{M}$ for each $\varphi \in M(\mathcal{H})$.

Let $\mathcal{M}$ be a scalar multiplier invariant subspace of $\mathcal{H}_{\mathcal{D}}$. For $\lambda \in \Omega$ we write

$$\mathcal{M}_\lambda = \operatorname{clos}\{f(\lambda): f \in \mathcal{M}\},$$

and we define the fiber dimension of $\mathcal{M}$ to be $\sup_{\lambda\in\Omega} \dim \mathcal{M}_\lambda$. We will be interested in invariant subspaces with finite fiber dimension m. In this case we write $Z(\mathcal{M}) = \{\lambda \in \Omega: \dim \mathcal{M}_\lambda < m\}$. Note that for the scalar case this agrees with the earlier definition of $Z(\mathcal{M})$.

Let $\lambda_0 \in \Omega\backslash Z(\mathcal{M})$, then if $m < \infty$ the set $\{f(\lambda_0): f \in \mathcal{H}_{\mathcal{D}}\}$ is closed, and there are $f_1, \ldots, f_m \in \mathcal{M}$ such that $f_1(\lambda_0), \ldots, f_m(\lambda_0)$ forms an orthonormal basis for $\mathcal{M}_{\lambda_0}$. Then

$$g(\lambda) = \det\big(\langle f_i(\lambda), f_j(\lambda_0)\rangle\big)_{1\leq i,j\leq m}$$

is an analytic function on Ω, and it is a standard fact from linear algebra that $\dim \mathcal{M}_\lambda = m$ whenever $g(\lambda) \neq 0$. Thus, the family of vector spaces $\{\mathcal{M}_\lambda\}_{\lambda\in\Omega\backslash Z(\mathcal{M})}$ defines a vector bundle over $\Omega\backslash Z(\mathcal{M})$ and $Z(\mathcal{M})$ is the intersection of at most a countably infinite number of zero sets of analytic functions. In particular $\Omega\backslash Z(\mathcal{M})$ is connected and it is dense in Ω.

Theorem (2.2.2):[17] Let $\mathcal{D}$ be a separable Hilbert space and let $\mathcal{M}$ be a nonzero scalar multiplier invariant subspace of $H_d^2(\mathcal{D})$ with finite fiber dimension m. Then

$$\partial\mathbb{B}_d \subseteq \sigma_e(M_z|\mathcal{M}) \subseteq \partial\mathbb{B}_d \cup Z(\mathcal{M})$$

and for every $\lambda \in \mathbb{B}_d\backslash\sigma_e(M_z|\mathcal{M})$ the tuple $(M_z - \lambda)|\mathcal{M}$ has Fredholm index $(-1)^d m$.

In fact for all $\lambda \in \mathbb{B}_d\backslash Z(\mathcal{M})$ we have

$$\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M} + \cdots + (z_d-\lambda_d)\mathcal{M}) = m.$$

Proof: If $\mathcal{N} \subset \mathcal{M}$ are two invariant subspaces of $H_d^2(\mathcal{D})$, then the fiber dimension of $\mathcal{N}$ is less than or equal to the fiber dimension of $\mathcal{M}$, so the theorem implies an inequality between

$\operatorname{ind}(M_z-\lambda)|\mathcal{N}$ and $\operatorname{ind}(M_z-\lambda)|\mathcal{M}$ for $\lambda \in \mathbb{B}_d\backslash(Z(\mathcal{M})\cup Z(\mathcal{N}))$. Thus for $d=1$ our theorem recaptures a well-known fact [30].

Recall from [18] that a commuting tuple $T=(T_1,\dots,T_d)$ of operators on a Hilbert space $\mathcal{K}$ is called a d-contraction if $\|T_1x_1+\cdots+T_dx_d\|^2 \le \|x_1\|^2+\cdots+\|x_d\|^2$ for all $x_1,\dots,x_d \in \mathcal{K}$. This condition is equivalent to $\sum_{i=1}^{d} T_iT_i^* \le I$ (see [25]). One then defines the defect operator $\Delta_T = \left(I-\sum_{i=1}^{d} T_iT_i^*\right)^{\frac{1}{2}}$, the defect space $\mathcal{D} = clos\, \Delta_T\mathcal{K}$, and one says that T has finite rank if $\mathcal{D}$ is finite dimensional. Furthermore, associated to each d-contraction is a completely positive map $\Psi: B(\mathcal{K}) \to B(\mathcal{K})$ defined by $\Psi(X)=\sum_{i=1}^{d} T_iXT_i^*$ [31] . The d-contraction is called pure if $\lim_{n\to\infty} \Psi^n(\mathrm{I}) = 0$ in the strong operator topology. In [18] it is shown that every pure d-contraction is the compression of $\left(M_z, H_d^2(\mathcal{D})\right)$ to the orthocomplement of some scalar multiplier invariant subspace of $H_d^2(\mathcal{D})$ [31,32] .

The curvature invariant $k(T)$ of a pure d-contraction of finite rank was defined in [24]. First we need to define a $\mathcal{B}(\mathcal{D})$-valued function on $\mathbb{B}_d$ by

$$k(\lambda) = (1-|\lambda|^2)\Delta_T(I-T(\lambda)^*)^{-1}\left(I-T(\lambda)\right)^{-1}\Delta_T,$$

where $T(\lambda)=\sum_{i=1}^{d}\bar{\lambda}_iT_i$. Arveson shows that for $\sigma - a.e.\ z \in \partial\mathbb{B}_d$ the nontangential limit of $k(\lambda)$ exists in the strong operator topology as λ approaches z. Here we have used σ to denote the rotationally invariant probability measure on $\partial\mathbb{B}_d$. We call this limit $k(z)$ and define the curvature invariant of T by

$$k(T) = \int_{\partial\mathbb{B}_d} \operatorname{trace} k(z)\, d\sigma(z).$$

It is clear that $0 \le k(T) \le \dim\mathcal{D}$. [33], it was shown that $k(T)$ is always an integer, in fact that

$$k(T) = \inf_{\lambda\in\mathbb{B}_d} \dim \bigcap_{i=1}^{d} \ker\left(T_i^*-\bar{\lambda}_i\right),$$

and that for $\sigma - a.e$. $z \in \mathbb{B}_d$ $k(T) = \operatorname{trace} k(z)$. Furthermore, if we write $K_\lambda = \bigcap_{i=1}^{d}\ker\left(T_i^*-\bar{\lambda}_i\right)$, and $E_T = \left\{\lambda \in \mathbb{B}_d : \dim K_\lambda > k(T) = \inf_{z\in\mathrm{B_d}} \dim K_z\right\}$, then it follows from [33] that E_T is contained in the zero set of a bounded analytic function. In Theorem (2.2.20) we will obtain the following new information about the value of $k(T)$ along with some spectral information of T. We write $\sigma(T)$ for the Taylor spectrum and $\sigma_p(T^*)^*$ for the set of d-tuples of complex conjugates of eigenvalues corresponding to the common eigenvectors of $T_1^*,\dots,T_d^*$.

Theorem (2.2.3):[17] If T is a pure d-contraction with finite rank, then $\sigma_e(T)\cap\mathbb{B}_d \subseteq E_T$ and for $\lambda \in \mathbb{B}_d\backslash\sigma_e(T)$,

$$k(T) = (-1)^d \operatorname{ind}(T-\lambda).$$

Furthermore, we have $\sigma(T) \cap \mathbb{B}_d = \sigma_p(T^*)^*$. Thus, if $k(T) = 0$, then $\sigma(T) \cap \mathbb{B}_d = \sigma_p(T^*)^* = E_T$.

We note that this theorem implies that if S and T are two pure d-contractions of finite rank such that $S_i - T_i$ is a compact operator for each $i = 1, \ldots, d$, then $k(T) = k(S)$ even though T and S may have different rank. We also mention that if each Ti is essentially normal, then one can show that $\sigma_p(T) \cap \mathbb{B}_d = \emptyset$, thus our theorem implies in this case that $k(T) = (-1)^d \operatorname{ind}(T)$. However, we will see that there are examples of pure finite rank d-contractions that are not Fredholm (i.e. $0 \in \sigma_e(T)$).[19,20,34,35].

Theorem (2.2.4):[17] Let $\mathcal{H}$ denote the Hardy or Bergman space of the ball or polydisc, let $\mathcal{D}$ be a separable Hilbert space, and let $\mathcal{M}$ be an invariant subspace of $\mathcal{H}_{\mathcal{D}}$ of finite fiber dimension m.

If $\lambda \in \Omega$ and there are bounded functions $f_1, \ldots, f_m \in \mathcal{M}$ such that the set $\{f_1(\lambda), \ldots, f_m(\lambda)\}$ is linearly independent, then the tuple $(M_z - \lambda)|\mathcal{M}$ is Fredholm with index $(-1)^d m$. In fact, for all such λ we have

$$\dim \mathcal{M}/((z_1 - \lambda_1)\mathcal{M} + \cdots + (z_d - \lambda_d)\mathcal{M}) = m.$$

See [19,37,38].

Lemma (2.2.5):[17] Let $T = (T_1, \ldots, T_d)$ and $S = (S_1, \ldots, S_d)$ be commuting tuples of operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively and let $X: \mathcal{K} \to \mathcal{H}$ and $Y: \mathcal{K} \to \mathcal{H}$ be bounded linear operators. If there exists an operator B on $\Lambda(\mathcal{H})$ such that for each $p = 0, \ldots, d$ we have $B\left(\Lambda^p(\mathcal{H})\right) \subseteq \Lambda^{p-1}(\mathcal{H})$ and

$$\partial_T B + B \partial_T = I \otimes E_0 - XY \otimes E_0$$

and if there exists an operator C on $\Lambda(\mathcal{K})$ such that for each $p = 0, \ldots, d$ we have

$C\left(\Lambda^p(\mathcal{K})\right) \subseteq \Lambda^{p-1}(\mathcal{K})$ and

$$\partial_S C + C \partial_S = I \otimes E_0 - YX \otimes E_0.$$

then for each $p = 0, \ldots, d$ the cohomology spaces $H^p(T)$ and $H^p(S)$ are isomorphic as vector spaces.

In particular, if T is a Fredholm tuple, then S is a Fredholm tuple and $\operatorname{ind} T = \operatorname{ind} S$.

If $\mathcal{H}$ is a Hilbert space of analytic functions on Ω such that for each $i = 1, \ldots, d$ multiplication by the coordinate functions defines a bounded linear operator on $\mathcal{H}$, then for each $\lambda \in \Omega$we will be interested in the Koszul complex $K(M_z - \lambda)$ for the d-tuple $M_z - \lambda$. The standard hypothesis on $\mathcal{H}$ will be that this complex is exact at every stage except at the last one, where its cohomology is one dimensional. This can be restated as saying that the augmented complex

$$K(M_z - \lambda, \mathbb{C}): 0 \to \Lambda^0(\mathcal{H}) \xrightarrow{\partial_0} \Lambda^1(\mathcal{H}) \xrightarrow{\partial_1} \ldots \xrightarrow{\partial_{d-1}} \Lambda^d(\mathcal{H}) \xrightarrow{\delta_\lambda} \mathbb{C} \to 0$$

is exact at every stage. Here we have written, for $k = 0,\dots,d-1, \partial k = \partial_{M_z-\lambda,k}$, and δ_λ is the evaluation map, $\delta_\lambda(f \otimes e_1 \wedge \dots \wedge e_d) = f(\lambda)$.

Similarly, if $\mathcal{H}$ is as above, $\mathcal{D}$ is a separable Hilbert space, and $\mathcal{M} \subseteq \mathcal{H}_{\mathcal{D}}$ is a scalar multiplier invariant subspace, then we will be interested in the augmented complex

$$K((M_z-\lambda)|\mathcal{M},\mathcal{M}_\lambda): 0 \to \Lambda^0(\mathcal{M}) \xrightarrow{\partial_0} \Lambda^1(\mathcal{M}) \xrightarrow{\partial_1} \dots \xrightarrow{\partial_{d-1}} \Lambda^d(\mathcal{M}) \xrightarrow{\delta_\lambda} \mathcal{M}_\lambda \to 0$$

where as above, for $k = 0,\dots,d-1, \partial_k = \partial_{(M_z-\lambda)|\mathcal{M},k}$, and δ_λ is the evaluation map,

$\delta_\lambda(f \otimes e_1 \wedge \dots \wedge e_d) = f(\lambda), f \in \mathcal{M}$. The purpose of introducing the augmented complex is that it will allow for a simple statement of the main results. We note that if $\lambda \in \Omega \backslash Z(\mathcal{M})$, then $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M})$ equals the fiber dimension of $\mathcal{M}$ is equivalent to saying the augmented Koszul complex is exact at the penultimate stage.

Lemma (2.2.6):[17] If $\lambda \in \Omega$ and $\mathcal{M}$ is a scalar multiplier invariant subspace of $H_{\mathcal{D}}$ of finite fiber dimension, then $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M}) = \dim \mathcal{M}_\lambda$ if and only if the augmented Koszul complex $K((M_z-\lambda)|\mathcal{M},\mathcal{M}_\lambda)$ is exact at the penultimate stage.

If the augmented complex $K((M_z-\lambda)|\mathcal{M},\mathcal{M}_\lambda)$ is exact, then $\lambda \notin \sigma_e(M_z|\mathcal{M})$ and $\operatorname{ind}\big((M_z-\lambda)|\mathcal{M}\big) = (-1)^d \dim \mathcal{M}_\lambda = \dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M})$.

Proof: Let $k = \dim \mathcal{M}_\lambda$ and $h_1,\dots,h_k \in \mathcal{M}$ such that $\mathcal{M}_\lambda$ equals the linear span of $h_1(\lambda),\dots,h_k(\lambda)$. It is clear that the cosets of $h_1,\dots,h_k$ are linearly independent in $\mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M}$. It follows that $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M}) = \dim \mathcal{M}_\lambda$, if and only if every $f \in \mathcal{M}$ is of the form $f(z) = \sum_{i=1}^k a_i h_i(z) + \sum_{i=1}^d (z_i-\lambda_i) g_i(z)$ for some $a_1,\dots,a_k \in \mathbb{C}$ and $g_1,\dots g_d \in \mathcal{M}$. Also note that for $f \in \mathcal{M}$ we have $f \otimes e_1 \wedge \dots \wedge e_d \in \operatorname{ran} \partial_{d-1}$ if and only if $f(z) = \sum_{i=1}^d (z_i-\lambda_i) g_i(z)$ for some $g_1,\dots g_d \in \mathcal{M}$.

Suppose $\dim \mathcal{M}/((z_1-\lambda_1)\mathcal{M}+\dots+(z_d-\lambda_d)\mathcal{M}) = \dim \mathcal{M}_\lambda$, and let $f \otimes e_1 \wedge \dots \wedge e_d \in \ker \delta_\lambda$. Then $f(\lambda) = 0$ and the hypothesis implies that there are $g_1,\dots g_d \in \mathcal{M}$ such that $f(z) = \sum_{i=1}^d (z_i-\lambda_i) g_i(z)$. Hence $\ker \delta_\lambda = \operatorname{ran} \partial_{d-1}$, i.e. the augmented complex is exact at this stage.

Conversely, suppose that the augmented complex is exact at this stage, i.e. $\ker \delta_\lambda = \operatorname{ran} \partial_{d-1}$. Let $f \in \mathcal{M}$, then since $h_1(\lambda),\dots,h_k(\lambda)$ is a basis for $\mathcal{M}_\lambda$, there are $a_1,\dots,a_k \in \mathbb{C}$ such that $f(\lambda) = \sum_{i=1}^k a_i h_i(\lambda)$. Set $f_1 = f - \sum_{i=1}^k a_i h_i$, then $f \otimes e_1 \wedge \dots \wedge e_d \in \ker \delta_\lambda = \operatorname{ran} \partial_{d-1}$. Hence there are $g_1,\dots g_d \in \mathcal{M}$ such that $f_1(z) = \sum_{i=1}^d (z_i-\lambda_i) g_i(z)$, and this implies $f(z) = \sum_{i=1}^k a_i h_i(z) + \sum_{i=1}^d (z_i-\lambda_i) g_i(z)$.

Lemma (2.2.7):[17] Let T be a d-tuple of commuting operators on a Hilbert space $\mathcal{H}$, let $A = (a_{ij})$ be an invertible $d \times d$ matrix, and set $S = (S_1,\dots,S_d), S_i = \sum_{j=1}^d a_{ij} T_j$.

Then the Koszul complex for T is isomorphic to the Koszul complex for S. Thus T is a Fredholm tuple if and only if S is a Fredholm tuple.

Proof: We have $\partial_S = \sum_{i=1}^{d} S_i \otimes E_i = \sum_{i,j=1}^{d} a_{ij} T_j \otimes E_i = \sum_{j=1}^{d} T_j \otimes D_j$, where $D_j = \sum_{i=1}^{d} a_{ij} E_i$, and it is a standard fact that there is an invertible linear map $L: \Lambda \to \Lambda$ such that each Λ^{p} is reducing for L and $LD_i = E_i L$ for $i = 1, \dots, d$ [39]. Thus for each p the space $\Lambda^{\mathrm{p}}(\mathcal{H})$ is reducing for $I \otimes L$ and $(I \otimes L)\partial_S = \partial_T (I \otimes L)$.

Lemma (2.2.8):[17] Let T be a d-tuple of commuting operators on a Hilbert space $\mathcal{H}$ and let $\lambda \in \mathbb{B}_d$ such that $I - \langle T, \lambda \rangle$, is invertible. Then the Koszul complexes of $T - \lambda$ and $\varphi_\lambda(T)$ are isomorphic. Thus $T - \lambda$ is a Fredholm tuple if and only if $\varphi_\lambda(T)$ is a Fredholm tuple.

Proof: We note that A_λ is invertible on $\mathbb{C}^d$. Hence it is easy to see that an isomorphism $K(T - \lambda) \to K\big(\varphi_\lambda(T)\big)$ is given by $U = (I - \langle T, \lambda \rangle)^{-1} \otimes L_\lambda$, where L_λ is the isomorphism from the proof of Lemma (2.2.7) applied with $T - \lambda$ and the matrix for A_λ.

For any d-tuple T of commuting operators on $\mathcal{H}$ and for any $\lambda \in \mathbb{B}_d$ the one-dimensional spectrum of the operator $\langle T, \lambda \rangle$ is contained in the disc of radius $|\lambda|$ whenever the Taylor spectrum of T is contained in clos $\mathbb{B}_d$. This follows from the spectral mapping property of the Taylor spectrum [38] since the function $f(z) = \langle T, \lambda \rangle$ maps $\mathbb{B}_d$ into the disc of radius $|\lambda|$.

The spaces H_d^2 , the Hardy, and Bergman spaces of the ball $\mathbb{B}_d$ are members of a family of Hilbert spaces of analytic functions. For $\alpha > 0$ we let $\mathcal{K}_\alpha$ be the space of analytic functions on $\mathbb{B}_d$ with reproducing kernel $k_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\alpha}$. Obviously, $\mathcal{K}_1 = H_d^2$, and it is also well-known that $\mathcal{K}_d = H^2(\mathbb{B}_d)$ is the Hardy space and $\mathcal{K}_{d+1} = L_d^2(\mathbb{B}_d)$is the Bergman space of the ball [40]. We shall need some spectral information about these three spaces and it will be convenient to treat all values of $\alpha > 0$ simultaneously.

Lemma (2.2.9):[17] Let $\alpha > 0$, and $\mathcal{H} = \mathcal{K}_\alpha$. Then for each $i = 1, .., d$ the self commutator $M_{z_i}^* M_{z_i} - M_{z_i} M_{z_i}^*$ is compact (i.e. M_{z_i} is essentially normal) and $\sum_{i=1}^{d} M_{z_i}^* M_{z_i} = 1 + K$ for some compact operator K. Furthermore, $\sigma(M_z) \subseteq$ clos $\mathbb{B}_d$.

Proof: Let $j = (j_1, j_2, \dots, j_d)$ be a multiindex of nonnegative integers, then $|j| = j_1 + j_2 + \dots + j_d, j! = j_1! j_2! \dots j_d!$, and for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathbb{C}^d, \lambda^j = \lambda_1^{j_1} \lambda_2^{j_2} \dots \lambda_d^{j_d}$, and the multinomial formula implies that for $z, \lambda \in \mathbb{B}_d$ and $n \geq 0$

$$\langle z, \lambda \rangle^n = \sum_{|j| = n} \frac{|j|!}{j!} z^j \bar{\lambda}_j.$$

Thus if we write $k_\lambda(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^\alpha} = \sum_{n=0}^{\infty} a_n (\langle z, \lambda \rangle)^n$, where $a_0 = 1$ and for $n \geq 1$

$a_n = \frac{\alpha(\alpha + 1) \dots (\alpha + n - 1)}{n!}$, then $k_\lambda(z) = \sum_j a_{|j|} \frac{|j|!}{j!} z^j \bar{\lambda}^j$, where the sum is taken over all multiindices j with entries in the integers. Since $k_\lambda(z) = \langle k_\lambda, k_z \rangle$ it follows that monomials in $\mathcal{K}_\alpha$ are mutually orthogonal and

$$\|z^j\|^2 = \frac{j!}{a_{|j|} |j|!} = \frac{j!}{\alpha(\alpha + 1) \dots (\alpha + |j| - 1)}.$$

Now for $1 \le i \le d$ let S_i denote the self-commutator of M_{z_i}, i.e. $S_i = M_{z_i}^* M_{z_i} - M_{z_i} M_{z_i}^*$, and let P_n denote the projection of $\mathcal{K}_\alpha$ onto the subspace of all polynomials of total degree less than n. We will show that $\|S_i - P_n S_i P_n\| \to 0$ as $n \to \infty$. It is clear that S_i is diagonalized by the monomials so that $S_i z^j = c_{i,j} z^j$ for each multiindex j and some $c_{i,j} \in \mathbb{R}$. Hence it will suffice to show that $\sup_{|j|\ge n} |c_{i,j}| \to 0$ as $n \to \infty$.

We write e_i for the multiindex with a 1 in the i-th spot and 0's otherwise. Then for any multiindex j and any $1 \le i \le d$ we have $M_{z_i}^* z^j = 0$ if $j_i = 0$ and $M_{z_i}^* z^j = \frac{\|z^j\|^2}{\|z^{j-e_i}\|^2} z^{j-e_i}$ otherwise. Hence if $j_i = 0$ we obtain $\langle S_i z^j, z^j\rangle = \frac{1}{\alpha+|j|}\|z^j\|^2$, while for $j_i > 0$ we compute

$$\langle S_i z^j, z^j\rangle = \left(\frac{\|z^{j+e_j}\|^2}{\|z^j\|^2} - \frac{\|z^j\|^2}{\|z^{j-e_i}\|^2}\right)\|z^j\|^2 = \frac{\alpha + |j| - j_i - 1}{(\alpha+|j|)(\alpha+|j|-1)}\|z^j\|^2.$$

Thus, if $n > 1$ and $|j| = n$, then

$$\left|\langle S_i z^j, z^j\rangle\right| \le \frac{\alpha + 2n}{(\alpha+n)(\alpha+n-1)}\|z^j\|^2 \le \frac{2}{\alpha+n-1}\|z^j\|^2.$$

Hence, for $|j| \ge n$ we have $\left|c_{i,j}\right| = \frac{|\langle S_i z^j, z^j\rangle|}{\|z^j\|^2} \le \frac{2}{\alpha+n-1} \to 0$ as $n \to \infty$. This implies that S_i is compact and M_{z_i} is essentially normal.

Similarly, we compute

$$\langle \sum_{i=1}^{d} M_{z_i}^* M_{z_i} z^j, z^j\rangle = \sum_{i=1}^{d} \|z_i z^j\|^2 = \left(1 + \frac{d-\alpha}{\alpha+|j|}\right)\|z^j\|^2.$$

This implies that $\sum_{i=1}^{d} M_{z_i}^* M_{z_i} - I$ is compact.

Finally we show that $\sigma(M_z) \subseteq \operatorname{clos} \mathbb{B}_d$, or equivalently that the spectral radius of M_z is less than or equal to 1. We set $\psi(X) = \sum_{i=1}^{d} M_{z_i}^* X M_{z_i}, X \in \mathcal{B}(\mathcal{K}_\alpha)$. By the spectral radius formula, [41], we must show that $\limsup_{n\to\infty} \|\psi^n(I)\|^{\frac{1}{2n}} \le 1$. It is easy to see that $\psi^n(I)$ is diagonalized by the monomials, and one calculates that for any multiindex j

$$\langle \psi^n(I) z^j, z^j\rangle = \sum_{i_1,\dots,i_n=1}^{d} \left\|M_{z_{i_1}} \dots M_{z_{i_n}}\right\|^2 = \prod_{k=0}^{n-1}\left(1 + \frac{d-\alpha}{\alpha+|j|+k}\right)\|z^j\|^2.$$

Thus for $\alpha \ge d$ we see that $\|\psi^n(I)\| \le 1$ and for $0 < \alpha \le d$ we have that $\|\psi^n(I)\| \le \prod_{k=0}^{n-1}\left(1 + \frac{d-\alpha}{\alpha+k}\right)$. [37].

Proposition(2.2.10):[17] Let $\alpha > 0$, and $\mathcal{H} = \mathcal{K}_\alpha$. Then $\sigma(M_z) = \operatorname{clos} \mathbb{B}_d, \sigma(M_z) = \partial\mathbb{B}_d$, and for each $\lambda \in \mathbb{B}_d$ the augmented Koszul complex for $M_z - \lambda$ is exact.

Proof: We know from Lemma (2.2.9) that

$$\sigma_e(M_z) \subseteq \sigma(M_z) \subseteq \text{clos } \mathbb{B}_d.$$

Next we let $\lambda = 0$, and we proceed as in the proof of [42], we know that M_z is a Fredholm tuple. Hence the operator ∂_{M_z} has closed range. Observe that $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \ldots$, where $\mathcal{H}_n$ is the space of homogeneous polynomials of degree n. Thus, for each p we get $\Lambda^{\mathrm{p}}(\mathcal{H}) = \Lambda^p(\mathcal{H}_0) \otimes \Lambda^p(\mathcal{H}_1) \otimes \ldots$. The definition of $\partial = \partial_{M_z}$ implies that for each p and $n\partial_p$ takes $\Lambda^{\mathrm{p}}(\mathcal{H}_n)$ into $\Lambda^{\mathrm{p+1}}(\mathcal{H}_{n+1})$.

Now let $0 \leq p \leq d$ and $x \in \ker \partial_p$. Then $x = \sum_{n=0}^{\infty} x_n$ for $x_n \in \Lambda^{\mathrm{p}}(\mathcal{H}_n)$, and it is clear that $x_n \in \ker \partial_p$ for each n. We must show that $x \in \text{ran } \partial_p$. We already know that ran ∂_{p-1} is closed, so it is enough to show that each $x_n \in \text{ran } \partial_{p-1}$. This is equivalent to exactness of the Koszul complex at stage p for the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$, and that is well known [43].

Similarly, the exactness of the augmented complex at the last stage is clear, because $1 \in \mathcal{H}$.

Next we claim that $\mathcal{K}_\alpha$ is automorphism invariant, i.e. for each $\lambda \in \mathbb{B}_d$ composition with φ_λ defines a bounded invertible operator on $\mathcal{K}_\alpha$. Fix $\lambda \in \mathbb{B}_d$, and choose a branch of $f(u) = (1 - |\lambda|^2)^{-\frac{\alpha}{2}}(1 - u)^\alpha$ that is analytic for $u \in \mathbb{D}$. For $z \in \mathbb{B}_d$ set $g(z) = f(\langle z, \lambda \rangle)$. It follows from Lemma (2.2.9) preceding it that $\sigma(\langle z, \lambda \rangle) \subseteq \mathbb{D}$. Thus it is clear that the operator $f(\langle M_z, \lambda \rangle)$ as defined by the Riesz-Dunford functional calculus equals the multiplication operator M_g, i.e. g is a multiplier of $\mathcal{K}_\alpha$. Notice that the well-known transformation formula for ball automorphism [29], shows that $k_{\varphi_{\lambda(\omega)}}(\varphi_\lambda(z)) = g(z)\overline{g(\omega)}k_\omega(z)$ for all $z, \omega \in \mathbb{B}_d$. Thus the linear transformation T defined on the reproducing kernels by $Tk_\omega = k_{\varphi_{\lambda(\omega)}}$ extends to be a bounded operator of norm $\|M_g^*\|$. Hence T^* is also bounded, and it is easy to verify that T^* is the operator of composition with φ_λ.

The automorphism invariance of $\mathcal{K}_\alpha$ implies that the tuples M_z and $\varphi_\lambda(M_z)$ are similar, and the result about the exactness of the augmented complex follows from Lemma (2.2.8). This implies that $\sigma(M_z) = \text{clos } \mathbb{B}_d$ and $\sigma_e(M_z) \cap \mathbb{B}_d = \emptyset$;. It also implies that the index of $M_z - \lambda$ is $(-1)^d$ for each $\lambda \in \mathbb{B}_d$. Thus the continuity property of the index on the components of the complement of the essential spectrum implies that $\sigma_e(M_z) = \partial\mathbb{B}_d$ [37,43].

Let $\mathcal{H}$ be a Hilbert space of complex-valued analytic functions on the open, connected and nonempty set $\Omega \subseteq \mathbb{C}^d$. We assume $1 \in \mathcal{H}$. Then $M(\mathcal{H}) \subseteq \mathcal{H}$. We use k_λ to denote the reproducing kernel of $\mathcal{H}$. For $\lambda \in \Omega$ it is defined by the relation $f(\lambda) = \langle f, k_\lambda \rangle$ for every $f \in \mathcal{H}$. In the scalar-valued version of the main theorem we assume that the invariant subspace $\mathcal{M}$ contains a multiplier ϕ (see Theorem (2.2.1)). For the vector-valued versions it will be convenient to use operator-valued multipliers.

Let $\mathcal{D}$ and $\mathcal{E}$ be two separable Hilbert spaces, and let $\phi: \mathbb{B}_d \to \mathcal{B}(\mathcal{E}, \mathcal{D})$ be an operator valued analytic function. For $\lambda \in \mathbb{B}_d$ and $f \in \mathcal{H}_\mathcal{E}$ we define $(\Phi f)(\lambda) = \phi(\lambda)f(\lambda)$. Then Φf is a $\mathcal{D}$-valued analytic function. If $\Phi f \in \mathcal{H}_\mathcal{D}$ for every $f \in \mathcal{H}_\mathcal{E}$, then Φf is called an operator-valued multiplier, and the closed graph theorem shows that the associated multiplication operator $\Phi: \mathcal{H}_{\mathcal{E}} \to \mathcal{H}_{\mathcal{D}}$ is bounded. One hypothesis on the invariant subspace $\mathcal{M}$ of $\mathcal{H}_{\mathcal{D}}$ in the main theorem will be that there exists a separable Hilbert space E and a multiplication operator $\Phi \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}, \mathcal{H}_{\mathcal{D}})$ such that ran $\Phi \subseteq \mathcal{M}$. We will then see that the augmented complex $K\big((M_z - \lambda)|\mathcal{M}, \mathcal{M}_\lambda\big)$ is exact at every $\lambda \in \Omega \backslash Z(\mathcal{M})$ with ran $\phi(\lambda) = \mathcal{M}_\lambda$. It is known that every scalar multiplier invariant subspace $\mathcal{M}$ of $H_d^2(\mathcal{D})$ is of the form $\mathcal{M} = \text{ran}\Phi$ for some multiplication operator Φ[24,25]. It is easy to construct a multiplication operator Φ with ran$\Phi \subseteq \mathcal{M}$ for a scalar multiplier invariant subspace $\mathcal{M}$ of the Hardy or Bergman space whenever $\mathcal{M}$ contains some bounded functions (see the proof of Theorem (2.2.21)).

We start Lemma 3.1 of [33]. For the rest we let $\mathcal{M}$ be a scalar multiplier invariant subspace of $\mathcal{H}_{\mathcal{D}}$ of finite fiber dimension m, i.e.

$$m = \sup_{\lambda \in \Omega} \dim \mathcal{M}_\lambda < \infty,$$

and we let $\Phi \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}, \mathcal{H}_{\mathcal{D}})$ be a multiplication operator with associated operator valued analytic function $\phi: \mathbb{B}_d \to \mathcal{B}(\mathcal{E}, \mathcal{D})$. We assume that ran$\Phi \subseteq \mathcal{M}$ and that $\sup_{\lambda \in \mathbb{B}_d} \dim \text{ran}\, \phi(\lambda) = m$.

For $\lambda \in \Omega$ write $\mathcal{D}_\lambda = \text{ran}\Phi \subseteq \mathcal{D}$. Then since $\mathcal{D}_\lambda \subseteq \mathcal{M}_\lambda$ we have $\dim \mathcal{D}_{\lambda_0} = \dim \mathcal{M}_\lambda$, and $\mathcal{D}_\lambda = \mathcal{M}_\lambda$ whenever $\dim \mathcal{D}_\lambda = m$. We fix a $\lambda_0 \in \Omega$ with $\dim \mathcal{D}_{\lambda_0} = m$. Let $\{e_n\}_{n=1}^m$ be an orthonormal basis for $\ker \phi(\lambda_0)^\perp \subseteq \mathcal{E}$, and $\{d_k\}_{k=1}^m$ be an orthonormal basis for $\mathcal{D}_{\lambda_0} = \text{ran}\phi(\lambda_0) \subseteq \mathcal{D}$. We define the $m \times m$ matrix

$$M(\lambda) = \big(\langle \phi(\lambda)_{e_n}, d_k \rangle_{\mathcal{D}}\big)_{1 \le n,k \le m},$$

and the analytic function φ,

$$\varphi(\lambda) = \det M(\lambda).$$

The choice of λ_0 implies that $\psi(\lambda_0) \neq 0$. It is easy to check that all entries of the matrix M are multipliers of $\mathcal{H}$, and it follows that ϕ is a multiplier of $\mathcal{H}$ also. Finally, we write $P_{\mathcal{D}_\lambda}$ for the orthogonal projection of $\mathcal{D}$ onto $\mathcal{D}_\lambda$.

Lemma (2.2.11):[17] If $f \in \mathcal{M}$ is such that for each $\lambda \in \Omega$ we have $P_{\mathcal{D}_{\lambda_0}}\big(f(\lambda)\big) = 0$, then $f = 0$.

Proof: Let $f \in \mathcal{M}$ be as in the hypothesis. We will show that $\varphi(\lambda) f(\lambda) = 0$ for all $\lambda \in \Omega$. Since $\varphi \neq 0$ this will imply that $f = 0$. Let $\lambda \in \Omega$ such that $\varphi(\lambda) \neq 0$. We must show $f(\lambda) = 0$.

Since $\varphi(\lambda) \neq 0$, the matrix $M(\lambda)$ has full rank and the set of vectors

$$\{\phi(\lambda)e_1, \dots, \phi(\lambda)e_m\}$$

is linearly independent in $\mathcal{D}_\lambda$. Thus, $\dim \mathcal{D}_\lambda = m$ and $f(\lambda) \in \mathcal{M}_\lambda = \mathcal{D}_\lambda = \text{ran}\phi(\lambda)$. Hence there must be $a_1(\lambda), a_2(\lambda), \dots, a_m(\lambda) \in \mathbb{C}$ such that $f(\lambda) = \sum_{n=1}^m a_n(\lambda)\phi(\lambda)e_n$.

Now the hypothesis on f implies that for each $k = 1,2,\ldots,m$ we have

$$0 = \langle f(\lambda), d_k\rangle = \sum_{n=1}^{m} a_n(\lambda)\,\langle \phi(\lambda)e_n, d_k\rangle.$$

Thus $\big(a_1(\lambda), a_2(\lambda), \ldots, a_m(\lambda)\big)M(\lambda) = 0$. But $M(\lambda)$ has full rank, hence $a_1(\lambda) = \cdots = a_m(\lambda) = 0$, and it follows that $f(\lambda) = 0$.

Lemma (2.2.12):[17] If $x \in \mathcal{D}_{\lambda_0}$ then there is a $g_x \in \operatorname{ran}\Phi \subseteq \mathcal{M} \subseteq \mathcal{H}_{\mathcal{D}}$ such that

$$P_{\mathcal{D}_{\lambda_0}}\big(g(\lambda)\big) = \varphi(\lambda)x \text{ for all } \lambda \in \Omega\,, \tag{34}$$

$$hg_x \in \mathcal{M} \text{ for all } \; h \in \mathcal{H} \tag{35}$$

Proof: For $\lambda \in \Omega$ and $1 \le i,j \le m$ and $m \ge 2$ we let $b_{i,j}(\lambda)$ equal $(-1)^{i+j}$ times the determinant of the $(m-1)\times(m-1)$ matrix obtained from $M(\lambda)$ by deleting the j-th row and the i-th column. If $m = 1$, we set $b_{1,1}(\lambda) = 1$. Then each $b_{i,j}$ is a multiplier of $\mathcal{H}$, and the matrix $M^{+}(\lambda) = \big(b_{i,j}(\lambda)\big)_{1\le i,j\le m}$ is the adjoint matrix of $M(\lambda)$. It satisfies

$$M^{+}(\lambda)M(\lambda) = M(\lambda)M^{+}(\lambda) = \varphi(\lambda)I_m,$$

where I_m denotes the $m \times m$ identity matrix.

Now let $x \in \mathcal{D}_{\lambda_0}$ and set $f_x(\lambda) = \sum_{i,j=1}^{m} b_{j,i}(\lambda)\,\langle x, d_j\rangle_{e_i}$ for $\lambda \in \Omega$. Since $M(\mathcal{H}) \subseteq \mathcal{H}$, it is clear that $f_x \in \mathcal{H}_{\mathcal{E}}$. Thus we may set $g_x = \Phi f_x \in \operatorname{ran}\Phi$ and we claim that $P_{\mathcal{D}_{\lambda_0}}\big(g_x(\lambda)\big) = \varphi(\lambda)$ for all $\lambda \in \Omega$.

Since $\{d_n\}$ is an orthonormal basis for $\mathcal{D}_{\lambda_0}$ we have

$$P_{\mathcal{D}_{\lambda_0}}\big(g_x(\lambda)\big) = \sum_{n=1}^{m}\langle g_x(\lambda), d_n\rangle d_n = \sum_{n=1}^{m}\langle \phi(\lambda) f_x(\lambda), d_n\rangle d_n$$

$$= \sum_{n,i,j=1}^{m}\langle \phi(\lambda) b_{j,i}(\lambda)\langle x, d_j\rangle e_i, d_n\rangle d_n$$

$$= \sum_{n,j=1}^{m}\langle x, d_j\rangle \sum_{i=1}^{m} b_{j,i}(\lambda)\langle \phi(\lambda)e_i, d_n\rangle d_n$$

$$= \sum_{n,j=1}^{m}\langle x, d_j\rangle \phi(\lambda)\delta_{n,j} d_n = \phi(\lambda)x.$$

Thus, g_x satisfies (34).

If $h \in \mathcal{H}$, then $hf_x = \sum_{i,j=1}^{m} hb_{j,i}(\lambda)\langle x, d_j\rangle_{e_i} \in \mathcal{H}_{\mathcal{E}}$. Thus, for each $\lambda \in \Omega$ we have $h(\lambda)(\Phi f_x)(\lambda) = h(\lambda)\Phi(\lambda)f_x(\lambda) = \Phi(\lambda)(hf_x)(\lambda)$, $hg_x = \Phi(hf_x) \in \mathrm{ran}\Phi \subseteq \mathcal{M}$ and (35) follows.

If $\lambda \in \Omega$, then $\mathcal{D}_\lambda \subseteq \mathcal{D}$ so we can think of $\mathcal{H}_{\mathcal{D}_\lambda}$ as a subspace of $\mathcal{H}_{\mathcal{D}}$. We will write P_λ for the orthogonal projection of $\mathcal{H}_{\mathcal{D}}$ onto $\mathcal{H}_{\mathcal{D}_\lambda}$. It satisfies $(P_\lambda f)(z) = P_{\mathcal{D}_\lambda}(f(z))$ for every $f \in \mathcal{H}_{\mathcal{D}}$ and every $z \in \Omega$. Thus it is clear that P_λ intertwines every scalar multiplication operator and Lemma (2.2.11) says that $P_{\lambda_0}$is $1-1$ when restricted to $\mathcal{M}$.

Lemma (2.2.13):[17] Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\Omega \subseteq \mathbb{C}^d$ with $1 \in \mathcal{H}$, let $\mathcal{E}$ and $\mathcal{D}$ be separable Hilbert spaces, and let $\mathcal{M}$ be a scalar multiplier invariant subspace of $\mathcal{H}_{\mathcal{D}}$ with finite fiber dimension m.

If $\Phi \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}, \mathcal{H}_{\mathcal{D}})$ is a multiplication operator with associated operator-valued multiplier ϕ such that $\mathrm{ran}\Phi \subseteq \mathcal{M}$ and if $\lambda_0 \in \Omega\backslash \mathrm{Z}(\mathcal{M})$ such that $\mathrm{rank}\phi(\lambda_0) = m$, , then there exists a $\varphi \in M(\mathcal{H})$ with $\varphi(\lambda_0) = 1$ and there is a multiplication operator $\Psi \in \mathcal{B}\left(\mathcal{H}_{\mathcal{D}_{\lambda_0}}, \mathcal{H}_{\mathcal{D}}\right)$ with $\mathrm{ran}\Psi \subseteq \mathcal{M}$ and such that

$$P_{\lambda_0}\Psi f = M_\varphi f \text{ for every } f \in \mathcal{H}_{\mathcal{D}_{\lambda_0}}, \text{and } \Psi P_{\lambda_0} f = M_\varphi f \text{ for all } f \in \mathcal{M}.$$

Proof: We fix $\lambda_0 \in \Omega\backslash \mathrm{Z}(\mathcal{M})$ such that $\mathrm{rank}\phi(\lambda_0) = m$,, and we note that it is sufficient to construct a function φ and an operator Ψ that satisfy the conclusions of the lemma with the weaker condition $\varphi(\lambda_0) \neq 0$ instead of $\varphi(\lambda_0) = 1$.

We will continue to use the notation that was introduced before Lemma (2.2.11) and in Lemma (2.2.12). In particular, we already have the function $\varphi \in M(\mathcal{H})$ with $\varphi(\lambda_0) \neq 0$. In order to construct Ψ let $g_1, g_2, \ldots, g_m \in \mathcal{H}_{\mathcal{D}}$ satisfy conditions (38) and (39) of Lemma (2.2.12) with $x = d_1, d_2, \ldots, d_m$. For $\lambda \in \Omega$ we set $\psi(\lambda) = \sum_{n=1}^{m} g_n(\lambda) \otimes d_n$, and if $f \in \mathcal{H}_{\mathcal{D}_{\lambda_0}}$, then

$$(\Psi f)(\lambda) = \psi(\lambda)f(\lambda) = \sum_{n=1}^{m} \langle f(\lambda), d_n\rangle g_n(\lambda).$$

Equation (39) of Lemma (2.2.12) implies that $\Psi f \in \mathcal{M}$ for each $f \in \mathcal{H}_{\mathcal{D}_{\lambda_0}}$ and a simple argument with the closed graph theorem shows that Ψ is bounded. Thus Ψ is a multiplication operator with $\mathrm{ran}\Psi \subseteq \mathcal{M}$.

If $f \in \mathcal{H}_{\mathcal{D}_{\lambda_0}}$, then $f(\lambda) = \sum_{n=1}^{m} \langle f(\lambda), d_n\rangle d_n$. Hence the choice of the g_n*'s* and condition (38) of Lemma (2.2.12) imply that $P_{\lambda_0}\Psi f = \varphi f$. Finally, if $f \in \mathcal{M}$, then the function $h = \varphi f - \Psi P_{\lambda_0} f$ satisfies $h \in \mathcal{M}$ and

$$h(\lambda) = \varphi(\lambda)f(\lambda) - \sum_{n=1}^{m} \langle f(\lambda), d_n\rangle g_n(\lambda).$$

Hence it follows from condition (38) of Lemma (2.2.12) and the fact that $\{d_n\}$ forms an orthonormal basis for $\mathcal{D}_{\lambda_0}$ that $P_{\mathcal{D}_{\lambda_0}}(h(\lambda)) = 0$ for each $\lambda \in \Omega$.

Theorem (2.2.14):[17] Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\Omega \subset \mathbb{C}^d$ with the properties that $1 \in \mathcal{H}$, the coordinate functions z_i are multipliers, one can solve Gleason's problem in the multiplier algebra of $\mathcal{H}$, and $M_z - \lambda$ is a Fredholm tuple with exact augmented Koszul complex $K(M_z - \lambda)$ for all $\lambda \in \Omega$.

Let $\mathcal{D}$ be a separable Hilbert space and let $\mathcal{M}$ be a nonzero scalar multiplier invariant subspace of $\mathcal{H}_{\mathcal{D}}$ of finite fiber dimension m such that there is a Hilbert space $\mathcal{E}$ and a bounded multiplication operator $\Phi \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}, \mathcal{H}_{\mathcal{D}})$ with associated operator valued multiplier ϕ such that $\operatorname{ran}\Phi \subseteq \mathcal{M}$.

Then for every $\lambda \in \Omega \backslash Z(\mathcal{M})$ such that rank $\phi(\lambda) = m$ the augmented complex

$$K\big((M_z - \lambda)|\mathcal{M}, \mathcal{M}_\lambda\big)$$

is exact.

In particular, we have

$$\sigma_e(M_z|\mathcal{M}) \cap \Omega \subseteq \{\lambda \in \Omega : \text{rank } \phi(\lambda) < m\},$$

and the tuple $(M_z - \lambda)|\mathcal{M}$ is Fredholm with index $(-1)^d m$ for every $\lambda \in \Omega \backslash \sigma_e(M_z)$ whenever $\{\lambda \in \Omega \backslash Z(\mathcal{M}) : \text{rank } \phi(\lambda) = m\}$ is nonempty.

Proof: Note that whenever $\{\lambda \in \Omega \backslash Z(\mathcal{M}) : \text{rank } \phi(\lambda) = m\}$ is nonempty, then it must be connected and dense in Ω. Hence the statement in the last sentence follows from the exactness of the augmented complex and Lemma (2.2.6).

Let $\lambda_0 = (\lambda_{0_1}, \dots, \lambda_{0_d}) \in \Omega \backslash Z(\mathcal{M})$ be such that rank $\phi(\lambda_0) = m$. We must show that the augmented Koszul complex $K\left((M_z - \lambda_0)|\mathcal{M}, \mathcal{M}_{\lambda_0}\right)$ is exact.

The definition and finite-dimensionality of $\mathcal{M}_{\lambda_0}$ imply that δ_{λ_0} is onto, and the complex is exact at the last stage.

The hypothesis implies that $\mathcal{D}_{\lambda_0} = \mathcal{M}_{\lambda_0}$. Also note that since $\mathcal{H}_{\mathcal{D}_{\lambda_0}}$ is isomorphic to a direct sum of m copies of $\mathbb{C}$ the augmented Koszul complex $K\left((M_z - \lambda_0)\middle|\mathcal{H}_{\mathcal{D}_{\lambda_0}}, \mathcal{D}_{\lambda_0}\right)$ is isomorphic to a direct sum of m copies of the augmented complex $K(M_z - \lambda_0, \mathbb{C})$. Hence it is exact. Since $\mathcal{M}$ and $\mathcal{H}_{\mathcal{D}_{\lambda_0}}$ are M_z-invariant subspaces of $\mathcal{H}_{\mathcal{D}}$ it follows that the boundary maps for the Koszul complexes $K\big((M_z - \lambda_0)|\mathcal{M}\big)$ and $K\left((M_z - \lambda_0)\middle|\mathcal{H}_{\mathcal{D}_{\lambda_0}}\right)$ are the restrictions to $\Lambda(\mathcal{H})$ and $\Lambda\left(\mathcal{H}_{\mathcal{D}_{\lambda_0}}\right)$ of the boundary map $\partial_{M_z - \lambda}$ for the complex $K\big((M_z - \lambda_0)|\mathcal{H}_{\mathcal{D}}\big)$. We will write ∂ in all cases.

We will use the multiplier φ and the operator Ψ from Lemma (2.2.13). The function $1-\varphi$ is a multiplier that vanishes at λ_0. Since we assume that one can solve Gleason's problem in the multiplier algebra, there are $\varphi_1,\ldots,\varphi_d \in M(\mathcal{H})$ such that

$$1-\varphi(z)=\sum_{i=1}^{d}\left(z_i-\lambda_{0_i}\right)\varphi_i(z).$$

We define an operator A on $\Lambda(\mathcal{H}_{\mathcal{D}})$ by $A=\sum_{i=1}^{d} M_{\varphi_i}\otimes E_i^*$. Then by the anticommutation relations of the creation operators one can see that

$$\partial A+A\partial=\sum_{i,j=1}^{d} M_{(z-\lambda_0)_i}M_{\varphi_i}\otimes\left(E_iE_j^*+E_j^*E_i\right)$$

$$=\sum_{i=1}^{d} M_{(z-\lambda_0)_i}M_{\varphi_i}\otimes E_0=\left(I-M_{\varphi}\right)\otimes E_0 \tag{36}$$

We will now apply Lemma (2.2.5) with $T=M_{z-\lambda_0}|\mathcal{M}$ and $S=M_{z-\lambda_0}\Big|\mathcal{H}_{\mathcal{D}_{\lambda_0}}$. We set $B=A\Big|\Lambda\left(\mathcal{H}_{\mathcal{D}_{\lambda_0}}\right)$ and since $\mathcal{M}$ is multiplier invariant we can let $C=A|\Lambda(\mathcal{M})$. Furthermore, we take $X=\Psi$ and $Y=P_{\lambda_0}|\mathcal{M}$. Then it follows from Lemma (2.2.13) and equation (40) that the hypotheses of Lemma (2.2.5) are satisfied. Therefore, the cohomology spaces corresponding to $K\left(\left(M_{z-\lambda_0}\right)|\mathcal{M}\right)$ and $K\left(\left(M_{z-\lambda_0}\right)\Big|\mathcal{H}_{\mathcal{D}_{\lambda_0}}\right)$ are isomorphic as vector spaces. Since we have shown that $K\left((M_z-\lambda_0)\Big|\mathcal{H}_{\mathcal{D}_{\lambda_0}}\right)$ is exact at all but the last stage we also have that $K\left((M_z-\lambda_0)|\mathcal{M}\right)$ is exact at all but the last stage. This means that we have now shown that the augmented complex $K\left(\left(M_{z-\lambda_0}\right)|\mathcal{M},\mathcal{M}_{\lambda_0}\right)$ is exact at all stages except perhaps at the penultimate stage.

To show exactness at the penultimate stage we must show that $\operatorname{ran}\partial_{d-1}=\ker\delta_{\lambda_0}$ where the maps are understood to act on $\Lambda(\mathcal{M})$. Since we know that $\operatorname{ran}\partial_{d-1}\subseteq\ker\delta_{\lambda_0}$ it suffices to show that

$$\dim\frac{\Lambda^{\mathrm{d}}(\mathcal{M})}{\ker\delta_{\lambda_0}}=\dim\frac{\Lambda^{\mathrm{d}}(\mathcal{M})}{\operatorname{ran}\partial_{d-1}}.$$

It is clear that $\dim\frac{\Lambda^{\mathrm{d}}(\mathcal{M})}{\ker\delta_{\lambda_0}}=\dim\mathcal{M}_{\lambda_0}=m$, the fiber dimension of $\mathcal{M}$. Lemma (2.2.6) and the exactness of the augmented complex $K\left((M_z-\lambda_0)\Big|\mathcal{H}_{\mathcal{D}_{\lambda_0}},\mathcal{D}_{\lambda_0}\right)$ imply that $m=\dim\frac{\Lambda^{\mathrm{d}}\left(\mathcal{H}\left(\mathcal{D}_{\lambda_0}\right)\right)}{\operatorname{ran}\partial_{d-1}}$, hence by the earlier part of the proof it follows that $\dim\frac{\Lambda^{\mathrm{d}}(\mathcal{M})}{\operatorname{ran}\partial_{d-1}}=m$ as well.

Corollary (2.2.15):[17] Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on $\Omega\subset\mathbb{C}^d$ with the properties that $1\in\mathcal{H}$, the coordinate functions z_i are multipliers, one can solve

Gleason's problem in the multiplier algebra of $\mathcal{H}$, and $M_z - \lambda$ is a Fredholm tuple with exact augmented Koszul complex $K(M_z - \lambda, \mathbb{C})$ for all $\lambda \in \Omega$.

Let $\mathcal{D}$ be a finite dimensional Hilbert space and let $\mathcal{M}$ be a nonzero scalar multiplier invariant subspace of $\mathcal{H}_{\mathcal{D}}$ of finite fiber dimension m such that there is a Hilbert space $\mathcal{E}$ and a bounded multiplication operator $\Phi \in \mathcal{B}(\mathcal{H}_{\mathcal{E}}, \mathcal{H}_{\mathcal{D}})$ with associated operator valued multiplier ϕ such that $\operatorname{ran} \Phi \subseteq \mathcal{M}$.

Let $S \coloneqq M_z|\mathcal{M}$ and $T := P_{\mathcal{M}^\perp} M_z|\mathcal{M}^\perp$. Then if $\lambda \in \Omega \backslash Z(\mathcal{M})$ and $\mathcal{M}_\lambda = \operatorname{ran}\phi(\lambda)$, the tuple $T - \lambda$ is Fredholm with index $(-1)^d (\dim \mathcal{D} - m)$ and

$$0 \to K(T - \lambda) \overset{\delta_\lambda}{\to} \mathcal{M}_\lambda^\perp \to 0$$

is an exact complex, where $\delta_\lambda : \Lambda^d(\mathcal{M}^\perp) \to \mathcal{M}_\lambda^\perp$ is defined by $\delta_\lambda(f \otimes e_1 \wedge \ldots \wedge e_d) = P_{\mathcal{M}_\lambda^\perp} f(\lambda)$.

If $\lambda \in \Omega \cap Z(\mathcal{M})$, then $\dim \bigcap_{i=1}^d \ker\left(T_i^* - \bar{\lambda}_i\right) > \dim \mathcal{D} - m$. In particular it follows that $Z(\mathcal{M}) \subseteq \sigma_p(T^*)^*$.

Proof: Let $\lambda \in \Omega$. We first note that

$$0 \to K(T - \lambda) \overset{\delta_\lambda}{\to} \mathcal{M}_\lambda^\perp \to 0$$

can be identified with the quotient of the complexes $K(S - \lambda, \mathcal{M}_\lambda)$ and $K(M_z - \lambda, \mathcal{D})$ and is therefore a complex which we will denote by $K(T - \lambda, \mathcal{M}_\lambda^\perp)$.

For $p = 0, 1, \ldots, d$ let $l_p : \Lambda^p(\mathcal{M}) \to \Lambda^p(\mathcal{H}_{\mathcal{D}})$ and $l_{d+1} : \mathcal{M}_\lambda \to \mathcal{D}$ be the natural inclusion maps. Similarly let $\pi_p : \Lambda^p(\mathcal{H}_{\mathcal{D}}) \to \Lambda^p(\mathcal{M}^\perp)$ and $\pi_{d+1} : \mathcal{D} \to \mathcal{M}_\lambda^\perp$ be the natural projections. With these definitions of l and π one can easily check that

$$0 \to K(S - \lambda, \mathcal{M}_\lambda) \overset{l}{\to} K(M_z - \lambda, \mathcal{D}) \overset{\pi}{\to} K\left(T - \lambda, \mathcal{M}_\lambda^\perp\right) \to 0$$

is a short exact sequence of Hilbert space complexes. Therefore, by the Fundamental Theorem of Homological Algebra, [39], , there exists an induced long exact sequence of cohomology spaces. The argument at the beginning of the proof of Theorem (2.2.14) shows that $K(M_z - \lambda, \mathcal{D})$ is exact. This means all of the corresponding cohomology spaces of this complex are $\{0\}$.

Now assume that $\lambda \in \Omega \backslash Z(\mathcal{M})$ and $\mathcal{M}_\lambda = \operatorname{ran}\phi(\lambda)$. From Theorem (2.2.14) we know that $K(S - \lambda, \mathcal{M}_\lambda)$ is exact, hence its corresponding cohomology spaces are $\{0\}$. So we have that for each $p = 0, 1, \ldots, d, d + 1$

$$\ldots \to 0 \to \frac{\ker \partial_p}{\operatorname{ran} \partial_{p-1}} \to 0 \to \cdots$$

is part of a long exact sequence. Therefore, these cohomology spaces are also all equal to $\{0\}$. This means that $K\left(T - \lambda, \mathcal{M}_\lambda^\perp\right)$ is exact and since $\mathcal{M}_\lambda^\perp$ is finite dimensional, it follows that $T - \lambda$ is Fredholm and $\operatorname{ind}(T - \lambda) = (-1)^d (\dim \mathcal{D} - m)$.

Finally, we assume $\lambda \in \Omega \cap Z(\mathcal{M})$. Then $\dim \mathcal{M}_\lambda < m$, hence $\dim \mathcal{M}_\lambda^\perp > \dim \mathcal{D} - m$. The statement follows, because

$$\mathcal{M}_\lambda^\perp = \{x \in \mathcal{D}: \langle f(\lambda), x\rangle = 0\ \forall f \in \mathcal{M}\}$$

$$= \{x \in \mathcal{D}: \langle f, k_\lambda x\rangle = 0\ \forall f \in \mathcal{M}\}$$

$$= \{x \in \mathcal{D}: k_\lambda x \in \mathcal{M}^\perp\}$$

and this is isomorphic to $\{k_\lambda x \in \mathcal{M}^\perp\} = \bigcap_{i=1}^d \ker\left(T_i^* - \bar{\lambda}_i\right)$.

Let $\mathcal{H} = H_d^2$ be the Hilbert space of analytic functions on the unit ball of $\mathbb{C}^d$ defined by the kernel $k_\lambda(z) = \frac{1}{1-\langle z,\lambda\rangle}$. If $\mathcal{D}$ is a separable Hilbert space, then we will write $H_d^2\ (\mathcal{D}) = \mathcal{H}_\mathcal{D}$ for the space of $\mathcal{D}$-valued H_d^2 -functions see [18,24,25,33]. Tn particular, the polynomials are dense in H_d^2 and each coordinate function z_i is a multiplier. The tuple $M_z = (M_{z_1}, \dots, M_{z_d})$ on $H_d^2\ (\mathcal{D})$ is called the d-shift of multiplicity $\dim \mathcal{D}$. We note that each subspace that is invariant for M_z is in fact a scalar multiplier invariant subspace of $H_d^2\ (\mathcal{D})$ [33].

It was shown in [42] that the augmented Koszul complex for (M_z, H_d^2) is exact, and in Proposition (2.2.10) we have used that argument to show that the same is true for $(M_z - \lambda, H_d^2)$) for each $\lambda \in \mathbb{B}_d$. Arveson showed that for every scalar multiplier invariant subspace $\mathcal{M}$ of $H_d^2\ (\mathcal{D})$ there exists a Hilbert space $\mathcal{E}$ and a bounded multiplier $\Phi \in \mathcal{B}\left(H_d^2(\mathcal{E}), H_d^2(\mathcal{D})\right)$ such that ran $\Phi = \mathcal{M}$ [25].

We show that Gleason's problem for the multiplier algebra of H_d^2 can be solved one will have verified all the hypotheses of Theorem (2.2.14) for $\mathcal{H} = H_d^2$, $\Omega = \mathbb{B}_d$, and $\mathcal{M}$ any nonzero M_z-invariant subspace of $H_d^2\ (\mathcal{D})$.[30,31,45,46,47].

Theorem (2.2.16):[17] Let $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ be complex Hilbert spaces and let $S \subseteq \mathbb{B}_d$ arbitrary. Suppose that $\alpha: S \to B(\mathcal{F}, \mathcal{G})$ and $\beta: S \to B(\mathcal{E}, \mathcal{G})$ are given operator-valued functions. Then there is a multiplier $\psi: \mathbb{B}_d \to B(\mathcal{E}, \mathcal{F})$ with associated multiplication operator Ψ such that

$$\|\Psi\|_{\mathcal{B}\left(H_d^2(\mathcal{E}), H_d^2(\mathcal{F})\right)} \le 1 \quad \text{and} \quad \alpha(z)\psi(z) = \beta(z) \quad z \in S$$

if and only if the mapping

$$K_{\alpha,\beta}: S \times S \to \mathcal{B}(\mathcal{G}),\ \ K_{\alpha,\beta}(z,\omega) = \frac{\alpha(z)\alpha(\omega)^* - \beta(z)\beta(\omega)^*}{1 - \langle z, \omega\rangle}$$

is positive definite.

Corollary (2.2.17):[17] Let $\lambda \in \mathbb{B}_d$ and $\phi \in M(H_d^2)$ with $\phi(\lambda) = 0$. Then for each $i = 1, \dots, d$ there exists $\psi_i \in M(H_d^2)$ such that

$$\phi(z) = \sum_{i=1}^{d} (z_i - \lambda_i)\, \psi_i(z)$$

for all $z \in \mathbb{B}_d$, i.e. one can solve Gleason's problem for $M(H_d^2)$.

Proof: We will first prove the theorem for the case that $\lambda = 0$, and we shall assum that $\|\phi\|_{M(H_d^2)} \leq 1$. We will use Theorem (2.2.16) with $\alpha: \mathbb{B}_d \to \mathcal{B}(\mathbb{C}^d, \mathbb{C}), \alpha(z)(c_1, \dots, c_d) = z_1 c_1, \dots z_d c_d$, and $\beta: \mathbb{B}_d \to \mathcal{B}(\mathbb{C}, \mathbb{C}), \quad \beta(z) = \phi(z)$. Then we have

$$K_{\alpha,\beta}(z, \omega) = \frac{\alpha(z)\alpha(\omega)^* - \beta(z)\beta(\omega)^*}{1 - \langle z, \omega \rangle} = \frac{\langle z, \omega \rangle - \phi(z)\overline{\phi(\omega)}}{1 - \langle z, \omega \rangle} = \frac{1 - \phi(z)\overline{\phi(\omega)}}{1 - \langle z, \omega \rangle} - 1.$$

Since ϕ is a multiplier of norm 1 we know that the kernel

$$k_\omega(z) = \frac{1 - \phi(z)\overline{\phi(\omega)}}{1 - \langle z, \omega \rangle}$$

is positive definite, and since $\phi(0) = 0$ we have $k_0(z) = 1$. Hence there exists a Hilbert space $\mathcal{H}$ with kernel $k_\omega(z)$ and an orthonormal basis $\{k_0\} \cup \{e_i\}_{i \geq 1}$ of $\mathcal{H}$ such that

$$k_\omega(z) = 1 + \sum_{j \geq 1} e_j(z)\overline{e_j(\omega)}.$$

This implies that $K_{\alpha,\beta}(z, \omega) = k_\omega(z) - 1$ is a positive definite kernel. Thus by Theorem (2.2.16) there are functions $\psi_i(z)$ such that $\psi: \mathbb{B}_d \to \mathcal{B}(\mathbb{C}, \mathbb{C}^d)$ defined by

$$\psi(z) := \begin{pmatrix} \psi_1(z) \\ \vdots \\ \psi_d(z) \end{pmatrix}$$

is a multiplier in $\mathcal{B}\left(H_d^2, H_d^2(\mathbb{C}^d)\right)$ of norm less than or equal to 1 and with $\phi(z) = \sum_{i=1}^d z_i \psi_i(z)$. It is then clear that ψ_i is a multiplier of norm ≤ 1, in fact for $f \in H_d^2$ we have $\sum_{i=1}^d \|\psi_i f\|^2 \leq \|f\|^2$.

If $\lambda \in \mathbb{B}_d$ is arbitrary we use the ball automorphism φ_λ which takes λ to 0. We have already noted that composition with φ_λ defines a bounded invertible operator on H_d^2 ; [42]. Thus since ϕ is a multiplier it follows that $\phi \circ \varphi_\lambda^{-1}$ is a multiplier with $\left(\phi \circ \varphi_\lambda^{-1}\right)(0) \neq \phi(\lambda) = 0$. Hence from the first part of the proof we obtain multipliers g_i such that $\left(\phi \circ \varphi_\lambda^{-1}\right)(z) = \sum_{i=1}^d z_i g_i(z)$.

Now recall that the ith component of $\varphi_\lambda(z)$ can be written as

$$\left(\varphi_\lambda(z)\right)_i = \sum_{j=1}^d (z_j - \lambda_j) \frac{a_{ij}}{1 - \langle z, \lambda \rangle}$$

for some $a_{ij} \in \mathbb{C}$. Therefore,

$$\phi(z) = \sum_{i=1}^d \left(\varphi_\lambda(z)\right)_i g_i\left(\varphi_\lambda(z)\right)$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}(z_j-\lambda_j)\frac{a_{ij}}{1-\langle z,\lambda\rangle}\,g_i(\varphi_\lambda(z))$$

$$= \sum_{j=1}^{d}(z_j-\lambda_j)\psi_j(z)$$

where the functionsψ_j are multipliers of H_d^2 .

Theorem (2.2.18):[17] Let $\mathcal{D}$ be a separable Hilbert space and let $\mathcal{M}$ be a scalar multiplier invariant subspace of $H_d^2(\mathcal{D})$ with finite fiber dimension m. Then $\sigma_e(M_z|\mathcal{M})\cap\mathbb{B}_d\subseteq Z(\mathcal{M})$, and for every $\lambda\in\mathbb{B}_d\backslash Z(\mathcal{M})$ the tuple $(M_z-\lambda)|\mathcal{M}$ has index $(-1)^d m$ and the augmented complex

$$K((M_z-\lambda)|\mathcal{M},\mathcal{M}_\lambda)$$

is exact.

Proof: This follows directly from Theorem (2.2.14) since we have $\mathcal{M}=\text{ran}\Phi$ for some multiplication operator Φ [24,25]. It is thus clear that $\{\lambda\in\mathbb{B}_d:\text{rank}\phi(\lambda)<m\}=Z(\mathcal{M})$.

We will take $d=2$ and $\mathcal{D}=\mathbb{C}$. We claim that we have the isometric and orthogonal decomposition

$$H_2^2=H^2(\mathbb{D})\oplus z_2\,L_a^2\oplus z_2^2\mathcal{K},$$

where $\mathcal{K}$ is some Hilbert space of analytic functions on $\mathbb{B}_2$. One can check this either by computing the norms of monomials in H_d^2 as in Lemma (2.2.9) or as follows. Note that for $x\neq 1$ and $x+y\neq 1$ we have

$$\frac{1}{1-(x+y)}=\frac{1}{1-x}+\frac{y}{(1-x)^2}+\frac{y^2}{(1-x)^2\big(1-(x+y)\big)}.$$

Thus the reproducing the kernel of H_2^2 is of the form

$$k_\omega(z)=\frac{1}{1-\langle z,\omega\rangle_2}=k_{\omega_1}^1(z_1)+z_2\overline{\omega_2}k_{\omega_1}^2(z_1)+(z_2\overline{\omega_2})^2k_\omega^3(z),$$

where $k_{\omega_1}^1(z_1)=\frac{1}{1-\overline{\omega}_1 z_1}$ is the Szeg¨o kernel and $k_{\omega_1}^2(z_1)=\frac{1}{(1-\overline{\omega}_1 z_1)^2}$ is the Bergman kernel on the open unit disc, and k^3 is some positive definite kernel on the ball $\mathbb{B}_2$. It now follows from standard results about reproducing kernels that this decomposition of the kernel implies the decomposition of the space, because it is clear that the orthogonal summands have (0) as their intersection [48] .

Let $\mathcal{N}$ be an invariant subspace of the Bergman space such that $\dim\mathcal{N}\ominus(z-\lambda)\mathcal{N}$ is infinite for every $\lambda\in\mathbb{D}$. It is well known that such subspaces exist [20,22] . Set

$$\mathcal{M}=\{0\}\oplus z_2\mathcal{N}\oplus z_2^2\mathcal{K}.$$

It is clear that $\mathcal{M}$ is a closed invariant subspace of H_2^2 and that $Z(\mathcal{M}) = \{(z_1,0): |z_1| < 1\}$. We will now show that for no $(\lambda, 0) \in Z(\mathcal{M})$ is the tuple $(M_{z_1} - \lambda, M_{z_2})|\mathcal{M}$ Fredholm. In fact for $\lambda \in \mathbb{D}$ we will see that the defect dimension at the last stage of the Koszul complex for $(M_{z_1} - \lambda, M_{z_2})|\mathcal{M}$ is infinite. We will accomplish this by showing that whenever $f \in \mathcal{N} \ominus (z-\lambda)\mathcal{N}$, then the function g defined by $g(z_1,z_2) = z_2 f(z_1)$ is orthogonal to $(z_1 - \lambda)\mathcal{M} + z_2\mathcal{M}$. The result will follow, since $\mathcal{N} \ominus (z-\lambda)\mathcal{N}$ is infinite dimensional.

Since $g \in z_2\mathcal{N}$ it is clear that g is orthogonal to $z_2\mathcal{M} \subseteq z_2^2\mathcal{K}$. Let $h \in \mathcal{M}$. Then $h(z_1,z_2) = z_2 f_1(z_1) + z_2^2 k(z_1,z_2)$ for some $f_1 \in \mathcal{N}$ and $k \in \mathcal{K}$, and we see

$$\langle g, (z_1 - \lambda)h\rangle_{H_2^2} = \langle z_2 f_1, (z_1-\lambda)f_1\rangle_{H_2^2} + \langle g, z_2^2(z_1-\lambda)k\rangle_{H_2^2} = \langle f, (z_1-\lambda)f_1\rangle_{L_a^2} = 0 .$$

Theorem (2.2.19):[17] ([19]). Let $T = (T_1, \ldots, T_d)$ be a d-contraction on $\mathcal{H}$ with defect operator $\Delta_T = \left(I - \sum_{i=1}^d T_i T_i^*\right)^{\frac{1}{2}}$ and defect space $\mathcal{D} = \overline{\Delta_T\mathcal{H}}$. Let M_z be the d-shift associated to the space $H_d^2(\mathcal{D})$. Then there exists a spherical unitary tuple $Z = (Z_1, \ldots, Z_d)$ on a Hilbert space $\mathcal{K}$, an $(M_z^* \oplus Z^*)$-invariant subspace $\mathcal{M}^\perp$ of $H_d^2(\mathcal{D}) \oplus \mathcal{K}$, and a unitary operator $U: \mathcal{M}^\perp \to \mathcal{H}$ such that

$$U^* T_i U = P_{\mathcal{M}^\perp}(M_{z_i} \oplus Z_i)|\mathcal{M}^\perp$$

for all $i = 1, \ldots, d$. If T is a pure d-contraction then $\mathcal{K} = \{0\}$.[19].

Theorem (2.2.20):[17] If T is a pure d-contraction with finite rank with representation $T_i = P_{\mathcal{M}^\perp} M_{z_i}|\mathcal{M}^\perp$ for $\mathcal{M} \subseteq H_d^2(\mathcal{D}), \dim\mathcal{D} < \infty$, then $\sigma_e(T) \cap \mathbb{B}_d = \sigma_e(M_z|\mathcal{M}) \cap \mathbb{B}_d \subseteq Z(\mathcal{M})$ and $\sigma(T) \cap \mathbb{B}_d = \sigma_P(T^*)^*$.

If $\lambda \in \mathbb{B}_d \backslash \sigma_e(T)$, then

$$k(T) = (-1)^d \text{ind}\,(T - \lambda) \tag{37}$$

Furthermore, if $k(T) \neq 0$, then $\sigma(T) \cap \mathbb{B}_d = \sigma_P(T^*)^* = \mathbb{B}_d$ and if $k(T) = 0$, then $\sigma(T) \cap \mathbb{B}_d = \sigma_P(T^*)^* = Z(\mathcal{M})$.

Proof: Let $\lambda \in \mathbb{B}_d$. As in the proof of Corollary (2.2.15) we use the natural inclusion and projection maps to obtain a short exact sequence of Koszul complexes

$$0 \to K\big((M_z - \lambda)|\mathcal{M}\big) \to K(M_z - \lambda) \to K(T - \lambda) \to 0.$$

Thus, as $M_z - \lambda$ is a Fredholm tuple, it follows from the Fundamental Theorem of Homological Algebra [40] that $T - \lambda$ is Fredholm if and only if $(M_z - \lambda)|\mathcal{M}$ is Fredholm. Thus, $\sigma_e(T) \cap \mathbb{B}_d = \sigma_e(M_z|\mathcal{M}) \cap \mathbb{B}_d$.

From Theorem (2.2.18) we have that for $\lambda \in \mathbb{B}_d \backslash Z(\mathcal{M})$ the augmented Koszul complexes $K(M_z - \lambda, \mathcal{D})$ and $K\big((M_z - \lambda)|\mathcal{M}, \mathcal{M}_\lambda\big)$ are exact. So we can apply the results of Corollary (2.2.15) to obtain that $T - \lambda$ is Fredholm and $\text{ind}(T - \lambda) = (-1)^d(\text{rank}T - m)$. Thus Equation (37) follows by use [33], which states that for $\lambda \in \mathbb{B}_d \backslash Z(\mathcal{M}), k(T) = \text{rank}(T) - m$, where $k(T)$ is the curvature invariant of T. Finally it follows that Equation (41) holds for all $\lambda \in \mathbb{B}_d \backslash \sigma_e(T)$, since the index of $T-\lambda$ is constant on connected components of $\mathbb{C}\backslash\sigma_e(T)$.

