Chapter One

Introduction

1.1 The History of Cosmology

The history of cosmology starts from the beginning of human life on the earth. People observe the sun moving and distributing light from the morning. At night the moon appear on the sky as a golden hemisphere surrounded by glittery beautiful objects some of them asks; what is the nature of these astronomical objects? Why some of them move and look brighter than others [1, 2, and 3].

The accumulated know ledge these objects led scientists like kepler to formulate some rules for some of the regular behaviors of these astronomical objects these rules are concerned with the motion of planets around the sun [4, 5]. Later on lsaag newton discovered that is called, the gravitational field is responsible for the motion planets around the sun [6].

Gravitational field is used to explain the flounce that a massive body spreads in to the space around it the so called a field, which produces a force on another massive body. Thus gravitational field phenomena, and is measured in new ton per kg (N/kg) [7, 8].

Newton Lows of gravitational succeeded in describing the motion of macroscopic objects [9] until the beginning of twentieth century, where Michelso-Morely experiment indicates the violation of the Newton Low of addition of velocity for light [10, 11, and 12].

This experiment shows that the speed of light in vacuum is always constant and is completely independent on the motion of the source or observer [13].

But unfortunately Newton gravitational Low suffers from noticeable setbacks. For instanle it fails to describe the preheating of mercury, beside the failure in describing the behavior of quasi staller objects and black holes [14, 15].

In 1915, Albert Einstein developed his theory of general relativity, having earlier shown that gravity dos influence light’s motion [16, 17].

Einstein’s theory of general relativity (GR) is one of the fundamental physical theories at the present time; it describes a number of gravitational physical phenomena, which agree with astronomical observations [18, 19].

Despite these successes GR suffers from being isolated from the main stream of physics.

This is since the equation of motion and the energy momentum tensor conservation Lows differs radically from that in other physical theories, which are derived from the action principle. It also suffers from the lack of a full expression for the energy-momentum tensor of the gravitational field [20, 21, 22,].

General Relativity Lows can’t easily explain the behavior of exotic astronomical objects, like black holes, pulsars, quasars and neutron stars [23, 24, and 25].

For instance it is difficult to explain the large red shift of quasars within the frame work of GR [26, 27, and 28].

The behavior of black holes is even more complex. A black holes is a place where gravity is very strong that even light can’t get out. The gravity is so strong because matter has been squeezed in to a tiny space this can happen when a star is dying. Because not light can get out, people can’t see black holes. They are invisible, space telescopes with special tools help find black holes. The special tools can see how stars that are very close to black holes act differently than other stars [29, 30].

Black holes can be big or small. Scion tests think the smallest black holes are as large mountain.

Another kind of black holes is called staller its mass can be up to 20 times than the mass of the sun. There may be many, many staller mass black holes in Earth's galaxy [31, 32].

Staller black holes are made when the center of Avery big star falls in upon itself, o collapse. When this happens, it causes a super move. A super move is an exploding star that blasts part of the star in to space [33].

All these gravitational pheromone seems to reed a full quantum gravitational theory as pointed out by many anthers [33, 34, 35, 36, and 37].

1.2 Research Problem

Einstein general relativity GR is isolated in its geometrical content from the main stream of physics. This is because there is no well-established quantum gravitational theory. Thus the behavior of black holes and neutron stars can't be explained fully. More over the unification of gravity with other field is not yet achieved.

1.3 Literature Review

Different attempts were made to find a full quantum gravitational theories [29, 30, 31, 32, 33, 34, 35].In same of them wheeler de Witt quantum wave function is used to describe the universe evolution [36, 37, 38, 39, 40].In to the approaches canonical quantization based on GR is also proposed [41, 42].A quantum model based on quantized general relativity (GGR) is also proposed by some people [43]. The quantum models based on GR does not based on the Hamiltonian which is not defined in GR. The ones based on GGR do not have a wave function and does not dens be isolated bodies.

1.4 Aim of the Work

Motivated by the successes of quantum semiclassical models [43], the aim of this work is devoted to find quantum mode to quantized static field generated by isolated stars.

This can describe the behavior of black holes and quasi-stellar and to remove singularity.

1.5 Presentation of the thesis

A part from introduction, the thesis consists of 3 chapters. Chapter2 is concerned with the theoretical black ground. Chaper3 is devoted for the literature review, while the contribution is exhibited in chaper4.

Chapter Two

The theory of General Relativity and Generalized General Relativity

2.1 Introduction

This chapter is can corn with exhibiting boric principles of general relativity (GR) including the mathematical structure and frame work of GR.

It also derives Einstein gravity equation. It is also concerns with the expression of equation motion and the energy and momentum equation for generalized general relativity (GGR).

2.2 Equivalence principle and Geometry

Consider anelevator in free space moving towards an object with acceleration g. An observer in this elevator, observer this particle falling with acceleration g. This situation can’t be distinguished from that observed by an observer existing in a gravitational field, where observer particles falling with speed g. This the laws of motion or the laws of nature takes the same from as in un accelerated in accelerated frame in space frame in the presence of gravitation. This statement is known as the principle of equivalence [0]; where or the laws of mechanics one talks about Einstein equivalence principle and for the laws of nature one talks about strong equivalence principle. This situation is deeply analogous to that where curved space can be locally regarded as being flat [44].

In view of the above-stated analogy the laws of gravitation can be symbolized in terms of curved Riemannian geometry and hence the relation between physics and geometry can be manifested. In this concept gravitation can be described by a curvature of space-time. The appropriate mathematical tool for implementing Riemannian geometry, which is concerned with curved space time, in this respect is tensor analysis. We shall review some useful tonsorial relation in the following sections [45].

2.3 The Equation Of Motion of a Particle in Gravity Field

In order to see how physical events take place in a gravitational field, consider the proper time interval $d\tau$ between two world points of a 4-space (x_0, x_1, x_2, x_3) of an inertial coordinate system i.e. [46].

$$d\tau^2 = d^2x_0 - [d^2x_1 + d^2x_2 + d^2x_3] \quad (2.3.2)$$

This is invariant under any transformation between inertial frames [6]. If we, however, express the interval in an arbitrary (non-inertial) coordinate system then its form would have a more general type of the coordinate differentials expressed by

$$d\tau^2 = -g_{\mu\nu}dx^{\mu} \quad (2.3.2)$$

$$\mu, \nu = 0{,}1{,}2{,}3$$

Where $g_{\mu\nu}$ is a function describing the space-time metric in for rectilinear motion of special Relativity in the Lorentz-Minkowskian flat geometry, the proper time interval becomes

$$d\tau^2 = -\eta_{\alpha\beta}d\xi^{\alpha}d\xi^{\beta} \quad (2.3.3)$$

Where $\eta_{\alpha\beta}$being a limiting value of $g_{\mu\nu}$in Euclidian pace. The invariance of $d\tau^2$ under transformations between these coordinates gives

$$-g_{\mu\nu}dx^{\mu}dx^{\nu} = d\tau^2 = -\eta_{\alpha\beta}d\xi^{\alpha}d\xi^{\beta} = -\eta_{\alpha\beta}\frac{\partial\xi^{\alpha}\partial\xi^{\beta}}{\partial x^{\mu}\partial x^{\nu}}dx^{\mu}dx^{\nu} \quad (2.3.4)$$

This yield

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^{\alpha}\partial\xi^{\beta}}{\partial\xi^{\mu}\partial\xi^{\nu}} = g_{\nu\mu} \quad (2.3.5)$$

Where $g_{\mu\nu}$is called metric tensor. The inverse form becomes

$$g^{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^{\mu}\partial x^{\nu}}{\partial\xi^{\alpha}\partial\xi^{\beta}} \quad (2.3.6)$$

These two forms with lower and upper indices are respectively called covariant and contra variant tensor. There fore

$$g_{\mu\nu}g^{\mu\lambda} = \delta_\nu^\lambda = \begin{Bmatrix} 1 & \nu = \lambda \\ 0 & \nu \neq \lambda \end{Bmatrix} \qquad (2.3.7)$$

The motion of a particle moving freely under the influence of a gravitational field can be described in a freely falling coordinate systemξ^α. The equation of motion of this particle this system reads

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \qquad (2.3.8)$$

With $d\tau$ the proper time. There fore

$$0 = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\mu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} \qquad (2.3.9)$$

Multiplying by ${\partial x^\lambda}/{\partial\xi^\alpha}$ and introducing the definition

$$\Gamma_{\mu\nu}^\lambda \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu} \qquad (2.3.10)$$

Yields

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \qquad (2.3.11)$$

Where we have used the identity

$$\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial\xi^\alpha} \equiv \delta_\mu^\lambda \qquad (2.3.12)$$

The above structure of the metric tensor components and its first derivatives yield

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\lambda}^{\rho} g_{\rho\nu} + \Gamma_{\lambda}^{\rho} g_{\rho\mu} \qquad (2.3.13)$$

Hence

$$\Gamma_{\lambda\mu}^\gamma \equiv \frac{1}{2}g^{\nu\gamma}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\nu}\right) \equiv \begin{Bmatrix} & \gamma & \\ \lambda & & \mu \end{Bmatrix} \qquad (2.3.14)$$

That is called the affine connection and occasionally Christoffel symbol, which doesn't transform as a tensor. It is obvious that in the absence of gravitation when then metric tensor becomes constant and the affine connection $\Gamma^{\gamma}_{\lambda\mu}$vanishes.

2.4 General Covariance and the Curvature Tensor

It states that an equation of physics holds in a general gravitational filed if it holds in the absence of gravitation and if it preserves its form under a general coordinate transformation, that is, if it is generally covariant. By this, one may learn that the principle of general covariance forms a mathematical description of the equivalence principle. From this principle it also follows that the equations which govern the gravitational field of arbitrary strength must be written in a tensor form. It is therefore very essential to introduce different tensor quantities built from the metric tensor and the relationships between them.

Useful expressions of tonsorial forms built from the metric tensor constitutes a mixed tensor of fourth rank, like

$$\partial_{\gamma}\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\gamma\eta} - \Gamma^{\eta}_{\mu\gamma}\Gamma^{\lambda}_{\nu\eta} \equiv R^{\lambda}_{\mu\nu\gamma} \qquad (2.4.1)$$

This is called Riemann-Christoffel curvature tensor. It is the only tensor that can be constructed from the metric tensor and its first derivatives and linearly from its second derivatives. This tensor expresses the presence or the absence of the gravitational field. Therefore if this tensor vanishes i.e.

$$R^{\lambda}_{\mu\nu\gamma} = 0 \qquad (2.4.2)$$

The gravitational field disappears. The contraction of the curvature tensor by the metric tensor yields the covariant fourth rank Riemann tensor

$$g_{\lambda\sigma}R^{\sigma}_{\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} \qquad (2.4.3)$$

With the following properties

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \qquad (2.4.4)$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \qquad (2.4.5)$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \tag{2.4.6}$$

And

$$g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = R_{\mu\kappa} = R_{\kappa\mu} g^{\lambda\mu} R_{\lambda\mu\nu\kappa} = 0 \tag{2.4.7}$$

$R_{\kappa\mu}$is a symmetric tensor called Ricci tensor which by contraction gives the scalar curvature

$$R = g^{\mu\kappa} R_{\mu\kappa} = g^{\mu\kappa} g^{\lambda\nu} R_{\lambda\mu\nu\kappa}$$

$$= \frac{g^{\mu\kappa} g^{\lambda\nu}}{2} \left(\partial_{\kappa\mu} g_{\lambda\nu} - \partial_{\kappa\lambda} g_{\mu\nu} - \partial_{\nu\lambda} g_{\lambda\kappa} + \partial_{\nu\lambda} g_{\mu\kappa}\right)$$

$$+ g_{\eta\sigma} \left(\Gamma^{\eta}_{\nu\lambda} \Gamma^{\eta}_{\nu\lambda} - \Gamma^{\eta}_{\nu\lambda} \Gamma^{\sigma}_{\mu\nu}\right) \tag{2.4.8}$$

Where

$$\partial_{\mu\nu} g_{\rho\sigma} = \frac{\partial^2 g_{\rho\sigma}}{\partial x^{\mu} \partial x^{\nu}} \tag{2.4.9}$$

Further, since the ordinary derivatives of tensors are not tensors, we introduce the following definitions of the covariant derivatives [8] for the first and the rank tensor

$$T_{\mu;\lambda} = \partial_{\lambda} T_{\mu} - \Gamma^{\rho}_{\mu\lambda} T_{\rho} \tag{2.4.10}$$

$$T_{\mu\nu;\lambda} = \partial_{\mu\nu} T_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda} T_{\rho\nu} - \Gamma^{\rho}_{\mu\lambda} T_{\mu\rho} \tag{2.4.11}$$

And

$$T^{\mu}_{;\lambda} = \partial_{\lambda} T^{\mu} + \Gamma^{\mu}_{\lambda\rho} T^{\rho} \tag{2.4.12}$$

$$T^{\mu\nu}_{;\lambda} = \partial_{\lambda} T^{\mu\nu} + \Gamma^{\mu}_{\lambda\rho} T^{\rho\nu} + \Gamma^{\nu}_{\lambda\rho} T^{\mu\rho} \tag{2.4.13}$$

Hence

$$T_{\mu;\kappa;\nu} - T_{\mu;\nu;\kappa} = R^{\lambda}_{\mu\nu\kappa} T_{\lambda} \tag{2.4.14}$$

$$T^{\lambda}_{;\nu;\kappa} - T^{\lambda}_{;\kappa;\nu} = T^{\mu} R^{\lambda}_{\mu\nu\kappa} \tag{2.4.15}$$

And

$$T^{\lambda}_{\mu;\nu;\kappa} - T^{\lambda}_{\mu;\kappa;\nu} = T^{\sigma}_{\mu} R^{\lambda}_{\sigma\nu\kappa} - T^{\lambda}_{\sigma} R^{\sigma}_{\mu\nu\kappa} \tag{2.4.16}$$

For the metric tensor

$$g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\lambda\mu} g_{\rho\nu} - \Gamma^\rho_{\lambda\nu} g_{\rho\mu} \tag{2.4.17}$$

This by (2.3.13) vanishes as a manifestation of covariance principle. Similar results hold for contra variant derivatives and we therefore have

$$g_{\mu\nu;\lambda} = g^{\mu\nu}_{;\lambda} = 0 \tag{2.4.18}$$

For the covariant derivative of tensor of rank higher than 2 the number of terms with T multiplied by Γ will be equal to the number of indices. Such terms can be set to vanish by adopting locally inertial coordinate system. Therefore for the tensor we obtain

$$R_{\lambda\mu\nu;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{2.4.19}$$

These are called Bianchi identities. By contracting (2.4.19) and due to (2.4.18) we get

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^\nu_{\mu\kappa\eta;\nu} = 0 \tag{2.4.20}$$

Further contraction yields

$$\left(R^\mu_\eta - \frac{1}{2}\delta^\mu_\eta R\right)_{;\mu} = 0 \tag{2.4.21}$$

Or equivalently

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} R\right)_{;\mu} \tag{2.4.22}$$

Known as contracted Bianchi identities

And

$$h_{00} = -2\phi + conctant \tag{2.4.23}$$

Further integration yields the Newtonian potential

$$\phi = -\frac{GM}{r} \tag{2.4.24}$$

At $r \to \infty, h = 0_{oo}, \emptyset = 0$, thus $g_{oo} = -1$and hence the constant in (2.4.23) is zero. Therefore $g_{00} = \eta_{00} + h_{00} = -1 - 2\phi$

Or

$$\phi = -\frac{1}{2}(g_{00} + 1) \tag{2.4.25}$$

2.5 The Equation of the Gravitational Field

Differently from the electromagnetic field which does not influence its source, the charge, and which is determined by linear partial differential equations, the gravitational field dose affect the mass producing it and therefore should be described by non-linear equations.

To obtain these equations Einstein started from the belief that they must have a generalized form of Newtonian gravitational equations where the scalar potential $\emptyset$ can be approximately expressed through the time component of the metric tensor by [47].

$$\emptyset \approx -\frac{1}{2}(g_{00} + 1) \tag{2.5.1}$$

The corresponding Poisson equation reads [10]

$$\nabla^2\emptyset = 4\pi G_\rho \tag{2.5.2}$$

Where $\rho$the non-relativistic mass density and G the known gravitational constant. Thus by (2.5.1) we obtain

$$\nabla^2 g_{00} = -8\pi G_\rho = -8\pi G T_{00} \tag{2.5.3}$$

Where in this case the mass density equals the energy T_{00} if we extend the right hand side of (2.5.3) so that $T_{00} \to T_{\alpha\beta}$, then by tensor analysis the left hand side should be equal to some second rank spatial tensor $G_{\alpha\beta}$.this means

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta} \tag{2.5.4}$$

$$\alpha, \beta = 1,2,3$$

Where $G_{\alpha\beta}$is a linear combination of$g_{\alpha\beta}$ and its and first and second derivatives. By the equivalence principle these equations can be further generalized to

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \qquad (2.5.5)$$

To obtain the equations that govern the behavior of the gravitational field we need to find the form of $G_{\mu\nu}$.We therefore set a number of requirements with regard to the properties of the gravitational field and which should be observed in constructing the sought equations. Thus the following requirements should be satisfied by $G_{\mu\nu}$;

(I) By definition it is a tensor consisting of the metric and its derivatives.

(II) This tenor should contain only terms that are either quadratic in the first derivatives of the metric tensor or linear in its second derivatives.

(III) It should be symmetric as $T_{\mu\nu}$.

(IV) Since $T_{\mu\nu}$ is conserved it should be equally so and vice versa.

(V) It should be reducible to the Newtonian limit.

By the fulfillment of these requirements and employing certain properties of the curvature tensor and its contractions presented in the previous section it can be seen [11] that the right hand side of equation (2.5.5) should have the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad (2.5.6)$$

This expression is called Einstein tensor. Thus equation (2.5.5) becomes

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \qquad (2.5.7)$$

Contracting with $g_{\mu\nu}$ yields

$$R = 8\pi G T_{\lambda}^{\lambda} \qquad (2.5.8)$$

Hence

$$R_{\mu\nu} = -8\pi G (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_{\lambda}^{\lambda} \qquad (2.5.9)$$

Equation (2.5.7) and (2.5.9) are the Einstein Field equations that describe the gravitational field and summarize the theory of General Relativity (GR). These equations can be alternatively derived by exploiting the variational action principle.

2.6 The Action Principle

The Principle of least action variations states that the action being a functional of the dynamical variables is stationary with respect to small variations of these variables. If the action is subjected to such a variation one can directly discover the connection between symmetry principles and conservation laws. Due to the symmetry of the action imposed by general covariance, the energy momentum tensor con be generally defined as a functional derivative of the action for any material system. Thus this tensor is certainly conserved. The total action $I$of a gravitational system which consists of a field and its source is given by.

$$I = I_M + I_G \tag{2.6.1}$$

Where I_M is the matter action and I_G is the gravitational one. The energy-momentum tensor of matter is defined as the functional derivatives ofI_M whose variation with respect to infinitesimal variation of $g_{\mu\nu}$ yields [48].

$$\delta I_M = \frac{1}{2}\int d^4x\sqrt{g(x)}T^{\mu\nu}(x)\delta g_{\mu\nu}(x) \tag{2.6.2}$$

Where

$$g = -Det g_{\mu\nu} \tag{2.6.3}$$

I.e. $-g$ is the determinant of the metric tensor. The coefficient $T^{\mu\nu}(x)$ is defined to be the energy-momentum tensor of this system. The gravitational action is defined by

$$I_G = -\frac{1}{16\pi G}\int \sqrt{g(x)}R(x)d^4x \tag{2.6.4}$$

Where the coefficient $1/16\pi G$ is introduced to satisfy the Newtonian limit. The gravitational action is defined by

$$\delta I_G = -\frac{1}{16\pi G}\int \delta\sqrt{gR}\, d^4x \tag{2.6.5}$$

Can be carried out by utilizing the following relations

$$\delta\left(\sqrt{gR}\right) = \delta\left(\sqrt{g}\, g^{\mu\nu} R_{\mu\nu}\right)$$

$$= R\delta\sqrt{g} + \sqrt{g} R_{\mu\nu}\delta g^{\mu\nu} + \sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} \tag{2.6.6}$$

With

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu}\delta g_{\mu\nu} \tag{2.6.7}$$

And

$$\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma}\delta g_{\rho\sigma} \tag{2.6.8}$$

And also

$$R\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\, g^{\mu\nu} R\delta g_{\mu\nu} \tag{2.6.9}$$

Hence

$$\sqrt{g} R_{\mu\nu}\delta g^{\mu\nu} = -\sqrt{g}\, g^{\mu\rho} g^{\nu\sigma} R_{\mu\nu}\delta g_{\rho\sigma} = -\sqrt{g} R^{\rho\sigma}\delta g_{\rho\sigma}$$

$$= -\sqrt{g} R^{\mu\nu}\delta g_{\mu\nu} \tag{2.6.10}$$

Further the variation of the Ricci tensor (2.4.7) yields the following relations [13]

$$\delta R = (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} - \left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda} \tag{2.6.11}$$

Substituting this in the last term of (2.6.6) gives

$$\sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{g}\left[\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\nu} - \left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right)_{;\lambda}\right] \tag{2.6.12}$$

Since I can be shown that [14]

$$\sqrt{g}\, T^{\mu}_{;\mu} = \partial_\mu\left(\sqrt{g}\, T^{\mu}\right) \tag{2.6.13}$$

Then we get

$$\sqrt{g}\, g^{\mu\nu}\delta R_{\mu\nu} = \frac{\partial}{\partial x^{\nu}}\left(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right) - \frac{\partial}{\partial x^{\lambda}}\left(\sqrt{g}\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right) \tag{2.6.14}$$

Owing to Gauss theorem this term vanishes when integrated over all space. Using the above given relations, the variation of the action I_G will be

$$\delta I_G = \frac{1}{16\pi G}\int \sqrt{g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu}d^4x \qquad (2.6.15)$$

The variation of the action can be obtained from equation (2.6.1)

$$\delta I = \delta I_M + \delta I_G \qquad (2.6.16)$$

If the total action is stationary with respect to an arbitrary variation in $g_{\mu\nu}$, then equations (2.6.2), (2.6.5) and (2.6.15), yield

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -8\pi G T^{\mu\nu} \qquad (2.6.17)$$

This is nothing but Einstein's field equations (2.5.9)

2.7 Einstein Equation In static Field

Stars, planets and any astronomical object produce static field around it. The field generated by there objects described by static isotropic metric. The interval for the static isotropic metric takes the form

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \qquad (2.7.1)$$

The metric has the no vanishing components

$$g_{rr} = A(r) \quad g_{\theta\theta} = r^2 \quad g_{\varphi\varphi} = r^2 sin^2\theta \quad g_{tt} = -B(r) \qquad (2.7.2)$$

With functions $A(r)$ and $B(r)$ that are to be determined by solving the field equations. Since $g_{\mu\nu}$ is easy to write down all the non vanishing components of its inverse:

$$g^{rr} = A^{-1}(r) \quad g^{\theta\theta} = r^{-2} \quad g^{\varphi\varphi} = r^{-2}(sin\theta)^{-2} \; g^{tt} = -B^{-1}(r) \qquad (2.7.3)$$

Furthermore, the determinant of the metric tensor is $-g$ where

$$g = r^4 A(r)B(r)sin^2\theta \qquad (2.7.4)$$

So the invariant volume element is

$$\sqrt{g}drd\theta d\varphi = r^2\sqrt{A(r)B(r)}sin\theta drd\theta d\varphi \quad (2.7.5)$$

The affine connection can be computed from usual formula

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)$$

Its only non vanishing components are

$$\Gamma^{r}_{rr} = \frac{\dot{A}}{2A}\frac{dA(r)}{dr} \qquad \Gamma^{r}_{\theta\theta} = -\frac{r}{A(r)}$$

$$\Gamma^{r}_{\varphi\varphi} = -\frac{rsin^2\theta}{A(r)} \qquad \Gamma^{r}_{tt} = \frac{1}{2A(r)}\frac{dB(r)}{dr}$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{\theta}_{\varphi\varphi} = -sin\theta\ cos\theta$$

$$\Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{r\varphi} = \frac{1}{r} \qquad \Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = cot\theta$$

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{1}{2B(r)}\frac{dB(r)}{dr} \quad (2.7.6)$$

We also need the Ricci tensor. It is given by (6.2.4) and (6.1.5) as

$$R_{\mu k} = \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{k}} - \frac{\partial\Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{k\eta} - \Gamma^{\eta}_{\eta k}\Gamma^{\lambda}_{\lambda\eta} \quad (2.7.7)$$

(Note that despite its appearance, the first term is symmetric in μ and$k$, because (4.7.6). gives $\Gamma^{\lambda}_{\mu\lambda}$ equal to $\frac{1}{2} \quad \partial\ln\frac{g}{\partial x^{\mu}}$.Inserting in (2.7.7) the components of the affine connection given by (2.7.6), we find

$$R_{rr} = \frac{B''(r)}{2B(r)} - \frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{A'(r)}{A(r)}\right)$$

$$R_{\theta\theta} = -1 + \frac{r}{2A(r)}\left(-\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) + \frac{1}{A(r)}$$

$$R_{\varphi\varphi} = sin^2\theta R_{\theta\theta}$$

$$R_{tt} = -\frac{B''(r)}{2A(r)}\frac{1}{4}\left(\frac{B'(r)}{B(r)}\right)\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) - \frac{1}{r}\left(\frac{B'(r)}{A(r)}\right) \quad (2.7.8)$$

$$R_{\mu v} = 0 \quad for\ \mu \neq \nu$$

2.8 The Schwarzschild Solution of Einstein Equation

The interval for the general static isotropic metric is given by

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2 - d\theta^2 + sin^2\theta d\varphi^2 \quad (2.8.1)$$

The field equations for empty space are

$$R_{\mu v} = 0 \quad (2.8.2)$$

The components the Ricci tensor are given by (2.7.8) thus

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) \quad (2.8.3)$$

Thus equation (2.8.3) requires

$$\frac{B'}{B} = -\frac{A'}{A}, or A(r)B(r) = constant \quad (2.8.4)$$

The boundary condition that for → ∞) requires the metric tensor approaching the Minkowski tensor in spherical coordinates, that is,

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \quad (2.8.5)$$

From (2.8.4) and (2.8.5) one has

$$A(r) = \frac{1}{B(r)} \quad (2.8.6)$$

Since (2.8.3) now vanishes, it remains to make $R_{rr}\ and\ R_{\theta\theta}$vavish using (2.8.6) in (2.7.8) yieios

$$R_{\theta\theta} = -1 + B'(r)r + B(r) \quad (2.8.7)$$

$$R_{rr} = \frac{B''(r)}{2B(r)} + \frac{B'}{rB(r)} = \frac{R'_{\theta\theta}}{2rB(r)} \quad (2.8.8)$$

So it is sufficient to set $R_{\theta\theta}$equal it zero, that is

$$\frac{d}{dr}\left(rB(r)\right) = rB'(r) + B(r) = 1$$

The solution is

$$rB(r) = r + constant \tag{2.8.9}$$

To fix the constant of integration we recall that at great distances from a central mass$M$, the component $g_{tt} \equiv -B$ must approach $-1 - 2\phi$ where ϕ, is the Newtonian potential$\phi - \frac{MG}{r}$. Hence the constant of integration is $-2MG$, and our final solution is

$$B(r) = \left[1 - \frac{2MG}{r}\right] \tag{2.8.10}$$

$$A(r) = \left[1 - \frac{2MG}{r}\right]^{-1} \tag{2.8.11}$$

The full metric in equation (2.8.1) is given by

$$d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 - d\theta^2$$
$$+ sin^2\theta d\varphi^2 \tag{2.8.12}$$

2.9 Equation of Motion of freely falling particles

Consider the motion of a freely falling material particle or photon in a static isotropic gravitational field. The interval is given by

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2 - d\theta^2 + sin^2\theta d\varphi^2 \tag{2.9.1}$$

The equations of free fall are

$$\frac{d^2x^\mu}{dp^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{dp}\frac{dx^\lambda}{dp} = 0 \tag{2.9.2}$$

Where p is parameter describing the trajectory. In general $d\tau$is proportional to dp, so for a material particle we could normalize $p = \tau$. However, for a photon the proportionality constant $\frac{d\tau}{dp}$ vanishes, and since we wish to treat photons as well as massive particles, we shall find it convenient to reserve the right to fix the normalization of p independently from that ofτ.