Finally, we prove the last part of the Theorem. First we recall from Corollary (2.2.15) that $Z(\mathcal{M}) \subseteq \sigma_P(T^*)^* \subseteq \sigma(T) \cap \mathbb{B}_d$. If $k(T) \neq 0$ and $\lambda \in \mathbb{B}_d \backslash Z(\mathcal{M})$, then by the first part of the proof $\mathcal{M}_\lambda^\perp \neq (0)$. Thus, as in the proof of Corollary (2.2.15) we see that $\lambda \in \sigma_P(T^*)^*$. Hence $\sigma_P(T^*)^* = \sigma(T) \cap \mathbb{B}_d = \mathbb{B}_d$ in this case. If $k(T) = (0)$ and $\lambda \in \mathbb{B}_d \backslash Z(\mathcal{M})$, then the first part of the proof implies that $\mathcal{M}_\lambda^\perp = 0$ and $K(T-\lambda)$ is exact, hence $\lambda \notin \sigma(T)$. Thus,

$$\sigma(T) \cap \mathbb{B}_d = \sigma_P(T^*)^* = Z(\mathcal{M}).$$

Corollary (2.2.21):[17] Let $\mathcal{H}$ denote the Hardy or Bergman space of the ball or polydisc, or $\mathcal{H} = \mathcal{K}_\alpha, \alpha \geq d$, let $\mathcal{D}$ be a separable Hilbert space, and let $\mathcal{M}$ be an invariant subspace of $\mathcal{H}_\mathcal{D}$ of finite fiber dimension m.

If $\lambda \in \Omega$ and there are bounded functions $f_1, \ldots, f_m \in \mathcal{M}$ such that the set $\{f_1(\lambda), \ldots, f_m(\lambda)\}$ is linearly independent, then the tuple $(M_z - \lambda)|\mathcal{M}$ is Fredholm with index $(-1)^d m$, and the augmented complex $K\big((M_z-\lambda)|\mathcal{M}, \mathcal{M}_\lambda\big)$ is exact.[49,50].

Proof: We know from Proposition (2.2.10) all the spaces $\mathcal{H}$ satisfy the hypothesis of Theorem (2.2.14). The multipliers are either H^∞ of the ball or the polydisc and in both cases it is known that one can solve Gleason's problem [29].

We let $\mathcal{E} = \mathbb{C}^m$ and for $g = (g_1, \ldots, g_m) \in \mathcal{H}_\mathcal{E}$ we set $\Phi g = \sum_{i=1}^m f_i g_i$. It is clear that Φ is a multiplication operator. Thus the corollary follows from Theorem (2.2.14).

If $\mathcal{H}$ is the Bergman space of a bounded region Ω in $\mathbb{C}^d$, then it easily follows from a theorem of Bercovici in [28] that there are invariant subspaces where $\overline{\Omega}$ is contained in the essential spectrum.

In the case $d = 1$ the existence of invariant subspaces with high index on the whole disc is connected with the nonexistence of nontangential limits of the functions in the space [23]. Thus, as far as we know, it is conceivable that for every invariant subspace $\mathcal{M}$ of the Hardy spaces on the ball or polydisc of $\mathbb{C}^d, d > 1$, $(M_z-\lambda)|\mathcal{M}$ is a Fredholm tuple for all λ in a large subset of the region. That the situation is more complicated than for H_d^2 can be illustrated by constructions that are similar to what we have done for H_2^2.

Indeed, the reproducing kernel for the Hardy space of $\partial\mathbb{B}_2$ is

$$\frac{1}{(1-\langle z,\omega\rangle)^2} = \frac{1}{(1-\bar{\omega}_1 z_1)^2} + \bar{\omega}_2 z_2 k_\omega(z)$$

for some positive definite kernel k on $\mathbb{B}_2$. Hence as before $H^2(\partial\mathbb{B}_2) = L_a^2 \oplus z_2\mathcal{K}$ for some space $\mathcal{K}$ of analytic functions on $\mathbb{B}_2$, and one can proceed as above and consider invariant subspaces of the type $\mathcal{M} = \mathcal{N} \oplus z_2\mathcal{K}$. The new feature is that the set of common zeros of $\mathcal{M}$ may be empty, and we note that in this case the essential Taylor spectrum of $M_z|\mathcal{M}$ has a part in the ball but outside the set of common zeros of functions in $\mathcal{M}$.

Similarly, for the Hardy space $H^2(\mathbb{D}_2)$ of the bidisc we consider the map $P: f \to g$, where $g(z) = f(z,z)$. [26], it is well known that this is a partial isometry of $H^2(\mathbb{D}_2)$ onto L_a^2. It then follows easily that for every invariant subspace $\mathcal{N}$ of L_a^2 the space $\mathcal{M} = P^*\mathcal{N} + \ker P$ is an invariant subspace of $H^2(\mathbb{D}_2)$. Furthermore, if $\dim \mathcal{N} \ominus z\mathcal{N}$ is infinite, then one shows that every λ on the diagonal of $\mathbb{D}_2$ is contained in the essential Taylor spectrum of $M_z|\mathcal{M}$, even though most of those points will not be in the set of common zeros of $\mathcal{M}$.[51].

Chapter 3

Similarity and Reducing Manifolds with Nearly Invariant Subspace

We show a new approach and extended a theorem of D_0 Hitt describing certain subspaces of H^2 that miss by one dimension being invariant under the backward shift operator.

Section(3.1): Unitary Equivalence of Volterra Operators

We are concerned with Volterra operators T_F where $T_F f(x) = \int_x^1 F(x,y)f(y)dy$ mapping $L_p[0,1]$ into itself $(1 < p < \infty)$ and study their similarity, reducing manifolds (i.e., subspaces $S \subset L_p[0,1]$ such that $T_F S \subset S$), and in the case $p = 2$ their unitary equivalence. These operators are continuous analogs of nilpotent n by n triangular matrices $M = (m_{ij})$ where $m_{ij} = 0$ for $i \geqq j$. The starting point of this investigation is provided by the following two simple theorems about matrices M of maximal index of nilpotency: (i) the complete set of reducing manifolds of M consists of the subspaces spanned by $e_1, \dots, e_i, (1 \le i \le n)$ where the e_i are a basis relative to which M is triangular; (ii) every such matrix M is unitarily equivalent to a triangular matrix where $m_{i,i+1} > 0$ and two such matrices are unitarily equivalent if and only if they are equal. Their similarity invariants are well known: any two such matrices are similar. The continuous analog of "maximal index of nilpotency" turns out to be the following type of condition: $F(x,y) = (y - x)m - G(x,y)$ where $G(x,x) \neq 0$. We deals with similarity properties of our operators and establishes what amounts to "canonical forms under similarity" of the functions F. We lean on results by Volterra and Volterra and Peres [54,55]. It should be noted that the result of Lemma (3.1.4) was improved somewhat by Lilsis[56]; this improvement would correspondingly improve slightly several of the results based on that Lemma. For the functions F considered the only reducing manifolds of T_F are the spaces $L_p[0,a]$ for all $a \in [0,1]$. We close that with two examples.

We are dealing with functions of the two variables x and y defined on the triangle $0 \leqq x \leqq y \leqq 1$. Such functions will be denoted by capital letters thus: $F(x,y)$. Unless a statement is made to the contrary, these functions will always be of the form $F(x,y) = (y - x)^{m-1}G(x,y)$ where the complex valued function G is continuously differentiable; $G(x,x)$ is real valued and different from 0. The positive integer m is called the order of F [55]. If F depends only on $y - x$ we generally use lower case letters: $F(x,y) = F(y - x) = f(y - x)$. We write

$$T_F\, g(x) = \int_x^1 F(x,y)g(x)dy.$$

To the product $T_{F_1} T_{F_2}$ of two transformations corresponds a third function F_3 such that $T_{F_1} T_{F_2} = T_{F_3}$ where F_3 is given by

$$F_3(x,y) = \int_x^y F_1(x,z)F_2(z,y)dz\,;$$

we use the notation $F_1^* F_2$ for F_3 and $F^{*,m}$ for $F^{*\cdots *}F$ (m factors). If $p(z) = \sum_0^N a_i z^i$, we write $p^*(F) = \sum_0^N a_i F^{*,i+1}$. Note that if $F_i(x,y) = f_i(y - x)(i = 1,2), f_1^* f_2$ is again a function of $y - x$ and

$$f_1^* f_2(y-x) = \int_0^{y-x} f_1(y-x-u) f_2(u) du = f_2^* f_1(y-x).$$

Lemma(3.1.1):[53] Let $F(x,y)$ be measurable and bounded such that $|F(x,y)| \leqq K$ for all x and y such that $0 \leqq x \leqq y \leqq 1$. Let $1 \leqq p < \infty$. Then T_F is a bounded, generalized nilpotent linear transformation mapping $L_p[0,1]$ into itself and $\|T_F\| \leq K$.

Proof: Except for generalized nilpotency, the assertions of this Lemma follow from the inequalities

$$|T_F f(x)| \leqq \int_x^1 |F(x,y) f(y)| dy \leqq K \int_x^1 |f(y)| dy \leqq K\|f\|_1 \leqq K\|f\|_p$$

(the last inequality results from the fact that the basic interval of integration has measure 1). Generalized nilpotency follows from the inequality

$$\left|F^{*,i}(x,y)\right| \leq \frac{K^i (y-x)^{i-1}}{(i-1)!}.$$

Lemma(3.1.2):[53] Let $1 \leqq p < \infty$. Let $k \in L_1[0,1], f \in L_p[0,1]$. Then T_k is a bounded linear transformation mapping $L_p[0,1]$ into itself such that $\|k^* f\|_p = \|T_k f\|_p \leqq \|k\|_1 \|f\|_p$ whence $\|T_k\| \leqq \|k\|_1$. Hence $k^{*,i} \in L_1[0,1]$ for all positive integral i and $k_i \to k$ in $L_1[0,1]$ implies that $T_{ki} \to T_k$ uniformly.

Proof: The assertions of the lemma are implied by the well-known inequality $\|k^* f\|_p \leqq \|k\|_1 \|f\|_p$. Thus $T_k f \in L_p[0,1]$ for $f \in L_p[0,1]$ and

$$\|T_k f\|_p = \|k^* f\|_p \leqq \|k\|_1 \|f\|_p$$

Two continuous linear transformations T_1 and T_2 mapping $L_p[0,1]$ into itself are called similar if there exists a continuous linear transformation P mapping $L_p[0,1]$ onto itself with the continuous linear inverse P^{-1} such that $T_1 = PT_2P^{-1}$. Two continuous linear transformations mapping $L_2[0,1]$ into itself are called unitarily equivalent if there exists a unitary linear transformation U such that $T_1 = UT_2U^* = UT_2U^{-1}$. We will in most instances be able to restrict the linear transformations P and U implementing similarity and unitary equivalence to products of linear transformations of the following three kinds: (1) multiplication by a measurable function $h(x)$: $M_h f(x) = h(x) f(x)$; (2) substitution (change of measure of $[0,1]$) using a monotone function $r(t)$ mapping $[0,1]$ onto itself such that $r(O) = 0$ and $r(1) = 1$: $S_r f(x) = f(r(x))$; (3) linear transformations of the type $I + T_M$ where I is the identity transformation and T_M is generalized nilpotent.

If $F(x,y) = (y - x)^{m-1}G(x,y)$ is of order m and $G \in C^1$ we say that F is canonical with constant c if

$$G(x,x) = c, \;\; G_x(x,x) = G_y(x,x) = 0$$

We use the standard notations L_p and L_q where $1/p + 1/q = 1$; if $f \in L_p[0,1]$ and $g \in L_q[0,1]$ and $p \neq 2$, we write $(f,g) = \int_0^1 f(x)g(x)\,dx$; if f and g are in $L_2[0,1]$, we write $(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx$. The set of all functions $f \in L_p[0,1]$ such that $f(x) = 0$ a.e. for all $x \geq a$ is called $L_p[0,1]$. The subset of $L_p[0,a]$ consisting of all functions which are in no $L_p[0,a']$ for $0 \leq a' < a$ will be called $L_p^d[0,a]$. The function identically equal to 1 will be denoted by E or simply by 1.

Lemma(3.1.3):[53] If $F(x,y) = (y - x)^{m-1}G(x,y)$ is of order m then there exists a function $H(x,y)$ of order m which is canonical with constant

$$c = sign\, G(x,x)\left(\int_0^1 |G(u,u)|^{\frac{1}{m}} du\right)^m$$

such that T_F is similar to $T_H = P^{-1}T_F P$. This is achieved by setting $P = S_r M_h$ where $r(t) = \int_0^1 \left(\frac{G(u,u)}{c}\right)^{\frac{1}{m}} du$ and where the function h is determined as follows: define $F_1(x,y) = (y-x)^{m-1}G_1(y,x)$ by $T_{F_1} = S_r^{-1}T_F S_r$. Then

$$h(t) = \exp\left(\left(\frac{1}{c}\right)\int_0^t G_{1x}(u,u)du\right).$$

Lemma(3.1.4):[53] If F is analytic in x and y in a suitable region and if it is of order m then there exists (exactly one) function $G(x,y)$ of order 1 and real for $y = x$ analytic in the same region such that $T_G^m = T_F$. The same conclusion holds if $F(x,y) = f(y-x) = (y-x)^{m-1}f_1(y-x)$ is of order m and $f_1 \in C^2$. The m^{th} root $G(x,y)$ of $f(y-x)$ is in C^2 and is of the form $G(y - x)$.[54].

Theorem (3.1.5):[53] Let $F \in C^2$ be of order 1. Then T_F is similar to cT_E where c is defined on Lemma (3.1.3) (with $m = 1$). More precisely, $cT_E = PP_1^{-1}T_F P_1 P^{-1}$ where $p_1 1$ is as in Lemma (3.1.3) so that $P_1^{-1}T_F P_1 = T_K$ where K is canonical with constant c (and of order 1) and $P = I + T_M$ and M is determined by the integral equation

$$K + M^* K = c(1 + 1^* M) \tag{1}$$

In fact if the continuous function $M(0,y)$ is prescribed, there exists a continuous solution of (1) which is unique and for which $(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})M(x,y)$ exists.

Proof: Note first of all that for the function M which we are going to find, P^{-1} will exist since T_M is generalized nilpotent by Lemma(3.1.1). Dividing (1) by c and setting $K/c = L$, we are led to consider

$$L(x,y) + \int_x^y M(x,\omega)L(\omega,y)d\omega = 1 + \int_x^y M(\omega,y)d\omega \tag{2}$$

This equation is equivalent to the following equation where $r = (y-x)/2, s = (y+x)/2$:

$$M(x,y) = \int_r^s \int_{v-r}^{v+r} M(v-r,\omega)L_{yy}(\omega,v+r)d\omega dv$$

$$-\int_x^y M(x,\omega)L_y(\omega,y)d\omega + \int_0^{2r} M(0,\omega)L_y(\omega,2r)d\omega \tag{3}$$

$$-\int_r^s L_{xy}(v-r,v+r)dv + h(2r)$$

where $h(t)$ is arbitrary.

Equation (3) can be derived from (2) under the assumption that $M \in C^1$. After showing this, we show that (2) and (3) are equivalent if we merely assume that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) M(x,y)$ exists. On differentiating (2) with respect to x and y we obtain

$$\frac{\partial}{\partial s} M(x,y) = M_x(x,y) + M_y(x,y)$$

$$= -\int_x^y M_x(x,\omega)L_y(\omega,y)d\omega + M(x,x)L_y(x,y) - L_{xy}(x,y)$$

whence

$$M(x,y) = -\int_r^s \int_{v-r}^{v+r} M_x(v-r,\omega)L_y(\omega,v+r)d\omega dv$$

$$+\int_r^s M(v-r,v-r)L_y(v-r,v+r)dv - \int_r^s L_{xy}(v-r,v+r)dv + h(2r)$$

If we now interchange the order of integration in the double integral, then integrate by parts in order to eliminate M_x, and then once more interchange the order of integration in the double integral, we obtain (3).

We shall obtain below the unique solution of (3) (with prescribed $M(0,y)$) which has the property that $\frac{\partial}{\partial s}M(x,y)$ exists. To establish the equivalence of (2) and (3) using only the assumption that $\frac{\partial}{\partial s}M(x,y)$ exists, we proceed as follows. On differentiation of (3) with respect to s we obtain

$$\frac{\partial}{\partial s}\left(M(x,y)+\int_x^y M(x,\omega)L_y(\omega,y)d\omega\right)=\int_x^y M(x,\omega)L_{yy}(\omega,y)d\omega-L_{xy}(x,y) \qquad (4)$$

Or $A(x,y)=B(x,y)$.We now from

$$\int_x^y\int_\xi^y A(\xi,\eta)d\eta d\xi=\int_x^y\int_\xi^y B(\xi,\eta)d\eta d\xi \qquad (5)$$

On the left we make the following change of variables: $\xi=\sigma-\rho$ and $\eta=\sigma+\rho$ and obtain after one integration

$$2\int_0^r\left(M(y-2\rho,y)+\int_{y-2\rho}^y M(y-2\rho,\omega)L_y(\omega,y)d\omega\right)d\rho$$

$$-2\int_0^r\left(M(x,2\rho+x)+\int_x^{2\rho+x} M(x,\omega)L_y(\omega,2\rho+x)d\omega\right)d\rho$$

or by appropriate changes of variables

$$\int_x^y M(\omega,y)d\omega+\int_x^y\int_v^y M(v,\omega)L_y(\omega,y)d\omega\, dv$$

$$-\int_x^y M(x,\omega)d\omega-\int_x^y\int_x^v M(x,\omega)L_y(\omega,v)d\omega\, dv$$

Finally, after integrating the last integral by parts, we obtain for the left hand side of (5)

$$\int_x^y M(\omega,y)d\omega+\int_x^y\int_v^y M(v,\omega)L_y(\omega,y)d\omega\, dv$$

$$-\int_x^y M(x,\omega)d\omega-\int_x^y M(x,\omega)L(\omega,y)d\omega+\int_x^y M(x,\omega)d\omega$$

The right hand side of (5) becomes after an integration by parts of its first integral $\int_x^y \int_v^y M(x,\omega)L_y(\omega,y)d\omega\, dv - 1 + L(x,y)$. On equating these two expressions, we obtain (2). In order to obtain (3) from (2) we note first that we can reverse the steps leading from (3) to (2) until we get (4). To go from (4) to (3) we observe that in (3), $M(0,y) = h(y)$. We now see that on integrating (4) with respect to s from r to s, we obtain (3).

We now proceed to solve (3), where we replace $M(0,\omega)$ by $h(\omega)$, i.e., we actually solve the following equation:

$$M(x,y) = \int_r^s \int_{v-r}^{v+r} M(v-r,\omega)L_{yy}(\omega,v+r)d\omega\, dv$$
$$-\int_x^y M(x,\omega)L_y(\omega,y)d\omega + \int_0^{2r} h(\omega)L_y(\omega,2r)d\omega \tag{6}$$
$$-\int_r^s L_{xy}(v-r,v+r)dv + h(2r)$$

Any solution of (6) provides one of (3), since if $M(x,y)$ satisfies (6), we see that $M(0,y) = -\int_0^y M(0,y)L_y(\omega,y)d\omega + \int_0^y h(\omega)L_y(\omega,y)d\omega + h(y)$ so that for M, $M(0,y) = h(y)$. We now rewrite (6) thus:

$$(I-T)M(x,y) = \int_0^{2r} h(\omega)L_y(\omega,2r)d\omega \ - \int_r^s L_{xy}(v-r,v+r)dv + h(2r)$$
$$= R(x,y) \tag{7}$$

where the linear transformation T is defined by

$$TG(x,y) = \int_r^s \int_{v-r}^{v+r} G(v-r,\omega)L_{yy}(\omega,v+r)d\omega\, dv - \int_x^y G(x,\omega)L_y(\omega,y)d\omega$$

If $\frac{\partial}{\partial s}G(x,y)$ exists, $\frac{\partial}{\partial s}T^n G(x,y)$ also exists for all positive integral n. Our hypotheses regarding $M(0,y) = h(y)$ and $L(x,y)$ imply the existence of a positive constant C independent of x and y such $\text{Max}\big(|R(x,y)|, |L_y(x,y)|, |L_{xy}(x,y)|, |L_{yy}(x,y)|, |h(x)|, 1\big) \leqq C$ .Then $|T^n R(x,y)| \leqq \frac{C(2C(y-x)^n)}{n!}$ and $\left|\frac{\partial}{\partial s}T^n R(x,y)\right| \leqq \frac{C(4C(y-x)^n)}{n!}$ These inequalities imply that (7) has the solution $\sum_0^\infty T^n R(x,y)$ and that $\frac{\partial}{\partial s}M(x,y)$ exists and equals $\sum_0^\infty \frac{\partial}{\partial s}T^n R(x,y)$ since our hypotheses imply that $\frac{\partial}{\partial s}R(x,y)$ exists.

The uniqueness of the solution $M(x,y)$ of (2) or (3) is established as follows: Suppose the two solutions M_1 and M_2 have M_0 as their difference and that $M_1(0,y) = M_2(0,y)$ so that $M_0(0,y) = 0$. Then M_0 satisfies

$$\int_x^y M_0(x,\omega)L(\omega,y)d\omega = \int_x^y M_0(\omega,y)d\omega \tag{8}$$

By making computations similar to those which relate (2) and (3), we see that M_0 must satisfy

$$(I-T)M_0(x,y) = h_0(2r) \tag{9}$$

The requirement that $M_0(0,y) = 0$ implies that $h_0(y) = 0$ so that (9) implies that $M_0(x,y) = 0$.

Corollary(3.1.6):[53] If F is analytic in x and y in a suitable region and if it is of order m then T_F is similar to $c\,T_E^m$ where c is the constant of Lemma(3.1.3). The same conclusion holds if $F(x,y) = f(y-x) = (y-x)^{m-1}f_1(y-x)$ is of order m and $f \in C^2$.

Lemma(3.1.7):[53] If T_1 and T_2 are continuous linear transformations mapping L_p into itself which are not nilpotent, then similarity of T_1 and T_2 implies that

$$\lim_n \left(\frac{\|T_1^n\|^{\frac{1}{n}}}{\|T_2^n\|^{\frac{1}{n}}} \right) = 1$$

Proof: This is implied by the inequalities $(\|P\|\|P^{-1}\|)^{-1}\|T_2^n\| \leq \|T_1^n\| \leqq \|P\|\|P^{-1}\|\|T_2^n\|$ if we write $T_2 = pT_1p^{-1}$. If $r \geq 1$ is a real number, let us define T_E^r as T_F with $F(x,y) = (y-x)^{r-1}/\Gamma(r)$.

Theorem(3.1.8):[53] If c_1 and c_2 are real numbers and r_1 and r_2 are real numbers such that $r_i \geq 1$, then $c_1T_E^{r_1}$ is similar to $c_2T_E^{r_2}$ if and only if $c_1 = c_2$ and $r_1 = r_2$.

Proof: Suppose that $r_1 > r_2$. Then

$$\operatorname{Lim}_n \left(\frac{\left\|\left(c_1T_E^{r_1}\right)^n\right\|^{\frac{1}{n}}}{\left\|\left(c_2T_E^{r_2}\right)^n\right\|^{\frac{1}{n}}} \right) = \left(\frac{|c_1|}{|c_2|}\right)\lim_n \left(\frac{\left\|\left(T_E^{r_1}\right)^n\right\|^{\frac{1}{n}}}{\left\|\left(T_E^{r_2}\right)^n\right\|^{\frac{1}{n}}} \right) \leqq \left(\frac{|c_1|}{|c_2|}\right)\lim_n \left\|\left(T_E^{r_1-r_2}\right)^n\right\|^{\frac{1}{n}} = 0$$

since T_E^r is generalized nilpotent for $r > 1$ (see Lemma(3.1.1)). Lemma(3.1.7) implies, therefore, that $r_1 = r_2$. Let us assume next that $r_1 = r_2 = r$. The equation $\operatorname{Lim}_n \left(\frac{\|(c_1T_E^r)^n\|^{\frac{1}{n}}}{\|(c_2r)^n\|^{\frac{1}{n}}} \right) = \left(\frac{|c_1|}{|c_2|}\right)$ implies by Lemma(3.1.7) that we must have $|c_1| = |c_2|$. Suppose finally that T_E^r is similar to $-T_E^r = pT_E^rp^{-1}$. Then P commutes with T_E^r. We now proceed to show that $T_E^r = \lim_{\mathrm{jp_j}}(T_E^{2r}\ )$uniformly for polynomials p_j which implies that P commutes also with T_E^r contradicting $PT_E^rP^{-1} = -T_E^r$. In view of Lemma (3.1.2) it suffices to show that

$t^{r-1}/\Gamma(r)$ is the L_1-limit of polynomials of the form $\sum_1^N a_j t^{2rj-1}/\Gamma(2rj)$ where we have written t for $y-x$. Let $\{e_i\}$ be a sequence of positive real numbers converging to 0. Let

$$f_i(t) = \begin{cases} b_i t & \text{for } 0 \leqq t \leqq e_i \\ t^{-r} & \text{for } e_i \leqq t \leqq 1 \end{cases}$$

be continuous functions (i.e., $b_i e_r^{r+1} = 1$). The Stone-Weierstrass theorem implies the existence of polynomials $p_i(t^{2r}) = \sum_{j=0}^{N_i} a_{ij} t^{2rj}$ such that $|p_i(t^{2r}) - f_i(t)| \leqq e_i$.

For $0 \leqq t \leqq 1$. It is now an easy matter to verify that $t^{2r-1} p_i(t^{2r}) \to t^{r-1}$ in L_1 or that $\sum_{k=1}^{N_i+1} a_{i,k-1} \left(\Gamma(2rk)/\ \Gamma(r)\right) t^{2rk-1}/\Gamma(2rk) \to t^{r-1}/\ \Gamma(r)$ in L_1.

Corollary(3.1.9):[53] If F is as described in Lemma (3.1.4) or Theorem (3.1.5), then T_F is similar to a unique operator cT_E^m where c is as in Lemma (3.1.3) and m is the order of F.

Lemma(3.1.10):[53] $T_{D\,a,h}$ is the uniform limit of polynomials in T_E^m without constant term for all $m \geqq 1$.

Proof: If $a > 0$, $t^{1-m} D_{a,h}(t)$ is the L_1 limit of polynomials of the form

$$\sum a_i t^{mi}/\Gamma(m),$$

hence $D_{a,h}(t)$ is the L_1 limit of polynomials of the form $\sum a_i t^{m(i+1)-1}/\Gamma(m)$. Lemma (3.1.2) then implies the truth of the present lemma in the case considered. If $a = 0$, observe that $D_{0,h}(t)$ is the L_1 limit of the functions $D_{e,h}(t)$ as $e \to 0$ and the same conclusion holds.

Lemma(3.1.11):[53] If $f \in L_p^d[0,c]$ then the functions $T_E^m f, T_E^{2m} f, \ldots$ and their linear combinations are dense in $L_p[0,c]$ for all $c \in [0,1]$.

Proof: Let $g \in L_q[0,c]$ and consider $\left(T_E^{mi} f, g\right)$ for $= 1, \ldots$. By Lemma(3.1.10), $T_{D_{a,h}}$ is a uniform limit of polynomials in T_E^m without constant term. Thus if we assume that $\left(T_E^{mi} f, g\right) = 0$ for all $i > 0$, we have

$$I_{a,h} = \left(T_{D_{a,h}} f, g\right) = 0 \tag{10}$$

for all a and h such that $0 \leqq a < c, 0 < h \leqq c - a$. We now apply Fubini's theorem to $I_{a,h} = \int_0^c \int_x^c D_{a,h}(y-x) f(y) g(x) dy dx$ and obtain

$$I_{a,h} = \int_a^c \int_{y-a-h}^{y-a} f(y) g(x) dy dx$$

where we put $g = 0$ outside $[0,c]$. Let

$$c_{a,h} = \frac{I_{a,h}}{h} - \int_a^c f(y) g(y-a) dy = \int_a^c \left(\left(\frac{1}{h}\right) \int_{y-a-h}^{y-a} g(x) dx - g(y-a) \right) f(y) dy$$

Then $\lim_{h=0} c_{a,h} = 0$ [57]. Since by (10) $I_{a,h} = 0$, we conclude that $\int_a^c f(y)g(y-a)dy = 0$ for all $a \in [0,c]$. A theorem of Titchmarsh's [58] implies that then $g = 0$ a.e.

Lemma(3.1.12):[53] Let $n(x,y) = -n(y - x) \in L_1[0,1]$ and $n = 0$ in the interval $\left[0,\frac{1}{k}\right]$ for some positive integer k. Let m be a positive integer. Then the only reducing manifolds of $T_E^m + T_n$, are the subspaces $L_p[0,c]$ of $L_p[0,1]$ for all $c \in [0,1]$.

Proof: We show first that T_E^m is a uniform limit of polynomials in $T_E^m + T_n$. without constant term; note that Lemma (3.1.2) implies that such polynomials are indeed continuous operators mapping $L_p[0,1]$ into itself. It is sufficient by Lemma (3.1.2) to construct polynomials in the functions corresponding to $T_E^m + T_n$ which converge in L_1 to $(y-x)^{m-1}/\Gamma(m)$. Let $p(z) = \sum_0^N a_i z^{*,i+1}$, $p^*(f) = \sum_0^N a_i f^{*,i+1}$ for $f(t) = 1^{*,m} + n(t) = \frac{t^{m-1}}{\Gamma(m)} + n(t)$ where we write t instead of $y - x$. Since our hypotheses concerning n imply that $n^{*,k} = 0$, we have the following formulas:

$$p^*(f) = \pi(t) + \sum_{i=0}^{k-2} a_i n^{*,i+1} + \sum_{i=1}^{k-1} d_i(t)^* n^{*,i}$$

where $\pi(t) = \sum_{i=0}^N a_i 1^{*,m(i+1)}$ or

$$\pi(t) = \sum_{i=0}^{N} a_i t^{m(i+1)-1}/(m(i+1)-1)! \tag{11}$$

and

$$d_i(t) = \sum_{r=i}^{N} a_r \binom{r+1}{i} t^{m(r-i+1)-1}/(m(r-i+1)-1)! \tag{12}$$

if p has a subscript s, we write correspondingly π_s and d_{si} .

We shall find it necessary in what follows to have expressions for the polynomials $d_i(t)$ which show explicitly their dependence on the polynomial $\pi(t)$ and its derivatives. This is done by the following formula

$$d_i(t) = \sum_{r=0}^{i} q_{ri}(t)\pi^{(mi+1)}(t) \tag{13}$$

where the polynomials $q_{ri}(t)$ are independent of π and where superscripts in parentheses indicate differentiation. Express the binomial coefficient in (12) as $\binom{r+1}{i} = \sum_{s=0}^{i} c_{is} r^s$; (12) becomes

$$d_i(t) = \sum_{r=i}^{N} \frac{a_r \sum_{s=0}^{i} c_{is} r^s t^{m(r-i+1)-1}}{(m(r-i+1)-1)!} = \sum_{s=0}^{i} c_{is} g_{is}(t)$$

where $g_{is}(t) = \sum_{r=i}^{N} \frac{a_r r^s t^{m(r-i+1)-1}}{(m(r-i+1)-1)!}$. Note that $g_{i0}(t) = \sum_{r=i}^{N} \frac{a_r t^{m(r-i+1)-1}}{(m(r-i+1)-1)!} = \pi^{(mi)}(t)$ and $g_{is}(t) = \left(\frac{1}{mt^{m(i-1)}}\right)\left(t^{m(i-1)+1} g_{i,s-1}(t)\right)'$ so that there exist polynomials $g_{isu}(t)$ independent of π such that $g_{is}(t) = \sum_{u=0}^{s} g_{isu}(t)\pi^{(mi+u)}(t)$ and

$$d_i(t) = \sum_{s=u}^{i} c_{is} \sum_{u=0}^{s} g_{isu}(t)\pi^{(mi+u)}(t) = \sum_{u=0}^{i} q_{ui}(t)\pi^{(mi+u)}(t)$$

where $q_{ui}(t) = \sum_{s=u}^{i} c_{is} g_{isu}(t)$ is independent of π and (13) is established.

Proposition(3.1.13):[53] Given a positive real number ε, a positive integer $k_{1m} - 1 = M$, a polynomial $\pi_0(t)$ of form (11) considered in the interval $[0,b]$ where $0 < b \leqq 1$, and a function $n \in L_1[O,b]$ such that $n = 0$ in the interval $[0,a]$ for some a such that $0 < a < b$. Then there exists a polynomial $\pi_1(t)$ of form (11) such that $\left|\pi_1^{(v)}(t) - \pi_0^{(v)}\right| \leqq \varepsilon$ for all $t \in [0,a]$ and for $v = 0,\dots,M$ and such that $\|\pi - n\|_1 \leqq \varepsilon$ where $\|\dots\|_1$ relates to the interval $[a,b]$.

Proof: Instead of approximating π_0 and n as described above, we approximate 0 and $n - \pi_0 = n_1$ by a polynomial π of form (11) such that $\left|\pi^{(v)}(t)\right| \leqq \varepsilon$ for all $t \in [0,a]$ and $v = 0,\dots,M$ and $\|\pi - n\|_1 \leqq \varepsilon$ in $L_1[a,b]$. The polynomial $\pi_1 = \pi + \pi_0$ will then have the desired properties. We show first that we can approximate n_1 in $L_1[a,b]$ by a polynomial $p(t) = (t-a)^{M+1} p_1(t)$ so that $p^{(v)}(a) = 0$ for $v = 0,\dots,M$. This can be done since the polynomials of the form $q((t-a)^{M+1})$ are uniformly dense in all (complex valued) continuous functions on $[a,b]$ by the Stone-Weierstrass theorem (by approximating the real and imaginary parts separately) and hence the polynomials $q_0((t-a)^{M+1})$ without constant term are uniformly dense in all continuous functions on $[a,b]$ vanishing at a. We now approximate n_1 in $L_1[a,b]$ by a function f continuous on $[a,b]$. This function f is L_1 limit of polynomials $q_0((t-a)^{M+1})$ without constant term: If

$$f_e(t) = \left(\frac{f(a+e)}{e}\right)(t-a)$$

on $[a, a+e]$ and $f_e(t) = f(t)$ on $[a+e, b]$ then $f(t)$ is L_1 limit of $f_e(t)$ as $e \to 0$. But the continuous function $f_e(t)$ vanishes at a and hence can be approximated uniformly by polynomials $q_0((t-a)^{M+1})$ without constant term so that $f(t)$ and hence n_1 is L_1 limit of such polynomials. Given ε, it is therefore possible to find a polynomial $p(t)$ such that $p^{(v)}(a) = 0$ for $v = 0,\dots,M$ and such that

$$\|p - n_1\|_1 \leq \varepsilon/2$$

on $L_1[a,b]$. Consider now the function $g(t) = 0$ on $[0,a]$ and $p(t)$ on $[a,b]$. Clearly $\|g - n\|_1 \leqq \varepsilon/2$ on $L_1[a,b]$ and $g \in C^M$ on $[0,b]$. We now approximate $(g, g', \dots, g^{(M)})$ on

$[0,b]$ uniformly by a polynomial of form (11) and its derivatives: The Stone-Weierstrass Theorem implies the existence of a polynomial $\pi_2(t) = \sum_{i=0}^{N'} a_i t^{mi}$ such that $\left|\pi_2(t) - g^{(M)}(t)\right| \leq \varepsilon/2$ for all $t \in [0,b]$. We now integrate π_2 and $g^{(M)}$ M times from 0 to t and obtain $\pi(t) = \sum_{i=0}^{N'} b_i t^{mi+M}$ and $g(t)$ respectively (observe that $g(v) = 0$ for all v). Since t lies in $[0,1]$ we get

$$\left|\pi^{(v)}(t) - g^{(M)}(t)\right| \leq \varepsilon/2$$

for all $t \in [0,b]$ and $v = 0, \dots, M$.

We shall show that given a positive real number ε there exists a polynomial $\pi(t)$ of form (10) such that for the corresponding polynomial $p^*(f)$

$$\left\| p^*(f) - \frac{t^{m-1}}{(m-1)!} \right\|_1 \leq \varepsilon \tag{14}$$

on $L_1[0,1]$. The polynomial $\pi(t)$ is obtained as the last of a sequence of k polynomials of form (11) $\left(\pi_1(t), \dots, \pi_k(t) = \pi(t)\right)$ whose construction is described below. Subdivide the interval $[0,1]$as follows: $I_j = \left[\frac{j-1}{k}, \frac{j}{k}\right], J_j = \cup_{s \leqq j} I_s$ $(j = 1, \dots, k)$. Let the positive integer k_1 be chosen so that $M = k_1 m - 1 \geqq (k-1)(m+1)$. We shall replace the inequality $\|f - g\|_1 \leqq \eta$ in L_1 of the interval I by $f = g$ in I; η is a positive real number to be determined later.

$$\pi_1(t) = t^{m-1}/(m-1)!$$

$$\pi_j(t) = \begin{cases} \pi_{j-1}(t) \quad in\ J_{j-1}\ \ such\, that \quad \left|\pi_j^{(v)}(t) - \pi_{j-1}^{(v)}(t)\right| \leq \zeta \\ \qquad\qquad \text{for } t \in J_{j-1} and\ v = 0, \dots, M \\ \dfrac{t^{m-1}}{(m-1)!} - n - \displaystyle\sum_{i=1}^{j-1} d_{j-1,i}(t)^* n^{*,i} \qquad in\ I_j \\ \qquad\qquad \text{for } j = 2, \dots, k \end{cases}$$

In order to verify (14) we examine $p_k^*(f) - \frac{t^{m-1}}{(m-1)!}$ in the various intervals I_j: observe that since $n = 0$ in I_1, $p_k^*(f) = \pi_k(t) + \sum_{i=0}^{j-2} a_j n^{*,i+1} + \sum_{i=1}^{j-1} d_{ki}(t)^{*,i}$ in I_j. Thus we obtain

$$p_k^*(f) - \frac{t^{m-1}}{(m-1)!} = \pi_k(t) - \pi_j(t) + \pi_j(t) + \sum_{i=0}^{j-2} a_i n^{*,i+1} + \sum_{i=1}^{j-1} d_{ki}(t)^* n^{*,i} - \frac{t^{m-1}}{(m-1)!}$$

$$= \pi_k(t) - \pi_j(t) + \frac{t^{m-1}}{(m-1)!} + (a_0 - 1)n + \sum_{i=1}^{j-2} \left(d_{ki}(t) - d_{j-1,i}(t)\right)^* n^{*,i}$$
$$- t^{m-1}/(m-1)!$$

in I_j where, for $j = 1$, the term $(a_0 - 1)n$ and the two sums are absent, and when $j = 2$, the first sum is absent; j ranges from 1 to k. We now make the following three estimates: (i) Our construction implies that for all $t \in I_j$, $\left|\pi_k(t)-\pi_j(t)\right| \leqq k\eta$. (ii) Since if we write $\pi_k(t)=\sum_{i=0}^{N}\frac{a_i t^{m(i+1)-1}}{(m(i+1)-1)!}$, $a_i=\pi_k^{(m(i+1)-1)}(0)$ our construction implies that $\left|a_i-\pi_k^{(m(i+1)-1)}(0)\right| \leq k\eta$, $|a_0-1| \leq k\eta$, $|a_i| \leq k\zeta \quad for\ i=2,\dots,k$ (since by our choice of M, $k-2 \leqq M$. Hence if C is a constant such that $\left\|n^{*,i}\right\|_1 \leq C \quad for\ i=1,\dots,k-1$, $\left\|(a_0-1)_n+\sum_{i=1}^{j-2} a_i n^{*,i+1}\right\| \leq k^2 C\eta$. (iii) Before making an estimate of the sums involving terms of the type $d_i(t)^* n^{*,i}$ in I_j, note that since $n=0$ in I_1, this function uses only those values of $d_i(t)$ which correspond to $t \in J_{j-i}$. Thus in order to estimate the contribution of $\sum_{i=1}^{j-1}\left(d_{ki}(t)-d_{j-1,i}(t)\right)^* n^{*,i}$ we proceed as follows: By (13)

$$d_{ki}(t)-d_{j-1,i}(t)=\sum_{r=0}^{i} q_{ri}(t)(\pi_k^{(mi+r)}(t)-\pi_{j-1}^{(mi+r)}(t))$$

for $t \in J_{j-i}$ where the polynomials $q_{ri}(t)$ are independent of the polynomials π_k and π_{j-1}. Again referring to our construction, we see that $\left|\pi_k^{(r)}(t)-\pi_{j-1}^{(v)}(t)\right| \leq k\eta$ for $t \in J_{j-i}$ (hence for $t \in J_{j-i}$since $i \geqq 1$) and for $v=m,\dots,(m+1)(k-1) \leqq M$ by our choice of M. Hence, if $\max_{r,i,t}|q_{ri}(t)|=Q$, $\left\|\sum_{i=1}^{j-1}\left(d_{ki}(t)-d_{j-1,i}(t)\right)^* n^{*,i}\right\|_1 \leqq k^3 CQ\eta$ in I_j so that finally $\left\|p^*(f)-\frac{t^{m-1}}{(m-1)!}\right\|_1 \leq (k+k^2+h^3C+k^4CQ)\eta$,in $L_1[0,1]$ which after choosing η properly implies (14).

Lemma(3.1.14):[53] If $N(x,y)$ be defined for $0 \leqq x \leq y \leqq 1$ such that T_N is a bounded linear transformation mapping $L_p[0,c]$ into itself and such that T_N commutes with T_E^m for some positive integral m then $N(x,y)=N(y-x)$.

Proof: Our hypotheses imply that $T_N T_E^{mi}=T_E^{mi}T_N$ for all positive integral i; Lemma(3.1.10) therefore implies that $T_N T_{D_{a,h}}=T_{D_{a,h}}T_N$ for all relevant a and h; we therefore have $\left(\frac{1}{h}\right)\int_{y-a-h}^{y-a} N(x,z)dz=(1/h)\int_{x+a}^{y+a+h} N(z,y)dz$. If we now restrict ourselves to the appropriate set of measure 2 in the triangle $0 \leqq x \leq y \leqq 1$, both sides converge to $N(x,y-a)$ and $N(x+a,y)$ respectively which proves the assertion of the lemma.

Theorem(3.1.15):[53] The following operators T_F have the property that their only reducing manifolds are the subspaces $L_p[0,c]$ of $L_p[0,1]$ for all $c \in [0,1]$:

(i) $F(x,y) \in C^2$ and F is of order 1;

(ii) $F(x,y)$ analytic in x and y in a suitable region and F is of order m where m is a positive integer;

(iii) $F(x,y)=F(y-x)=(y-x)^{m-1}k(y-x)+n(y-x)$ where F is of order m, $k \in C^2$, $n \in L_1[0,1]$ and $n=0$ in a neighborhood of $x=y$.

Proof: (i) and (ii): Theorem (3.1.5) and Corollary(3.1.6) imply that T_F is similar to cT_E^m which, by Lemma(3.1.12), has as its only reducing manifolds the spaces $L_p[0,c]$, $c \in [0,1]$.

The similarity of T_F and cT_E^m is implemented by a product of linear operators of the form S_r, M_h , and $I + T_M$ (see Theorem(3.1.5) and Corollary(3.1.6)) which map the reducing subspaces of T_F and cT_E^m onto each other in a $1-1$ order preserving manner. (iii): The Corollary(3.1.6) implies that $T_F = T_{(y-x)^{m-1}k} + T_n$ is similar to $cT_E^m + T_{n_1}$. Since T_n commutes with $T_{(y-x)^{m-1}k}$,T_{n_1} commutes with cT_E^m. A simple computation which relates n and n_1 shows that the hypotheses concerning n imply that T_{n_1} satisfies the hypotheses of Lemma(3.1.14). We can therefore conclude that $n_1(x,y) = n_1(y - x)$. The computation relating n and n_1 also shows that the remaining hypothesis of part (iii) of the theorem is satisfied: $n_1 = 0$ in a neighborhood of $y = x$. We now apply Lemma(3.1.12) and conclude as in the end of the proof of (i) and (ii) that the only reducing manifolds of T_F are indeed the spaces $L_p[0,c], c \in [0,1]$.

Theorem(3.1.16):[53] Let T_{F_1} and T_{F_2} be two continuous linear operators of $L_2[0,1]$ into itself whose only reducing manifolds are the spaces $L_2[0,c]$, $c \in [0,1]$ (e.g., like the operators described in Theorem(3.1.13)). Then if T_{F_1}, is unitarily equivalent to $T_{F_2} = UT_{F_1}U^*$, there exists (i) a measurable function $h(x)$ defined on the interval $[0,1]$ such that $|h(x)| \equiv 1$, (ii) a function$r(t)$; such that $U = M_h U_r^*$ and $F_2(x,y) = h(x)\big(h(y)\big)^{-1}\big(s'(x)s'(y)\big)^{\frac{1}{2}}$

$F_1(s(x),s(y))$ Suppose that conversely this equation is satisfied for some functions h and r satisfying (i) and (ii), then if we set $U = M_h U_r^*, T_{F_2} = UT_{F_1}U^*$, i.e., T_{F_1} and T_{F_2} are unitarily equivalent.

Proof: Suppose that $T_{F_2} = UT_{F_1}U^*$; since the operators T_{F_i} are supposed to have the same reducing manifolds, there exists a monotone increasing function r such that $r(0) = 0$ and $r(1) = 1$ and such that $E_{r(t)} = UE_tU^*$ where E_t is the projection on $L_2[0,t]$. We now show that r is univalent and absolutely continuous: Let us consider $\big(E_{r(t)}f, g\big) = (UE_tU^*f, g) = (E_tf_1, g_1)$ or

$$\int_0^{r(t)} f(x)\overline{g(x)}dx = \int_0^t f_1(x)\overline{g_1(x)}dx$$

If we set $f = g = I$ we see that $(t) = \int_0^t f_1(x)\overline{g_1(x)}dx$: r is absolutely continuous, and if we set $f_1 = g_1 = 1$ we see that $t = \int_0^{r(t)} f(x)\overline{g(x)}dx$: r is univalent. A simple calculation shows that $U_r^*E_tU_r = E_{r(t)}$; but $E_{r(t)} = UE_tU^*$ so that U_rU commutes with all E_t . A standard theorem of spectral theory [59] shows that $U_rU = M_k$ for a measurable function $k(x)$ which, since U_rU is unitary, has the property that $|k(x)| \equiv 1$. Thus $U = U_r^*M_k$ and if we set $h(x) = k(s(x))$ a simple calculation shows that then $U = M_hU_r^*$ as desired. The equation relating $F_1(x,y)$ and $F_2(x,y)$ follows.

Theorem(3.1.17):[53] Let T_F be a bounded linear operator mapping $L_2[0,1]$ into itself such that $F(x,y) = (y-x)^{m-1}G(x,y)$ where $G \in C^1$ in a neighborhood of $y = x$, $G(x,x)$ is real and different from 0 and further such that the only reducing manifolds of T_F are the spaces $L_2[0,c], c \in [0,1]$ (e.g., like the operators described in Theorem(3.1.15)). Then T_F is unitarily equivalent to a unique operator $T_{F_1} = UT_FU^*$ where

$$F_1(x,y) = (y-x)^{m-1} G_1(x,y)$$
$$G_1(x,x) = c \quad (c \text{ real and different from } 0) \tag{15}$$
$$Im\big(G_{1x}(x,x)\big) = Im\left(G_{1y}(x,x)\right) = 0$$

The constant c in (15) is the same as in Lemma(3.1.3):$c = sign\, G(x,x)\left(\int_0^1 |G(u,u)|^{\frac{1}{m}} du\right)^m$

This is achieved by setting $U = M_h U_r^*$ where $r(t) = \int_0^t \left(\frac{G(u,u)}{c}\right)^{1/m}$ and where the function h is determined as follows: $F_0(x,y) = (y-x)^{m-1} G_0(x,y)$ by $T_{F_0} = U_r^* T_F U^*$ and we write $G_0(x,y) = H_0(x,y) + iK_0(x,y)$ for real H_0 and K_0. Then

$$h(x) = \exp((-i/c)\int_0^x K_{0x}(u,u)du).$$

Section (3.2): Invariant Subspace of the Backward Shift

Let S denote the unilateral shift operator on the Hardy space H^2 of the unit disk, D_o A subspace M of H^2 will be called nearly invariant under S^* if it is S^*-invariant modulo the one-dimensional subspace of constant functions, that is, if S^*h is in M whenever h is and $h(0) = 0$. These subspaces arose in the work and were characterized by D. Hitt [61] (who called them weakly invariant rather than nearly invariant).

To describe Hitt's result we note that, if the subspace M is nearly S^*-invariant and nontrivial, then M cannot be contained in H_0^2, so that $M \cap H_0^2$ has unit codimension in M. There exists therefore a unique function g in M that is orthogonal to $M \cap H_0^2$, has unit norm, and is positive at the origin. Hitt's theorem then states that if h is any function in M, the quotient h/g is in H^2 and has the same norm as does h; moreover, the subspace M' consisting of all such quotients is S^*-invariant. Thus, the Toeplitz operator T_g , the operator on H^2 of multiplication by g, maps the S^*-invariant subspace M' isometrically onto the given nearly S^*-invariant subspace M. From the famous theorem of A. Beurling and subsequent work of many others, one has a good picture of the structure of the S^*-invariant subspaces. Hitt's theorem thus refocuses that picture so as to provide a picture of the nearly S^*-invariant subspaces. Given a function g of unit norm in H^2, what are the S^*-invariant subspaces M' that can arise along with g in Hitt's theorem? [62,63,64].

Nearly S^*-invariant subspaces arise, in particular, as the kernels of Toeplitz operators, and this special case of Hitt's theorem was independently discovered, and established through different methods, by E. Hayashi [65].

We shall have to deal below with certain unbounded Toeplitz operators. If x is any function in L^2 of the unit circle, then by T_x we shall understand the operator on H^2 that sends the function h to the function

$$(T_x h)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x(e^{i\theta})h(e^{i\theta})}{1 - ze^{-i\theta}} d\theta \qquad (|z| < 1);$$

in other words, $T_x h$ is the standard Fourier projection of the L^1 function xh. The preceding definition clearly reduces to the usual one when x is bounded. The range of T_x is just a certain space of holomorphic functions in D; it is contained in H^P for $p < 1$. To follow T_x by an analytic Toeplitz operator, with a bounded symbol, say. Such products will arise below, and the ones of interest turn out to be bounded operators on H^2.

If B is a bounded operator acting in H^2 then, following L. de Branges, we define $\mathcal{M}(B)$ to be the space BH^2 with the Hilbert space structure that makes B into a coisometry of H^2 onto $\mathcal{M}(B)$. Thus, for example, if the function h in H^2 is orthogonal to $\ker B$, then the norm of Bh in $\mathcal{M}(B)$ equals $\|h\|_2$ the norm of h in H^2. The space $\mathcal{M}(B)$ does not determine B, but a simple argument shows that two such spaces, say $\mathcal{M}(B)$ and $\mathcal{M}(B_1)$ are identical as Hilbert spaces if and only if $BB^* = B_1 B_1^*$. If B is a contraction operator then $\mathcal{H}(B)$, the so-called complementary space of $\mathcal{M}(B)$, is defined to be $\mathcal{M}((1 - BB^*)^{\frac{1}{2}})$.

Here is the case $B = T_b$, where b is a function in the unit ball of H^∞. The corresponding spaces $\mathcal{M}(T_b)$ and $\mathcal{H}(T_b)$ will be denoted by $\mathcal{M}(b)$ and $\mathcal{H}(b)$. The norm in $\mathcal{H}(b)$ will be denoted by $\| \|_b$.

The kernel function in H^2 for the point w of D will be denoted by k_w ($k_w(z) = (1 - \bar{w}z)^{-1}$). A simple argument [62,63] shows that the kernel function in $\mathcal{H}(b)$ for the point w is the function $k_w^b = \left(1 - \overline{b(w)}b\right)k_w$.

The space $\mathcal{H}(b)$ is invariant under S^* and S^* acts as a contraction in it [62,63]. The restriction operator $S^*|\mathcal{H}(b)$ will be denoted by x_b .

Two special cases. The first is where b is an inner function. Then $\mathcal{M}(b)$ is just the S-invariant subspace H^2 , and $\mathcal{H}(b)$ is its ordinary orthogonal complement, an S^*-invariant subspace. The other is where b is not an extreme point of the unit ball of H^∞. In that case, as was proved in [63], the kernel functions k_w belong to $\mathcal{H}(b)$ and, as was proved in [64], actually span $\mathcal{H}(b)$.

Let g be a function of unit norm in H^2, and let f be the outer factor of g, normalized (for definiteness) by the condition $f(0) > 0$. Let F be the Herglotz integral of $|f|^2$:

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left|f(e^{i\theta})\right|^2 d\theta \qquad (|z| < 1).$$

We note that $F(0) = 1$ and define the functions b and a by

$$b = \frac{F - 1}{F + 1}, \qquad a = \frac{2f}{F + 1}$$

Then $b(0)=0$ and, as the function F has a positive real part, the function b is in the unit ball of H^{∞}. The function a is an outer function, being the quotient of two outer functions, and almost everywhere on ∂D we have

$$|a|^2+|b|^2=\frac{4|f|^2+|F-1|^2}{|F+1|^2}=\frac{4\,Re\,F+|F-1|^2}{|F+1|^2}=1$$

Thus a is also in the unit ball of H^{∞}, and because $1-|b|^2=|a|^2$ on ∂D,we see that $\log(1-|b|^2)$ is integrable, so that b is not an extreme point of the unite ball of H^{∞}.

Lemma(3.2.1):[60] For z and w in D,

$$\langle fk_w, fk_z\rangle=\left(1-b(z)\right)^{-1}\left(1-\overline{b(w)}\right)^{-1}k_w^b(z)$$

Proof: The inner product on the left side equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left|f\left(e^{i\theta}\right)\right|^2}{(1-\overline{w}e^{i\theta})(1-ze^{-i\theta})}d\theta$$

which can be rewritten as

$$\frac{1}{2\pi(1-\overline{w}z)}\int_{-\pi}^{\pi}\frac{1}{2}\left[\frac{e^{-i\theta}+\overline{w}}{e^{-i\theta}-\overline{w}}+\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]\left|f\left(e^{i\theta}\right)\right|^2d\theta$$

in other words, as

$$\frac{F(z)+\overline{F(w)}}{2(1-\overline{w}z)}$$

an expression that is easily reduced to the right side.

Lemma(3.2.2):[60] The operator $T_{1-b}T_{\bar{f}}$ is bounded and is in fact an isometry of H^2 onto $\mathcal{H}(b)$. Hence, the operator $T_{1-b}T_{\bar{g}}$ is a coisometry of H^2 onto $\mathcal{H}(b)$; its null space equals $\mathcal{H}(v)$, where v is the inner factor of g.

Corollary(3.2.3):[60] $\left(T_{1-b}T_{\bar{g}}\right)\left(T_{1-b}T_{\bar{g}}\right)^*=1-T_bT_{\bar{b}}$.

Proof: The adjoint referred to here is the adjoint of $T_{1-b}T_{\bar{g}}$ as an operator of H^2 into itself. The equality follows from the identity of the two spaces $\mathcal{M}(T_{1-b}T_{\bar{g}})$ and $\mathcal{M}((1-T_bT_{\bar{b}})^{\frac{1}{2}})$, as explained. We note here the equality $(1-T_bT_{\bar{b}})^*=T_gT_{1-\bar{b}}$, which is valid even when g is unbounded, as a simple argument shows.

Lemma(3.2.4):[60] $T_{1-b}T_{\bar{g}}S^*g=S^*b$

Proof: In fact, we have seen in the proof of Lemma(3.2.2) that $T_{1-b}T_{\bar{f}}$ maps f to the constant function 1 (i.e., the function $\left(1-\overline{b(0)}\right)^{-1}k_0^b$); hence also $T_{1-b}T_{\bar{g}}g = 1$. Thus, because $T_{\bar{g}}$ and S^* commute,

$$T_{1-b}T_{\bar{g}}S^*g = T_{1-b}S^*T_{\bar{g}}g = S^*T_{1-b}T_{\bar{g}}g + (S^*T_{b-1} - T_{b-1}S^*)T_{\bar{g}}g$$

$$= (S^*T_{b-1} - T_{b-1}S^*)T_{\bar{g}}g$$

As $S^*T_{b-1} - T_{b-1}S^*$ equals the rank-one operator $S^*b\otimes 1$, one easily verifies that $(S^*T_{b-1} - T_{b-1}S^*)T_{\bar{g}}$ is the bounded operator $S^*b\otimes g$, and the desired equality follows.

Lemma(3.2.5):[60] The operator $T_{1-b}T_{\bar{g}}$ intertwines the operator R_g with the operator X_b: $T_{1-b}T_{\bar{g}}R_g = X_bT_{1-b}T_{\bar{g}}$.

Proof: Letting v, as before, denote the inner factor of g, we note that $T_{\bar{g}}$ annihilates the subspace $\mathcal{H}(v)$. So does $g\otimes g$ so that R_g coincides with S^* in $\mathcal{H}(v)$, implying that $\mathcal{H}(v)$ is R_g-invariant. The desired equality thus holds in $\mathcal{H}(v)$, and it only remains to show that it holds in $\mathcal{M}(v)$. For that it will suffice to show that it holds for the functions vk_w with w in D. We have

$$R_g vk_w = S^*vk_w - \langle vk_w, g\rangle S^*g = \bar{w}vk_w + S^*v - \overline{f(w)}S^*g.$$

As $T_{\bar{g}}$ annihilates S^*v we obtain, using Lemma(3.2.4),

$$T_{1-b}T_{\bar{g}}R_g vk_w = \bar{w}T_{1-b}T_{\bar{g}}vk_w - \overline{f(w)}T_{1-b}T_{\bar{g}}S^*g = \bar{w}T_{1-b}T_{\bar{f}}k_w - \overline{f(w)}S^*g$$

$$= \bar{w}\overline{f(w)}(1-b)k_w - \overline{f(w)}S^*g = \overline{f(w)}S^*((1-b)k_w)$$

$$= S^*T_{1-b}T_{\bar{f}}k_w = X_bT_{1-b}T_{\bar{g}}vk_w;$$

as desired.

Lemma(3.2.6):[60] $R_g^n \to 0$ strongly as $n \to \infty$.

Proof: We note first that $X_g^n \to 0$ strongly as $n \to \infty$. In fact, for w in D we have $X_b k_w = \bar{w}k_w$, implying that $X_b^n k_w \to 0$ as $n \to \infty$. Because, as was mentioned, the functions k_w span $\mathcal{H}(b)$, the desired conclusion follows.

We continue to let v denote the inner factor of g. Let h be any function in H^2, and fix a positive number ϵ. There is a positive integer m such that $\left\|X_b^m T_{1-b}T_{\bar{g}}h\right\|_b < \epsilon$. Let h_0 and h_1 be the components of $R_g^m h$ in $\mathcal{H}(v)$ and $\mathcal{M}(v)$, respectively. By Lemmas (3.2.1) and (3.2.5), $\|h_1\|_2 = \left\|X_b^m T_{1-b}T_{\bar{g}}h\right\|_b < \epsilon$. Since R_g coincides with S^* in $\mathcal{H}(v)$ we obtain, for any positive integer n,

$$\left\|R_g^{m+n}h\right\|_2 \leqq \left\|R_g^n h_0\right\|_2 + \left\|R_g^n h_1\right\|_2 \leqq \|S^{*n}h_0\|_2 + \left\|R_g^n\right\|\|h_1\|_2 < \|S^{*n}h_0\|_2 + \epsilon$$

As $S^{*n} \to 0$ strongly we conclude that $\lim\max\left\|R_g^n h\right\|_2 < \epsilon$, and as ϵ is arbitrary.

Theorem(3.2.7):(Hitt's theorem)[60] Let M be a nontrivial nearly invariant subspace of S^*, and let g be the function of unit norm in M that is orthogonal to $H \cap H_0^2$ and positive at the origin. Then $M = T_g M'$, where M' is an S^* -invariant subspace on. which T_g acts isometrically.

Proof:. Let h be any function in M, and let $c_0 = \langle h, g \rangle$. Then $R_g h = S^*(h - c_0 g)$, which implies that

$$h = c_0 g + SR_g h,$$

because $h - c_0 g$ vanishes at the origin. The function $SR_g h$ is thus in M, and as it is also in H_0^2 it is orthogonal to g, implying that

$$\|h\|_2^2 = |c_0|^2 \|g\|_2^2 + \|SR_g h\|_2^2 = |c_0|^2 + \|R_g h\|_2^2$$

Similarly, for any positive integer n,

$$R_g^n h = c_n g + SR_g^{n+1} h$$

where $c_n = \langle R_g^n h, g \rangle$, and

$$\|R_g^n h\|_2^2 = |c_n|^2 + \|R_g^{n+1} h\|_2^2$$

We can thus iterate to obtain

$$h = (c_0 + c_1 S + \cdots + c_n S^n) g + S^{n+1} R_g^{n+1} h$$

for any positive integer n, with

$$\|h\|_2^2 = |c_0|^2 + |c_1|^2 + \cdots + |c_n|^2 + \|R_g^{n+1} h\|_2^2$$

Departing now from Hitt's line of reasoning, we let $n \to \infty$ and use Lemma(3.2.6); our conclusion is that h has the factorization gq where the H^2 function $q(z) = \sum_0^\infty c_n z^n$ has the same norm as does h. We have thus shown that $M' = \{\frac{h}{g} : h \in M\}$ is a subspace of H^2 and that T_g maps M' isometrically onto M. Moreover, with h, c_0 and q as above, we have $c_0 = q(0)$, and accordingly,

$$R_g h = S^*(gq) - q(0) S^* g = g S^* q + q(0) S^* g - q(0) S^* g = g S^* q$$

showing that M' is S^*-invariant.

Theorem(3.2.8):[60] Let g be a function of unit norm in H^2 and let b be as defined. Let u be an inner function with $u(0) = 0$. The following conditions are equivalent:

(i) T_g acts isometrically on $\mathcal{H}(u)$
(ii) u divides b

(iii) $\mathcal{H}(u)$ is contained isometrically in $\mathcal{H}(b)$.

Proof: (ii) → (i). Assume u divides b, and let h be any function in $\mathcal{H}(u)$. Then $T_{\bar{b}}h = 0$, so, by the Corollary (3.2.3)

$$\left\|T_g h\right\|_2^2 = \left\|T_g T_{1-\bar{b}} h\right\|_2^2 = \langle (T_{1-b}T_{\bar{b}})(T_{1-b}T_{\bar{b}})^* h, h\rangle$$

$$= \langle (1 - T_b T_{\bar{b}})h, h\rangle = \langle h, h\rangle = \|h\|_2^2$$

as desired.

(i)→(ii). Assume T_g acts isometrically on $\mathcal{H}(u)$, and let h be any function in $\mathcal{H}(u)$. Because $\mathcal{H}(u)$ is invariant under $T_{\bar{b}}$ we obtain, using Lemma(3.2.2) and its corollary(3.2.3),

$$\|T_{1-\bar{b}}h\|_2 = \left\|T_g T_{1-\bar{b}} h\right\|_2 = \left\|T_{1-b}T_{\bar{g}}T_g T_{1-\bar{b}} h\right\|_b$$

$$= \|(1 - T_b T_{\bar{b}})h\|_b = \left\|(1 - T_b T_{\bar{b}})^{(1/2)}\, h\right\|_2$$

in other words,

$$\langle T_{1-b}T_{1-\bar{b}}h, h\rangle = \langle (1 - T_b T_{\bar{b}})h, h\rangle$$

which can be rewritten

$$2\|T_{\bar{b}}h\|_2^2 = \langle h, T_{\bar{b}}h\rangle + \langle T_{\bar{b}}h, h\rangle$$

Since $u(0) = 0$ the space $\mathcal{H}(u)$ contains the constant functions, and since $b(0) = 0$ the operator $T_{\bar{b}}$ annihilates the constant functions. We can therefore replace h in the last equality by $h + c$, where c is any constant, obtaining

$$2\|T_{\bar{b}}h\|_2^2 = 2Re\bar{c}(T_{\bar{b}}h)(0) + \langle h, T_{\bar{b}}h\rangle + \langle T_{\bar{b}}h, h\rangle$$

This cannot possibly be true for all constants c unless $(T_{\bar{b}}h)(0) = 0$. But the last equality for all h in $\mathcal{H}(u)$ implies $T_{\bar{b}}h = 0$ for all h in $\mathcal{H}(u)$ (since $\mathcal{H}(u)$ is S^*-invariant). Thus $T_{\bar{b}}$ annihilates $\mathcal{H}(u)$, which implies that u divides b.

(ii) → (iii). If u divides b and h is in $\mathcal{H}(u)$, then $T_{\bar{b}}h = 0$, which implies that $(1 - T_b T_{\bar{b}})^{(1/2)}\, h = h$ and hence that h is in $\mathcal{H}(b)$ with

$$\|h\|_b = \left\|(1 - T_b T_{\bar{b}})^{\left(\frac{1}{2}\right)}\, h\right\|_b = \|h\|_2$$

(iii) → (ii) If $\mathcal{H}(u)$ is contained isometrically in $\mathcal{H}(b)$ and h is in $\mathcal{H}(u)$, then $h = (1 - T_b T_{\bar{b}})^{(1/2)}\, h_1$ for some h_1 in H^2, and

$$\|h_1\|_2 = \|h\|_b = \|h\|_2 = \left\|(1 - T_b T_{\bar{b}})^{\left(\frac{1}{2}\right)}\, h_1\right\|_2$$

The last equality implies $(1-T_bT_{\bar{b}})^{\left(\frac{1}{2}\right)}h_1 = h_1$ (since $(1-T_bT_{\bar{b}})^{\left(\frac{1}{2}\right)}$ is a positive contraction), which means that $h_1 = h$ and so $(1-T_bT_{\bar{b}})h = h$, in other words, $T_{\bar{b}}h = 0$. That means $T_{\bar{b}}$ annihilates $.\mathcal{H}(u)$, so that u divides b.

The hypothesis $u(0) = 0$ in Theorem(3.2.8) was used only for the implication (i) →(ii), and that implication can fail in its absence. For example, suppose u is the Blaschke factor $(z - w)/(1-\bar{w}z)$, where w is in D and $w \neq 0$. Then $\mathcal{H}(u)$ is spanned by the kernel function k_w. One easily checks that $\|gk_w\|_2 = \|k_w\|_2$ if and only if $Re\,F(w) = 1$ (where F is as defined), but that u divides b if and only if $F(w) = 1$.

In [64] it is shown that, for b as in Theorem(3.2.8), the proper invariant subspaces of the operator X_b are the subspaces $\mathcal{H}(u)\cap\mathcal{H}(b)$. with u an inner function. As the operator $T_{1-b}T_{\bar{f}}$ (where f is the outer factor of g) implements a unitary equivalence between the operators R_f and X_b, the proper invariant subspaces of R_f are the inverse images under $T_{1-b}T_{\bar{f}}$ of the subspaces $\mathcal{H}(u)\cap\mathcal{H}(b)$. In case u divides b, Theorem(3.2.8) says that $\mathcal{H}(u)$ sits isometrically in $\mathcal{H}(b)$ and so is by itself an invariant subspace of X_b . Then, as one would expect, the inverse image of $\mathcal{H}(u)$ under $T_{1-b}T_{\bar{f}}$ is just the nearly S^*-invariant subspace $T_f\mathcal{H}(u)$ of Hitt. In fact, denoting $T_f\mathcal{H}(u)$ by M and using the corollary(3.2.3) to Lemma(3.3.2) plus the inclusion . $\mathcal{H}(u) \subset \ker T_{\bar{b}}$) ,we obtain $T_{1-b}T_{\bar{f}}M = T_{1-b}T_{\bar{f}}T_fT_{1-\bar{b}}\mathcal{H}(u) = (1-T_bT_{\bar{b}})\mathcal{H}(u) = \mathcal{H}(u)$.We see from this that $T_{1-b}T_{\bar{f}}$ agrees on M with $T_{1/f}$.

Suppose X is a function in L^∞ , not identically zero, such that the operator T_X has a nontrivial kernel. Then $\ker T_X$ is a nontrivial nearly S^*-invariant subspace so, according to Theorem(3.2.7), it equals T_gM', where g has unit norm in H^2 and M' is an S^*-invariant subspace, containing the constant functions, on which T_g acts isometrically. Since $\ker T_X$ clearly contains the outer factor of each of its members, the function g must be an outer function. The case $M' = H^2$ is thus excluded by the assumption that X is not identically 0, implying that $M' = \mathcal{H}(u)$ for an inner function u that vanishes at the origin; thus $\ker T_X = T_g\mathcal{H}(u)$. As noted, the preceding description of the kernel of a Toeplitz operator is due to E. Hayashi [65], who used different methods (deriving from prediction theory). Hayashi in fact proved more, namely, he showed that g^2 is an exposed point of the unit ball of H^1 [66]. His argument could be incorporated without alterations into the present treatment of his result.

Because the function X multiplies a nonzero function in H^2 into $\bar{H}_0^2$ it must be log-integrable, so it can be written as $\bar{x}_1y$, where y is unimodular and X_1 is the outer function with modulus $|X|$. Then $T_X = T_{\bar{x}_1}T_y$ and as T_X has a trivial kernel, the kernel of T_y coincides with that of T_X. In fact, $y = \overline{ug}/g$. This equality from [65,66] can be established as follows. Because the function g is in $\ker T_y$, the product yg is in $\bar{H}_0^2$ so (since g is outer) it can be written as $\bar{u}_1\bar{g}$ where u_1 is an inner function that vanishes at the origin. Thus $y = \bar{u}_1\bar{g}/g$. Suppose h is any function in $\mathcal{H}(u)$. Then gh is in $\ker T_y$ implying that $\bar{u}_1\bar{g}h$ is in $\bar{H}_0^2$. Since g is outer it follows that $\bar{u}_1h$ is in $\bar{H}_0^2$ which shows that $T_{\bar{u}_1}$ annihilates $\mathcal{H}(u)$ and hence that u divides u_1 . Suppose, on the other hand, that h is a bounded function in $\mathcal{H}(u_1)$. Then $ygh = \overline{gu_1}h$, a function in $\bar{H}_0^2$ implying that gh is in $\ker T_y$, that is, in $T_g\mathcal{H}(u)$, and hence h is in $\mathcal{H}(u)$. As $\mathcal{H}(u_1)$ is spanned by its bounded functions (for example, by its kernel functions), we can conclude that $\mathcal{H}(u_1) \subset \mathcal{H}(u)$, so that u_1 divides u. Therefore u_1 is a constant multiple of u, the desired conclusion.

From the results of Hayashi and those one sees that the Toeplitz operator with unimodular symbol y has a nontrivial kernel if and only if y has the form $\overline{ug}/g$, where g^2 is an exposed point of the unit ball of H^1, and u is an inner function that vanishes at the origin and divides the function b associated with g in the way specified.

Chapter 4

Analytic Continuability and Complemented Invariant Subspaces with Linear Graph Transformations

We show that the invariant subspace of the Bergman space A^p of the unit disk, generated by either an A^p-interpolating sequence or a singular inner function with a single point mass on the unit circle, is complemented in A^p. We investigate the existence of non-trivial common invariant subspaces of operator algebras of the type $\mathcal{A}_{\mathcal{M}} = \{A \in \mathcal{B}(\mathcal{H}): A\mathcal{D} \subseteq \mathcal{D}: AT_i f = T_i A f \ \ \forall f \in \mathcal{D}\}$. For the Bergman space L_a^2 we exhibit examples of invariant graph subspaces of fiber dimension 2 such that $\mathcal{A}_{\mathcal{M}}$ does not have any nontrivial invariant subspaces that are defined by linear relations of the graph transformations for $\mathcal{M}$.

Section (4.1): Bergman Inner Functions

For $0 < p < \infty$, the Bergman space L_a^p consists of those function f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ for which

$$\|f\|_{L_a^p}^p = \iint_{\mathbb{D}} |f(z)|^p \frac{dA(z)}{\pi} < \infty.$$

If $p \geq 1$ then $\|.\|_{L_a^p}$ is a norm making L_a^p a Banach space, and if $0 < p < 1$ then $d(f, g) = \|f - g\|_{L_a^p}^p$ is a metric making L_a^p a nonlocally convex complete metric topological vector space.

Let $\{\alpha_1, \alpha_2, ...\}$ be an L_a^p zero sequence that is, the sequence of zeros, repeated according to multiplicity, of some nonidentically vanishing L_a^p function and let M be the set of L_a^p functions that vanish on the sequence $\{\alpha_n\}$ to at least the prescribed multiplicity. We let N denote the number of times that 0 appears in the sequence $\{\alpha_n\}$ and consider the following extremal problem:

$$\sup\{Re\, f^{(N)}(0): f \in M, \|f\|_{L_a^p} \leq 1\} \tag{1}$$

It is shown in [68;69] that there is a unique extremal function φ for this problem, and that φ satisfies the following properties:

$$\iint_{\mathbb{D}} |\varphi(z)|^p u(z) \frac{dA(z)}{\pi} = u(0) \tag{2}$$

if u is a bounded harmonic function in $\mathbb{D}$

$$\text{if } f \in M \text{ then } \frac{f}{\varphi} \in L_a^p \text{and } \left\|\frac{f}{\varphi}\right\|_{L_a^p} \leq \|f\|_{L_a^p} \tag{3}$$

in particular (3) says that function φ vanishes at each point of the sequence to the sequence to exactly the prescribed multiplicity; that is, it has no “extra zeros”.