Using the non vanishing components of the affine connection equation (2.9.2) reads

$$0 = \frac{d^2r}{dp^2} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A(r)}\left(\frac{d\theta}{dp}\right)^2 - r\frac{sin^2\theta}{A(r)}\left(\frac{d\varphi}{dp}\right)^2 + \frac{B'(r)}{2A(r)}\left(\frac{dt}{dp}\right)^2 \quad (2.9.3)$$

$$0 = \frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{d\theta}{dp}\frac{dr}{dp} - sin\theta\ cos\theta\left(\frac{d\varphi}{dp}\right)^2 \quad (2.9.4)$$

$$0 = \frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{d\varphi}{dp}\frac{dr}{dp} + 2cot\frac{d\varphi}{dp}\frac{d\theta}{dp} \quad (2.9.5)$$

$$0 = \frac{d^2t}{dp^2} + \frac{B'(r)}{B(r)}\frac{dt}{dp}\frac{dr}{dp} \quad (2.9.6)$$

For isotropic field consider the orbit of a particle to be confined to the equatorial plane, i.e.

$$\theta = \frac{\pi}{2} \quad (2.9.7)$$

Dividing (2.9.5) and (2.9.6) by $\frac{d\varphi}{dp}$and$\frac{dt}{dp}$, respectively it follows that

$$\frac{d}{dp}\left\{\ln\frac{d\varphi}{dp} + \ln r^2\right\} = 0 \quad (2.9.8)$$

$$\frac{d}{dp}\left\{\ln\frac{dt}{dp} + \ln B\right\} = 0 \quad (2.9.9)$$

The solution of (2.9.9)

$$\frac{dt}{dp} = \frac{1}{B(r)} \tag{2.9.10}$$

Since $B(r)$ is close to unity, p is equal to$t$. The other constant is obtained from (2.9.8) and plays the role of an angular momentum per unit mass

$$r^2 \frac{d\varphi}{dp} = J = constant \tag{2.9.11}$$

Inserting (2.9.7), (2.9.10) and (2.9.11) in (2.9.3) gives

$$\frac{d^2r}{dp^2} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{r^3 A(r)} + \frac{B'(r)}{2A(r)B^2(r)} \tag{2.9.12}$$

Multiplying this equation with $2A(r)\frac{dr}{dp}$ one finds

$$\frac{d}{dp}\left\{A(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)}\right\} = 0$$

$$A(r)\left(\frac{dr}{dp}\right)^2 = \frac{J^2}{r^2} - \frac{1}{B(r)} = \frac{-E}{constant} \tag{2.9.13}$$

The proper timeτ may now be determined from (2.9.1), (2.9.7), (2.9.10), (2.9.11) and (2.9.12) to get

$$d\tau^2 = Edp^2 \tag{2.9.14}$$

E Must take the values

$$\begin{gathered} E > 0\, for\ material\ particles \\ E = 9\, for\ photons \end{gathered} \tag{2.9.15}$$

Since $A(r)$ is always positive, so (2.9.13) tells us that a particle can reach radius $r$only if

$$\frac{J^2}{r^2} + E \leq \frac{1}{B(r)} \tag{2.9.16}$$

The parameter $p$may be eliminated everywhere by using (2.9.10) in (2.9.11), (2.9.13) and (2.9.14) to have

$$r^2 \frac{d\varphi}{dp} = JB(r)$$

$$\frac{A(r)}{B^2(r)}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E \tag{2.9.17}$$

$$d\tau^2 = EB^2(r)dt^2 \tag{2.9.18}$$

For a slowly moving particle in a weak field $\frac{J^2}{r^2}$, $\left(\frac{dr}{dp}\right)^2$, $A - 1, and$

$B - 1 \simeq 2\phi$Will all be small, and to first order in these quantities the above equations of motion become

$$r^2 \frac{d\varphi}{dp} \simeq J \tag{2.9.19}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2r^2} + \phi \simeq \frac{1-E}{2} \tag{2.9.20}$$

Particle in a circular orbit at R.since $\frac{dr}{dt}$ vanishes, equation.(2.9.19) became

$$\frac{J^2}{R^2} - \frac{1}{B(R)} + E = 0 \tag{2.9.21}$$

Also, for equilibrium at this radius, the derivative at R of the left-hand side must also vanish, so

$$-\frac{2J^2}{R^3} + \frac{B'(R)}{B^2(R)} = 0 \tag{2.9.22}$$

If one regard a circle as the limit of an ellipse with perihelia $R - \delta$ and aphelia $R + \delta$, then (2.9.19) shows that $\frac{J^2}{r^2} - \frac{1}{B(r)} + E$ must vanish at $r = R \pm \delta$, and this gives (2.9.21) and (2.9.22) in the limit $\delta \to 0$). From (2. 9.21) and (2.9.22) we find

$$E = \frac{1}{B(R)}\left(1 - \frac{RB'(R)}{2B(R)}\right) \tag{2.9.23}$$

$$J^2 = \frac{B'(R)R^3}{2B^2(R)} \tag{2.9.24}$$

Thus equation (2.9.24) and (2.9.18) gives the rate of revolution as

$$\frac{d\varphi}{dp} = \left(\frac{B'(R)}{2R}\right)^{\frac{1}{2}} \tag{2.9.25}$$

Whereas (2.9.23) and (2.9.2) give the proper time as

$$\frac{d\tau}{dt} = \sqrt{B(R) - \frac{1}{2}RB'(R)} \tag{2.9.26}$$

By using the Robertson expansion one gets

$$\frac{d\varphi}{dp} = \left(\frac{MG}{R^3}\right)^{\frac{1}{2}}\left[1 - \frac{(\beta-\gamma)MG}{R} + \cdots\right] \tag{2.9.27}$$

$$\frac{d\tau}{dt} = \left[1 - \frac{3MG}{R} + \cdots\right] \tag{2.9.28}$$

In most applications the shape of the orbits is needed, that is r as a function ofφ. The orbit shape can be obtained directly by eliminating $dp$from (2.9.11) and (2.9.13); this gives

$$\frac{A(r)}{r^2}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2B(r)} = \frac{E}{J^2} \tag{2.9.29}$$

The solution is thus gives by

$$\varphi = \pm\int \frac{A^{\frac{1}{2}}(r)\,dr}{r^2\left(\frac{1}{J^2B(r)} - \frac{E}{J^2} - \frac{1}{r^2}\right)^{\frac{1}{2}}} \tag{2.9.30}$$

2.10 Deflection of light by the sun

Consider a photon approaching the sun from very great distances. At infinity the metric becomes Minkowskian, that is, $A(\infty) = B(\infty) = 1$, The motion ∞ on a straight line at constant velocity V, that is.

$$b \simeq rsin(\varphi - \varphi_\infty) \simeq r(\varphi - \varphi_\infty)$$

$$-V \simeq \frac{d}{dt}(rcos(\varphi - \varphi_\infty) \simeq \frac{dr}{dt}$$

Where b is "impact parameter" and φ_∞ is the incident direction. Inserting these in (2.9.18) and (2.9.19) it is clear that they do satisfy the equations of motion at infinity where$A = B = 1$, and that the constants of the motion are

$$J = bV^2 \tag{2.10.1}$$

$$E = 1 - V^2 \tag{2.10.2}$$

One can express J in terms of the distance $r_\circ$ of closest approach to the sun rather than the impact parameter$b$. At $r_\circ$, $\frac{dr}{d\varphi}$ vanishes, so (2.9.29) and (2.10.2)

$$J = r_\circ \left(\frac{1}{B(r_\circ)} - 1 - V^2\right)^{\frac{1}{2}} \tag{2.10.3}$$

The orbit is described by (2.9.30), that is.

$$\varphi(r) = \varphi_\infty + \int_r^\infty \frac{A^{\frac{1}{2}}(r)\, dr}{r^2\left(\frac{1}{(r_\circ)}\left[\frac{1}{B(r)} - 1 - V^2\right]\left[\frac{1}{B(r_\circ)} - 1 - V^2\right]^{-1} - \frac{1}{r^2}\right)^{\frac{1}{2}}} \tag{2.10.4}$$

The total change in φ as r decreases from infinity to its minimum value $r_\circ$ and then increases again to infinity is just twice its change from ∞ to $r_\circ$, that is, $2|\varphi(r_\circ) - \varphi_\infty|$. if the trajectory were a straight line, this would equal just π; hence the deflection of the orbit from a straight line is

$$\Delta_\varphi = 2|\varphi(r_\circ) - \varphi_\infty| - \pi \tag{2.10.5}$$

If this is positive, then the angle φ changes by more that 180°, that is the trajectory is bent toured the sun; if Δ_φ is negative then the trajectory is bent away from the sun.

For photon $V^2 = 1$ and, (2.104) gives

$$\varphi(r) - \varphi_\infty = \int_r^\infty A^{\frac{1}{2}}(r) \left[\left(\frac{r}{r_\circ}\right)^2 \left(\frac{B(r_\circ)}{B(r)}\right) - 1\right]^{-\frac{1}{2}} \frac{dr}{r} \qquad (2.10.6)$$

By using the values of $A(r)$ and $B(r)$ gives by the Schwarzschild solution then one would obtain $\varphi(r)$ and Δ_φ as elliptic integrals which can be evaluated numerically by expanding in the small parameters $\frac{MG}{r_\circ}$ and $\frac{MG}{r}$. It is easier to expand before integrating, using for $A(r)$ and $B(r)$ the Robertson expansions

$$A(r) = 1 + 2\gamma\frac{MG}{r} + \cdots$$

$$B(r) = 1 - 2\frac{MG}{r} + \cdots$$

Thus

$$\left[\left(\frac{r}{r_\circ}\right)^2 \left(\frac{B(r_\circ)}{B(r)}\right) - 1\right] = \left(\frac{r}{r_\circ}\right)^2 \left[1 + MG\left(\frac{1}{r} - \frac{r}{r_\circ}\right) + \cdots\right] - 1$$

$$= \left[\left(\frac{r}{r_\circ}\right)^2 - 1\right]\left[1 - \frac{2MGr}{r_\circ(r + r_\circ)} + \cdots\right]$$

So (2.10.6) gives

$$\varphi(r) - \varphi_\infty = \int_r^\infty \frac{dr}{r\left[\left(\frac{r}{r_\circ}\right)^2 - 1\right]^{\frac{1}{2}}} \left[1 + \frac{\gamma MGr}{r} + \frac{MGr}{r_\circ(r + r_\circ)} \cdots\right]$$

The integral thus gives

$$\varphi(r) - \varphi_\infty$$

$$= sin^{-1}\left(\frac{r_\circ}{r}\right) + \frac{MG}{r_\circ}(1 + \gamma - \gamma\sqrt{1 - \left(\frac{r_\circ}{r}\right)^2} - \sqrt{\left(\frac{r - r_\circ}{r + r_\circ}\right)}$$

$$+ \cdots \qquad (2.10.7)$$

Hence to first order in $\frac{MG}{r_\circ}$, the deflection (2.10.5) is

$$\Delta_{\varphi} = \frac{4MG}{r_{\circ}} \left(\frac{1+\gamma}{2} \right) \tag{2.10.8}$$

For a light ray deflected by the sun $\frac{MG}{r_{\circ}} M = M_{\odot} = 1.97 \times 10^{33} g$, that is, $MG = M_{\odot} G = 1.475\ km$, and the minimum value of $r_{\circ}$ is $R_{\odot} = 6.95 \times 10^{5} km$, so (2.10.8) gives

$$\Delta_{\varphi} = \left(\frac{R_{\odot}}{r_{\circ}} \right) \theta_{\odot} \tag{2.10.9}$$

Where

$$\theta_{\odot} = \frac{4M_{\odot}G}{R_{\odot}} \left(\frac{1+\gamma}{2} \right) = 1.75'' \left(\frac{1+\gamma}{2} \right) \tag{2.10.10}$$

Furthermore, general relativity gives $\gamma = 1$, so deflection toward the sun, with $\theta_{\odot} = 1.75''$.

(Six month earlier) from φ (eclipse) then, in principle, should give Δ_{φ}. However there is an unavoidable change in the scale of the photographs over a six-month interval, owing partly to small changes in the temperature and in the mechanical configuration of the telescope and camera over so long a time. A change in the scale of the photograph would give an apparent deflection of any star toward or away from the sun by an angle proportional to the distance $r_{\circ}$ at which its light passes the sun; hence what is done in practice is to compare observations with a theoretical curve

$$\Delta_{\varphi} = \theta_{\odot} \left(\frac{R_{\odot}}{r_{\circ}} \right) + s \left(\frac{r_{\circ}}{R_{\odot}} \right) \tag{2.10.11}$$

Where S is the unknown scale constant (often called α) and θ is an angle to be compared with the theoretical value $1.75''$. There are other effects that could contribute to Δ_{φ}, such as refraction of the starlight in the solar corona or as it enters the colder air in the moon’s shadow, but none of these is believed to play an important role.

Observations cannot be carried closer to the sun's disk then$r_{\circ} \approx 2R_{\odot}$, but they can still be used to determine $_{\odot}$ by fitting the observed Δ_{φ} values to the theoretical curve (2.10.11). The difficulty with this program is just that Δ_{φ} is very difficult to measure accurately in the brief time available during an eclipse. In 1919 eclipse expeditions were sent to two small islands, sobral, off the northeast coast of Brazil, and Principe, in the gulf of guinea. About a dozen stars in all were studied, and yielded $values^{2}$ $1.98 \pm 0.12''$and$1.61 \pm 0.0.31''$, in substantial agreement with Einstein's prediction$\theta_{\odot} = 1.75''$. it was perhaps this dramatic result more than any other success that brought general relativity to the attention of the general public in the 1920's.

Since 1919 there have been measurements on about 380 stars observed during the eclipses of 1922, 1929, 1936, 1947, and 1952, which we summarize in table 8.1 (taken from the summary of von$Kluber^{3}$). The values obtained for $\theta_{\odot}$vary form$1.3''to2.7''$, but mostly lie between 1.7 and$2''$. The most recent of these results is$\Delta_{\varphi} = 1.70 \pm 0.10''$, in very good agreement with Einstein's prediction, but it is not clear that the systematic error here is really smaller than for previous observations. From all this we can conclude that there definitely is a deflection of light greater than the value $\theta_{\odot} = 0.875''$. That would be predicted for $\gamma = 0$(i.e, A(r) =1), but as to its precise value we can say little more then that $\theta_{\odot}$is somewhere between 1.6 and $2.2''$;that is γ is between about 0.9 and 1.3. it may become possible to improve the accuracy of this determination in the near future by using photoelectric techniques to monitor star positions without waiting for an eclipse.

Recent developments in radio $astronomy^{4}$ have made it possible to measure the deflection of radio signals by the sun with potentially for greater accuracy than is possible in optical astronomy. The angular accuracy of optical observations is limited by in homogeneities in the earth's atmosphere to about$0.1''$. Whereas a radio interferometer with wavelength $\lambda$and baseline D can in principle measure.

2.11 The Cosmological General Relatwistic Model

The modern cosmological theory is built on the cosmological principle, which treat the universe an spatially homogeneous and isotropic. The space-time metric of such a universe is given by the Robertson-Walker metric.

$$d\tau^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right] \quad (2.11.1)$$

Where $a(t)$an unknown function of time is called the cosmic scale factor and k is a constant known as the spatial curvature, which by a suitable choice of units for r can be set to take only three values $1, 0, or - 1$ for a closed, spatially flat and open universes respectively. The spatial polar coordinates $r, \theta\ and\ \phi$ form a co-moving system in the sense that typical galaxies have constant spatial coordinates $r, \theta\ , \phi$.

The energy momentum tensor that describes the cosmic matter is the same form as for a perfect fluid.

$$T_{\mu\nu} = (\rho + p)U_\nu U_\mu + p g_{\mu\nu} \quad (2.11.2)$$

Where for commoving coordinates

$$U_t = 1 \ \ U_i = 0$$

$$T_{tt} = \rho(t), T_{it} = 0, and, T_{ij} = \ g_{ij}p \quad (2.11.3)$$

With

$$i, j = r, \theta, \phi$$

p is the proper pressure, ρ is the proper total energy density, and U_μ is the velocity four-vector satisfying the relation

$$g^{\mu\nu}U_\mu U_\nu = -1 \quad (2.11.4)$$

The conservation of the energy momentum tensor is given by.

$$T^{\mu\nu}; \nu = 0 \quad (2.11.5)$$

This equation is trivially satisfied for $\mu = r.\theta, \phi$ while for $\mu = t$ it reads:

$$a^3(t)\frac{dp(t)}{dt} = \frac{d}{dt}(a^3(t)[\rho(t) + p(t)]) \tag{2.11.6}$$

If the pressure of the cosmic matter is negligible then equation (2.11.6) reduces to

$$\rho(t)a^3(t) = constant \tag{2.11.7}$$

The proper distance between galaxies, one at the origin and the other r_1, θ_1, ϕ_1 is given by [7[:

$$d_{prop}(t) = \int_0^{r_1} \sqrt{g_{rr}}\, dr = a(t) \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} \tag{2.11.8}$$

Therefore galaxies move apart when $a(t)$ increases or become closet when $a(t)$ decreases.

Information about $a(t)$ comes from the observation of shifts in frequency of light emitted by distant sources. To find such frequency shifts, consider an electromagnetic wave travelling towards us along$-r$ direction with $\theta\ and \phi$ fixed. The equation of motion of a given wave crest is then.

$$0 = d\tau^2 = dt^2 - a^2(t)\frac{dr^2}{1 - kr^2} \tag{2.11.9}$$

Hence if the wave leaves a typical galaxy, located at r_1, θ_1, ϕ_1at $t_1$then it will reach us at a time t_0 given by:

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = f(r_1) \tag{2.11.10}$$

Where:

$$f(r_{1)} = \int_0^{r_1} \frac{dr}{\sqrt{1 - kr^2}} = \begin{Bmatrix} sin^{-1}r_1 & k = 1 \\ r_1 & k = 0 \\ sinh^{-1}r_1 & k = -1 \end{Bmatrix} \tag{2.11.11}$$

If the next wave crest leaves r_1 at time$t_1 + \delta t_1$, it will arrive to us at time$t_0 + \delta t_0$, which is again by

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = f(r_1) \qquad (2.11.12)$$

Noting that $a(t)$ does not change much during the periods δt_0 and δt_1we obtain

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \qquad (2.11.13)$$

If $\delta t_0$and $\delta t_1$are periodic times then the emitted frequency v_1 and the observed frequency v_0 are given by:

$$\frac{v_0}{v_1} = \frac{\delta t_1}{t_0} = \frac{a(t_1)}{a(t_0)} \qquad (2.11.14)$$

The red shift parameter z which is defined as the fractional increase in the wave length λ is given by:

$$\frac{\lambda_0 - \lambda_1}{a(t_1)} = \frac{v_0}{v_1} - 1 = \frac{a(t_0)}{a(t_1)} - 1 \qquad (2.11.15)$$

Red shifts are observed when $\lambda_0 > \lambda_1$and hence$z > 1$, while blue shifts are observed when $\lambda_0 < \lambda_1$and hence$z < 1$. If the universe is expanding then $a(t_0) > a(t_1)$and as a result red shift should be observed. Such a frequency shift might be due to a Doppler Effect which results from the relative motion of the source and the observer. If it happens that two relatively close galaxies move away from or towards the Milky Way, then the radial velocity $v_r$is given from equation (2.11.8) by:

$$v_r = \frac{d}{dt} d_{prop} \approx \frac{d}{dt} a(t) \int_0^{r_1} dr = r_1 \, a(t) \qquad (2.11.16)$$

With the dot meaning time differentiation. For $r_1 \to 0, t_1 \to t_0$ the frequency shift z is given by:

$$z = \frac{a(t_0) - a(t_1)}{a(t_1)} \approx \frac{\dot{a}(t_0)(t_0 - t_1)}{a(t_0)} \qquad (2.11.17)$$

On the other hand by (2.11.9) for$k \to 0$

$$\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} \approx \int_0^{r_1} dr = \int_{t_1}^{t_0} \frac{dt}{a(t)} \approx \frac{1}{a(t_0)} \int_0^{r_1} dt$$

$$r_1 = \frac{(t_0 - t_1)}{a(t_0)}$$

$$z \approx a(t_0) r_1 = v_r \qquad (2.11.18)$$

In 1922 Vesto Melvin Slipher gave data for 41 spiral nebulae, of which 36 had a absorption lines shifted to the red by amounts up to $z \approx 0.006$. these frequency shifts were interpreted as due to the Doppler effect. The observation of red shifts in all parts of the sky suggests that it is due a general recession of galaxies. In a series of papers Wirt and K. Lundmark showed that Slipper's red shifts could most easily be understood in terms of a general recession of distant galaxies. In 1929 Edwin Hubble showed that the speed of distant galaxies increases linearly with their distance from us. The relation between the radial distance r of a given star and its red shift $\Delta\lambda$ known as Huddle"s Law is given by:

$$\frac{\Delta\lambda}{\lambda} = z = \frac{\dot{a}(t_0)}{a(t_0)} (t_0 - t_1) = H_0 r \qquad (2.11.19)$$

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \qquad (2.11.20)$$

$H_0$is called Hubble's constant.

To describe the universe one use standard big bang (SBB) model. This model is based on the cosmological principle and Einstein's filed equations. According to the cosmological principle the energy-momentum tensor in the field equation is that of a perfect fluid, while the space time metric is given by the Robertson-Walker metric. The Robertson-Walker metric components given by:

$$g_{tt} = -1, \qquad g_{it} = 0, \quad i = r, \theta, \phi$$

$$g_{rr} = a^2(t)(1 - kr^2), \qquad g_{\theta\theta} = a^2(t) r,$$

$$g_{\phi\phi} = a^2(t) r^2 sin^2\theta \qquad (2.11.21)$$

The only non vanishing components of the affine connection for this metric are

$$\Gamma_{ij}^{t} = \frac{\dot{a}}{a} g_{ij} \Gamma_{ij}^{i} = \frac{\dot{a}}{a} \delta_j^i$$

Where according to

$$\Gamma_{jk}^{i} = \frac{1}{2} g^{il} \left[\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{lk}}{\partial x^l} \right] \tag{2.11.22}$$

The Ricci tensor components are shown to be

$$R_{tt} = 3\frac{\ddot{a}}{a}, R_{ti} = 0, R_{ij} = -\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2} \right) g_{ij} \tag{2.11.23}$$

$$i = r, \theta, \phi$$

The curvature scalar is then given by:

$$R = g^{tt} R_{tt} + g^{ii} R_{ii}$$

$$= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \tag{2.11.24}$$

Using Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

The time-time component rears

$$R_{tt} - \frac{1}{2} g_{tt} R = -8\pi G T_{tt}$$

Then by (2.11.23),(2.11.24),(22.11.22) and (2.11.3)

$$\dot{a}^2 + k = \frac{8\pi G}{3} \rho a^2$$

$$\rho - \rho_c = \frac{3}{8\pi G} \frac{k}{a^2} \tag{2.11.25}$$

$$\rho_c = \frac{3\dot{a}^2}{8\pi G a^2} \tag{2.11.26}$$

Is called the critical density. The space-space components are given by:

$$R_{ii} - \frac{1}{2} g_{ii} R = -8\pi G T_{ii}$$

$$2\ddot{a}a + \dot{a}^2 + k = -8\pi G p a^2 \qquad (22.11.27)$$

The term $\dot{a}^2$ can be eliminated by subtracting equation (2.11.25) from equation (2.11.27) to get

$$3\ddot{a} = -4\pi G(\rho + 3p)a \qquad (2.11.28)$$

Multiplying equation (2.11.25) by 3 and adding equation (2.11.27) yields

$$\ddot{a}a + 2\dot{a}^2 + 2k = 4\pi G(\rho + p)a^2 \qquad (2.11.29)$$

Beside the field equations we have the equation (2.11.6) of energy conservation

$$\dot{p}a^3 = \frac{d}{dt}[a^3(\rho + p)]$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \qquad (2.11.30)$$

The energy conservation equation can be used to find the density ρas a function of$a(t)$. In a radiation dominated era, where the energy density is dominated by ultra-relativistic particles, the equation of state is given by.

$$p \approx \frac{\rho}{3} \qquad (2.11.31)$$

Hence equation (2.11.30) yields

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0$$

This is satisfied by

$$\rho = c_1 a^{-4} \qquad (2.11.32)$$

In a matter dominated era where non relativistic matter with negligible pressure is present$p \ll \rho, p \approx 0$, the energy equation yields

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0$$

And this is satisfied by

$$\rho = c_2 a^{-3} \tag{2.11.33}$$

Where c_2 is a constant. It is possible to extract some information about the past and future of the universe from the field equation [13]. At present we know that $a(t) > 0$. According to equation (2.11.28) $\ddot{a}$ is negative as long as $\rho + 3p$ remains positive. Since we observe red shifts it follows that $\dot{a} > 0$. This means that the curve of $a(t)$ versus t must be concave downward, and should have reached $a(t) = 0$ at some finite time in the past. Let us set this time at $t = 0$ so that

$$a(0) = 0 \tag{2.11.34}$$

This equation states that the universe has started with a sin gularity of infinite density as shown by using equation (2.11.34) in (2.11.32) and (2.11.33) where

$$a(t = 0) = 0, \quad \rho = \infty \tag{2.11.35}$$

The future of the universe depends on its curvature. From equations (2.11.32) and (2.11.33), we see that the density ρ must decrease with increasing $a(t)$, at least as fast as a^{-3}, so that for $a(t) \to \infty$, the right hand side of equation (2.11.25) vanishes at least as fast as $a^{-1} for\ k = -1$

$$\dot{a}^2 = -k = 1$$

i.e.

$$a(t) = t. \qquad hence\ a(t) \to \infty, as\ t \to \infty$$

$$a(t) = t, \qquad t \to \infty, k = -1$$

i.e. $a(t)$ goes on increasing forever. For $k = 0$

$$\dot{a}^2 = \frac{8\pi G}{3} \rho a^2$$

Hence $\dot{a}^2$ remains positive, so $a(t)$ goes on increasing, more slowly than t.

For $k = +1$.

$$\dot{a}^2 = -1 + \frac{8\pi G}{3} \rho a^2$$

$\dot{a}^2$Will reach zero when ρa^2 drops to the value$\frac{3}{8\pi G}$. Since according to (2.11.28) $\ddot{a}$ is negative it follows that $\dot{a}$ will be negative and as a result $a(t)$ will begin to decrease again until it reaches the initial value $a = 0$ at some finite time in the future. Hence the cosmic history of the universe depends on the sign of the spatial curvature k and the relation between the actual density ρ and the critical density ρ_c(see equation (2.11.26)). If$k = -1, \rho < \rho_c$ or $k = 0, \rho = \rho_c, \quad \rho > \rho_c$, then the expansion will eventually cease and will be followed by a contraction back to a singular state with $a(t) = 0$ [4].

The dynamical equation of the universe can be also derived by determining the total energy of a co-moving sphere. We can think of the universe as consisting as consisting of a Newtonian gas in a state of an everywhere-uniform expansion. Any given gas particle will have a trajectory

$$X(t) = X(t_0) = \frac{a(t)}{a(t_0)}$$

The gravitational potential energy V, of such a particle gust arises from the matter within a sphere of radius $|X(t)|$ and centre at the origin, reads

$$V(t) = -\frac{4\pi}{3}|x(t)|^3\rho(t)\frac{mG}{|x(t)|} = -\frac{4\pi}{3}mG|x(t_0)|^2\frac{a(t)^2}{a(t_0)^2}$$

Where $m$the particle is mass and $\rho(t)$ is the mass density. The kinetic energy of this is

$$\frac{a(t)}{a(t_0)} = \frac{1}{2}m|\dot{x}(t)|^2 = \frac{1}{2}m|x(t_0)|^2\frac{\dot{a}^2(t)}{a^2(t_0)}$$

The total energy of the particle is then given by:

$$E = T(t) + V(t) = \frac{1}{2}m\frac{|x(t_0)|^2}{a^2(t_0)}\left(\dot{a}^2(t) - \frac{8\pi G}{3}\rho(t)a^2(t)\right) \quad (2.11.36)$$

Using equation (2.11.25) yields

$$E = -\frac{1}{2}m\frac{|x(t_0)|^2}{\square^2(t_0)}k \qquad (2.11.37)$$

For $k = -1$, E is positive and the gas will expand for ever. For $k = 0$, E vanishes and the gas is just barely able to expand indefinitely. For $k = +1$ E is negative and the expansion will cease and be followed by a collapse.

2.12 Matter-Dominated Era in This era

The density of radiation is much less than the density of matter-hence we are now in a matter dominated era. Using equation (2.11.25) and (2.11.20) the present density ρ_0 is given by [15].

$$\rho_0 = \frac{3}{8\pi G}\left(\frac{k}{a_0^2} + H_0^2\right) \qquad (2.12.1)$$

The present pressure p_0 can also be obtained in terms of the deceleration parameter q_0 which is defined as

$$q_0 = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} \qquad (2.12.2)$$

Combining equation (2.12.1) and (2.12.2) yields

$$p_0 = -\frac{1}{8\pi G}\left[\frac{k}{a_0^2} + H_0^2(1 - 2q_0)\right] \qquad (2.12.3)$$

Where a_0 is the present value of the cosmic scale factor, and H_0 and q_0 are the Hubble constant and the deceleration parameter at present. According to equation (2.12.1) the sign of the spatial curvature depends on whether ρ_0 is greater or less than a critical density

$$\rho_c = \frac{3H_0^2}{8\pi G} = \frac{(1.1)10^{-29}g}{cm^3} \qquad (2.12.4)$$

Where H_0 is estimated to satisfy $H_0^{-1} = 13 \times 10^9 years$. then by (2.11.25)

$$\rho_0 - \rho_c\frac{3}{8\pi G}\frac{k}{a_0^2} \qquad (2.12.5)$$

In a matter dominated era$p_0 \approx 0 \ll \rho_0$. In this case, equation (2.12.3) yields

$$\frac{k}{a_0^2} = (2q_0 - 1)H_0^2 \tag{2.12.6}$$

Using equations (2.12.4), (2.12.5) and (2.12.6) yields

$$\frac{\rho_0}{\rho_c} = 2q_0 \tag{2.12.7}$$

Hence if $q_0 > \frac{1}{2}$ $then$ $\rho_0 > \rho_c$ and$\rho_c k = +1$, while if $q_0 < \frac{1}{2}, \rho_0 < \rho_c$ and$\rho_c k = -1$. When$q_0 = \frac{1}{2}$ $then$ $\rho_0 = \rho_c$ and$\rho_c k = 0$.