We take (2) to be the defining property of L_a^p inner function. . There has been much interest in these function in recent years, starting with [7] in which he estalished, among other facts, that(3) holds in the case $p = 2$ see [68,69,70].

If $p = 2$ and the zero sequence $\{\alpha_n\}$ is finite, then an easy argument shows that the extremal function φ is rational function with poles at the points $1/\bar{\alpha}_n$, and hence it continues analytically across $\partial\mathbb{D}$. Suppose $I \subset \partial\mathbb{D}$ is an open arc that dose not meet clos $\{\alpha_n\}$. Then it is a consequence of theorem of [71,72] that the associated extremal function φ extends analytically across I.

In [68,69] it is shown that, for general p,the extremal function associated to a finite zero sequence extends analytically across $\partial\mathbb{D}$. One asked whether it were also true for general p that the extremal function associated to a zero sequence extends analytically across any $I \subset \partial\mathbb{D}$ not meeting the closure of the sequence [73,74].

Let $\{\alpha_1, \ldots, \alpha_n\}$ be finite sequence of points in $\mathbb{D}$ and denote by φ its associated L_a^p inner function.

Let g denote the Cauchy transform of $|\varphi|^p \mathcal{X}_{\mathbb{D}}$; that is,

$$g(z) = \iint_{\omega \in \mathbb{D}} \frac{|\varphi(\omega)|^p}{\omega - z} \frac{dA(\omega)}{\pi} \tag{4}$$

We will use the following facts about g:

$$\bar{\partial} g = -|\varphi|^p \mathcal{X}_{\mathbb{D}} \quad \text{in the sense of distributions} \tag{5}$$

$$g \text{ is continuous in all of } \mathbb{C} \tag{6}$$

$$g(z) = -\frac{1}{z} \quad \text{for} \quad |z| \geq 1 \tag{7}$$

(Here $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$ is the standard Cauchy-Riemann partial operator). (5) is standard [76]; (6) follows from the boundedness of φ, which in turn is a consequence of the analytic continuability of φ across $\partial\mathbb{D}$ [68, 69]; and (7) follows for $|z| > 1$ from (2) and for $|z| = 1$ from continuity.

We next cut out of $\mathbb{D}$ a set of nonintersecting curves $\gamma_1, \ldots, \gamma_n$, each γ_j conecting α_j to a point $\beta_j \in \partial\mathbb{D}$ (if $\alpha_j = \alpha_k$ we assume that $\gamma_j = \gamma_k$). We denote by Ω the resulting simply connected region. Because, as mentioned previously, φ has no “extra zeros”, φ dose not vanish in Ω and we can define $\varphi^{\frac{p}{2}}$ in Ω. For definiteness choosing it so that $\varphi^{\frac{p}{2}}(0) > 0$. We can then define its integral

$$\Phi(z) = \int_{\sigma_z} \varphi^{\frac{p}{2}}(\omega) d\omega \qquad \text{for } z \in \Omega \tag{8}$$

Where σ_z is some rectifiable path connecting 0 to z in Ω.

We use all the functions just constructed to define

$$h(z) = g(z) + \overline{\Phi(z)}\, \varphi^{\frac{p}{2}}(z) \tag{9}$$

The following properties of h are important for us:

$$h \text{ is analytic in } \Omega \tag{10}$$

$$h \text{ is continuous in } \Omega \cup (\partial\mathbb{D} \setminus \{\beta_1, \dots, \beta_n\}) \tag{11}$$

$$h(z) = -\frac{1}{z} + \overline{\Phi(z)}\, \varphi^{\frac{p}{2}}(z) \quad \text{for } z \in \partial\mathbb{D} \setminus \{\beta_1, \dots, \beta_n\} \tag{12}$$

the first of these properties is a consequence of Weyl's lemma [76],(5), and the definition of Φ, which implies that $\bar{\partial}\left(\overline{\Phi}\, \varphi^{\frac{p}{2}}\right) = \overline{\Phi'}\, \varphi^{\frac{p}{2}} = |\varphi|^p$. The second property is consequence of (6) and the analytic continuability of φ across $\partial\mathbb{D}$. Finally, (12) follows from (7) and the continuity of $\varphi^{\frac{p}{2}}$ and Φ up to $\partial\mathbb{D}$.

We can now state a formula giving an expression for the analytic continuation of φ. Let $\Omega^* = \{\frac{1}{\bar{z}} : z \in \Omega\}$ and define

$$\Phi(z) = \frac{\overline{z + h(\frac{1}{\bar{z}})}}{\overline{\varphi^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right)}}, \quad z \in \Omega^* \tag{13}$$

The continuity of h and φ up to $\partial\mathbb{D}\setminus\{\beta_1, \dots, \beta_n\}$, together with the fact that $|\varphi(z)| \geq 1$ for $z \in \partial\mathbb{D}$ [69;70], shows that we can extend (13) continuously to $z \in \partial\mathbb{D}\setminus\{\beta_1, \dots, \beta_n\}$. By (12) this extension agrees with the original definition of Φ there, and so by (10) we see that (13) gives an analytic continuation of Φ to $\widetilde{\Omega} = \Omega \cup \Omega^* \cup (\partial\mathbb{D}\setminus\{\beta_1, \dots, \beta_n\})$. The nonvanishing of φ near $\partial\mathbb{D}$ now shows that the formula $\varphi = (\varphi')^{\frac{2}{p}}$ yields an analytic continuation of φ across each arc of $\partial\mathbb{D}\setminus\{\beta_1, \dots, \beta_n\}$.

The estimate

$$|f(z)| \leq \frac{1}{(1-|z|)^{\frac{2}{p}}} \|f\|_{L_a^p}, \quad f \in L_a^p \tag{14}$$

is will know and elementary; it follows from the sub harmonicity of $|f|^p$ after integration over $\{\omega : |\omega - z| < 1 - |z|\}$. Suppose now that the curves γ_j have been chosen such that, for any $z \in \Omega$, a rectifiable path σ_z from 0 to z within Ω can be chosen along which $|d\omega| \leq 2d|\omega|$. Then, by (14),

$$\Phi(z) = \left| \int_{\sigma_z} \varphi^{\frac{p}{2}}(\omega)\, d\omega \right| \leq \int_{\sigma_z} \frac{|d\omega|}{1 - |\omega|} \leq 2 \int_0^{|z|} \frac{dr}{1 - r}$$

yielding

$$|\Phi(z)| \le 2\log\frac{1}{1-|z|}, \quad z \in \Omega \tag{15}$$

The estimate

$$|g(z)| \le \frac{6}{1-|z|}, \quad z \in \Omega \tag{16}$$

Follows from (14) and simple estimates on the defining integral (4) of g obtained by dividing this integral into the integral over $|\omega - z| < \frac{1}{2}(1-|z|)$ and that over $|\omega - z| > \frac{1}{2}(1-|z|)$. Finally, we combine (14),(15), and (16) to obtain

$$|h(z)| \le \frac{8}{(1-|z|)^2}, \quad z \in \Omega \tag{17}$$

Lemma(4.1.1):[67] Let $U \subset \mathbb{C}$ be open and let $\mathcal{F}$ be a family of functions analytic in U. suppose there exists a $\rho \in L^1_{loc}(U)$ such that $\log^+|f(z)| \le \rho(z)$ for any $f \in \mathcal{F}$ and $z \in U$. Then $\mathcal{F}$ is a normal family.

Proof: Let $K \subset U$ be compact, and pick a $\delta > 0$ such that $K_\delta = \{z \in \mathbb{C}: \operatorname{dist}(z,K) \le \delta\} \subset U$. Then if $f \in \mathcal{F}$ and $z \in K$, the subharmonicity of $\log^+|f|$ implies that

$$\log^+|f(z)| \le \frac{1}{\pi\delta^2}\iint_{|\omega-z|\le\delta} \log^+|f(\omega)|\, dA(\omega) \le \frac{1}{\pi\delta^2}\iint_{K_\delta} \rho(\omega)dA(\omega)$$

Thus

$$|f(z)| \le \exp\left[\frac{1}{\pi\delta^2}\iint_{K_\delta} \rho(\omega)dA(\omega)\right] \text{ for any } f \in \mathcal{F}, z \in K$$

so an application of Montel's theorem [78,79] proves the lemma.

Theorem(4.1.2):[67] Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is an L^p_a zero sequence and that $I \subset \partial\mathbb{D}$ is an arc not meeting clos $\{\alpha_n\}$. Then the associated L^p_a inner function φ has an analytic continuation across I.

Proof: If $z_0 \in \partial\mathbb{D}$ is not a limit point of $\{\alpha_n\}$ then φ has an analytic continuation to a neighborhood of z_0. Given such a z_0,we construct nonintersecting curves γ_n in $\mathbb{D}$, each γ_n connecting α_n to point $\beta_n \in \partial\mathbb{D}$ (again, if $\alpha_j = \alpha_k$ we set $\gamma_j = \gamma_k$) in such a way that if Ω_n is the simply connected open set $\mathbb{D}\backslash\{\gamma_1, \dots, \gamma_n\}$, then the following properties hold:

$$\text{the closure of } \cup\gamma_n \text{ dose not contian } z_0 \tag{18}$$

$$\Omega = \cap\Omega_n \quad \text{is open} \tag{19}$$

each $z \in \Omega_n$ can be conncted to 0 by a rectifiable path σ_z in Ω_n along which

$$|d\omega| \le 2d|\omega| \tag{20}$$

we now let φ_n denote the L_a^p extremal function corresponding to the finite set $\{\alpha_1, \dots, \alpha_n\}$, and use φ_n as define functions g_n in $\mathbb{C}$ and functions Φ_n, h_n in Ω_n. The formula

$$\Phi_n(z) = \frac{z + \overline{h_n(1/\bar{z})}}{\overline{\varphi_n^{\frac{p}{2}}(1/\bar{z})}}, \quad z \in \Omega_n^* \tag{21}$$

gives an analytic continuation of Φ_n to $\widetilde{\Omega}_n = \Omega_n \cup \Omega_n^* \cup (\partial\mathbb{D} \backslash \{\beta_1, \dots, \beta_n\})$. Since $\varphi_n(z) \to \varphi(z)$ for every $z \in \mathbb{D}$ [69;70], our theorem will be proved if we can show that the functions $\{\Phi_n\}$ from a normal family of functions analytic in $\widetilde{\Omega} = \Omega \cup \Omega^* \cup (\partial\mathbb{D} \backslash \mathrm{clos}\{\beta_n\})$.[71,72,77].

We will now show that the family $\{\varphi_n\}$ satisfies the hypothesis of Lemma (4.1.1) in the open set $\widetilde{\Omega}$. Write (21) in the form

$$\Phi_n(z) = \left[z + \overline{h_n\left(\frac{1}{\bar{z}}\right)}\right] \frac{\overline{\varphi^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right)}}{\overline{\varphi_n^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right)}} \frac{1}{\overline{\varphi^{\frac{p}{2}}(1/\bar{z})}}, \quad z \in \Omega^* \tag{22}$$

By (3) and (14), $\left|\varphi^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right) / \varphi_n^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right)\right| \leq |z|/(|z| - 1)$. We combine this estimate with (17) and use (22) and a little manipulation to derive the estimate

$$|\varphi_n(z)| \leq \frac{8|z|^2}{(|z| - 1)^2} \frac{1}{\left|\varphi\left(\frac{1}{\bar{z}}\right)\right|^{\frac{p}{2}}}, \quad z \in \Omega^* \tag{23}$$

This estimate, together with (15), shows that the function

$$\rho(z) = \begin{cases} \log^+\left(2\log\frac{1}{1 - |z|}\right), & |z| < 1 \\ \log\frac{8|z|^2}{(|z| - 1)^2} + \frac{p}{2}\log^+\frac{1}{|\varphi(1/\bar{z})|}, & |z| > 1 \end{cases} \tag{24}$$

dominates $\log^+|\varphi_n(z)|$ for all n. It remains only to show that $\rho \in L_{loc}^1(\mathbb{C})$, and this is trivial except for the term $\log^+(1/|\varphi_n(1/\bar{z})|$. To handle this we write

$$\iint\limits_{1<|z|<R} \log^+\frac{1}{\left|\varphi\left(\frac{1}{\bar{z}}\right)\right|} dA(z) = \iint\limits_{\frac{1}{R}<|\omega|<1} \frac{1}{|\omega|^4}\log^+\frac{1}{|\varphi(\omega)|} dA(\omega)$$

$$\leq R^4 \iint\limits_{|\omega|<1} \log^+\frac{1}{|\varphi(\omega)|} dA(\omega)$$

$$= R^4 \iint\limits_{\mathbb{D}} \log^+|\varphi(\omega)|\, dA(\omega) - R^4 \iint\limits_{\mathbb{D}} \log|\varphi(\omega)|\, dA(\omega)$$

$$\leq R^4 \frac{3\pi}{2p} - R^4 \pi \log|\varphi(0)|,$$

where the first term comes from (14) and an integration; the second term comes from the inequality $\log|\varphi(0)| \leq \iint_{\mathbb{D}} \log|\varphi(\omega)|\, dA(\omega)/\pi$, which follows from the sub harmonicity of $\log|\varphi|$.

Corollary(4.1.3):[67] Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two L_a^p zero sequences such that $clos\{\alpha_n\} \cap clos\{\beta_n\} \cap \partial\mathbb{D} = \phi$. Then $\{\alpha_n\} \cup \{\beta_n\}$ is an L_a^p zero sequence.

Proof: Let φ_1 and φ_2 be the L_a^p inner functions associated with $\{\alpha_n\}$ and $\{\beta_n\}$, respectively. Then, by theorem (4.1.1) and an easy compactness argument, $\varphi_1\varphi_2 \in L_a^p$.

In [74], give the following formula for the L_a^p inner function φ associated with a finite zero sequence $\{\alpha_1, \dots, \alpha_n\}$:

$$\varphi(z) = B(z)\left[b - \sum_k \frac{a_k}{1 - \bar{\alpha}_{jk} z}\right]^{\frac{2}{p}} \tag{25}$$

Here

$$B(z) = \prod_{j=1}^{n} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \bar{\alpha}_j z}$$

is the Blaschke product associated with $\{\alpha_j\}$ (where we adopt the convention $\frac{|0|}{0} = -1$), and $\{\alpha_{j_1}, \dots, \alpha_{j_s}\}$ is a listing of the distinct nonzero elements of $\{\alpha_1, \dots, \alpha_n\}$. We here give an alternate proof of this formula based on the ideas. It is easy to see from (9) and (8) that

$$h(z) = h_j(z) + \overline{\Phi(\alpha_j)}\varphi^{\frac{p}{2}}(z) \tag{26}$$

where h_j is analytic in a neighborhood of α_j. It follows from (26) and (13) that, if $\alpha_j \neq 0$'

$$\Phi(z) = \Phi(\alpha_j) + \frac{z + \overline{h_j(1/\bar{z})}}{\overline{\varphi^{\frac{p}{2}}(1/\bar{z})}\,(1 - \bar{\alpha}_j z)} \tag{27}$$

and hence

$$\varphi^{\frac{p}{2}}(z) = \varphi'(z) = \frac{H_j(z)}{\overline{\varphi^{\frac{p}{2}}\left(\frac{1}{\bar{z}}\right)}(1 - \bar{\alpha}_j z)} \tag{28}$$

Where H_j is analytic in a neighborhood of $1/\bar{\alpha}_j$. Thus $\left(\frac{\varphi}{B}\right)^{p/2}$ is meromorphic near $1/\bar{\alpha}_j$ with at worst a simple pole there. Similar reasoning shows that $\frac{\varphi}{B}$ is analytic and nonvanishing in a neighborhood of ∞. Since $\frac{\varphi}{B}$ dose not vanish in clos$\mathbb{D}$, we can finally conclude that $\left(\frac{\varphi}{B}\right)^{p/2}$ is rational with simple poles or removable singularities at those α_j that are nonzero (in fact, it is not difficult to show it must have poles at these point). The formula (25) follows.

It should be mentioned that the method dose proved a little more information than that of MacGregor and Stessin, namely that $\varphi^{p/2}$ possesses a primitive in Ω. This is of course equivalent to the statement that

$$\int_{\Gamma_j} \varphi^{\frac{p}{2}}(z)dz = 0$$

If Γ_j is any rectifiable simple closed curve enclosing α_j and $1/\bar{\alpha}_j$,and not enclosing α_k or $1/\bar{\alpha}_k$. In fact, it can be shown that this condition, together with $\|\varphi\|_{L_a^p} = 1$, determines the coefficients b and a_k.

Suppose $\{\alpha_n\}$ is an L_a^2 zero sequence of distinct points in $\mathbb{D}$ and that $\{\omega_n\}$ is a sequence of points in $\mathbb{C}$ such that there exists an $f \in L_a^2$ such that $f(\alpha_n) = \omega_n$ for all n. Let ψ be the L_a^2 function of minimal norm accomplishing this interpolation. Then [71] that ψ continues analytically across any boundary arc not meeting clos$\{\alpha_n\}$. It is natural to ask if this result holds in L_a^p for $p \neq 2$. The following example shows that it dose not, even in the case of two interpolation points.

Let $0 < r < 1$ and set

$$\psi(z) = 1 - \frac{(1-r)^2}{(1-rz)^2}$$

Since 1 and $1/(1-rz)^2$ are the reproducing kernels for the points 0 and r, we see that ψ is the minimal L_a^2 interpolating function taking 0 to $1-(1-r)^2$ and r to $1-1/(1+r)^2$. The function ψ has a simple zero at $z = 1$ and no other zeros in clos$\{\mathbb{D}\}$. Let $B(z) = z[(r - z)/(1-rz)]$. By minimality,

$$\frac{d}{dt}\bigg|_{t=0} \|\psi + tF\psi B\|_{L_a^2} = 0 \qquad \forall F \in L_a^2,$$

which leads to

$$\iint_D |\psi|^2 BF \frac{dA}{\pi} = 0 \qquad \forall F \in L_a^2 \tag{29}$$

For $p > 1$ we now argue as in the proof of Theorem (4.2.1) of [80]: if F is a polynomial, then

$$\left\|\psi^{\frac{2}{p}}\right\|_{L_a^p}^p = \|\psi\|_{L_a^2}^2 = \iint_{\mathbb{D}} (\psi^{\frac{2}{p}} + \psi^{\frac{2}{p}} FB) \frac{|\psi|^2}{\psi^{\frac{2}{p}}} \frac{dA}{\pi} \qquad (by\ 29)$$

$$\leq \left[\iint_{\mathbb{D}} \left|\psi^{\frac{2}{p}} + \psi^{\frac{2}{p}} FB\right|^p \frac{dA}{\pi}\right]^{\frac{1}{p}} \times \left[\iint_{\mathbb{D}} |\psi|^{\left(2-\frac{2}{p}\right)\left(\frac{p}{p-1}\right)} \frac{dA}{\pi}\right]^{\frac{p-1}{p}}$$

$$= \left\|\psi^{\frac{2}{p}} + \psi^{\frac{2}{p}} FB\right\|_{L_a^p} \left\|\psi^{\frac{2}{p}}\right\|_{L_a^p}^{p-1}$$

Dividing by $\left\|\psi^{\frac{2}{p}}\right\|_{L_a^p}^{p-1}$, we see that

$$\left\|\psi^{\frac{2}{p}}\right\|_{L_a^p} \leq \left\|\psi^{\frac{2}{p}} + \psi^{\frac{2}{p}} FB\right\|_{L_a^p}$$

For any polynomial F. Since the functions of the form $\psi^{\frac{2}{p}} F$ (F a polynomial) are clearly dense in L_a^p, this shows that $\psi^{\frac{2}{p}}$ is the L_a^p minimal interpolating function taking 0 to $[1-(1-r)^2]^{2/p}$ and r to $[1-1/(1+r)^2]^{2/p}$. Of course, $\psi^{\frac{2}{p}}$ has a zero of order $2/p$ at 1 and hence does not extend analytically around 1 if $p > 1$ and $p \neq 2$.

Section(4.2): Complemented Invariant Subspaces in Bergman Spaces

A closed subspace I of A^p is said to be invariant if $hf \in I$ for all $f \in I$, where $h(z) = z$ is the identity function on $\mathbb{D}$. It is easy to show that a closed subspaces I of A^p is invariant if and only if $hf \in I$ for all $f \in I$ and all polynomials h if and only if $hf \in I$ for all $f \in I$ and all $h \in H^\infty$.

For any $f \in A^p$, the A^p-closure of the set of all polynomial multiples of f is clearly an invariant subspace of A^p, it is called the invariant subspace generated by f and will be denoted by I_f^p. In particular, if $\varphi \in H^\infty$, then I_φ^p is well-defined for all $0 < p < \infty$ and is the A^p-closure of φA^p.

A sequence $Z = \{a_n\}$ of points in $\mathbb{D}$ is called an A^p zero set if there exists a non-zero function $f \in A^p$ such that f vanishes on Z, counting multiplicity. Every A^p zero set Z gives rise to an invariant subspace I_Z^p consisting of all functions in A^p that vanish on Z, counting multiplicities again. It is known that $I_Z^p = I_G^p$, where G is the extremal function of I_Z^p; [82] for example.

We will consider a class of special A^p zero sets, namely, A^p interpolating sequences. Recall that a sequence $Z = \{a_n\}$ of distinct points in $\mathbb{D}$ is called A^p interpolating of if for every sequence $\{\omega_n\}$ of complex numbers satisfying

$$\sum |\omega_n|^p (1-|a_n|^2)^2 < \infty$$

There exists a function $f \in A^p$ such that $f(a_n) = \omega_n$ for all n. Such sequence have been geometrically characterized in [10].

Recall that I is complemented in A^p if there exists another closed subspace X of A^p such that $A^p = I \oplus X$. It is easy to see that a closed subspace I of A^p is complemented if and only if there exists a bounded linear operator$Q$ from A^pon to I such that $Q^2 = Q$. Such an operator Q is naturally called a projection of A^p onto I. Note that the projection Q, and so the complemental subspace X, is highly non-unique. When $p = 2$, every closed subspace I of $A^p$is complemented; and among all the projections of A^2 onto I,there exists a unique orthogonal projection.

In the case of Hardy spaces, which we denote by H^p, of the unit disk, every invariant subspace is complemented, at least when $1 < p < \infty$. In fact, if I is an invariant subspace of H^p, then there exists an inner function φ such that $I = \varphi H^p$.It follows from the M. Riesz theorem [83] that the operator Q defined by

$$Qf(z) = \frac{\varphi(z)}{2\pi i} \int_{|\zeta|=1} \frac{\bar{\varphi}(\zeta) f(\zeta)}{\zeta - z} d\zeta$$

is bounded projection from H^p onto I, proved that $1 < p < \infty$. Note that the operator Q is simply the orthogonal projection from H^2 onto φH^2.

Theorem(4.2.1):[81] Suppose that $0 < p < \infty$ and Z is an A^p interpolating sequence. Then the invariant subspace I_Z^p is complemented in A^p.

Theorem(4.2.2):[81] Suppose that $1 < p < \infty$ and $0 < \sigma < \infty$. Then the invariant subspace I_a^p generated by the singular inner function

$$S_\sigma(z) = \exp\left(-\sigma \frac{1+z}{1-z}\right)$$

is complemented in A^p.

To illustrate the notion of a complemented invariant subspace here, consider the case where the zero set Z has only one simple zero at $a \in \mathbb{D}$. It is obvious that the decomposition

$$f = \big(f - f(a)\big) + f(a)$$

induces a direct sum decomposition of A^p, namely,

$$A^p = I_Z^p \oplus \mathbb{C}.$$

Slightly more general is the case when Z consists of a single zero at $a \in \mathbb{D}$ of multiplicity n. Then the decomposition

$$f = (f - f_n) + f_n,$$

where f_n is the Taylor polynomial of f at a of order $n-1$, gives rise to the splitting

$$A^p = I_Z^p \oplus P_n$$

where P_n is the space of all polynomial of order n or less. This clearly works for all n and all $p \in (0,\infty)$.

If Z is a finite sequence with possible multiplicities, the space I_Z^p has finite codimension in A^p, and it follows from general functional analysis that I_Z^p is complemented in A^p.

Lemma(4.2.3):[81] Suppose that Z consists of distinct points $a_1, \dots, a_n$. Then for any $a \in \mathbb{D}$ the reproducing kernel I_Z^2 at a is of the form

$$K_Z(z,a) = \frac{1}{(1-z\bar{a})^2} - \sum_{k=1}^{n} \frac{A_k}{(1-z\bar{a}_k)^2},$$

where $A_1, \dots, A_n$ are constants chosen so that

$$\sum_{k=1}^{n} \frac{A_k}{(1-a_m\bar{a}_k)^2} = \frac{1}{(1-a_m\bar{a})^2}$$

for $m = 1, \dots, n$.

Proof: The existence of constants $A_1, \dots, A_n$ follows from the well-known fact that the matrix

$$\left(\frac{1}{(1-a_m\bar{a}_k)^2}\right)_{n\times n}$$

is non-singular.

The function $h(z) = K_z(z,a)$ defined above is clearly in A^2 and vanishes at $a_1, \dots, a_n$, and so belongs to I_Z^2. For any $f \in I_Z^2$, the reproducing property of the Bergman kernel gives

$$\langle f, h\rangle = f(a) - \sum_{k=1}^{n} A_k f(a_k) = f(a)$$

By uniqueness, the function $K_Z(z,a)$ defined above is indeed the reproducing kernel of I_Z^2 at a.

Lemma(4.2.4):[81] Suppose that Z consists of distinct points $a_1, \dots, a_n$. Then the reproducing kernel of I_Z^2 is of the form

$$K_Z(z,\omega) = \frac{1}{(1-z\bar{\omega})^2} - \sum_{k=1}^{n} \frac{\overline{\varphi_k(\omega)}}{(1-z\bar{a}_k)^2}$$

where

$$\varphi_k(z) = \frac{K_{Z_k}(z,a_k)}{K_{Z_k}(a_k,a_k)}, \quad 1 \le k \le n$$

and $Z_k = Z - \{a_k\}$ for $1 \le k \le n$.

Theorem(4.2.5):[81] Suppose that $0 < p < \infty$. If Z is either finite (with possible multiplicities) or A^p interpolating, then the space I_Z^p is complemented in A^p.

Proof: First assume that Z consists of a finite number of distinct points, $a_1, \dots, a_n$. Let $Q_Z: A^2 \to I_Z^2$ be the orthogonal projection. Then by Lemma(4.2.4) and the reproducing property of the Bergman kernel, we have

$$Q_Z f(z) = f(z) - \sum_{k=1}^{n} f(a_k)\varphi_k(z)$$

Since φ_k belongs to A^p for $1 \le k \le n$ and $0 < p < \infty$ (by Lemma(4.2.3)), and since each point-evaluation in $\mathbb{D}$ is a bounded linear functional on A^p, this formula clearly defines a bounded linear operator $Q_Z$on A^p. The functions φ_k satisfy $\varphi_k(a_k) = 1$ and $\varphi_k(a_m) = 0$ for all k and m with $k \neq m$. It follows that $Q_Z f \in I_Z^p$ for all $f \in A^p$ and $Q_Z f = f$ for all $f \in I_Z^p$, so Q_Z is a projection from A^p onto I_Z^p.

If Z is finite but has zeros of higher multiplicity, then the formula for $K_Z(z, \omega)$ not only involves the Bergman kernels at the points of Z but also their derivatives. For example, if a_k appears N times in Z, where $N > 1$, then the functions

$$\frac{1}{(1 - z\bar{a}_k)^2}, \frac{z}{(1 - z\bar{a}_k)^3}, \dots, \frac{z^{N-1}}{(1 - z\bar{a}_k)^{N+1}}$$

will show up in the formula for $K_Z(z, \omega)$, and accordingly, the values

$$f(a_k), f'(a_k), \dots, f^{(N-1)}(a_k)$$

will show up in the formula for $Q_Z f$, but it is clear that the resulting operator Q_Z is still bounded on all A^p.

When $Z = \{a_n\}$ is A^p interpolating, all the zeros are simple. Moreover, there exists a sequence $\{\varphi_n\}$ of functions in A^p satisfying the following conditions.

(i) $\varphi_n(a_n) = 1$ for all n.
(ii) $\varphi_n(a_k) = 0$ for all n and k with $n \neq k$.
(iii) There exists a positive constant M such that whenever $\{\omega_n\}$ is a sequence of complex numbers satisfying

$$\sum |\omega_n|^p (1 - |a_n|^2)^2 < \infty$$

then the series

$$f(z) = \sum \omega_n \varphi_n(z)$$

converges in A^p and

$$\|f\|_p^p \le M \sum |\omega_n|^p (1 - |a_n|^2)^2$$

[85] for the existence of such a sequence $\{\varphi_n\}$.

Now define an operator Q_Z on A^p as follows:

$$Q_Z f(z) = f(z) - \sum f(a_n)\varphi_n(z), \quad f \in A^p$$

Since an A^p interpolating sequence is separated in the hyperbolic metric, there existes a constant $M_1 > 0$ such that

$$\sum |f(a_n)|^p (1 - |a_n|^2)^2 \leq M_1 \int_{\mathbb{D}} |f(z)|^p dA(z)$$

For all $f \in A^p$. This along with property (iii) above shows that Q_Z is a bounded linear operator on A^p. Also, it is clear from properties (i) and (ii) that Q_Z maps A^p into I_Z^p, and Q_Z leaves all functions in I_Z^p fixed. In other words, $Q_Z$maps A^p onto I_Z^p and acts as the identety opesorator on I_Z^p, Q_Z is abounded projection from A^p onto I_Z^p.

For any $r \in (0, \infty)$ the equation

$$\frac{1 - |z|^2}{|1 - z|^2} = r$$

defines a circle C_r internally tangent to the unit circle at the point 1. In fact, the above equation can easily be transformed to the standard from of a circle,

$$\left| z - \frac{r}{1 + r} \right|^2 = \frac{1}{(1 + r)^2}.$$

Such circles are called orocycles, [89].

As r runs from 0 to ∞, the orocycles C_r non-overlappingly fill up the whole disk $\mathbb{D}$. Since the orocycle C_r has Euclidean center at $\frac{r}{1+r}$ and Euclidean radius $\frac{1}{1+r}$, we can parameterize the unit disk as follows,

$$z = z(r, \theta) = \frac{r}{1 + r} + \frac{1}{1 + r} e^{i\theta}, \quad 0 \leq r < \infty, 0 \leq \theta \leq 2\pi$$

This parameterization will be called the oro-coordinates of $\mathbb{D}$.

Lemma(4.2.6):[81] Suppose that g is Lebesgue measurable on $\mathbb{D}$. If g is non-negative or belongs to $L^1(\mathbb{D}, dA)$, then

$$\int_{\mathbb{D}} g(z) dA(z) = \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} g(r, \theta) \frac{1 - \cos\theta}{(1 + r)^3} dr d\theta$$

Where $g(r, \theta) = g(z(r, \theta))$ is the function g in oro-coordinates.

Oro-coordinates are especially suitable for studying the singular inner functions

$$S_\sigma(z) = \exp\left(-\sigma\frac{1+z}{1-z}\right),$$

Where σ is any positive constant (called the mass of S_σ). By definition, the orocycles C_r are the level curves of S_σ. More specifically, we have

$$|S_\sigma(z)| = e^{-\sigma r}, \quad z = z(r,\theta)$$

The function $1 - z$ also plays a special role in oro-coordinates, namely,

$$|1-z| = \frac{|1-e^{i\theta}|}{1+r}, \quad z = z(r,\theta),$$

and

$$|1-z|^2 = \frac{2(1-\cos\theta)}{(1+r)^2}, \quad z = z(r,\theta).$$

Lemma(4.2.7):[81] Suppose that g is Lebesgue integrable on $\mathbb{D}$. If g is non-negative or belongs $L^1(\mathbb{D}, dA)$, then

$$\int_{\mathbb{D}} g(z)dA(z) = \frac{1}{2\pi}\int_0^\infty dr \int_{C_r} g(\zeta)|1-\zeta|^2|d\zeta|.$$

Lemma(4.2.8):[81] Let f be analytic on the closed unit disk $\overline{\mathbb{D}}$ except at $z = 1$. Suppose that $|f(z)| \le M$ for all $|z| = 1$ with $z \ne 1$,

$$\limsup_{z\to 1}|1-z|^2 \log|f(z)| \le 0$$

and

$$\limsup_{x\to 1^-}(1-x)\log|f(x)| \le 0$$

Then $|f(z)| \le M$ for all $z \in \mathbb{D}$.

Proof: The fractional liner transformation

$$\omega = \frac{1+z}{1-z}$$

maps the unit disk to the right half-plane and the unit circle to the imaginary axis. It also transforms the function f to function g on the right half-plane.

The assumptions on f translate into the following assumptions on g: $|g(\omega)| \le M$ for all ω on the imaginary axis,

$$\lim_{\omega\to\infty}\sup\frac{\log|g(\omega)|}{|\omega|^2} \le 0$$

where $Re\ \omega > 0$, and

$$\lim_{x\to+\infty} \sup \frac{\log|g(x)|}{|x|^2} \leq 0$$

Fix any small $\epsilon > 0$ and consider the function

$$g_\epsilon(\omega) = g(\omega)e^{-\epsilon\omega}$$

on the first quadrant Ω of the ω-plane, which is between two straight lines making an angle of $\pi/2$ at the origin.

It is easy to see that the assumptions on g imply that $|g_\epsilon(\omega)| \leq M$ for all ω on the imaginary axis,

$$g_\epsilon(\omega) = O(\exp(\delta|\omega|^2)), \quad \omega \in \Omega$$

where δ is any positive number, and

$$\lim_{x\to+\infty} g_\epsilon(x) = 0$$

In particular, the function g_ϵ is also bounded on the positive real axis. Now using the Phrame'n-Lindelof theorem quoted before this lemma (with $\alpha = 2$) and mimicking the proof in [86,87], we obtain $|g_\epsilon(\omega)| \leq M$ for all $\omega \in \Omega$.letting $\epsilon \to 0$, we conclude that $|g(\omega)| \leq M$ for all $\omega \in \Omega$.

Exactly the same argument shows that $|g(\omega)| \leq M$ for all ω in the fourth quadrant of the ω-plane.

Proposition(4.2.9):[81] Suppose that $0 < p < \infty$ and $0 < \sigma < \infty$. A function $f \in H^p$ belongs to $S_\sigma H^p$ if and only if

$$\limsup_{x\to1^-}(1-x)\log|f(x)| \leq -2\sigma$$

Proof: It is easy to see that

$$\lim_{x\to1^-}(1-x)\log|S_\sigma| = -2\sigma$$

If g is an arbitrary function in H^p, then [83]

$$|g(z)| \leq \frac{M}{(1-|z|^2)^{\frac{1}{p}}}$$

for all $z \in \mathbb{D}$, where M is a positive constant, and so

$$\limsup_{x\to1^-}(1-x)\log|f(x)| \leq 0$$

This shows that if $f = S_\sigma g$, where $g \in H^p$, then

$$\limsup_{x\to 1^-}(1-x)\log|f(x)| = \limsup_{x\to 1^-}(1-x)\,[\log|S_\sigma(x)| + \log|g|] \le -2\sigma$$

To prove the other direction, we assume that f belong to H^p and satisfies the condition

$$\limsup_{x\to 1^-}(1-x)\log|f(x)| \le -2\sigma$$

Then for each $n \ge 1$ the function

$$g_n(z) = \frac{f(z)}{S_\sigma(z)}\left[\frac{n(1-z)}{n(1-z)+1}\right]^{\frac{2}{p}}$$

satisfies

$$\limsup_{x\to 1^-}(1-x)\log|g_n(x)| \le 0$$

since $|f(z)|(1-|z|^2)^{\frac{1}{p}}$ is bounded in $\mathbb{D}$ and the functions $1-|z|^2$ and $|1-z|^2$ are comparable on any orocycle, we see that each function g_n is bounded on C_r (recall that S_σ has constant modulus on C_r). Inside C_r, we have

$$|1-z|^2\log\left|\frac{1}{S_\sigma(z)}\right| = \sigma(1-|z|^2)$$

and

$$|1-z|^2\log|f(z)| \le \frac{1-|z|^2}{r}\log|f(z)|\,;$$

which clearly implies that

$$\limsup_{z\to 1}|1-z|^2\log|g_n(z)| \le 0$$

By Lemma(4.2.8) and the remark following it, each function g_n is bounded inside C_r. Since $|g_n|$ is dominated by $|f|$ outside C_r, we can find a positive constant M_n such that

$$|g_n(z)| \le M_n(1+|f(z)|)$$

for all $z \in \mathbb{D}$. In particular, each g_n belongs to H^p.

Let $f_n = S_\sigma g_n$. Then $|f_n(z)| \le |f(z)|$ and $f_n(z) \to f(z)$ for all $z \in \mathbb{D}$. This shows that $\|f_n - f\|_{H^p} \to 0$ as $n \to \infty$. Since each f_n is in $S_\sigma H^p$ and $S_\sigma H^p$ is closed in H^p, we conclude that $f \in S_\sigma H^p$.

A consequence of the above result is that an inner function φ is divisible by S_σ if and only if

$$\limsup_{x\to 1^-}(1-x)\log|\varphi(x)| \le -2\sigma$$

In particular, if B is Blaschke product, then

$$\limsup_{x\to 1^-}(1-x)\log|B(x)| = 0$$

Now consider the mapping

$$z = \frac{r}{1+r} + \frac{1}{1+r}\omega$$

which maps the unit circle $|\omega| = 1$ to the orocycle C_r, and the unit disk $|\omega| < 1$ onto the interior of C_r. The invers of the above mapping is given by

$$\omega = (1+r)\left(z - \frac{r}{1+r}\right)$$

Using these conformal mappings, and using the well-known theory of Hardy spaces of the unit disk $\mathbb{D}$, we easily realize $H^p(C_r)$ as a closed subspace of $L^p(C_r, dm_r)$, where dm_r is the normalized arc-length measure on C_r. In particuler, the following norm estimate for the Cauchy transform is a consequence of the above change of variables and the classical M. Riesz theorem [83] for the unit circle.

Lemma(4.2.10):[81] For each $1 < p < \infty$ the Cauchy transform

$$Qf(z) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\zeta)d\zeta}{\zeta - z}$$

is a projection from $L^p(C_r, dm_r)$ onto $H^p(C_r)$. Morever, there exists a positive constant M_p, independent of r, such that

$$\int_{C_r} |Qf(z)|^p|dz| \le M_p \int_{C_r} |f(z)|^p|dz|$$

for all $f \in L^p(C_r, dm_r)$.

Lemma(4.2.11):[81] If $f \in A^p$, then the function $(1-z)^{\frac{2}{p}}f(z)$ belongs to $H^p(C_r)$.

Proof: Since $|1-z|^2$ is comparable to $1-|z|^2$ on C_r, it is easy to see that the measure $|1-z|^2 dm_r(z)$ is Carleson-type measure for A^p; [88].

This shows that the function

$$g(z) = (1-z)^{\frac{2}{p}}f(z)$$

is in $L^p(C_r, dm_r)$. Inside C_r, $|1-z|^2$ is dominated by $1-|z|^2$ and $f(z)$ grows at a maximum rate of $(1-|z|^2)^{-\frac{2}{p}}$ [85], so the function $(1-z)^N f(z)$ is bounded in C_r when $N > 4/p$. It follows that g can be represented as a bounded analytic function in C_r divided by a certain power of $1-z$. In particular, g is in $H^q(C_r)$ when q is small enough. By [89] and the conclusion of the previous paragraph, g belongs to Hardy space $H^p(C_r)$.

Lemma(4.2.12):[81] Every function $f \in H^p(C_r)$ satisfies

$$\limsup_{x \to 1^-} (1-x) \log|f(x)| \leq 0$$

Moreover, a function $f \in H^p(C_r)$ belongs to the closed subspace $S_\sigma H^p(C_r)$ if and only if

$$\limsup_{x \to 1^-} (1-x) \log|f(x)| \leq -2\sigma$$

Proof: This follows from proposition(4.2.9) via the conformal mapping

$$z = \frac{r}{1+r} + \frac{1}{1+r}\omega$$

from the unit disk to the interior of C_r. Note that the point mass σ at $z = 1$ on C_r transforms to the point mass $\sigma(1+r)$ at $z = 1$ on the unit circle. More specifically, if z and ω are related as above, then

$$S_{\sigma(1+r)}(\omega) = cS_\sigma(z)$$

where $c = e^{\sigma r}$ is a constant.

Proposition(4.2.13):[81] Suppose that $0 < p < \infty$ and $f \in A^p$. Then $f \in I_\sigma^p$ if and only if

$$\limsup_{x \to 1^-} (1-x) \log|f(x)| \leq -2\sigma$$

Proof: It is well known [90,85] that the extremal problem

$$\sup\left\{Re\, f(0): f \in I_\sigma^p, \|f\|_p \leq 1\right\}$$

has a unique solution which is given by the formula

$$G(z) = (1+p\sigma)^{-\frac{1}{p}} \left(1 + \frac{p\sigma}{1-z}\right)^{\frac{2}{p}} S_\sigma(z)$$

This is called the extremal function of I_σ^p. Furthermore, if $f \in I_\sigma^p$, then $\left\|\frac{f}{G}\right\|_p \leq \|f\|_p$.

Now if a function $f \in A^p$ belongs to I_σ^p, then $f = Gg$, where G is the extremal function mentioned in the previous paragraph and $g \in A^p$. It is well known [85] that every function g in A^p satisfies

$$|g(z)|^p \leq \frac{M}{(1-|z|^2)^2}, \quad z \in \mathbb{D},$$

where M is a positive constant, so

$$\limsup_{x\to 1^-}(1-x)\log|g(x)| \le 0$$

Since

$$\limsup_{x\to 1^-}(1-x)\log|G(x)| = -2\sigma$$

we conclude that

$$\limsup_{x\to 1^-}(1-x)\log|f(x)| \le -2\sigma$$

The proof of the other direction is similar to the corresponding part in Proposition (4.2.9), except here we use the functions

$$g_n(z) = \frac{f(z)}{S_\sigma\,(z)}\left[\frac{n(1-z)}{n(1-z)+1}\right]^{\frac{4}{p}}.$$

Corollary(4.2.14):[81] Suppose that $0 < p < \infty$ and $f \in A^p$. Then the following conditions are equivalent.

(i) f belongs to I_σ^p.

(ii) The function $(1-z)^{\frac{2}{p}} f(z)$ belongs to $S_\sigma H^p(C_r)$ for every $r > 0$.

(iii) The function $(1-z)^{\frac{2}{p}} f(z)$ belongs to $S_\sigma H^p(C_r)$ for some $r > 0$.

Lemma(4.2.15):[81] For $1 \le p < \infty$ we define an operator Q_p acting on A^p as follows.

$$Q_p f(z) = (1-z)^{-\frac{2}{p}} S_\sigma(z) \int\limits_{C_r} \frac{f(\zeta)(1-\zeta)^{\frac{2}{p}}}{S_\sigma(\zeta)(\zeta-z)}\,\frac{d\zeta}{2\pi i},$$

where r is any positive number such that z lies inside C_r. Then the integra above is independent of r so long as z lies inside C_r.

Theorem(4.2.16):[81] For any $1 < p < \infty$ the operator Q_p above is a bounded porjection from A^p onto I_σ^p. Furthermor, Q_2 is the orthogonal projection from A^2 onto I_σ^p.

Proof: We first prove that Q_p is bounded on A^p. To this end, fix $f \in A^p$ and let

$$\tilde{f}(z) = \int\limits_{C_r} \frac{f(\zeta)(1-\zeta)^{\frac{2}{p}}}{S_\sigma(\zeta)(\zeta-z)}\,\frac{d\zeta}{2\pi i}, \qquad z \in \mathbb{D},$$

where C_r is any orocycle such that z is inside it. By Lemma(4.2.7), we have

$$\int\limits_{\mathbb{D}} \left|Q_p f(z)\right|^p dA(z) = \frac{1}{2\pi}\int\limits_0^\infty e^{-p\sigma r}\,dr \int\limits_{C_r} \left|\tilde{f}(\zeta)\right|^p |d\zeta|.$$

For any fixed $r > 0$, the values of $\tilde{f}$ in the disk enclosed by C_r can be computed using the same circle C_r and the resulting function is simply the cauchy transform of the following function in $L^p(C_r, |dz|)$.

$$\frac{f(\zeta)(1-\zeta)^{\frac{2}{p}}}{S_\sigma(\zeta)}, \qquad \zeta \in C_r.$$

By Lemma(4.2.1), there exists a constant M_p, independent of r and f, such that

$$\int_{C_r} \left|\tilde{f}(\zeta)\right|^p |d\zeta| \leq M_p \int_{C_r} \left|\frac{f(\zeta)(1-\zeta)^{\frac{2}{p}}}{S_\sigma(\zeta)}\right|^p |d\zeta|$$

It follows that

$$\int_{\mathbb{D}} \left|Q_p f(z)\right|^p dA(z) \leq \frac{M_p}{2\pi}\int_0^\infty e^{-p\sigma r} dr \int_{C_r} e^{p\sigma r} |f(\zeta)|^p |1-\zeta|^2 |d\zeta| = M_p \int_{\mathbb{D}} |f(z)|^p dA(z),$$

where Lemma(4.2.7) was used again (for the last equality above). This shows that Q_p is bounded operator on A^p.

Next we show that Q_p maps A^p into I_σ^p. Fix any $r > 0$ and consider the function $Q_p f$ in the orocycle C_r. Thus for z inside C_r we have

$$Q_p f(z) = (1-z)^{-\frac{2}{p}} S_\sigma(z) \tilde{f}(z),$$

where

$$\tilde{f}(z) = \int_{C_r} \frac{f(\zeta)(1-\zeta)^{\frac{2}{p}}}{S_\sigma(\zeta)(\zeta - z)} \frac{d\zeta}{2\pi i}.$$

Since $\tilde{f}$ belongs to $H^p(C_r)$, Lemma (4.2.12) gives us

$$\limsup_{x \to 1^-} (1-x) \log\left|\tilde{f}(x)\right| \leq 0$$

This then implies that

$$\limsup_{x \to 1^-} (1-x) \log\left|Q_p f(x)\right| \leq -2\sigma$$

By Proposition (4.2.13), we have $Q_p f \in I_\sigma^p$.

That Q_p acts as the identity operator on I_σ^p is a consequence of Corollary (4.2.14) and the reproducing property of the Cauchy transform on $H^p(C_r)$.

Finally, observe that Cauchy transform is orthogonal projection from $L^2(C_r, dm_r)$ onto $H^2(C_r)$. Examining the arguments in the first paragraph of this proof, we realize that the projection Q_2 has norm 1 on A^2, which forces Q_2 to be an orthogonal projection.

If Q is any projection on a Hilbert space H, then it is an orthogonal projection if and only if $\|Q\| = 1$. It is of coures well known that every orthogonal has norm 1. When Q, whose associated direct sum decomposition is qiven by

$$H = \mathrm{ran}(Q) + \ker(Q);$$

is not orthogonal, there must exist unit vectors $f \in \mathrm{ran}(Q)$ and $g \in \ker(Q)$ such that $\langle f, g\rangle > 0$. For any positive ϵ the vector $h = f - \epsilon g$ satisfies $\|Qh\| = \|f\|$ and

$$\|f\|^2 - \|h\|^2 = \epsilon(2\langle f, g\rangle - \epsilon).$$

If we choose ϵ such that $0 < \epsilon < 2\langle f, g\rangle$, then $\|Qh\| > \|h\|$, so the norm of Q is qreater than 1.

from A^2 onto I^p_σ can be used to calculate the extremal function G of I^p_σ. In fact, it is elementary to see that

$$G(z) = \frac{Q_2(1)(z)}{\sqrt{Q_2(1)(0)}}, \qquad z \in \mathbb{D}.$$

When calculating $Q_1(1)(z)$ we may let $r \to 0^+$ and obtain

$$Q_2(1)(z) = \frac{S_\sigma(z)}{2\pi i(1-z)} \int_{|\zeta|=1} \frac{\bar{S}_\sigma(\zeta)(1-\zeta)}{\zeta - z} d\zeta = \frac{S_\sigma(z)}{1-z}[S_\sigma(0) - S_\sigma(0)z - \acute{S}_\sigma(0)]$$
$$= e^{-\sigma} S_\sigma(z)\left(1 + \frac{2\sigma}{1-z}\right).$$

It follows that

$$G(z) = \frac{1}{\sqrt{1+2\sigma}}\left(1 + \frac{2\sigma}{1-z}\right) S_\sigma(z).$$

In general, if $K_\sigma(z, \omega)$ is the reproducing kernel of I^p_σ, then

$$K_\sigma(z, \omega) = Q_2(K_\omega)(z), \;\; K_\omega(z) = (1 - z\overline{\omega})^{-2}$$

An explicit formula for $K_\sigma(z, \omega)$ is bounded in [91] by a different method.

Section(4.3): Spaces of Analytic Functions

Let $d \geq 1$, $\Omega \subseteq \mathbb{C}^d$ be an open, connected, and nonempty set, and let $\mathcal{H} \subseteq Hol(\Omega)$ be a reproducing kernel Hilbert space. If $\varphi \in \mathrm{Hol}(\Omega)$ such that $\varphi f \in \mathcal{H}$ for all $f \in \mathcal{H}$, then φ is called a multiplier and $M_\varphi f = \varphi f$ defines a bounded linear operator on $\mathcal{H}$. We use $\mathcal{M}(\mathcal{H})$ to denote the multiplier algebra of $\mathcal{H}$,$\mathcal{M}(\mathcal{H}) = \{M_\varphi \in \mathcal{B}(\mathcal{H}) : \varphi \text{ is a multiplier}\}$.

A sub algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is called a transitive algebra if it contains the identity operator and if it has no nontrivial common invariant subspaces. It is a longstanding open question (due to Kadison), called the transitive algebra problem, to decide whether every transitive algebra is dense in $\mathcal{B}(\mathcal{H})$ in the strong operator topology. If that were the case, then, as is well-known, it would easily follow that every $T \in \mathcal{B}(\mathcal{H})$ which is not a scalar multiple of the identity has a nontrivial hyper invariant subspace [93]. Recall that a subspace $\mathcal{M}$ is called hyperinvariant for an operator A, if it is invariant for every bounded operator that commutes with A.

Arveson was the first to systematically study the transitive algebra problem. We say that an operator A (respectively an algebra $\mathcal{A}$) has the transitive algebra property, if every transitive algebra that contains A(respectively $\mathcal{A}$) is strongly dense in $\mathcal{B}(\mathcal{H})$. Arveson showed that any maximal abelian self-adjoint subalgebra and the unilateral shift have the transitive algebra property. [93].

Arveson's approach requires a detailed knowledge of the invariant subspace structure of the operator or the algebra that is to be shown to have the transitive algebra property. Thus based on information about the invariant subspaces of the Dirichlet space Richter was able to use Arveson's approach to establish that the Dirichlet shift has the transitive algebra property, [94]. Then more generally Chong, Guo and Wang, [97], followed a similar strategy to show among other things that $\mathcal{M}(\mathcal{H})$ has the transitive algebra property, whenever $\mathcal{H}$ has a complete Nevanlinna–Pick kernel, i.e. if the reproducing kernel $k_\lambda(z)$ for $\mathcal{H}$ is of the form $k_\lambda(z) = \frac{\overline{f(\lambda)}f(z)}{1-u_\lambda(z)}$, where f is an analytic function and $u_\lambda(z)$ is positive definite and sesquianalytic.

We was motivated by the desire to decide which other multiplier algebras have the transitive algebra property. Although we did not obtain any specific answers, the investigations lead us to consider some interesting questions related to the invariant subspace structure of $\mathcal{M}(\mathcal{H})$ see [95].

For its statement we need to define invariant graph subspaces. If $N > 1$ then $\mathcal{H}^{(N)}$denotes the direct sum of N copies of $\mathcal{H}$, and for an operator $A \in \mathcal{B}(\mathcal{H})A^{(N)}$ is the N-fold ampliation of A, $A^{(N)}: \mathcal{H}^{(N)} \to \mathcal{H}^{(N)}$, $A^{(N)}(x_1, \ldots, x_N) = (Ax_1, \ldots, Ax_N)$.

If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is an algebra of bounded operators on $\mathcal{H}$, then a closed subspace $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ is called an invariant graph subspace for $\mathcal{A}$ if there is a linear manifold $\mathcal{D} \subseteq \mathcal{H}$ and linear transformations $T_1, \ldots, T_{N-1}: \mathcal{D} \to \mathcal{H}$ such that

$$\mathcal{M} = \{(x, T_1x, \ldots, T_{N-1}x): x \in \mathcal{D}\} \quad (30)$$

and such that $A^{(N)}\mathcal{M} \subseteq \mathcal{M}$ for every $A \in \mathcal{A}$. The transformations $T_1, \ldots, T_{N-1}$ are called linear graph transformations for $\mathcal{A}$. Note that if a linear manifold $\mathcal{D}$ and linear transformations $T_1, \ldots, T_{N-1}: \mathcal{D} \to \mathcal{H}$ are given, then (30) defines an invariant graph subspace for $\mathcal{A}$, if and only if $\mathcal{M}$ is closed, $A\mathcal{D} \subseteq \mathcal{D}$ for every $A \in \mathcal{A}$, and $AT_i = T_iA$ on $\mathcal{D}$ for each $i = 1, \ldots, N-1$. Thus the graph transformations for $N = 2$ correspond to the closed linear transformations that commute with $\mathcal{A}$. Arveson's Lemma states that a transitive algebra $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$ if and only if the only linear graph transformations for $\mathcal{A}$ are multiples of the identity operator, [96].

Theorem(4.3.1):[92] Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space. $\mathcal{M}(\mathcal{H})$ has the transitive algebra property if and only if the following condition is satisfied:

Whenever $N > 1$ and

$$\mathcal{M} = \{(f, T_1 f, \ldots, T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$$

is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ such that for each $\alpha = (\alpha_0, \ldots, \alpha_{N-1}) \in \mathbb{C}^N$, $\alpha \neq (0, \ldots, 0)$ the linear transformation

$$L_\alpha : \mathcal{D} \to \mathcal{H}, L_\alpha = \bar{\alpha}_0 I + \sum_{i=1}^{N-i} \bar{\alpha}_i T_i$$

is $1-1$ and has dense range, then

$$\mathcal{A}_{\mathcal{M}} = \{A \in \mathcal{B}(\mathcal{H}) : A\mathcal{D} \subseteq \mathcal{D} : AT_i f = T_i A f \quad \forall f \in \mathcal{D}\}$$

has nontrivial invariant subspaces.

It is easy to see that for any invariant graph subspace $\mathcal{M}$ the collection $\mathcal{A}_{\mathcal{M}}$ is a strongly closed algebra, contains $\mathcal{M}(\mathcal{H})$, and that $\mathcal{M}$ is an invariant graph subspace for $\mathcal{A}_{\mathcal{M}}$. $\mathcal{A}_{\mathcal{M}}$ is the largest algebra that has $\mathcal{M}$ as an invariant graph subspace. It is clear that for any $\alpha \in \mathbb{C}^n$ the closures of $\ker L_\alpha$ and ran L_α are invariant subspaces for $\mathcal{A}_{\mathcal{M}}$. We will say that $\mathcal{A}_{\mathcal{M}}$ does not have any nontrivial invariant subspaces that are determined by linear relations of the graph transformations, if for each $\alpha \in \mathbb{C}^n$ we have $\overline{\ker L_\alpha}, \overline{\mathrm{ran} L_\alpha} \in \{(0), \mathcal{H}\}$. With this terminology one easily checks that the condition in Theorem (4.3.1)is equivalent to the two conditions:

(i) the set $\{I, T_1, \ldots, T_{N-1}\}$ is linearly independent, and
(ii) $\mathcal{A}_{\mathcal{M}}$ does not have any nontrivial invariant subspaces that are determined by linear relations of the graph transformations.

We note that $\mathcal{D} = \mathrm{ran} L_\alpha$ for $\alpha = (1, 0, \ldots, 0)$. Thus condition (ii) implies that $\mathcal{D}$ is dense in $\mathcal{H}$.

A useful invariant in the study of invariant subspaces $\mathcal{M} \subseteq \mathcal{H}^N$ is the fiber dimension of $\mathcal{M}$. It is defined as follows. If $\lambda \in \Omega$, if $N \geq 1$, and if $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ is a subspace, then the fiber of $\mathcal{M}$ at λ is

$$\mathcal{M}_\lambda = \{(f_1(\lambda), \ldots, f_N(\lambda)) : (f_1, \ldots, f_N) \in \mathcal{M}\} \subseteq \mathbb{C}^N.$$

The fiber dimension of $\mathcal{M}$ is

$$\mathrm{fd}\mathcal{M} = \sup_{\lambda \in \Omega} \dim \mathcal{M}_\lambda .$$

A simple argument using determinants shows that $\mathrm{fd}\mathcal{M} = \dim \mathcal{M}_\lambda$ for all $\lambda \in \Omega \backslash E$, where E is the zero set of some nontrivial analytic function on Ω, [17].

If $\mathcal{M} \subseteq \mathcal{H}^N$ is an invariant graph subspace, then it is easy to see that

$$\mathcal{M}_\lambda^\perp = \{\alpha \in \mathbb{C}^N : k_\lambda \perp \text{ ran } L_\alpha\},$$

see Lemma (4.3.10). Thus, the condition that $\text{ran}L_\alpha$ is dense implies that $\mathcal{M}$ has full fiber dimension at each point, i.e. $\mathcal{M}_\lambda = \mathbb{C}^N$ for all $\lambda \in \Omega$ such that $k_\lambda \neq 0$. It follows that the invariant graph subspaces $\mathcal{M}$ considered in Theorem (4.3.1) all have fiber dimension $N > 1$.

We will see that whenever $\text{fd}\mathcal{M} > 1$, then $\mathcal{A}_\mathcal{M} \neq \mathcal{B}(\mathcal{H})$, see Proposition (4.3.8). In particular, we note that any $\mathcal{A}_\mathcal{M}$ as above that is transitive would be a counterexample to the transitive algebra problem.

If $\mathcal{H}$ has a complete Nevanlinna–Pick kernel then every nonzero invariant graph subspace of $\mathcal{M}(\mathcal{H})$ has fiber dimension one. Thus the condition of the theorem is trivially satisfied, because there is no invariant graph subspace of $\mathcal{M}(\mathcal{H})$ that satisfies the hypothesis of the condition [97].

This means that it becomes a question of interest to decide for which spaces $\mathcal{H}$ one can construct examples of invariant graph subspaces which satisfy the condition of Theorem (4.3.1). We will outline a strategy for constructing such invariant graph subspaces (in the case $N = 2$), and we will discuss what other nontrivial invariant subspaces the algebra $\mathcal{A}_\mathcal{M}$ may have. We will show that this can be carried out for the Bergman space L_a^2.

Example(4.3.2):[92] Let $\mathcal{H} \subseteq \text{Hol}(\Omega)$ be a reproducing kernel Hilbert space, let φ, ψ be multipliers such that $\frac{1}{\varphi-\psi}$ is a multiplier, and let $\mathcal{L}, \mathcal{N} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$.Then with $\mathcal{D} = \mathcal{N} + \mathcal{L}$ and $T(f+g) = \varphi f + \psi g$ the space $\mathcal{M} = \{(h, Th) : h \in \mathcal{D}\}$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ of fiber dimension 2.

Examples of invariant subspaces with $\mathcal{N} \cap \mathcal{L} = (0)$ can be based on zero sets. Recall that a set $E \subseteq \Omega$ is called a zero set for $\mathcal{H}$ if $I(E) = \{f \in \mathcal{H} : f(\lambda) = 0\ \forall \lambda \in E\} \neq (0)$. Then if $A, B \subseteq \Omega$ are zero sets for $\mathcal{H}$ such that $A \cup B$ is not a zero set for $\mathcal{H}$, one checks that $I(A)$ and $I(B)$ are invariant subspaces with $I(A) \cap I(B) = (0)$. [9], for a concrete example of this. For $S \subseteq \mathcal{H}$ let $Z(S) = \{\lambda \in \mathbb{D} : f(\lambda) = 0\ \forall f \in S\}$. It turns out that if in Example(4.3.2) $\lambda \in Z(\mathcal{N}) \cup Z(\mathcal{L})$, then $\dim \mathcal{M}_\lambda < 2$.

Theorem(4.3.3):[92] Let $\mathcal{H} \subseteq \text{Hol}(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, $\text{ran}(M_z - \lambda)$ is closed for all $|\lambda| < 1$, and $\dim \mathcal{H}/z\mathcal{H} = 1$. Let $\varphi, \psi \in H^\infty$ such that $1/(\varphi - \psi) \in H^\infty$ and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be $\mathcal{M}(\mathcal{H})$-invariant subspaces such that

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ is dense in $\mathcal{H}$,
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$,
(iv) the inner–outer factorizations of $\varphi - \lambda$ and $\psi - \lambda$ have no singular inner factor for any $\lambda \in \mathbb{C}$,
(v) neither φ nor ψ is a constant function,

then $\mathcal{M}$ as in Example(4.3.2) satisfies the hypothesis of Theorem(4.3.1).

Theorem(4.3.4):[92] There are two closed subspaces $\mathcal{N}, \mathcal{L} \subseteq L_a^2$ which are invariant for $\mathcal{M}(L_a^2)$ and such that

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ is dense in L_a^2, and
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$,

The Bergman shift has a complicated invariant subspace structure. Thus the above result may not come as a surprise. For these perceived complications is the existence of invariant subspaces $\mathcal{N} \subseteq L_a^2$ of high index, i.e. with $\dim \mathcal{N} \ominus z\mathcal{N} > 1$, [8,4,22]. The construction is independent of the high index phenomenon. We will exhibit a space $\mathcal{H} \subseteq$Hol($\mathbb{D}$) with no invariant subspaces of high index, but still admitting the above type of example Theorem(4.3.5).

For the Bergman space it is a result of Horowitz that there are zero sets whose union is not a zero set, [9]. We start with Horowitz's example and apply a result of Korenblum, which shows how to "push" zeros to the boundary $\partial\mathbb{D}$, [99]. Then we show that if this is done often enough one can end up with the required example.

In the constructed examples the algebras $\mathcal{A}_{\mathcal{M}}$ have no nontrivial invariant subspaces that are defined by linear relations of the graph transformations. Can one show that they have others? We will see that for many choices of φ and ψ one or both of the subspaces $\mathcal{N}$ and $\mathcal{L}$ that were used in the construction of the example turn out to be invariant for $\mathcal{A}_{\mathcal{M}}$.

Theorem(4.3.5):[93] Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq H$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example(4.3.2).

If

$$\varphi(\mathbb{D}) \backslash \overline{\psi(\mathbb{D})} \neq \emptyset,$$

then $\mathcal{N}$ is an invariant subspace for $\mathcal{A}_{\mathcal{M}}$.

In particular, $\mathcal{A}_{\mathcal{M}}$ has a non-trivial invariant subspace.

Similarly, if $\psi(\mathbb{D}) \backslash \overline{\varphi(\mathbb{D})} \neq \emptyset$, then $\mathcal{L}$ is invariant for $\mathcal{A}_{\mathcal{M}}$.

This will be Theorem(4.3.18). It raises the question whether the distinguished subspaces $\mathcal{N}$ and $\mathcal{L}$ of Example1(4.3.2) are always invariant for $\mathcal{A}_{\mathcal{M}}$, but we will give an example of carefully chosen zero-based invariant subspaces of the Bergman space and H^∞-functions φ and ψ that satisfy the hypothesis of Example(4.3.1), but such that neither $\mathcal{N}$ nor $\mathcal{L}$ are invariant for $\mathcal{A}_{\mathcal{M}}$ (see Example(4.3.24)).

A simple way to construct functions φ and ψ that satisfy the hypothesis of Example(4.3.2) and Theorem(4.3.3), but do not satisfy the hypothesis of Theorem(4.3.5) is to let φ be an analytic function that takes the unit disc onto an annulus centered at 0 and to take $\psi = e^{2\pi i t}\varphi$

for some $t \in (0,1)$. In the case that t is rational the following theorem implies that $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces.

Theorem(4.3.6):[92] Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H}))$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example(4.3.2).

If there is a $u \in \mathrm{Hol}(\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})})$ such that $u \circ \varphi = u \circ \psi$, then $\mathcal{A}_{\mathcal{M}}$ has a non-trivial invariant subspace.

Lemma(4.3.7):[92] $\mathcal{M}(\mathcal{H})$ has the transitive algebra property, if and only if the following condition holds:

Whenever $\mathcal{M} = \{(x, T_1 x, \ldots, T_{N-1} x) : x \in \mathcal{D}\}$ is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ such that $\mathcal{D}$ is dense in $\mathcal{H}$ and at least one of the T_i's is not a multiple of the identity, then $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces.

Proof: We start by showing that the condition is sufficient for the transitive algebra property of $\mathcal{M}(\mathcal{H})$. Let $\mathcal{A}$ be a transitive algebra that contains $\mathcal{M}(\mathcal{H})$. We need to show that $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. By Arveson's Lemma it suffices to prove that the only linear graph transformations for $\mathcal{A}$ are multiples of the identity operator, [93]. Thus let $\mathcal{M} = \{(x, T_1 x, \ldots, T_{N-1} x) : x \in \mathcal{D}\}$ be an invariant graph subspace of $\mathcal{A}$ and suppose that there is an $i, 1 \le i \le N-1$such that T_i is not a multiple of the identity. Then clearly $D \neq (0)$ and since $\mathcal{A}$ is transitive we must have that $\mathcal{D}$ is dense in $\mathcal{H}$. Note that we have $\mathcal{M}(\mathcal{H}) \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}}$. Thus $\mathcal{M}$ is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ and hence the hypothesis implies that $\mathcal{A}_{\mathcal{M}}$ is not transitive. But since $\mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}}$ this would imply that $\mathcal{A}$ is not transitive, a contradiction. Hence all T_i have to be multiples of the identity, and hence $\mathcal{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.

For the converse we suppose that the condition is not satisfied and we will show that $\mathcal{M}(\mathcal{H})$ then does not have the transitive algebra property. Thus our hypothesis now says that there is an invariant graph subspace $\mathcal{M}$ of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{D}$ is dense in $\mathcal{H}$, such that one of the graph transformations is not a multiple of the identity, and such that $\mathcal{A}_{\mathcal{M}}$ is transitive. Since $\mathcal{A}_{\mathcal{M}}$ contains $\mathcal{M}(\mathcal{H})$ it will be the required example, if we show that $\mathcal{A}_{\mathcal{M}}$is not strongly dense in $\mathcal{B}(\mathcal{H})$. But all the T_i's are linear graph transformations for $\mathcal{A}_{\mathcal{M}}$, so the result follows from the easy direction of Arveson's Lemma.

The most obvious linear graph transformations are multiplications by meromorphic functions. For $f \in \mathcal{H}$ we let $[f]$ be the smallest $\mathcal{M}(\mathcal{H})$ invariant subspace containing f. Let $f, g \in \mathcal{H}, g \neq 0$ and

$$\mathcal{D} = \left\{h \in [g] : \frac{fh}{g} \in [f]\right\},$$

then one easily checks that $T = M_{\frac{f}{g}}$ is a closed linear transformation that commutes with M_φ for all $\varphi \in \mathcal{M}(\mathcal{H})$. Note that $\mathcal{D}$ contains $\{\varphi g : \in \mathcal{M}(\mathcal{H})\}$, thus T will be densely defined whenever g is cyclic in $\mathcal{H}$, i.e. whenever $[g] = \mathcal{H}$.[93,94,96,97].

Proposition(4.3.8):[92] Let $N \geq 2$ and $\mathcal{M} = \{(f, T_1 f, \ldots, T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ such that $\mathcal{D} \neq (0)$.

(i) Then $\mathcal{M}$ has fiber dimension one, if and only if every T_i is a multiplication.
(ii) If the fiber dimension of $\mathcal{M}$ is one, then either every T_i is a multiple of the identity and $\mathcal{A}_{\mathcal{M}} = \mathcal{B}(\mathcal{H})$ or $\mathcal{A}_{\mathcal{M}}$ has a nontrivial invariant subspace which is defined by a linear relation of the graph transformations.
(iii) If the fiber dimension of M is > 1, then $\mathcal{A}_{\mathcal{M}} \neq \mathcal{B}(\mathcal{H})$.

Proof: (i) Suppose for each i we have $T_i = M_{\varphi_i}$ for some meromorphic function φ_i. Let $f_0 \in \mathcal{D}$ with $f_0 \neq 0$. For $\lambda \in \Omega$ such that $f_0(\lambda) \neq 0$ and λ being not a pole of any of the φ_i set

$$u_\lambda = \big(f_0(\lambda), \varphi_1(\lambda) f_0(\lambda), \ldots, \varphi_{N-1}(\lambda) f_0(\lambda)\big) \in \mathbb{C}^N.$$

Then one easily checks that for any $f \in \mathcal{D}$ we have

$$(f(\lambda), (T_1 f)(\lambda), \ldots, (T_{N-1} f)(\lambda)) = f(\lambda)/f_0(\lambda) u_\lambda$$

Hence $\mathcal{M}_\lambda = \mathbb{C} u_\lambda$ and $\dim \mathcal{M}_\lambda = 1$. This is true for all λ in an open subset of Ω, hence the fiber dimension of $\mathcal{M}$ must be one.

Conversely, suppose that $\mathcal{M}$ has fiber dimension one, and let $f_0 \in \mathcal{D}$ with $f_0 \neq 0$. For $i = 1, \ldots, N-1$ set $\varphi_i = T_i f_0 / f_0$. Then φ_i ismeromorphic.

Let S_0 be the set of zeros of f_0 and let $\lambda \in \mathbb{D} \setminus S_0$. Set

$$u_\lambda = \big(f_0(\lambda), (T_1 f_0)(\lambda), \ldots, (T_{N-1} f_0)(\lambda)\big).$$

Then $0 \neq u_\lambda \in \mathcal{M}_\lambda$. Thus the hypothesis implies that $\dim \mathcal{M}_\lambda = 1$, and for each $f \in \mathcal{D}$ there is $c_\lambda \in \mathbb{C}$ such that

$$\big(f(\lambda), (T_1 f)(\lambda), \ldots, (T_{N-1} f)(\lambda)\big) = c_\lambda u_\lambda.$$

Hence $c_\lambda = f(\lambda)/f_0(\lambda)$ and for $i = 1, \ldots, N-1$ we have

$$(T_i f)(\lambda) = c_\lambda (T_i f_0)(\lambda) = \varphi_i(\lambda) f(\lambda).$$

Since $T_i f \in \mathcal{H}$ for each i we conclude that for every $f \in \mathcal{D}$ the function $\varphi_i f$ extends to be analytic in Ω and that T_i is multiplication by φ_i.

(ii) It follows from (i) that each T_i is a multiplication. Let $E = \{\lambda \in \Omega : k_\lambda = 0\}$, where k_λ is the reproducing kernel for $\mathcal{H}$. Since $\mathcal{M} \neq (0)$ it is clear that $\Omega \setminus E$ is a nonempty open set. If one of the T_i is not a multiple of the identity, then $T_i = M_\varphi$ where φ is not constant on $\Omega \setminus E$. Let $\lambda_0 \in \Omega \setminus E$, then $T_i - \varphi(\lambda_0)$ is not identically equal to 0 and $k_{\lambda_0} \perp \operatorname{ran} T_i - \varphi(\lambda_0)$. Thus the closure of $\operatorname{ran} T_i - \varphi(\lambda_0)$ is a nontrivial invariant subspace of $\mathcal{A}_{\mathcal{M}}$. We would say that $\mathcal{A}_{\mathcal{M}}$ has a nontrivial invariant subspace that is defined by a linear relation of the graph transformations.

(iii) If $\mathcal{A}_{\mathcal{M}} = \mathcal{B}(\mathcal{H})$, then $\mathcal{M}$ is an invariant graph subspace of $\mathcal{B}(\mathcal{H})$. It follows that each linear transformation T_i is a multiple of the identity, and this implies that the fiber dimension of $\mathcal{M}$ is one.

Corollary(4.3.9):[92] $\mathcal{M}(\mathcal{H})$ has the transitive algebra property if and only if the following condition holds:

Whenever $\mathcal{M}$ is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ of fiber dimension > 1, then $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces.

We will now restrict the class of the invariant graph subspaces that need to be checked by excluding the ones where $\mathcal{A}_{\mathcal{M}}$ has nontrivial invariant subspaces defined by linear relations of the graph transformations.

Lemma(4.3.10):[92] Let $\mathcal{M} = \{(f, T_1 f, \dots, T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$, and let $\lambda \in \Omega$, then

$$\mathcal{M}_\lambda^\perp = \{\alpha \in \mathbb{C}^N : k_\lambda \perp \operatorname{ran} L_\alpha\}.$$

Here as before for $\alpha \in CN$ we defined $L_\alpha = \overline{\alpha_0} I + \sum_{i=1}^{N-1} \overline{\alpha_i} T_i$.

In particular it follows that if $\operatorname{ran} L_\alpha$ is dense in $\mathcal{H}$ for all non zero $\alpha \in \mathbb{C}^N$, then $\mathcal{M}_\lambda = \mathbb{C}^N$ for all $\lambda \in \Omega$, $k_\lambda \neq 0$.

Lemma(4.3.11):[92] Let $\mathcal{M} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$. If $\mathcal{A}_{\mathcal{M}}$ has no nontrivial invariant subspaces defined by linear relations of the graph transformations, then there is a subspace $\mathcal{K} \subseteq \mathbb{C}^N$ such that $\mathcal{M}_\lambda = \mathcal{K}$ for all $\lambda \in \Omega$ with $k_\lambda \neq 0$.

Proof: Suppose that all invariant subspaces of $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$, and let $\lambda_1, \lambda_2 \in \Omega$ such that $k_{\lambda_1}, k_{\lambda_2} \neq 0$. The lemma will follow, if we show that $\mathcal{M}_{\lambda_1} = \mathcal{M}_{\lambda_2}$.

Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{N-1}) \in \mathcal{M}_{\lambda_1}^\perp$ then by the previous lemma k_{λ_1} is orthogonal to $\operatorname{ran} L_\alpha$. The closure of $\operatorname{ran} L_\alpha$ is an invariant subspace of $\mathcal{A}_{\mathcal{M}}$ that is defined by a linear relation of the graph transformations, and it does not equal $\mathcal{H}$ since $k_{\lambda_1} \neq 0$. Hence the hypothesis implies $\operatorname{ran} L_\alpha = (0)$. This implies that $L_\alpha = 0$ whenever $\alpha \in \mathcal{M}_{\lambda_1}^\perp$. This means $\alpha \in \mathcal{M}_\lambda^\perp$ and hence $\mathcal{M}_\lambda \subseteq \mathcal{M}_{\lambda_1}$ for all $\lambda \in \Omega$. In particular then $\mathcal{M}_{\lambda_2} \subseteq \mathcal{M}_{\lambda_1}$, and in fact by symmetry we conclude $\mathcal{M}_{\lambda_1} = \mathcal{M}_{\lambda_2}$.

Lemma(4.3.12):[92] Let $\mathcal{M} = \{(f, T_1 f, \dots, T_{N-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(N)}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ such that all invariant subspaces of $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$.

If $\mathcal{M}$ has fiber dimension $1 \leq k \leq N$, then there are linear graph transformations $S_1, \dots, S_{k-1} : \mathcal{D} \to \mathcal{H}$ such that each S_i is a linear combination of I and $T_1, \dots, T_{N-1}$ and such that

$$\mathcal{N} = \{(f, S_1 f, \dots, S_{k-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(k)}$$

is an invariant graph subspace for $\mathcal{M}(\mathcal{H})$ with $\mathcal{A}_{\mathcal{N}} = \mathcal{A}_{\mathcal{M}}$, and $L_\alpha^{\mathcal{N}} = \overline{\alpha_0} I + \sum_{i=1}^{k-1} \overline{\alpha_\iota} S_i$ is 1-1 and has dense range for all nonzero $\alpha \in \mathbb{C}^k$.

Proof: The hypothesis and Lemma (4.3.11) imply that there is a k-dimensional subspace $\mathcal{L} \subseteq \mathbb{C}^N$ such that $\mathcal{M}_\lambda = \mathcal{L}$ for all $\lambda \in \Omega$ with $k_\lambda \neq 0$. Write $T_0 = I$, then as in the proof of Lemma (4.3.11) we have $\sum_{i=1}^{N-1} \overline{\alpha_\iota} T_i = 0$ for all $\alpha = (\alpha_0, \ldots, \alpha_{N-1}) \in \mathcal{L}^\perp$. This implies that $\{I, T_1, \ldots, T_{N-1}\}$spans a k-dimensional subspace of the linear transformations $\mathcal{D} \to \mathcal{H}$. Let $\{S_0, \ldots, S_{k-1}\}$be a basis for this space. Since the space contains I we may assume that $S_0 = I$. It is now easy to check that

$$\mathcal{N} = \{(f, S_1 f, \ldots, S_{k-1} f) : f \in \mathcal{D}\} \subseteq \mathcal{H}^{(k)}$$

satisfies the conclusion of the lemma. Indeed, it is immediate that $\mathcal{N}$ is a closed invariant graph subspace of $\mathcal{M}(\mathcal{H})$ and that $\mathcal{A}_{\mathcal{M}} = \mathcal{A}_{\mathcal{N}}$.

Note that $\mathcal{A}_{\mathcal{N}}$ satisfies that all invariant subspaces of $\mathcal{A}_{\mathcal{M}}$ that are defined by linear relations of the graph transformations are either (0) or $\mathcal{H}$, since any linear combination of I and $S_1, \ldots, S_{k-1}$ is a linear combination of I and $T_1, \ldots, T_{N-1}$. Since $I, S_1, \ldots, S_{k-1}$ are linearly independent we conclude that for each non zero $\alpha \in \mathbb{C}^k, L_\alpha^{\mathcal{N}} \neq 0$. Thus $\ker L_\alpha^{\mathcal{N}} = (0)$ and $\operatorname{ran} L_\alpha^{\mathcal{N}}$ is dense.

Theorem(4.3.13):[92] Let $\mathcal{M}$ be an invariant graph subspace for $\mathcal{M}(\mathcal{H})$, and suppose that there is a non-constant meromorphic function u on Ω and a non-zero linear subspace $\mathcal{D}_1$ such that multiplication by $u, M_u : \mathcal{D}_1 \to \mathcal{H}$ commutes with every $A \in \mathcal{A}_{\mathcal{M}}$, i.e. whenever $A \in \mathcal{A}_{\mathcal{M}}$, then $A\mathcal{D}_1 \subseteq \mathcal{D}_1$ and $AM_u = M_u A$ on $\mathcal{D}_1$.Then $\mathcal{A}_{\mathcal{M}}$ has non-trivial invariant subspaces.

Proof: Let $\lambda \in \Omega$ such that λ is not a pole of u and $k_\lambda \neq 0$. Then $k_\lambda \perp (M_u - u(\lambda)I)f$ for every $f \in \mathcal{D}_1$, and hence the closure of $(M_u - u(\lambda)I)\mathcal{D}_1$ is a non-trivial invariant subspace for $\mathcal{A}_{\mathcal{M}}$. Another way to look at the previous theorem is to note that if $\mathcal{M}_1$ is the closure of $\{(f, uf) : f \in \mathcal{D}_1\}$, then $\mathcal{M}_1$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ with fiber dimension 1 and $\mathcal{A}_{\mathcal{M}} \subseteq \mathcal{A}_{\mathcal{M}_1}$. Thus the existence of non-trivial invariant subspaces follows from Proposition (2.3.8).

Example (4.3.14):[92] [98]. A densely defined closed linear transformation T that is not a multiplication, but commutes with $\mathcal{M}(\mathcal{H})$. Thus by Proposition (4.3.8) the invariant graph subspace $\mathcal{M} = \{(f, Tf) : f \in \mathcal{D}\}$ has fiber dimension 2.

This can be modified to apply to more general situations where one has index 2 invariant subspaces.

Let $\mathcal{L}, \mathcal{N}$ be index 1 invariant subspaces of the Bergman space L_a^2 such that they are at a positive angle, assume that $\mathcal{N}$ is a zero set based invariant subspace. As was observed by Hedenmalm [4] the existence of such subspaces follows from [2].

Then $\mathcal{L} \vee \mathcal{N} = \mathcal{L} + \mathcal{N}$. Let $f \in \mathcal{L}, f \neq 0$ and let

$$\mathcal{D} = \{h + g : h \in L_a^2, hf \in \mathcal{L}, g \in \mathcal{N}\},$$

then $\mathcal{D}$ contains the polynomials and hence is dense in L^2_a. Note that if $h+g=0$ with $h\in L^2_a, hf\in\mathcal{L}, g\in\mathcal{N}$, then $hf=-fg\in\mathcal{L}\subseteq L^2_a$. Thus $fg\in\mathcal{N}$, because it has the correct zeros. This implies $hf, fg\in\mathcal{L}\cap\mathcal{N}$, hence $hf=fg=0$, i.e. $h=g=0$. This implies that $T:\mathcal{D}\to L^2_a, T(h+g)=hf+g$ is well-defined.

It is closed also: Indeed, if $h_n+g_n\in\mathcal{D}$ such that $h_n+g_n\to u$ and $h_nf+g_n\to v$, then because of the positive angle condition we have $g_n\to v_1\in\mathcal{N}$ and hence $h_n\to u-v_1$ and $h_nf\to v-v_1$. This implies that $(u-v_1)f=v-v_1\in\mathcal{L}$, and hence $u=(u-v_1)+v_1\in\mathcal{D}$ and $Tu=(u-v_1)f+v_1=v$. Thus we have the invariant graph subspace

$$\mathcal{M}=\{(h+g,hf+g): h\in L^2_a, hf\in\mathcal{L}, g\in\mathcal{N}\}$$

We already observed that T is densely defined, but the range of Twill not be dense since $T\mathcal{D}\subseteq\mathcal{L}+\mathcal{N}$ which has index 2. Furthermore, for all points λ in the common zero set of $\mathcal{N}$ the space $\mathcal{M}_\lambda$ is only one-dimensional.

Example(4.3.15):[92] Let $H\subseteq \mathrm{Hol}(\Omega)$ be a reproducing kernel Hilbert space, let φ,ψ be multipliers such that $\frac{1}{\varphi-\psi}$ is a multiplier, and let $\mathcal{N},\mathcal{L}\subseteq\mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N}\cap\mathcal{L}=(0)$.

Then with $\mathcal{D}=\mathcal{N}+\mathcal{L}$ and $T(f+g)=\varphi f+\psi g$ the space $\mathcal{M}=\{(h,Th): h\in\mathcal{D}\}$ is an invariant graph subspace of $\mathcal{M}(\mathcal{H})$ of fiber dimension 2.

Clearly T is well-defined, and $M_u\mathcal{D}\subseteq\mathcal{D}$ and $M_uT=TM_u$ for every multiplier u. If $f_n\in\mathcal{L}$, $g_n\in\mathcal{N}$ such that $f_n+g_n\to u$ and $\varphi f_n+\psi g_n\to v$, then $(\varphi-\psi)gn\to\varphi u-v$. Hence by the hypothesis on $\varphi-\psi$ we have $g_n\to u_1=\frac{\varphi u-v}{\varphi-\psi}\in\mathcal{N}$.Then $f_n\to u_2=u-\frac{\varphi u-v}{\varphi-\psi}\in\mathcal{L}$, and $v=\varphi u_1+\psi u_2=T(u_1+u_2)$. Thus, T is closed and hence we obtain the invariant s graph subspace

$$\mathcal{M}=\{(f+g,\varphi f+\psi g): f\in\mathcal{L}, g\in\mathcal{N}\}$$

We have $\mathcal{M}_\lambda=\mathbb{C}^2$ whenever $\lambda\in\mathbb{D}\setminus(Z(\mathcal{L})\cup Z(\mathcal{N}))$. In this case we have $(1,\varphi(\lambda))\in\mathcal{M}_\lambda$ and $(1,\psi(\lambda))\in\mathcal{M}_\lambda$. These vectors are linearly independent since the hypothesis implies that $\varphi(\lambda)\neq\psi(\lambda)$ for all $\lambda\in\mathbb{D}$. However, it is clear that the dimension of $\mathcal{M}_\lambda<2$ at every $\lambda\in Z(\mathcal{L})\cup Z(\mathcal{N})$. Thus, in order to have an example satisfying the condition of Theorem (4.3.1) we will at least need that $Z(\mathcal{L})=Z(\mathcal{N})=\emptyset$. If neither φ nor ψ is a constant function, then $\ker(T-\lambda)=(0)$ for all $\lambda\in\mathbb{C}$. Suppose $f\in\mathcal{L}, g\in\mathcal{N}$ such that $(T-\lambda)(f+g)=0$. Then $(\varphi-\lambda)f=-(\psi-\lambda)g\in\mathcal{L}\cap\mathcal{N}$. Thus $(\varphi-\lambda)f=-(\psi-\lambda)g=0$, hence $f=g=0$. For $\alpha=(\alpha_0,\alpha_1)$ we have $L_\alpha=\alpha_0I+\alpha_1T$, this L_α has dense range for all non zero $\alpha\in\mathbb{C}^2$, if and only if $\mathcal{L}+\mathcal{N}$ and $(\varphi-\lambda)\mathcal{L}+(\psi-\lambda)\mathcal{N}$are dense in $\mathcal{H}$ for every $\lambda\in\mathbb{C}$.

Lemma (4.3.16):[92] Let $\mathcal{H}\subseteq\mathrm{Hol}(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H})=\{M_u: u\in H^\infty\}$ with equivalence of norms, and $\mathrm{ran}(M_z-\lambda)$ is closed for all $|\lambda|<1$, and $\dim H/zH=1$.

Let $\mathcal{K}\subseteq\mathcal{H}$ be an $\mathcal{M}(\mathcal{H})$-invariant subspace with $Z(\mathcal{K})=\emptyset$. If there is a Blaschke product B such that $B\mathcal{H}\subseteq\mathcal{K}$, then $\mathcal{K}=\mathcal{H}$.

Proof: Let $\lambda\in\mathbb{D}$ and let $f\in\mathcal{K}$ with $f(\lambda)=0$. We claim that $f/(z-\lambda)\in\mathcal{K}$.[3]

First suppose that $B(\lambda) \neq 0$. As in [3] it follows from the hypothesis on $\mathcal{H}$ that $\frac{f}{z-\lambda} \in \mathcal{H}$. Hence by hypothesis $\frac{Bf}{z-\lambda} \in \mathcal{K}$. Note that $\frac{B-B(\lambda)}{z-\lambda} \in H^\infty$, thus $\frac{B-B(\lambda)}{z-\lambda} f \in \mathcal{K}$ and this implies $B(\lambda)f/(z-\lambda) \in \mathcal{K}$. Since $B(\lambda) \neq 0$ we conclude that $\frac{f}{z-\lambda} \in \mathcal{K}$.

If $B(\lambda) = 0$, then let $\lambda_n \in \mathbb{D}$ with $B(\lambda_n) \neq 0$ and $\lambda_n \to \lambda$. By hypothesis there is a $g \in \mathcal{K}$ with $g(\lambda) \neq 0$. Then for each n we have $h_n = f_n - \frac{f}{g}(\lambda_n)g \in \mathcal{K}$ and $h_n(\lambda_n) = 0$. By what we have already shown, it follows that $\frac{h_n}{z-\lambda_n} \in \mathcal{K}$ for each n. The hypothesis on $\mathcal{H}$ implies that $M_z - \lambda I$ is bounded below, then $M_z - \lambda_n I$ will be bounded below with a similar constant for large n. That can be used to show that $h_n/(z-\lambda_n) \to f/(z-\lambda)$. Thus $\frac{f}{z-\lambda} \in \mathcal{K}$.

In particular, if $f \in \mathcal{H}$, then since $Bf \in \mathcal{K}$ we conclude that $\frac{Bf}{z-\lambda} \in \mathcal{K}$ for every $\lambda \in \mathbb{D}$ with $B(\lambda) = 0$. This easily implies that $\frac{Bf}{B_n} \in \mathcal{K}$, where B_n is the finite Blaschke product determined by the first n simple factors of B. As $n \to \infty$ the hypothesis implies that $\frac{Bf}{B_n} \to f$ weakly, hence $f \in \mathcal{K}$. Thus $\mathcal{K} = \mathcal{H}$.

Proposition (4.3.17):[92] Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, and $\mathrm{ran}(M_z - \lambda)$ is closed for all $|\lambda| < 1$, and $\dim \mathcal{H}/z\mathcal{H} = 1$. Let $\varphi, \psi \in H^\infty$ such that $1/(\varphi - \psi) \in H^\infty$ and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be $\mathcal{M}(\mathcal{H})$-invariant subspaces such that

(i) $\mathcal{N} \cap \mathcal{L} = (0)$,
(ii) $\mathcal{N} + \mathcal{L}$ is dense in $\mathcal{H}$,
(iii) $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$, and
(iv) the inner–outer factorizations of $\varphi - \lambda$ and $\psi - \lambda$ have no singular inner factor for any $\lambda \in \mathbb{C}$, then $(\varphi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N}$ is dense in $\mathcal{H}$ for every $\lambda \in \mathbb{C}$.

Proof: Let $\lambda \in \mathbb{C}$ and write

$$\mathcal{K} = \overline{(\phi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N}}.$$

We must show that $\mathcal{K} = \mathcal{H}$.

Note that if $z_0 \in \mathbb{D}$, then either $\varphi(z_0) \neq \lambda$ or $\psi(z_0) \neq \lambda$. In either case the hypothesis (iii) implies that there is a function $f \in \mathcal{K}$ such that $f(z_0) \neq 0$, i.e. $Z(\mathcal{K}) = \emptyset$.

It follows from the hypothesis (iv) that there exist Blaschke products B_1, B_2 and bounded outer functions f_1, f_2 such that $\varphi - \lambda = B_1 f_1$ and $\psi - \lambda = B_2 f_2$. Then

$$\mathcal{K} \supseteq (\varphi - \lambda)\mathcal{L} + (\psi - \lambda)\mathcal{N} \supseteq \mathrm{B_1 f_1 B_2 f_2}(\mathcal{L} + \mathcal{N}) = \mathrm{Bf}(\mathcal{L} + \mathcal{N})$$

for some Blaschke product B and some bounded outer function f. Since f is outer, there exists a sequence of polynomials p_n such that $p_n f \to 1$in the weak$*$-topology of H^∞, hence $M_{p_n} f \to I$ in the weak operator topology. Thus combining this observation with hypothesis (ii) we obtain $\mathcal{K} \supseteq \overline{B\mathcal{H}}$. Hence $\mathcal{K} = \mathcal{H}$ follows from Lemma (4.3.17).

Now let $\mathcal{H}, \mathcal{L}, \mathcal{N}, \varphi, \psi$ be as in Proposition (4.3.16), set $\mathcal{D} = \mathcal{L} + \mathcal{N}$, and let $\|f + g\|_{\mathcal{D}}$ be the graph norm on $\mathcal{D}$,

$$\|f + g\|_{\mathcal{D}}^2 = \|f + g\|^2 + \|\varphi f + \psi g\|^2.$$

Then one easily checks that $\mathcal{L}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{D}$ which satisfy $\mathcal{L} \cap \mathcal{N} = 0$ and $\mathcal{L} + \mathcal{N} = \mathcal{D}$. Thus there is a projection $P \in \mathcal{B}(\mathcal{D})$ with $\operatorname{ran} P = \mathcal{L}$ and $\ker P = \mathcal{N}$. Let $Q = I - P$.

Theorem (4.3.18):[92] Let $\mathcal{H} \subseteq \operatorname{Hol}(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example (4.3.2).

If

$$\varphi(\mathbb{D}) \backslash \overline{\psi(\mathbb{D})} \neq \emptyset,$$

then $\mathcal{N}$ is an invariant subspace for $\mathcal{A}_{\mathcal{M}}$.

In particular, $\mathcal{A}_{\mathcal{M}}$ has a non-trivial invariant subspace.

Similarly, if $\psi(\mathbb{D}) \backslash \overline{\varphi(\mathbb{D})} \neq \emptyset$, then $\mathcal{L}$ is invariant for $\mathcal{A}_{\mathcal{M}}$.

Proof: Let $A \in \mathcal{A}_{\mathcal{M}}$. We will show that $A \in \mathcal{B}(\mathcal{D})$ and $PAQ = 0$.

From the definition of $\mathcal{A}_{\mathcal{M}}$ we have $A\mathcal{D} \subseteq \mathcal{D}$ and

$$\|Ah\|_{\mathcal{D}}^2 = \|Ah\|^2 + \|TAh\|^2 = \|Ah\|^2 + \|ATh\|^2$$

$$\leq \|A\|^2(\|h\|^2 + \|Th\|^2) = \|A\|^2 \|h\|_{\mathcal{D}}^2$$

Thus $A, PAQ, M_\varphi, M_\psi \in \mathbb{B}(\mathcal{D})$. For $f \in \mathcal{L}$ and $g \in \mathcal{N}$ we have

$$PAQM_\psi(f + g) = PAQ(\psi f + \psi g) = PA\psi g = PATg = PTAg$$

$$= PT(P + Q)AQ(f + g) = PM_\varphi PAQ(f + g) + PM_\psi QAQ(f + g)$$

$$= M_\varphi PAQ(f + g).$$

Thus $PAQM_\psi = M_\varphi PAQ$ and hence $(PAQ)^* M_\varphi^* = M_\psi^* (PAQ)^*$.

The hypothesis implies that there is a $\lambda_0 \in \mathbb{D}$ such that

$$\operatorname{dist}\big(\varphi(\lambda_0), \psi(\mathbb{D})\big) > 0.$$

Then by continuity there is an open neighborhood $\mathcal{U}$ of λ_0 in $\mathcal{D}$ and a $\delta > 0$ such that for all $\lambda \in \mathcal{U}$ and all $z \in \mathbb{D}$ we have $|\psi(z) - \varphi(\lambda)| \geq \delta$, hence $M_\psi - \varphi(\lambda)I$ is invertible. This implies $\ker(M_\psi^* - \overline{\varphi(\lambda)}) = (0)$ for all $\lambda \in \mathcal{U}$.

Let $\lambda \in \mathcal{U}$ and let k_λ be the reproducing kernel for $\mathcal{D}$. We have

$$\left(M_\psi^* - \overline{\varphi(\lambda)}\right)(PAQ)^* k_\lambda = (PAQ)^*\left(M_\varphi^* - \overline{\varphi(\lambda)}\right) k_\lambda = 0$$

This implies that $(PAQ)^* k_\lambda = 0$ for all $\lambda \in \mathcal{U}$. Since finite linear combinations of $k_\lambda, \lambda \in \mathcal{U}$ are dense in $\mathcal{D}$ we obtain $PAQ = 0$.

Thus if $f \in \mathcal{N} \subseteq \mathcal{D}$, then $f = Qf$ and $Af = (P+Q)Af = PAQf + QAf = QAf \in \mathcal{N}$, i.e. $A\mathcal{N} \subseteq \mathcal{N}$.

Theorem (4.3.19):[92] Let $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ be such that $\mathcal{M}(\mathcal{H}) = \{M_u : u \in H^\infty\}$ with equivalence of norms, let $\varphi, \psi \in H^\infty$ such that $\frac{1}{\varphi - \psi} \in H^\infty$, and let $\mathcal{N}, \mathcal{L} \subseteq \mathcal{H}$ be closed nonzero invariant subspaces of $\mathcal{M}(\mathcal{H})$ such that $\mathcal{N} \cap \mathcal{L} = (0)$. Let $\mathcal{M}$ be the invariant graph subspace as in Example (4.3.2).

If there is a $u \in \mathrm{Hol}\overline{(\varphi(\mathbb{D}) \cup \psi(\mathbb{D}))}$ such that $u \circ \varphi = u \circ \psi$, then $\mathcal{A}_\mathcal{M}$ has a non-trivial invariant subspace.

Proof: Let $v = u \circ \varphi = u \circ \psi$, then $v \in H^\infty(\mathbb{D})$. We will show that $M_v : \mathcal{D} \to \mathcal{H}$ commutes with $\mathcal{A}_\mathcal{M}$. Then the result will follow from Theorem (4.3.13) We will use a special property of our example, namely that $T\mathcal{D} \subseteq \mathcal{D}$.

If $\lambda \in \mathbb{C}, \lambda \notin \overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$, then $\frac{1}{\varphi - \lambda} f \in \mathcal{N}$ and $\frac{1}{\psi - \lambda} g \in \mathcal{L}$ for all $f \in \mathcal{N}$ and $g \in \mathcal{L}$. Thus one easily checks that $(T - \lambda)^{-1}(f + g) = \frac{1}{\varphi - \lambda} f + \frac{1}{\psi - \lambda} g$ and for every $A \in \mathcal{A}_\mathcal{M}$ we have $A(T-\lambda)^{-1} = (T-\lambda)^{-1}A$. It follows that $r(T)A = Ar(T)$ for every rational function r with poles outside of $\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$. The hypothesis on u implies that there is a sequence of rational functions r_n such that $r_n \to u$ uniformly in a neighborhood of $\overline{\varphi(\mathbb{D}) \cup \psi(\mathbb{D})}$. Then $r_n \circ \varphi$ and $r_n \circ \psi$ are bounded sequences in H^∞ that converge pointwise to v. Thus for every $f \in \mathcal{N}$ and $g \in \mathcal{L}$ we have $r_n(T)(f+g) = r_n \circ \varphi f + r_n \circ \psi g \to v(f+g)$ weakly. Hence $Ar_n(T)(f+g) \to AM_v(f+g)$ and $r_n(T)A(f+g) \to M_v A(f+g)$ weakly for each $f \in \mathcal{N}$ and $g \in \mathcal{L}$. Thus $M_v A = AM_v$.

A simple way to satisfy the hypothesis that $1/(\varphi - \psi)$ is a multiplier is if $\varphi = \psi + c$ for some constant $c \neq 0$. Then for appropriate $\mathcal{H}$ it is easy to see that the hypotheses of both of the previous theorems are satisfied, thus $\mathcal{A}_\mathcal{M}$ has non-trivial invariant subspaces. For the u in the previous theorem we can take $u(z) = e^{\frac{2\pi i}{c} z}$. Thus $\mathcal{A}_\mathcal{M}$ commutes with M_v, where $v(z) = e^{\left(\frac{2\pi i}{c}\right)\varphi(z)}$. Actually in this case one can verify directly that $\mathcal{A}_\mathcal{M}$ commutes with M_φ.

$$AM_\varphi(f + g) = AM_\varphi f + AM_\psi g + cAg = AT(f + g) + cAg$$

$$= TAf + TAg + cAg = M_\varphi Af + M_\psi Ag + cAg = M_\varphi A(f + g).$$

This implies that $AM_\varphi = M_\varphi A$ on $\mathcal{H}$.

If $\varphi(z) = z$, then under the hypothesis of Theorem (4.3.18) the relation $AM_z = M_zA$ implies $A \in \mathcal{M}(\mathcal{H})$, hence $\mathcal{A}_{\mathcal{M}} = \mathcal{M}(\mathcal{H})$. Thus it seems worthwhile to point out that it can happen that $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example (4.3.20):[92] Take $\mathcal{H} = L_a^2$, $\varphi(z) = z^2$, $\psi = \varphi + c$, for $c \neq 0$, and choose the two subspaces $\mathcal{L}$ and $\mathcal{N}$ as above such that they are invariant under $(Uf)(z) = f(-z)$. For example, take two zero sets A and B such that the union is not a zero set and such that they both accumulate only on a small arc near1. Then let $A' = A \cup (-A)$ and $B' = B \cup (-B)$. It is well-known that the extremal function for $I(A)$ has an analytic continuation across any arc $I \subseteq \partial\mathbb{D}$ that does not contain any accumulation points of A [71]. Thus, if $f_1$is the extremal function for $I(A)$ and f_2 is the extremal function. for $I(-A)$, then it follows easily that $f_1 f_2 \in I(A')$. Hence both $A'$and B' are zero sets for $\mathcal{H}$ and their union is not a zero set. Now set $\mathcal{L} = I(A')$ and $\mathcal{N} = I(B')$.

One verifies easily that in this case $U \in \mathcal{A}_{\mathcal{M}}$, thus $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example (4.3.21):[92] Let $\varphi \in \mathrm{Hol}(\mathbb{D})$, $t \in \mathbb{R}\backslash\mathbb{Z}$, $\alpha = e^{2\pi i t} \neq 1$and such that $\varphi(\mathbb{D}) = \{z \in \mathbb{C}: r < |z| < R\}$, and $\psi = \alpha\varphi$. For example, φ could be the composition of aconformal map of the disc onto a vertical strip and the exponential function,

$$\varphi(z) = \exp\left(i \log \frac{1-z}{1+z}\right).$$

Then $|\varphi(z) - \psi(z)| = |1 - \alpha||\varphi(z)| > c$. Furthermore, we check that for no $\lambda \in \mathbb{C}$ the function $\varphi - \lambda$ can have a singular inner factor. Since φ has an analytic continuation at every point except $+1$ or -1, it is clear that the only possible singular inner factors of $\varphi - \lambda$ are determined by point masses at 1 or -1. If $\varphi - \lambda$ had a singular inner factor at1, then we would have $\varphi(r) - \lambda \to 0$ as $r \to 1^-$. But $\varphi(r) - \lambda$ does not converge as $r \to 1^-$. Similarly we see that there is no singular inner factor with mass at -1. Thus this provides an example of the situation of Theorem (4.3.3), and since $\varphi(\mathbb{D}) = \psi(\mathbb{D})$ Theorem (4.3.18) does not apply. Theorem (4.3.19) applies only if $t = \frac{n}{m}$ is rational, $u(z) = z^m$. Thus if t is irrational we don't know of any non-trivial invariant subspaces of $\mathcal{A}_{\mathcal{M}}$.

Example (4.3.22):[92] Can one show that $\mathcal{A}_{\mathcal{M}}$ has non-trivial invariant subspaces in the previous example if t is irrational?

Example (4.3.23):[92] Let $\varphi(z) = \exp\left(i \log \frac{1-z^2}{1+z^2}\right)$, $\psi(z) = \alpha\varphi(z)$ and assume that $f(z) \in \mathcal{N}$ if and only if $f(-z) \in \mathcal{N}$ and $g(z) \in \mathcal{L}$ if and only if $g(-z) \in \mathcal{L}$. One can achieve this as in Example (4.3.20). By combining the approach of Example (4.3.20) with the construction of one can also achieve this with the added property that $Z(\mathcal{N}) = Z(\mathcal{L}) = \emptyset$. As in Example (4.3.20) the operator $Uf(z) = f(-z)$ will be in $\mathcal{A}_{\mathcal{M}}$. Thus, $\mathcal{A}_{\mathcal{M}} \neq \mathcal{M}(\mathcal{H})$.

Example (4.3.24):[92] We will construct zero set based invariant subspaces $\mathcal{N}$ and $\mathcal{L}$ of L_a^2 with $\mathcal{N} \cap \mathcal{L} = (0)$ and a disc automorphism u such that $C_u\mathcal{N} = \mathcal{L}$ and $C_u\mathcal{L} = \mathcal{N}$ and an H^∞-function φ such that $1/\varphi \in H^\infty$ and $C_u\varphi = -\varphi$. Here C_u is the composition operator with symbol u.

Then we set $\psi = -\varphi = C_u\varphi$. As above $|\varphi - \psi| = 2|\varphi|$ is bounded below, thus with $\mathcal{D} = \mathcal{N} + \mathcal{L}$ this provides an example satisfying the hypothesis of Example (4.3.2). Furthermore, one now easily checks that $C_u\mathcal{D} \subseteq \mathcal{D}$ and $TC_u = C_uT$ on $\mathcal{D}$. Thus $C_u \in \mathcal{A}_{\mathcal{M}}$ and hence $\mathcal{N}, \mathcal{L} \notin \mathrm{Lat}\mathcal{A}_{\mathcal{M}}$.

To get started we recall the definitions of interpolating and sampling sequences of a space $\mathcal{H}$ of analytic functions on $\mathbb{D}$.

For a sequence $\{\lambda_n\}$ of distinct points in $\mathbb{D}$ we define $T: \mathcal{H} \to l^\infty$ by $Tf = \left\{\frac{f(\lambda_n)}{\|k_{\lambda_n}\|}\right\}_n$. Then $\{\lambda_n\}$ is called an interpolating sequence for $\mathcal{H}$, if T is a bounded operator from $\mathcal{H}$ into and onto l^2, and $\{\lambda_n\}$ is called a sampling sequence for $\mathcal{H}$, if there is a constant $c > 0$ such that $c\|f\| \le \|Tf\|_{l^2} \le \frac{1}{c}\|f\|$ for all $f \in \mathcal{H}$.

Lemma(4.3.25):[92] If $\Gamma \subseteq \mathbb{D}$ is a sampling sequence for $\mathcal{H}$, if $\overline{\mathbb{D}} = D_+ \cup D_-$, where D_+ and D_- are closed semi-discs, then

$$\Gamma_+ = \Gamma \cap D_+$$

is not a zero-sequence for $\mathcal{H}$.

Proof: Suppose that $f \in \mathcal{H}$ is a non-zero function with $f(\lambda) = 0$ for all $\lambda \in \Gamma_+$. Since Γ is a sampling sequence, there must be a $c > 0$ such that

$$c\|pf\|^2 \le \sum_{\lambda \in \Gamma \cap D_-} \frac{|pf(\lambda)|^2}{\|k_\lambda\|^2} \le \|p\|^2_{\infty, D_-} \sum_{\lambda \in \Gamma \cap D_-} \frac{|f(\lambda)|^2}{\|k_\lambda\|^2} \le \left(\frac{1}{c}\right) \|p\|^2_{\infty, D_-} \|f\|^2$$

for all polynomials p. Fix $\lambda_0 \in \mathbb{D} \setminus D_-$ with $f(\lambda_0) \neq 0$. By Runge's theorem we may choose a sequence of polynomials p_n such that p_n converges to 0 uniformly on D_- and $p_n(\lambda_0) \to 1$. Then the inequality above implies that $\|p_nf\| \to 0$. This contradicts $p_nf(\lambda_0) \to f(\lambda_0) \neq 0$. Thus Γ_+ is not a zero set for $\mathcal{H}$.

Now let $S = \{z \in \mathbb{C} : -1 < \mathrm{Re}z < 1\}$ and let $\mathbb{H}^+$ denote the upper half plane of $\mathbb{C}$. The function $f(z) = ie^{-\frac{i\pi z}{2}}$ is a conformal map from S onto $\mathbb{H}^+$ with $f(0) = i$. We note that f takes $\{z: 0 < Re z < 1\}$ onto the first quadrant and $f^{-1}: \mathbb{H}^+ \to S$ takes rays emanating from 0 to vertical lines in S. If we further let $g(z) = i\frac{1+z}{1-z}$ be a conformal map of $\mathbb{D}$ onto $\mathbb{H}^+$, then $h = f^{-1} \circ g$ is a conformal map from $\mathbb{D}$ onto S. The function $\varphi = e^{ih}$ is bounded and bounded below as required for Example (4.3.24).

For $a > 1$ and $b > 0$ define the lattice

$$\Lambda(a,b) = \{a^m(bn + i) : m, n \in \mathbb{Z}\}$$

of points in $\mathbb{H}^+$, and consider the corresponding set $\Gamma(a,b) = g^{-1}(\Lambda(a,b))$ in $\mathbb{D}$. [100] states that $\Gamma(a,b)$ is interpolating for $\mathcal{H} = L_a^2$ if $\frac{2\pi}{b \log a} < \frac{1}{2}$ and $\Gamma(a,b)$ is sampling for L_a^2 if

$\frac{2\pi}{b\log a} > \frac{1}{2}$. Now set $a = e^{\frac{\pi^2}{2}}$ so that $f(z + i\pi) = af(z)$ for all $z \in S$, and choose bsuch that $\frac{2\pi}{b\log a^2} < \frac{1}{2} < \frac{2\pi}{b\log a}$. Then $\Gamma(a^2, b)$ is interpolating and $\Gamma(a,b)$is sampling for L_a^2.

Set $\Lambda_1 = \{a^{2m}(bn + i) : m, n \in \mathbb{Z}, n \geq 0\}$, $\Lambda_2 = \{a^{2m+1}(bn + i) : m, n \in \mathbb{Z}, n \geq 0\}$ and for $j = 1, 2$ set $\Gamma_j = g^{-1}(\Lambda_j)$. Then Γ_1 and Γ_2 are subsets of interpolating sets for L_a^2, hence they both are zero sets for L_a^2. Furthermore, $\Gamma_1 \cup \Gamma_2 = g^{-1}(\{a^m(bn + i) : m, n \in \mathbb{Z}, n \geq 0\})$ and it follows from the choice of a and b and Lemma (4.3.25) that $\Gamma_1 \cup \Gamma_2$ is not a zero set for L_a^2. Thus, $\mathcal{N} = I(\Gamma_1)$ and $\mathcal{L} = I(\Gamma_2)$ are nontrivial invariant subspaces with $\mathcal{N} \cap \mathcal{L} = (0)$.

For $z \in \mathbb{D}$ set $u(z) = g^{-1}(ag(z))$, then u is a disc automorphism with $u(\Gamma_1) = \Gamma_2$ and $u(\Gamma_2) = \Gamma_1$. This implies that $C_u\mathcal{N} = \mathcal{L}$ and $C_u\mathcal{L} = \mathcal{N}$. Furthermore one checks that $h\big(u(z)\big) = h(z) + i\pi$ for all $z \in \mathbb{D}$. Thus $C_u\varphi = -\varphi$ and this concludes the construction for Example (4.3.12).

Let μ be a positive discrete measure on the unit circle $\mathbb{T}$, given by a sequence of points $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{T}$ with corresponding masses $0 < \omega_k < \infty$ such that

$$\mu = \sum_{k=1}^{\infty} \omega_k \delta_{\lambda_k}$$

We shall refer to $\{\lambda_k\}$ as the a-support of μ.

When $\|\mu\| = \sum_k \omega_k < \infty$, μ is associated with the singular inner function

$$S_\mu(z) = \exp\left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right)$$

and by $I_\mu = [S_\mu]$ we denote the invariant subspace of $L_a^2(\mathbb{D})$ generated by S_μ. For non-finite measures μ we define I_μ instead by

$$I_\mu = \cap\{[S_\nu]: 0 \leq \nu \leq \mu, \|\nu\| < \infty\}.$$

We say that μ is admissible when $I_\mu \neq \{0\}$. Since singly generated invariant subspaces have index 1, it follows from [3] that I_μ has index one whenever μ is admissible. Thus I_μ is generated by its extremal function. We note that a routine argument with contractive zero divisors shows that the extremal function for I_μ is nonzero in $\mathbb{D}$. In conclusion, I_μ is zero free whenever μ is admissible.

Proposition (4.3.26):[92] Suppose $f \in L_a^2$ is zero free. Then

(i) $\lim_{r\to 1}(1 - r^2)\log\frac{1}{|f(r\lambda)|} \geq 0$ exists for all $\lambda \in \mathbb{T}$.

(ii) For $\lambda \in \mathbb{T}$ and $\omega > 0$, we have that $f \in I_{\omega\delta_\lambda}$ if and only if $\lim_{r\to 1}(1 - r^2)\log\frac{1}{|f(r\lambda)|} \geq 4\omega$.

Proof: Let $D_\lambda \subset \mathbb{D}$ be the disc of radius ½ that is tangent to $\mathbb{T}$ at λ and note that $f|_{D_\lambda}$ is in the Smirnov class N^+ of D_λ. Standard arguments of Nevanlinna theory now give the validity of (i). A proof of (ii) appears in [81].

Lemma (4.3.27):[92] Let $\mu = \sum_k \omega_k \delta_{\lambda_k}$ be admissible. If $\lambda \in \mathbb{T}\backslash\{\lambda_k\}$ and $\omega > 0$, then $I_\mu \not\subset I_{\omega\delta_\lambda}$.

Proof: Suppose on the contrary that $I_\mu \subset I_{\omega\delta_\lambda}$. Let ϕ_μ and $\phi_{\omega\delta_\lambda}$ be the respective extremal functions for I_μ and $I_{\omega\delta_\lambda}$, so that $\phi_\mu \in [\phi_{\omega\delta_\lambda}]$. Then $\phi\mu/\phi_{\omega\delta_\lambda} \in L_a^2$, $\left\|\frac{\phi\mu}{\phi_{\omega\delta_\lambda}}\right\|_{L_a^2} \le 1$, and $\frac{\phi\mu}{\phi_{\omega\delta_\lambda}}(0) > \phi_\mu(0)$.

We are now going to demonstrate that $\frac{\phi\mu}{\phi_{\omega\delta_\lambda}} \in I_\mu$, contradicting the extremality of ϕ_μ.

To this end we first note that we may write down $\phi_{\omega\delta_1}$ explicitly using the method for proving Formula (15) in [69],

$$\phi_{\omega\delta_1}(z) = \frac{1 + \frac{2w}{1-z}}{(1 + 2w)^{\frac{1}{2}}} S_{\omega\delta_1}(z)$$

from which we deduce that for all k,

$$\lim_{r\to 1}(1 - r^2)\log\left|\phi_{\omega\delta_\lambda}(r\lambda_k)\right| = 0.$$

Hence, by Proposition (4.3.26),

$$\lim_{r\to 1}(1 - r^2)\log\left|\frac{\phi_{\omega\delta_\lambda}(r\lambda_k)}{\phi_\mu(r\lambda_k)}\right| = \lim_{r\to 1}(1 - r^2)\log\left|\frac{1}{\phi_\mu(r\lambda_k)}\right| \ge 4\sum_{\lambda_\ell=\lambda_k}\omega_\ell$$

Applying Proposition (4.3.26) once more we obtain $\frac{\phi\mu}{\phi_{\omega\delta_\lambda}} \in I_\mu$.

Theorem (4.3.28):[92] There exist two positive discrete admissible measures μ and ν such that

(i) $I_\mu \cap I_\nu = \{0\}$, and
(ii) $I_\mu + I_\nu$ is dense in L_a^2.

Proof: The non-admissibility of $\mu + \nu$ is equivalent to the fact that $I_\mu \cap I_\nu = \{0\}$. It remains to prove that $I_\mu + I_\nu$ is dense in L_a^2.

From the existence of a non-zero $f \in I_\mu$ extending analytically across a subarc of $\mathbb{T}$ it follows that $\text{clos}(I_\mu + I_\nu)$ is an index-one invariant subspace of L_a^2, [21]. Hence $\text{clos}(I_\mu + I_\nu)$ is generated by its extremal function ϕ, which clearly has no zeros in $\mathbb{D}$. Denote by ϕ_μ and ϕ_ν

the respective extremal functions for I_μ and I_ν, and let $f = \phi\mu/\phi$ and $g = \phi\nu/\phi$, recalling that $f, g \in L_a^2$[70].

We claim that $f \in I_\mu$. To see this note that

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi_\nu(r\lambda_k)|} = 0, \lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} \geq 0, \forall k \geq 1,$$

by Proposition (4.3.26) and Lemma (4.3.27) So for every $k \geq 1$ we have

$$0 \leq \lim_{r\to 1}(1-r^2)\log\frac{1}{|g(r\lambda_k)|} = -\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} \leq 0,$$

whence

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi(r\lambda_k)|} = 0.$$

Therefore

$$\lim_{r\to 1}(1-r^2)\log\frac{1}{|f(r\lambda_k)|} = \lim_{r\to 1}(1-r^2)\log\frac{1}{|\phi_\mu(r\lambda_k)|} \geq 4\sum_{\lambda_\ell=\lambda_k}\omega_\ell;$$

proving that $f \in I_\mu$, by Proposition (4.3.26). Similarly one shows that $g \in I_\nu$.

Now let $\{p_n\}_n$ and $\{q_n\}_n$ be two sequences of polynomials such that $p_n\phi_\mu + q_n\phi_\nu \to \phi$ in L_a^2 as $n \to \infty$. By the contractive divisor property of ϕ we obtain that $p_n f + q_n g = \frac{p_n\phi_\mu + q_n\phi_\nu}{\phi} \in I_\mu + I_\nu$ is a Cauchy sequence, hence $p_n f + q_n g \to 1$. That is, $I_\mu + I_\nu$ is dense in L_a^2.

Lemma(4.3.29):[92] I_{μ_0} contains a nonzero function that continues analytically across the open arc $J = \{z \in \mathbb{C} : |z| = 1 \text{and } \operatorname{Re} z < 0\} \subseteq \mathbb{T}$.[101].

Proof: Since the zero set Λ is contained in $\{z \in \mathbb{D}: Rez > 0\}$ it is known that the extremal function G for the zero-based invariant subspace $I(\Lambda)$ continues analytically across J, [71,101]. For $\alpha \in \mathbb{D}$ set $b_\alpha(z) = \frac{\bar{\alpha}}{|\alpha|}\frac{\alpha - z}{1-\bar{\alpha}z}$ and

$$S_\alpha(z) = e^{-2\frac{1-|\alpha|}{1+|\alpha|}\frac{\frac{\alpha}{|\alpha|}+z}{\frac{\alpha}{|\alpha|}-z}}$$

In [99] Korenblum shows that if $\alpha \in \mathbb{D}$ and if $f \in L_a^2$ satisfies $f(\alpha) = 0$, then $\left\|\frac{S_\alpha}{b_\alpha}f\right\| \leq \|f\|$.

An easy calculation shows that if $K \subseteq \mathbb{C}$ is a compact set such that $K \cap [1,\infty) = \emptyset$, then there is a $c > 0$ such that

$$\left|1-\frac{\frac{r-z}{1-rz}}{e^{-2\frac{1-r}{1+r}\frac{1+z}{1-z}}}\right|\leq c(1-r)^2$$

for all $z\in K$ and all $0\ \leq r<1$.

Since Λ is an L^2_a-zero set we have $\sum_{\alpha\in\Lambda}(1-|\alpha|)^2<\infty$ [9]. Thus the above estimate shows that the product

$$P(z)=\prod_{\alpha\in\Lambda}\frac{b_\alpha}{S_\alpha}$$

converges uniformly on each compact subset of $\mathbb{D}\cup\{Rez<0\}$ with $P(z)\neq 0$ for all z with $Re\,z<0$. Thus the function $f=G/P$ has an analytic continuation across J. Let $\{P_m\}$ be the sequence of partial products of P, then by iterating Korenblum's inequality we have $\|G/P_m\|\leq\|G\|$, so $G/P_m\to f$ weakly L^2_a and it follows that $f\in I_{\mu_0}$.

For a fixed $J\geq 1$, pick angles $\theta_1,\ldots,\theta_J$ such that $\frac{\theta_1}{2\pi},\ldots,\frac{\theta_J}{2\pi}$ are linearly independent over the rational numbers. Then the a-supports of $\mu_1,\ldots,\mu_J$ are pairwise disjoint, where μ_j is the rotation of μ_0 by the angle θ_j, $1\ \leq j\leq J$. We also introduce some further notation;

$$\mu_{N,j}=\sum_{\substack{3\leq n\leq N\\ |k|<\frac{3^n}{4}}}\omega_n\delta_{e^{i\left(\frac{2\pi k}{3^n}+\theta_j\right)}}\ ,\qquad \mu^N=\sum_{j=1}^{J}\mu_{N,j}$$

letting F_N denote the a-support of μ^N. For later reference we note that $\|\mu^N\|\sim JN$.

Lemma (4.3.30):[92] Let $h(z)=P[\delta_1](z)=\frac{1-|z|^2}{|1-z|^2}$ and define for integers $K\geq 27$

$$H_K(z)=\sum_{k=0}^{K-1}h(e^{\frac{i2\pi k}{K}}z),\qquad z\in\mathbb{D}$$

Then $H_K(z)=Kh(z^K)$ and there exists a constant $C>0$, independent of K, such that $H_K(re^{i\theta})\ <C$ whenever $1\ -r<\theta^2$ and $|\theta|\ \leq\pi/K$.[103,104].

Let $h:[0,1]\to\mathbb{R}$ be defined by $h(t)\ =\frac{1}{2\pi^2}t^2(1-t)^2$. For $\varepsilon\ \in(0,2\pi]$ and $t\ \in[0,\varepsilon]$ set $r_\varepsilon(t)\ =1-\varepsilon^2h(\frac{t}{\varepsilon})$. Then $0\ <r_\varepsilon(t)\ \leq 1$ and $|r_\varepsilon'(t)|$and $|r_\varepsilon''(t)|$ are bounded uniformly for all $\varepsilon\in(0,2\pi]$ and $t\ \in[0,\varepsilon]$. Note also that $r_\varepsilon'(0)=r_\varepsilon'(\varepsilon)\ =0$ and $r_\varepsilon''(0)\ =r_\varepsilon''(\varepsilon)\ =\frac{1}{\pi 2}$.

Now let $\emptyset\neq F\subseteq T$ be finite and define the closed path $\gamma_F:[0,2\pi]\ \to\overline{\mathbb{D}}$ as follows: If $t\ \in[0,2\pi]$ is such that $e^{it}\in F$, then set $\gamma_F(t)\ =e^{it}$. Otherwise $e^{it}\in I$, where I is some complementary arc of F with endpoints $e^{it_0}$and e^{it_1}. Then we set $\gamma_F(t)=r_{|I|}(t-t_0)e^{it}$, where $|I|$ is the length of I. The curve Γ_F is defined as the range of γ_F. It is clear that $\Gamma_F\subseteq\overline{\mathbb{D}}$ is a Jordan curve such that $\Gamma_F\cap T=F$. The properties of the functions r_ε imply that each Γ_F

is C^2-smooth and there is a $C > 0$ such that $\|\gamma_F''\|_\infty \le C$ for all finite nonempty sets $F \subseteq T$. Furthermore one checks that the Jordan region bounded by $\Gamma_1$is contained in the Jordan region bounded by $\Gamma_2$whenever $F_1 \subseteq F_2$, and that we have the estimate

$$\frac{1}{8\pi^2}\operatorname{dist}\left(\frac{z}{|z|},F\right)^2 \le 1-|z| \le \frac{1}{2\pi^2}\operatorname{dist}\left(\frac{z}{|z|},F\right)^2, \qquad z \in \Gamma_F \tag{31}$$

where dist refers to the geodesic distance along $\mathbb{T}$.

Lemma (4.3.31):[92] There are constants $c, C > 0$ such that for all finite nonempty sets $F \subseteq T$ we have $c < |\varphi'_F(z)| < C$ for all $z \in \Gamma_F$. Furthermore, if ω_F denotes harmonic measure at 0 on Γ_F, then $d\omega_F = |\varphi'_F| \frac{|dz|}{2\pi}$ and hence

$$\frac{c}{2\pi}\int_{\Gamma_F} h(z)|dz| \le \int_{\Gamma_F} h(z)d\omega_F(z) \le \frac{C}{2\pi}\int_{\Gamma_F} h(z)|dz|$$

for all nonnegative Borel measurable functions h on Γ_F. Here $|dz|$denotes arc length measure.[105].

Lemma (4.3.32):[92] There exists a constant $D > 0$, independent of J such that for every $N \ge 3$ we have $log|S_{\mu^N}(z)| \ge -DJ$for $z \in \Gamma_N \cap \mathbb{D}$.

Proof: For this proof we introduce the set $\tilde{F}_N \supset F_N$,

$$\tilde{F}_N = \left\{e^{i\left(\frac{2\pi k}{3^n}+\theta_j\right)}: 3 \le n \le N, 1 \le j \le J, 0 \le k \le 3^n - 1\right\},$$

and let $\tilde{\Gamma}_N = \Gamma_{\tilde{F}_N}$be the curve defined by use of the complementary arcs of $\tilde{F}_N$.

Fix for the moment n and j. For a point $z = re^{i\theta} \in \tilde{\Gamma}_N \cap \mathbb{D}$, let k_0 be a minimizer of

$$\min_{0\le k\le 3^n-1} \operatorname{dist}\left(e^{i\theta}, e^{i\left(\frac{2\pi k}{3^n}+\theta_j\right)}\right),$$

and let $z_0 = ze^{-i\left(\frac{2\pi k_0}{3n}+\theta j\right)} = re^{i\theta_0}$. Note that $|\theta_0| \le \pi/3^n$ and $1-r \le \theta_0^2$ by (31). Hence, by Lemma (4.3.30)

$$\sum_{|k|<\frac{3^n}{4}} P\left[\delta_{e^{i\left(\frac{2\pi k}{3^n}+\theta_j\right)}}\right](z) < H_{3^n}(z_0) < C \tag{32}$$

Since the domain enclosed by $\tilde{\Gamma}_N$ contains the domain enclosed by Γ_N, it follows by the maximum principle for harmonic functions that (32) holds also for $z \in \Gamma_N \cap \mathbb{D}$. Noting now that

$$\log\frac{1}{|S_{\mu^N}(z)|} = \sum_{\substack{3\le n\le N, |k|<\frac{3^n}{4}\\ 1\le j\le J}} \omega_n P\left[\delta_{e^{i\left(\frac{2\pi k}{3^n}+\theta_j\right)}}\right](z);$$

with $\omega_n \sim 1/3n$.

Example(4.2.33):[92] Proof that $\sum_{j=1}^{J} \mu j$ is not admissible for sufficiently large J.

Suppose that $\sum_{j=1}^{J} \mu_j$ is admissible. We will now argue that J has to be smaller than a certain universal constant. Fix $N \geq 3$ and note first that the admissibility of $\sum_{j=1}^{J} \mu_j$ implies that there exists an $\eta > 0$, independent of N, such that there exists a polynomial p such that $f = pS_{\mu^N}$ satisfies $\|f\|_{L_a^2} \leq 1$ and $|f(0)| \geq \eta$. In what follows there will be several implied constants that are all independent of both N and J.

With $f = pS_{\mu^N}$ as above and ω_N denoting harmonic measure on Γ_N with pole at 0 we write

$$\int_{\Gamma_N} \log|f(z)|\, d\omega_N(z) = \int_{\Gamma_N} \log|p(z)|\, d\omega_N(z) + \int_{\Gamma_N} \log\left|S_{\mu^N}(z)\right| d\omega_N(z) \tag{33}$$

Since $\|f\|_{L_a^2} \leq 1$we find by (31) and the estimate $|f(z)| \leq (1 - |z|^2)^{-1}$that

$$|f(z)| \leq \frac{8\pi^2}{\operatorname{dist}\left(\frac{z}{|z|}, F_N\right)^2}$$

Letting $\{I_h\}$be the collection of complementary arcs on $\mathbb{T}$ to F_N, we obtain

$$\int_{\Gamma_N} \log|f(z)|\, d\omega_N(z) \lesssim \int_{\Gamma_N} \log \frac{2\pi}{\operatorname{dist}\left(\frac{z}{|z|}, F_N\right)} |dz| + \log 2$$

$$\lesssim \int_{\Gamma_N} \log \frac{2\pi}{\operatorname{dist}(\omega, F_N)} |d\omega| + \log 2 \sim \sum_h |I_h| \log \frac{2\pi}{|I_h|} \lesssim 1 + \log|F_N| \lesssim N + \log J \;\; R \tag{34}$$

where $|I_h|$ denotes the length of I_h and $|F_N| \leq 3^N J$ the number of points in F_N. We have used the fact that the entropy $\sum_h |I_h| \log \frac{2\pi}{|I_h|}$ for a fixed number of intervals is maximized when all intervals are of equal size.

We also note that

$$\int_{\Gamma_N} \log|p(z)|\, d\omega_N(z) \geq \log|p(0)| = \log|f(0)| + \log \frac{1}{\left|S_{\mu^N}(0)\right|}$$

$$= \log|f(0)| + \|\mu_N\| \gtrsim \log \eta + NJ, \tag{35}$$

and by Lemma (4.3.32) that

$$\int_{\Gamma_N} \log\left|S_{\mu^N}(z)\right| d\omega_N(z) \gtrsim -J \tag{36}$$

Combining (33), (34), (35), and (36), we find

$$N + \log J \gtrsim \log \eta + NJ - J$$

Letting $N \to \infty$ we conclude that Jmust be smaller than some universal constant $A, J \leq A$.

Theorem (4.3.34):[92] There is a space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$such that every invariant graph subspace $\mathcal{M}$ has the property that ind$\mathcal{M} = \mathrm{fd}\mathcal{M}$, and such that there are index1 invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ of $(M_z, \mathcal{H})$ such that $\mathcal{M} \cap \mathcal{N} = (0)$ and $\mathcal{M} + \mathcal{N}$ is dense in $\mathcal{H}$.

Proof: It follows from the construction in the proof of Theorem (4.3.28) that the measures μ and ν can be chosen in such a way that the union of their a-supports is disjoint from some non-empty closed arc $I \subseteq \mathbb{T}$ (just take I to be a small arc centered at $-1$and choose all θ_j to be sufficiently small). Let σ be the measure defined by $d\sigma = \chi_I|dz| + dA|\mathbb{D}$ and consider the space $P^2(\sigma)$, the closure of the polynomials in $L^2(\sigma)$. Then one verifies that $P^2(\sigma)$is irreducible and clearly every point of $\mathbb{D}$ defines a bounded point evaluation for $P^2(\sigma)$, i.e. $P^2(\sigma)$ is an analytic P^2-space in the sense of [23] and [106]. For such spaces it was shown that every non-empty M_z -invariant subspace has index 1 [23], and in fact, Carlsson [107]showed that every $M_z^{(N)}$ -invariant subspace of $P^2(\mu)^{(N)}$satisfies that its index equals its fiber dimension. In particular, the index of each invariant graph subspace equals its fiber dimension.

Chapter 5

Algebraic Properties and Index of Invariant Subspaces with Fiber Dimension

We show that if S is a bounded below operator, then $\operatorname{ind} M + \operatorname{ind} N \geq \operatorname{ind}(M \cap N) + \operatorname{ind}(M \vee N)$. If, in addition, $\operatorname{ind} M = \operatorname{ind} N = 1$ and $M \cap N \neq \{0\}$ then $\operatorname{ind}(M \vee N) = 1$. We show that the natural counterpart to this statement in Hilbert spaces of $\mathbb{C}^n$-valued analytic functions is false and show a correct generalization of the theorem. We obtain new information on the boundary behavior of functions in such spaces, thereby improving the result. Other new findings include: a lattice-additive formula and its applications ; a new concept of "absorbance" which describes a rough containment relation for invariant subspaces; the existence of a unique, smallest CF subspace containing an arbitrary invariant subspace and preserving the fiber dimension.

Section (5.1): The Index of Invariant Subspaces of Operators on Banach Spaces

If S is an operator on a Banach space X, then a closed subspace M of X is called invariant for S if $SM \subset M$. The collection of invariant subspaces of an operator S is denoted by $\operatorname{Lat}(S, X)$. It forms a complete lattice with respect to intersections and closed spans. One of the important notions in the general theory of operators, such as bounded below operators, is the index of an element in $\operatorname{Lat}(S, X)$, which is defined as follows. [109].

Definition (5.1.1):[108] The map

$$\operatorname{ind}: \operatorname{Lat}(S, X) \to \{0\} \cup \mathbb{N} \cup \{\infty\}$$

is defined as $\operatorname{ind} M = \dim(M/SM)$ and $\operatorname{ind} M = 0$ if and only if $M = \{0\}$. We say that M has index n if $\operatorname{ind} M = n$.

The index function plays an essential role in the study of invariant subspaces of Banach space. See[3]. We give various algebraic properties of the index function. Amongst others, we show that if $M, N \in Lat(S, X), indM = indN = 1$ and $M \cap N \neq \{0\}$ then $ind(M \vee N) = 1$, where $M \vee N$ denotes the closed span of M and N. (Equivalently, $M \vee N$ is the closure of $M + N$). [3].

Theorem (5.1.2):[108] Let R be a commutative ring with identity and let A, A', B' be free unitary R-modules such that A' and B' are free submodules of A. Then

$$rank\left(\frac{A}{A'}\right) + rank\left(\frac{A}{B'}\right) = rank\left(\frac{A}{A' \cap B'}\right) + rank\left(\frac{A}{A' + B'}\right)$$

Proof: Consider the following sequence

$$0 \to \frac{A}{A' \cap B'} \xrightarrow{f} (A/A' \oplus A/B') \xrightarrow{g} A/(A' + B') \to 0$$

Where $f([y]) = ([y], [y])$, $g([x], [y]) = [x - y]$ and $[.]$ denotes the equivalence class in the appropriate quotient module. We claim that the sequence above is exact.

To prove the claim we first show that f and g are well-defined homomorphism. Letting $[y] \in \frac{A}{A' \cap B'}$ and $x \in A' \cap B'$, we obtain that $f([y+x]) = ([y+x],[y+x]) = ([y],[y])$. Hence, f is well defined. Moreover, f is homomorphism, since

$$f([y]+[z]) = ([y]+[z],[y]+[z]) = ([y],[y]) + ([z],[z])$$

$$f(r[y]) = (r[y], r[y]) = r([y],[y]), r \in R.$$

Similarly, if $([x],[y]) \in A/A' \oplus A/B$, and $x_1 \in A', x_2 \in B'$, then $g([x+x_1],[y+y_1]) = [(x+x_1)-(y+y_1)] = [(x-y)+(x_1-y_1)] = [x-y]$, Since $x_1 - y_1 \in A' + B'$. Thus, g is well defined. Moreover, g is homomorphism, since

$$g\big(([x],[y]) + ([x'],[y'])\big) = g([x]+[x'],[y]+[y']) = g([x+x'],[y+y'])$$

$$= [(x+x')-(y+y')] = [x-y+x'-y'] = [x-y] + [x'-y']$$

And

$$g\big(r([x],[y])\big) = g([rx],[ry]) = [rx - ry] = r[x-y], r \in R$$

It remains to show that $\ker g = \operatorname{im} f$. For this let $([x],[y]) \in A/A' \oplus A/B'$ be such that $g([x],[y]) = 0$. Then $[x-y] = 0$, and thus $x - y \in A' + B'$. This implies that $x + A' = y + B'$, i.e.,$[x]_{\frac{A}{A'}} = [y]_{\frac{A}{B'}}$ wherefore $\left([x]_{\frac{A}{A'}}, [y]_{\frac{A}{B'}}\right) \in \operatorname{im} f$, and hence $\ker g \subset im\, f$.

Conversely, if $([x],[y]) \in \operatorname{im} f$ then $x + A' = y + B'$ and hence $x + A' + B' = y + A' + B'$. It follows that $g([x],[y]) = [x-y] = 0$ so that $\operatorname{im} f \subset \ker g$.

Since $A/(A'+B')$ is a free module, it is in particular projective, and hence the above exact sequence splits [110]. Therefore

$$\frac{A}{A'} \oplus \frac{A}{B'} = \frac{A}{A' \cap B'} \oplus \frac{A}{A'+B'}.$$

This immediately implies that

$$\operatorname{rank}\left(\frac{A}{A'}\right) + \operatorname{rank}\left(\frac{A}{B'}\right) = \operatorname{rank}\left(\frac{A}{A' \cap B'}\right) + \operatorname{rank}\left(\frac{A}{A'+B'}\right).$$

Corollary (5.1.3):[108] If X is a Banach space and S an operator on X, for all $M, N \in \operatorname{Lat}(S, X)$

$$\operatorname{ind} M + \operatorname{ind} N = \operatorname{ind}(M \cap N) + \operatorname{ind}(M+N)$$

In the case when S is a bounded below operator, like the shift operator on Banach spaces of analytic functions, the following holds.

Lemma (5.1.4):[108] Suppose $M, N \in \operatorname{Lat}(S, X)$, where S is a bounded below operator on a Banach space X. Then

$$\mathrm{ind}(M \vee N) \leq \mathrm{ind}(M + N) \leq \mathrm{ind}M + \mathrm{ind}N$$

Proof: If either $\mathrm{ind}M$ or $\mathrm{ind}N$ is infinite, then there is nothing to prove. So we may assume that $\mathrm{ind}M < \infty$ and $\mathrm{ind}N < \infty$. Thus there are finite-dimensional subspaces M_1 and N_1 of M and N, respectively, such that $M = SM + M_1, N = SN + N_1$, where $\dim M_1 = indM$ and $\dim N_1 = indN$. We find that

$$M + N = SM + M_1 + SN + N_1 = S(M + N) + M_1 + N_1$$

$$\subseteq S(M \vee N) + (M_1 + N_1) \subseteq M \vee N.$$

Since S is abounded below operator, its range is closed [111], and hence the second to last expression is the sum of a closed and a finite-dimensional subspace, hence it is closed. Since $M + N$ is dense in $M \vee N$ we obtain that the last inclusion in above is actually an equality. From this it follows that

$$\mathrm{ind}(M \vee N) \leq \dim(M_1 + N_1) = \mathrm{ind}(M + N) \leq \mathrm{ind}M + \mathrm{ind}N.$$

Theorem (5.1.5):[108] If X is a Banach space and S a bounded below operator on X then, for all $M, N \in \mathrm{Lat}(S, X)$,

$$\mathrm{ind}M + \mathrm{ind}N \geq \mathrm{ind}(M \cap N) + \mathrm{ind}(M \vee N).$$

Corollary (5.1.6):[108] Suppose that $M_1, M_2 \in \mathrm{Lat}(S, X)$ are such that $\mathrm{ind}M_1 = \mathrm{ind}M_2 = 1$, where S, X are as in the previous theorem. If $M_1 \cap M_2 \neq \{0\}$ then $\mathrm{ind}(M_1 \vee M_2) = 1$.

Section (5.2): Hilbert Spaces of Vector-Valued Analytic Functions

Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$, let $\mathbb{T}$ be its boundary and let m denote the normalized arc-length measure on $\mathbb{T}$. Moreover let z denote the identity function on $\mathbb{D}$, i.e. $z(\zeta) = \zeta$ for $\zeta \in \mathbb{D}$ and let $n \in \mathbb{N}$ be fixed. We will consider Hilbert spaces $\mathcal{H}$ of $\mathbb{C}^n$-valued analytic functions on $\mathbb{D}$ such that $zf \in \mathcal{H}$ for all $f \in \mathcal{H}$, and the corresponding operator of multiplication by z will be denoted M_z , that is, $(M_z f)(\zeta) = \zeta f(\zeta)$.

We shall always assume that the spaces $\mathcal{H}$ satisfy the following conditions

$\forall\lambda \in \mathbb{D}$ the evaluation map $f \to f(\lambda)$ is continuous and surjective from $\mathcal{H}$ onto $\mathbb{C}^n$. (1)

If $f \in \mathcal{H}$ and $f(\lambda) = 0$, then $f \in \mathrm{Ran}(M_z - \lambda)$. (2)

By virtue of the Closed Graph Theorem, (1) implies that M_z is a bounded operator. We shall only consider $\mathcal{H}$ such that

$$\|M_z\| \leq 1 \tag{3}$$

and such that there exists a constant $c > 0$ with

$$\left\|\frac{z - \lambda}{1 - \bar{\lambda} z} f\right\| \geq c\|f\| \tag{4}$$

for all $\lambda \in \mathbb{D}$ and all $f \in \mathcal{H}$. For examples and an introduction to such spaces [106,107]. We concerned with the index of M_z-invariant subspaces and the boundary behavior of the functions in $\mathcal{H}$.

Let $\mathcal{M}$ be a closed M_z-invariant subspace of $\mathcal{H}$. The index of $\mathcal{M}$, denoted ind$\mathcal{M}$, is then defined as

$$\mathrm{ind}\mathcal{M} = \mathrm{codimRan}(M_z|\mathcal{M}),$$

where $M_z|\mathcal{M}$ denotes the restriction of M_z to $\mathcal{M}$. Note that $\mathrm{ind}\mathcal{H} = n$ and that M_z has closed range by conditions (1) and (2), so $\mathrm{ind}\mathcal{M} = -\mathrm{ind}(M_z|\mathcal{M})$ where $\mathrm{ind}(M_z|\mathcal{M})$ denotes the Fredholm index of $M_z|\mathcal{M}$. The origin of [8], from which it follows that if

$$\lim_{k\to\infty} \left\|M_z^k f\right\| = 0 \tag{5}$$

for some element $f \in \mathcal{H}, f \not\equiv 0$, then one can find invariant subspaces $\mathcal{M}$ with arbitrary index. The standard example of a Hilbert space of $\mathbb{C}$-valued analytic functions with the above property is the Bergman space L_a^2 , which is easily verified using the dominated convergence theorem.[106].

Theorem (5.2.1):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}$-valued analytic functions that satisfies (3) and (4). Then the following are equivalent:

(i) $\mathrm{ind}\mathcal{M} = 1$ for all invariant subspaces $\{0\} \neq \mathcal{M} \subset \mathcal{H}$.
(ii) There is a measurable set $\Sigma \subset \mathbb{T}$ with $m(\Sigma) > 0$ such that the quotient f/g has non-tangential limits a.e. on Σ for any $f, g \in H$ with $g \not\equiv 0$.
(iii) $\exists\, f \in \mathcal{H}$ such that $\lim_{k\to\infty} \left\|M_z^k f\right\| \neq 0$.

Note that combined with the theorem of Apostol, Bercovici, Foias¸ and Pearcy this implies that, given a Hilbert space $\mathcal{H}$ of $\mathbb{C}$-valued analytic functions that satisfies (3) and (4), the following dichotomy holds: Either

$$\lim_{k\to\infty} \left\|M_z^k f\right\| \neq 0 \text{ for all } f \in \mathcal{H}, f \not\equiv 0. \tag{6}$$

or

$$\lim_{k\to\infty} \left\|M_z^k f\right\| = 0 \text{ for all } f \in \mathcal{H}. \tag{7}$$

We find the appropriate extension of Theorem (5.2.1) to the case when $\mathcal{H}$ is a space of $\mathbb{C}^n$-valued analytic functions. For this purpose (i), (ii) and (iii) needs to be modified. The reason why (i) needs to be changed is that it is very easy to see that for any $m \leq n$ there always exists an invariant subspace $\mathcal{M} \subset \mathcal{H}$ with $ind\mathcal{M} = m$. Therefore the natural counterpart to condition (i) is

(iv) $\mathrm{ind}\mathcal{M} \leq n$ for all invariant subspaces $\mathcal{M} \subset \mathcal{H}$.

The problem with (ii) is that f/g is not even defined for $\mathbb{C}^n$-valued functions. To overcome this difficulty we proceed as follows. Fix any element $F = (f_1, \ldots, f_n) \in \mathcal{H}^n$, where $\mathcal{H}^n$ stands for the direct sum of n copies of $\mathcal{H}$. We will think of F as a matrix-valued analytic function with columns $f_1,\dots,f_n$. Assume that $F(\lambda)$ is invertible at some $\lambda \in \mathbb{D}$ so that the determinant $det(F(\cdot))$ becomes a non-zero analytic function, and let $Z(F)$ denote its zero-set. Instead of the quotient f/g in (ii) we will consider the $\mathbb{C}^n$-valued meromorphic function $C(f, F, \cdot)$ defined by

$$C(f, F, \lambda) = \left(F(\lambda)\right)^{-1} f(\lambda) \tag{8}$$

The analogue of condition (ii) is:

(v) There is a measurable set $\Sigma \subset \mathbb{T}$ with $m(\Sigma) > 0$ such that for all $f \in \mathcal{H}$ and $F \in \mathcal{H}^n$ with $\det(F(\cdot)) \not\equiv 0, C(f, F, \cdot)$ has non-tangential limits a.e. on Σ.

It is easy to see that (iii) is not equivalent to neither (iv) nor (v). Just take the Hardy space H^2 and the Bergman space L_a^2 and identify $\mathcal{H} = H^2 \oplus L_a^2$ with a Hilbert space of $\mathbb{C}^2$-valued analytic functions in the natural way. It can be shown that conditions (1) to (4) holds, and it is not hard to see that neither (iv) nor (v) holds although clearly (iii) is satisfied by the constant function $f(\cdot) = (1,0)$. However, the example $\mathcal{H} = H^2 \oplus L_a^2$ is ruled out if we replace (iii) with condition (6), so this would be a natural candidate for (vi).

This has the following consequence for Hilbert spaces of $\mathbb{C}$-valued analytic functions. Note that condition (2) is equivalent to

$$\operatorname{codim}\operatorname{Ran}(M_z - \lambda) = 1, \forall \lambda \in \mathbb{D}.$$

If we instead consider spaces $\mathcal{H}$ with $\operatorname{codim}\operatorname{Ran}(M_z - \lambda) = n$ for all $\lambda \in \mathbb{D}$. and some $n \in \mathbb{N}$, then the same phenomenon as above occurs, i.e. there may be invariant subspaces with index larger than n, even if $\lim_{k\to\infty} \|M_z^k f\| \neq 0$ holds for all $f \in \mathcal{H}$ with $f \not\equiv 0$. That the results mentioned above can be applied in this situation is a consequence of the Cowen–Douglas model. (See [9] or Theorem 1.1 in [6], where it is shown that the adjoint of each operator in the Cowen–Douglas class $\mathcal{B}_n(\mathbb{D})$ is unitarily equivalent to M_z on some Hilbert space of $\mathbb{C}^n$-valued analytic functions.

To find the proper replacement for (iii) we need to use multiplicity theory for a certain unitary operator associated to M_z see [107]. Set

$$\mathcal{M} = \left\{ f \in \mathcal{H} : \lim_{k\to\infty} \|M_z^k f\| = 0 \right\}$$

and let P denote the orthogonal projection on $\mathcal{M}^\perp$. It turns out that one can define a new norm on $\mathcal{M}^\perp$ via the formula

$$\|f\|_* = \lim_{k\to\infty} \|M_z^k f\|$$

and that $(\mathcal{M}^\perp, \|.\|_*)$ is a pre-Hilbert space. Let $\mathcal{K}$ denote its completion. The continuous operator $S : \mathcal{K} \to \mathcal{K}$ defined by $Sf = PM_z f$ for $f \in \mathcal{M}^\perp$ is then easily seen to be isometric, and hence it has a minimal unitary extension V on some Hilbert space $\widetilde{\mathcal{K}}$ that include $\mathcal{K}$ as a subspace. V then has a multiplicity function M_V which by [107] satisfies $M_V \leq n$. Thus M_V can be written as

$$M_V(.) = \sum_{i=1}^{n} \mathcal{X}\sigma_i(.) \quad (9)$$

where $\sigma_n \subset \sigma_{n-1} \subset \cdots \subset \sigma_1 \subset \mathbb{T}$ and $\mathcal{X}\sigma_i$ denotes the characteristic function of σ_i. (If $\lim_{k\to\infty}\|M_z^k f\| = 0$ for all $f \in \mathcal{H}$ we set $\sigma_i = \emptyset$ for all $i \geq 0$).