To obtain the functional form of the cosmic scale factor $a(t)$ in a matter dominated era the equation of state is substituted in the field equation. Thus we can write equation (2.11.25) in the form

$$\dot{a} = \pm\left(\frac{2mG}{a} - k\right)^{\frac{1}{2}}$$

Where in view of (2.11.33)

$$m = \frac{4\pi}{3}a^3\rho = \frac{4\pi}{3}c_2 \tag{2.12.8}$$

Hence

$$d\dot{a} = \pm\frac{mGda}{a\sqrt{2mGa - ka^2}} \tag{2.12.9}$$

On the other hand since $p = 0$ in a matter era, hence by (2.11.33) equation (2.11.28) becomes.

$$\frac{d\dot{a}}{dt} = -\frac{4\pi G\rho a}{3} = -\frac{mG}{a^2}$$

By using equation (2.12.9) we get,

$$d\dot{a} = -\frac{mG}{a^2}dt = \frac{mGda}{a\sqrt{2mGa - ka^2}}$$

$$t = \int \frac{mGda}{a\sqrt{2mGa - ka^2}} \qquad (2.12.10)$$

If $k = 0$ this equation gives

$$t = \pm\sqrt{\frac{2}{9mG}}\, a^{3/2}\; a = \left(\frac{9mG}{2}\right)^{1/3} t^{2/3} \qquad (2.12.11)$$

In this era the universe expands forever.

If $k = +1$then equation (2.12.10) yields

$$t = mG\sqrt{\frac{a}{2mG}(1 - \frac{a}{2mG})} - sin^{-1}\left(1 - \frac{a}{2mG}\right) \qquad (2.12.12)$$

And this indicates that the expansion will eventually cease and be followed by contraction. While if $k = -1$

$$t = mG\sqrt{\frac{a}{2mG}(1 + \frac{a}{2mG})} + \frac{1}{2}\ln\left[\frac{\sqrt{\left(1 + \frac{a}{2mG}\right)} - \sqrt{\frac{a}{2mG}}}{\sqrt{\left(1 + \frac{a}{2mG}\right)} + \sqrt{\frac{a}{2mG}}}\right] \qquad (212.13)$$

And the universe will expand forever.

2.13 Radiation Dominated Era

A weak cosmic microwave radiation background signal was observed in 1965 [17]. In the early universe when the temperature was very high matter and radiation were in thermal equilibrium. The radiation would dominate and is called black-body spectrum throughout the early universe. The radiation has a black body spectrum, thus energy density is given by:

$$\rho = \sigma T^4 \qquad (2.13.1)$$

Where σ is a constant. During the ear when the temperature was bounded by $10^{12}K\; and 10^{9}K$ radiation energy dominated and it follows that the equation describing this state is given by:

$$p = \frac{\rho}{3} \qquad (2.13.2)$$

As a result, the energy density is thus given by equation (2.11.32)

$$\rho = c_1 a^{-4} \tag{2.13.3}$$

Using equation (2.13.1) and equation (2.13.3) the relation between the cosmic scale factor and the temperature is given by:

$$T(t) = \frac{c_1}{\sigma}\frac{1}{a(t)} \tag{2.13.4}$$

At present the temperature of the cosmic microwave background radiation is$T_0 = 2.7^0 K$. if there had been no scattering of the background radiation since the recombination of hydrogen at about $T = 4000^0 K$, then the time t_R corresponding to a red shift $z_R$should be given by:

$$1 + z_R = \frac{a_0}{a(t_R)} = \frac{T(t_R)}{T_0} = \frac{400}{2.7} \approx 1500 \tag{2.13.5}$$

In the early universe, and in view of (2.13.1) and (2.13.3), the right hand side of the equation just before (2.11.25) $8\pi G\rho a^2$ varies as $\frac{1}{a^2} \sim T^2$, hence it was large in the early universe. On the other hand according to the same equation $\dot{a}$ is also large in the early universe, hence the curvature term can be dropped from equation (2.11.25) which becomes

$$\dot{a}^2 = \frac{8\pi G c_1 a^2}{3} \tag{2.13.6}$$

In view of equation (2.3.13) one gets

$$\dot{a}^2 = \frac{8\pi G c_1}{3} a^{-2}$$

Thus

$$a = \left[\left(\frac{32\pi G c_1}{3}\right)^{1/2} t + c_3\right]^{1/2}$$

But at

$$t = 0 \quad a = 0 \quad c_3 = 0$$

$$a = \left(\frac{32\pi G c_1}{3}\right)^{1/4} t^{1/2} \qquad (2.13.7)$$

Chapter Three

Generalized General relativity

3.1 Introduction

The afore noted reducibility of the GFE to Einstein's gravitational equation implies that they allow not only Schwarz child's but other possible solutions. Of cosmological significance. We restrict ourselves to solutions with a globally constant scalar curvature to investigate these possibilities.

3.2 Generalized General Relativistic Gravitational Equation

Using the general action pineapple in a general Lagrangian a fourth-order equation field the metric tensor, which is treated as field variables, was obtained by Ali-Eltahir[49].

This equation was obtained Later also by Lahzcos but the field variables are not selected puberty [50].

The generalized general relativity [GGR] is expressed in terms of a general Lagranigian in the form

$$\mathcal{L}'''\left(R_{;\mu}R_{;\nu} - g_{\mu\nu}g^{\rho\sigma}R_{;\rho}R_{;\sigma}\right) + \mathcal{L}'''\left(R_{;\mu;\nu} - g_{\mu}\Box^{2}R\right) + \mathcal{L}'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L} \qquad (3.2.1)$$

The Lagranigianl here is dependent on the scalar curvature R which in term dependent on the metric tensor; i.e.

$$\mathcal{L} = \mathcal{L}(R) \qquad (3.2.2)$$

$$R = R\left(g_{\mu\nu}, g_{\mu\nu;\gamma}, g_{\mu\nu;\gamma,k}\right) \qquad (3.2.3)$$

This situation conforms to the electro nag retie and other fields in the selection of field variables.

It is very interesting to note that when this Lagrangian is Linear, it reduces to (GR), i.e. when

$$\mathcal{L} = \beta R + \gamma \tag{3.2.4}$$

It follows that

$$\mathcal{L}' = \beta \qquad \mathcal{L}'' = 0 \quad \mathcal{L}''' = 0 \tag{3.2.5}$$

Thus equation (3.2.1) becomes

$$\beta\, R_{\mu\nu} - \frac{1}{2}\beta g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} \tag{3.2.6}$$

By salting

$$\beta = \frac{1}{16\pi G}$$

$$\frac{1}{2} g_{\mu}\gamma = -T_{\mu\nu} \tag{3.2.7}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{3.2.8}$$

Which is the ordinary GR equation (2.5.7).

3.3 A Static Isotropic Lagrangian-Dependent Metric

In reference [6] the GFE have been applied to the case of a static isotropic metric given by [51]

$$g_{rr} = \mathrm{A(r)}, g_{\theta} = r^2, g_{\emptyset\emptyset} r^2 \sin^2\theta, g_{tt} = -\mathrm{B(r)} \tag{3.31}$$

Where the proper time interval is given by

$$ds^2 = dr^2 = -\, g_{\mu\nu} dx^{\mu} dx^{\nu} \tag{3.3.2}$$

$$dr^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\emptyset^2 \tag{3.3.3}$$

This metric describes the behavior of the gravitational field assumed to be generated by a single star. The non-vanishing components of the affine connections are given by [1],

$$\Gamma^r_{rr} = \frac{\dot{A}}{2A}, \Gamma^r_{\theta\theta} = -\frac{r}{A}, \Gamma^r_{\emptyset\emptyset} = -\frac{r sin^2\theta}{A} \tag{3.3.4}$$

$$\Gamma_{tt}^{r} = \frac{\dot{B}}{2A}, \Gamma_{r\theta}^{\theta} = \Gamma_{\theta r}^{\theta} = \frac{1}{r}, \Gamma_{\emptyset\emptyset}^{\theta} = -sin\theta con\theta \quad (3.3.5)$$

$$\Gamma_{\emptyset r}^{\emptyset} = \Gamma_{r\emptyset}^{\emptyset} = \frac{1}{r} \quad (3.3.6)$$

$$\Gamma_{\emptyset\theta}^{\emptyset} = \Gamma_{\theta\emptyset}^{\emptyset} = cot\theta \quad (3.3.7)$$

$$\Gamma_{tr}^{t} = \Gamma_{rt}^{t} = \frac{\dot{B}}{2B} \quad (3.3.8)$$

And where

$$\dot{A} = \frac{dA(r)}{dr} \dot{B} = \frac{dB(r)}{dr} \quad (3.3.9)$$

The components of the Ricci tensor are given by

$$R_{rr} = \frac{\ddot{B}}{2B} - \frac{\dot{B}}{4B}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) - \frac{1}{R}\frac{\dot{A}}{A} \quad (3.3.10)$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \frac{1}{A} \quad (3.3.11)$$

$$R_{\emptyset\emptyset} = sin^2\theta R_{\theta\theta} \quad (3.3.12)$$

And

$$R_{tt} = \frac{\ddot{B}}{2A} + \frac{\dot{B}}{4A}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) - \frac{\dot{B}}{rA} \quad (3.3.13)$$

$$R_{\mu\nu} = 0, for\ \mu \neq \nu \quad (3.3.14)$$

And the covariant derivatives of R are then given by

$$R_{;\mu} = \partial_{\mu}R, R_{;\mu;\nu} = \partial_{\nu}R_{;\mu} - \Gamma_{\mu\nu}^{\lambda};_{\lambda} \quad (3.3.15)$$

$$R_{;r} = \dot{R}, R_{;\theta} = R_{;\phi} = R_{;t} = 0 \quad (3.3.16)$$

$$R_{;r;r} = \ddot{R} - \frac{\dot{A}}{2A}\dot{R}, R_{;\theta;\theta} = \frac{r}{A}\dot{R} = \frac{R_{;\phi;\phi}}{sin^2\theta} \quad (3.3.17)$$

$$R_{;t;t} = -\frac{\dot{B}}{2A}\dot{R}, and\ R_{;\mu;\nu} = 0, for\ \mu \neq \nu \quad (3.3.18)$$

By using the above relation and denoting the L.H.S of by $H_{\mu\nu}$ we get [2]

$$H_{rr} = -\mathcal{L}''\dot{R}\left(\frac{\dot{B}}{2B} + \frac{2}{r}\right) + \mathcal{L}'R_{rr} - \frac{A}{2}\mathcal{L} = 0 \qquad (3.3.19)$$

$$H_{\theta\theta} = -\mathcal{L}'''\frac{\dot{R}^2 r^2}{A} - \frac{\mathcal{L}'' r^2}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{1}{r}\right)\right] + \mathcal{L}'R_{\theta\theta} - \frac{r^2}{2}\mathcal{L}$$

$$= 0 \qquad (3.3.20)$$

$$H_{\phi\phi} = sin^2\theta H_{\theta\theta} = 0 \qquad (3.3.21)$$

$$H_{tt} = \mathcal{L}'''\frac{B}{A}\dot{R}^2 + \mathcal{L}''\frac{B}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{2}{r}\right)\right] + \mathcal{L}'R_{tt} + \frac{B}{2}\mathcal{L}$$

$$= 0 \qquad (3.3.22)$$

$$H = -3\frac{\mathcal{L}'''\dot{R}^2}{A} - 3\frac{\mathcal{L}''}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{2}{r}\right)\right] + \mathcal{L}'R - 2\mathcal{L}$$

$$= 0 \qquad (3.3.23)$$

$$H_{tt} + \frac{B}{r^2}H_{\theta\theta} = \frac{\mathcal{L}''\dot{R}B}{2A}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) + \mathcal{L}'\left(R_{tt} + \frac{B}{r^2}R_{\theta\theta}\right)$$

$$= 0 \qquad (3.3.24)$$

And

$$H_{rr} + \frac{A}{B}H_{tt}$$

$$= \mathcal{L}'''\dot{R}^2 + \mathcal{L}''\ddot{R} - \frac{\mathcal{L}''\dot{R}}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)$$

$$+ \mathcal{L}'\left(R_{rr} + \frac{A}{B}\right)R_{tt} \qquad (3.3.25)$$

The following relation can also be obtained from equations (3.3.14), (3.3.23), and (3.3.25).

$$\mathcal{L}'''\dot{R}^2 + \mathcal{L}''\ddot{R} - \left[\frac{\mathcal{L}''\dot{R}}{2} + \frac{\mathcal{L}'}{r}\right]\left[\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right] = 0 \qquad (3.3.26)$$

And

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{2\mathcal{L}'''\dot{R}}{\mathcal{L}''} + 2\frac{\ddot{R}}{\dot{R}} + \frac{4}{r} + \frac{2A}{3\dot{R}\mathcal{L}''}(2\mathcal{L} - R\mathcal{L}') \tag{3.3.27}$$

By substituting $\ddot{R}$ from (3.3.26) in (3.3.27) one gets

$$\frac{\dot{A}}{A} = -\frac{\dot{B}}{B} - \frac{\mathcal{L}''\dot{R}}{\mathcal{L}'}\left(\frac{\dot{B}r}{B} + 2\right) + \frac{A}{3\mathcal{L}'}(R\mathcal{L}' - 2\mathcal{L}) \tag{3.3.28}$$

Moreover using the relation (3.3.25), with the help of (3.3.14) will lead to

$$AR = \frac{\mathcal{L}''\dot{R}}{2\mathcal{L}'}\left(\frac{2}{r} - \frac{\dot{B}}{B}\right) + \frac{3}{r^2}AR_{\theta\theta} + R_{rr} \tag{3.3.29}$$

$$A\left(\frac{Rr}{4} + \frac{1}{r}\right) = \frac{\mathcal{L}''\dot{R}r}{4\mathcal{L}'}\left[\frac{2}{r} - \frac{\dot{B}}{B}\right] + \frac{1}{r} - \frac{3\dot{A}}{4A} + \frac{\dot{B}}{4B} \tag{3.3.30}$$

Further multiplying $H_{\theta\theta}$ by $\frac{-3}{r^2}$ in equation (3.3.22) then by using (3.3.23) we get

$$\frac{2r\dot{R}}{A} + \frac{2\mathcal{L}'R_{\theta\theta}}{\mathcal{L}''} - \frac{r^2\mathcal{L}}{\mathcal{L}''} + \frac{2r}{3\mathcal{L}''}(2\mathcal{L} - \mathcal{L}'R) = 0 \tag{3.3.31}$$

Multiplying H_{tt} by $\frac{3}{B}$ in (3.3.22) and using (3.3.23) we will have

$$-\frac{2r\dot{R}}{A} + \frac{4r\mathcal{L}'R_{tt}}{\mathcal{L}''\dot{B}} + \frac{2r\mathcal{L}B}{\mathcal{L}''\dot{B}} - \frac{4rB}{3\dot{B}\mathcal{L}''}(2\mathcal{L} - \mathcal{L}'R) = 0 \tag{3.3.32}$$

Now adding (3.3.30) and (3.3.32) and using (3.3.13) we obtain

$$\frac{1}{A}\left[-\ddot{B} + \frac{\dot{B}^2}{B} - \frac{\dot{B}}{r}\right] - \frac{\dot{B}}{R} + (2B - r\dot{B})\left(\frac{R}{3} - \frac{\mathcal{L}}{6\mathcal{L}'}\right) = 0 \tag{3.3.33}$$