Theorem (5.2.2):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions that satisfies (3) and (4). Then $C(f, F, \cdot)$ has non-tangential limits a.e. on $\sum\mathcal{H}$ for any $f \in \mathcal{H}$ and any $F \in \mathcal{H}^n$ with $det\, F(\cdot) \not\equiv 0$.

(vi) $m(\sum(\mathcal{H})) > 0$

implies (v). It is also shown in [107], that (vi) implies (iv). We show that the reverse implications hold as well, and hence that the behavior of the index of the invariant subspaces in $\mathcal{H}$ is determined by (vi). We shall also improve Theorem (5.2.2) by showing that $\sum(\mathcal{H})$ is optimal.

Corollary (5.2.3):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions such that (3) and (4) hold and $m(\Sigma(\mathcal{H})) \neq 0$. If $\mathcal{M}_1$ and $\mathcal{M}_2$are two invariant subspaces with$\mathcal{M}_1 \subset \mathcal{M}_2$, then

$$\operatorname{ind}(\mathcal{M}_1) \leq \operatorname{ind}(\mathcal{M}_2)$$

We now give an example that demonstrates the consequences of Theorem (5.2.4). After Apostol, Bercovici, Foias¸ and Pearcy proved that there are subspaces of any given index in Hilbert spaces of $\mathbb{C}$-valued analytic functions with

$$\lim_{k\to\infty}\|M_z^k f\| = 0 \quad (10)$$

for some $f \in \mathcal{H}, f \not\equiv 0$, Håkan Hedenmalm was the first to actually construct a "natural" invariant subspace of the Bergman space L_a^2 with index 2 [4]. Since then several people have constructed various methods to find invariant subspaces with large indices in Hilbert spaces of $\mathbb{C}$-valued analytic functions that have the property (10). Recall that by Theorem (5.2.1), (10) is also a necessary property for such subspaces to exist. Below we will construct a Hilbert space of $\mathbb{C}^2$-valued analytic functions such that

$$\lim_{k\to\infty}\|M_z^k f\| \neq 0$$

for all $f \in \mathcal{H}, f \not\equiv 0$, that have invariant subspaces with any given index in $\mathbb{N} \cup \{\infty\}$.

We need the "$P^2(\mu)$-spaces", where μ is a finite positive Borel measure on $\overline{\mathbb{D}}$ and $P^2(\mu)$ is defined as the closure of the polynomials in $L^2(\mu)$. see [23] and [107]. Here we will simply state the facts necessary for the example. If $d\mu = dA + \chi_\sigma\, dm$, where A denotes area measure on$\mathbb{D}$, and σ is a (measurable) subset of $\mathbb{T}$, then

(i) For any $f \in P^2(\mu)$, $f\,|_{\mathbb{D}}$ is (a.e. equal to) an analytic function. (When working with an element $f \in P^2(\mu)$, we shall sloppily think of f as a given representative of the equivalence class, and we shall assume that this is chosen such that $f\,|_{\mathbb{D}}$ is analytic.)

(ii) The set $\{ f |_{\mathbb{D}}: f \in P^2(\mu)\}$ is a Hilbert space of analytic functions that satisfies conditions (1)–(4). We will denote this space by $P^2(\mu)$ as well.
(iii) For each $f \in P^2(\mu)$ and a.e. $\xi \in \sigma$, $f |_{\mathbb{D}}$ has the non-tangential limit $f(\xi)$ at ξ. In fact, $\Theta(P^2(\mu)) = \sigma$.

Example (5.2.4):[112] Let σ_1, σ_2 be subsets of $\mathbb{T}$, $d\mu_i = dA + \chi_{\sigma_i}\, dm$ for $i = 1,2$ and consider the Hilbert space $\mathcal{H}$ of $\mathbb{C}^2$-valued analytic functions defined in the obvious way as $\mathcal{H} = P^2(\mu_1) \oplus P^2(\mu_2)$. It is not hard to prove that $\Sigma(H) = \sigma_1 \cap \sigma_2$, in fact,

$$M_V = \chi_{\sigma_1} + \chi_{\sigma_2}$$

Hence by Theorem (5.2.3) we have that $\Theta(H) = \sigma_1 \cap \sigma_2$, but this can also be verified by direct calculations. By Theorem (5.2.4) however, we have that if $\sigma_1 \cap \sigma_2 = \emptyset$, then there are subspaces of $\mathcal{H}$ with any given index $k \in \mathbb{N} \cup \{\infty\}$, although clearly

$$\lim_{k\to\infty} \left\| M_z^k f \right\| \neq 0$$

for all non-zero $f \in \mathcal{H}$.

Fix $F \in \mathcal{H}^n$ with $det\, F \not\equiv 0$. Given $f \in \mathcal{H}$ let $c_1(f, F, \cdot), \dots, c_n(f, F, \cdot)$ denote the components of the $\mathbb{C}^n$-valued function $C(f, F, \cdot)$, i.e. the meromorphic functions such that

$$C(f,F,.) = \begin{pmatrix} c_1(f,F,.) \\ . \\ . \\ . \\ c_n(f,F,.) \end{pmatrix} \tag{11}$$

These functions are called “the canonical coefficients of f with respect to ” and are studied in detail in [107]. For $i \in \{1, \dots, n\}$ let Θ_F^i be the largest set where all functions in $\{c_i(f, F, \cdot): f \in \mathcal{H}\}$ have non-tangential limits a.e.

Example (5.2.5):[112] Consider the same space $\mathcal{H}$ as Example (5.2.4) and recall that the multiplicity function M_V is given by

$$M_V = \chi_{\sigma_1} + \chi_{\sigma_2}$$

Let f be an arbitrary element in $\mathcal{H}$ and let f_1, f_2 be its components in $P^2(\mu_1)$ and $P^2(\mu_2)$ respectively. With $F = (e_1, e_2)$ (8), we then get $c_i(f, F) = f_i$ so

$$\Theta_F^i = \sigma_i$$

for $i = 1, 2$. To see this just apply Theorem (5.2.6) to each $P^2(\mu_i)$ separately. Thus we see that

$$M_V = \chi_{\Theta_F^1} + \chi_{\Theta_F^2}$$

so in this particular case there is a stronger connection between M_V and the boundary behavior of the canonical coefficients. On the other hand, the above conclusion clearly relies on the particular choice of F and on the fact that $\mathcal{H}$ is a direct sum of two subspaces with simpler structure.

Thus it is natural to ask whether is some stronger link between the multiplicity function M_V and the boundary behavior of the canonical coefficients for certain choices of F. Theorems (5.2.16) and (5.2.17) below show that the situation in Example (5.2.5) is typical, i.e. the answer is no in general but yes if the space can be decomposed in a direct sum of cyclic subspaces.

Given $f \in \mathcal{H}$ let $[f]$ denote the closed linear span of the set $\{M_z^k f : k \geq 0\}$. For subspaces $A_1, \ldots, A_n \subset \mathcal{H}$ we will use the notation

$$A_1 + \cdots + A_n = \mathcal{H}$$

to mean that each $f \in \mathcal{H}$ can be written in a unique way as $f = \sum f_i$ with $f_i \in A_i$. By standard functional analysis there is a constant $C > 0$ such that

$$C^{-1}\|f\| \leq \sum \|f_i\| \leq C\|f\|$$

A Hilbert space of $\mathbb{C}^n$-valued analytic functions will be called decomposable if there are $f_1, \ldots, f_n \in \mathcal{H}$ such that

$$[f_1] + \cdots + [f_n] = \mathcal{H}$$

These consist mainly in observing that by the Cowen–Douglas model, Theorems (5.2.6)–(5.2.17) can be applied in a more general setting. We shall show that Theorems (5.2.6)–(5.2.17) hold under slightly weaker conditions than (1) and (2). We also obtain a result which, in the case $n = 1$, implies that if $\mathcal{M} \subset \mathcal{H}$ is a nontrivial invariant subspace, then

$$\Theta(\mathcal{H}) = \Theta(\mathcal{M}).$$

Thus in order to find $\Theta(\mathcal{H})$ it suffices to find the corresponding set for any cyclic invariant subspace.

Theorem (5.2.6):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions that satisfies (3) and (4). Then

$$\Sigma(\mathcal{H}) = \Theta(\mathcal{H}) = \Delta_F(\mathcal{H}) \ a.e.$$

for any $F \in \mathcal{H}^n$ with $\det(F(\cdot)) \not\equiv 0$.

The proof is structured as follows. We will show each of the inclusions

$$\Delta_F(\mathcal{H}) \supset \Theta(\mathcal{H}) \ a.e., \tag{13}$$

$$\Theta(\mathcal{H}) \supset \Sigma(\mathcal{H}) a.e., \tag{14}$$

$$\Sigma(\mathcal{H}) \supset \Delta_F(\mathcal{H}) a.e. \tag{15}$$

Once the appropriate definitions have been made, (13) follows without major modifications from the methods developed by Aleman, Richter and Sundberg in [106]. Therefore we will just state the necessary lemmas without proofs. Eq. (14) is simply a restatement of Theorem (5.2.2) which was proved in [107], so the only part where essentially new ideas are required is (15). $\mathcal{H}$ will denote a Hilbert space of $\mathbb{C}^n$-valued analytic functions that satisfies (3) and (4), f will be an arbitrary function in $\mathcal{H}$ and $F \in \mathcal{H}^n$ be such that $det(F) \not\equiv 0$.

By a functional of evaluation on $\mathcal{H}$, we mean a functional $e_{\lambda,a}$ of the form

$$e_{\lambda,a}(f) = \sum_{i=1}^{n} a_i f_i(\lambda),$$

where $a = (a_i) \in \mathbb{C}^n$ and $\lambda \in \mathbb{D}$. For a finite or countable sequence Λ we will write $\lambda \in \Lambda$ to denote that λ is an entry in the sequence and moreover we will write $(h_\lambda)_{\lambda\in\Lambda}$ for a sequences of numbers $h_\lambda \in C$ indexed by the sequence Λ. Let l^2_Λ denote the space of such sequences that are finite in the norm $\|(h_\lambda)\|^2 = \sum_{\lambda\in\Lambda}|h_\lambda|^2$.

Definition (5.2.7):[112] A finite or countable sequence $\Lambda = (e_i)_i$ of functionals of evaluation on $\mathcal{H}$ is called interpolating for $\mathcal{H}$ if the operator $T_\Lambda : \mathcal{H} \to l^2_\Lambda$ given by

$$T_\Lambda(f) = \left(\frac{e_i(f)}{\|e_i\|}\right)_{e_i\in\Lambda}$$

is surjective.

Note that $\Lambda = (e_i)_i$ is interpolating for $\mathcal{H}$ if and only if there exists an $M > 0$ such that

$$M^{-1}\left\|\left(b_{e_i}\right)\right\|^2_{l^2_\Lambda} \leq \left\|\sum_{e_i\in\Lambda} b_{e_i}\frac{e_i}{\|e_i\|}\right\|^2_{\mathcal{H}} \leq M\left\|\left(b_{e_i}\right)\right\|^2_{l^2_\Lambda} \qquad (16)$$

for all $\left(b_{e_i}\right) \in l^2_\Lambda$, and that for $n = 1$ this definition coincides with the standard one. Any M such that (16) holds will be called an interpolating constant for Λ. For a sequence Λ as above we will write $\tilde{\Lambda}$ for the set of points in $\mathbb{D}$ corresponding to the functionals of evaluation in the obvious way, and we will use the notation $Ntl\,\tilde{\Lambda}$ for the set of non-tangential accumulation points of $\tilde{\Lambda}$, i.e. the set of points $\xi \in \mathbb{T}$ such that there exists a subsequence of $\tilde{\Lambda}$ that converge non-tangentially to ξ .

Let $F \in \mathcal{H}^n$ such that $detF(\cdot) \not\equiv 0$ be fixed and recall that $Z(F)$ denotes the zero-set of the function $detF(\cdot)$, that $k^i_\lambda \in \mathcal{H}$ denotes the element such that $\langle f, k^i_\lambda\rangle = c_i(f,F,\lambda)$ and note that k^i_λ, due to (8), is a point evaluation. Finally, given $f_1,\ldots,f_k \in H$ we use the notation

$$[f_1,\ldots,f_k] = cl\left(\text{span}\left\{M^i_z f_j : i \geq 0, 1 \leq j \leq k\right\}\right) \qquad (17)$$

where cl stands for the closure.

Definition (5.2.8):[112] A finite subset $A \subset \mathbb{D}$ is called a V-set if the collection $\{I_\lambda\}_{\lambda\in A}$ consists of mutually disjoint intervals. Let $E \subset \mathbb{T}$ be closed. If, in addition, $\{\tilde{I}_\lambda\}_{\lambda\in A}$ covers E, then we say that A is a V-set for E.

From now on, Λ will always denote a (finite or infinite) sequence of the form $\left(k_{\lambda_j}^{i_j}\right)_j$ where $\lambda_j \in \mathbb{D}\backslash Z(F)$ and $1 \le i_j \le n$. Set

$$L(\Lambda) = \inf_{k_{\lambda_j}^{i_j} \in \Lambda} \left\{ \sqrt{1 - |\lambda_j|^2} \left\| k_{\lambda_j}^{i_j} \right\| \right\}.$$

Lemma (5.2.9):[112] Let $0 < r < 1, L > 0$ and $E \subset E_{\sigma'}$ be closed. Then there is a finite sequence Λ such that $L(\Lambda) > L$, $\tilde{\Lambda}$ is a V-set for E and $\tilde{\Lambda} \cap r\mathbb{D} = \emptyset$.

Lemma (5.2.10):[112] There is a constant $K > 0$, depending only on σ , with the property that if $\Lambda = (k_{\lambda_j}^{i_j})$ is a finite sequence such that $\tilde{\Lambda}$ is a V-set and $\lambda_{j_1} \ne \lambda_{j_2}$ whenever $j_1 \ne j_2$, then Λ is also interpolating for $\mathcal{H}$ with interpolating constant K.

Lemma (5.2.11):[112] Let $\Lambda = (k_{\lambda_j}^{i_j})$ be a finite sequence such that $\tilde{\Lambda}$ is a V-set and $\lambda_{j_1} \ne \lambda_{j_2}$ whenever $j_1 \ne j_2$. Moreover let $\{\zeta_{\lambda_j}\}$ be given numbers in $\mathbb{T}$. Then there exists an $f \in$ SpanΛ such that $\|f\| \le K\sqrt{2/\sigma}$, $\left|\langle f, k_{\lambda_j}^{i_j}\rangle\right| > L(\Lambda)$ and $\frac{\langle f, k_{\lambda_j}^{i_j}\rangle}{\zeta_{\lambda_j}} > 0$ for all $k_{\lambda_j}^{i_j} \in \Lambda$.

Lemma (5.2.12):[112] Suppose that $[f_1, \dots, f_n] = \mathcal{H}$. Given $\delta > 0$ and Λ_1 such that $\tilde{\Lambda}_1$ is a V-set, there exists an $L > 0$ with the property that whenever Λ_2 is such that $\tilde{\Lambda}_2$ is a V-set and $L(\Lambda_2) > L$, then $|\langle u, v\rangle| \le \delta \|u\| \|v\|$ for all $u \in Span\Lambda_1$ and $v \in Span\Lambda_2$.[106].

Proposition (5.2.13):[112] Let $F = (f_1, \dots, f_n) \in \mathcal{H}^n$ be fixed.

(i) There exists an $f \in H$ such that

$$nt - \lim_{\lambda \to \xi} \sup |C(f, F, \lambda)| = \infty$$

for $a.e.$ $\xi \in \mathbb{T}\backslash \Delta_F(\mathcal{H})$.

(ii) Assume that F is such that

$$[f_1, \dots, f_n] = \mathcal{H}$$

Then there is a sequence

$$\Lambda = \left(k_{\lambda_j}^{i_j}\right)$$

(where $1 \le i_j \le n$ and $\lambda_j \in \mathbb{D}\backslash Z(F)$), that is interpolating for $\mathcal{H}$ and satisfies $Ntl\tilde{\Lambda} = \mathbb{T}\backslash \Delta_F(\mathcal{H})\ a.e.$

Proof: (i) First pick a sequence of closed sets $E_q \subset E_{\sigma'}$ with $\bigcup_{q \ge 1} E_q = E_{\sigma'}$ a.e. We will inductively choose a sequence of finite sequences $\Lambda_q = \left(k_{\lambda_j^q}^{i_j^q}\right)_j$ (where $q \ge 1$, $\lambda_j^q \in \mathbb{D}$ and

$i_j^q \in \{1,\dots,n\}$) and functions $f_q \in Span\Lambda_q$, such that the properties (i)–(iii) listed below hold. To each q we associate the numbers

$$r_q = \frac{1+\sup\{|\lambda| : \lambda \in \tilde{\Lambda}_q\}}{2} \tag{18}$$

$$M_q = \sup\{\|k_\lambda^i\| : k_\lambda^i \in \Lambda_q \text{ for } p \le q\} \tag{19}$$

$$a_1 = \frac{1}{2} \text{ and } \mathrm{a_q} = \min\left(2^{-q}, \left(2^q K \sqrt{\frac{2}{\sigma}} M_{q-1}\right)^{-1}\right) \text{ for } g > 1 \tag{20}$$

We also define functions h_q (for $q \ge 1$), via $h_q = \sum_{p=1}^{q} a_p f_p$. The induction process will ensure that (for $q > 1$), the following conditions hold:

(i) $\tilde{\Lambda}_q$ is a V-set for E_q, $L(\Lambda_q) > \frac{q}{a_q}$ and $\tilde{\Lambda}_q \cap (r_{q-1}\mathbb{D}) = \emptyset$,

(ii) $\|f_q\| \le K\sqrt{\frac{2}{\sigma}}$,

(iii) for all $k_\lambda^i \in \Lambda_q$ we have $|\langle f_q, k_\lambda^i\rangle| > L(\Lambda_q)$ and

$$\frac{\langle f_q, k_\lambda^i\rangle}{\langle h_{q-1}, k_\lambda^i\rangle} > 0 \quad (\text{or } \langle f_q, k_\lambda^i\rangle > 0 \text{ if } \langle h_{q-1}, k_\lambda^i\rangle = 0\ ).$$

Indeed, it is clear that at we can use Lemmas (5.2.9) and (5.2.11) to choose Λ_1 and f_1 such that the applicable parts of (i)–(iii) hold. Likewise, at the qth step of the induction process, the existence of a Λ_q satisfying (i) is guaranteed by Lemma (5.2.14) and the existence of an f_q satisfying (ii) and (iii) is guaranteed by Lemma (5.2.16).

Given Λ_q's and f_q's satisfying (i)–(iii) the desired function f is given by

$$f = \sum_{p=1}^{\infty} a_p f_p$$

That the sum converges is guaranteed by (ii) and the fact $a_p \le 2^{-p}$. Given $q \in \mathbb{N}$ and $k_\lambda^i \in \Lambda_q$ we get

$$\left|\langle \sum_{p=1}^{\infty} a_p f_p , k_\lambda^i\rangle\right| = |\langle h_{g-1} + a_p f_p, k_\lambda^i\rangle| \ge |\langle a_p f_q, k_\lambda^i\rangle| \ge a_q L(\Lambda_q) \ge q$$

by (i) and (iii). Moreover for $p > q$ we have

$$|\langle a_p f_p, k_\lambda^i\rangle| \le a_p\|f_p\|\|k_\lambda^i\| \le \left(2^p K \sqrt{\frac{2}{\sigma}} M_{p-1}\right)^{-1} \|k_\lambda^i\| \le 2^{-p}$$

by (ii) and the definition of a_p. Combining these two inequalities we easily obtain $|\langle f, k_\lambda^i\rangle| \geq q-1$ which implies that

$$|C(f,F,\lambda)| \geq q-1$$

for all $\lambda \in \tilde{\Lambda}_q$. Finally, by the fact that $\tilde{\Lambda}_q$ is a V-set for E_q and that $\lim_{q\to\infty} r_q = 1$ it follows that for every $\xi \in \bigcup_{q\geq 1} E_q$ there is a sequence in $\bigcup_{q\geq 1} \tilde{\Lambda}_q$ that converges non-tangentially to ξ.

(ii): Let $G = (g_i)$ and $H = (h_i)$ denote arbitrary elements of $\mathcal{H}^n$ and define on $(\mathcal{H}^n) \oplus (\mathcal{H}^n)$ the function

$$I(G,H) = \det\begin{pmatrix} \langle g_1, h_1\rangle & \dots & \langle g_n, h_1\rangle \\ \vdots & \ddots & \vdots \\ \langle g_n, h_1\rangle & \dots & \langle g_n, h_n\rangle \end{pmatrix} \tag{21}$$

I satisfies $I(G,G) \geq 0$ and

$$|I(G,H)|^2 \leq I(G,G)I(H,H)$$

To see this one has to show that I extends to a sesquilinear positive form on the wedge product $\Lambda_1^n\mathcal{H}$. The above inequality is then just a special case of the Cauchy–Schwartz inequality.

Let $F = (f_1, \dots, f_n) \in \mathcal{H}^n$ be as usual and set $K_\lambda = \left(k_\lambda^1, \dots, k_\lambda^n\right) \in \mathcal{H}^n$. Then for all $\lambda \in \mathbb{D}$ we have

$$\begin{aligned} 1 = \left(I(F,K_\lambda)\right)^2 &= (1-|\lambda|^2)^{2n}\left(I\left(\frac{F}{1-\bar{\lambda}z}, K_\lambda\right)\right)^2 \\ &\leq (1-|\lambda|^2)^n I\left(\frac{F}{1-\bar{\lambda}z}, \frac{F}{1-\bar{\lambda}z}\right)(1-|\lambda|^2)^n I(K_\lambda, K_\lambda) \end{aligned} \tag{22}$$

where $\frac{F}{1-\bar{\lambda}z}$ stands for the element $\left(\frac{f_1}{1-\bar{\lambda}z}, \dots, \frac{f_n}{1-\bar{\lambda}z}\right)$. Moreover, $I(K_\lambda, K_\lambda)$ consists of a sum of $n!$ terms, each of the form

$$(-1)^{sgn(\sigma)}\prod_{i=1}^n \langle k_\lambda^i, k_\lambda^{\sigma(i)}\rangle$$

where σ is a permutation of $\{1, \dots, n\}$ and $sgn(\sigma)$ denotes the sign of σ. By the Cauchy–Schwartz inequality we conclude that

$$(1-|\lambda|^2)^n I(K_\lambda, K_\lambda) \leq n!\prod_{i=1}^n (1-|\lambda|^2)\left\|k_\lambda^i\right\|^2$$

Combining this with (23), we see that if we can show that

$$\mathrm{nt}-\lim_{\lambda\to\xi}(1-|\lambda|^2)^n I\left(\frac{F}{1-\bar{\lambda}z}, \frac{F}{1-\bar{\lambda}z}\right) \tag{23}$$

for a.e. $\xi \in \mathbb{T} \setminus \Sigma(\mathcal{H})$, then it follows that for each such ξ there exist at least one $i \in \{1, \ldots, n\}$ with

$$nt - \lim_{\lambda \to \xi} \sup(1 - |\lambda|^2)\left\|k_\lambda^i\right\|^2 = \infty$$

i.e. $\Delta_F(\mathcal{H}) \subset \Sigma(\mathcal{H})$ a.e. as desired.

To prove (23) we recall a few facts from [106] and [107]. We first assume that

$$\lim_{k \to \infty} \left\|M_z^k f\right\| \neq 0$$

for some element of $\mathcal{H}$. Recall the space $\widetilde{K}$ and its associated objects like V, σ_i etc. that were defined, and let E denote the spectral measure for V . Define the measures m_i on $\mathbb{T}$ by $dm_i = \chi_{\sigma_i}\, dm$.

By [107], it follows that E is absolutely continuous and that we can take $\widetilde{K} = L^2(m_1) \oplus \cdots \oplus L^2(m_n)$ with V being the operator of multiplication by the independent variable. Moreover, for any $f \in \mathcal{H}$ let $\left(f^i\right)_{i=1}^n$ denote the element corresponding to f in $L^2(m_1) \oplus \cdots \oplus L^2(m_n)$. If we treat the elements of $L^2(m_i)$ as functions on T that are identically 0 in $\setminus \sigma_i$, then we have that

$$\frac{d\langle E(.)f, g\rangle}{dm} = \sum_{i=1}^{n} f^i \overline{g^i}$$

where $\frac{d\langle E(.)f,g\rangle}{dm}$ denotes the Radon–Nikodym derivative of the measure $\langle E(.)f, g\rangle$ with respect to m. If $\lim_{k\to\infty}\left\|M_z^k\right\| = 0$ for all elements of $\mathcal{H}$, then we simply define $f^i \equiv 0$ on $\mathbb{T}$ for all $f \in \mathcal{H}$ and $i = 1, \ldots, n$. By a slight modification of [106].

Lemma (5.2.14):[112] For any $f, g \in \mathcal{H}$ we have

$$\mathrm{nt} - \lim_{\lambda \to \xi}(1 - |\lambda|^2)\left\langle \frac{f}{1 - \bar{\lambda}z}, \frac{g}{1 - \bar{\lambda}z}\right\rangle = \sum_{i=1}^{n} f^i(\xi)\overline{g^i(\xi)}$$

for a.e. $\xi \in \mathbb{T}$.

Set

$$W_F(\xi) = \begin{pmatrix} f_1^1(\xi) & \cdots & f_n^1(\xi) \\ \vdots & \ddots & \vdots \\ f_1^n(\xi) & \cdots & f_n^n(\xi) \end{pmatrix}$$

where $f_j^i = \left(f_j\right)^i \in L^2(m_i)$. By Lemma (5.2.18) the limit in (23) exists for a.e. $\xi \in \mathbb{T}$ and

$$\text{nt}-\lim_{\lambda\to\xi}(1-|\lambda|^2)^n I\left(\frac{F}{1-\bar{\lambda}z},\frac{F}{1-\bar{\lambda}z}\right)=\det\begin{pmatrix}\sum_{j=1}^{n} f_1^j(\xi)\overline{f_1^j(\xi)} & \cdots & \sum_{j=1}^{n} f_n^j(\xi)\overline{f_1^j(\xi)}\\ \vdots & \ddots & \vdots\\ \sum_{j=1}^{n} f_1^j(\xi)\overline{f_n^j(\xi)} & \cdots & \sum_{j=1}^{n} f_n^j(\xi)\overline{f_n^j(\xi)}\end{pmatrix}$$

$$=\det\big(W_F(\xi)^*W_F(\xi)\big)=|\det W_F(\xi)|^2.$$

Recall that $\Sigma(\mathcal{H})=\sigma_n$ and that the support of f_i^n is included in σ_n. Therefore if $\xi\in\mathbb{T}\setminus\Sigma(\mathcal{H})$ then the last row of $W_F(\xi)$ is identically 0 and hence $\det(W_F(\xi))=0$.

Theorem (5.2.15):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions that satisfies (3) and (4).

(i) If $m\big(\Sigma(\mathcal{H})\big)>0$ then $\operatorname{ind}\mathcal{M}\le n$ for all invariant subspaces $\mathcal{M}\subset\mathcal{H}$. In fact, $\operatorname{ind}\mathcal{M}=\sup_{\lambda\in\mathbb{D}}(\dim\{f(\lambda): f\in M\})$.

(ii) If $m\big(\Sigma(\mathcal{H})\big)=0$ then given any $k\in\mathbb{N}\cup\{\infty\}$ the exists an invariant subspace $\mathcal{M}\subset\mathcal{H}$ with $\operatorname{ind}\mathcal{M}=k$.

Proof: The first part is proved in [107], so we turn immediately to the second part. First, if $0\le k\le n$ then the problem is easily solved. For $k=0$ we simply take $\mathcal{M}=\{0\}$ and otherwise we take $f_1,\dots,f_k\in\mathcal{H}$ such that $f_k(0)=e_k$ and put $\mathcal{M}=[f_1,\dots,f_k]$ (recall (17)). It is then easily verified that

$$\operatorname{Ran} M_z|\mathcal{M}+\operatorname{Span}\{f_1,\dots,f_k\}=\mathcal{M}$$

So that $\operatorname{ind}\mathcal{M}=k$ as desired. To see this, let $g\in\mathcal{M}$ be arbitrary and pick polynomials p_j^i such that

$$g=\lim_{j\to\infty}\sum_{i=1}^{k} p_i^j f_i$$

As point evaluations are continuous we infer that there are numbers $a_1,\dots,a_k$ such that $\lim_{j\to\infty} p_i^j(0)=ai$. Therefore $\left(\sum_{i=1}^{k}\left(p_i^j-p_i^j(0)\right)f_i\right)_j$ is a Cauchy sequence and as M_z is bounded below we get

$$g-\sum_{i=1}^{k} a_i f_i=\lim_{j\to\infty}\sum_{i=1}^{k}\left(p_i^j-p_i^j(0)\right)f_i=M_z\left(\lim_{j\to\infty}\sum_{i=1}^{k}\frac{p_i^j-p_i^j(0)}{z}f_i\right)$$

So that $g-\sum_{i=1}^{k} a_i f_i\in RanM_z|\mathcal{M}$, as desired.

We now assume that $k>n$ and that $m\big(\Sigma(\mathcal{H})\big)=0$. The following argument has been taken from [21]. If $\mathcal{M}\subset\mathcal{H}$ is an M_z-invariant subspace, then $\mathcal{M}^\perp$ is M_z^*-invariant. We can decompose the operator M_z with respect to $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$ as

$$M_z \cong \begin{pmatrix} M_z|\mathcal{M} & p_{\mathcal{M}} M_z|_{\mathcal{M}^\perp} \\ 0 & p_{\mathcal{M}} M_z|_{\mathcal{M}^\perp} \end{pmatrix}$$

Put $S = M_z^*|_{\mathcal{M}^\perp}$ and observe that $S^* = P_{\mathcal{M}^\perp}(M_z^*)^*|_{\mathcal{M}^\perp} = P_{\mathcal{M}^\perp} M_z|_{\mathcal{M}^\perp}$ and then that

$$M_z \cong \begin{pmatrix} M_z|\mathcal{M} & p_{\mathcal{M}} M_z|_{\mathcal{M}^\perp} \\ 0 & S^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & S^* \end{pmatrix} \begin{pmatrix} I & p_{\mathcal{M}} M_z|_{\mathcal{M}^\perp} \\ 0 & I \end{pmatrix} \begin{pmatrix} M_z|_{\mathcal{M}} & 0 \\ 0 & 0 \end{pmatrix}$$

Thus we get

$$-n = \text{ind}\, M_z = \text{ind} M_z|_{\mathcal{M}} + 0 + \text{ind}\, S^* = \text{ind}\, M_z|_{\mathcal{M}} - \text{ind}\, S$$

If either $\text{ind} M_z|\mathcal{M}$ or ind S is finite, and hence

$$\text{ind} S - n = \text{ind} M_z|_{\mathcal{M}} \tag{24}$$

Holds even for $\text{ind} M_z|_{\mathcal{M}} = -\infty$. As $S = M_z^*|_{\mathcal{M}^\perp}$ this implies that if we can find a subspace $\mathcal{M}$ such that

$$\text{ind} M_z^*|_{\mathcal{M}^\perp} = n - k \tag{25}$$

then we are done.

We will first prove that such a subspace $\mathcal{M}$ can be found under the additional assumption that there exist $f_1, \ldots, f_n \in \mathcal{H}$ such that

$$[f_1, \ldots, f_n] = \mathcal{H} \tag{26}$$

By Proposition (5.2.13) we then get that there exists a sequence

$$\Lambda = \left(k_{\lambda_j}^{i_j}\right)$$

(where $1 \leq i_j \leq n$ and $\lambda_j \in \mathbb{D} \setminus Z(F)$), that is interpolating for $\mathcal{H}$ and satisfies $Ntl\,\tilde{\Lambda} = \mathbb{T}$ a.e. Moreover, by the proof it follows that we can assume that $\lambda_{j_1} \neq \lambda_{j_2} \neq 0$ whenever $j_1 = j_2$. Put $\mathcal{N} = cl(\text{Span}\left\{k_{\lambda_j}^{i_j}\right\})$. For any $f \in \mathcal{H}$ we have

$$\langle M_z^* k_{\lambda_j}^{i_j}, f\rangle = \langle k_{\lambda_j}^{i_j}, M_z f\rangle = \overline{\lambda_j}\, \langle k_{\lambda_j}^{i_j}, f\rangle,$$

so $\mathcal{N}$ is an M_z^* -invariant subspace of $\mathcal{H}$. We will identify l_Λ^2 with the standard space $l^2(\mathbb{N})$ in the obvious way. As $\mathcal{N} = Ran\, T_\Lambda^* = (Ker\, T_\Lambda)^\perp$ Definition (5.2.11), it is easily verified that $T_\Lambda|_N$ is a bijection from $\mathcal{N}$ onto l^2, and the above calculation implies that

$$M_z^*|_{\mathcal{N}}\, T_\Lambda^* = T_\Lambda^* D$$

where D denotes the operator on l^2 such that $D(a_i) = (\bar{\lambda}_i a_i)$. Now, it is a known fact that there are D-invariant subspaces $\mathcal{L}$ of l^2 such that

$$ind\, D|\mathcal{L} = n - k \tag{27}$$

(even when $k = \infty$), and hence (25) holds with $\mathcal{M} = \left(T_\Lambda^*(\mathcal{L})\right)^\perp$ and so by (24) we are done (in the case when (26) holds).

We now outline a proof of how to see that D has the desired invariant subspace. The crucial fact here is that $Ntl\,\tilde{\Lambda} = \mathbb{T}$ a.e. By [114] it follows that there exist elements $x, y \in l^2$ such that

$$\langle D^i x, y\rangle = \begin{cases} 1 & \text{if} \quad \mathrm{i} = 0 \\ 0 & \text{if} \quad \mathrm{i} > 0 \end{cases}$$

Put $\mathcal{L} = cl(\mathrm{Span}\{D^i x : i \geq 0\})$. As $\lambda_i \neq 0 \quad \forall i \in \mathbb{N}$ and $\tilde{\Lambda}$ has no accumulation points in D we see that $D|_{\mathcal{L}}$ is bounded below and as x is a cyclic vector we deduce that $\mathrm{ind}\, D|\mathcal{L}$ is either 0 or -1. But it cannot be 0 because this would imply that $D|_{\mathcal{L}}$ is invertible, which contradicts the fact that y is orthogonal to $D|_{\mathcal{L}}$ but not to $\mathcal{L}$. Thus $\mathrm{ind}\, D|\mathcal{L} = -1$ so (27) holds for $k = n + 1$. To produce subspaces with $\mathrm{ind}\, D|_{\mathcal{L}} < -1$ one splits the set $\tilde{\Lambda}$ into several subsets and then use the above construction on each. The argument goes as follows. For each $\lambda \in \mathbb{D}$ let $I_\lambda \subset \mathbb{T}$ be the open interval centered at $\lambda/|\lambda|$ with $m(I_\lambda) = 1 - |\lambda|$. It follows from [115] that a discrete relatively closed sequence $\Gamma = (\gamma_i)_{i=1}^\infty$ in $\mathbb{D}$ satisfies $Ntl\Gamma = \mathbb{T}$ if and only if

$$\bigcup_{i > i_0} I_{\gamma_i} = \mathbb{T} \ \text{a. e.}$$

for all $i_0 \in \mathbb{N}$. By this observation it follows that there are numbers $i_1, i_2, \ldots$ such that

$$m\left(\bigcup_{i_{j+1} \geq i > i_j} I_{\lambda_i}\right) \geq 1 - 1/j$$

and using these subsets it is easy to find $k - n$ disjoint subsets $\tilde{\Lambda}_j \subset \tilde{\Lambda}$ with $Ntl\,\tilde{\Lambda}_j = \mathbb{T}$ and

$$\bigcup_{j=1}^{k-n} \tilde{\Lambda}_j = \tilde{\Lambda}.$$

Given a sequence $\Gamma = (\gamma_i)$ we let D_Γ denote the diagonal operator on l^2 given by $D(a_i) = (\overline{\gamma_i} a_i)$, and we shall think of the sets $\tilde{\Lambda}_j$ and $\tilde{\Lambda}$ as sequences ordered so that the sub-indices of the λ_i's are increasing. It is easily seen that $D = D_{\tilde{\Lambda}}$ is unitarily equivalent to

$$\bigoplus_{j=1}^{k-n} D_{\tilde{\Lambda}_j}.$$

Let $\mathcal{L}_j \subset l^2$ be $D_{\tilde{\Lambda}_j}$-invariant subspaces such that $ind\ D_{\tilde{\Lambda}_j}|_{\mathcal{L}_j} = -1$ and note that a short argument shows that

$$\operatorname{ind}\left(\left(\bigoplus_{j=1}^{k-n} D_{\bar{\lambda}_j}\right)\Bigg|_{\bigoplus_{j=1}^{k-n}\mathcal{L}_j}\right) = \sum_{j=1}^{k-n} \operatorname{ind} D_{\bar{\lambda}_j}\mid_{\mathcal{L}_j} = \sum_{j=1}^{k-n} -1 = n-k$$

This immediately implies that we can find a subspace $\mathcal{L} \subset l^2$ such that (27) holds, as desired.

It remains to prove that the assumption (26) can be removed. Fix $(f_1,\ldots,f_n) = F \in \mathcal{H}^n$ such that $\det F(0) \neq 0$. If a subspace M with $indM = k$ can be found as a subspace of $[f_1,\ldots,f_n]$, then we are obviously done. The already proved results might however not be applicable because the space $[f_1,\ldots,f_n]$ may not satisfy conditions (1) and (2). This problem can be overcome as follows. By the argument in the first part of this proof we have that $ind\,[f_1,\ldots,f_n] = n$, which by [116] implies that there exists a Hilbert space of $\mathbb{C}^n$-valued analytic functions $\mathcal{H}'$that does satisfy (1) and (2), and a unitary map $U : [f_1,\ldots,f_n] \to \mathcal{H}'$ such that $M_z U = U M_z$. If we denote by $\mathcal{K}'$and $\widetilde{\mathcal{K}'}$ the spaces corresponding to $\mathcal{H}'$as $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ correspond to $\mathcal{H}$, then it is clear that we may consider $\mathcal{K}'$as a subspace of $\mathcal{K}$ which implies that $\widetilde{\mathcal{K}'}$can be taken as a subspace of $\widetilde{\mathcal{K}}$. Thus

$$\Sigma(\mathcal{H}') = \left\{\xi \in \mathbb{T}: M_{V|_{K'}}(\xi) \geq n\right\}_{a.e.} \subset \{\xi \in \mathbb{T}: M_V(\xi) \geq n\} = \Sigma(\mathcal{H})$$

We conclude that

$$m\big(\Sigma(\mathcal{H}')\big) = 0$$

and hence it follows from what we have already proven that there are M_z-invariant subspaces of $\mathcal{H}'$ with index k. As U is unitary and $M_z U = U M_z$.

Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions. Given $F \in \mathcal{H}^n$ with $\det(F(\cdot)) \not\equiv 0$, we set

$$\Theta_F^i = \Theta(\{c_i(f,F,\cdot): f \in \mathcal{H}\})$$

i.e. Θ_F^i is the “largest” subset of $\mathbb{T}$ where the ith canonical coefficient has non-tangential limits a.e. for all $f \in \mathcal{H}$. Recall that a Hilbert space of $\mathbb{C}^n$-valued analytic functions is called decomposable if there are $f_1,\ldots,f_n \in \mathcal{H}$ such that

$$[f_1] + \cdots + [f_n] = \mathcal{H} \tag{28}$$

Theorem (5.2.16):[112] Let $\mathcal{H}$ be a Hilbert space of $\mathbb{C}^n$-valued analytic functions that satisfies (3) and (4) and assume that $\mathcal{H}$ is decomposable, i.e. there are $f_1,\ldots,f_n$ such that (28) holds. Put $F = (f_1,\ldots,f_n)$. Then

$$M_V = \sum_{i=1}^{n} \chi_{\Theta_F^i} \quad a.e$$

Proof: Fix $i \in \{1, \ldots, n\}$. Note that any $f \in [f_i]$ is of the form ϕf_i where ϕ is a holomorphic function in $\mathbb{D}$. Moreover it is easy to see that $\phi = c_i(f, F, \cdot)$ and that given $g = g_1 + \cdots + g_n \in \mathcal{H}$ with $g_i \in [f_i]$ we have $c_i(g, F, \cdot) = c_i(g_i, F, .)$. Let $\mathcal{C}_i$ denote the set of functions

$$\{c_i(f, F, .) : f \in [f_i]\}$$

with the norm $\|c_i(f, F, \cdot)\| = \|f\|$. It is easy to check that $\mathcal{C}_i$ becomes a Hilbert space of $\mathbb{C}$-valued analytic functions that satisfies (3) and (4). In particular, by Theorem (5.2.19) we have that

$$\Theta_F^i = \Sigma(\mathcal{C}_i) \qquad \text{a.e.}$$

so in order to prove theorem (5.2.20) we have to show that

$$M_V = \sum_{i=1}^{n} \chi_{\Sigma(\mathcal{C}_i)} \quad a.e.$$

Recall the definition of $\mathcal{M}, \mathcal{K}, S, \widetilde{\mathcal{K}}$ and V associated to $\mathcal{H}$, as defined in the introduction. Analogously, for each $i \in \{1, \ldots, n\}$ let $\mathcal{M}_i \subset \mathcal{C}_i$ denote the subspace

$$\mathcal{M}_i = \left\{ f \in \mathcal{C}_i : \lim_{k \to \infty} \left\| M_z^k f \right\| = 0 \right\}$$

and let $P_{\mathcal{M}_i^{\perp}}$ denote the orthogonal projection onto $\mathcal{M}_i^{\perp}$. Note that $\mathcal{M} = \sum_{i=1}^{n} \mathcal{M}_i$ and that due to the dichotomy mentioned after Theorem (5.2.1) we either have $\mathcal{M}_i = \{0\}$ or $\mathcal{M}_i = \mathcal{C}_i$.

When $\mathcal{M}_i^{\perp} = \mathcal{C}_i$ we define a new norm on $\mathcal{C}_i$ via the formula

$$\|f\|_* = \lim_{k \to \infty} \left\| M_z^k f \right\|$$

and denote by $\mathcal{K}_i$ the completion of the pre-Hilbert space $(\mathcal{C}_i, \|.\|_*)$. Let $S_i : \mathcal{K}_i \to k_i$ denote the isometric operator such that $S_i f = M_z f$ for all $f \in \mathcal{C}_i$, and let V_i be its minimal unitary extension on the Hilbert space $\widetilde{\mathcal{K}}_i$ When $\mathcal{M}_i^{\perp} = \{0\}$ we define $\widetilde{\mathcal{K}}_i = \{0\}$ and let both V_i and S_i to be equal to the operator that maps 0 to 0.

By [108] it follows that $M_{V_i} \leq 1$ a.e. and thus by the definition of $\Sigma(\mathcal{C}_i)$ we get $M_{V_i} = \chi_{\Sigma(\mathcal{C}_i)}$ a.e. Hence (29) can be reformulated as

$$M_V = \sum_{i=1}^{n} M_{V_i} \qquad a.e. \tag{30}$$

Let $\mathcal{L}_i$ be the closure of $P_{\mathcal{M}^{\perp}}[f_i]$ in $\mathcal{K}$. As $M_z(\mathcal{M}) \subset \mathcal{M}$ it easily follows that $S(\mathcal{L}_i) \subset \mathcal{L}_i$. Let $S|_{\mathcal{L}_i}$ denote the operator S restricted to $\mathcal{L}_i$. Moreover let $\widetilde{\mathcal{L}_i} \subset \widetilde{\mathcal{K}}$ denote the closure of the linear span of the sets $V^{-k} \mathcal{L}_i, k = 0,1, \ldots, \infty$. By standard results about unitary extensions it follows that $V|_{\widetilde{\mathcal{L}_i}}$ is a minimal unitary extension of $|_{\mathcal{L}_i}$.

We shall now define a natural unitary map $R_i\colon \mathcal{K}_i \to \mathcal{L}_i$. First, if $\mathcal{M}_i^{\perp} = \{0\}$, then $[f_i] \subset \mathcal{M}$ and thus $\mathcal{L}_i = \mathcal{K}_i = \{0\}$. In this case we define $R_i(0) = 0$. Otherwise, i.e. when $\mathcal{M}_i^{\perp} = \mathcal{C}_i$, we set

$$R_i(c_i(f,F,.)) = P_{\mathcal{M}^\perp} f$$

for all $f \in [f_i]$. Clearly R_i maps $\mathcal{C}_i$ onto a dense subset of $P_{\mathcal{M}^\perp}[f_i]$. Moreover, the calculation

$$\|c_i(f,F,.)\|_* = \lim_{k\to\infty} \left\|M_z^k f\right\| = \lim_{k\to\infty} \left\|M_z^k P_{\mathcal{M}^\perp} f\right\| = \|P_{\mathcal{M}^\perp} f\|_*$$

shows that it is isometric. Thus R_i extends by continuity to a unitary operator from $\mathcal{K}_i$ to $\mathcal{L}_i$, as desired.

Under this map the operators S_i and $S|_{\mathcal{L}_i}$ are unitarily equivalent. If $\mathcal{M}_i^{\perp} = \{0\}$ then the statement is trivial, so we assume that $\mathcal{M}_i^{\perp} = \mathcal{C}_i$. For $f \in [f_i]$ we then get

$$\begin{aligned} R_i\big(s_i c_i(f,F,.)\big) &= R_i\big(M_z c_i(f,F,.)\big) = R_i\big(c_i(M_z f,F,.)\big) \\ &= P_{\mathcal{M}^\perp} M_z f = P_{\mathcal{M}^\perp} M_z P_{\mathcal{M}^\perp} f = S R_i\big(c_i(f,F,.)\big) \end{aligned}$$

which easily implies that $R_i\, S_i = S|_{\mathcal{L}_i}\, R_i$.

Now, minimal unitary extensions are unique up to unitary equivalence and thus we have shown that $M_{V_i} = M_{V|_{\tilde{\mathcal{L}}_i}}$ a.e. In order to prove (30), it is thus sufficient to show that

$$M_V = \sum_{i=1}^{n} M_V|_{\mathcal{L}_i} \qquad a.e. \tag{31}$$

But (31) does hold if we can show that $\tilde{\mathcal{L}}_1 + \dots + \tilde{\mathcal{L}}_n = \tilde{\mathcal{K}}$, because due to a theorem by Putnam [116], similar normal operators are automatically unitarily equivalent.

It thus remains to prove that $\tilde{\mathcal{L}}_1 + \dots + \tilde{\mathcal{L}}_n = \tilde{\mathcal{K}}$. By standard results about minimal unitary extensions, the sets $V^{-k}\,\mathcal{K}, k \in \mathbb{N}$, are dense in $\tilde{\mathcal{K}}$. It follows that $\tilde{\mathcal{L}}_1 + \dots + \tilde{\mathcal{L}}_n$ is dense in ˜K, so we only have to show that for any two subsets $I_1, I_2 \subset \{1,\dots,n\}$ with $I_1 \cap I_2 = \emptyset$ the two planes $\sum_{i\in I_1} \tilde{\mathcal{L}}_i$ and $\sum_{i\in I_2} I_2$ have a positive angle, i.e. that

$$\sup\left\{\frac{\langle x,y\rangle}{\|x\|\|y\|} : x \in \sum_{i\in I_1} \tilde{\mathcal{L}}_i\,, y \in \sum_{i\in I_2} \tilde{\mathcal{L}}_i\right\} < 1$$

By the fact that V is unitary, the definition of $\tilde{\mathcal{L}}_i$ and that $P_{\mathcal{M}^\perp}[f_i]$ is dense in $\mathcal{L}_i$ we get

$$\sup\left\{\frac{\langle x,y\rangle}{\|x\|\|y\|} : x \in \sum_{i\in I_1} \tilde{\mathcal{L}}_i\,, y \in \sum_{i\in I_2} \tilde{\mathcal{L}}_i\right\} \quad = \sup\left\{\frac{\langle x,y\rangle}{\|x\|\|y\|} : x \in \sum_{i\in I_1} \mathcal{L}_i\,, y \in \sum_{i\in I_2} \mathcal{L}_i\right\}$$

$$= \sup\left\{\frac{|\langle f,g\rangle_*|}{\|f\|_*\|g\|_*}: f\in\sum_{i\in I_1}P_{\mathcal{M}^\perp}[f_i], g\in\sum_{i\in I_2}P_{\mathcal{M}^\perp}[f_i]\right\}$$

$$\leq \sup\left\{\lim_{k\to\infty}\frac{|\langle M_z^k f', M_z^k g'\rangle|}{\|M_z^k f'\|\,\|M_z^k g'\|}: f'\in\sum_{i\in I_1}[f_i], g'\in\sum_{i\in I_2}[f_i]\right\}$$

$$\leq \sup\left\{\frac{|\langle f',g'\rangle|}{\|f'\|\|g'\|}: f'\in\sum_{i\in I_1}[f_i], g'\in\sum_{i\in I_2}[f_i]\right\}.$$

But the last supremum is indeed less than 1 by the assumption that $[f_1]+\cdots+[f_n]=\mathcal{H}$.

Theorem (5.2.17):[112] There is a Hilbert space $\mathcal{H}$ of $\mathbb{C}^2$-valued analytic functions that satisfies (3) and (4) such that

$$\chi_{\Theta_F^1}+\chi_{\Theta_F^2}< M_V \ a.e.$$

for all choices of $F\in\mathcal{H}^2$ with $\det(F(.))\not\equiv 0$.

Proof: Let τ_i denote subset of $\mathbb{T}$ which lies in the i-th quadrant of the plane, $i=1,\ldots,4$. We define the measures μ,ν and ω by

$$d\mu = dA+\chi_{\tau_1\cup\tau_2}\,dm,$$

$$d\nu = dA+\chi_{\tau_1\cup\tau_3}\,dm,$$

$$d\omega = dA+\chi_{\tau_4}\,dm.$$

let $\mathcal{H}$ be the set

$$\{(f^1,f^2): f^1\in P^2(\mu), f^2\in P^2(\nu), f^1+f^2\in P^2(\omega)\}$$

endowed with the norm

$$\|(f^1,f^2)\|^2=\|f^1\|^2_{P^2(\mu)}+\|f^2\|^2_{P^2(\nu)}+\|f^1+f^2\|^2_{P^2(\omega)}$$

Using (i) and (ii) one easily sees that $\mathcal{H}$ can be identified with a Hilbert space of $\mathbb{C}^2$-valued analytic functions. We keep the notation from the introduction of the corresponding objects like $\mathcal{M},S,\mathcal{K},V$ *etc*. Clearly $\mathcal{M}^\perp=\mathcal{H}$ and by the dominated convergence theorem it follows that

$$\|(f^1,f^2)\|_*^2=\int_{\tau_1\cup\tau_2}|f^1|^2dm+\int_{\tau_1\cup\tau_3}|f^2|^2dm+\int_{\tau_4}|f^1+f^2|^2dm$$

Let m_1 denote the measure given by $dm_1=\chi_{\tau_1}\,dm$ and let $J:\mathcal{H}\to L^2(m)\oplus L^2(m_1)$ be given by

$$J\big((f^1,f^2)\big) = (f^1\chi_{\tau_1\cup\tau_2} + f^2\chi_{\tau_3} + (f^1+f^2)\chi_{\tau_4}, f^2\chi_{\tau_3})$$

Clearly, J is isometric with respect to the $*$-norm on $\mathcal{H}$. Let N denote the operator on $L^2(m)\oplus L^2(m_1)$ of multiplication by the independent variable, i.e. the operator such that for a.e. $\xi\in\mathbb{T}$ we have $N(f^1,f^2)(\xi) = (\xi f^1(\xi), \xi f^2(\xi))$, and let $\mathcal{K}'$ denote the closure of $J(\mathcal{H})$ in $L^2(m)\oplus L^2(m_1)$. Obviously $JM_z = NJ$, and this implies that S is unitarily equivalent with $N|\mathcal{K}'$. Using the Stone–Weierstrass theorem, it is not hard to see that $\bigcup_{k>0} N^{-k}\mathcal{K}'$ is dense in $L^2(m)\oplus L^2(m_1)$. It follows that N is a minimal unitary extension of $N|\mathcal{K}'$. Summing up we have proved that

$$M_v = M_N = \chi_{\mathbb{T}} + \chi_{\tau_1} \qquad a.e.$$

To conclude the theorem, we will show that for any choice of $F\in\mathcal{H}^2$ we have

$$\chi_{\Theta_F^1}(\xi) + \chi_{\Theta_F^2}(\xi) = 0$$

for a.e. ξ in some non-trivial subset of $\mathbb{T}$.

Let $F = (f^1,f^2)$ where $f_1 = (f_1^1, f_1^2)$ and $f_2 = (f_2^1, f_2^2)$. Note that for any $g = (g^1,g^2)\in\mathcal{H}$ and $\lambda/\in Z(F)$, we have that

$$c_1(g,F,\lambda) = \frac{f_2^2(\lambda)}{\det F(\lambda)} g^1(\lambda) - \frac{f_2^1(\lambda)}{\det F(\lambda)} g^2(\lambda)$$

and a similar equation holds for c_2. First assume that at least 3 of the functions f_i^j are non-vanishing, say $f_2^1\not\equiv 0$ and $f_2^2\not\equiv 0$. By choosing $g^2\equiv 0$ and $g^1\in P^2(\mu+\omega)$ such that $\tau_3\cap\Theta(g^1) = \tau_3\setminus\Theta(\frac{f_2^2}{\det F})$ a.e., we deduce that

$$m(\Theta_F^1(\mathcal{H})\cap\tau_3 = 0$$

In a similar way we deduce that

$$m(\Theta_F^1(\mathcal{H})\cap\tau_2 = 0$$

and that

$$m(\Theta_F^2(\mathcal{H})\cap\tau_k) = 0 \tag{32}$$

for at least one value of $k\in\{2,3\}$, which implies that

$$\chi_{\Theta_F^1}(\xi) + \chi_{\Theta_F^2}(\xi) = 0$$

for a.e. $\xi\in\tau_k$, where k is such that (32) holds. Thus $\chi_{\Theta_F^1} + \chi_{\Theta_F^2} < M_V$ a.e. as desired.

Now assume that two of the f_j^i's are identically zero, say, $f_1^2\equiv 0$ and $f_2^1\equiv 0$. Then

$$c_1(g,F,\lambda) = \frac{g^1(\lambda)}{f_1^1(\lambda)},\ \ c_2(g,F,\lambda) = \frac{g_2(\lambda)}{f_1^1(\lambda)}.$$

By choosing $g^1 = -g^2 \in P^2(\mu + \nu)$ such that $\tau_4 \cap \Theta(g^1) = \tau_4 \setminus \Theta\left(\frac{1}{f_1^1}\right)$, we deduce that

$$m(\Theta_F^1(\mathcal{H}) \cap \tau_4) = 0$$

Similarly we see that also

$$m(\Theta_F^2(\mathcal{H}) \cap \tau_4) = 0$$

and thus we get that

$$\chi_{\Theta_F^1}(\xi) + \chi_{\Theta_F^2}(\xi) = 0$$

for a.e. $\xi \in \tau_4$.

Let T be a contraction on some Hilbert space $\mathcal{X}$ such that $T - \lambda$ is bounded below for each $\lambda \in \mathbb{D}, \bigcap_{i=1}^{\infty} T^i \chi = \{0\}$ and ind $T = -n$ for some $n \in \mathbb{N}$. It can be shown that the above conditions are equivalent to demanding that T^* is a contraction which lies in the Cowen–Douglas class $B_n(\mathbb{D})$. By [117], it then follows that there exists a Hilbert space $\mathcal{H}$ of $\mathbb{C}^n$-valued analytic functions that satisfies (1) and (2) and a unitary map $U: \mathcal{X} \to \mathcal{H}$ such that $UT = M_z U$. [115] Throughout, $\mathcal{H}$ and $\mathcal{X}$ will be related in this way. Analogously with the definition of $\Sigma(\mathcal{H})$ we may define $\Sigma(\mathcal{X}, T)$ and obviously we then get

$$\Sigma(\mathcal{H}) = \Sigma(\mathcal{X}, T)$$

Moreover, for any $x \in \mathcal{X}$ and $(x_1, \ldots, x_n) = X \in \mathcal{X}^n$ such that

$$\text{Span}\{x_1, \ldots, x_n\} + \text{Ran}(T - \lambda_0) = \mathcal{X} \tag{33}$$

for some $\lambda_0 \in \mathbb{D}$, we may define meromorphic functions

$$c_1(x, X, \mathcal{X}, T, \cdot), \ldots, c_n(x, X, \mathcal{X}, T, \cdot),$$

by the equation

$$\sum_{i=1}^{n} c_i(x, X, \mathcal{X}, T, \cdot) x_i \in \text{Ran}(T - \lambda) \tag{34}$$

That this equation defines unique meromorphic functions follows from the following simple observations.

(i) The condition (33) is equivalent to

$$\det((Ux_1)(\lambda_0), \ldots, (Ux_n)(\lambda_0)) \not\equiv 0.$$

(ii) The c_i's are invariant under unitary transformations, i.e.

$$c_i(x, X, \mathcal{X}, T, \cdot) = c_i(Ux, UX, \mathcal{H}, M_z, \cdot),$$

where $UX = (Ux_1, \ldots, Ux_n) \in \mathcal{H}^n$.

(iii) In $\mathcal{H}$, the "new" $c_i's$ defined via (33) coincide with the "old" $c_i's$ that were defined via (8) and (11), i.e.

$$c_i(Ux, UX, \mathcal{H}, M_z, \cdot) = c_i(Ux, UX, \cdot),$$

and the "old" $c_i's$ are clearly meromorphic.

Corollary (5.2.18):[112] Let T be a contraction on some Hilbert space $\mathcal{X}$ such that $T^* \in B_n(\mathbb{D})$. Assume that there exists a $c > 0$ such that

$$\left\|(T-\lambda)\left(1-\bar{\lambda}T\right)^{-1}x\right\| \geq c\|x\|$$

for all $\lambda \in \mathbb{D}$ and all $x \in \mathcal{X}$. Then

$$\Theta(\mathcal{X}, T) = \Sigma(\mathcal{X}, T)$$

and moreover

(i) if $m\big(\Sigma(\mathcal{X}, T)\big) = 0$ then given any $k \in \mathbb{N} \cup \{\infty\}$ the exists a T-invariant subspace $\mathcal{M} \subset \mathcal{X}$ with $ind\, T|_{\mathcal{M}} = -k$;
(ii) if $m\big(\Sigma(\mathcal{X}, T)\big) > 0$ then $ind\, T|_{\mathcal{M}} \geq -n$ for all T-invarian subspaces $\mathcal{M} \subset \mathcal{X}$.

The evaluation map $f \to f(\lambda)$ from $\mathcal{H}$ onto $\mathbb{C}^n$ is continuous for all $\lambda \in \mathbb{D}$ and it is surjective for some $\lambda_0 \in \mathbb{D}$. (35)

$$\operatorname{ind}(M_z - \lambda) = -n \text{ for all } \lambda \in \mathbb{D} \tag{36}$$

Corollary (5.2.19):[112] Let $\mathcal{X}$ and T be such that Corollary (5.2.22) applies assume that $\mathcal{M}$ is a T-invariant subspace, then

$$\Sigma(\mathcal{M}, T|_{\mathcal{M}}) \supset_{a.e.} \Sigma(\mathcal{X}, T).$$

Proof: By [115] we may assume that $\mathcal{X}$ is a Hilbert space of $\mathbb{C}^n$-valued analytic functions and that $T = M_z$. We may also assume that $m\big(\Sigma(\mathcal{X}, M_z)\big) > 0$, because otherwise there is nothing to prove. Set $k = \operatorname{ind}\mathcal{M}$. By Theorem (5.2.20) we get that $k \leq n$ and that we may take $F = (f_1, \dots, f_n) \in \mathcal{X}^n$ such that $\det F(.) \not\equiv 0$ and $f_1, \dots, f_k \in \mathcal{M}$. It is not hard to see that

$$c_i(f, (f_1, \dots, f_k), \mathcal{M}, M_z|_{\mathcal{M}}, \cdot) = c_i(f, F, \mathcal{H}, M_z, \cdot)$$

for $1 \leq i \leq k$ and $f \in \mathcal{M}$. Thus

$$\Theta(\mathcal{M}, M_z|_{\mathcal{M}}) \supset_{a.e.} \Theta(\mathcal{X}, M_z)$$

so the result follows by Corollary (5.2.18).

Corollary (5.2.20):[112] Let $\mathcal{X}$ and T be such that Corollary (5.2.22) applies assume that $\mathcal{M}$ is a T-invariant subspace such that $\operatorname{ind} T|_{\mathcal{M}} = -n$. Then

$$\Theta(\mathcal{M}, T|_{\mathcal{M}}) = \Theta(\mathcal{X}, T)\, a.e.$$

Corollary (5.2.25):[112] Let $\mathcal{X}$ and T be such that Corollary (5.2.22) applies assume that

$$\mathrm{m}\big(\Sigma(\mathcal{X},\mathrm{T})\big) \neq 0$$

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two T-invariant subspaces with $\mathcal{M}_1 \subset \mathcal{M}_2$, then

$$\operatorname{ind}(T|_{\mathcal{M}_1}) \geq \operatorname{ind}(T|_{\mathcal{M}_2}).$$

Section (5.3): Invariant Subspaces and Fiber Dimension

For a given linear space $\mathcal{M}$ consisting of analytic, $\mathbb{C}^N$-valued functions $(N \in \mathbb{N})$ over a domain $\Omega \subseteq \mathbb{C}$, the fiber dimension of $\mathcal{M}$ is defined by

$$\operatorname{fd}(\mathcal{M}) = \sup_{\lambda\in\Omega} \dim \mathcal{M}(\lambda) \tag{37}$$

where the fiber space $\mathcal{M}(\lambda)$ at λ is given by

$$\mathcal{M}(\lambda) = \{f(\lambda): f \in \mathcal{M}\} \subseteq \mathbb{C}^N$$

A point λ in Ω is called a maximal point, or an m-point for short, for $\mathcal{M}$ if $\dim \mathcal{M}(\lambda) = \operatorname{fd}(\mathcal{M})$, and is called a degenerate point if $\dim \mathcal{M}(\lambda) < \operatorname{fd}(M)$. It is not hard to see that the collection of degenerate points forms a discrete subset in Ω whose Lebesgue area measure is 0. The set of m-points and degenerate points of $\mathcal{M}$ will be denoted by $\operatorname{mp}(\mathcal{M})$ and $\mathcal{Z}_{dg}(\mathcal{M})$, respectively. The fiber dimension has proved to be a fruitful tool to several problems in operator theory. To the notorious transitive algebra problem [119], to the cellular indecomposable property [120], to multi-variable Fredholm index [17], to Samuel multiplicity [121,122], to general structure of invariant subspaces [123].

We fix Ω to be an open, connected, and bounded subset in the complex plane $\mathbb{C}$. Moreover, for convenience, we assume $0 \in \Omega$. We also fix n, $N \in \mathbb{N}$. We denote by $\mathcal{A}_n(\Omega)$ the collection of analytic operators which are defined to be the adjoints of operators in the Cowen–Douglas class $B_n(\Omega^*)$ [124], where $\Omega^* = \{\bar{z}: z \in \Omega\}$. By well known constructions in operator theory [125,126], any $T \in \mathcal{A}_n(\Omega)$ can be represented as the coordinate multiplication operator M_z on a Hilbert space H satisfying the following:

(i) H consists of $\mathbb{C}^N$-valued analytic functions over the domain Ω;

(ii) The evaluation functional at $\lambda: f \in H \to f(\lambda) \in \mathbb{C}^N$ is a continuous map from H to $\mathbb{C}^N$ for each $\lambda \in \Omega$;

(iii) If $f \in H$, then so is zf, where z is the coordinate function; moreover, the multiplication operator $M_z - \lambda$ is bounded below for each $\lambda \in \Omega$;

(iv) H satisfies the condition $\operatorname{cod}(H) = \operatorname{fd}(H)$,where $\operatorname{cod}(H) = \dim(H \ominus zH)$.

Definition (5.3.1):[118] Let $\mathcal{M}$ be a linear space of $\mathbb{C}^N$-valued analytic functions over Ω invariant under multiplication by z. We say that $\mathcal{M}$ has the division property at $\lambda \in \Omega$, if for any $f \in \mathcal{M}$ vanishing at λ, there is a $g \in \mathcal{M}$ such that $f = (z - \lambda)g$.

Definition (5.3.2):[118] For an analytic operator $T \in B(K)$ acting on a Hilbert space K, we say that $U: K \to H$ is a CF representation of T if U is a unitary module map from K onto a Hilbert space H satisfying the above (i)–(iv). Here, by a module map we mean that the unitary operator U satisfies $UT = M_z U$.

Definition (5.3.3):[118] For an invariant subspace $\mathcal{M} \subseteq K$of an analytic operator $T \in B(K)$, the fiber dimension of $\mathcal{M}$ in K, denoted by $\mathrm{fd}(\mathcal{M})$, is defined by

$$\mathrm{fd}(\mathcal{M}) = \mathrm{fd}\big(U(\mathcal{M})\big)$$

for some CF representation U of T.

Lemma (5.3.4):[118] Let $\mathcal{M}$ be an M_z invariant subspace of a Hilbert space H satisfying (i)–(iv). The following are equivalent.

(i) $\mathcal{M}$ has the division property at one m-point.
(ii) $\mathcal{M}$ has the division property at all m-points.
(iii) $\mathrm{cod}(\mathcal{M}) = \dim(\mathcal{M} \ominus T\mathcal{M}) = \mathrm{fd}(M)$.

Proof: For an m-point $\lambda \in \Omega$, let E_λ be the evaluation functional from H to $\mathbb{C}^N$ restricted to $\mathcal{M}$. By the definition above, $\mathcal{M}$ has division property at λ iff $\ker E_\lambda = (z-\lambda)\mathcal{M}$. Observe that $(z-\lambda)\mathcal{M} \subseteq \ker E_\lambda$, $\ker E_\lambda = (z-\lambda)\mathcal{M}$ iff $\dim(\mathcal{M}/\ker E_\lambda) = \dim(\mathcal{M}/(z-\lambda)\mathcal{M})$. On the other hand, $\dim(\mathcal{M}/\ker E_\lambda) = \dim \mathcal{M}(\lambda)$ while $\dim(\mathcal{M}/(z-\lambda)\mathcal{M})$ does not depend on λ, from which the lemma follows immediately.

Lemma (5.3.5):[118] Given two Hilbert spaces H_1, H_2 of vector-valued analytic functions satisfying the above (i)–(iii) and an operator $\Phi: H_1 \to H_2$ satisfying $\Phi M_z = M_z \Phi$, if H_1satisfies (iv), then for any invariant subspace $\mathcal{M} \subseteq H_1$,

$$\mathrm{fd}(\mathcal{M}) \geq \mathrm{fd}\big(\Phi(\mathcal{M})\big).$$

So if Φ is invertible and H_2 also satisfies (iv), $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}\big(\Phi(\mathcal{M})\big)$.

Proof: Since degenerate points of a fixed subspace are contained in a subset of zero Lebesgue area measure, we can choose a common m-point λ for $\mathcal{M}, \Phi(\mathcal{M})$ and H_1, and we define a map $\Phi(\lambda): \mathcal{M}(\lambda) \to \big(\Phi(\mathcal{M})\big)(\lambda)$ by

$$\Phi(\lambda)\big(f(\lambda)\big) = \big(\Phi(f)\big)(\lambda),$$

for $f \in \mathcal{M}$. If we can show that this map is well-defined, then it is automatically linear and surjective, hence $\mathrm{fd}(\mathcal{M}) \geq \mathrm{fd}(\Phi(\mathcal{M}))$ follows immediately.

In fact, if $f(\lambda) = g(\lambda)$ for f, g in H_1, then since H_1satisfies (iv), there exists $h \in H_1$ such that $f - g = (z-\lambda)h$ (see Lemma (5.3.4)) hence

$$\big(\Phi(f-g)\big)(\lambda) = \Big(\Phi\big((z-\lambda)h\big)\Big)(\lambda) = \big((z-\lambda)\Phi(h)\big)(\lambda) = 0$$

This verifies that $\Phi(\lambda)$ is well-defined and we are done.

Definition (5.3.6):[118] Let $\mathcal{M}$ be an invariant subspace of an analytic operator T. $\mathcal{M}$ is called CF if its codimension $\mathrm{cod}(\mathcal{M}) = \dim(\mathcal{M} \ominus T\mathcal{M})$ is equal to its fiber dimension, i.e., $\mathrm{cod}(\mathcal{M}) = \mathrm{fd}(\mathcal{M})$.

For convenience we say that $\mathcal{M} = \{0\}$ is CF.

The CF property is clearly a generalization of the codimension-one property which holds for an invariant subspace $\mathcal{M} \subseteq H^2(\mathbb{D})$ of the Hardy space over the unit disc: $\mathrm{cod}(\mathcal{M}) = 1$. The first systematic investigation of the codimension-one property is Richter's thesis [3]. Examples of CF subspaces can be found in [112,119,121,17,94]. All invariant subspaces of Nevanlinna–Pick spaces [17] are CF.

Definition (5.3.7):[118] Let T be a Fredholm operator on a Hilbert space K and $\mathcal{M} \subseteq K$ be an invariant subspace of T. We define the fiber dimension of $\mathcal{M}$ at the origin by

$$\mathrm{fd}^{\mathrm{I}}(\mathcal{M}) = \lim_{\mathrm{k}\to\infty} \frac{\dim(P_k\mathcal{M})}{k},$$

where P_k is the orthogonal projection from K onto $K \ominus T^k K$.

Lemma (5.3.8):[118] The above limit exists and is an integer.

Proof: For any $k \geq 1$, let $E_k = T^{k-1}K \ominus T^k K$. Then it is sufficient to show that the limit

$$\lim_{\mathrm{k}\to\infty} \dim\left(P_{E_k}(\mathcal{M})\right)$$

exists and is an integer. Here P_{E_k} is the orthogonal projection from K onto E_k.

We first claim that $\{\dim E_k\}_k \geq 1$is a decreasing sequence, since the following natural map T_k induced by T

$$T^{k-1}K/T^k K \xrightarrow{T_k} T^k K/T^{k+1}K$$

is well defined and is surjective. In particular, it follows that $\lim_{k\to\infty} \dim E_k$ exists and is a finite integer. Next we apply the following elementary fact to $E_k = T^{k-1}K \ominus T^k K$.

Fact(5.3.9):[118] For any (closed, finite dimensional) vector space E in a Hilbert space L, and another closed subspace $M \subseteq L$, we have $\dim E = \dim(P_E(\mathcal{M})) + \dim(M^\perp \cap E)$.

Now with $E = E_k$ and $M = \mathcal{M}$, we have $\dim E_k = \dim(P_{E_k}(\mathcal{M})) + \dim(\mathcal{M}^\perp \cap E_k)$,

Note that $\mathcal{M}^\perp \cap E_k$ is just the collection of those vectors in $T^{k-1}K$ which are orthogonal to both $\mathcal{M}$ and $T^k K$. In terms of quotient, it is naturally isomorphic as vector spaces to

$$F_k = \frac{T^{k-1}K + \mathcal{M}}{T^k K + \mathcal{M}}.$$

Their dimensions form a decreasing sequence because the natural maps T_k' induced by T

$$\frac{T^{k-1}K+\mathcal{M}}{T^kK+\mathcal{M}} \xrightarrow{T_k'} \frac{T^kK+\mathcal{M}}{T^{k+1}K+\mathcal{M}}$$

are well defined and are surjective. In particular, it follows that $\lim_{k\to\infty} \dim(F_k)$ exists and is a finite integer. Now our desired limit follows from

$$\dim\left(P_{E_k}(\mathcal{M})\right) = \lim_{k\to\infty} E_k - \lim_{k\to\infty} \dim F_k$$

where both limits on the right side exist.[127,128,129].

Definition (5.3.10):[118] Let T be a Fredholm operator on a Hilbert space K and $\mathcal{M} \subseteq K$ be an invariant subspace of T. Define

$$\mathrm{fd}^{\mathrm{II}}(\mathcal{M}) = \chi(\widehat{\mathcal{M}}_0)$$

Now we will show that the two definitions $\mathrm{fd}^{\mathrm{I}}(\mathcal{M}), \mathrm{fd}^{\mathrm{II}}(\mathcal{M})$ agree with the original one via CF representation in the case of analytic operators. We shall use Serre's theorem [130] that the Euler characteristic $\chi(\widehat{\mathcal{M}}_0)$ is equal to the Samuel multiplicity $e(\widehat{\mathcal{M}}_0)$ with respect to the maximal ideal $m = m_0$ in $\mathcal{O}_0$.

Theorem (5.2.11):[118] Let T be an analytic operator in $B(K)$ and $\mathcal{M}$ be any invariant subspace of T, then $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}^{\mathrm{I}}(\mathcal{M}) = \mathrm{fd}^{\mathrm{II}}(\mathcal{M})$.

Theorem (5.3.12):[118] Let T be a Fredholm operator in $B(K)$ and $\mathcal{M}$ be any invariant subspace, then

$$\mathrm{fd}^{\mathrm{II}}(\mathcal{M}) + e(\mathcal{M}^{\perp}) = e(K).$$

Recall that for any bounded operator $A \in B(K)$ on a Hilbert space K such that $\dim(K/AK) < \infty$, the Samuel multiplicity of A is defined to be

$$e(A) = \lim_{k\to\infty} \frac{\dim(K/A^kK)}{k},$$

which always exists and is an integer[131]. In the sequel, we sometimes write $e(K)$ instead of $e(A)$ when a particular operator acting on K is specified. Here we assume that the natural operator acting on $\mathcal{M}^{\perp}$ is the compression of T onto $\mathcal{M}^{\perp}$.

Proof: Let $S = P_{\mathcal{M}^{\perp}} T|_{\mathcal{M}^{\perp}}$ be the compression of T onto $\mathcal{M}^{\perp}$. Then we have the Samuel multiplicities

$$e(\mathcal{M}^{\perp}) = \lim_{k\to\infty} \frac{\dim(\mathcal{M}^{\perp}/S^k\mathcal{M}^{\perp})}{k}$$

and

$$e(K) = \lim_{k\to\infty} \frac{\dim(K/T^kK)}{k}$$

We first show that for any Fredholm operator $T \in B(L)$ acting on a Hilbert space L, and for any invariant subspace $\mathcal{M} \subseteq L$, we have

$$\mathrm{fd}^{\mathrm{I}}(\mathcal{M}) + e(\mathcal{M}^{\perp}) = e(L) \tag{38}$$

This follows by considering several exact sequences. We start by

$$0 \to \mathcal{M} \to L \to \mathcal{M}^{\perp} \to 0.$$

Let I be the maximal ideal of the polynomial ring $\mathbb{C}[z]$ at the origin, and regard $\mathcal{M}, L$ and $\mathcal{M}^{\perp}$ as Hilbert modules with the natural module action, then

$$\frac{\mathcal{M}}{I^k\mathcal{M}} \to \frac{L}{I^kL} \to \frac{\mathcal{M}^{\perp}}{I^k\mathcal{M}^{\perp}} \to 0$$

is exact.

Complete it to get

$$0 \to \mathcal{M} + \frac{I^kL}{I^kL} \to \frac{L}{I^kL} \to \frac{\mathcal{M}^{\perp}}{I^k\mathcal{M}^{\perp}} \to 0$$

Now observe that $\dim\left(\mathcal{M} + \frac{I^kL}{I^kL}\right) = \dim(P_{L\ominus I^kL}\mathcal{M}) \ = \dim(P_{L\ominus T^kL}\mathcal{M})$ (which, divided by k, converges to $\mathrm{fd}^{\mathrm{I}}(\mathcal{M})$), combined with the definition of the Samuel multiplicity, we can complete the proof of (38).

Now we show that for any analytic operator $T \in B(K)$ and for any invariant subspace $\mathcal{M} \subseteq K$, we have

$$\mathrm{fd}(\mathcal{M}) = \mathrm{fd}^{\mathrm{I}}(\mathcal{M}) \tag{39}$$

Now we come to the main step in the proof and we show that for any Fredholm operator $T \in B(L)$ acting on a Hilbert space L, and for any invariant subspace $\mathcal{M} \subseteq L$, we have

$$\mathrm{fd}^{\mathrm{II}}(\mathcal{M}) + e(\mathcal{M}^{\perp}) = e(L) \tag{40}$$

Consider the quotient module Q corresponding to the submodule $\widehat{\mathcal{M}}_0$ in $\tilde{L}_0$, that is,

$$Q = \frac{\tilde{L}_0}{\widehat{\mathcal{M}}_0} = \frac{\mathcal{O}_0(L)}{\mathcal{O}_0(\mathcal{M})} + (z-T)\mathcal{O}_0(L)$$

By the additively of Samuel multiplicity for Noetherian modules over the ring $\mathcal{O}_0$ [132], which is a classical result in commutative algebra?

$$e(\widehat{\mathcal{M}}_0) + e(Q) = e(\tilde{L}_0)$$

By the identification of the Hilbert polynomials of a Hilbert module and its sheaf model in [133], we have $e(\tilde{L}_0) = e(L)$. So it suffices to show that $e(Q) = e(\mathcal{M}^{\perp})$. For this we will show that there exist natural, surjective module homomorphism's i, j such that

$$\frac{L}{I^k L} + \mathcal{M} \xrightarrow{i} \frac{Q}{m^k Q} \xrightarrow{j} \frac{\mathcal{M}^\perp}{I^k \mathcal{M}^\perp}$$

Then by[133]again, the natural module homomorphism

$$\frac{L}{I^k L} \to \frac{\tilde{L}_0}{m^k \tilde{L}_0} \cong \frac{\mathcal{O}_0(L)}{z^k \mathcal{O}_0(L)} + (z-T)\mathcal{O}_0(L)$$

given by sending $h \in H$ to the class represented by the constant function $f(z) = h$ is always an isomorphism. So it induces a natural surjective homomorphism

$$\frac{L}{I^k L} + \mathcal{M} \to \frac{\mathcal{O}_0(L)}{z^k \mathcal{O}_0(L)} + \mathcal{O}_0(\mathcal{M}) + (z-T)\mathcal{O}_0(L) \tag{41}$$

Since

$$\frac{Q}{m^k Q} \cong \frac{\mathcal{O}_0(L)}{z^k \mathcal{O}_0(L)} + \mathcal{O}_0(\mathcal{M}) + (z-T)\mathcal{O}_0(L) \tag{42}$$

so we obtain the first natural homomorphism i.

Next we consider another natural isomorphism established in [133]

$$\frac{\mathcal{M}^\perp}{I^k \mathcal{M}^\perp} \cong \frac{\mathcal{O}_0(\mathcal{M}^\perp)}{(z-S)\mathcal{O}_0(\mathcal{M}^\perp)} + z^k \mathcal{O}_0(\mathcal{M}^\perp),$$

where $S = P_{\mathcal{M}^\perp} T|_{\mathcal{M}^\perp}$. Since each element $x = \mathcal{O}_0(L)$ can be uniquely decomposed as $y + z$ for $y \in \mathcal{O}_0(\mathcal{M})$ and $z \in \mathcal{O}_0(\mathcal{M}^\perp)$, we have

$$\frac{\mathcal{M}^\perp}{I^k \mathcal{M}^\perp} \cong \mathcal{O}_0(\mathcal{M}^\perp) + \frac{\mathcal{O}_0(\mathcal{M})}{(z-S)\mathcal{O}_0(\mathcal{M}^\perp)} + z^k \mathcal{O}_0(\mathcal{M}^\perp) + \mathcal{O}_0(\mathcal{M}) \tag{43}$$

By comparing (42) and (43) we conclude that there exists a natural map from $\frac{Q}{m^k Q}$ to $\frac{\mathcal{M}^\perp}{I^k \mathcal{M}^\perp}$, by taking the quotient homomorphism, hence surjective, since

$$(z-S)\mathcal{O}_0(\mathcal{M}^\perp) + \mathcal{O}_0(\mathcal{M}) \supseteq (z-T)\mathcal{O}_0(L).$$

So we have the existence of a natural homomorphism $j \circ i \colon \frac{L}{I^k L} + \mathcal{M} \to \frac{\mathcal{M}^\perp}{I^k \mathcal{M}^\perp}$. Because such a natural homomorphism must be the identity, we can conclude that $e(Q) = e(\mathcal{M}^\perp)$ by the definition of e(·).

Definition (5.3.13):[118] Let T be an analytic operator in $B(K)$. An isometric module map U from K to H is called a CF embedding of T if H is a Hilbert space of vector-valued analytic functions satisfying (i)–(iv).

Again, by a module map we mean that U intertwines T on K and M_z on H by $UT = M_z U$.

Definition (5.3.14):[118] For an analytic operator $T \in B(K)$, its fiber dimension range fr(T) is defined to be the following subset of positive integers

$$\mathrm{fr}(T) = \{\mathrm{fd}(U(K)) : U \text{ is a CF embedding of } T\}.$$

By [126], we see that $\mathrm{cod}(K) = \dim(K \ominus TK) \in \mathrm{fr}(T)$. On the other hand, since $\mathrm{fd}(U(K)) \le \mathrm{cod}(U(K)) = \mathrm{cod}(K)$, we know that $\mathrm{fr}(T) \subseteq \{1, 2, \dots, \dim(K \ominus TK)\}$.

Hence a natural question is when a subset of $\mathbb{N}$ is the fiber dimension range of an analytic operator.

Conjecture(5.3.15):[118] There exists an analytic operator $T \in B(K)$ such that $1 \neq \mathrm{fr}(T)$.

Conjecture(5.3.16): Let $\mathfrak{S} = \{T|_{\mathcal{M}} : T \in \mathcal{A}_1 \text{ and } \mathcal{M} \text{ is invariant for} T\}$ and $\mathfrak{S}' = \{T \in \mathcal{A}_n \text{ for } n = 1, 2, \dots\}$, then $\mathfrak{S} \neq \mathfrak{S}'$.

Theorem (5.3.17):[118] For an analytic operator $T \in B(K)$, we have $1 \in \mathrm{fr}(T)$ if and only if there exists a Hilbert space H of scalar-valued analytic functions, satisfying (i)–(iv), and an invariant subspace $\mathcal{M} \subseteq H$ such that T is unitarily equivalent to $M_z|_{\mathcal{M}}$.

The collection of analytic operators $\mathfrak{S} = \{T : 1 \in \mathrm{fr}(T)\}$ is unitarily equivalent to the collection $\mathfrak{S}' = \{S|_{\mathcal{M}}\}$, where $S \in B(H)$ and $S \in \mathcal{A}_1(\Omega)$ for some Ω around the origin, and $\mathcal{M} \subseteq H$ is an invariant subspace. We mention that similar result holds analogously when the assumption $1 \in \mathrm{fr}(T)$ is replaced by $k \in \mathrm{fr}(T)$.

Proof : One direction is trivial. For the other direction, we assume $1 \in \mathrm{fr}(T)$. That is, $\mathrm{fd}(U(K)) = 1$ for some $U : K \to H_1$, where H_1 is a Hilbert space of vector-valued analytic functions over a domain Ω. By Lemma (5.3.33), we can find another CF subspace $H_2 \subseteq H_1$ containing $U(K)$ with $\dim(H_2 \ominus zH_2) = \mathrm{fd}(H_2) = 1$.

It remains to realize H_2 as a Hilbert space H of scalar-valued analytic functions. To this end, fix a nonzero element $h \in H_2$, then since $\mathrm{fd}(H_2) = 1$, an element $g \in H_2$ other than h can be written as $g = g'h$ for a scalar-valued analytic function g' over any domain Ω' that does not meet the zeros of h. Now we can take $H = \{g' : g = g'h \in H_2\}$ as a function space over Ω' equipped with the obvious Hilbert space structure via that on H_2 and let $\mathcal{M} = \{g' : g = g'h \in U(K)\}$.

Conjecture(5.3.18):[118] A subset $A \subseteq \mathbb{N}$ is equal to the fiber dimension range of an analytic operator if and only if A is a continuous block.

Here by a continuous block Awe mean that if $k < t$, and $k, t \in A$, then any integer between k and t is also in A. An illustrative example is $T = M_z|_{\mathcal{M}}$ for an invariant subspace $\mathcal{M} \subseteq L_a^2(\mathbb{D})$ of the Bergman space with codimension three, $\dim(\mathcal{M} \ominus z\mathcal{M}) = 3$. Then we clearly have $1, 3 \in \mathrm{fr}(T)$. Conjecture (5.3.18) says that we should have $2 \in \mathrm{fr}(T)$ also.

Lemma (5.3.19):[118] Let $T \in B(K)$ be any bounded operator and let $\lambda \in \rho_F(T)$ be any point in its Fredholm domain. Then there is a positive number $\epsilon > 0$ such that $\dim(K/(T - \mu)K) = e(T; \lambda)$ for any $0 < |\mu - \lambda| < \epsilon$ and where $e(T; \lambda)$ denotes the Samuel multiplicity of Tat λ.

Theorem (5.3.20):[118] For an invariant subspace $\mathcal{M}$ of an analytic operator, the additivity formula of Samuel multiplicity holds for $\mathcal{M}$ if and only if $\mathcal{M}$ is CF.

Proof: By fixing a CF representation, we may regard $\mathcal{M}$ as an invariant subspace of M_z on a Hilbert space H satisfying (i)–(iv) over a domain Ω.

If $\mathcal{M}$ is CF, then $\mathcal{M}$ has the division property at any m-point λ by Lemma (5.3.4), which implies that if $(z-\lambda)f \in \mathcal{M}$, then $f \in \mathcal{M}$. So $\ker(S_{\mathcal{M}}-\lambda)=\{0\}$ for such a λ. Hence,

$$\sigma_p(S_{\mathcal{M}}) \cap \Omega \subseteq Z_{dg}(\mathcal{M})$$

by the definition of m-points. Moreover, observe that $Z_{dg}(\mathcal{M})$ is a discrete subset of Ω. It follows that

$$\ker(S_{\mathcal{M}}-\lambda)=\{0\} \tag{44}$$

for almost all $\lambda \in \Omega$.

On the other hand, by the above lemma from [131], for each of $T=R_{\mathcal{M}}, S_{\mathcal{M}}$, and M_z, and $K=\mathcal{M}, \mathcal{M}^{\perp}$ and H, respectively,

$$e(T)=\dim(K/(T-\mu)K) \tag{45}$$

for almost all $\lambda \in \Omega$. To see this, we can first conclude, by a direct application of Lemma(5.3.19),the claim (45) for almost all λ in a small neighborhood. Then Lemma(5.3.19) also implies that the Samuel multiplicity $e(T;\lambda)$ is locally constant in any connected component of the Fredholm domain. Since we assume that Ω is connected, we can show claim (45) for almost all λ in any neighborhood in Ω. That is, we have (45) for almost all $\lambda \in \Omega$.

For almost all $\lambda \in \Omega$, both (44) and (45) hold. In particular, we can find at least one point for these two equalities to be true. Let us fix such a $\lambda_0 \in \Omega$.

Next we consider the short exact sequence

$$0 \to \mathcal{M} \to H \to \mathcal{M}^{\perp} \to 0$$

and the associated exact sequence for any $\lambda \in C$

$$0 \to \ker(S_{\mathcal{M}}-\lambda) \to \frac{\mathcal{M}}{(R_{\mathcal{M}}-\lambda)\mathcal{M}} \to \frac{H}{(z-\lambda)H} \to \frac{\mathcal{M}^{\perp}}{(S_{\mathcal{M}}-\lambda)\mathcal{M}^{\perp}} \to 0 \tag{46}$$

which can be obtained by considering the so-called snake lemma [132].

Now choose $\lambda=\lambda_0$ in (46) and count the dimensions of all terms, we obtain the Samuel additivity formula.

If $\mathcal{M}$ is not CF, then still by Lemma(5.3.4), it is not divisible at any m-point λ. So there exists a nonzero vector $f \in H$ such that $(z-\lambda)f \in \mathcal{M}$ but $f \notin \mathcal{M}$, which implies that $P_{\mathcal{M}^{\perp}}f \in \ker(S_{\mathcal{M}}-\lambda)$. We conclude that $\ker(S_{\mathcal{M}}-\lambda)$ is non-zero almost everywhere. Pick any $\lambda_0 \in \Omega$ such that (1) $\ker(S_{\mathcal{M}}-\lambda_0)$ is non-zero, and (38) (45) holds for all $R_{\mathcal{M}}, S_{\mathcal{M}}$, and M_z. Now we count dimensions in (46) again, it follows that

$$e(R_{\mathcal{M}})+e(S_{\mathcal{M}})-e(M_z)=\dim\ker(S_{\mathcal{M}}-\lambda_0) \geq 1$$

So the Samuel additively formula fails.

Corollary (5.3.21):[118] Let $\mathcal{M}$ be an invariant subspace of an analytic operator. Then the inter-section $\sigma_p(S_{\mathcal{M}}) \cap \Omega$ either contains the whole domain Ω or is contained in a discrete subset. In the former case, $\mathcal{M}$ is not CF; in the latter, $\mathcal{M}$ is CF.

Proof: By the proof of the above theorem, $\ker(S_{\mathcal{M}} - \lambda) = 0$ if and only if $\mathcal{M}$ has division property at λ. If $\mathcal{M}$ is CF, then $\mathcal{M}$ has division property at all λ except for a discrete subset, which implies $\ker(S_{\mathcal{M}} - \lambda) = 0$ for all λ except for a discrete subset. If $\mathcal{M}$ is not CF, then $\mathcal{M}$ does not has division property at λ when λ is an m-point. On the other hand, by considering the Fredholm index, M does not have division property at any degenerate point, so $\ker(S_{\mathcal{M}} - \lambda) \neq 0$ for every λ.

We contains three results (Theorem (5.3.23), Theorem (5.3.32) and Theorem (5.3.30)) related to the formula [120]

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2). \quad (47)$$

Lemma (5.3.22):[118] Let $\mathcal{M}_1$ and $\mathcal{M}_2$be two linear subspaces in a linear space of vector-valued analytic functions, then

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) \geq \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2). \quad (48)$$

Proof: Let $\lambda \in \Omega$ be a common m-point of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1 \cap \mathcal{M}_2$, and $\mathcal{M}_1 \vee \mathcal{M}_2$. Set

$$E = \mathcal{M}_1(\lambda) \cap \mathcal{M}_2(\lambda), \;\; E' = (\mathcal{M}_1 \cap \mathcal{M}_2)(\lambda), \quad E_i = \mathcal{M}_i(\lambda) \ominus E, \;\; i = 1,2.$$

Then $\dim \mathcal{M}_i(\lambda) = \dim E_i + \dim E$, $i = 1,2$. hence

$$\begin{aligned} \mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) &= \dim E_1 + \dim E_2 + 2\dim E \\ &\geq \dim E_1 + \dim E_2 + \dim E + \dim E' \\ &= \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2), \end{aligned}$$

where the inequality follows from the fact that $E' \subseteq E$ and the last equality follows from

$$(\mathcal{M}_1 \vee \mathcal{M}_2)(\lambda) = \mathcal{M}_1(\lambda) \vee \mathcal{M}_2(\lambda).$$

Theorem (5.3.23):[118] Let $H(k)$be a functional Hilbert space over the domain Ω, determined by a complete Nevanlinna–Pick reproducing kernel k. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two invariant subspaces of $H(k) \otimes \mathbb{C}^N, N \in \mathbb{N}$. Then

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2). \quad (49)$$

Now a natural question one may ask is the corresponding problem for codimension. Along this direction Chailos [108] proved

$$\mathrm{cod}(\mathcal{M}_1) + \mathrm{cod}(\mathcal{M}_2) \geq \mathrm{cod}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{cod}(\mathcal{M}_1 \cap \mathcal{M}_2). \quad (50)$$

under quite general conditions.

Lemma (5.3.24):[118] Given a family of linear subspaces $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ in a linear space of vector-valued analytic functions H satisfying (i)–(iv), there exists a finite subset S_0 of Γ such that

$$\mathrm{fd}\left(\bigvee_{\gamma\in S_0}\mathcal{M}_\gamma\right)=\mathrm{fd}\left(\bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma\right)$$

Proof: Let $\mathcal{M}=\bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma$ and let $\mathcal{L}$ denote the space of all finite linear combinations of functions in $\mathcal{M}_\gamma,\gamma\in\Gamma$. Then $\bar{\mathcal{L}}=\mathcal{M}$, and for each $\lambda\in\Omega$,

$$\bar{\mathcal{L}}(\lambda)=\mathcal{L}(\lambda)=\mathcal{M}(\lambda) \tag{51}$$

Assume that $\mathrm{fd}(\mathcal{M})=d$ and let λ_0 be an m-point of $\mathcal{M}$. Moreover, let

$$d'=\sup\dim\left(\bigvee_{\gamma\in S}\mathcal{M}_\gamma\right)(\lambda_0),$$

where the supremum is taken over all finite subsets S of Γ. Note that d' is always achieved. So assume that it is achieved at some finite subset $S=S_0$ of Γ.

If $d'<d$, then there exists a vector $v\in\mathcal{M}(\lambda_0)$, which is not in $\left(\bigvee_{\gamma\in S}\mathcal{M}_\gamma\right)(\lambda_0)$. Then by (54), $v\in\left(\bigvee_{\gamma\in S}\mathcal{M}_\gamma\right)(\lambda_0)$ for another finite subset $S_1\subseteq\Gamma$. In this case,

$$\dim\left(\bigvee_{\gamma\in S_0\vee S_1}\mathcal{M}_\gamma\right)(\lambda_0)>d',$$

contradicting the definition of d'.

Lemma (5.3.25):[118] For a family of CF subspaces $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ of an analytic operator, if the span of any finite sub-family is CF, then the span of the whole family $\bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma$ is CF.

Proof: Let $\mathcal{M}=\bigvee_{\gamma\in\Gamma}\mathcal{M}_\gamma$. Now fix a CF representation. By Lemma (5.3.23), we can find a finite subset S_0 of Γ such that

$$\mathrm{fd}(\mathcal{M}')=\mathrm{fd}(\mathcal{M}), \tag{52}$$

where $\mathcal{M}'=\bigvee_{\gamma\in S_0}\mathcal{M}_\gamma$. This implies that

$$\mathrm{mp}(\mathcal{M}')\subseteq\mathrm{mp}(\mathcal{M}) \tag{53}$$

Through similar arguments, the inclusion (53) can be strengthened as the following

Claim(5.3.26):[118] If $\mathcal{N}$ is a subspace between $\mathcal{M}'$ and $\mathcal{M}$, then

$$\mathrm{mp}(\mathcal{M}')\subseteq\mathrm{mp}(\mathcal{N})\subseteq\mathrm{mp}(\mathcal{M}) \tag{54}$$

Let $\mathcal{L}$ denote the space of all finite linear combinations of functions in $\mathcal{M}_\gamma$, $\gamma \in \Gamma$. Then $\bar{\mathcal{L}} = \mathcal{M}$. Let $\lambda \in \Omega$ be an m-point of $\mathcal{M}'$, then λ is also an m-point of $\mathcal{M}$ by (53).

Suppose that $(z-\lambda)h \in \mathcal{L}$ for some h, it suffices to show that $h \in \mathcal{M}$. From the definition of $\mathcal{L}$, there exists a finite subset S_1 of Γ such that $(z-\lambda)h \in \mathcal{M}'' = \bigvee_{\gamma \in S_1} \mathcal{M}_\gamma$. Clearly,

$$(z-\lambda)h \in \mathcal{M}' \vee \mathcal{M}'' = \bigvee_{\gamma \in S_0 \vee S_1} \mathcal{M}_\gamma$$

By assumption, we know that $\mathcal{M}' \vee \mathcal{M}''$ is CF. So $\mathcal{M}' \vee \mathcal{M}''$ is divisible at λ since λ is also an m-point of $\mathcal{M}' \vee \mathcal{M}''$ by the above claim. This implies that

$$h \in \mathcal{M}' \vee \mathcal{M}'' \subseteq \mathcal{M}$$

as desired.

Lemma (5.3.27):[118] Let $\mathcal{M}$ be an invariant subspace of a Hilbert space H satisfying (i)–(iv) and $\lambda \in \Omega$ be any m-point of $\mathcal{M}$. Then the following are equivalent.

(i) $\mathcal{M}$ is CF.
(ii) There is an invariant linear manifold $\mathcal{L} \subseteq \mathcal{M}$ with $\bar{\mathcal{L}} = \mathcal{M}$, such that if $(z-\lambda)h \in \mathcal{L}$ for some $h \in H$, then $h \in \mathcal{M}$.

Claim(5.3.28):[118] There exists a sequence $\{h_n\}$ in $\mathcal{M}$ such that $(z-\lambda)h_n \to (z-\lambda)h$. Since $\mathcal{L}$ is dense, we take a sequence $\{f_n\}$in $\mathcal{L}$ with $f_n \to (z-\lambda)h$.

Subclaim(5.3.29):[118] There exists a sequence $\{k_n\}$ in $\mathcal{L}$ such that $k_n(\lambda) = 0$ and $k_n - f_n \to 0$.

Proof: Suppose $\mathrm{fd}(\mathcal{M})$ $(= \mathrm{fd}(\mathcal{L}))$ $= s$, we take $g_1, \ldots, g_s$ in $\mathcal{L}$ such that $\{g_1(\lambda), \ldots, g_s(\lambda)\}$ form a base of $\mathcal{M}(\lambda)$. Then for any fixed n, there exists $\{c_{ni}\}_{i=1}^s$ such that $f_n(\lambda) = \sum_{i=1}^s c_{ni} g_i(\lambda)$. Note that $f_n(\lambda) \to 0$ by the choice of $\{f_n\}$, we have $c_{ni} \to 0$ as $n \to \infty$ for every $1 \le i \le s$. Finally, set $k_n = f_n - \sum_{i=1}^s c_{ni} g_i(\lambda)$, then $\{k_n\}$ satisfies all requirements.

Theorem (5.3.30):[118] Let $\mathcal{M}_1, \mathcal{M}_2$ be two CF subspaces of an analytic operator. if

$$\mathrm{fd}(\mathcal{M}_1) + \mathrm{fd}(\mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2), \qquad (55)$$

then the span $\mathcal{M}_1 \vee \mathcal{M}_2$ is CF.

Proof: Assume $\mathrm{fd}(\mathcal{M}_1 \cap \mathcal{M}_2) = n$. Let $\lambda \in \Omega$ be a common m-point of $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_1 \cap \mathcal{M}_2, \mathcal{M}_1 \vee \mathcal{M}_2$, and H with respect to a fixed CF representation of T.

Under our assumption on fiber dimensions, it is easy to see, by checking the proof of Lemma (5.3.18), that $\mathcal{M}_1(\lambda) \cap \mathcal{M}_2(\lambda) = (\mathcal{M}_1 \cap \mathcal{M}_2)(\lambda)$, which enables us to choose $f_1, \ldots, f_n \in \mathcal{M}_1 \cap \mathcal{M}_2$, such that

$$E' = \mathrm{span}\{f_1(\lambda), \ldots, f_n(\lambda)\}.$$

For further discussion, we take $\{f_1^i, \ldots, f_{n_i - n}^i \in \mathcal{M}_i, i = 1,2\}$, such that

$$E_i = \text{span}\{f_1^i(\lambda), \dots, f_{n_i-n}^i(\lambda)\}.$$

Then

$$\mathcal{M}_i(\lambda) = \text{span}\{f_1^i(\lambda), \dots, f_{n_i-n}^i(\lambda), f_1(\lambda), \dots, f_n(\lambda)\}$$

when $n_i = \text{fd}(\mathcal{M}_i)$ and $f_1^i, \dots, f_{n_i-n}^i, f_1, \dots, f_n$ are linearly independent in $\mathcal{M}_i$, $i = 1,2$. Let

$$k_1^i = P_\lambda^i(f_1^i), \dots, k_{n_i-n}^i = P_\lambda^i(f_{n_i-n}^i), l_1^i = P_\lambda^i(f_1), \dots, l_n^i = P_\lambda^i(f_n),$$

where P_λ^i is the projection from $\mathcal{M}_i$ onto $((z-\lambda)\mathcal{M}_i)^\perp$. Then

$$k_1^i(\lambda) = f_1^i(\lambda), \dots, k_{n_i-n}^i(\lambda) = f_{n_i-n}^i(\lambda), l_1^i(\lambda) = f_1(\lambda), \dots, l_n^i(\lambda) = f_n(\lambda)$$

Hence

$$k_1^i(\lambda), \dots, k_{n_i-n}^i(\lambda), l_1^i(\lambda), \dots, l_n^i(\lambda)$$

are linearly independent, and so are

$$k_1^i, \dots, k_{n_i-n}^i, l_1^i, \dots, l_n^i$$

Moreover, they form the base for $\mathcal{M}_i \ominus (z-\lambda)\mathcal{M}_i$ since $\mathcal{M}_i$ is CF.

Claim(5.3.31):[118] $\mathcal{M}_1 + \mathcal{M}_2$ has the division property at λ.

Indeed, if $(z-\lambda)h \in \mathcal{M}_1 + \mathcal{M}_2$, then we write

$$(z-\lambda)h = g_1 + g_2, g_i \in \mathcal{M}_i, i = 1,2.$$

Then there exist constants $c_1^i, \dots, c_{n_i-n}^i, d_1^i, \dots, d_n^i$ and $h_i \in \mathcal{M}_i$ such that

$$g_i = c_1^i k_1^i + \cdots + c_{n_i-n}^i k_{n_i-n}^i + d_1^i l_1^i + \cdots + d_n^i l_n^i + (z-\lambda)h_i, i = 1,2.$$

Obviously, $g_1(\lambda) + g_2(\lambda) = 0$. Meanwhile,

$$k_1^i(\lambda), \dots, k_{n_i-n}^i(\lambda), \quad i = 1,2, \quad l_1^1(\lambda) = l_1^2(\lambda), \dots, l_n^1(\lambda) = l_n^2(\lambda)$$

are linearly independent, so

$$c_1^i = 0, \dots . c_{n_i-n}^i = 0, d_1^1 + d_1^2 = 0, \dots, d_n^1 + d_n^2 = 0$$

By the construction of $l_1^i, l_2^i, \dots, l_n^i, i = 1,2$, there exist functions $v_1^i, v_2^i, \dots, v_n^i \in \mathcal{M}_i$ such that

$$l_j^1 + (z-\lambda)v_j^1 = f_j = l_j^2 + (z-\lambda)v_j^2, \qquad j = 1,2, \dots, n$$

Therefore,

$$l_j^i = f_j - (z-\lambda)v_j^i, \quad i = 1,2; \; j = 1,2,\dots,n.$$

Hence

$$\begin{aligned}(z-\lambda)h &= g_1 + g_2 \\ &= d_1^1(l_1^1 - l_1^2) + \cdots + d_n^1(l_n^1 - l_n^2) + (z-\lambda)(h_1 + h_2) \\ &= d_1^1(z-\lambda)(v_1^2 - v_1^1) + \cdots + d_n^1(z-\lambda)(v_n^2 - v_n^1) + (z-\lambda)(h_1 + h_2)\end{aligned}$$

which implies that $h = d_1^1(v_1^2 - v_1^1) + \cdots + d_n^1(v_n^2 - v_n^1) + h_1 + h_2 \in \mathcal{M}_1 + \mathcal{M}_2$, as desired.

Theorem (5.3.32):[118] For a given analytic operator, if any two of its invariant subspaces $\mathcal{M}_1, \mathcal{M}_2$ satisfy

$$\text{fd}(\mathcal{M}_1) + \text{fd}(\mathcal{M}_2) = \text{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) + \text{fd}(\mathcal{M}_1 \cap \mathcal{M}_2). \tag{56}$$

then any two of its invariant subspaces $\mathcal{N}_1, \mathcal{N}_2$ also satisfy

$$\text{cod}(\mathcal{N}_1) + \text{cod}(\mathcal{N}_2) \geq \text{cod}(\mathcal{N}_1 \vee \mathcal{N}_2) + \text{cod}(\mathcal{N}_1 \cap \mathcal{N}_2). \tag{57}$$

Proof: We will indeed prove that every invariant subspace is CF. We first observe that it is not hard to show that every invariant subspace generated by one element is CF and we leave out the proof here. Then, equipped with Theorem (5.3.30) whose proof is given a little later, we can use an easy induction argument to show that any finitely generated subspace is CF. Again, the details are skipped. Next, Lemma (5.3.25) means that every invariant subspace is CF, hence (56) implies (57).

Definition (5.3.33):[118] Let $\mathcal{M}_1, \mathcal{M}_2$ be two invariant subspaces of an analytic operator. We say that $\mathcal{M}_2$ is absorbed by $\mathcal{M}_1$, denoted by $\mathcal{M}_2 \prec \mathcal{M}_1$, if

$$\text{cod}(\mathcal{M}_1 \vee \mathcal{M}_2) = \text{fd}(\mathcal{M}_1). \tag{58}$$

Note that in case $\mathcal{M}_2 \prec \mathcal{M}_1, \mathcal{M}_1 \vee \mathcal{M}_2$ is necessarily CF since

$$\text{fd}(\mathcal{M}_1) = \text{cod}(\mathcal{M}_1 \vee \mathcal{M}_2) \geq \text{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) \geq \text{fd}(\mathcal{M}_1).$$

So

$$\text{cod}(\mathcal{M}_1 \vee \mathcal{M}_2) = \text{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) = \text{fd}(\mathcal{M}_1). \tag{59}$$

Lemma (5.3.34):[118] Let $\mathcal{M}_2, \mathcal{M}_1$ be two CF subspaces of an analytic operator and $\lambda \in \Omega$ be a common m-point of $\mathcal{M}_2 \prec \mathcal{M}_1$ and $\mathcal{M}_1 \vee \mathcal{M}_2$ with respect to a fixed CF representation. The following are equivalent.

(i) $\mathcal{M}_2 \prec \mathcal{M}_1$.

(ii) $\mathcal{M}_2(\lambda) \subseteq \mathcal{M}_1(\lambda)$, and for any $v \in \mathcal{M}_2$, there are sequences $\{g_n^i\}_{n=1}^{\infty}$ in $\mathcal{M}_i$ such that $g_n^i(\lambda) = v, i = 1,2$, and $\|g_n^1 - g_n^2\| \to 0$ as $n \to \infty$.

Proof: (i) ⇒(ii). If $\mathcal{M}_2 \prec \mathcal{M}_1$, then $\mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1)$ hence $\mathcal{M}_2(\lambda) \subseteq \mathcal{M}_1(\lambda)$. Now for any $v \in \mathcal{M}_2(\lambda)$, we can find $f_i \in \mathcal{M}_i, i = 1,2,$ such that $f_i(\lambda) = v$. Since $\mathcal{M}_1 \vee \mathcal{M}_2$ is CF, $f_1 - f_2 = (z-\lambda)h$ for some $h \in \mathcal{M}_1 \vee \mathcal{M}_2$ by the division property. Now take $h_n^i \in \mathcal{M}_i, i = 1,2,$ such that $h_n^1 - h_n^2 \to h$. Then it is easy to check that $g_n^i = f_i - (z-\lambda)h_n^i, i = 1,2,$ satisfy the requirement of (ii).

(ii) ⇒(i). Since $\mathcal{M}_2(\lambda) \subseteq \mathcal{M}_1(\lambda)$, one has $\mathrm{fd}(\mathcal{M}_1 \vee \mathcal{M}_2) = \mathrm{fd}(\mathcal{M}_1)$. So it remains to show that $\mathcal{M}_1 \vee \mathcal{M}_2$ is CF.

Suppose that

$$(z-\lambda)h = f_1 - f_2 \in \mathcal{M}_1 + \mathcal{M}_2$$

for some $h \in H$ and $f_i \in \mathcal{M}_i, i = 1,2$. We need only to show that $h \in \mathcal{M}_1 \vee \mathcal{M}_2$. Now $f_1(\lambda) = f_2(\lambda) = v \in \mathcal{M}_2(\lambda)$, choose g_n^i as in condition (ii) with $g_n^i(\lambda) = v$, then

$$(z-\lambda)h = (f_1 - g_n^1) - (f_2 - g_n^2) + (g_n^1 - g_n^2).$$

By the CF property of $\mathcal{M}_1$ and $\mathcal{M}_2$ we have

$$f_i - g_n^i = (z-\lambda)h_n^i \quad \text{for some } h_n^i \in \mathcal{M}_i, i = 1,2.$$

Hence

$$(z-\lambda)h = (z-\lambda)h_n^1 - (z-\lambda)h_n^2 + (g_n^1 - g_n^2). \tag{60}$$

Note that $M_z - \lambda$ is bounded below and $g_n^1 - g_n^2 \to 0$, as $n \to \infty$. Multiplying $\frac{1}{z-\lambda}$ to both sides of Eq.(60), one has that $h_n^1 - h_n^2 \to h$ so $h \in \mathcal{M}_1 \vee \mathcal{M}_2$.

Lemma (5.3.35):[118] Let $\mathcal{M}, \mathcal{N}, \mathcal{L}$ be CF subspaces of an analytic operator such that $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(\mathcal{N})$ and $\mathcal{M} \subseteq \mathcal{N}$. If $\mathcal{M} \prec \mathcal{L}$, then $\mathcal{N} \prec \mathcal{L}$.

Theorem (5.3.36):[118] Given a family of CF invariant subspaces $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$ and another CF sub-space $\mathcal{N}$ of an analytic operator, if each $\mathcal{M}_\gamma$ is absorbed by $\mathcal{N}$, then so is the span $\bigvee_{\gamma\in\Gamma} \mathcal{M}_\gamma$.

In particular, if in a family of CF subspaces $\{\mathcal{M}_\gamma\}_{\gamma\in\Gamma}$, there exists $\mathcal{M}_{\gamma_0}$ such that every $\mathcal{M}_\gamma$ other than $\mathcal{M}_{\gamma_0}$ is absorbed by $\mathcal{M}_{\gamma_0}$, then $\bigvee_{\gamma\in\Gamma} \mathcal{M}_\gamma$ is also CF.

Proof: We first treat the case that Γ is finite, then the general case.

STEP I: Γ is a finite set. Without loss of generality, assume that $\Gamma = \{1, 2, \ldots, n\} (n \in \mathbb{N})$. We shall show by induction that $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_i \prec \mathcal{N}$, $i = 1, \ldots, n$. Note that the case $i = 1$ is trivial.

Now assume that $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_{i-1}$ is absorbed by $\mathcal{N}$, which implies that

$$\mathcal{N}_{i-1} = \mathcal{N} \vee \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_{i-1}$$

is CF with

$$\mathrm{cod}(\mathcal{N}_{i-1}) = \mathrm{fd}(\mathcal{N}_{i-1}) = \mathrm{fd}(\mathcal{N})$$

By the definition of absorbance, it is easy to check that $\mathcal{N} \prec \mathcal{N}_{i-1}$. Then by (59) we have

$$\mathrm{fd}(\mathcal{N} \vee \mathcal{M}_i) = \mathrm{fd}(\mathcal{N})$$

since $\mathcal{M}_i \prec \mathcal{N}, i = 1, \ldots, n$. Next by lemma (5.3.35), if $\mathcal{N}$ is absorbed by $\mathcal{N}_{i-1}$, then so is $\mathcal{N} \vee \mathcal{M}_i$. That is, $\mathcal{N} \vee \mathcal{M}_i \prec \mathcal{N}_{i-1}$, which implies that

$$\mathrm{cod}(\mathcal{N} \vee \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_i) = \mathrm{fd}(\mathcal{N}_{i-1}) = \mathrm{fd}(\mathcal{N}).$$

Hence $\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_i \prec \mathcal{N}$ by definition.

STEP II: Γ is a general index set. Let

$$\mathcal{N}_\gamma = \mathcal{M}_\gamma \vee \mathcal{N}, \quad \gamma \in \Gamma$$

Then each $\mathcal{N}_\gamma$ is CF since $\mathcal{M}_\gamma \prec N$. The conclusion of STEP I implies that the span of any finite sub-family $\{\mathcal{N}_\gamma\}_{\gamma\in\Gamma}$ is CF. So the whole span $\bigvee_{\gamma\in\Gamma} \mathcal{N}_\gamma$ is also CF by Lemma (5.3.27).

By lemma (5.3.25), we can take a finite subset S_0 of Γ such that

$$\mathrm{fd}\left(\bigvee_{\gamma\in S_0} \mathcal{N}_\gamma\right) = \mathrm{fd}\left(\bigvee_{\gamma\in\Gamma} \mathcal{N}_\gamma\right) \tag{61}$$

By STEP I, we can check that

$$\bigvee_{\gamma\in S_0} \mathcal{N}_\gamma \prec \mathcal{N}.$$

By the above (61) and Lemma (5.3.35), we can extend the absorbance from S_0 to Γ,

$$\bigvee_{\gamma\in\Gamma} \mathcal{N}_\gamma \prec \mathcal{N},$$

which, by the definition of absorbance, implies that

$$\bigvee_{\gamma\in S_0} \mathcal{M}_\gamma \prec \mathcal{N}.$$

as desired.[3].

Theorem (5.3.37):[118] Let $\mathcal{M}$ be an invariant subspace of an analytic operator on a Hilbert space H, then:

(i) There exists a unique, smallest invariant CF subspace $E(\mathcal{M})$ containing $\mathcal{M}$;

(ii) $E(\mathcal{M})$ preserves the fiber dimension of $\mathcal{M}$,

$$\mathrm{fd}(\mathrm{E}(\mathcal{M})) = \mathrm{fd}(\mathcal{M});$$

(iii) The interior approximate spectrum is preserved in the following sense:
$$\sigma_{ap}(S^*_{\mathcal{M}}) \cap \Omega^* = \sigma_{ap}\left(S^*_{E(\mathcal{M})}\right) \cap \Omega^*,$$
where $S_{\mathcal{M}}$ is the compression of T to $\mathcal{M}^{\perp}$ and Ω^* is the conjugate set for Ω. Here Ω is the underlying domain for the analytic operator.

Corollary (5.3.38):[118] For any analytic operator T, the collection of CF invariant subspaces for T forms a complete lattice with respect to intersection and enveloping.

Claim (5.3.39):[118] The intersection of any family of CF invariant subspaces is still CF.

Lemma (5.3.40):[118] For any family of CF subspaces $I_i, i \in \mathcal{F}$, let $I = \bigcap_{i\in\mathcal{F}} I_i$. If there exists one common m-point for I and all I_i, then I is CF.

Lemma (5.3.41):[118] For any family of CF subspaces $\{I_i\}_{i\in\mathcal{F}}$, let $I = \bigcap_{i\in\mathcal{F}} I_i$. If $\mathrm{fd}(I_i) = \mathrm{fd}(I)$ for any $i \in \mathcal{F}$, then I is CF.

Lemma (5.3.42):[118] For any invariant subspace $\mathcal{M}$ of an analytic operator there is an invariant CF subspace $\mathcal{M}'$ containing $\mathcal{M}$ with $\mathrm{fd}(\mathcal{M}') = \mathrm{fd}(\mathcal{M})$.

Proof: We will use an iteration algorithm to construct the CF subspace $\mathcal{M}'$ with desired properties. To this end, it suffices, without loss of generality, to regard $\mathcal{M}$ as an invariant subspace of a Hilbert space H satisfying (i)–(iv) via a fixed CF representation.

Take an arbitrary m-point $\lambda_0 \in \Omega$ of $\mathcal{M}$. Let

$$\mathcal{M}_1 = \{f \in H: (z-\lambda_0)f \in \mathcal{M}\}$$

Then it is easy to check that $\mathcal{M}_1$ is closed, hence is an invariant subspace containing $\mathcal{M}$. Note that if $\mathcal{M}$ is CF, then $\mathcal{M}_1 = \mathcal{M}$ by Lemma (5.3.4), and we can stop our algorithm. Moreover, it is easy to check, by definition of $\mathcal{M}_1$, that for any point λ other than λ_0,

$$\mathcal{M}(\lambda) = \mathcal{M}_1(\lambda).$$

Therefore $\mathrm{fd}(\mathcal{M}_1) = \mathrm{fd}(\mathcal{M})$, and by basic properties of fiber dimension that λ_0 is actually a common m-point for $\mathcal{M}$ and $\mathcal{M}_1$ (hence $\mathcal{M}(\lambda_0) = \mathcal{M}_1(\lambda_0)$. as well).

Inductively, we can construct an increasing sequence of invariant subspaces by letting

$$\mathcal{M}_n = \{f \in H: (z-\lambda_0)f \in \mathcal{M}_{n-1}\}$$

then we clearly have

$$\mathcal{M} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \cdots \subseteq \mathcal{M}_{\mathrm{n}} \ldots.$$

Each $\mathcal{M}_n$ is a closed invariant subspace, with two further properties

(i) $\mathrm{fd}(\mathcal{M}_n) = \mathrm{fd}(\mathcal{M})$.
(ii) λ_0 is an m-point for $\mathcal{M}_n$.

If $\mathcal{M}_n$ is CF for some n, we set $\mathcal{M}' = \mathcal{M}$ and stop the algorithm. Otherwise, we set

$$\mathcal{M}' = \bigvee_{n=1}^{\infty} \{\mathcal{M}_n\}$$

Here $\bigvee$ denotes the closed span. In the latter case, a moment thought shows that $\mathrm{fd}(\mathcal{M}') = \mathrm{fd}(\mathcal{M})$ and since $\mathcal{M} \subset \mathcal{M}'$, λ_0 is a common m-point of $\mathcal{M}$ and $\mathcal{M}'$.

Next we show that $\mathcal{M}'$ is CF in order to finish the proof of Lemma (5.3.42). Observe that $\mathcal{M}'$ admits a dense linear manifold $\mathcal{L}$ consisting of finite linear combinations of functions in $\mathcal{M}_n$. By Lemma (5.3.27), it suffices to show that whenever $(z-\lambda_0)h \in \mathcal{L}, h \in \mathcal{M}'$. In fact, suppose that $(z-\lambda_0)h \in \mathcal{L}$ for some $h \in H$, then there exists an n_0 such that $(z-\lambda_0)h \in \mathcal{M}_{n_0}$, hence $h \in \mathcal{M}_{n_0+1} \subseteq \mathcal{M}'$ by the construction of $\mathcal{M}_{n_0+1}$, so $\mathcal{M}'$ is CF.

Claim (5.3.43):[118] For any invariant subspace $\mathcal{N}$,

$$\sigma_{ap}(S_{\mathcal{N}}^*) \cap \Omega^* = \{\lambda \in \Omega : \dim \mathcal{N}(\lambda) < \mathrm{fd}(H)\}^*.$$

Given Claim (5.3.43), we distinguish two cases. If $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(E(\mathcal{M})) < \mathrm{fd}(H)$, it follows by Claim (5.3.43) that $\sigma_{ap}(S_{\mathcal{M}}^*) \cap \Omega^* = \sigma_{ap}\left(S_{E(\mathcal{M})}^*\right) \cap \Omega^* = \Omega^*$. If $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(E(\mathcal{M})) = \mathrm{fd}(H)$, then by Claim (5.3.43), $\sigma_{ap}(S_{\mathcal{M}}^*) \cap \Omega^* = Z_{dg}(\mathcal{M})^*$ and $\sigma_{ap}\left(S_{E(\mathcal{M})}^*\right) \cap \Omega^* = Z_{dg}\left(E(\mathcal{M})\right)^*$. So it suffices to show that $Z_{dg}(\mathcal{M}) = Z_{dg}(E(\mathcal{M}))$ and this reduces to $Z_{dg}(\mathcal{M}) \subseteq Z_{dg}(E(\mathcal{M}))$ since the other direction is trivial. However, we have shown in the proof of Lemma (5.3.41) that for any λ in Ω, $\mathcal{M}'(\lambda) = \mathcal{M}(\lambda)$, where $\mathcal{M}'$ is a CF subspace containing $\mathcal{M}$ hence $Z_{dg}(\mathcal{M}) = Z_{dg}(\mathcal{M}')$. But $\mathcal{M}' \supseteq E(\mathcal{M})$ hence $Z_{dg}(\mathcal{M}') \subseteq Z_{dg}(E(\mathcal{M}))$.

For the other direction, one considers the natural short exact sequence of Hilbert modules

$$0 \to \mathcal{N} \to H \to \mathcal{N}^{\perp} \to 0.$$

Let I_λ be the maximal ideal of the polynomial ring $A = \mathbb{C}[z]$ at λ. Applying the tensor product factor $\cdot \otimes_A A/I_\lambda$, which is right half-exact, one has the following exact sequence

$$\frac{\mathcal{N}}{I_\lambda \mathcal{N}} \xrightarrow{i_\lambda} \frac{H}{I_\lambda H} \to \frac{\mathcal{N}^{\perp}}{I_\lambda \mathcal{N}^{\perp}} \to 0. \tag{62}$$

Now the assumption that $\dim \mathcal{N}(\lambda) = \mathrm{fd}(H)$ forces i_λ to be surjective, hence $\frac{\mathcal{N}^{\perp}}{I_\lambda \mathcal{N}^{\perp}} = \{0\}$. Therefore, $S_{\mathcal{N}} - \lambda$ is surjective, which implies that $\bar{\lambda} \notin \sigma_{ap}(S_{\mathcal{N}}^*)$.

Definition (5.3.44):[118] Suppose that $\{\mathcal{M}_i\}_{i\in I}$ is a family of invariant subspaces of an analytic operator, then the subspace

$$E(\{\mathcal{M}_i\}_{i\in I}) = \bigcap_{\gamma} \mathcal{M}_\gamma,$$

where the intersection is taken over all CF subspaces $\mathcal{M}_\gamma$ such that $\bigvee_{i\in I} \mathcal{M}_i \subseteq \mathcal{M}_\gamma$, is called the CF-envelope of $\{\mathcal{M}_i\}_{i\in I}$.

That $E(\{\mathcal{M}_i\}_{i\in I})$ is always CF follows from Claim (5.3.39) in the proof of Theorem (5.3.37).[12].

Corollary (5.3.45):[118] Under the hypothesis of Theorem (5.3.37),

(i) If $\mathcal{M}$ is CF, then $\sigma_{ap}(S_M^*) \cap \Omega^* = \sigma(S_M^*) \cap \Omega^*$;

(ii) If $\mathcal{M}$ is not CF, then $(\bar{\Omega})^* \subseteq \sigma(S_{\mathcal{M}}^*)$.

Proof : (i) We have to show the inclusion $\sigma(S_{\mathcal{M}}^*) \cap \Omega^* \subseteq \sigma_{ap}(S_{\mathcal{M}}^*) \cap \Omega^*$. The case $\mathrm{fd}(\mathcal{M}) < \mathrm{fd}(H)$ follows directly from Claim (5.3.43). For the case $\mathrm{fd}(\mathcal{M}) = \mathrm{fd}(H)$, assume that $\lambda \in \Omega^* \backslash \sigma_{ap}(S_{\mathcal{M}}^*)$, then inclusion amounts to that $\mathrm{ran}(S_{\mathcal{M}}^* - \lambda)$ is dense in $\mathcal{M}^\perp$ as can be easily seen. While the density of $\mathrm{ran}(S_{\mathcal{M}}^* - \lambda)$ in $\mathcal{M}^\perp$ reduces to that $\mathcal{M}$ has the division property at $\bar{\lambda}$, which is also a consequence of Claim (5.3.43) in the proof of Theorem(5.3.37).

Chapter 6

Volterra Invariant Subspaces

We show that the result can be applied to derive complete characterizations of such subspaces in a large class of Banach spaces of analytic functions in the unit disc containing the usual Bergman and Dirichlet spaces. Each invariant subspace of parabolic non-automorphism composition operator always consists of the closed span of a set of eigen functions. As a consequence, such composition operators have no non-trivial reducing subspaces. We also include a characterization of the closed ideals of the Banach algebra $W^{1,2}[0,\infty)$. Although such a characterization is known, the proof we provide here is somehow different. Inspired by Sarason's results, we find the lattice of closed invariant subspaces of the shift plus complex Volterra operator acting on the Hardy space.

Section (6.1): Volterra Invariant Subspaces of H^p

The Volterra integral operator

$$V_a f(z) = \int_a^z f(t)\,dt$$

is well-defined for functions f in the Hardy space H^1 and for all $|a| \le 1$. It maps H^1 into the disc algebra and its spectrum on every Hardy space $H^p (p \ge 1)$ consists of a single point $\lambda = 0$.The resolvent of V_a can be expressed as

$$(\lambda - V_a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} V_a^n , \qquad \lambda \neq 0,$$

where the series converges in the operator norm. A closed subspace $\mathcal{M}$ of H^p is V_a -invariant if $V_a\mathcal{M} \subset \mathcal{M}$. The lattice of all V_a-invariant subspaces of H^p was described by [135] in the case when $p = 2$ and $a = 0$. Donoghue's method is pure operator theory, and hardly adaptable to other values of p and especially if $|a| = 1$.

Theorem (6.1.1):[134] (i) A proper subspace $\mathcal{M}$ of H^p $(p > 1)$ is V_a-invariant, where $|a| < 1$ if and only if there exists a positive integer N such that

$$\mathcal{M} = b_a^N H^p$$

Where $b_a(z) = \frac{z-a}{1-\bar{a}z}$;

(ii) A proper subspace $\mathcal{M}$of $H^p (p > 1)$ is V_a-invariant, where $|a| = 1$ if and only if there exists a $t > 0$ such that

$$\mathcal{M} = S_a^t H^p$$

where $S_a(z) = \exp \frac{z+a}{z-a}$.

Corollary (6.1.2):[134] The lattice of V_a-invariant subspaces of $H^p(p > 1)$is linearly ordered by inclusion.

The description of the invariant subspaces for the operator $V: L^2(0,1) \to L^2(0,1)$,

$$Vf(x) = \int_0^x f(t)dt,$$

is essentially the problem posed by [136] and first solved by [137] who showed that all V-invariant subspaces of $L^2(0,1)$ have the form

$$\mathcal{M}_t = \chi_{(t,1)} L^2(0,1), \qquad 0 < t < 1,$$

and thus form a linearly ordered lattice. The result has been extended to a larger class of integral operators by [53], [138] found a different approach to the problem by identifying a resolvent of the Volterra operator with the restriction of the backward shift to one of its invariant subspaces.

The method we use for proving Theorem (6.1.1) bears strong resemblance to the classical methods going back to Wiener, Carleman, et al. in the study of invariant subspaces and also to Sarason's ideas mentioned above, even if in the case considered here, we encounter a different situation.To be more specific, our approach is based on a combination of duality between H^p and H^q, $\frac{1}{p} + \frac{1}{q} = 1$, and some harmonic analysis based on Borel transforms of complex conjugates of H^p-functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$, where $\mathbb{D}$ denotes the unit disc. Given $h \in \overline{H^p}$ its Borel transform is the entire function defined by

$$\tilde{h}(\lambda) = \int_{\mathbb{T}} e_\lambda \, h \, dm,$$

where $e_\lambda(z) = e^{\lambda z}$and $dm = \frac{|dz|}{2\pi}$is the normalized Lebesgue measure on $\mathbb{T}$. The space of entire functions which are Borel transforms of elements of $\overline{H^p}$ is denoted by $\mathcal{V}_p$. Now if we start with V_a-invariant subspace $\mathcal{M} \subset H^p$then it is not hard to verify that the Borel transforms of the complex conjugates of the functions in $\mathcal{M}^\perp \subset H^q, \frac{1}{p} + \frac{1}{q} = 1$ form a closed subspace of $\mathcal{V}_q$that is invariant under a rank-one perturbation of a backward shift depending on the point $a \in \overline{\mathbb{D}}$. Both objects involved here, the space $\mathcal{V}_q$and the rank-one perturbations of the backward shift acting on it are not well understood. However, all invariant subspaces of these rank-one perturbations of backward shifts share a slightly more general property called nearly invariance. A closed subspace $\mathcal{N}$of $\mathcal{V}_q$ is called nearly invariant if for every $f \in \mathcal{N}$ and every $\lambda \in \mathbb{C}$ which is a zero of f, but not a common zero of $\mathcal{N}$, we have $\frac{f}{(\zeta-\lambda)} \in \mathcal{N}$. Nearly invariance plays a crucial role in a number of important problems related to invariant subspaces for various operators, like for example, the shift on the Hardy space over a multiply connected domain [62,61], or the differentiation operator on C^∞ [139]. The main result about nearly invariant subspaces of $\mathcal{V}_q$ is proved and essentially asserts that a nearly invariant subspace where multiplication by the independent variable is densely defined must be invariant for differentiation on $\mathcal{V}_q$ and thus it is the Borel transform of the complex conjugate of a space of the form $(\theta H^p)^\perp$, where θ is an inner function.

We apply this result in order to show a similar structure theorem for V_a-invariant subspaces in every Banach space X which consists of analytic functions in $\mathbb{D}$ and has the following natural properties

(i) Point evaluations are continuous functionals on X,
(ii) Multiplication by the independent variable is bounded from above and below on X,
(iii) X is invariant for composition with analytic selfmaps $\varphi: \mathbb{D} \to \mathbb{D}$of the form $\varphi(z) = cz + d,\ c, d \in \mathbb{C}$, and the norm of such a composition operator is dominated by a negative power of $1 - |\varphi(0)|$,
(iv) There exists a nonnegative integer m such that the Banach algebra of analytic functions in $\mathbb{D}$ whose mth derivative belongs to the disc algebra (with the usual $C^m(\overline{\mathbb{D}})$-norm) is continuously contained and dense in X.

When applied to concrete cases, this result yields an extension of Theorem (6.1.1) to H^1. The first part of this result holds true in all spaces $H^p, 0 < p < \infty$ and is proved with a similar method. We obtain a complete characterization of all V_a-invariant subspaces in the standard weighted Bergman spaces as well as in most standard weighted Dirichlet spaces.

We shall focus on Borel transforms of complex conjugates of H^p-functions on the unit circle. Given $h \in \overline{H^p}$ with $p > 1$, its Borel transform $\tilde{h}$ is the entire function given by

$$\tilde{h}(\lambda) = \int_{\mathbb{T}} e_\lambda\, h\, dm.$$

We shall denote by $\mathcal{V}_p$ the space of entire functions obtained this way endowed with the induced norm

$$\left\|\tilde{\bar{h}}\right\|_{V_p} = \|h\|_p, \qquad h \in H^p.$$

Clearly, $\mathcal{V}_2$ is a Hilbert space and if $f \in \mathcal{V}_2$ with $f(z) = \sum_{n\geq 0} f_n z^n$ then

$$\|f\|^2 = \sum_{n=0}^{\infty} |f_n|^2 (n!)^2 < \infty.$$

This space of functions has been considered by [140] in connection with infinite differential equations. In [140] it is observed that the norm in $\mathcal{V}_2$ actually is a weighted L^2-normand this is due to the fact that $(n!)^2$ are the moments of a measure on $[0, \infty)$. There is a lot of information available about sequences of the form $((n!)^c)$, [141].

Proposition (6.1.3) :[134] If $f \in \mathcal{V}_2$ then

$$\|f\|_{V_2}^2 = \pi^{-2} \int_{\mathbb{C}} \int_{\mathbb{C}} |f(zw)|^2 e^{-|z|^2 - |w|^2}\, dA(z)\, dA(w) = \int_{\mathbb{C}} |f(u)|^2 v(u)\, dA(u),$$

Where

$$v(u) = \frac{1}{\pi}\int_0^\infty e^{-\frac{|u|^2}{t^2}-t^2}\frac{dt}{t}.$$

Proof: The first equality follows by a direct calculation with Parseval's formula. To see the second, note that for $w \neq 0$

$$\int_{\mathbb{C}} |f(zw)|^2 e^{-|z|^2}\, dA(z) = \frac{1}{|w|^2}\int_{\mathbb{C}} |f(u)|^2 e^{-|u|^2/|w|^2}\, dA(u)$$

and integrate this identity on $\mathbb{C}$ against the measure $e^{-|w|^2} dA(w)$. Then the result follows by Fubini's theorem.

We shall denote throughout by ζ the identity function on $\mathbb{C}$, that is, $\zeta(z) = z$.

Proposition (6.1.4): [134] (i) The spaces $\mathcal{V}_p, p > 1$ consist of functions of exponential type at most one. Moreover, if $f \in \mathcal{V}_p$ then

$$|f(\lambda)| = o\left(e^{|\lambda|}\right)$$

when $|\lambda| \to \infty$.

(ii) If $f \in \mathcal{V}_p$ with $f = \tilde{\tilde{h}}, h \in H^p$ then $f' \in \mathcal{V}_p$ and $f' = \widetilde{\widetilde{Bh}}$ where B denotes the backward shift on H^p. Consequently, the differentiation operator $Df = f$ is a bounded linear operator on $\mathcal{V}_p$.

(iii) For $a \in \overline{\mathbb{D}}$ and $\lambda \in \mathbb{C}$ denote by $R_{a,\lambda}$ the integral operator defined on $H^q, \frac{1}{p} + \frac{1}{q} = 1$ by

$$R_{a,\lambda} g(z) = \int_a^z e^{\lambda(z-t)} g(t) dt.$$

If $f \in \mathcal{V}_p$ with $f = \tilde{\tilde{h}}, h \in H^p$ and $f(\lambda) = 0$ for some $\lambda \in \mathbb{C}$ then $\frac{f}{(\zeta-\lambda)} \in \mathcal{V}_p$ with

$$\frac{f}{\zeta - \lambda} = \widetilde{\widetilde{R^*_{a,\lambda} h}}.$$

(iv) An entire function f satisfies $\zeta f \in \mathcal{V}_p$ if and only if $f = \tilde{\tilde{h}}$ with $h' \in H^p$.

Proof: (i). By Hölder's inequality we have for $\frac{1}{p} + \frac{1}{q} = 1$

$$\left|\tilde{\tilde{h}}(\lambda)\right| = \left|\int_{\mathbb{T}} e_\lambda\, h\, dm\right| \leq \|h\|_p \|e_\lambda\|_q \leq e^{|\lambda|} \left\|\tilde{\tilde{h}}\right\|_{V_p}.$$

To see the second part, note that the estimate is obvious when $f = \tilde{\tilde{h}}$ where h is a polynomial. Then if $f = \tilde{\tilde{h}} \in \mathcal{V}_p$ is arbitrary we can apply the first estimate to $f - \tilde{\tilde{g}}$, where g is a polynomial to obtain

$$\limsup_{|\lambda|\to\infty} e^{-|\lambda|}|f(\lambda)| \leq \left\|\widetilde{\overline{h-g}}\right\|_{V_p},$$

and the result follows from the fact that polynomials are dense in H^p.

(ii) is immediate since

$$f'(\lambda) = \int_{\mathbb{T}} z\, e_\lambda(z)\overline{h(z)}dm(z) = \int_{\mathbb{T}} e_\lambda(z)\overline{Bh(z)}\, dm(z).$$

(iii). Note that

$$R_{a,\lambda}e_\alpha = \frac{e_\alpha - e^{(\alpha-\lambda)a}e_\lambda}{\alpha - \lambda}.$$

Since h annihilates e_λ we obtain that

$$\widetilde{\overline{R^*_{a,\lambda}h}}(\alpha) = \int_{\mathbb{T}} R_{a,\lambda}e_\alpha(z)\overline{h(z)}\, dm(z) = \frac{1}{\alpha-\lambda}\int_{\mathbb{T}} e_\alpha(z)\overline{h(z)}\, dm(z) = \frac{f(\alpha)}{\alpha-\lambda}.$$

(iv). If $\zeta f \in \mathcal{V}_p$ we have by (iii) that $f = \frac{\zeta f}{\zeta} \in \mathcal{V}_p$ and if $\zeta f = \tilde{\tilde{g}}$ then for any $a \in \overline{\mathbb{D}}$ we have

$$f = \widetilde{\overline{R^*_{a,0}g}}$$

We shall show that $\left(R^*_{a,0}g\right)' \in H^p$ whenever $a \in \mathbb{D}$ and $g \in H^p$. Since

$$\left(R^*_{a,0}g\right)'(\lambda) = \int_{\mathbb{T}} g\, \overline{R_{a,0}\frac{\zeta}{\left(1-\bar{\lambda}\zeta\right)^2}}dm$$

and

$$R_{a,0}\frac{\zeta}{\left(1-\bar{\lambda}\zeta\right)^2}(z) = \int_a^z \frac{tdt}{\left(1-\bar{\lambda}t\right)^2} = \frac{1}{\bar{\lambda}^2}\log\frac{1-\bar{\lambda}z}{1-\bar{\lambda}a} + \frac{1}{\bar{\lambda}^2}\left(\frac{1}{1-\bar{\lambda}z} - \frac{1}{1-\bar{\lambda}a}\right)$$

we obtain

$$\left(R^*_{a,0}g\right)'(\lambda) = \frac{1}{\lambda^2}\int_{\mathbb{T}} g\log\frac{1-\lambda\bar{a}}{1-\lambda\bar{\zeta}}dm + \frac{g(\lambda)}{\lambda^2} - \frac{g(0)}{\lambda^2(1-\lambda\bar{a})},$$

i.e $\left(R^*_{a,0}g\right)' \in H^p$ whenever $a \in \mathbb{D}$ and $g \in H^p$. The converse follows directly from the equality

$$\tilde{\tilde{h'}} = \zeta\, \tilde{\tilde{h}}.$$

The exponential type in Proposition (5.1.4) (i) cannot be improved since all of these spaces contain the exponential functions $e_\alpha, |\alpha| < 1$, as the simple identity below shows

$$e^{\alpha\lambda} = \int_{\mathbb{T}} e_\alpha(z) \frac{1}{1-\lambda\bar{z}} dm(z).$$

The main objects under investigation are the nearly invariant subspaces of $\mathcal{V}_p, p > 1$. Recall that, by definition, a closed subspace $\mathcal{N}$ of $\mathcal{V}_p$ is nearly invariant if whenever $f \in \mathcal{N}$ and $\lambda \in \mathbb{C}$ is a zero of f, but not a common zero of $\mathcal{N}$, we have $\frac{f}{(\zeta-\lambda)} \in \mathcal{N}$.

Lemma (6.1.5):[134] For every $p > 1$ and $0 < \epsilon < \frac{1}{2p}$ there exists a positive constant $C_{p,\epsilon} > 0$ such that whenever $f \in \mathcal{V}_p$ and $\lambda \in \mathbb{C}$with $f(\lambda) = 0$

$$\left\|\frac{f}{\zeta-\lambda}\right\|_{V_p} \leq C_{p,\epsilon} \frac{\|f\|_{V_p}}{1+|\lambda|^{\frac{1}{2p}-\epsilon}}. \tag{1}$$

Proof: Assume that $\lambda \neq 0$ and let $z_\lambda = \frac{\bar{\lambda}}{|\lambda|}$. By Proposition (6.1.4) (iii) we have that if $f = \tilde{\tilde{h}}$ with $h \in H^p$ and $\lambda \in \mathbb{C}$ with $f(\lambda) = 0$ then

$$\frac{f}{\zeta-\lambda} = \widetilde{\overline{R^*_{z_\lambda,\lambda}}h},$$

so that, the result will follow once we prove the appropriate estimate for the operator norms $\|R_{z_\lambda,\lambda}\|$. To this end, we integrate along the line segment from z to z_λ to obtain for every $g \in H^q$, where $q = \frac{p}{p-1}$

$$R_{z_\lambda,\lambda}g(z) = (z_\lambda - z)\int_0^1 e^{-t(|\lambda|-\lambda z)} g\big(z + t(z_\lambda - z)\big)dt.$$

Note also that $R_{z_\lambda,\lambda}g$ belongs to the disc algebra, hence, it will suffice to work with the boundary values of these functions. It is useful to recall that if $|z| = 1$ then

$$|z - z_\lambda|^2 = 2Re(1 - \overline{z_\lambda}z)$$

which implies that

$$1 - |z + t(z_\lambda - z)|^2 = t(1-t)|z - z_\lambda|^2.$$

Now use the standard estimate

$$\big|g\big(z + t(z_\lambda - z)\big)\big| \leq 2^{\frac{2}{q}}(1 - |z + t(z_\lambda - z)|^2)^{-\frac{1}{q}}\|g\|_q$$

$$= 2^{\frac{2}{q}} t^{-\frac{1}{q}} (1-t)^{-\frac{1}{q}} |z - z_\lambda|^{-\frac{2}{q}} \|g\|_q$$

in order to obtain for $|z| = 1$

$$\left|R_{z_\lambda,\lambda} g(z)\right| \le \|g\|_q 2^{\frac{2}{q}} |z - z_\lambda|^{1-\frac{2}{q}} \int_0^1 t^{-\frac{1}{q}} (1-t)^{-\frac{1}{q}} \exp[-t|\lambda| Re(1 - \overline{z_\lambda} z)] dt.$$

For $1 < r < q, r' = \frac{r}{r-1}$ we apply Hölder's inequality to the above integral

$$\int_0^1 t^{-\frac{1}{q}} (1-t)^{-\frac{1}{q}} \exp[-t|\lambda| Re(1 - \overline{z_\lambda} z)]\, dt$$

$$\le \left(\int_0^1 \big(t(1-t)\big)^{-\frac{r}{q}} dt \right)^{\frac{1}{r}} \left(\int_0^1 \exp[-r' t|\lambda| Re(1 - \overline{z_\lambda} z)]\, dt \right)^{\frac{1}{r'}} \le C_{q,r} |z - z_\lambda|^{-\frac{2}{r'}} |\lambda|^{-\frac{1}{r'}},$$

where the constant $C_{q,r} > 0$ depends only on q and r. This leads to the estimate

$$\left|R_{z_\lambda,\lambda} g(z)\right| \le 2^{\frac{2}{q}} C_{q,r} |\lambda|^{-\frac{1}{r'}} |z - z_\lambda|^{1-\frac{2}{q}-\frac{2}{r'}} \|g\|_q.$$

For $r' > 2p$, i.e. when $1 < r < \frac{2q}{q+1}$ we have

$$q\left(1 - \frac{2}{q} - \frac{2}{r'}\right) > q - 2 - \frac{q}{p} = -1$$

which shows that $(\zeta - z_\lambda)^{1-\frac{2}{q}-\frac{2}{r'}} \in H^q$. Moreover, we have $r' \to 2p$ when $r \to \frac{2q}{q+1}$ and the result follows.

Theorem (6.1.6):[134] Let $\mathcal{N}$ be a nearly invariant subspace of $\mathcal{V}_p, p > 1$ without common zeros.

(i) If $f \in \mathcal{N}$ and $\zeta f \in \mathcal{V}_p$ then $f' \in \mathcal{N}$.

(ii) If the set of functions $f \in \mathcal{N}$ with $\zeta f \in \mathcal{V}_p$ is dense in $\mathcal{N}$ then $\mathcal{N}$ is invariant for the differentiation operator on $\mathcal{V}_p$ and there exists an inner function θ such that $\mathcal{N}$ is the Borel transform of $\overline{(\theta H^q)^\perp}$, where $q = \frac{p}{p-1}$.

Proof: (i) We start with the following identity which is valid for all functions of finite exponential type, and actually is a reformulation of Hadamard's factorization theorem. If f is a nonzero entire function of exponential type then

$$f'(z) = f(z)\left(a + \frac{m}{z} + \sum_{\substack{f(\lambda)=0 \\ \lambda\neq 0}} \frac{z}{\lambda(z-\lambda)}\right), \tag{2}$$

where $a \in \mathbb{C}, m \in \mathbb{N} \cup \{0\}$. Moreover, the series

$$\sum_{\substack{f(\lambda)=0 \\ \lambda\neq 0}} \frac{z}{\lambda(z-\lambda)} f(z) \tag{3}$$

converges uniformly on compact subsets of $\mathbb{C}$. Consider a function $f \in \mathcal{N}$ such that $\zeta f \in \mathcal{V}_p$. Then by Lemma (6.1.5) we have that

$$\left\|\frac{\zeta f}{(\lambda(\zeta-\lambda))}\right\| = O\left(|\lambda|^{-1-\frac{1}{2p}+\epsilon}\right)$$

when $|\epsilon| \to \infty$ and $f(\lambda) = 0,$ hence, by standard results about functions of exponential type, we can conclude that the series in (3) converges in $\mathcal{V}_p$ and the result follows. Under the assumption in (ii), $\mathcal{N}$ is differentiation-invariant and by Proposition (6.1.4) (ii) $\mathcal{N}$ is the Borel transform of a backward shift invariant subspace. Then by Beurling's theorem, $\mathcal{N}$ has the form in the statement.[142].

Proposition (6.1.7):[**134**] (i) For $\lambda \in \mathbb{C}\backslash\{0\}$ the resolvent operator $(\lambda - V_a)^{-1}: H^p \to H^p$ satisfies for every $b \in \mathbb{D}$

$$(\lambda - V_a)^{-1} f(z) = [(\lambda - V_a)^{-1} f](b) e^{(z-b)/\lambda} + \frac{1}{\lambda} e^{\frac{z}{\lambda}} \int_b^z e^{-\frac{t}{\lambda}} f'(t)\, dt$$

$$= [(\lambda - V_a)^{-1}](b) e^{(z-b)/\lambda} + \frac{f(z)}{\lambda} - \frac{f(b)}{\lambda} e^{\frac{(z-b)}{\lambda}} + \frac{1}{\lambda^2} e^{\frac{z}{\lambda}} \int_b^z e^{-\frac{t}{\lambda}} f(t) dt.$$

(ii) If $\mathcal{M}$ is a closed subspace of H^P which is invariant for V_a then $\mathcal{M}$ is invariant for $(\lambda - V_a)^{-1}$ for all $\lambda \in \mathbb{C}\backslash\{0\}$.

(iii) Every exponential function e_α is a cyclic vector for V_a.

Proof: (i) is a direct computation and will be omitted. To see (ii) note that since V_a is quasinilpotent we have that $\|V_a^n\|^{\frac{1}{n}} \to 0$ when $n \to \infty$ which implies that for $\lambda \in \mathbb{C}\backslash\{0\}$

$$(\lambda - V_a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} V_a^n,$$

where the series converges in the operator norm. Since $\mathcal{M}$ is invariant for V_a^n the result follows.

(iii). From (i) we see that for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\lambda\alpha \neq 1$ and all $b \in \mathbb{D}$ we have

$$(\lambda - V_a)^{-1}e_\alpha - \frac{\alpha e_\alpha}{\lambda\alpha - 1} = e_{\frac{1}{\lambda}}e^{-\frac{b}{\lambda}}\left[[(\lambda - V_a)^{-1}e_\alpha](b) - \frac{\alpha e_\alpha(b)}{\lambda\alpha - 1}\right]$$

By (ii) the left hand side belongs to the V_a-invariant subspace generated by e_α and cannot vanish identically because V_a has no eigen values. Then the V_a-invariant subspace generated by e_α contains $e_{1/\lambda}$ for all $\lambda \in \mathbb{C}\backslash\{0\}$ with $\lambda\alpha \neq 1$ and the result follows.