These yields

$$\frac{d}{dr}\left[\frac{\dot{B}r}{B}\right] + \frac{A}{r}\left[\frac{\dot{B}r}{B} - r^2\left(2 - \frac{\dot{B}r}{B}\right)\xi\right] = 0 \tag{3.3.34}$$

Where

$$\xi = \xi(R) = \frac{1}{3}\left(R - \frac{\mathcal{L}}{2\mathcal{L}'}\right) \tag{3.3.35}$$

Finally by using equation (3.3.34) the metric coefficient a can then be written as

$$A(r) = \frac{-r\frac{d}{dr}[\frac{\dot{B}r}{B}]}{\frac{\dot{B}r}{B} - r^2[2 - \frac{\dot{B}r}{B}]\xi(R)} = \frac{-\frac{d}{dr}\ln\frac{\dot{B}r}{B}}{\frac{1}{r} + r\left[1 - \frac{2}{\frac{\dot{B}}{B}r}\right]\xi} \qquad (3.3.36)$$

When we use equation (3.3.22) and (3.3.23) we get

$$H_{\theta\theta} - \frac{r^2 H}{3} = 0 \qquad (3.3.37)$$

And by the use of (3.3.14) we get

$$\frac{\dot{A}}{A} = \frac{\dot{B}}{B} + \frac{2}{r} - \frac{2A}{r} - \frac{2ArR}{3} + \frac{Ar\mathcal{L}}{3\mathcal{L}'} + \frac{2\mathcal{L}''\dot{R}}{\mathcal{L}'} \qquad (3.3.38)$$

Also equation (3.3.28) together with (3.3.30) led to

$$\frac{\dot{A}}{A} = \frac{1}{R} - \frac{A}{r} - \frac{ARr}{4} - \frac{\mathcal{L}''\dot{R}r\dot{B}}{2\mathcal{L}'B} + \frac{Ar}{12}(R\mathcal{L}' - 2\mathcal{L}) \qquad (3.3.39)$$

I.e.

$$\frac{d}{dr}\left(\frac{r}{A}\right) = 1 + \frac{r^2}{12}(3R + 2\mathcal{L} - R\mathcal{L}') + \frac{\dot{R}\mathcal{L}''\dot{B}r^2}{2A\mathcal{L}'B} \qquad (3.3.40)$$

If we add and subtract (3.3.28) and (3.3.28) with each other the resulting equation takes the form

$$\frac{\dot{A}}{A} = \frac{1}{r} - \frac{A}{R} - \frac{Ar}{6}\left(R + \frac{\mathcal{L}}{\mathcal{L}'}\right) - \frac{\mathcal{L}''\dot{R}\dot{B}r}{2\mathcal{L}'B} \qquad (3.3.41)$$

To find the functional dependence of the metric A on the scalar curvature the term $\frac{\mathcal{L}''\dot{R}\dot{B}r}{4\mathcal{L}'B}$ is eliminated from equation (3.3.28) and (3.3.30) to get

$$A\left(\frac{Rr}{4} + \frac{1}{r}\right) = \frac{\dot{B}}{2B} - \frac{\dot{A}}{2A} + \frac{1}{r} + \frac{\mathcal{L}''\dot{R}}{\mathcal{L}'} - \frac{Ar}{12\mathcal{L}'}(R\mathcal{L}' - 2\mathcal{L}) \qquad (3.3.42)$$

To eliminate $\dot{A}/_A$ and $\dot{B}/_B$ and equation (3.3.27) is utilized to get

$$A(r) = \frac{12\frac{d}{dr}\ln[\mathcal{L}'(r\dot{R}\mathcal{L}'')^{-1}]}{\left\{3rR + \frac{12}{r} - (R\mathcal{L}' - (R\mathcal{L}' - 2\mathcal{L})\left[\frac{4}{R\mathcal{L}'' - \frac{r}{\mathcal{L}'}}\right]\right\}} \qquad (3.3.43)$$

A useful expression for B can be obtained from equation (3.3.27) to be

$$\dot{R} = \frac{c}{r^2}\sqrt{\frac{A}{B}}\psi(r) \tag{3.3.44}$$

With

$$\psi(r) = (\mathcal{L}'')^{-1}\exp\left[\int \frac{A(\mathcal{L}'R - 2\mathcal{L})}{3\dot{R}\mathcal{L}''}dr\right] \tag{3.2.45}$$

And

$$B(r) = \frac{c^2}{r^4\dot{R}^2\mathcal{L}''^2}\exp\left[\frac{2}{3}\int A(\mathcal{L}'R - 2\mathcal{L})\left(\dot{R}\mathcal{L}''^{-1}\right)dr\right] \tag{3.3.46}$$

Where $c$an arbitrary constant is these equations express the metric components in terms of r, R(r), $\mathcal{L}$ and their derivatives.

3.4 Solution with a Globally Constant Scalar Curvature

When the scalar curvature is constant i.e. R=R_0, then equation (3.3.38) and (3.2.39) yield [52].

$$B_0 = B(R_0) = \frac{c_0A_0}{r^2}\exp\left[\int \left(\frac{2}{r} + \frac{2rR_0}{3} - \frac{r\mathcal{L}_0}{3\mathcal{L}_0'}\right)A_0\,dr\right] \tag{3.4.1}$$

And

$$A_0 = A(R_0) = [1 + \kappa_0 r^{-1} + \gamma_0(R_0)r^2]^{-1} \tag{3.4.2}$$

Substituting equation (3.4.2) in (3.4.1) one gets

$$B_0 = \frac{c_0}{r^2(1 + \kappa_0 r^{-1} + \gamma_0 r^2)}\exp\left[\int \frac{2 + \lambda_0 r^2}{\kappa_0 + r + \gamma_0 r^3}\right]dr \tag{3.4.3}$$

Where $c_0$and $\kappa_0$are some constants and

$$\gamma_0 = \gamma(R_0) = \frac{1}{36}(3R_0 + 2\mathcal{L} - R_0\mathcal{L}_0') \tag{3.4.4}$$

With

$$\mathcal{L}_0 = \mathcal{L}(R_0) \tag{3.4.5}$$

And

$$\lambda_0 = \lambda(R_0) = \frac{1}{3}\left[2R_0 - \mathcal{L}_0\left(\mathcal{L}_0'^{-1}\right)\right] \tag{3.4.6}$$

It is interesting to note that for static and stationary universe equation (3.4.2) serves as a good solution where γ will represent the cosmological constant which turns out to be Lagrangian-dependent. Also if we set κ_0=0 and γ_0=-$\frac{A}{3}$ in equation (3.4.1) and (3.4.2) we have the de sitter universe

$$g_{tt} = g_{rr}^{-1} = 1 - \frac{1}{3}Ar^2 \tag{3.4.7}$$

For $\mathcal{L}_0$=R_0 in virtue of (3.4.6) equation (3.4.2) and (3.4.1) reduce to

$$A_0 = (1 + \kappa_0 r^{-2})^{-1} \tag{3.4.8}$$

And

$$B_0 = A_0^{-1} \exp\left(\frac{R_0}{3}\left[\frac{r^2}{3} - \kappa_0 r + \kappa_0^2 \ln(r + \kappa_0) + const\right]\right) \tag{3.4.9}$$

And if we set $R_0 = 0$ this reduces to Schwarzschild metric with singularities at r=-κ_0=2MG and r=0 [4].

3.5 The Generalized Field Equation in Static Spherically Symmetric Field

The GFE (lanczos, 1932; Ali, 1987), which is a fourth order equation, takes the form

$$\mathcal{L}'''\left[R_{;\mu}R_{;\nu} - g_{\mu\nu}g^{\rho\sigma}R_{;\rho}R_{;\sigma}\right] + \mathcal{L}''\left[R_{;\mu;\nu} - g_{\mu\nu}\nabla^2 R\right] + \mathcal{L}'R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{L} = 0 \tag{3.5.1}$$

Contracting this equation yields

$$\nabla^2 R = g^{\rho\sigma}R_{;\rho;\sigma} = \frac{\mathcal{L}'R - 2\mathcal{L}}{3\mathcal{L}''} - \frac{\mathcal{L}'''}{\mathcal{L}''}g^{\rho\sigma}R_{;\rho}R_{;\sigma} \tag{3.5.2}$$

Assuming the field of a star to be static, the metric then given by

$$g_{rr} = A(r)\,, g_{\theta\theta} = r^2, g_{\emptyset\emptyset} = r^2 sin^2\theta, g_{tt} = -B(r) \tag{3.5.3}$$

Where the proper time interval is given by

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\emptyset^2 \quad (3.5.4)$$

Substituting equation (3.5.4) in (3.5.1) we get the following relations (Ali, 1992)

$$B(r) = \frac{c^2 A}{r^4 \dot{R}^2 \mathcal{L}''^2} \exp[\frac{2}{3}\int A(\mathcal{L}'R - 2\mathcal{L})\left(\dot{R}\mathcal{L}''\right)^{-1} dr] \quad (3.5.5)$$

$$A(r) = \frac{12\frac{d}{dr}\ln[\mathcal{L}'\left(r\dot{R}\mathcal{L}''\right)^{-1}}{\left\{3rR + \frac{12}{r} - (R\mathcal{L}' - 2\mathcal{L})\left[\frac{4}{\dot{R}\mathcal{L}''} - \frac{r}{\mathcal{L}'}\right]\right\}} \quad (3.5.6)$$

And

$$\dot{R} = \frac{c}{r^2}\sqrt{\frac{A}{B}\frac{1}{\mathcal{L}''}} \exp\left[\int \frac{A(\mathcal{L}'R - 2\mathcal{L})}{3\dot{R}\mathcal{L}''} dr\right] \quad (3.5.7)$$

Where c is an arbitrary constant. These equations express the metric components in terms of r, R(r), $\mathcal{L}$ and their derivatives. Since the GFE is very complex and highly non-linear, it is difficult to obtain exact solution. For the sake of simplicity, let us take the non-linear lageangiag.

$$\mathcal{L} = -\propto R^2 + \beta R + \gamma \quad (3.5.8)$$

In this case the contracted equation (3.5.2) reduces to

$$\nabla^2 R = \frac{1}{A}\left[\left(\ddot{R} - \dot{R} - \frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{2}{r}\right)\right] = \frac{\beta}{6\alpha}R + \frac{\gamma}{3\alpha}\left(\nabla^2 - \frac{\beta}{6\alpha}\right)R = \frac{\gamma}{3\alpha}$$

(3.5.9)

Where γ is assumed to represent the source term. A further simplification can be achieved by assuming the space to be nearly flat, i.e.

$A \to 1 \quad , \; B \to 1$ (3.5.10)

Equation (3.5.9) thus becomes

$$\ddot{R} + \frac{2}{r}\dot{R} = \frac{\beta}{6\alpha}R + \frac{\gamma}{3\alpha} \quad (3.5.11)$$

If we are outside the source $\gamma = 0$, and by setting $\beta = 0$ then equation (3.5.12) reduces to

$$\frac{d\dot{R}}{dr} + \frac{2}{r}\dot{R} = 0 \qquad (3.5.12)$$

Therefore

$$\frac{d\dot{R}}{\dot{R}} = \frac{2}{r}dr \qquad (3.5.13)$$

Integrating both sides yields

$$\dot{R} = \frac{c}{r^2} \qquad (3.5.14)$$

And hence

$$R = \frac{c}{r} + c_1 \qquad (3.5.15)$$

When we are for away from the source the space is flat, i.e. R→ 0 as r→ ∞ and as result $c_1 = 0$. the scalar curvature is thus given by

$$R = \frac{c}{r} \qquad (3.5.16)$$

Using equation (3.5.13), (3.5.8) and the expression for A in a weak field, i.e.

$$A = \left(1 - \frac{2MG}{r}\right)^{-1} \qquad (3.5.17)$$

In equation (3.5.5) yields

$$B = \frac{A}{4\alpha^2}\exp\left[\frac{\beta}{6\alpha MG}\int R^2 dr\right] = \frac{A}{4\alpha^2}\exp\left[\frac{\beta}{18\alpha MG}r^3\right]$$

$$= 1 + 2\emptyset \qquad (3.5.18)$$

This expression indicates the existence of a short range gravitational field. If we consider a field near the surface of small radius super massive star then r→ 0, and

$$B = \frac{A}{4\alpha^2} \qquad (3.5.19)$$

The red shift then becomes

$$\mathcal{Z} = B^{-\frac{1}{2}} - 1 = 2\alpha\left(1 - \frac{2MG}{r}\right)^{\frac{1}{2}} - 1 \qquad (3.5.20)$$

Since (Weinberg, 1972)$\frac{MG}{R} < \frac{4}{g}$ and if $\alpha = 6$. Therefore, for the maximum value of this ratio $\mathcal{Z}$ is given by

$$\mathcal{Z} = 2\alpha\left(\frac{1}{3}\right) - 1 = 3 \qquad (3.5.21)$$

An alternative approach can also lead to the same result by seeking a general solution for equation (3.5.11). For mstances, let

$$R = \frac{c_1}{r} exp c_2 r + R_0 \qquad (3.5.22)$$

A direct substitution in equation (3.5.11) yields

$$\frac{c_1 c_2^2}{r} exp C_2 r = R_0 + \frac{B c_1}{6\alpha r} exp C_2 r + \frac{\gamma}{3\alpha} \qquad (3.5.23)$$

$$c_2 = \pm\sqrt{\frac{\beta}{6\alpha}}, R_0 = -\frac{\gamma}{3\alpha}, R = \frac{c_1}{r} exp - \sqrt{\frac{\beta}{6\alpha}} r - \frac{\gamma}{3\alpha} \qquad (3.5.24)$$

To relate the potential to R, we suppose the metric to be close to Minkowskian metric i.e.

$$g_{\mu\upsilon} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (3.5.25)$$

Where we raise and lower indices using $\eta^{\mu\nu}$ as long as we restrict ourselves to first order in h, therefore (Carmela, 1982)

$$R = g^{\mu\kappa} g^{\lambda\nu} R_{\lambda\mu\nu\kappa} = g^{00} g^{ii} R_{i0i0} \qquad (3.5.26)$$

$$g^{ii} \to \eta^{ii}, g^{00} = \eta^{00}\eta^{00} g_{00} = -(1 + 2\emptyset) \qquad (3.5.27)$$

$$R_{i0i0} = \frac{1}{2}\nabla^2 g_{00} = -4\pi G_\rho \qquad (3.5.28)$$

$$R = 8\pi G_{\rho\emptyset} + 4\pi G_\rho \qquad (3.5.29)$$

Comparing equation (3.5.22) and (3.5.25) yields

$$\frac{\gamma}{3\alpha} = -4\pi G_\rho, \emptyset = \frac{c_1}{8\pi G_{\rho r}} exp - \sqrt{\frac{\beta}{6\alpha} r} \qquad (3.5.30)$$

This indicates again the existence of a short range force or a possible link with strong nuclear force. If we set

$$C_1 = 8\pi G_\rho \qquad (3.5.31)$$

Then the red shift becomes

$$\mathcal{Z} = (B)^{-\frac{1}{2}} - 1 = (1 + \frac{2}{r}\mathrm{exp}\left(-\sqrt{\frac{\beta}{6\alpha}} r\right)^{-\frac{1}{2}} - 1$$

$$\approx \left(\frac{r}{2}\right)^{\frac{1}{2}} \mathrm{exp}\left[\frac{1}{2}\sqrt{\frac{\beta}{6\alpha}} r\right] - 1 \qquad (3.5.32)$$

When we are just outside the star $\rho = 0$ and one of the possible ways to do this is to set $\frac{1}{\alpha} \rightarrow 0$ and for r=32 (Weinberg, 1972)

$$\mathcal{Z} \approx 3 \qquad (3.5.33)$$

Thus the origin of a large red shift of quasars can be explained.

Chapter Fore

Literature Review

4.1 Introduction

This Chapter is concerned with the attempts made to construct new Hamiltonians the different from that of GR. Many attempts were done [53, 54, 55, and 56].

The attempts made to quantize GR are also presented.

4.2 Energy Momentum Tensor Based on General Gerval Relativity

The energy momentum tensor expression can be simplified by using the contracted form of (GGR) [57].

Thus the energy momentum tensor is given by

$$T_{\mu\nu} = 2\mathcal{L}'\left[\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}\right] + \mathcal{L}''\left[g_{\mu\nu}\nabla^2 R - R_{;\mu;\nu}\right]$$
$$+ \mathcal{L}'''\left(g_{\mu\nu}R_{;\sigma}R_{;\rho} - R_{;\mu}R_{;\nu}\right) \qquad (4.2.1)$$

The Hamiltonians for Gravitational field to see what the expression for the energy momentum tensor represents. It's Hamiltonian is compared with that of the electromagnetic field substituting $L = \alpha R^2$ in the field equation yields $HR^2 = o$

Hence equation becomes

$$T_{\mu\nu} = -\alpha g_{\mu\nu}R^2 - 2\alpha R_{\mu\nu} \qquad (4.2.2)$$

The Hamiltonian is thus given by

$$H = -T_0^0 = \alpha R^2 - 2\alpha g^{00}R_{00} \qquad (4.2.3)$$

On the other hand the field equation and The Hamiltonian of the electromagnetic field are given by equation

$$\nabla^2 E = 0 \qquad (4.2.4)$$

$$H = \frac{\epsilon}{2}E^2 \qquad (4.2.5)$$

Comparing the field equation and Hamiltonian it is found that Hamiltonian in equation represents the gravitational energy.

4.3 Gravitational Self Energy Mass

In the work done by (M. Dirar) [58]. The gravitational constant is quantized by using the GGR equation of motion (3.4.1)

For the Lagrangian

$$L = \alpha R^2 - \beta R + \gamma \tag{4.3.1}$$

Equation becomes

$$R = \frac{\beta R}{6\alpha} + \frac{\gamma}{3\alpha} \tag{4.3.2}$$

By setting

$$R = R(r, t) g_{rr} = A(r, t) g_{tt} = B(r, t) \tag{4.3.3}$$

And

$$dx = \sqrt{A} dr, \qquad d\tau = \sqrt{B} dt \tag{4.3.4}$$

And gets

$$\partial_{xx} R + \partial_{ii} R = \frac{\beta R}{6\alpha} + \frac{\gamma}{3\alpha} \tag{4.3.5}$$

Using the method of separation of variable now let R to be a product of two functions$T$ which depends on time$t$, beside X which depends on x.

$$R = T(\times) \tag{4.3.6}$$

Inserting (4.3.6) in(4.3.5) one gets

$$T \partial_{xx} X + \frac{1}{T} \partial_{ii} T = \frac{B}{6\alpha} \tag{4.3.7}$$

Dividing both sides by TX yields

$$\frac{1}{X} \partial_{xx} X + \frac{1}{T} \partial_{ii} T = \frac{B}{6\alpha} T \tag{4.3.8}$$

This means that the first term and the second term (4.3.5) is constants. Set the time dependent part to be

$$\frac{1}{T}\partial_{ii}T = -\mu^2 \tag{4.3.9}$$

To solve this equation

$$T = C_1 e^{i\omega t} \tag{4.3.10}$$

Whereas

$$-\omega^2 = -\mu^2 \;\; and \;\; \omega^2 = \mu^2$$

$$\frac{1}{X}\partial_{xx}X = \frac{B}{6\alpha} = \mu^2 \tag{4.3.11}$$

$$X = C_2 sinkx \tag{4.3.12}$$

$$K = \sqrt{\frac{1}{3\sqrt{3}\sqrt{6}} - \mu^2} \tag{4.3.13}$$

$$dx = \sqrt{A}dr\,,$$

$$f = \frac{2y}{\beta} \tag{4.3.14}$$

To take Y in consideration β insert (4.3.13) and (4.3.14)

$$\partial_{xx}\mathrm{f} + \partial_{ii}\mathrm{f} = \frac{\beta}{6\alpha}\mathrm{f} \tag{4.3.15}$$

By setting

$$f = \sigma T \tag{4.3.16}$$

After comparing (4.3.16) with (4.4.7) to get

$$T = C_3 e^{i\omega t},\;\; \sigma C_4 sinkx \tag{4.3.17}$$

$$K = \sqrt{\frac{1}{3\sqrt{3}\sqrt{6}} - \omega^2} \tag{4.3.18}$$

Hence (4.3.15)

$$f = R + \frac{2\gamma}{\beta} = \sigma T = C_3 C_4 e^{i\omega t} \tag{4.3.19}$$

$$R = C_3 C_4 e^{i\omega t} sinkx - \frac{2\gamma}{\beta} \tag{4.3.20}$$

R, K*Is real when*

$$\frac{1}{3\sqrt{3}\sqrt{6}} > \omega^2 \tag{4.3.21}$$

One can quantize the gravitational field know that outside of universe both gravity R and matter Y

Vanishes outside the unit near bound dries i.e.

$$R = 0 \;\; Y = 0 \;\; at\; x = x_0 \tag{4.3.22}$$

Then

$$0 = C_3 C_4 e^{i\omega t} sinkx_0 = 0 \tag{4.3.23}$$

This can be satisfied if

$$Kx_0 = n\pi \tag{4.3.24}$$

$$n = 0,1,2,3, \ldots.$$

$$-k^2 = \frac{\beta}{6\alpha} + \mu^2 = \sqrt{\frac{-\beta}{6\alpha} - \omega^2} \tag{4.3.25}$$

The term α found negative strictly by work of Dirar[ ]α was found to be

$$\alpha = \frac{-\sqrt{\beta}}{\sqrt{24}} = \frac{-\sqrt{\beta}}{2\sqrt{6}}, \frac{-\beta}{6\alpha} = \frac{\sqrt{\beta}\sqrt{6}}{3} \tag{4.3.26}$$

$$\frac{\sqrt{6}}{3} \times \frac{1}{\sqrt{6}\sqrt{3\pi}\sqrt{6}} = \frac{1}{3\sqrt{36}} \tag{3.4.26}$$

Hence

$$K = \frac{n\pi}{x}\sqrt{\frac{1}{3\sqrt{3}\sqrt{6}} - \omega^2} = \frac{n\pi}{x_0} \tag{4.3.27}$$

$$\frac{1}{3\sqrt{3}\sqrt{6}} = \frac{n^2\pi^2}{x_0^2} + \omega^2 \tag{4.3.28}$$

$$\sigma = \frac{1}{\left(\frac{n^2\pi^2}{x_0^2} + \omega^2\right)^2} \tag{4.3.29}$$

Gravitational coupling is constant quantized at the early universe x_0 thus quantized takes place.

Haw ever at present $x_0 \to \infty$ hence

$$\sigma = \frac{1}{27\omega^4} \tag{4.3.30}$$

Thus on quantized is observes.

4.4 Quantum Model for Oscillating Field Based on Generalized General

Recently Ibrahim. H. Hassan quantized time and space by using GGR [59].

And

$$L = -\alpha R^2 + \beta R + \gamma_\nu \tag{4.4.1}$$

Where α and β constant parameters and γ_ν is the vacuum energy.

According to static metric the equation takes the form [60]

$$\ddot{R} + \frac{2}{r}\dot{R} = \frac{\beta}{6\alpha}\mathrm{R} + \frac{\gamma_\nu}{3\alpha} \tag{4.4.2}$$

The solution of this equation is

$$R = \frac{C_1}{r} e^{\left[-\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2r}}\right]} - \frac{\gamma_\nu}{3\alpha} \tag{4.4.3}$$

For Euclidian space $\gamma_\nu = 0$ so equation becomes:

$$\partial_x^2 R - \frac{\partial_x^2}{C_1} R = \frac{\beta}{6\alpha} R \tag{4.4.4}$$

Solution is [55]

$$R = R_0 \sin(\omega t - kx) \tag{4.4.5}$$

If Ris only function of $t$as expected for cosmological model thus

$$R = R_0 \sin \omega t \tag{4.4.6}$$

Since at $r \to \infty \quad R = 0$

$$\omega t = n\pi \tag{4.4.7}$$

$$n = 0, \pm 1 \pm 2 \pm \cdots$$

$$r = ct = \frac{n\pi}{\omega} = n\pi \left(\frac{12\alpha}{\beta}\right)^{\frac{1}{2}} \tag{4.4.8}$$

Thus time and space are quantized.

4.5 Quantum Correction Hawking Radiation Spectrum

It is very well-known that a black hole must emit particles; the radiation spectrum must follow the Plank radiation spectrum. However, strictly speaking, this may not be the case, as Hawking's calculation was semi-classical rather than full-quantum. Perhaps, from these considerations. Bekenstein and Mukhanov showed that the Hawking radiation spectrum would be discrete if the allowed area is the integer multiples of unit area. In this work, by closely reviewing how Planck's black body radiation formula is derived, the Hawking radiation is shown to be discrete contrary to what Barreira, Carfora and Rovelli argued, and kerasnov argued even in the case that the allowed area is not simply the integer multiples of unit area, as long as the area spectrum is quantized as loop quantum gravity predicts. [61]

The black hole has the following area eigenvalues (i.e. the unit areas):

$$A_{unit} = A_1, A_2, A_3, A_4, A_5, A_6 \ldots \tag{4.5.1}$$

Then, the black hole area A must be given by the following formula.

$$A = \sum_j N^j A_j \tag{4.5.2}$$

Where N^js are non-negative integers. One can consider black having holes as partitions, each of which has A_{unit}(i.e. the unit areas) as its area. Change of area krasnov argues. The following:

$$\Delta A = A_j - A_i \tag{4.5.3}$$

For some A_i bigger thanA_j. In other words, the partition with the area $A_j$on the black. A black hole of mass $M$and radius r, having temperature T satisfies

$$r = 2M$$

$$A = 4\pi r^2 = 16\pi M^2$$

$$KT = \frac{1}{8\pi M} \tag{4.5.4}$$

As a photon is emitted, the black hole loses energy, and thus its area decreases by a certain amount. From this consideration, we. The energy of the emitted photon$E_p$is related to the mass of the black hple which decrease by an amount

$$\Delta M = -E_{photon}$$

Thus the area of the black hole decreases as follows

$$\Delta A = 32\pi M\Delta M = -32\pi M E_{photon} = -\frac{4E_{photon}}{KT} = -A_{unit} \tag{4.5.5}$$

Where A_{unit} is the unit area. The fact that the black hole area must be decreased by $A_{unit}$the unit area is predicted by loop quantum gravity. Therefore

$$E_{photon} = \frac{A_{unit}}{4} KT \tag{4.5.6}$$

In case of isolated horizon, the minimum area is given by $4\pi\sqrt{3\gamma}$ where γ is the Immirzi parameter. Therefore, the minimum energy of emitted photon is given by

$$E_{min} \approx 1.49KT \tag{4.5.7}$$

In case of Tanaka-Tamaki scenario the minimum area is given by$4\pi\gamma$ where γ is the Lmmirzi parameter. This given the minimum energy of emitted photon to be

$$E_{min} \approx 2.462KT \tag{4.5.8}$$

In case of Kong-Yoon scenario the minimum area is given by $4\pi\sqrt{2}$. In this case, one has

$$E_{min} \approx 4.44KT \tag{4.5.9}$$

Therefore, the Hawking radiation is truncated below this energy. An other simple derivation can be made by using the entropy. Heat relation

$$\Delta Q = T\Delta S \tag{4.5.10}$$

Using the followings, relations

$$\Delta Q = -E_{photon} \tag{4.5.11}$$

$$\Delta S = -\frac{KA_{unit}}{4} \tag{4.5.12}$$

One gets

$$E_{photon} = \frac{KA_{unit}}{4} \tag{4.5.13}$$

In his famous textbook, Griffiths considers a statistical mechanics problem where he considers an arbitrary potential, for which the one-particle energies are $E_1, E_2 E_3, \ldots$, with degeneracies $d_1, d_2, d_3, \ldots$ Suppose one put N particles, for which there are N_1 particles with energy E_1, N_2 particles with energy E_2 and so on. Then he shows the following in case of bosons.

$$Q = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!} \tag{4.5.14}$$

With the following two conditions:

$$\sum_{n=1}^{\infty} N_n = N, \quad \sum_{n=1}^{\infty} N_n E_n = E \tag{4.5.15}$$

The first condition shows that the total number of particles is N. The second condition shows that the total energy is E.

Then, to find the most probable configuration $(N_1, N_2, N_3, \ldots)$, he maximizes in Q as follows:

$$G \equiv \ln Q + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right] \qquad (4.5.16)$$

Where G is to be maximized and $\alpha\ and\ \beta$ are Lagrange multipliers. Thus one has

$$N_n = \frac{d_n}{e^{\alpha+\beta E_n} - 1} \qquad (4.5.17)$$

Of course, in case photon, the number isn't conserved, which implies that we can set$\alpha = 0$. There fore

$$N_n = \frac{d_n}{e^{\frac{E_n}{(KT)}} - 1} \qquad (4.5.18)$$

Using another standard method, we blackbody radiation spectrum for a photon with frequency $f$is given by

$$N_f = \frac{8\pi f^2 df}{e^{\frac{hf}{KT}} - 1} \qquad (4.5.19)$$

Comparing equations (18) and (19) yields

$$hf = E_n \qquad (4.5.20)$$

Recalling that the black hole (or any black body) loses hfupon emission of a photon with frequency$f$, one can thus get

$$\Delta E = -hf \qquad (4.5.21)$$

Plugging (23) to the above equation, we conclude:

$$\Delta E = E_n \qquad (4.5.22)$$

Now, suppose a hypothetical case in which the area deduction is given by $\Delta A = A_j - A_i$as Krasnov argued, and see why that doesn't make any sense. In such a case, we would have $\Delta E = E_j - E_i$which implies energy of emitted photon is given by$hf = E_i - E_j$. Given this, let's compare the black body radiation formula in this hypothetical case with (18). Denominator doesn't match as (18)'s denominator is$e^{\frac{E_n}{(KT)}} - 1$ while Krasnov's hypothetical one would be $e^{\frac{(E_i-E_j)}{(KT)}} - 1$.they are clearly different. Furthermore, the numerator doesn't match at all either. In the case of (18), we have the degeneracy of nth quanta, given asd_n. In Krasnov's hypothetical case one never knows whether it should be$d_i or d_j o \ \ d_i d_j$. Perhaps, there is no consistent way to assign a value to the numerator in such a way that it reduces to$d_n$in the case that$E_i = E_n$and$E_j = 0$, but still different from$d_n$when $E_i = E_n$but$E_j \neq 0$, in conclusion, Kresnov's area deduction condition is wrong as it cannot reproduce (18).