Lemma (6.1.8):[134] A nonzero function $f \in H^p$ belongs to $S_a^t H^p$ if and only if

$$\limsup_{r\to 1^-}(1-r)\log|f(ra)| \leq -2t.$$

Proof: We recall first a well-known fact about Poisson integrals of finite measures on the unit circle. If u is a harmonic function in $\mathbb{D}$ of the form

$$u(z) = \int_{\mathbb{T}} P_z\, d\mu, \qquad z \in \mathbb{D},$$

where $P_z(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$ is the Poisson kernel and μ is a finite measure on $\mathbb{T}$ then [143] for every $a \in \mathbb{T}$ we have

$$\lim_{r\to 1^-}(1-r)u(ra) = 2\mu(\{a\}). \tag{4}$$

Our second observation is that if B is a Blaschke product then

$$\limsup_{r\to 1^-}(1-r)\log|B(ra)| = 0$$

for all $a \in \mathbb{T}$. This is a direct consequence of the Phragmen–Lindelof principle. Indeed, if we assume the contrary،

$$\limsup_{r\to 1^-}(1-r)\log|B(ra)| = -2\tau < 0$$

then the Phragmen–Lindelöf principle immediately implies that $BS_a^{-\tau}$ is bounded on $\mathbb{D}$, which gives a contradiction.

Now if $f \in H^p$ is not identically zero, we use the canonical factorizations of such functions to write

$$\log|f| = \log|B| + u,$$

where B is a Blaschke product and u is the Poisson integral of a finite measure on the unit circle and the result follows by the above considerations.

Proposition (6.1.9):[134] (i) If $a \in \mathbb{D}$ and $b_a(z) = \frac{z-a}{1-\bar{a}z}$ denotes the Blaschke factor with a zero at a then for every positive integer $N$the subspace $b_a^N H^p$ is invariant for V_a.

(ii) If $|a| = 1$ and $S_a(z) = \exp\frac{z+a}{z-a}$ is the atomic singular inner function with singularity at a, then for every $t > 0$ the subspace $S_a^t H^p$ is invariant for V_a.

Proof: Part (i) is obvious, while (ii) follows by Lemma (6.1.8). Indeed, according to this result we only need to show that

$$\limsup_{r\to 1^-}(1-r)\log|V_a f(ra)| \le -2t$$

whenever $f \in H^p$ is not identically zero and

$$\limsup_{r\to 1^-}(1-r)\log|f(ra)| \le -2t,$$

and this is immediate.[135].

Theorem (6.1.10):[134] Let $\mathcal{M}$ be a nontrivial invariant subspace of V_a on $H^p, p > 1$.

(i) If $a \in \mathbb{D}$ then there exists a positive integer N such that $\mathcal{M} = b_a^N H^p$.

(ii) If $|a| = 1$ then there exists $t > 0$ such that $\mathcal{M} = S_a^t H^p$.

Proof: Note first that for every V_a-invariant subspace $\mathcal{M}$, the Borel transform $\mathcal{N}$ of $\overline{\mathcal{M}^\perp}$ is a closed subspace of $\mathcal{V}_q$, $\frac{1}{p}+\frac{1}{q} = 1$ which is nearly invariant and has no common zeros in $\mathbb{C}$. Indeed, by Proposition(6.1.7) (ii), $\mathcal{M}$ is invariant for $\left(\frac{1}{\alpha} - V_a\right)^{-1}$, $\alpha \ne 0$, hence, $\mathcal{M}^\perp$ is invariant for the adjoints of these operators. Then if $f \in \mathcal{N}$ with $f = \tilde{\bar{h}}, h \in \mathcal{M}^\perp$, and for every $\alpha \in \mathbb{C}\backslash\{0\}$ with $f(\alpha) = 0$ we can apply Proposition (6.1.4) (iii) to conclude that $\frac{f}{\zeta-\alpha} \in \mathcal{N}$. By continuity we see that this property holds for $\alpha = 0$ as well, and the claim follows.

Given a subspace $\mathcal{M}$ as in the statement, let $\mathcal{M}_1$ be the V_a-invariant subspace defined by

$$\mathcal{M}_1 = (V_a^*\mathcal{M}^\perp)^\perp.$$

Clearly, $\mathcal{M}_1$ is V_a-invariant and if $g \in \mathcal{M}_1$ then $V_a\, g \in \mathcal{M}$. Moreover, $\mathcal{M}_1^\perp$ is the closure of $V_a^* M^\perp$ in H^q. The Borel transform $\mathcal{N}_1$ of the complex conjugate space $\overline{\mathcal{M}_1^\perp}$ is a nearly invariant subspace of $\mathcal{V}_p$ without common zeros in $\mathbb{C}$ and, in addition, since $V_a^*\mathcal{M}^\perp$ is dense in $\mathcal{M}_1^\perp$ we can apply Proposition(6.1.4) (iii) and (iv) to conclude that the set of functions $f \in \mathcal{N}_1$ with $\zeta f \in \mathcal{V}_p$ is dense in $\mathcal{N}_1$. Thus by Theorem (6.1.6) (ii) we have that $\mathcal{M}_1 = \theta H^p$ for some inner function $\theta \in H^\infty$. Let Λ_θ be the union of the zero set of θ and the support of the singular measure corresponding to its singular inner factor. For every $g \in \mathcal{M}_1$ and every integer $n \ge 1$ the function $V_a^n g$ belongs to the disc algebra and hence, its extension to $\overline{\mathbb{D}}$ must vanish at all points of the set Λ_θ. Since

$$V_a^n\, g(z) = \frac{1}{(n-1)!}\int_a^z (z-w)^{n-1}\, g(w)dw,$$

we conclude that for every $b \in \Lambda_\theta$, every $f \in \mathcal{M}$and every integer $n \geq 1$ we have

$$0 = \int_a^b (b-w)^{n-1}\, g(w)dw = (b-a)^n \int_0^1 t^{n-1}\, g\big(b+t(a-b)\big)dt.$$

Clearly, if $a \neq b$ this implies that $g = 0$ by the Weierstrass approximation theorem and hence, $\mathcal{M} = \{0\}$. Consequently, we have $b = a$ and if $a \in \mathbb{D}$ then $\theta \in b_a^N$ for some positive integer N, while if $a \in \mathbb{T}, \theta = S_a^t$ for some $t > 0$.

If $a \in \mathbb{D}$ consider the set of analytic functions g in $\mathbb{D}$ with a zero of order $N+1$ at a and with$g' \in H^p$.Since$g = V_a g'$ and $g' \in \theta H^p = \mathcal{M}_1$, we obtain that $g \in \mathcal{M}$, i.e. $\mathcal{M}$ contains all functions g from above. Consequentl,

$$b_a^{N+1} H^p \subseteq \mathcal{M} \subseteq b_a^N H^p$$

and at least one of these inclusions must hold with equality. Similarly, if $a \in \mathbb{T}$ we can consider the set of analytic functions of the form $h = S_a^t (1-\zeta)^2 g$ where $g' \in H^p$. Each such function h satisfies $h = V_a h'$ and $h' \in S_a^t H^p = \mathcal{M}_1$ which implies that $h \in \mathcal{M}$. We conclude that $\mathcal{M} = S_a^t H^p$.

We are going to show that our main result actually implies a similar structure theorem for V_a-invariant subspaces in a large class of Banach spaces of analytic functions in the unit disc. To be more precise, let us consider Banach spaces $(X, \|\cdot\|)$ which consist of analytic functions in $\mathbb{D}$ such that:

(i) For each $\lambda \in \mathbb{D}$ the point evaluation

$$f \to f(\lambda), \qquad f \in X$$

is continuous on X.

(ii) The operator $\mathcal{M}_\zeta$ defined by $\mathcal{M}_\zeta f = \zeta f$ is bounded on X and $\mathcal{M}_\zeta - \lambda I$ is bounded below on X for all $\lambda \in \mathbb{D}$, where I denotes the identity operator on X.

(iii) For every analytic function $\varphi: \mathbb{D} \to \mathbb{D}$ of the form $\varphi(z) = cz + d$, the composition operator C_φ defined on X by

$$C_\varphi f = f \circ \varphi$$

is bounded on X and there exist positive constants K, γ such that

$$\|C_\varphi\| \leq K(1 - |\varphi(0)|)^{-\gamma} \tag{5}$$

for all such maps φ.

(iv) If A_0 denotes the disc algebra and A_m is the Banach algebra of analytic functions in $C^m(\overline{\mathbb{D}})$ which are analytic on D, then there exists a nonnegative integer m such that A_m is continuously contained and dense in X.

The Banach algebras $A_m, m \geq 0$, the Hardy spaces $H^p, 1 \leq p < \infty$, all standard weighted Bergman and Dirichlet spaces, the little Bloch space, the analytic VMO and the Besov spaces satisfy these assumptions.

It follows easily that for every $\lambda \in \mathbb{D}$ the backward shift $L_\lambda f = \frac{f - f(\lambda)}{\zeta - \lambda}$ is a bounded linear operator on X. Indeed, by (iv) this operator is densely defined on A_m, and by (i) and (ii) it satisfies an inequality of the form

$$\|L_\lambda f\| \leq C_\lambda \|f\|, \qquad f \in A_m,$$

with respect to the norm in X. Then L_λ has a bounded extension to X and the claim follows by another application of (i).

For every $\lambda \in \mathbb{D}$ and every positive integer N, the subspace $X_{\lambda,N}$ of all functions in X with a zero of order N at λ can be written in the form

$$X_{\lambda,N} = (\zeta - \lambda)^N X = \left(M_\zeta - \lambda\right)^N X = \overline{\left(M_\zeta - \lambda\right) A_m} \tag{6}$$

whenever (iv) holds for the nonnegative integer m.

Proposition (6.1.11):[134] Suppose that the space X satisfies (i)–(iv). Then:

(i) If $\gamma > \gamma(X)$ there exists $K > 0$ such that for each $\lambda \in \mathbb{D}$

$$|f(\lambda)| \leq K(1 - |\lambda|)^{-\gamma} \|f\|, \qquad f \in X.$$

(ii) If $\gamma(X) < 1$, the Volterra operators $V_a, |a| \leq 1$, are well-defined and bounded on X. Moreover, $V_a X \subset A_0 \cap X$.

Proof: (i) If $\varphi_{t,\lambda}(z) = t\lambda + (1-t)z, t \in (0,1), \lambda \in \mathbb{D}$, then by (i) and (iii) we have for all $f \in X, \lambda \in \mathbb{D}$ and $t \in (0,1)$

$$|f(t\lambda)| = \left|C_{\varphi_{t,\lambda}} f(0)\right| \leq K_0 \left\|C_{\varphi_{t,\lambda}}\right\| \|f\| \leq K_0 K_\gamma (1 - t|\lambda|)^{-\gamma} \|f\|,$$

where the constants $K_0, K_\gamma > 0$ are independent of λ and f. To see (ii) write

$$V_a f(z) = \int_0^1 f\big(a + t(z-a)\big)(z-a)dt = \left(M_\zeta - a\right) \int_0^1 C_{\Psi_{t,a}} f(z) dt,$$

where $\Psi_{t,a}(z) = a + t(z-a)$. Since $\Psi_{t,a}(0) = (1-t)a$ for $a \neq 0$ and $\gamma(X) < \gamma < 1$, we have by (iii)

$$\int_0^1 \left\|C_{\Psi_{t,a}}\right\| dt \leq K \int_0^1 t^{-\gamma}\, dt < \infty. \tag{7}$$

Moreover, if $f \in A_m$ then $V_a f \in A_m$ and if we choose m such that (iv) holds, then by the above estimates we obtain the following inequality which involves the norm on X

$$\|V_a f\| \leq K \|f\|, \qquad f \in A_m,$$

where K is independent of f. This implies the first part of the assertion. The second part is also a direct consequence of the estimate (7).

Theorem (6.1.12):[**134**] Let X be a Banach space of analytic functions which satisfies (i)–(iv) and assume that the operators $V_a, |a| \leq 1$ are well-defined and bounded on X with $V_a X \subset A_0 \cap X$.

(i) A proper subspace $\mathcal{M}$ of X is V_a-invariant, where $|a| < 1$ if and only if there exists a positive integer N such that

$$\mathcal{M} = (\zeta - a)^N X.$$

(ii) If the proper subspace $\mathcal{M}$ of X is V_a-invariant, where $|a| = 1$, and m is any non negative integer such that (iv) holds, then one and only one of the following alternatives must occur. Either there exists $t > 0$ such that for every $0 < r < \frac{1}{2}$

$$S_a^t H^\infty \cap A_m \subset \mathcal{M} \subset \overline{S_a^t H^r \cap X},$$

where $S_a(z) = \exp\frac{z+a}{z-a}$, or there exists $1 \leq k \leq m+1$ such that

$$\mathcal{M} \cap A_m = A_m^k(a).$$

Proof: Consider first the space $V_a\mathcal{M} \subset A_0 \cap X$, let $p > 1$ be fixed but arbitrary, and apply Theorem (6.1.10) to conclude that the closure of $V_a\mathcal{M}$ in H^p equals $b_a^N H^p$ for some $N \geq 0$, if $|a| < 1$, and $S_a^t H^p$ for some $t \geq 0$, if $|a| = 1$. Let m be a nonnegative integer such that (iv) holds. If (f_n) is a sequence in $\mathcal{M}$ such that $(V_a f_n)$ converges to $g \in H^p$ then $(V_a^{m+2} f_n)$ converges to $V_a^{m+1} g$ in A_m, hence by (iv), it converges also in X. Thus $\mathcal{M}$ contains $V_a^{m+1}\mathcal{M}_p$, where $\mathcal{M}_p$ is the closure of $V_a\mathcal{M}$ in H^p. This gives

$$b_a^{N+m+1} A_m \subset \mathcal{M}$$

If $|a| < 1$,

$$S_a^t H^\infty \cap A_m \subset \mathcal{M}$$

If $|a| = 1, t > 0$ and

$$V_a^{m+1} H^p \subset \mathcal{M}$$

if $|a| = 1$ and $t = 0$. If $|a| < 1$ we know that the functions in $\mathcal{M}$ have a common zero of order N at a. From the equality above and (6) we have $(\zeta - a)^{N+m} X \subset \mathcal{M}$. If $f_0 \in \mathcal{M}$ is such that $f_0^{(N+1)}(a) \neq 0$ then every function $f \in (\zeta - a)^N X$ can be written in the form

$$f = \sum_{n=0}^{m+1} c_n V_a^n f_0 + g$$

with scalars c_n and $g \in (\zeta - a)^{N+m+1}$ which proves (i).

A similar argument shows that if $|a| = 1$ and $t = 0$ then the second alternative in (ii) occurs. Indeed, it is easy to verify that the closure of $V_a^{m+1} H^p$ in A_m equals $A_m^{m+1}(a)$. If $k \leq m+1$ is the order of the common zero of the functions in $\mathcal{M} \cap A_m$ at a then by (iv) k must be positive. If $k=m+1$ the statement follows from above. If $k<m+1$ we choose again $f_0 \in \mathcal{M} \cap A_m$ with $f_0^{(k+1)}(a) \neq 0$ and write an arbitrary function in $A_m^k(a)$ in the form

$$f=\sum_{n=0}^{m-k} c_n V_a^n f_0 + g$$

with scalars c_n and $g \in A_m^{m+1}(a) \in \mathcal{M}$ and the result follows.

It remains to prove that if $|a|=1$ and $t>0$ then

$$\mathcal{M} \subset \overline{S_a^t H^r \cap X}.$$

Recall from the beginning of the proof that for every $p>1$, the closure of $V_a\mathcal{M}$ in H^p equals $S_a^t H^p$. Thus it suffices to prove that any function $f \in X$ with $V_a f \in S_a^t H^p$ can be approximated in X by functions in $S_a^t H^r \cap X$ for every $r<\frac{p}{2+2p}$.

To this end, we consider for $0<\epsilon<1$ the functions $\varphi_\epsilon : \mathbb{D} \to \mathbb{D}$ with

$$\varphi_\epsilon(z)=\epsilon a+(1-\epsilon)z.$$

A simple computation yields for $|z|=1$

$$1-|\varphi_\epsilon(z)|^2=\epsilon(1-\epsilon)|z-a|^2. \tag{8}$$

Composition with φ_ϵ has the following properties.

(i) If h is analytic in $\mathbb{D}$ and satisfies for some $\alpha>0$ the growth restriction

$$|h(z)|=O((1-|z|)^{-\alpha}), \quad |z| \to 1^-$$

then by (8) we have that $(\zeta-a)^{2\alpha} h \circ \varphi_\epsilon \in H^\infty$. Consequently, $h \circ \varphi_\epsilon \in H^s$ for all $\zeta<\frac{1}{2\alpha}$
.

(ii) The composition operators C_{φ_ϵ} on X satisfy $\left\|C_{\varphi_\epsilon}\right\| \leq K_\gamma(1-\epsilon)^{-\gamma}$ for every $\gamma>\gamma(X)$ and also, if $f \in A_m$ then

$$\lim_{\epsilon\to 0} C_{\varphi_\epsilon} \mathrm{f}=f,$$

in A_m. Then by (iv) we have that $C_{\varphi\epsilon}$ converges strongly to the identity on X.

Now if $f \in X$ with $V_a f \in S_a^t H^p$ write $V_a f=S_a^t F$ with $F \in H^p$. Then

$$f=S_a^t\left(F'-\frac{2t}{(\zeta-a)}F\right)$$

hence, for $0<\epsilon<1$

$$f \circ \varphi_\epsilon=e^{-\frac{t\epsilon}{1-\epsilon}} S_a^{\frac{t}{1-\epsilon}}\left(F' \circ \varphi_\epsilon-\frac{2t}{(1-\epsilon)^2(\zeta-a)^2} F \circ \varphi_\epsilon\right).$$

Since $F \in H^p$ we have $F \circ \varphi_\epsilon (\zeta - a)^{-2} \in H^s$ for all $s < \frac{p}{2p+1}$. Moreover,

$$|F'(z)| = O\left((1 - |z|^{-1})^{-1-\frac{1}{p}}\right), \quad |z| \to 1^-,$$

hence, by property (i) we obtain $F' \circ \varphi_\epsilon \in H^s$ for all $s < \frac{p}{2p+2}$. Then $f \circ \varphi_\epsilon \in H^s$ for all $s < \frac{p}{2p+2}$, the claim follows by property (ii).

Corollary (6.1.13):[**134**] The proper V_a-invariant subspaces of H^1 are precisely those of the form $b_a^N H^1, N \in \mathbb{N}$, if $|a| < 1$, and $S_a^t H^1, t > 0$, if $|a| = 1$.

For $0 < p < 1$ the operator V_a is bounded on H^p if and only if $|a| < 1$. The invariant sub spaces of these operators can be determined with the same methods.

Corollary (6.1.14):[134] For $|a| < 1$and $0 < p < 1$the proper V_a-invariant subspaces of $H^p$are preciselythose of the form $b_a^N H^p, N \in \mathbb{N}$.

Proof: If $|a| < 1$and $\mathcal{M}$is a proper V_a-invariant subspace of $H^p, 0 < p < 1$, then there existsa positive integer k such that $V_a^k \mathcal{M} \subset H^1$. By Corollary (6.1.13) the closure of $V_a^k \mathcal{M}$ in H^1 has the form $b_a^m H^1$ which implies that $\mathcal{M}$contains $b_a^m H^p$. The rest of the proof is identical to the argument used in the proof of Theorem (6.1.12).

Corollary (6.1.15):[134] The proper V_a-invariant subspaces of A_m are precisely those of the form $b_a^N A_m, N \in \mathbb{N}$, if $|a| < 1$, and $S_a^t H^\infty \cap A_m, t > 0$, or $A_m^k(a), 1 \le k \le m+1$ if $|a| = 1$.[144].

Corollary (6.1.16):[134] Let $p \ge 1$ and $\alpha > -1$ be such that $\frac{\alpha+2}{p} < 1$. If $|a| < 1$ the proper V_a-invariant subspaces of $L_a^{p,\alpha}$are precisely those of the form $b_a^N L_a^{p,\alpha}, N \in \mathbb{N}$. If $|a| = 1$ the V_a-invariant subspaces of $L_a^{p,\alpha}$ coincide with the $\mathcal{M}_\zeta$-invariant subspaces of $L_a^{p,\alpha}$ generated by S_a^t for some $t > 0$.

We note also that for $\frac{\alpha+2}{p} > 1$ the operators $V_a, |a| = 1$ are unbounded on $L_a^{p,\alpha}$. Moreover, the $\mathcal{M}_\zeta$-invariant subspaces generated by $S_a^t, t > 0$, are always strictly contained in $L_a^{p,\alpha}$ [104] .

We consider the standard weighted Dirichlet spaces $D^{p,\alpha}, \alpha > -1$, which consist ofanalytic functions in $\mathbb{D}$whose derivative belongs to $L_a^{p,\alpha}$. The norm on $D_{p,\alpha}$is defined by

$$\|f\|_{D_{p,\alpha}}^p = |f(0)|^p + \|f'\|_{p,\alpha}^p.$$

It is well known [104] that $D^{p,\alpha} = L_a^{p,\alpha}$ when ever $\alpha > p - 1$. Here we shall only consider the case when $p > 1$ and $\alpha > p - 1$.

The verification of the assumptions (i), (ii) and (iv) (with $m = 1$) for $D^{p,\alpha}$ is again straightforward. To see (iii) let $\varphi: \mathbb{D} \to \mathbb{D}$ with $\varphi(z) = cz + d$ and write

$$\|C_\varphi f\|_{D^{p,\alpha}}^p = |f(\varphi(0))|^p + \|f' \circ \varphi \varphi'\|_{p,\alpha}^p \le |f(\varphi(0))|^p + \|f' \circ \varphi\|_{p,\alpha}^p,$$

where the last inequality follows from the fact that $|\varphi'(z)| = |c| \leq 1$. Clearly, the norm of the point evaluation at $\varphi(0)$ satisfies an estimate of the type required in (iii) and then the claim follows from the considerations made for the weighted Bergman spaces $L_a^{p,\alpha}$.

Proposition (6.1.17):[134] Let $p > 1$, let $-1 < \alpha < p-1$ and set $\beta = -\frac{\alpha}{p-1}$.

(i) Every continuous linear functional l on $D^{p,\alpha}$ can be represented uniquely in the form

$$l(f) = \lim_{r\to 1^-} \int_{\mathbb{T}} f(rz)\overline{g_l(rz)}dm(z),$$

where $g_l \in L_a^{q,\beta}, \frac{1}{p} + \frac{1}{q} = 1$. The linear map $l \to g_l$ from the dual of $D^{p,\alpha}$ into $L_a^{p,\beta}$ is continuous and bijective. Moreover, $D^{p,\alpha}$ is reflexive.

(ii) $D^{p,\alpha}$ is continuously contained in H^p.

(iii) If θ is an inner function and $f \in D^{p,\alpha}$ satisfies $\frac{f}{\theta} \in H^p$, then f/θ belongs to $D^{p,\alpha}$ and there is a constant $K > 0$ independent of f such that

$$\left\|\frac{f}{\theta}\right\|_{D^{p,\alpha}} \leq K\|f\|_{D^{p,\alpha}}.$$

Proof: (i) Using Parseval's formula we can write for f, g analytic on $\mathbb{D}^{p,\alpha}$ and $0 < r < 1$

$$\int_{\mathbb{T}} f(rz)\overline{g(rz)}dm(z) = \int_{\mathbb{D}} (\zeta f)'(rz)\overline{g(rz)}dA(z).$$

The linear map $f \to (\zeta f)'$ from $D^{p,\alpha}$ into $L_a^{p,\alpha}$ is continuous and invertible, so that, all we need to show is that the dual of $L_a^{p,\alpha}$ can be identified with $L_a^{q,\beta}$ via the pairing

$$\langle h, g\rangle = \lim_{r\to 1^-} \int_{\mathbb{D}} h(rz)\overline{g(rz)}dA(z).$$

To prove reflexivity, we have to show that via the above pairing the dual of $L_a^{q,\beta}$ is $L_a^{p,\alpha}$. These are particular cases of the results obtained in [145]. Part (ii) follows directly from (i) since $H^q, \frac{1}{p} + \frac{1}{q} = 1$ is continuously contained in $L_a^{q,\beta}$. (iii) asserts that $D^{p,\alpha}$ has the so-called (F)-property [146]. If θ is inner then the operator M_θ of multiplication by θ is a bounded linear operator on $L_a^{q,\beta}$. Since $D^{p,\beta}$ is reflexive, its adjoint M_θ^* is bounded on $D^{p,\alpha}$. If $\frac{f}{\theta} \in H^p$ and $g \in H^q$ then

$$\lim_{r\to 1^-} \int_{\mathbb{T}} f(rz)\overline{M_\theta g(rz)}dm(z) = \int_{\mathbb{T}} \left(\frac{f}{\theta}\right)(z)\overline{g(z)}dm(z).$$

From the fact that H^q is dense in $L_a^{q,\beta}$ we obtain that $M_\theta^* f = \frac{f}{\theta}$ and the result follows.

As a direct application of part (ii) we obtain that the Volterra operators $V_a, |a| \le 1$ are bounded on $D^{p,\alpha}$ and satisfy $V_a D^{p,\alpha} \subset A_0 \cap D^{p,\alpha}$. Indeed, by the Fejer–Riesz inequality we have that

$$|V_a f(0)| \le K\|f\|_{H^p} \le K'\|f\|_{D^{p,\alpha}}$$

for some constant $K' > 0$ and all $f \in D^{p,\alpha}$. The inequality

$$\int_{\mathbb{D}} |(V_a f)'(z)|^p (1-|z|)^\alpha \, dA(z) = \int_{\mathbb{D}} |f(z)|^p (1-|z|)^\alpha dA(z) \le K''\|f\|_{D^{p,\alpha}}^p$$

follows also by standard estimates [9], but can also be obtained by a direct application of Minkowski's inequality.

Lemma (6.1.18):[134] Let $p > 1$ and $\alpha > -1$ such that $\frac{\alpha+1}{p} < 1$. Then for $t > 0$ and $|a| = 1, S_a^t H^\infty \cap A_1$ is dense in $S_a^t H^p \cap D^{p,\alpha}$.

Proof: Let $f \in S_a^t H^p \cap D^{p,\alpha}$. By Proposition(6.1.17) (iii) we can approximate $f S_a^{-t} \in D^{p,\alpha}$ in $D^{p,\alpha}$ by a sequence (f_n) of functions in A_1. Then it is a simple matter to show that $(a-\zeta)^2 S_a^t f_n \to (a-\zeta)^2 f$ in $D^{p,\alpha}$. Thus $(a-\zeta)^2 f$ belongs to the closure of $S_a^t H^\infty \cap H_1$ when ever $f \in S_a^t H^p \cap D^{p,\alpha}$. To finish the proof it suffices to show that for such f we have

$$\lim_{r\to 1^-} \frac{(a-\zeta)^2}{(a-r\zeta)^2} f = f \tag{9}$$

in the norm of $D^{p,\alpha}$. Since $f S_a^{-t} \in D^{p,\alpha}$ it follows that

$$\|f S_a^{-t}\|_{D^{p,\alpha}}^p \ge \int_{\mathbb{D}} \left| f'(z) + \frac{2t}{(z-a)^2} f(z) \right|^p (1-|z|)^\alpha \, dA(z)$$

and this implies that

$$\int_{\mathbb{D}} |z-a|^{-2p} |f(z)|^p (1-|z|)^\alpha \, dA(z) < \infty.$$

But from this inequality and the dominated convergence theorem we obtain that

$$\lim_{r\to 1^-} \int_{\mathbb{D}} \left| \left(\frac{(a-\zeta)^2}{(a-r\zeta)^2} f \right)' (z) \right|^p (1-|z|)^\alpha \, dA(z) = \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^\alpha \, dA(z).$$

Then (9) follows by a standard argument and the proof is complete.

Lemma (6.1.19):[134] For $p > 1$ and $|a| = 1$ the closure of $A_1^2(a)$ in $D^{p,\alpha}$ equals $D^{p,\alpha}$ if $\frac{\alpha+2}{p} \ge 1$ and $D^{p,\alpha}(a)$ if $\frac{\alpha+2}{p} \ge 1$.

Proof: Since polynomials are dense in $D^{p,\alpha}$ it follows easily that the set of polynomials which vanish at a is dense in $D^{p,\alpha}(a)$. Thus it suffices to show that $1, \zeta - a$ belong to the closure of $A_1^2(a)$ in $D^{p,\alpha}$ when $\frac{\alpha+2}{p} \geq 1$, and that $\zeta - a$ belongs to the closure of $A_1^2(a)$ in $D^{p,\alpha}$ when $\frac{\alpha+2}{p} < 1$. For $r > 1$ consider the functions

$$f_r(z) = (z-a)^2(z-ra)^{-1}, \quad g_r(z) = (z-a)(z-ra)^{-1}, \quad z \in \mathbb{D}$$

Clearly, $f_r \in A_1^2(a), f_r(z) \to z - a$ when $r \to 1^+$, and $|f_r'(z)| \leq 3$ for all $z \in \mathbb{D}$. Using the dominated convergence theorem it is a standard matter to show that $f_r \to \zeta - a$ in $D^{p,\alpha}$ for all $p \geq 1$ and all $\alpha > -1$. The functions g_r satisfy $g_r \in A_1^2(a), g_r(z) \to 1$ when $r \to 1^-$ for all $z \in \mathbb{D}$ and

$$|g_r'(z)| = \frac{r-1}{|z-ra|^2} + 2\frac{(r-1)^2}{|z-ra|^3}, \qquad z \in \mathbb{D}.$$

By standard estimates [104] it follows that $\|g_r\|_{D^{p,\alpha}}$ stay bounded when $r \to 1^+$. Since $D^{p,\alpha}$ is reflexive, we conclude that $g_r \to 1$ weakly in $D^{p,\alpha}$.

Corollary (6.1.20):[134] Let $p > 1$ and $\alpha > -1$ such that $\frac{\alpha+1}{p} < 1$. If $|a| < 1$ the proper V_a-invariant subspaces of $D^{p,\alpha}$ are precisely those of the form $b_a^N D^{p,\alpha}, N \in \mathbb{N}$. If $|a| = 1$ and $\frac{\alpha+2}{p} \geq 1$ then every V_a-invariant subspace of $D^{p,\alpha}$ has the form $S_a^t H^p \cap D^{p,\alpha}$ for some $t > 0$. If $|a| = 1$ and $\frac{\alpha+2}{p}$ then a V_a-invariant subspace of $D^{p,\alpha}$ is either equal to $D^{p,\alpha}(a)$, or it has the form $S_a^t H^p \cap D^{p,\alpha}$ for some $t > 0$.

Corollary (6.1.21):[168] If $f_j \in \mathcal{V}_2$ then

$$\sum_j \|f_j\|_{\mathcal{V}_2}^2 = \pi^{-1} \int_{\mathbb{C}} \int_{\mathbb{C}} \sum_j \left|f_j\left(z_n z_{(n+1)}\right)\right|^2 e^{-\left(|z_n|^2 + \left|z_{(n+1)}\right|^2\right)} dA(z_n)\, dA\left(z_{(n+1)}\right)$$

$$= \int_{\mathbb{C}} \sum_j \left|f_j\left(z_{(n+2)}\right)\right|^2 v\left(z_{(n+2)}\right) dA\left(z_{(n+2)}\right),$$

Where

$$v\left(z_{(n+2)}\right) = \frac{1}{\pi} \int_0^\infty e^{-\left(\frac{|z_{n+2}|^2 + (1-\epsilon)^4}{(1-\epsilon)^2}\right)} \frac{d(1-\epsilon)}{1-\epsilon}.$$

Proof: The first equality follows by a direct calculation with Parseval's formula. To see the second, note that for $z_{(n+1)} \neq 0$

$$\int_{\mathbb{C}} \sum_j \left|f_j\left(z_n z_{(n+1)}\right)\right|^2 e^{-|z_n|^2} dA(z_n)$$

$$= \frac{1}{\left|z_{(n+1)}\right|^2} \int_{\mathbb{C}} \sum_j \left|f_j\left(z_{(n+2)}\right)\right|^2 e^{-\left(\frac{\left|z_{(n+2)}\right|^2}{\left|z_{(n+1)}\right|^2}\right)} dA\left(z_{(n+2)}\right)$$

and integrate this identity on $\mathbb{C}$ against the measure $e^{-|z_{(n+1)}|^2}dA(z_{(n+1)})$. Then the result follows by Fubini's theorem.

Corollary (6.1.22):[168] (i) The spaces $\mathcal{V}_{(1+\epsilon)}, \epsilon > 0$ consist of functions of exponential type at most one. Moreover, if $f_j \in \mathcal{V}_{(1+\epsilon)}$ then

$$|f_j(\lambda^2 - 1)| = o(e^{|\lambda^2-1|})$$

when $|\lambda^2 - 1| \to \infty$.

(ii) If $f_j \in \mathcal{V}_{(1+\epsilon)}$ with $f_j = \tilde{\tilde{h}}_j, h_j \in H^{(1+\epsilon)}$ then $f_j{}' \in \mathcal{V}_{(1+\epsilon)}$ and $f_j{}' = \widetilde{\widetilde{Bh_j}}$ where B denotes the backward shift on $H^{(1+\epsilon)}$. Consequently, the differentiation operator $Df_j = f_j{}'$ is a bounded linear operator on $\mathcal{V}_{(1+\epsilon)}$.

(iii) For $(a^2 - 1) \in \overline{\mathbb{D}}$ and $(\lambda^2 - 1) \in \mathbb{C}$ denote by $R_{(a^2-1,\lambda^2-1)}$ the integral operator defined on $H^{\left(\frac{1+\epsilon}{\epsilon}\right)}, \epsilon > 0$ by

$$\sum_j R_{(a^2-1,\lambda^2-1)} g_j(z_n) = \int_{a^2-1}^{z_n} \sum_j e^{(\lambda^2-1)(z_n-(1-\epsilon))} g_j(1-\epsilon)\, d(1-\epsilon).$$

If $f_j \in \mathcal{V}_{(1+\epsilon)}$ with $f_j = \tilde{\tilde{h}}_j, h_j \in H^{(1+\epsilon)}$ and $f_j(\lambda^2 - 1) = 0$ for some $(\lambda^2 - 1) \in \mathbb{C}$ then $\frac{f_j}{(\zeta-\lambda^2+1)} \in \mathcal{V}_{(1+\epsilon)}$ with

$$\frac{f_j}{\zeta - \lambda^2 + 1} = \widetilde{\widetilde{R^*_{(a^2-1,\lambda^2-1)} h_j}}.$$

(iv) An entire function f_j satisfies $\zeta f_j \in \mathcal{V}_{(1+\epsilon)}$ if and only if $f_j = \tilde{\tilde{h}}_j$ with $h_j{}' \in H^{(1+\epsilon)}$.

Proof: (i). By Hölder's inequality we have for $\epsilon > 0$

$$\sum_j \left|\tilde{\tilde{h}}_j(\lambda^2 - 1)\right| = \sum_j \left|\int_{\mathbb{T}} e_{(\lambda^2-1)}\, h_j\, dm\right| \le \sum_j \|h_j\|_{(1+\epsilon)} \|e_{(\lambda^2-1)}\|_{\left(\frac{1+\epsilon}{\epsilon}\right)}$$
$$\le e^{|\lambda^2-1|} \sum_j \left\|\tilde{\tilde{h}}_j\right\|_{\mathcal{V}_{(1+\epsilon)}}.$$

To see the second part, note that the estimate is obvious when $f_j = \tilde{\tilde{h}}_j$ where h_j is a polynomial. Then if $f_j = \tilde{\tilde{h}}_j \in \mathcal{V}_{(1+\epsilon)}$ is arbitrary we can apply the first estimate to $f_j - \tilde{\tilde{g}}_j$, where g_j is a polynomial to obtain

$$\lim_{|\lambda^2-1|\to\infty} \sum_j \sup e^{-|\lambda^2-1|} |f_j(\lambda^2 - 1)| \le \sum_j \left\|\widetilde{\widetilde{h_j - g_j}}\right\|_{\mathcal{V}_{(1+\epsilon)}},$$

and the result follows from the fact that polynomials are dense in $H^{(1+\epsilon)}$.

(ii) is immediate since

$$\sum_j f_j{}'(\lambda^2-1) = \int_{\mathbb{T}} \sum_j z_n\, e_{(\lambda^2-1)}(z_n)\overline{h_j(z_n)}\, dm(z_n)$$

$$= \int_{\mathbb{T}} \sum_j e_{(\lambda^2-1)}(z_n)\overline{Bh_j(z_n)}\ dm(z_n).$$

(iii). Note that for $\alpha \neq \lambda$

$$R_{(\alpha^2-1,\lambda^2-1)}e_{(\alpha^2-1)} = \frac{e_{(\alpha^2-1)} - e^{\left((\alpha^2-\lambda^2)(\alpha^2-1)\right)}e_{(\lambda^2-1)}}{\alpha^2-\lambda^2}.$$

Since h_j annihilates $e_{(\lambda^2-1)}$ we obtain that

$$\sum_j \overline{\widetilde{R^*_{(\alpha^2-1,\lambda^2-1)}h}}_j(\alpha^2-1) = \int_{\mathbb{T}} \sum_j R_{(\alpha^2-1,\lambda^2-1)}e_{(\alpha^2-1)}(z_n)\overline{h_j(z_n)}\ dm(z_n)$$
$$= \frac{1}{\alpha^2-\lambda^2}\int_{\mathbb{T}} \sum_j e_{(\alpha^2-1)}(z_n)\overline{h_j(z_n)}\ dm(z_n) = \frac{\sum_j f_j(\alpha^2-1)}{\alpha^2-\lambda^2}.$$

(iv). If $\zeta f_j \in \mathcal{V}_{(1+\epsilon)}$ we have by (iii) that $f_j = \left(\frac{\zeta f_j}{\zeta}\right) \in \mathcal{V}_{(1+\epsilon)}$ and if $\zeta f_j = \widetilde{\overline{g_j}}$ then for any $(a^2-1) \in \overline{\mathbb{D}}$ we have

$$f_j = \overline{\widetilde{R^*_{(a^2-1,0)}g_j}}$$

We shall show that $\left(R^*_{(a^2-1,0)}g_j\right)' \in H^{(1+\epsilon)}$ whenever $(a^2-1) \in \mathbb{D}$ and $g_j \in H^{(1+\epsilon)}$. Since

$$\left(R^*_{(a^2-1,0)}g_j\right)'(\lambda^2-1) = \int_{\mathbb{T}} \sum_j g_j\, \overline{R_{(a^2-1,0)}\frac{\zeta}{\left(1-\overline{(\lambda^2-1)}\zeta\right)^2}}\, dm$$

And

$$R_{(a^2-1,0)}\frac{\zeta}{\left(1-\overline{(\lambda^2-1)}\zeta\right)^2}(z_n) = \int_{a^2-1}^{z_n} \frac{(1-\epsilon)d(1-\epsilon)}{\left(1-\overline{(\lambda^2-1)}(1-\epsilon)\right)^2}$$
$$= \frac{1}{\overline{(\lambda^2-1)^2}}\log\frac{1-\overline{(\lambda^2-1)}z_n}{1-\overline{(\lambda^2-1)}(a^2-1)}$$
$$+\frac{1}{\overline{(\lambda^2-1)^2}}\left(\frac{1}{1-\overline{(\lambda^2-1)}z_n} - \frac{1}{1-\overline{(\lambda^2-1)}(a^2-1)}\right)$$

we obtain

$$\sum_j \left(R^*_{(a^2-1,0)} g_j\right)'(\lambda^2-1) = \frac{1}{(\lambda^2-1)^2}\int_{\mathbb{T}} \sum_j g_j \log \frac{1-(\lambda^2-1)\overline{(a^2-1)}}{1-(\lambda^2-1)\bar{\zeta}}\, dm$$

$$+\frac{\sum_j g(\lambda^2-1)}{(\lambda^2-1)^2} - \frac{\sum_j g_j(0)}{(\lambda^2-1)^2\left(1-(\lambda^2-1)\overline{(a^2-1)}\right)},$$

i.e $\left(R^*_{(a^2-1,0)} g_j\right)' \in H^{(1+\epsilon)}$ whenever $(a^2-1) \in \mathbb{D}$ and $g_j \in H^{(1+\epsilon)}$. The converse follows directly from the equality

$$\widetilde{\widetilde{h_j{}'}} = \zeta\, \tilde{\tilde{h}}_j.$$

The exponential type in Corollary (6.1.22) (i) cannot be improved since all of these spaces contain the exponential functions $e_{(\alpha^2-1)}, \alpha \le \sqrt{2}$, as the simple identity below shows

$$e^{\left((\alpha^2-1)(\lambda^2-1)\right)} = \int_{\mathbb{T}} e_{(\alpha^2-1)}(z_n) \frac{1}{1-(\lambda^2-1)\overline{z_n}}\, dm(z_n).$$

The main objects under investigation are the nearly invariant subspaces of $\mathcal{V}_{(1+\epsilon)}, \epsilon > 0$. Recall that, by definition, a closed subspace $\mathcal{N}$ of $\mathcal{V}_{(1+\epsilon)}$ is nearly invariant if whenever $f_j \in \mathcal{N}$ and $(\lambda^2-1) \in \mathbb{C}$ is a zero of f_j, but not a common zero of $\mathcal{N}$, we have $\frac{f_j}{(\zeta-\lambda^2+1)} \in \mathcal{N}$.

Corollary (6.1.23):[168] For every $\epsilon > 0$ and $0 < 2\epsilon(1+\epsilon) < 1$ there exists a positive constant $C_{(1+\epsilon,\epsilon)} > 0$ (depending only on $(1+\epsilon)$ and ϵ) such that whenever $f_j \in \mathcal{V}_{(1+\epsilon)}$ and $(\lambda^2-1) \in \mathbb{C}$ with $f_j(\lambda^2-1) = 0$,

$$\sum_j \left\|\frac{f_j}{\zeta-\lambda^2+1}\right\|_{\mathcal{V}_{(1+\epsilon)}} \le C_{(1+\epsilon,\epsilon)} \frac{\sum_j \|f_j\|_{\mathcal{V}_{(1+\epsilon)}}}{1+|\lambda^2-1|^{\left(\frac{1-2\epsilon(1-\epsilon)}{2(1+\epsilon)}\right)}}.$$

Proof: Assume that $\lambda \ne \pm 1$ and let $z_n(\lambda^2-1) = \frac{\overline{(\lambda^2-1)}}{|\lambda^2-1|}$. By Corollary (6.1.22) (iii) we have that if $f_j = \tilde{\tilde{h}}_j$ with $h_j \in H^{(1+\epsilon)}$ and $(\lambda^2-1) \in \mathbb{C}$ with $f_j(\lambda^2-1) = 0$ then

$$\sum_j \frac{f_j}{\zeta-\lambda^2+1} = \sum_j \overline{\widetilde{R^*_{(z_n(\lambda^2-1),\lambda^2-1)} h_j}},$$

so that, the result will follow once we prove the appropriate estimate for the operator norms $\left\|R_{\left((z_n)_{(\lambda^2-1)},\lambda^2-1\right)}\right\|$. To this end, we integrate along the line segment from z_n to $(z_n)_{(\lambda^2-1)}$ to obtain for every $g_j \in H^{\left(\frac{1+\epsilon}{\epsilon}\right)}$, where $\epsilon > 0$

$$\sum_j R_{\left((z_n)_{(\lambda^2-1)},(\lambda^2-1)\right)} g_j(z_n)$$

$$= \left((z_n)_{(\lambda^2-1)} - z_n\right) \int_0^1 e^{-(1-\epsilon)(|\lambda^2-1|-(\lambda^2-1)z_n)} \sum_j g_j \left((1-\epsilon)(z_n)_{(\lambda^2-1)} + \epsilon z_n\right) d(1 - \epsilon).$$

Then we found that $R_{\left((z_n)_{(\lambda^2-1)},\lambda^2-1\right)} g_j$ belongs to the disc algebra, hence, it will suffice to work with the boundary values of these functions. It is useful to recall that if $|z_n| = 1$ then

$$\left|z_n - (z_n)_{(\lambda^2-1)}\right|^2 = 2Re\left(1 - \overline{(z_n)_{(\lambda^2-1)}} z_n\right)$$

which implies that

$$1 - \left|(1-\epsilon)(z_n)_{(\lambda^2-1)} + \epsilon z_n\right|^2 = \epsilon(1-\epsilon)\left|z_n - (z_n)_{(\lambda^2-1)}\right|^2.$$

Now use the standard estimate

$$\sum_j \left|g_j\left((1-\epsilon)(z_n)_{(\lambda^2-1)} + \epsilon z_n\right)\right|$$
$$\leq 2^{\left(\frac{2\epsilon}{1+\epsilon}\right)} \left((1-\epsilon)(z_n)_{(\lambda^2-1)} + \epsilon z_n\right)^{-\left(\frac{\epsilon}{1+\epsilon}\right)} \sum_j \|g_j\|_{\left(\frac{\epsilon}{1+\epsilon}\right)}$$
$$= 2^{\left(\frac{2\epsilon}{1+\epsilon}\right)} (\epsilon(1-\epsilon))^{-\left(\frac{\epsilon}{1+\epsilon}\right)} \left|z_n - (z_n)_{(\lambda^2-1)}\right|^{-\left(\frac{2\epsilon}{1+\epsilon}\right)} \sum_j \|g_j\|_{\left(\frac{1+\epsilon}{\epsilon}\right)}$$

in order to obtain for $|z_n| = 1$

$$\sum_j \left|R_{\left((z_n)_{(\lambda^2-1)},\lambda^2-1\right)} g_j(z_n)\right| \leq \sum_j \|g_j\|_{\left(\frac{1+\epsilon}{\epsilon}\right)} 2^{\left(\frac{2\epsilon}{1+\epsilon}\right)} \left|z_n - (z_n)_{(\lambda^2-1)}\right|^{\left(\frac{1-\epsilon}{1+\epsilon}\right)}$$
$$\int_0^1 (\epsilon(1-\epsilon))^{-\left(\frac{\epsilon}{1+\epsilon}\right)} \exp\left[(\epsilon-1)|\lambda^2-1| Re\left(1 - \overline{(z_n)_{(\lambda^2-1)}} z_n\right)\right] d(1-\epsilon).$$

For $0 < \epsilon(r-1) < 1$ we apply Hölder's inequality to the above integral

$$\int_0^1 (\epsilon(1-\epsilon))^{-\left(\frac{\epsilon}{1+\epsilon}\right)} \exp\left[(\epsilon-1)|\lambda^2-1| Re\left(1 - \overline{(z_n)_{(\lambda^2-1)}} z_n\right)\right] d(1-\epsilon)$$
$$\leq \left(\int_0^1 \left((1-\epsilon)(\epsilon)\right)^{-\left(\frac{r\epsilon}{1+\epsilon}\right)} d(1-\epsilon)\right)^{\frac{1}{r}}$$

$$\left(\int_0^1 \exp\left[\left(\frac{r}{1-r}\right)(1-\epsilon)|\lambda^2-1|Re\left(1-\overline{(z_n)_{(\lambda^2-1)}}z_n\right)\right]d(1-\epsilon)\right)^{\left(\frac{r-1}{r}\right)}$$
$$\leq C_{\left(\frac{1+\epsilon}{\epsilon},r\right)}\left|z_n-(z_n)_{(\lambda^2-1)}\right|^{\left(\frac{2(1-r)}{r}\right)}|\lambda^2-1|^{\left(\frac{1-r}{r}\right)},$$

where the constant $C_{\left(\frac{1+\epsilon}{\epsilon},r\right)}>0$ depends only on $\frac{1+\epsilon}{\epsilon}$ and r. This leads to the estimate

$$\sum_j\left|R_{\left((z_n)_{(\lambda^2-1)},(\lambda^2-1)\right)}g_j(z_n)\right|$$
$$\leq 2^{\left(\frac{2\epsilon}{1+\epsilon}\right)}C_{\left(\frac{1+\epsilon}{\epsilon},r\right)}|\lambda^2-1|^{\left(\frac{1-r}{r}\right)}\left|z_n-(z_n)_{(\lambda^2-1)}\right|^{\left(\frac{r(1-\epsilon)+2(1+\epsilon)(1-r)}{r(1+\epsilon)}\right)}\sum_j\|g_j\|_{\left(\frac{1+\epsilon}{\epsilon}\right)}.$$

For $1<r<\frac{2(1+\epsilon)}{1+2\epsilon}$ we have

$$2(1+\epsilon)-r(1+2\epsilon)>0$$

which shows that $\left(\zeta-(z_n)_{(\lambda^2-1)}\right)^{\left(\frac{r(1-\epsilon)+2(1+\epsilon)(1-r)}{r(1+\epsilon)}\right)}\in H^{\left(\frac{1+\epsilon}{\epsilon}\right)}$. Moreover, we have $r\to\frac{2(1+\epsilon)}{1+2\epsilon}$ and the result follows.

Corollary (6.1.24):[168] Let $\mathcal{N}$ be a nearly invariant subspace of $\mathcal{V}_{(1+\epsilon)},\epsilon>0$ without common zeros.

(i) If $f_j\in\mathcal{N}$ and $\zeta f_j\in\mathcal{V}_{(1+\epsilon)}$ then $f_j'\in\mathcal{N}$.
(ii) If the set of functions $f_j\in\mathcal{N}$ with $\zeta f_j\in\mathcal{V}_{(1+\epsilon)}$ is dense in $\mathcal{N}$ then $\mathcal{N}$ is invariant for the differentiation operator on $\mathcal{V}_{(1+\epsilon)}$ and there exists an inner function θ such that $\mathcal{N}$ is the Borel transform of $\overline{\left(\theta\, H^{\left(\frac{1+\epsilon}{\epsilon}\right)}\right)}^{\perp}$, where $\epsilon>0$.

Proof: (i) We start with the following identity which is valid for all functions of finite exponential type, and actually is a reformulation of Hadamard's factorization theorem. If f_j is a nonzero entirefunction of exponential type then

$$\sum_j f_j'(z_n)=\sum_j f_j(z_n)\left((a^2-1)+\frac{m}{z_n}+\sum_{\substack{f_j(\lambda^2-1)=0\\ \lambda\neq\pm1}}\frac{z_n}{(\lambda^2-1)(z_n-\lambda^2+1)}\right)$$

where $(a^2-1)\in\mathbb{C},m\in\mathbb{N}\cup\{0\}$. Moreover, the series

$$\sum_{\substack{f_j(\lambda^2-1)=0\\ \lambda\neq\pm1}} \frac{z_n}{(\lambda^2-1)(z_n-\lambda^2+1)} \sum_j f_j(z_n)$$

Converges uniformly on compact subsets of $\mathbb{C}$. Consider a function $f_j \in \mathcal{N}$ such that $\zeta f_j \in \mathcal{V}_{(1+\epsilon)}$. Then by Corollary (6.1.23) we have that

$$\left\|\frac{\zeta f_j}{\left((\lambda^2-1)(\zeta-\lambda^2+1)\right)}\right\| = O\left(|\lambda^2-1|^{\frac{2\epsilon^2-3}{2(1+\epsilon)}}\right)$$

when $|\epsilon| \to \infty$ and $f_j(\lambda^2-1)=0$, hence, by standard results about functions of exponential type, we can conclude that the series in (3) converges in $\mathcal{V}_{(1+\epsilon)}$ and the result follows. Under the assumption in (ii), $\mathcal{N}$ is differentiation-invariant and by Corollary (6.1.22) (ii) $\mathcal{N}$ is the Borel transform of a backward shift invariant subspace. Then by Beurling's theorem, $\mathcal{N}$ has the form in the statement.

Corollary (6.1.25):[168] (i) For $(\lambda^2-1) \in \mathbb{C}\backslash\{0\}$ the resolvent operator

$\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1} : H^{(1+\epsilon)} \to H^{(1+\epsilon)}$ satisfies for every $(a^2-1+\epsilon) \in \mathbb{D}$

$$\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1}\left(\sum_j f_j(z_n)\right)$$

$$= \left[\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1}\left(\sum_j f_j\right)\right](a^2-1+\epsilon)e^{\left(\frac{z_n-a^2-1+\epsilon}{\lambda^2-1}\right)}$$

$$+\frac{1}{\lambda^2-1}e^{\left(\frac{z_n}{\lambda^2-1}\right)}\int_{a^2-1+\epsilon}^{z_n} e^{-\left(\frac{1-\epsilon}{\lambda^2-1}\right)}\sum_j f_j'(1-\epsilon)\; d(1-\epsilon)$$

$$= \left[\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1}\right](a^2-1+\epsilon)e^{\left(\frac{z_n-a^2-1+\epsilon}{\lambda^2-1}\right)} + \frac{\left(\sum_j f_j(z_n)\right)}{\lambda^2-1}$$

$$-\frac{\sum_j f_j(a^2-1+\epsilon)}{\lambda^2-1}e^{\left(\frac{z_n-a^2-1+\epsilon}{\lambda^2-1}\right)}$$

$$+\frac{1}{(\lambda^2-1)^2}e^{\left(\frac{z_n}{\lambda^2-1}\right)}\int_{2-a^2-\epsilon}^{z_n} e^{-\frac{(1-\epsilon)}{\lambda^2-1}}\sum_j f_j(1-\epsilon)\,d(1-\epsilon).$$

(ii) If $\mathcal{M}$ is a closed subspace of $H^{(1+\epsilon)}$ which is invariant for $V_{(a^2-1)}$ then $\mathcal{M}$ is invariant for $\left((\lambda^2-1)-V_{a^2-1}\right)^{-1}$ for all $(\lambda^2-1) \in \mathbb{C}\backslash\{0\}$.

(iii) Every exponential function $e_{(a^2-1)}$ is a cyclic vector for $V_{(a^2-1)}$.

Proof: (i) is a direct computation and will be omitted. To see (ii) note that since $V_{(a^2-1)}$ is quasinilpotent we have that $\left\|V_{(a^2-1)}^{(n)}\right\|^{\frac{1}{n}} \to 0$ when $n \to \infty$ which implies that for $(\lambda^2-1) \in \mathbb{C}\backslash\{0\}$

$$\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1} = \sum_{n=0}^{\infty}(\lambda^2-1)^{-(n+1)}V_{(a^2-1)}^{(n)},$$

where the series converges in the operator norm. Since $\mathcal{M}$ is invariant for $V_{(a^2-1)}^{(n)}$ the result follows.

(iii). From (i) we see that for $(\lambda^2-1) \in \mathbb{C}\backslash\{0\}$ with $(\lambda^2-1)(\alpha^2-1) \neq 1$ and all $a \in \mathbb{D}$ we have

$$\begin{aligned}\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1} e_{(a^2-1)} &- \frac{(\alpha^2-1)e_{(\alpha^2-1)}}{\lambda^2(\alpha^2-1)-\alpha^2}\\ &= e_{\left(\frac{1}{\lambda^2-1}\right)}e^{-\left(\frac{\alpha^2-1+\epsilon}{\lambda^2-1}\right)}\left[\left[\left((\lambda^2-1)-V_{(a^2-1)}\right)^{-1} e_{\alpha^2-1}\right](a^2-1+\epsilon)\right.\\ &\left.-\frac{(\alpha^2-1)e_{\alpha^2-1}(a^2-1+\epsilon)}{\lambda^2(\alpha^2-1)-\alpha^2}\right]\end{aligned}$$

By (ii) the left hand side belongs to the $V_{(a^2-1)}$-invariant subspace generated by $e_{(\alpha^2-1)}$ and cannot vanish identically because $V_{(a^2-1)}$ has no eigenvalues. Then the $V_{(a^2-1)}$-invariant subspace generated by $e_{(\alpha^2-1)}$ contains $e_{\left(\frac{1}{\lambda^2-1}\right)}$ for all $(\lambda^2-1) \in \mathbb{C}\backslash\{0\}$ with $(\lambda^2-1)(\alpha^2-1) \neq 1$ and the result follows.

Corollary (6.1.26):[168] A nonzero function $f_j \in H^{(1+\epsilon)}$ belongs to $S_{(a^2-1)}^{(1-\epsilon)}H^{(1+\epsilon)}$ if and only if

$$\lim_{r\to 1^-}\sum_j \sup(1-r)\log\left|f_j(r(a^2-1))\right| \leq -2(1-\epsilon).$$

Proof: We recall first a well-known fact about Poisson integrals of finite measures on the unit circle. If u is a harmonic function in $\mathbb{D}$ of the form

$$u(z_n) = \int_{\mathbb{T}} P_{z_n}\, d\mu, \qquad z_n \in \mathbb{D},$$

Where $P_{z_n}\left(e^{i(1-\epsilon)}\right) = \frac{1-|z_n|^2}{\left|e^{i(1-\epsilon)}-z_n\right|^2}$ is the Poisson kernel and μ is a finite measure on $\mathbb{T}$ then (see [144]) for every $(a^2-1) \in \mathbb{T}$ we have

$$\lim_{r\to 1^-}(1-r)u(r(a^2-1)) = 2\mu(\{a^2-1\}).$$

Our second observation is that if B is a Blaschke product then

$$\lim_{r\to1^-}\sup(1-r)\log|B(r(a^2-1))|=0$$

for all $(a^2-1)\in\mathbb{T}$. This is a direct consequence of the Phragmen–Lindelöf principle. Indeed, if we assume the contrary،

$$\lim_{r\to1^-}\sup(1-r)\log|B(r(a^2-1))|=-2\tau<0$$

then the Phragmen–Lindelöf principle immediately implies that $BS^{-\tau}_{(a^2-1)}$ is bounded on $\mathbb{D}$, which gives a contradiction.

Now if $f_j\in H^{(1+\epsilon)}$ is not identically zero, we use the canonical factorizations of such functions to write

$$\sum_j \log|f_j|=\log|B|+u,$$

where B is a Blaschke product and u is the Poisson integral of a finite measure on the unit circle and the result follows by the above considerations.

Corollary (6.1.27):[168] Suppose that the space X satisfies (i)–(iv). Then:

(i) If $\gamma>\gamma(X)$ there exists $K>0$ such that for each $(\lambda^2-1)\in\mathbb{D}$

$$\sum_j|f_j(\lambda^2-1)|\le K(1-|\lambda^2-1|)^{-\gamma}\sum_j\|f_j\|,\qquad f_j\in X.$$

(ii) If $\gamma(X)<1$, the Volterra operators $V_{(a^2-1)}, a\le\sqrt{2}$, are well-defined and bounded on X. Moreover, $V_{(a^2-1)}X\subset A_0\cap X$.

Proof: (i) If $\varphi_{(1-\epsilon,\lambda^2-1)}(z_n)=(1-\epsilon)(\lambda^2-1)+\epsilon z_n, 0<\epsilon<1,(\lambda^2-1)\in\mathbb{D}$, then by (i) and (iii) we have for all $f_j\in X,(\lambda^2-1)\in\mathbb{D}$ and $0<\epsilon<1$

$$\sum_j|f_j((1-\epsilon)(\lambda^2-1)|=\sum_j\left|C_{\varphi_{(1-\epsilon,\lambda^2-1)}}f_j(0)\right|\le K_0\left\|C_{\varphi_{(1-\epsilon,\lambda^2-1)}}\right\|\sum_j\|f_j\|$$
$$\le K_0K_\gamma(1-(1-\epsilon)|\lambda^2-1|)^{-\gamma}\sum_j\|f_j\|,$$

where the constants $K_0,K_\gamma>0$ are independent of (λ^2-1) and f_j. To see (ii) write

$$V_{a^2-1}\left(\sum_j f_j(z_n)\right)=\int_0^1\sum_j f_j\left(z_n(1-\epsilon)-\epsilon(1-a^2)\right)(z_n-a^2+1)d(1-\epsilon)$$
$$=\left(M_\zeta-a^2+1\right)\int_0^1 C_{\Psi_{(1-\epsilon,a^2-1)}}\sum_j f_j(z_n)\,d(1-\epsilon),$$

where $\Psi_{(1-\epsilon,a^2-1)}(z_n) = z_n(1-\epsilon) - \epsilon(1-a^2)$. Since $\Psi_{(1-\epsilon,a^2-1)}(0) = \epsilon(a^2-1)$ for $a \neq \pm 1$ and $\gamma(X) < \gamma < 1$, we have by (iii)

$$\int_0^1 \left\| C_{\Psi_{(1-\epsilon,a^2-1)}} \right\| d(1-\epsilon) \leq K \int_0^1 (1-\epsilon)^{-\gamma} \, d(1-\epsilon) < \infty.$$

Moreover, if $f_j \in A_m$ then $V_{(a^2-1)}\left(\sum_j f_j\right) \in A_m$ and if we choose m such that (iv) holds, then by the above estimates we obtain the following inequality which involves the norm on X

$$\sum_j \left\| V_{(a^2-1)} f_j \right\| \leq K \sum_j \left\| f_j \right\|, \qquad f_j \in A_m,$$

where K is independent of f_j. This implies the first part of the assertion. The second part is also a direct consequence of the estimate (9).

Corollary (6.1.28):[168] Let X be a Banach space of analytic functions which satisfies (i)–(iv) and assume that the operators $V_{(a^2-1)}, a \leq \sqrt{2}$ are well-defined and bounded on X with $V_{(a^2-1)}X \subset A_0 \cap X$.

(i) A proper subspace $\mathcal{M}$ of X is $V_{(a^2-1)}$-invariant, where $a \leq \sqrt{2}$ if and only if there exists a positive integer N such that
$$\mathcal{M} = (\zeta - a^2 + 1)^N X.$$
(ii) If the proper subspace $\mathcal{M}$ of X is $V_{(a^2-1)}$-invariant, where $|a^2-1| = 1$, and m is any non negative integer such that (iv) holds, then one and only one of the following alternatives must occur. Either there exists $\epsilon < 1$ such that for every $0 < r < \frac{1}{2}$
$$S_{(a^2-1)}^{(1-\epsilon)} H^\infty \cap A_m \subset \mathcal{M} \subset \overline{S_{(a^2-1)}^{(1-\epsilon)} H^r \cap X},$$
where $S_{(a^2-1)}(z_n) = \exp\left(\frac{z_n + a^2 - 1}{z_n - a^2 + 1}\right)$, or there exists $1 \leq k \leq m+1$ such that
$$\mathcal{M} \cap A_m = A_m^k(a^2-1).$$

Proof: Consider first the space $V_{(a^2-1)}\mathcal{M} \subset A_0 \cap X$, let $\epsilon > 0$ be fixed but arbitrary, and apply Theorem (6.1.10) to conclude that the closure of $V_{(a^2-1)}\mathcal{M}$ in $H^{(1+\epsilon)}$ equals $(a^2-1+\epsilon)_{(a^2-1)}^N H^{(1+\epsilon)}$ for some $N \geq 0$, if $a \leq \sqrt{2}$, and $S_{(a^2-1)}^{(1-\epsilon)} H^{(1-\epsilon)}$ for some $\epsilon < 1$, if $|a^2-1| = 1$. Let m be a nonnegative integer such that (iv) holds. If $\left(f_j\right)_n$ is a sequence in $\mathcal{M}$ such that $\left(V_{(a^2-1)}\left(f_j\right)_n\right)$ converges to $g_j \in H^{(1+\epsilon)}$ then $\left(V_{(a^2-1)}^{(m+2)}\left(f_j\right)_n\right)$ converges to $V_{(a^2-1)}^{(m+1)} g_j$ in A_m, hence by (iv), it converges also in X. Thus $\mathcal{M}$ contains $V_{(a^2-1)}^{(m+1)}\mathcal{M}_{(1+\epsilon)}$, where $\mathcal{M}_{(1+\epsilon)}$ is the closure of $V_{(a^2-1)}\mathcal{M}$ in $H^{(1+\epsilon)}$. This gives

$$(a^2-1+\epsilon)_{(a^2-1)}^{(N+m+1)} A_m \subset \mathcal{M}$$

If $a \le \sqrt{2}$,

$$S_{a^2-1}^{(1-\epsilon)} H^{\infty} \cap A_m \subset \mathcal{M}$$

If $|a^2 - 1| = 1, \epsilon < 1$ and

$$V_{(a^2-1)}^{(m+1)} H^{(1-\epsilon)} \subset \mathcal{M}$$

If $|a^2 - 1| = 1$ and $\epsilon = 1$. If $a \le \sqrt{2}$ we know that the functions in $\mathcal{M}$ have a common zero of order N at $a^2 - 1$. From the equality above and (6) we have $(\zeta - a^2 + 1)^{N+m} X \subset \mathcal{M}$. If $(f_j)_0 \in \mathcal{M}$ is such that $(f_j)_0^{(N+1)}(a^2 - 1) \neq 0$ then every function $f_j \in (\zeta - a^2 + 1)^N X$ can be written in the form

$$\sum_j f_j = \sum_{n=0}^{m+1} c_n V_{(a^2-1)}^n (f_j)_0 + g_j$$

with scalars c_n and $g_j \in (\zeta - a^2 + 1)^{(N+m+1)}$ which proves (i).

A similar argument shows that if $|a^2 - 1| = 1$ and $\epsilon = 1$ then the second alternative in (ii) occurs .Indeed, it is easy to verify that the closure of $V_{(a^2-1)}^{(m+1)} H^{(1+\epsilon)}$ in A_m equals $A_m^{m+1}(a^2 - 1)$. If $k \le m + 1$ is the order of the common zero of the functions in $\mathcal{M} \cap A_m$ at a then by (iv) k must be positive. If $k = m + 1$ the statement follows from above. If $k < m + 1$ we choose again $(f_j)_0 \in \mathcal{M} \cap A_m$ with $(f_j)_0^{(k+1)}(a^2 - 1) \neq 0$ and write an arbitrary function in $A_m^k(a^2 - 1)$ in the form

$$\sum_j f_j = \sum_{n=0}^{m-k} c_n V_{(a^2-1)}^n (f_j)_0 + g_j$$

with scalars c_n and $g_j \in A_m^{(m+1)}(a^2 - 1) \in \mathcal{M}$ and the result follows.

It remains to prove that if $|a^2 - 1| = 1$ and $\epsilon < 1$ then

$$\mathcal{M} \subset \overline{S_{(a^2-1)}^{(1-\epsilon)} H^r \cap X}.$$

Recall from the beginning of the proof that for every $\epsilon > 0$, the closure of $V_{(a^2-1)}\mathcal{M}$ in $H^{(1+\epsilon)}$ equals $S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)}$. Thus it suffices to prove that any function $f_j \in X$ with $V_{(a^2-1)}(f_j) \in S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)}$ can be approximated in X by functions in $S_{(a^2-1)}^{(1-\epsilon)} H^r \cap X$ for every $r < \frac{1+\epsilon}{4+2\epsilon}$.

To this end, we consider for $0 < \epsilon < 1$ the functions $\varphi_\epsilon \colon \mathbb{D} \to \mathbb{D}$ with

$$\varphi_\epsilon(z_n) = \epsilon(a^2 - 1) + (1 - \epsilon) z_n.$$

A simple computation yields for $|z_n| = 1$

$$1 - |\varphi_\epsilon(z_n)|^2 = \epsilon(1-\epsilon)|z_n - a^2 + 1|^2.$$

Composition with φ_ϵ has the following properties.

(i) If h_j is analytic in $\mathbb{D}$ and satisfies for some $\alpha < \pm 1$ the growth restriction

$$\sum_j |h_j(z_n)| = O\big((1-|z_n|)^{1-\alpha^2}\big), \quad |z_n| \to 1^-$$

then by (8) we have that $(\zeta - a^2 + 1)^{2(\alpha^2-1)} h_j \circ \varphi_\epsilon \in H^\infty$. Consequently, $h_j \circ \varphi_\epsilon \in H^s$ for all $s < \frac{1}{2(\alpha^2-1)}$.

(ii) The composition operators C_{φ_ϵ} on X satisfy $\|C_{\varphi_\epsilon}\| \le K_\gamma (1-\epsilon)^{-\gamma}$ for every $\gamma > \gamma(X)$ and also, if $f_j \in A_m$ then

$$\lim_{\epsilon \to 0} C_{\varphi_\epsilon} f_j = f_j,$$

in A_m. Then by (iv) we have that C_{φ_ϵ}converges strongly to the identity on X.Now if $f_j \in X$with $V_{(a^2-1)}\left(\sum_j f_j\right) \in S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)}$ write $V_{(a^2-1)}\left(\sum_j f_j\right) = S_{(a^2-1)}^{(1-\epsilon)}\left(\sum_j F_j\right)$ with $F_j \in H^{1+\epsilon}$. Then

$$\sum_j f_j = S_{(a^2-1)}^{(1-\epsilon)} \sum_j \left(F_j' - \frac{2(1-\epsilon)}{(\zeta - a^2 + 1)} F_j\right)$$

hence, for $0 < \epsilon < 1$

$$\sum_j f_j \circ \varphi_\epsilon = e^{-\epsilon} S_{(a^2-1)}^{1} \sum_j \left(F_j' \circ \varphi_\epsilon - \frac{2}{(1-\epsilon)(\zeta - a^2 + 1)^2} F_j \circ \varphi_\epsilon\right).$$

Since $F_j \in H^{(1+\epsilon)}$ we have $F_j \circ \varphi_\epsilon (\zeta - a^2 + 1)^{-2} \in H^s$ for all $s < \frac{1+\epsilon}{3+2\epsilon}$. Moreover,

$$\sum_j |F_j'(z_n)| = O\left((1-|z_n|^{-1})^{\left(\frac{2+\epsilon}{1+\epsilon}\right)}\right), \quad |z_n| \to 1^-,$$

hence, by property (i) we obtain $F_j{}' \circ \varphi_\epsilon \in H^s$ for all $s < \frac{1+\epsilon}{4+2\epsilon}$. Then $f_j \circ \varphi_\epsilon \in H^s$ for all $s < \frac{1+\epsilon}{4+2\epsilon}$, the claim follows by property (ii) and the proof is complete.

Corollary (6.1.29):[168] Let $\epsilon > 0$, let $0 < \alpha < \sqrt{\epsilon + 1}$.

(i) Every continuous linear functional l on $D^{(1+\epsilon, \alpha^2-1)}$ can be represented uniquely in the form

$$l\left(\sum_j f_j\right) = \lim_{r\to 1^-}\int_{\mathbb{T}}\sum_j f_j(rz_n)\overline{\left(g_j\right)_l(rz_n)}\,dm(z_n),$$

where $\left(g_j\right)_l \in L_{(a^2-1)}^{(1+\epsilon,\alpha^2-1)}, \epsilon > 0$. The linear map $l \to \left(g_j\right)_l$ from the dual of $D^{(1+\epsilon,\alpha^2-1)}$ into $L_{(a^2-1)}^{\left(1+\epsilon,\frac{1-\alpha^2}{\epsilon}\right)}$ is continuous and bijective. Moreover, $D^{(1+\epsilon,\alpha^2-1)}$ is reflexive.

(ii) $D^{(1+\epsilon,\alpha^2-1)}$ is continuously contained in $H^{(1+\epsilon)}$.

(iii) If θ is an inner function and $f_j \in D^{(1+\epsilon,\alpha^2-1)}$ satisfies $\frac{f_j}{\theta} \in H^{(1+\epsilon)}$, then $\frac{f_j}{\theta}$ belongs to $D^{(1+\epsilon,\alpha^2-1)}$ and there is a constant $K > 0$ independent of f_j such that

$$\sum_j \left\|\frac{f_j}{\theta}\right\|_{\left(D^{1+\epsilon,\alpha^2-1}\right)} \le K\sum_j \|f_j\|_{D^{(1+\epsilon,\alpha^2-1)}}.$$

Proof: (i) Using Parseval's formula we can write for f_j, g_j analytic on $D^{(1+\epsilon,\alpha^2-1)}$ and $0 < r < 1$

$$\int_{\mathbb{T}}\sum_j f_j(rz_n)\overline{g_j(rz_n)}\,dm(z_n) = \int_{\mathbb{D}}\sum_j \left(\zeta f_j\right)'(rz_n)\overline{g_j(rz_n)}\,dA(z_n).$$

The linear map $f_j \to \left(\zeta f_j\right)'$ from $D^{(1+\epsilon,\alpha^2-1)}$ into $L_{(a^2-1)}^{(1+\epsilon,\alpha^2-1)}$ is continuous and invertible, so that, all we need to show is that the dual of $L_{(a^2-1)}^{(1+\epsilon,\alpha^2-1)}$ can be identified with $L_{(a^2-1)}^{\left(\frac{1+\epsilon}{\epsilon},\frac{1-\alpha^2}{\epsilon}\right)}$ via the pairing

$$\sum_j \langle h_j, g_j\rangle = \lim_{r\to 1^-}\int_{\mathbb{D}}\sum_j h_j(rz_n)\overline{g_j(rz_n)}\,dA(z_n).$$

To prove reflexivity, we have to show that via the above pairing the dual of $L_{(a^2-1)}^{\left(\frac{1+\epsilon}{\epsilon},\frac{1-\alpha^2}{\epsilon}\right)}$ is $L_{(a^2-1)}^{(1+\epsilon,\alpha^2-1)}$ [146]. Part (ii) follows directly from (i) since $H^{\left(\frac{1+\epsilon}{\epsilon}\right)}, \epsilon > 0$ is continuously contained in $L_{(a^2-1)}^{\left(\frac{1+\epsilon}{\epsilon},\frac{1-\alpha^2}{\epsilon}\right)}$. (iii) asserts that $D^{(1+\epsilon,\alpha^2-1)}$ has the so-called (F)-property (see [147]). The proof follows with the method in [147]. If θ is inner then the operator M_θ of multiplication by θ is a bounded linear operator on $L_{(a^2-1)}^{\left(\frac{1+\epsilon}{\epsilon},\frac{1-\alpha^2}{\epsilon}\right)}$. Since $D^{\left(1+\epsilon,\frac{1-\alpha^2}{\epsilon}\right)}$ is reflexive, its adjoint M_θ^* is bounded on $D^{(1+\epsilon,\alpha^2-1)}$.If $\frac{f_j}{\theta} \in H^{(1+\epsilon)}$ and $g_j \in H^{\left(\frac{1+\epsilon}{\epsilon}\right)}$ then

$$\lim_{r\to 1^-}\int_{\mathbb{T}}\sum_j f_j(rz_n)\overline{M_\theta g_j(rz_n)}\,dm(z_n) = \int_{\mathbb{T}}\sum_j \left(\frac{f_j}{\theta}\right)(z_n)\overline{g_j(z_n)}\,dm(z_n).$$

From the fact that $H^{\left(\frac{1+\epsilon}{\epsilon}\right)}$ is dense in $L_{(a^2-1)}^{\left(\frac{1+\epsilon}{\epsilon},\frac{1-\alpha^2}{\epsilon}\right)}$ we obtain that $M^*_{\theta f_j} = f_j/\theta$ and the result follows.

As a direct application of part (ii) we obtain that the Volterra operators $V_{(a^2-1)}, a \leq \sqrt{2}$ are bounded on $D^{(1+\epsilon,\alpha^2-1)}$ and satisfy $V_{(a^2-1)}D^{(1+\epsilon,\alpha^2-1)} \subset A_0 \cap D^{(1+\epsilon,\alpha^2-1)}$. Indeed, by the Fejer–Riesz inequality we have that

$$\sum_j \left|V_{a^2-1}\left(f_j(0)\right)\right| \leq K \sum_j \|f_j\|_{H^{(1+\epsilon)}} \leq K' \sum_j \|f_j\|_{D^{(1+\epsilon,\alpha^2-1)}}$$

for some constant $K' > 0$ and all $f_j \in D^{(1+\epsilon,\alpha^2-1)}$. The inequality

$$\int_{\mathbb{D}} \sum_j \left|\left(V_{(a^2-1)}f_j\right)'(z_n)\right|^{(1+\epsilon)} (1-|z_n|)^{(\alpha^2-1)}\, dA(z_n)$$
$$= \int_{\mathbb{D}} \sum_j \left|f_j(z_n)\right|^{(1+\epsilon)} (1-|z_n|)^{(\alpha^2-1)}\, dA(z_n) \leq K'' \sum_j \|f_j\|^{(1+\epsilon)}_{D^{(1+\epsilon,\alpha^2-1)}}$$

follows also by standard estimates (see [105]), but can also be obtained by a direct application of Minkowski's inequality.

Corollary (6.1.30):[168] Let $\epsilon > 0$ and $\alpha > 0$ such that $\alpha < \sqrt{\epsilon}$. Then for $\epsilon < 1$ and $|a^2-1| = 1$, $S_{(a^2-1)}^{(1-\epsilon)} H^\infty \cap A_1$ is dense in $S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)} \cap D^{(1+\epsilon,\alpha^2-1)}$.

Proof: Let $f_j \in S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)} \cap D^{(1+\epsilon,\alpha^2-1)}$. By Corollary (6.1.29) (iii) we can approximate $f_j S_{(a^2-1)}^{(\epsilon-1)} \in D^{(1+\epsilon,\alpha^2-1)}$ in $D^{(1+\epsilon,\alpha^2-1)}$ by a sequence $\left(\left(f_j\right)_n\right)$ of functions in A_1. Then it is a simple matter to show that $(a^2-1-\zeta)^2 S_{(a^2-1)}^{(1-\epsilon)}\left(f_j\right)_n \to (a^2-1-\zeta)^2 f_j$ in $D^{(1+\epsilon,\alpha^2-1)}$. Thus $(a^2-1-\zeta)^2 f_j$ belongs to the closure of $S_{(a^2-1)}^{(1-\epsilon)} H^\infty \cap H_1$ whenever $f_j \in S_{(a^2-1)}^{(1-\epsilon)} H^{(1+\epsilon)} \cap D^{(1+\epsilon,\alpha^2-1)}$. To finish the proof it suffices to show that for such f_j we have

$$\lim_{r\to 1^-} \sum_j \frac{(a^2-1-\zeta)^2}{(a^2-1-r\zeta)^2} f_j = \sum_j f_j$$

in the norm of $D^{(1+\epsilon,\alpha^2-1)}$. Since $f_j S_{(a^2-1)}^{(\epsilon-1)} \in D^{(1+\epsilon,\alpha^2-1)}$ it follows that

$$\sum_j \left\| f_j S_{(a^2-1)}^{(\epsilon-1)} \right\|^{(1+\epsilon)}_{D^{(1+\epsilon,\alpha^2-1)}}$$
$$\geq \int_{\mathbb{D}} \sum_j \left| f_j{}'(z_n) + \frac{2(1-\epsilon)}{(z_n-a^2+1)^2} f_j(z_n) \right|^{(1+\epsilon)} (1-|z_n|)^{(\alpha^2-1)}\, dA(z_n)$$

and this implies that

$$\int_{\mathbb{D}} \sum_j |z_n - a^2 + 1|^{-2(1+\epsilon)} \left|f_j(z_n)\right|^{(1+\epsilon)} (1 - |z_n|)^{(\alpha^2-1)} \ dA(z_n) < \infty.$$

But from this inequality and the dominated convergence theorem we obtain that

$$\lim_{r\to 1^-} \int_{\mathbb{D}} \sum_j \left| \left(\frac{(a^2-1-\zeta)^2}{(a^2-1-r\zeta)^2} f_j \right)' (z_n) \right|^{(1+\epsilon)} (1 - |z_n|)^{(\alpha^2-1)} \, dA(z_n)$$
$$= \int_{\mathbb{D}} \sum_j \left|f_j{}'(z_n)\right|^{(1+\epsilon)} (1 - |z_n|)^{(\alpha^2-1)} \ dA(z_n).$$

Then (9) follows by a standard argument and the proof is complete.

Corollary (6.1.31):[168] For $\epsilon > 0$ and $|a^2 - 1| = 1$ the closure of $A_1^2(a^2-1)$ in $D^{(1+\epsilon,\alpha^2-1)}$ equals $D^{(1+\epsilon,\alpha^2-1)}$ if $\alpha \geq \sqrt{\epsilon}$ and $D^{(1+\epsilon,\alpha^2-1)}(a^2-1)$ if $\alpha^2 \geq \sqrt{\epsilon}$.

Proof: Since polynomials are dense in $D^{(1+\epsilon,\alpha^2-1)}$ it follows easily that the set of polynomials which vanish at a is dense in $D^{(1+\epsilon,\alpha^2-1)}(a^2-1)$. Thus it suffices to show that $1, \zeta - a^2 + 1$ belong to the closure of $A_1^2(a^2-1)$ in $D^{(1+\epsilon,\alpha^2-1)}$ when $\alpha^2 \geq \sqrt{\epsilon}$, and that $\zeta - a^2 + 1$ belongs to the closure of $A_1^2(a^2-1)$ in $D^{(1+\epsilon,\alpha^2-1)}$ when $\alpha^2 < \sqrt{\epsilon}$. For $r > 1$ consider the functions

$$\left(f_j\right)_r(z_n) = (z_n - a^2 + 1)^2 \left(z_n - r(a^2-1)\right)^{-1},$$

$$\left(g_j\right)_r(z_n) = (z_n - a^2 + 1)(z_n - r(a^2-1))^{-1}, z_n \in \mathbb{D}$$

Clearly, $\left(f_j\right)_r \in A_1^2(a^2-1), \left(f_j\right)_r(z_n) \to z_n - a^2 + 1$ when $r \to 1^+$, and $\left|\left(f_j\right)_r'(z_n)\right| \leq 3$ for all $z_n \in \mathbb{D}$. Using the dominated convergence theorem it is a standard matter to show that $\left(f_j\right)_r \to \zeta - a^2 + 1$ in $D^{(1+\epsilon,\alpha^2-1)}$ for all $\epsilon > 0$ and all $\alpha > 0$. The functions $\left(g_j\right)_r$ satisfy $\left(g_j\right)_r \in A_1^2(a^2-1), \left(g_j\right)_r(z_n) \to 1$ when $r \to 1^-$ for all $z_n \in \mathbb{D}$ and

$$\left|\left(g_j\right)_r'(z_n)\right| = \frac{r-1}{|z_n - r(a^2-1)|^2} + \frac{2(r-1)^2}{|z_n - r(a^2-1)|^3}, \qquad z_n \in \mathbb{D}.$$

By standard estimates (see [105]) it follows that $\left\|\left(g_j\right)_r\right\|_{D^{(1+\epsilon,\alpha^2-1)}}$ stay bounded when $r \to 1^+$. Since $D^{(1+\epsilon,\alpha^2-1)}$ is reflexive, we conclude that $\left(g_j\right)_r \to 1$ weakly in $D^{(1+\epsilon,\alpha^2-1)}$ and the proof is complete.

Section (6.2): Parabolic Self-maps in the Hardy Space

The problem of giving a precise description of the lattice of invariant subspaces of abounded linear operator on Hilbert space is one of the most interesting and difficult in operator theory. Very few operators admit a useful description of the lattice of invariant subspaces. Understanding the lattice of a particular operator can solve the invariant subspace problem. This was done by, [148,149]. They consider the composition operator C_φ acting on the Hardy space, where φ is an automorphism of the disk fixing ± 1. They show that if every invariant subspace of C_φ of infinite dimension has a non-trivial invariant subspace, then the general conjecture is true.

Beurling's Theorem provides a complete description of the invariant subspaces of the shift operator acting on $\mathcal{H}^2$ the lattice of the invariant subspace shift operator acting on the Bergman space is not completely understood, [150,70,104].

We will describe the invariant subspaces of the composition operators C_φ acting on the Hardy space $\mathcal{H}^2$ where φ is a parabolic on-automorphism that takes $\mathbb{D}$ into itself, which has the formula

$$\varphi_a(z) = \frac{(2-a)z + a}{-az + 2 + a}, \qquad where \;\; \mathcal{R}a > 0 \tag{10}$$

Since $\varphi_a(\mathbb{D})$ is contained in $\mathbb{D}$, Littlewoods Subordination Principle implies, the composition operator $\left(C_{\varphi_a} f\right) = f\left(\varphi_a(z)\right)$ acts bounded on $\mathcal{H}^2$ [144].

If T is an operator on Hilbert space $\mathcal{H}$ and x is a vector in $\mathcal{H}$, then the smallest invariant subspace of T that contains x is the closure of the linear span of the orbit of x under T. If the minimal subspace is $\mathcal{H}$, then x is called a cyclic vector. We describe all cyclic vector for C_{φ_a}. The family of all composition operators induced by parabolic non-automorphism have common dynamics, since they have common cyclic vector, Corollary (6.2.2). Each orbit of any vector under all composition operators induced by parabolic non-automorphism has a common closure. See[144,151].

If $\mathcal{R}a > 0$, the spectrum $\sigma\left(C_{\varphi_a}\right)$ is the spiral

$$\sigma\left(C_{\varphi_a}\right) = \{0\} \cup \{e^{-at} : t \in [0, \infty)\}.$$

Indeed, C_{φ_a} has a well-known family of inner functions as its eigenfunctions,

$$C_{\varphi_a} e_t = e^{-at} e_t, \;\; \text{where } e_t(z) = \exp\left(t \frac{z+1}{z-1}\right) \text{for each } t \geq 0. \tag{11}$$

All invariant subspaces we consider will be closed. Let Lat T denote the lattice of invariant subspaces of the bounded linear operator T and let $\mathbb{F}[0, \infty)$ denote the set of closed subsets of $[0, \infty)$. As usual, the closed span of the empty set is the trivial subspace consisting of just the zero vector.

Corollary(6.2.1):[147] Composition operators induced by parabolic non automorphism that take the unit disk into itself have the same lattice of invariant subspaces and the same cyclic vectors.

Recall that a subspace that is invariant for an operator as well as for its adjoint is called a reducing subspace.

Theorem(6.2.2):[147] The map Ψ is an isometric isomorphism from $L^2(\mathbb{T})$ onto $W^{1,2}(\mathbb{R})$. In addition, $\Psi(z\mathcal{H}^2) = W_0^{1,2}[0,\infty)$ and $\Psi\left(\bar{z}\overline{\mathcal{H}^2}\right) = W_0^{1,2}(-\infty,0]$.

Proof: For each f in $L^2(\mathbb{T})$, we have

$$(\Psi f)(t) = \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)\exp\left(t\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)d\theta, \qquad t \in \mathbb{R}.$$

The change of variables $x = \frac{i(1+e^{i\theta})}{(1-e^{i\theta})}$ yields

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{1+x^2}\,dx, \qquad t \in \mathbb{R}. \tag{12}$$

Therefore, $\Psi = \mathcal{F}MT$, where $\mathcal{F}$ denotes the Fourier transform,

$$(Mg)(y) = \frac{1}{\sqrt{\pi}}\frac{g(y)}{\sqrt{1+y^2}} \quad \text{and} \quad (Tf)(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+x^2}}\, f\left(\frac{x-i}{x+i}\right).$$

The obvious change of variables shows that T is an isometric isomorphism from $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$. In addition, the properties of the Fourier transform along Plancherel's Theorem show that $\mathcal{F}M$ is an isometric isomorphism from $L^2(\mathbb{R})$ onto $W^{1,2}(\mathbb{R})$, which proves the first statement of the proposition.

Now, let f be in $z\mathcal{H}^2$, that is, $f(z) = zg(z)$ with g in $\mathcal{H}^2$. Using (10), we obtain

$$(\Psi f)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} g\left(\frac{x-i}{x+i}\right)\frac{e^{-itx}}{(x+i)^2}\,dx, \quad \text{for each } t \in \mathbb{R}.$$

Since the map

$$h \to \frac{1}{\sqrt{\pi}(x+i)}h\left(\frac{x-i}{x+i}\right)$$

is an isometric isomorphism from $\mathcal{H}^2$ onto $\mathcal{H}^2(\Pi)$, [153], and multiplication by $(w+i)^{-1}$ is bounded on $\mathcal{H}^2(\Pi)$, we find that Ψf is the Fourier transform of a function of $\mathcal{H}^2(\Pi)$. Thus, the Paley-Wiener Theorem, [152], shows that Ψf, which is continuous, must vanish on $(-\infty, 0]$ and, therefore, $\Psi(z\mathcal{H}^2) \subset W_0^{1,2}[0,\infty)$. Similarly, $\Psi(\bar{z}\overline{\mathcal{H}}^2) \subset W_0^{1,2}(-\infty,0]$. The fact that $\Psi(z\mathcal{H}^2) = W_0^{1,2}[0,\infty)$, and $\Psi(\bar{z}\overline{\mathcal{H}}^2) = W_0^{1,2}[-\infty,0)$ follows immediately from the orthogonal decomposition $W^{1,2}(\mathbb{R}) = W_0^{1,2}(-\infty,0]\oplus\left[e^{-|t|}\right]\oplus W_0^{1,2}[0,\infty)$, which in turns follows, being Ψ an isometric isomorphism, from the orthogonal decomposition $L^2(\mathbb{T}) = \bar{z}\overline{\mathcal{H}}^2\oplus[1]\oplus z\mathcal{H}^2$ and the fact that $\Psi_1 = e^{-|t|}$, where $[f]$ denotes the one-dimensional linear space spanned by the vector f.

Corollary(6.2.3):[147] The operator Φ defines an isomorphism from $\mathcal{H}^2$ onto $W^{1,2}[0,\infty)$. Indeed, $\|\Phi f\|_{1,2}^2 = \|f\|_{\mathcal{H}^2}^2 - |f(0)|^2/2$.

Proof: Upon applying Theorem (6.2.4), Φ and Ψ coincide on $z\mathcal{H}^2$ and, therefore, Φ defines an isometric isomorphism from $z\mathcal{H}^2$ on to $W_0^{1,2}[0,\infty)$. Since $e^{-|t|}$ is orthogonal to $W_0^{1,2}[0,\infty]$, so is $e^{-t}\chi_{[0,\infty)}$. Thus $W^{1,2}[0,\infty) = [e^{-t}\chi_{[0,\infty)}] \oplus W_0^{1,2}[0,\infty) = (\Phi 1) \oplus \Phi(z\mathcal{H}^2) = \Phi(\mathcal{H}^2)$, which proves that Φ is an isomorphism.The formula for the norm is trivial.

Proposition(6.2.4):[147] Let φ_a, with $\mathcal{R}a \geq 0$, be as in (10). Then the adjoint of C_{φ_a} acting on $\mathcal{H}^2$ is similar under Φ to the multiplication operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$, acting on $W^{1,2}[0,\infty)$.

Proof: Using the eigenvalues equation (11), for each $f \in \mathcal{H}^2$, we have

$$\left(\Phi C^{\star}_{\varphi_a} f\right)(t) = \langle C^{\star}_{\varphi_a} f, e_t\rangle_{\mathcal{H}^2} = \langle f, C_{\varphi_a} e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}\langle f, e_t\rangle_{\mathcal{H}^2} = e^{-\bar{a}t}(\Phi f)(t),$$

for each $t \geq 0$. Thus $M_\psi = \Phi C^{\star}_{\varphi_a} \Phi^{-1}$.

Proposition(6.2.5):[147] The operator M_ψ, where $\psi(t) = e^{-\bar{a}t}$ and $\mathcal{R}a > 0$, acting on $W^{1,2}[0,\infty)$ is cyclic with cyclic vector ψ.

Proof: Let $k_\alpha(z) = (1 - \bar{\alpha}z)^{-1}$, where $\alpha = \frac{(a-1)}{(a+1)}$, be the reproducing kernel at $\alpha \in \mathbb{D}$ in the Hardy space $\mathcal{H}^2$. Since $\Phi k_\alpha = \psi$, by Proposition (6.2.4), it is enough to show k_α is cyclic for $C^{\star}_{\varphi_a}$. Suppose that f in $\mathcal{H}^2$ is orthogonal to the orbit of k_α under $C^{\star}_{\varphi_a}$. Then, for each $n \geq 0$, we have

$$0 = \langle C^{\star n}_{\varphi_a} k_\alpha, f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C^n_{\varphi_a} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, C_{\varphi_{na}} f\rangle_{\mathcal{H}^2} = \langle k_\alpha, f \circ \varphi_{na}\rangle_{\mathcal{H}^2} = f\left(\varphi_{na}(\bar{\alpha})\right).$$

Since $\{\varphi_{na}(\bar{\alpha})\}$ is not a Blaschke sequence, the function f and the result follows.

An interesting consequence of Corollary(6.2.3) is a summability theorem for the Laguerre polynomials. Set $u_n(z) = z^n$. Then $\tilde{u}_n(t) = (\Phi u_n)(t) = L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$, where $L_n^{(-1)}(t)$ is the Laguerre polynomial of degree n and of index -1. Indeed, since $\tilde{u}_n = \langle z^n, e_t(z)\rangle_{\mathcal{H}^2}$ is the n-th coefficient of the Taylor series of $e_t(z)$, by definition of the Laguerre polynomials see [155], we have

$$e_t(z) = e^{-t}\exp\left(-\frac{2tz}{1-z}\right) = \sum_{n=0}^{\infty} e^{-t} L_n^{(-1)}(2t) z^n. \tag{13}$$

Therefore, the following follows immediately.

Corollary(6.2.6):[147] Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers. Then the series $\tilde{f}(t) = \sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}\chi_{[0,\infty)}$ converges in $W^{1,2}[0,\infty)$ if and only if $\{a_n\}$ is in the sequence space ℓ^2. Indeed, $\|\tilde{f}\|_{1,2}^2 = -\frac{|a_0|^2}{2} + \|\{a_n\}_{n\geq 1}\|_2^2$. [156].

Corollary(6.2.7):[147] Each f in $W^{1,2}[0,\infty)$ satisfies $\|f\|_\infty \leq \sqrt{2}\|f\|_{1,2}$ and $\sqrt{2}$ is the best imbedding constant.

Proof: By Corollary (6.2.6), we can write $f(t) = \sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}$, where $\{a_n\}$ is in ℓ^2.The Cauchy-Schwarz inequality and Corollary (6.2.8), for each $t \geq 0$, yields

$$|f(t)| = \left|\sum_{n=0}^{\infty} a_n L_n^{(-1)}(2t)e^{-t}\right| \leq \|f\|_{1,2}\left(2e^{-2t} + \sum_{n=1}^{\infty}\left(L_n^{(-1)}(2t)\right)^2 e^{-2t}\right)^{\frac{1}{2}}.$$

Since $\|e_t\|_{\mathcal{H}^2} = 1$, using (13), one easily checks that the quantity into the brackets above equals to $1 + e^{-2t} \leq 2$ and, therefore, $\|f\|_{\infty} \leq \sqrt{2}\|f\|_{1,2}$. The fact that $\sqrt{2}$ is the best imbedding constant is straight forward.

Proposition (6.2.8):[147] The space $W^{1,2}[0,\infty)$ with the pointwise multiplication is a Banach algebra without identity.

An element a in a Banach algebra $\mathcal{A}$ is called cyclic, if it is cyclic for the bounded multiplication operator M_a that assigns to each b in $\mathcal{A}$ the element ab.

Proposition (6.2.9):[147] let $\mathcal{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are the closed ideal of $\mathcal{A}$.

Proof: First, since $\mathcal{A}$ has a cyclic element, it is commutative. Let a be a cyclic element of $\mathcal{A}$ and let $\mathcal{L}$ be an invariant subspace of M_a. Clearly,

$$\mathcal{M}_{\mathcal{L}} = \{b \in \mathcal{A}: bx \in \mathcal{L} \text{ for all } x \in \mathcal{L}\}$$

is a closed subalgebra of $\mathcal{A}$. Since $\mathcal{L}$ is an invariant subspace of M_a, we find that $a \in \mathcal{M}_{\mathcal{L}}$ and, therefore, $\mathcal{M}_{\mathcal{L}}$ contains the subalgebra generated by a and, being $\mathcal{M}_{\mathcal{L}}$ closed and a cyclic, it follows that $\mathcal{M}_{\mathcal{L}} = \mathcal{A}$. Hence, $\mathcal{L}$ is a left ideal and thus, being $\mathcal{A}$ commutative, an ideal of $\mathcal{A}$. On the other hand, each ideal of $\mathcal{A}$ is invariant with respect to M_a.

Theorem (6.2.10):[147] The closed ideals of $W^{1,2}[0,\infty)$ are

$$\mathcal{I}_F = \{f \in W^{1,2}[0,\infty): f \text{ vanishes on } F\}, \text{where } F \in \mathbb{F}[0,\infty).$$

Theorem (6.2.11):[147] Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then

$$\operatorname{Lat} C_{\varphi} = \{\overline{span}\{e_t: t \in F\}: F \in \mathbb{F}[0,\infty)\}.$$

Proof: By Proposition (6.2.5), the symbol ψ is a cyclic element of the Banach algebra $W^{1,2}[0,\infty)$. Thus, from Proposition (6.2.9) and Theorem (6.2.10) it follows that

$$\operatorname{Lat} M_{\psi} = \left\{\{\hat{f} \in W^{1,2}[0,\infty): \hat{f} \text{ vanishes on } F\}, \text{where } F \in \mathbb{F}[0,\infty)\right\},$$

Since $M_{\psi} = \Phi C_{\varphi}^{\star}\Phi^{-1}$, we have

$$\operatorname{Lat} C_{\psi}^{\star} = \left\{\{f \in \mathcal{H}^2: \langle f, e_t\rangle_{\mathcal{H}^2} = 0 \text{ for each } t \in F\}, \text{where } F \in \mathbb{F}[0,\infty)\right\}.$$

Since Lat C_φ consists of the orthogonal complements of Lat$C_\varphi^\star$, the statement of Theorem (6.2.11) follows immediately.

Theorem (6.2.12):[147] Let φ be a parabolic non-automorphism that takes the unit disk into itself. Then C_φ has no non-trivial reducing subspace.

Proof: Let F be in $\mathbb{F}[0,\infty)$ such that $N_F = \overline{\text{span}}\{e_t : t \in F\}$ is non-trivial. We must show that its orthogonal complement $N_F^\perp$ is not invariant under C_φ. We need the following formula, which is easily checked

$$\langle e_t, e_s\rangle = e^{-|t-s|}, \quad \text{for each} \quad t, s \geq 0. \tag{14}$$

First assume that 0 is not in F. Set $t_0 = \min F$. Since $f_{t_0} = 1 - e^{-t_0} e_{t_0}$ is orthogonal to e_t for each $t \geq t_0$, we find that t_{t_0} is in $N_F^\perp$. If $N_F^\perp$ is invariant under C_φ, then $f_{t_0} - C_\varphi f_{t_0}$ is in $N_F^\perp$. But $f_{t_0} - C_\varphi f_{t_0} = e^{t_0}(1 - e^{-at_0})$ is also in N_F, which means that $f_{t_0} - C_\varphi f_{t_0} = 0$. Hence, $f_{t_0} \equiv 1$, a contradiction.

Assume now that 0 is in F. Let M_{ϵ_1}denote the multiplication by e_1. We have

$$M_{e_1}(N_F) = e_1 \overline{\text{span}}\{e_t : t \in F\} = \overline{\text{span}}\{e_{1+t} : t \in F\} = N_{1+F}. \tag{15}$$

Clearly, $M_{e_1}$is a Hilbert space isometry preserving inner products. Therefore,

$$M_{e_1}(N_F^\perp) = M_{e_1}(N_F)^\perp \tag{16}$$

Proceeding by contradiction, assume that $N_F^\perp$ is also invariant under C_φ. Then

$$M_{e_1}\left(C_\varphi(N_F^\perp)\right) \subseteq M_{e_1}(N_F^\perp).$$

Since, for f in $\mathcal{H}^2$, we have $C_\varphi(M_{e_1} f) = C_\varphi(e_1 f) = e^{-a} e_1 C_\varphi f = e^{-a} M_{e_1}(C_\varphi f)$, from the above display, it follows that $C_\varphi\left(M_{e_1}(N_F^\perp)\right)$ is included in $M_{e_1}(N_F^\perp)$. Therefore, from (15) and (16), we immediately see that $C_\varphi(N_{1+F}^\perp) \subseteq N_{1+F}^\perp$, which is a contradiction because 0 is not in $1 + F$.

Proposition (6.2.13):[147] The spectrum of the Banach algebra $W^{1,2}[0,\infty)$ is

$$\Omega\big(W^{1,2}[0,\infty)\big) = \{\delta_t : t \geq 0\}.$$

Furthermore, the mapping that to each t assigns δ_t is a homeomorphism from $[0,\infty)$ onto $\Omega\big(W^{1,2}[0,\infty)\big)$.

Proof: Clearly, for each $t \geq 0$, the functional δ_t is a character on $W^{1,2}[0,\infty)$ that is, $\delta_t$is in $\Omega = \Omega\big(W^{1,2}[0,\infty)\big)$. To prove that each character on $W^{1,2}[0,\infty)$ is one of the δ_t's, we begin by considering the Banach algebra $\mathcal{C}^1[0,1]$, with point wise multiplication endowed with the norm $\|f\| = \max\{\|f\|_\infty, \|f'\|_\infty\}$. Consider also its Banach sub algebra $\mathcal{A}_0 = \{f \in$

$\mathcal{C}^1[0,1]: f(1) = 0\}$. Then, it is easy to check that $(Tf)(x) = f\left(\frac{x}{1+x}\right)$ defines a bounded operator from $\mathcal{A}_0$ in to $W^{1,2}[0,\infty)$ which is also an algebra homomorphism. Now, if $\varkappa$ is a character of $W^{1,2}[0,\infty)$, then it is easy to see that the functional $\bar{\varkappa}$ on $\mathcal{C}^1[0,1]$ defined by $\bar{\varkappa}(f) = \varkappa\left(T\big(f - f(1)\big)\right) + f(1)$ is also a character. Since the characters of $\mathcal{C}^1[0,1]$ are the point evaluations $f \to f(s)$, with $0 \le s \le 1$, [159], there is $0 \le s \le 1$ such that $\tilde{\varkappa}(f) = f(s)$ for each f in $\mathcal{C}[0,1]$. if $s = 1$, it follows immediately that $\varkappa(Tf) = 0$ for each f in $\mathcal{A}_0$. Hence $\varkappa$ vanishes on the range of T, which is dense because it contains $\mathcal{C}_c^\infty[0,\infty)$, therefore, $\varkappa$ is the zero functional. If $s \ne 1$, then set $t = \frac{s}{(1-s)} \ge 0$ and observe that $\varkappa(Tf) = (Tf)(t)$ for each $f \in \mathcal{A}_0$. Hence $\varkappa$ and δ_t coincide on a dense set, which implies that $\varkappa = \delta_t$. Thus we have shown that $\Omega = \{\delta_t : t \ge 0\}$.

Next, since each f in $W^{1,2}[0,\infty)$is continuous, so is the mapping $t \to \delta_t$ from $[0,\infty)$ onto Ω. Since $\|\delta_t\|_{1,2} \le \|\Phi^{-1}\| \|e_t\|_{\mathcal{H}^2} = \|\Phi^{-1}\|$, we find that Ω is norm bounded on the dual space. Since the weak topology of a separable Hilbert space is metrizable on bounded sets, it follows that Ω is metrizable. Thus, to prove that $t \to \delta_t$is a homeomorphism, it suffices to show that $t_n \to t_0$ whenever $\delta_{t_n} \to \delta_{t_0}$. Suppose that this is not the case, then there is $\epsilon > 0$ such that $|t_n - t_0| > \epsilon$ for each positive integer n. Consider the $W^{1,2}[0,\infty)$-function defined for $t \ge 0$ by

$$f(t) = \begin{cases} \epsilon - |t_0 - s|, & if \quad |t_0 - s| \le \epsilon; \\ \quad 0, & \quad \text{otherwise} \end{cases}$$

Since $\delta_{t_n}(f) = 0$ and $\delta_{t_0}(f) = \epsilon$, we find that δ_{t_n}cannot converge to δ_{t_0}. Therefore, the mapping $t \to \delta_t$ is a homeomorphism.

Lemma (6.2.14):[147] Let $\mathcal{A}$ be a semisimple regular commutative Banach algebra. Then the closed of $\mathcal{A}$ are

$$\mathcal{I}_F = \left\{\bigcap_{\aleph \in F} \ker \aleph \ : F \text{ is closed in } \Omega(\mathcal{A})\right\}$$

If and only if for each $x \in \mathcal{A}$ there exists a sequence $\{x_n\}$ tending to x in $\mathcal{A}$ and $\hat{x}_n$ vanishes on a neighborhood U_n of $h(x)$ with compact complement.

Proposition (6.2.15):[147] The Banach algebra $W^{1,2}[0,\infty)$is semisimple and regular and the mapping $F \to \cap_{t \in F} \ker \delta_t$ is one-to-one from $\mathbb{F}[0,\infty)$ onto the set of closed ideals of $W^{1,2}[0,\infty)$.

Proof: Since the characters $\delta_t{}'$s separate points, the Banach algebra $W^{1,2}[0,\infty)$ is semisimple. To prove that $W^{1,2}(\infty,0]$ is also regular, we have to show that for each closed F in Ω and each maximal regular ideal $\mathcal{M} \notin F$ there exist f in $W^{1,2}[0,\infty)$such that $\hat{f} = 0$ on F and $\hat{f}(\mathcal{M}) \ne 0$. By Proposition(6.2.13) $F \subseteq [0,\infty)$ and each point $t_0 \in [0,\infty) \backslash F$ there exists f in $W^{1,2}[0,\infty)$ such that f vanishes on F and $f(t_0) \ne 0$, which is obvious.

Section (6.3): Shift Plus Complex Volterra Operator

Let $\mathbb{D}$ be the unit disk of the complex plane and $H(\mathbb{D})$ be the space of holomorphic functions on the unit disk. We say that a holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disk belongs to the Hardy space $\mathcal{H}^2(\mathbb{D})$, if its sequence of power series coefficients is square-summable:

$$\mathcal{H}^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

We define a norm on $\mathcal{H}^2(\mathbb{D})$ by

$$\|f\|_{\mathcal{H}^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2. \quad (17)$$

It is well known that $\mathcal{H}^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{\mathcal{H}^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(x) = \sum_{n=0}^{\infty} b_n z^n$.

The operator defined on $\mathcal{H}^2(\mathbb{D})$

$$(M_z f)(z) = z f(z) \quad \text{for } f \in \mathcal{H}^2(\mathbb{D}) \text{ } and \text{ } z \in \mathbb{D}$$

is called the shift operator. The lattice of the shift operator acting on the Hardy space is completely described by Beurling's Theorem [2], and it is one of the most celebrated and widely used results. Let $L^2(0,1)$ be the space of square integrable functions on $(0,1)$. Sarason [138] characterized all closed invariant subspaces of the Volterra operator

$$(Vf)(x) = \int_0^x f(y)dy \text{ for } f \in L^2(0,1) \text{ } and \text{ } 0 < x \leq y < 1.$$

Aleman and Korenblum studied the complex Volterra operator in $\mathcal{H}^2(\mathbb{D})$ defined by

$$(\mathcal{V}f)(z) = \int_0^z f(w)dw,$$

then they characterized the lattice of closed invariant subspaces of $\mathcal{V}$ in [134]. While doing so they used the Beurling's Theorem. Sarason [163] studied the lattice of closed invariant subspaces of multiplication by x plus Volterra operator, $M_x + V$ acting on $L^2(0,1)$. Montes-Rodriguez, Ponce-Escudero and Shkarin [147] and Cowen, Gunatillake and Ko [164] used the idea of Sarason to study the invariant subspaces of certain classes of composition operators on Hardy spaces.

Following Sarason's work we are interested in characterizing the lattice of closed invariant subspaces of the shift plus complex Volterra operator on the Hardy space. Denote by T the operator

$$(Tf)(z) = zf(z) + \int_0^z f(w)dw, \quad \text{for } f \in \mathcal{H}^2(\mathbb{D}) \text{ and } z \in \mathbb{D}. \tag{18}$$

Since the shift operator is an isometry and the complex Volterra operator is a contraction, T is clearly bounded operator on the Hardy space. To show the main result we use the space $\mathcal{S}^2(\mathbb{D})$ defined by

$$\mathcal{S}^2(\mathbb{D}) = \{f \in H(\mathbb{D}): Df \in \mathcal{H}^2(\mathbb{D})\},$$

where $D = \frac{d}{dz}$ is the differential operator. It is clear that if Df is in $\mathcal{H}^2(\mathbb{D})$, then f belongs to $\mathcal{H}^2(\mathbb{D})$. The norm of $\mathcal{S}^2(\mathbb{D})$ is defined by

$$\|f\|^2_{\mathcal{S}^2(\mathbb{D})} = \|Df\|^2_{\mathcal{H}^2(\mathbb{D})} + \|f\|^2_{\mathcal{H}^2(\mathbb{D})}. \tag{19}$$

Corresponding inner product is given by

$$\langle f, g\rangle_{\mathcal{S}^2(\mathbb{D})} = \langle Df, Dg\rangle_{\mathcal{H}^2(\mathbb{D})} + \langle f, g\rangle_{\mathcal{H}^2(\mathbb{D})}.$$

We work describe the lattice of closed invariant subspaces of T acting on $\mathcal{H}^2(\mathbb{D})$.[165,166].

Proposition (6.3.1):[162] The following statements are true:

(i) $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.
(ii) $\mathcal{S}^2(\mathbb{D})$ is Banach algebra.
(iii) Polynomials are dense in $\mathcal{S}^2(\mathbb{D})$.

Proof: (i) Let $f \in \mathcal{S}^2(\mathbb{D})$, then $Df \in \mathcal{H}^2(\mathbb{D})$ and hence

$$|Df(z)| \leq \frac{\|Df\|_{\mathcal{H}^2(\mathbb{D})}}{\sqrt{1-|z|^2}}, \qquad \text{for all } z \in \mathbb{D}.$$

Now،

$$|f(z) - f(0)| = \left|\int_0^1 zDf(tz)dt\right| \leq \int_0^1 |zDf(tz)|\,dt \leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{1-|tz|^2}}dt$$
$$\leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{(1-|tz|)(1+t|z|)}}dt.$$

Since $1 + t|z| \geq 1$, we have $\frac{1}{\sqrt{1+t|z|}} \leq 1$.

$$|f(z) - f(0)| \leq \|Df\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z|}{\sqrt{(1-|tz|)}} dt \leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})} \tag{20}$$

This clearly shows that f belongs to H^∞, and hence $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.

(ii) To show $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the above norm, suppose $\{g_n\}_n^\infty$ is Cauchy in $\mathcal{S}^2(\mathbb{D})$ norm. Then, clearly $\{g_n\}$ and $\{Dg_n\}$ are Cauchy on $\mathcal{H}^2(\mathbb{D})$ norm. Since $\mathcal{H}^2(\mathbb{D})$ is a Banach space, $\{g_n\}$ converges to a holomorphic function $g \in \mathcal{H}^2(\mathbb{D})$ and $\{Dg_n\}$ converges to the function $Dg \in \mathcal{H}^2(\mathbb{D})$. Therefore $g \in \mathcal{S}^2(\mathbb{D})$, and hence $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the given norm.

Pointwise multiplication on $\mathcal{S}^2(\mathbb{D})$ form an algebra. For this, suppose that f and g are in $\mathcal{S}^2(\mathbb{D})$.

$$\begin{aligned}\|fg\|^2_{\mathcal{S}^2(\mathbb{D})} &= \|D(fg)\|^2_{\mathcal{H}^2(\mathbb{D})} + \|fg\|^2_{\mathcal{H}^2(\mathbb{D})} \\ &= \|gDf + fDg\|^2_{\mathcal{H}^2(\mathbb{D})} + \|fg\|^2_{\mathcal{H}^2(\mathbb{D})} \\ &= \|gDf\|^2_{\mathcal{H}^2(\mathbb{D})} + 2\|gDf\|_{\mathcal{H}^2(\mathbb{D})}\|fDg\|_{\mathcal{H}^2(\mathbb{D})} \\ &\quad + \|fDg\|^2_{\mathcal{H}^2(\mathbb{D})} + \|fg\|^2_{\mathcal{H}^2(\mathbb{D})}. \end{aligned} \tag{21}$$

Using (20), we see that for any $f \in \mathcal{S}^2(\mathbb{D})$

$$\|f\|_\infty \leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})} + |f(0)| \leq 2\|Df\|_{\mathcal{H}^2(\mathbb{D})} + 2\|f\|_{\mathcal{H}^2(\mathbb{D})} = 2\|f\|^2_{\mathcal{S}^2(\mathbb{D})}.$$

Hence using (21)

$$\begin{aligned}\|fg\|^2_{\mathcal{S}^2(\mathbb{D})} &= \|g\|^2_\infty \|Dg\|^2_{\mathcal{H}^2(\mathbb{D})} + 2\|f\|_\infty \|g\|_\infty \|Df\|_{\mathcal{H}^2(\mathbb{D})}\|Dg\|_{\mathcal{H}^2(\mathbb{D})} \\ &\quad + \|f\|^2_\infty \|Dg\|^2_{\mathcal{H}^2(\mathbb{D})} + \|g\|^2_\infty \|f\|^2_\infty \\ &\leq 4\|g\|^2_{\mathcal{S}^2(\mathbb{D})}\|f\|^2_{\mathcal{S}^2(\mathbb{D})} + 8\|g\|^2_{\mathcal{S}^2(\mathbb{D})}\|f\|^2_{\mathcal{S}^2(\mathbb{D})} + 4\|g\|^2_{\mathcal{S}^2(\mathbb{D})}\|f\|^2_{\mathcal{S}^2(\mathbb{D})} \\ &\leq 16\|f\|^2_{\mathcal{S}^2(\mathbb{D})}\|g\|^2_{\mathcal{S}^2(\mathbb{D})}.\end{aligned}$$

(iii) We want to show polynomials are dense in $\mathcal{S}^2(\mathbb{D})$. Given $f \in \mathcal{S}^2(\mathbb{D})$, let $f_q(z) = f(qz)$ be its dilation and $Df_q(z) = qDf(qz) = q(Df)_q(z)$ be derivative of dilation where $0 < q < 1$. Each function f_q is analytic in a larger disk, so itcan be approximated uniformly on $D$by a sequence of holomorphic polynomials P_q^n, and hence Df_q can be approximated uniformly on $\mathbb{D}$ by holomorphic polynomials DP_q^n. So it will be enough to prove that f can be approximated in $\mathcal{S}^2(\mathbb{D})$ by its dilation. That is to say $\|f - f_g\|_{\mathcal{S}^2(\mathbb{D})} \to 0$ as $q \to 1$. This means that $\|f - f_q\|^2_{\mathcal{H}^2(\mathbb{D})} + \|Df - Df_q\|^2_{\mathcal{H}^2(\mathbb{D})} \to 0$ as $q \to 1$. So finally it is enough to prove that $\|f - f_q\|^2_{\mathcal{H}^2(\mathbb{D})} \to 0$ and $\|Df - Df_q\|_{\mathcal{H}^2(\mathbb{D})} \to 0$ as $q \to 1$. For this, let us assume

$$f(z)=\sum_{n=0}^{\infty}a_n z^n.$$

Since $f\in\mathcal{H}^2(\mathbb{D})$, for all $\epsilon>0$, we can choose a natural number N large enough such that

$$\sum_{n=N+1}^{\infty}|a_n|^2<\frac{\epsilon}{2}.$$

Now choose $q_\epsilon\in(0,1)$ such that

$$(1-q_\epsilon^N)^2\sum_{n=0}^{N}|a_n|^2<\frac{\epsilon}{2}.$$

Then, since

$$\left\|f-f_q\right\|_{\mathcal{H}^2(\mathbb{D})}^2=\left\|\sum_{n=0}^{\infty}a_n z^n(1-q^n)\right\|^2=\sum_{n=0}^{\infty}|a_n(1-q^n)|^2,$$

it follows that for all $q\geq q_\epsilon$

$$\left\|f-f_q\right\|_{\mathcal{H}^2(\mathbb{D})}^2=\sum_{n=0}^{N}|a_n(1-q^n)|^2+\sum_{n=N+1}^{\infty}|a_n(1-q^n)|^2$$

$$\leq(1-q^N)\sum_{n=0}^{N}|a_n|^2+\sum_{n=N+1}^{\infty}|a_n|^2\leq\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

This shows that $\left\|f-f_q\right\|_{\mathcal{H}^2(\mathbb{D})}^2\to 0$ as $q\to 1$. On the other hand, we have

$$\left|Df(z)-Df_q(z)\right|=\left|\sum_{n=1}^{\infty}na_n z^{n-1}-\sum_{n=1}^{\infty}na_n q^n z^{n-1}\right|$$

$$=\left|\sum_{n=1}^{\infty}na_n(1-q^n)z^{n-1}\right|.$$

Similarly, we can show that $\left\|Df-Df_q\right\|^2\to 0$ as q approaches to 1.

Definition (6.3.3):[162] Define

$$\mathcal{S}_0^2(\mathbb{D})=\{f\in\mathcal{S}^2(\mathbb{D}):f(0)=0\}.$$

Corollary (6.3.4):[162] $\mathcal{S}_0^2(\mathbb{D})\subset\mathcal{S}^2(\mathbb{D})$ is a Banach algebra with the norm defined for $\mathcal{S}^2(\mathbb{D})$ and $\mathcal{S}^2(\mathbb{D})=[1]\oplus\mathcal{S}_0^2(\mathbb{D})$, and hence

$$\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{z^n : n \in \mathbb{N}\}.$$

Proof: For any f and g in $\mathcal{S}_0^2(\mathbb{D}) \subset \mathcal{S}^2(\mathbb{D})$, we immediately see that

$$\|fg\|_{\mathcal{S}^2(\mathbb{D})} \le \|f\|_{\mathcal{S}^2(\mathbb{D})}\|g\|_{\mathcal{S}^2(\mathbb{D})}.$$

Also, we have $f(0) = 0$ and $g(0) = 0$ so $(fg)(0) = 0$ and hence fg belongs to $\mathcal{S}_0^2(\mathbb{D})$.

To show $\mathcal{S}_0^2(\mathbb{D})$ is a closed subalgebra of $\mathcal{S}^2(\mathbb{D})$, assume $g_n \in \overline{\mathcal{S}_0^2(\mathbb{D})}$. That means there exists a sequence $S_n \in \mathcal{S}_0^2(\mathbb{D}), n \in \mathbb{N}$ such that g_n converges to g in $\mathcal{S}^2(\mathbb{D})$ norm.This implies g_n converges to g in $\mathcal{H}^2(\mathbb{D})$ norm. Since $g_n(0) = 0$ for all n, it follows that $g(0) = 0$.

Theorem (6.3.5):[162] Let $\mathcal{V}$ be the Volterra operator on $\mathcal{H}^2(\mathbb{D})$. Then the following statements are true:

(i) Range of $\mathcal{V} = \mathcal{S}_0^2(\mathbb{D})$.
(ii) $\mathcal{V}$ is a bounded isomorphism from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$, and its inverse is D.
(iii) The operator T acting on $\mathcal{H}^2(\mathbb{D})$ is similar under $\mathcal{V}$ to the multiplication operator M_z acting on $\mathcal{S}_0^2(\mathbb{D})$.

Proof:

(i) Let g be in the range of $\mathcal{V}$, then there exists $f \in \mathcal{H}^2(\mathbb{D})$ such that

$$g(z) = (\mathcal{V}f)(z) = \int_0^z f(w)dw$$

then $Dg = f \in \mathcal{H}^2(\mathbb{D})$ and $g(0) = 0$. Hence $g \in \mathcal{S}_0^2(\mathbb{D})$. Conversely, suppose that g belongs to $\mathcal{S}_0^2(\mathbb{D})$, then

$$(\mathcal{V}Dg)(z) = \int_0^z (Dg)(w)\,dw = g(z) - g(0) = g(z).$$

Therefore g belongs to the range of $\mathcal{V}$.

(ii) First we want to show $\mathcal{V}$ is a bounded operator on $\mathcal{H}^2(\mathbb{D})$. Let us assume f is in the Hardy space.

$$\|\mathcal{V}(f)\|_{\mathcal{S}^2(\mathbb{D})} = \|\mathcal{V}(f)\|_{\mathcal{H}^2(\mathbb{D})} + \left\|D\big(\mathcal{V}(f)\big)\right\|_{\mathcal{H}^2(\mathbb{D})} \le \|f\|_{\mathcal{H}^2(\mathbb{D})} + \|f\|_{\mathcal{H}^2(\mathbb{D})} = 2\|f\|_{\mathcal{H}^2(\mathbb{D})}.$$

Hence the map $\mathcal{V}$ from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ is bounded. Clearly $\mathcal{V}$ is linear. Now to show $\mathcal{V}$ is one-one, assume that f_1 and f_2 belong the Hardy space, and also assume that

$$\int_0^z f_1(w)dw = \int_0^z f_2(w)dw, \qquad \text{for all } z \in \mathbb{D}.$$

Differentiating both sides we see that $f_1 = f_2$ and hence $\mathcal{V}$ is one-one. From part (i) we have $\mathcal{V}Dg = g$ and clearly $D\mathcal{V}f = f$. This shows that $\mathcal{V}$ is abounded bijective linear operator from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ and $\mathcal{V}^{-1} = D$.

(iii) Suppose f belongs to $\mathcal{H}^2(\mathbb{D})$ and also suppose $\mathcal{V}f = g$, for some $g \in \mathcal{S}_0^2(\mathbb{D})$. Therefore we have $f(z) = (\mathcal{V}^{-1}g)(z) = (Dg)(z)$.

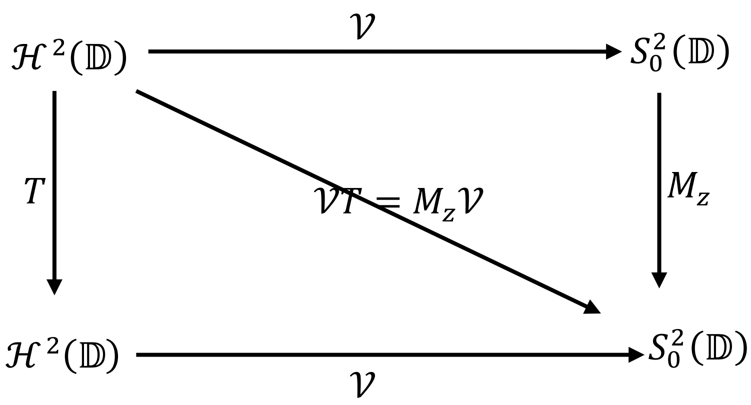
$$(Tf)(z) = zf(z) + (\mathcal{V}f)(z) = z(Dg)(z) + g(z) = D(zg(z)).$$

Now applying $\mathcal{V}$ on the both side, we see that

$$(\mathcal{V}Tf)(z) = \mathcal{V}D\big(zg(z)\big) = zg(z) = z(\mathcal{V}f)(z).$$

So, $\mathcal{V}T = M_z\mathcal{V}$ and $\mathcal{V}T\mathcal{V}^{-1} = M_z$. That is to say $\mathcal{V}$ transforms the operator T into the operator multiplication by z on $\mathcal{S}_0^2(\mathbb{D})$.

We can summarize the theorem by the following commutative diagram



Definition(6.3.6):[162] An element a in Banach algebra $\mathcal{A}$ is called cyclic if the subalgebra generated by a is dense in $\mathcal{A}$.

Proposition(6.3.7):[162] Let $\mathcal{A}$ be a Banach algebra. Then the invariant subspaces of multiplication by a cyclic element are exactly the closed ideals of $\mathcal{A}$.[147].

Lemma(6.3.8):[162] J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$ if and only if J is an ideal of $\mathcal{S}^2(\mathbb{D})$ contained in $\mathcal{S}_0^2(\mathbb{D})$.

Proof: Using of Corollary (6.3.4), we see that for any $h \in \mathcal{S}^2(\mathbb{D})$ there exists $h_1 \in \mathcal{S}_0^2(\mathbb{D})$ such that $h = c + h_1$.

Suppose J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$, then for any $h \in \mathcal{S}^2(\mathbb{D})$ and $j \in J$,

$$hj = (c + h_1)j = c_j + h_1 j \in J.$$

Since norm on both spaces are the same, J is a closed ideal of $\mathcal{S}^2(\mathbb{D})$. On the other hand, if J is an ideal of $\mathcal{S}^2(\mathbb{D})$ contained in $\mathcal{S}_0^2(\mathbb{D})$, then it is clear that J is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$.

Definition(6.3.9):[162] Let K be a closed subset of the unit circle $\partial\mathbb{D}$. For and inner function$G$ we say G is associated with K if

(i) if $a_1, a_2, \ldots$, are the zeros of $G(z)$ in the open disk, then all the limit points of $\{a_k\}$ belong to K,
(ii) the measure determining the singular part of G is supported on K.

Theorem (6.3.10):[162] Suppose K is a is closed subset of $\partial\mathbb{D}$ and let G be an inner function associated with K. Let $I_{\mathcal{S}^2(\mathbb{D})}(G;K)$ be the set of all $f \in \mathcal{S}^2(\mathbb{D})$ which are divisible by G and which vanish on K. Then $I_{\mathcal{S}^2(\mathbb{D})}(G;K)$ is a closed ideal of $\mathcal{S}^2(\mathbb{D})$. Moreover, every closed ideal of $\mathcal{S}^2(\mathbb{D})$. is obtained in this manner.[165].

Corollary (6.3.11):[162] Suppose K is a is closed subset of $\partial\mathbb{D}$ and let G be an inner function associated with K. Let $I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)$ be the set of all $f \in \mathcal{S}_0^2(\mathbb{D})$ which are divisible by G and which vanish on K. Then $I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)$ is a closed ideal of $\mathcal{S}_0^2(\mathbb{D})$. Moreover, every closed ideal of $\mathcal{S}_0^2(\mathbb{D})$ is obtained in this manner.

Theorem (6.3.12):[162] Let T be an operator

$$(Tf)(z) = zf(z) + \int_0^z f(w)dw.$$

defined on $\mathcal{H}^2(\mathbb{D})$. Then the lattice of closed invariant subspaces is

$$\text{Lat}T = \left\{S \subset \mathcal{H}^2(\mathbb{D}) : S = \left\{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right\}\right\}$$

for G, K defined in Definition (6.3.9).

Proof: From Corollary (6.3.4),

$$\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{z^n : n \in \mathbb{N}\},$$

so z is a cyclic element of the Banach algebra $\mathcal{S}_0^2(\mathbb{D})$. Thus from the Proposition (6.3.7), closed invariant subspaces of M_z on $\mathcal{S}_0^2(\mathbb{D})$ are exactly the closed ideals of $\mathcal{S}_0^2(\mathbb{D})$. Using Corollary (6.3.11), the lattice of closed invariant subspace of M_z acting on $\mathcal{S}_0^2(\mathbb{D})$ is given by

$$\text{Lat } M_z = \{I_{\mathcal{S}_0^2(\mathbb{D})}(G;K) : G, K \text{ defined in Definition (6.3.9)}\}.$$

Since $\mathcal{V}T\mathcal{V}^{-1}$, we see that $\mathcal{V}^{-1}\left(I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right)$ is a closed invariant subspace of T. From Theorem (6.3.5), we know that $\mathcal{V}^{-1}(f) = Df$. So, $S = \left\{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G,K)\right\}$ is a closed invariant subspace of T. Hence,

$$\text{Lat } M_z = \left\{S \subset \mathcal{H}^2(\mathbb{D}) : S = \left\{Df : f \in I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right\}\right\}$$

for G,K defined in Definition (6.3.9).

Corollary (6.3.13):[168] The following statements are true:

(i) $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.
(ii) $\mathcal{S}^2(\mathbb{D})$ is Banach algebra.

(iii) Polynomials are dense in $\mathcal{S}^2(\mathbb{D})$.

Proof:

(i) Let $f_r \in \mathcal{S}^2(\mathbb{D})$, then $Df_r \in \mathcal{H}^2(\mathbb{D})$ and hence

$$\sum_r |Df_r(z-\epsilon)| \le \sum_r \frac{\|Df_r\|_{\mathcal{H}^2(\mathbb{D})}}{\sqrt{1-|z-\epsilon|^2}}, \qquad \text{for all } (z-\epsilon) \in \mathbb{D}.$$

Now‘

$$\sum_r |f_r(z-\epsilon) - f_r(0)| = \sum_r \left| \int_0^1 (z-\epsilon) Df_r\big(t(z-\epsilon)\big) dt \right|$$

$$\le \sum_r \int_0^1 \left|(z-\epsilon)Df_r\big(t(z-\epsilon)\big)\right| dt \le \sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z-\epsilon|}{\sqrt{1-|t(z-\epsilon)|^2}} dt$$

$$\le \sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z-\epsilon|}{\sqrt{(1-|t(z-\epsilon)|)(1+t|z-\epsilon|)}} dt.$$

Since $1+t|z-\epsilon| \ge 1$, we have $\frac{1}{\sqrt{1+t|z-\epsilon|}} \le 1$.

$$\sum_r |f_r(z-\epsilon) - f_r(0)| \le \sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} \int_0^1 \frac{|z-\epsilon|}{\sqrt{(1-|t(z-\epsilon)|)}} dt \le 2\sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})}.$$

This clearly shows that f_r belongs to H^∞, and hence $\mathcal{S}^2(\mathbb{D}) \subset H^\infty$.

(ii) To show $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the above norm, suppose the double sequence $\{(g_r)_n\}_0^\infty$ is Cauchy in $\mathcal{S}^2(\mathbb{D})$ norm. Then, clearly $\{(g_r)_n\}$ and $\{D(g_r)_n\}$ are Cauchy on $\mathcal{H}^2(\mathbb{D})$ norm. Since $\mathcal{H}^2(\mathbb{D})$ is a Banach space, $\{(g_r)_n\}$ converges to a holomorphic function $g_r \in \mathcal{H}^2(\mathbb{D})$ and $\{D(g_r)_n\}$ converges to the function $Dg_r \in \mathcal{H}^2(\mathbb{D})$. Therefore $g_r \in \mathcal{S}^2(\mathbb{D})$, and hence $\mathcal{S}^2(\mathbb{D})$ is a Banach space under the given norm. Pointwise multiplication on $\mathcal{S}^2(\mathbb{D})$ form an algebra. For this, suppose that f_r and g_r are in $\mathcal{S}^2(\mathbb{D})$.

$$\sum_r \|f_r g_r\|^2_{\mathcal{S}^2(\mathbb{D})} = \sum_r \|D(f_r g_r)\|^2_{\mathcal{H}^2(\mathbb{D})} + \sum_r \|f_r g_r\|^2_{\mathcal{H}^2(\mathbb{D})}$$

$$= \sum_r \|g_r Df_r + f_r Dg_r\|^2_{\mathcal{H}^2(\mathbb{D})} + \sum_r \|f_r g_r\|^2_{\mathcal{H}^2(\mathbb{D})}$$

$$\le \sum_r \|g_r Df_r\|^2_{\mathcal{H}^2(\mathbb{D})} + 2\sum_r \|g_r Df_r\|_{\mathcal{H}^2(\mathbb{D})} \|f_r Dg_r\|_{\mathcal{H}^2(\mathbb{D})} + \sum_r \|f_r Dg_r\|^2_{\mathcal{H}^2(\mathbb{D})}$$
$$+ \sum_r \|f_r g_r\|^2_{\mathcal{H}^2(\mathbb{D})}.$$

Using (i), we see that for any $f_r \in \mathcal{S}^2(\mathbb{D})$

$$\sum_r \|f_r\|_\infty \le 2\sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} + \sum_r |f_r(0)| \le 2\sum_r \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} + 2\sum_r \|f_r\|_{\mathcal{H}^2(\mathbb{D})} = 2\sum_r \|f_r\|^2_{\mathcal{S}^2(\mathbb{D})}.$$

Hence

$$\begin{aligned}\sum_r \|f_r g_r\|^2_{\mathcal{S}^2(\mathbb{D})} &\le \sum_r \|g_r\|^2_\infty \|Df_r\|^2_{\mathcal{H}^2(\mathbb{D})} + 2\sum_r \|f_r\|_\infty \, \|g_r\|_\infty \|Df_r\|_{\mathcal{H}^2(\mathbb{D})} \|Dg_r\|_{\mathcal{H}^2(\mathbb{D})} \\ &+ \sum_r \|f_r\|^2_\infty \|Dg_r\|^2_{\mathcal{H}^2(\mathbb{D})} + 2\sum_r \|g_r\|^2_\infty \|f_r\|^2_\infty \\ &\le 4\sum_r \|g_r\|^2_{\mathcal{S}^2(\mathbb{D})} \|f_r\|^2_{\mathcal{S}^2(\mathbb{D})} + 8\sum_r \|g_r\|^2_{\mathcal{S}^2(\mathbb{D})} \|f_r\|^2_{\mathcal{S}^2(\mathbb{D})} \\ &+ 4\sum_r \|g_r\|^2_{\mathcal{S}^2(\mathbb{D})} \|f_r\|^2_{\mathcal{S}^2(\mathbb{D})} \le 16\sum_r \|f_r\|^2_{\mathcal{S}^2(\mathbb{D})} \|g_r\|^2_{\mathcal{S}^2(\mathbb{D})}.\end{aligned}$$

(iii) We want to show polynomials are dense in $\mathcal{S}^2(\mathbb{D})$. Given $f_r \in \mathcal{S}^2(\mathbb{D})$, let $f_{(1-\epsilon)}(z-\epsilon) = f_r\big((1-\epsilon)(z-\epsilon)\big)$ be its dilation and $D(f_r)_{(1-\epsilon)}(z-\epsilon) = (1-\epsilon)Df_r\big((1-\epsilon)(z-\epsilon)\big) = (1-\epsilon)(Df_r)_{(1-\epsilon)}(z-\epsilon)$ be derivative of dilation where $0 < \epsilon < 1$. Each function $(f_r)_{(1-\epsilon)}$ is analytic in a larger disk, so it can be approximated uniformly on D by a sequence of holomorphic polynomials $P^n_{(1-\epsilon)}$, and hence $D(f_r)_{(1-\epsilon)}$ can be approximated uniformly on $\mathbb{D}$ by holomorphic polynomials $DP^n_{(1-\epsilon)}$. So it will be enough to prove that f_r can be approximated in $\mathcal{S}^2(\mathbb{D})$ by its dilation. That is to say $\left\|f_r - (f_r)_{(1-\epsilon)}\right\|_{\mathcal{S}^2(\mathbb{D})} \to 0$ as $\epsilon \to 0$. This means that $\left\|f_r - (f_r)_{(1-\epsilon)}\right\|^2_{\mathcal{H}^2(\mathbb{D})} + \left\|Df_r - D(f_r)_{(1-\epsilon)}\right\|^2_{\mathcal{H}^2(\mathbb{D})} \to 0$ as $\epsilon \to 0$. So finally it is enough to prove that $\left\|f_r - (f_r)_{(1-\epsilon)}\right\|^2_{\mathcal{H}^2(\mathbb{D})} \to 0$ and $\left\|Df_r - D(f_r)_{(1-\epsilon)}\right\|_{\mathcal{H}^2(\mathbb{D})} \to 0$ as $\epsilon \to 0$. For this, let us assume

$$f_r(z-\epsilon) = \sum_{n=0}^{\infty} a^r_n (z-\epsilon)^n.$$

Since $f_r \in \mathcal{H}^2(\mathbb{D})$, for all $\epsilon > 0$, we can choose a natural number N large enough such that

$$\sum_r \sum_{n=N+1}^{\infty} |a^r_n|^2 < \frac{\epsilon}{2}.$$

Now choose $(1-\epsilon)_\epsilon \in (0,1)$ such that

$$(1-(1-\epsilon)_\epsilon^N)^2 \sum_r \sum_{n=0}^{N} |a_n^r|^2 < \frac{\epsilon}{2}.$$

Then, since

$$\sum_r \left\| f_r - (f_r)_{(1-\epsilon)} \right\|_{\mathcal{H}^2(\mathbb{D})}^2 = \sum_r \left\| \sum_{n=0}^{\infty} a_n^r (z-\epsilon)^n (1-(1-\epsilon)^n) \right\|^2$$
$$= \sum_r \sum_{n=0}^{\infty} |a_n^r (1-(1-\epsilon)^n)|^2,$$

it follows that for all $(1-\epsilon) \geq (1-\epsilon)_\epsilon$

$$\sum_r \left\| f_r - (f_r)_{(1-\epsilon)} \right\|_{\mathcal{H}^2(\mathbb{D})}^2 = \sum_r \sum_{n=0}^{N} |a_n^r (1-(1-\epsilon)^n)|^2 + \sum_r \sum_{n=N+1}^{\infty} |a_n^r (1-(1-\epsilon)^n)|^2$$
$$\leq (1-(1-\epsilon)^N) \sum_r \sum_{n=0}^{N} |a_n^r|^2 + \sum_r \sum_{n=N+1}^{\infty} |a_n^r|^2 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\left\| f_r - (f_r)_{(1-\epsilon)} \right\|_{\mathcal{H}^2(\mathbb{D})}^2 \to 0$ as $\epsilon \to 0$. On the other hand, we have

$$\sum_r \left| Df_r(z-\epsilon) - D(f_r)_{(1-\epsilon)}(z-\epsilon) \right|$$
$$= \sum_r \left| \sum_{n=1}^{\infty} n a_n^r (z-\epsilon)^{n-1} - \sum_{n=1}^{\infty} n a_n^r (1-\epsilon)^n (z-\epsilon)^{n-1} \right|$$
$$= \sum_r \left| \sum_{n=1}^{\infty} n a_n^r (1-(1-\epsilon)^n)(z-\epsilon)^{n-1} \right|.$$

Similarly, we can show that $\left\| Df_r - D(f_r)_{(1-\epsilon)} \right\|^2 \to 0$ as ϵ approaches to 0.

Corollary (6.3.14):[168] $\mathcal{S}_0^2(\mathbb{D}) \subset \mathcal{S}^2(\mathbb{D})$ is a Banach algebra with the norm defined for $\mathcal{S}^2(\mathbb{D})$ and $\mathcal{S}^2(\mathbb{D}) = [1] \bigoplus \mathcal{S}_0^2(\mathbb{D})$, and hence

$$\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{(z-\epsilon)^n : n \in \mathbb{N}\}.$$

Proof: For any f_r and g_r in $\mathcal{S}_0^2(\mathbb{D}) \subset \mathcal{S}^2(\mathbb{D})$, we immediately see that

$$\sum_r \|f_r g_r\|_{\mathcal{S}^2(\mathbb{D})} \leq \sum_r \|f_r\|_{\mathcal{S}^2(\mathbb{D})} \|g_r\|_{\mathcal{S}^2(\mathbb{D})}.$$

Also, we have $f_r(0) = 0$ and $g_r(0) = 0$ so $(f_r g_r)(0) = 0$ and hence $f_r g_r$ belongs to $\mathcal{S}_0^2(\mathbb{D})$.

To show $\mathcal{S}_0^2(\mathbb{D})$ is a closed subalgebra of $\mathcal{S}^2(\mathbb{D})$, assume $(g_r)_n \in \overline{\mathcal{S}_0^2(\mathbb{D})}$. That means there exists a double sequence $(g_r)_n \in \mathcal{S}_0^2(\mathbb{D}), n \in \mathbb{N}$ such that $(g_r)_n$converges to g_r in $\mathcal{S}^2(\mathbb{D})$ norm.This implies $(g_r)_n$ converges to g_r in $\mathcal{H}^2(\mathbb{D})$ norm. Since $(g_r)_n(0) = 0$ for all n, it follows that $g_r(0) = 0$.

Corollary (6.3.15):[168] Let $\mathcal{V}$ be the Volterra operator on $\mathcal{H}^2(\mathbb{D})$. Then the following statements are true:

(i) Range of $\mathcal{V} = \mathcal{S}_0^2(\mathbb{D})$.
(ii) $\mathcal{V}$ is a bounded isomorphism from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$, and its inverse is D.
(iii) The operator T acting on $\mathcal{H}^2(\mathbb{D})$ is similar under V to the multiplication operator $M_{(z-\epsilon)}$ acting on $\mathcal{S}_0^2(\mathbb{D})$.

Proof:

(i) Let g_r be in the range of $\mathcal{V}$, then there exists $f_r \in \mathcal{H}^2(\mathbb{D})$ such that

$$\sum_r g_r(z-\epsilon) = \sum_r (\mathcal{V}f_r)(z-\epsilon) = \int_0^{z-\epsilon} \sum_r f_r(w)dw$$

then $Dg_r = f_r \in \mathcal{H}^2(\mathbb{D})$ and $g_r(0) = 0$. Hence $g_r \in \mathcal{S}_0^2(\mathbb{D})$. Conversely, suppose that g_r belongs to $\mathcal{S}_0^2(\mathbb{D})$, then

$$\sum_r (\mathcal{V}Dg_r)(z-\epsilon) = \int_0^{z-\epsilon} \sum_r (Dg_r)(w)\,dw = \sum_r \big(g_r(z-\epsilon) - g_r(0)\big) = \sum_r g_r(z-\epsilon).$$

Therefore g_r belongs to the range of $\mathcal{V}$.

(ii) First we want to show $\mathcal{V}$ is a bounded operator on $\mathcal{H}^2(\mathbb{D})$. Let us assume f_r is in the Hardy space.

$$\begin{aligned}\sum_r \|\mathcal{V}(f_r)\|_{\mathcal{S}^2(\mathbb{D})} &= \sum_r \|\mathcal{V}(f_r)\|_{\mathcal{H}^2(\mathbb{D})} + \sum_r \big\|D\big(\mathcal{V}(f_r)\big)\big\|_{\mathcal{H}^2(\mathbb{D})} \\ &\leq \sum_r \|f_r\|_{\mathcal{H}^2(\mathbb{D})} + \sum_r \|f_r\|_{\mathcal{H}^2(\mathbb{D})} = 2\sum_r \|f_r\|_{\mathcal{H}^2(\mathbb{D})}.\end{aligned}$$

Hence the map $\mathcal{V}$ from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ is bounded. Clearly $\mathcal{V}$ is linear. Now to show $\mathcal{V}$ is one-one, assume that $(f_r)_1$ and $(f_r)_2$ belong to the Hardy space, and also assume that

$$\int_0^{z-\epsilon} \sum_r (f_r)_1(w)\,dw = \int_0^{z-\epsilon} \sum_r (f_r)_2(w)\,dw, \qquad \text{for all } (z-\epsilon) \in \mathbb{D}.$$

Differentiating both sides we see that $(f_r)_1 = (f_r)_2$ and hence $\mathcal{V}$ is one-one. From part (i) we have $VDg_r = g_r$ and clearly $D\mathcal{V}f_r = f_r$. This shows that $\mathcal{V}$ is abounded bijective linear operator from $\mathcal{H}^2(\mathbb{D})$ onto $\mathcal{S}_0^2(\mathbb{D})$ and $\mathcal{V}^{-1} = D$.

(iii) Suppose f_r belongs to $\mathcal{H}^2(\mathbb{D})$ and also suppose $\mathcal{V}f_r = g_r$, for some $g_r \in \mathcal{S}_0^2(\mathbb{D})$. Therefore we have

$$\sum_r f_r(z-\epsilon) = \sum_r (\mathcal{V}^{-1}g_r)(z-\epsilon) = \sum_r (Dg_r)(z-\epsilon).$$

$$\sum_r (Tf_r)(z-\epsilon) = (z-\epsilon)\sum_r f_r(z-\epsilon) + \sum_r (\mathcal{V}f_r)(z-\epsilon)$$

$$= (z-\epsilon)\sum_r (Dg_r)(z-\epsilon) + \sum_r g_r(z-\epsilon) = D\left((z-\epsilon)\sum_r g_r(z-\epsilon)\right).$$

Now applying $\mathcal{V}$ on both sides, we see that

$$\sum_r (\mathcal{V}Tf_r)(z-\epsilon) = \mathcal{V}D\left((z-\epsilon)\sum_r g_r(z-\epsilon)\right) = (z-\epsilon)\sum_r g_r(z-\epsilon)$$

$$= (z-\epsilon)\sum_r (\mathcal{V}f_r)(z-\epsilon).$$

So, $\mathcal{V}T = M_{(z-\epsilon)}\mathcal{V}$ and $\mathcal{V}T\mathcal{V}^{-1} = M_{(z-\epsilon)}$. That is to say $\mathcal{V}$ transforms the operator T into the operator multiplication by $(z-\epsilon)$ on $\mathcal{S}_0^2(\mathbb{D})$.

Corollary (6.3.16):[168] Let T be an operator

$$\sum_r (Tf_r)(z-\epsilon) = (z-\epsilon)\sum_r f_r(z-\epsilon) + \int_0^{z-\epsilon} \sum_r f_r(w)dw.$$

defined on $\mathcal{H}^2(\mathbb{D})$. Then the lattice of closed invariant subspaces is

$$\text{Lat } T = \left\{S \subset \mathcal{H}^2(\mathbb{D}): S = \left\{Df_r : f_r \in I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right\}\right\}$$

for G, Kdefined in Definition (6.3.9).

Proof: From Corollary (6.3.14),

$$\mathcal{S}_0^2(\mathbb{D}) = \overline{\text{span}}\{(z-\epsilon)^n : n \in \mathbb{N}\},$$

so $(z-\epsilon)$ is a cyclic element of the Banach algebra $\mathcal{S}_0^2(\mathbb{D})$. Thus from the Proposition (6.3.7), closed invariant subspaces of $M_{(z-\epsilon)}$ on $\mathcal{S}_0^2(\mathbb{D})$ are exactly the closed ideals of $\mathcal{S}_0^2(\mathbb{D})$. Using Corollary (6.3.11), the lattice of closed invariant subspace of $M_{(z-\epsilon)}$ acting on $\mathcal{S}_0^2(\mathbb{D})$ is given by

$$\text{Lat } M_{(z-\epsilon)} = \left\{I_{\mathcal{S}_0^2(\mathbb{D})}(G;K) : G, K \text{ defined in Definition (6.3.9)}\right\}.$$

Since $\mathcal{V}T\mathcal{V}^{-1} = M_{(z-\epsilon)}$, we see that $\mathcal{V}^{-1}\left(I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right)$ is a closed invariant subspace of T. From Corollary (6.3.15), we know that $\mathcal{V}^{-1}(f_r) = Df_r$. So, $S = \left\{Df_r : f_r \in I_{\mathcal{S}_0^2(\mathbb{D})}(G,K)\right\}$ is a closed invariant subspace of T. Hence,

$$\operatorname{Lat} T = \left\{\begin{matrix} S \subset \mathcal{H}^2(\mathbb{D}) : S = \left\{Df_r : f_r \in I_{\mathcal{S}_0^2(\mathbb{D})}(G;K)\right\} \\ \text{for } G, K \text{ defined in Definition (6.3.9)} \end{matrix}\right\}$$

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