4.6 Scalar Particles Tunneling and Effect of Quantum Gravity

Recently, the quantum gravity theory came into a period of rapid development the best application model of the quantum gravity is black hole model. More and more evidences imply that the generalized uncertainty principle (GUP) can be modified by the modified fundamental commutation relation; therefore the momentum operator will be corrected with it. Finally, the dynamics equation of particles in black hole can be modified by the quantum gravity, and the Hawking radiation is method and the GUP, the tunneling behavior of the scalar particle of Schwarzschild black hole has been studied by K Nozari. And many other studies. The aim of this work is to study the tunneling radiation of scalar particles in the Gibbons-Maeda-Dilation black hole with the Klein-Gordon equation near the horizon. The generalized uncertainty principle (GUP) can describe the minimum measurable length. Based on the modified fundamental commutation relation [62]

$$\left[x_i, p_j\right] = i\hbar\delta_{ij}[1 + \beta p^2] \tag{4.6.1}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}[1 + \beta(\Delta p)^2] \tag{4.6.2}$$

Where $M_p$is the Planck mass, $\beta = \frac{\beta_0}{M_p^2}$, β_0

Is a dimensionless parameter and $\beta_0 \leq 10^{34}$, $x_i \ and p_i$ can be found in the reference [17].

$$x_i = x_{0i}, p_i = p_{0i}(1 + \beta p^2) \tag{4.6.3}$$

The canonical commutation relation express as $[x_{0\mu}, p_{0\nu}] = i\hbar\delta_{\mu\nu}$should be satisfied the Klein-Gordon equation without the electromagnetic field is given by

$$-P_\mu P^\mu = m^2 \tag{4.6.4}$$

To study the effect which the quantum gravity has on the Klein-Gordon equation, we expand the Klein-Gordon equation as two parts. Thus the generalized expression of energy is given by

$$\bar{E} = E[1 - \beta(p^2 + m^2)] \tag{4.6.5}$$

Therefore, the modified Klein-Gordon equation takes the form

$$-(i\hbar)^2\partial^t\partial_t\psi = \left[(-i\hbar)^2\partial^i\partial_i + m^2\right]\left\{1 - 2\beta\left[(-i\hbar)^2\partial^i\partial_i + m^2\right]\right\}\psi \tag{4.6.6}$$

The modified Klein-Gordon equation tells us that the quantum gravity has an important influence on the dynamic equation of scalar particles. One can use equation (6) to study tunneling radiation of scalar particles of the Gibbons-Maeda-Dilaton black hole by.

Chapter Five

Quantum Static Gravity Model

5.1 Introduction

This Chapter is concerned with constructing equation gravity model based on GGR in static. It aims to equalize the gravitational field.

5.2 Radial Quantum Gravity Energy

The quantum equations in the Schrodinger picture are based on the fact that free particles can be soared as a wave having displacement

$$\psi = Ae^{\frac{i}{\hbar}(px-Et)} \tag{5.2.1}$$

Differentiating $w.r.t$ time displacement yields

$$i\hbar\frac{\partial\psi}{\partial t} = \in\psi\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = p\psi \tag{5.2.2}$$

Where panda E are the momentum and energy respectively beside the wave equation the quantum equation of the static field, which takes the form

$$E = H = \alpha R^2 + 2\alpha g^{tt}R_{;t;t} = \alpha R^2 + \frac{\alpha\dot{B}}{AB}\dot{R} \tag{5.2.3}$$

The redial momentum is given by

$$P_r = P_r^r = -T_r^r = \alpha R^2 + 2\alpha g^{rr}R_{;r;r} = \alpha R^2 + \frac{2\alpha}{A}\left[\ddot{R} - \frac{\dot{A}}{2A}\dot{R}\right]$$

$$= \alpha R^2 + \frac{2\alpha\ddot{R}}{A} - \frac{\alpha\dot{A}}{A^2}\dot{R} \tag{5.2.4}$$

$$E = \dot{P}_r - 2\frac{\alpha\ddot{R}}{A} + \frac{\dot{A}}{A^2}\dot{R} + \frac{\alpha\dot{B}}{AB}\dot{R} \tag{5.2.5}$$

This can be written in the form yields

$$E = P_r + f(x) \tag{5.3.6}$$

Where

$$f(r) = -2\frac{\alpha\ddot{R}}{A} + \frac{\dot{A}}{A^2}\dot{R} + \frac{\alpha\dot{B}}{AB}\dot{R} \quad (5.2.7)$$

To find the gravity quantum equation the GGR energy expression can multiplied by ψ to get

$$E\psi = P\psi + f(r)\psi \quad (5.2.8)$$

Substituting equation (5.2.2) in equation (5.2.8)

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{h}{i}\frac{\partial\psi}{\partial\psi} + f(r)\psi \quad (5.2.9)$$

This equation holds for the system of units in which

$$c = 1 \quad (5.2.10)$$

But in the ordinary system of units equation can be written in the form

$$E = cP_r + f(r) \quad (5.2.11)$$

Using equation (5.2.2) in r-direction one gets

$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \quad (5.2.12)$$

$$P\psi = \frac{\hbar}{i}\frac{\partial\psi}{\partial t} \quad (5.2.13)$$

To derive the equation of quantum mechanics of static field multiply (5.2.11) by ψ to get

$$E\psi = CP_r\psi + f(r)\psi \quad (5.2.14)$$

Substituting (5.2.12) and (5.2.13) and (5.2.14) yields

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{c\hbar}{i}\frac{\partial\psi}{\partial r} + f(r)\psi \quad (5.2.15)$$

This equation can be simplified by separating variables, where

$$\psi(r,t) = \cup\,(r)\,\mathcal{V}(t) \quad (5.2.16)$$

Inserting (5.2.16) in (5.2.15) yields (5.2.16)

$$i\hbar = u\frac{\partial v}{\partial t} = \frac{c\hbar}{i}v(t)\frac{\partial u}{\partial r}(r) + f(r)u(r)\,v(t) \quad (5.2.17)$$

Dividing both sides by uv yields

$$\frac{i\hbar}{v}\frac{\partial v}{\partial t} = \frac{c\hbar}{iu}\frac{\partial}{\partial r}u(r) + f(r) = E = constant \quad (5.2.18)$$

The time dependent part can thus be given by

$$\frac{i\hbar}{v}\frac{\partial v}{\partial t} = Ei\hbar\frac{\partial v}{\partial t} = Ev \quad (5.2.19)$$

While the spatial dependent pates given by

$$\frac{c\hbar}{iu}\frac{\partial u}{\partial r} + f(r) = E$$

$$\frac{c\hbar}{i}\frac{du}{dr} + f(r) = Eu \quad (5.2.20)$$

The solution for equation (5.2.19) can be obtained by direct integration where

$$i\hbar = \frac{\partial v}{\partial t} = Ev$$

$$i\hbar\int\frac{dv}{v} = E\int dt + C_0$$

$$i\hbar \ln v = Et + C_0$$

$$\ln v = \frac{E}{i\hbar}t + C_1 \quad v = v_0 e^{\frac{E}{i\hbar}t} \quad (5.2.21)$$

Where

$$v_0 = e^{C_1}$$

Consider now the behavior of any particle, free Euclidian space, where on gravity exists. For such space, accruing to equations.

$$B = 1 \ , A = 1 \ , R = R_0 = const \quad (5.2.22)$$

Thus in view of equation (5.2.5)

$$f(r) = 0 \quad (5.2.23)$$

Substituting equation (4.2.23) in equation (4.2.20) yields

$$\frac{du}{u} = \frac{i}{c\hbar}Edr \quad (5.2.25)$$

$$\frac{c\hbar}{i}\frac{du}{dr} = Eu \tag{5.2.24}$$

A direct integration given

$$\int \frac{du}{u} = \frac{iE}{c\hbar}\int dr + C_2$$

$$\ln u = \frac{iE}{c\hbar}r + C_2$$

$$u = e^{\frac{iE}{c\hbar}r+C_2}$$

$$u = e^{C_2}\, e^{i\frac{E}{C\hbar}r}$$

$$u_0 = e^{i\frac{E}{C\hbar}r} \tag{5.2.25}$$

With

$$u_0 = e^{C_2}$$

For relativistic particle with negligible mass.

$$E = PC \tag{5.2.26}$$

$$U = U_0 e^{\frac{iE}{C\hbar}r} = U_0 e^{\frac{iP}{\hbar}r} \tag{5.2.27}$$

Thus in flat free space the particle propagate as a pure wave which conforms with that predicted by ordinary Schrodinger equation

Form (5.2.20) in r- dimension

$$\frac{C\hbar}{i}\frac{du}{dr} + f\,(r)u = E\psi$$

Hence

$$\frac{c\hbar}{i}\frac{du}{dr} = [E - f(r)]u$$

$$\frac{du}{u} = i[E - f(r)]\frac{dx}{c\hbar} = \frac{i}{c\hbar}[E - f(r)]dx$$

Thus

$$\int \frac{du}{u} = \frac{i}{c\hbar}\int [E - f(r)]dr + C_2 \int \frac{du}{u}$$

$$= \frac{i}{c\hbar}\int [E - f(r)]dr$$

$$+ C_3 \qquad (5.2.28)$$

Where C_3 is the constant of Integration.

Integration both sides yields

$$\ln u = \frac{i}{c\hbar}\int [E - f(r)]dr + C_2$$

For constant$E$

$$U = e^{C_0} e^{\frac{i}{c\hbar}\int [E-f(r)]dr + C_2} = U_0 e^{\frac{i}{c\hbar}[Er - \int f(r)dr]} = U_0 e^{\frac{i}{\hbar}Pr} \quad (5.2.29)$$

A gain this equation indicates that the particle behave like a wave with momentum and wave number

$$P = \hbar k = \frac{E}{c} - \frac{\int f(r)dr}{cr}$$

$$k = \frac{E}{c\hbar} - \frac{\int f(r)dr}{c\hbar r} \qquad (5.2.30)$$

The momentum is no Longer a constant but depends on r.

5.3 The θ Dependent Part

The Hamiltonian that depends on the angular part θ of the momentum takes the form.

$$H = \alpha R^2 + \alpha \frac{\ddot{B}}{AB}\dot{R} \qquad (5.3.1)$$

The momentum is given by

$$P_\theta^\theta = T_\theta^\theta = \alpha R^2 + 2\alpha g^{\theta\theta} R_{;\theta;\theta} \qquad (5.3.2)$$

According to equation one gets

$$P_\theta^\theta = \alpha R^2 + \frac{\dot{R}}{rA} \qquad (5.3.3)$$

Thus

$$\alpha R^2 = P_\theta^\theta - \frac{\dot{R}}{rA} \tag{5.3.4}$$

Hence the Hamiltonian is given by in the ordinary system of unites by multiplying ploys to get

$$H = CP_\theta^\theta - \frac{\dot{R}}{rA} + \frac{\alpha \dot{B}}{AB}\dot{R} \tag{5.3.5}$$

To obtain quantum equation one multiplies makes the replacement

$$H \rightarrow \hat{H} \quad P_\theta^\theta = \hat{P}_\theta^\theta \tag{4.3.6}$$

I.e. the Hamiltonian and the momentum are replaced by their corresponding operator in (5.3.5) to get

$$H\psi = cP_\theta^\theta \psi - \frac{\dot{R}}{rA}\psi + \frac{\alpha \dot{B}}{AB}\dot{R}\psi \tag{5.3.7}$$

$$\hat{H}\psi = c\hat{P}_\theta^\theta \psi - \frac{\dot{R}}{rA}\psi + \frac{\alpha \dot{B}\dot{R}}{AB}\psi \tag{5.3.8}$$

But the operators of hand Hamiltonian *and momentum*are given by

$$\hat{H} = i\hbar\frac{\partial}{\partial t}\hat{P}_\theta^\theta \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial t} \tag{5.3.9}$$

Inserting equation (5.3.9) in (5.3.8) yields

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{c\hbar}{i}\frac{1}{r}\frac{\partial}{\partial t}\psi - \frac{\dot{R}}{rA}\psi + \frac{\alpha \ddot{B}}{AB}\psi \tag{5.3.10}$$

The compact form of (5.3.10)

$$i\hbar\frac{\partial}{\partial t}\psi = \frac{c\hbar}{i}\frac{1}{r}\frac{\partial \psi}{\partial t} + g(r)\psi \tag{5.3.11}$$

Where

$$g(r) = -\frac{\dot{R}}{rA} + \frac{\alpha \dot{B}}{AB}\dot{R} \tag{5.3.12}$$

To solve equation (5.3.11) one makes the separation of the wave function in the form

$$\psi(r,\theta,t) = u(r,\theta)v(t) = uv \qquad (5.3.13)$$

Substituting (4.3.13) in (4.3.11) yields

$$i\hbar\frac{udv}{\partial t} = \frac{c\hbar}{i}\frac{v}{r sin\theta}\frac{\partial u}{\partial\theta} + g(r)uv \qquad (5.3.14)$$

Divide both sides by uv to get

$$i\frac{\hbar}{v}\frac{dv}{dt} = \frac{c\hbar}{i}\frac{1}{r}\frac{1}{u} + g(r) = C_3 = E \qquad (5.3.15)$$

Thus the time part is given by

$$i\hbar\frac{dv}{dt} = C_3 v = Ev \qquad (5.3.16)$$

To solve (5.3.16) divide both sides by v

$$\int\frac{dv}{v} = \frac{E}{i}\int dt + C_4$$

Hence

$$\ln v = \frac{E}{i\hbar}\int dt + C_4 = \frac{E}{i\hbar}t + C_4$$

Therefore one gets

$$v = e^{\frac{Et}{i\hbar}} + C_4 = e^{C_3}e^{\frac{Et}{i\hbar}} = v_0 e^{\frac{Et}{i\hbar}} \qquad (5.3.16)$$

The spatial part is given according to (5.3.15) by

$$c\frac{\hbar}{i}\frac{1}{r}\frac{du}{d\theta} + g(r)u = C_3 u = EU \qquad (5.3.18)$$

In free space and for cans tan scalar curvature in general

$$R = R_0\dot{R} = 0 \quad \ddot{R} = 0 \qquad (5.3.19)$$

Thus equation (5.3.12) and (5.3.19) given

$$g(r) = 0 \qquad (5.3.20)$$

Inserting equation (5.3.20) in equation (5.3.18) yield

$$c\frac{\hbar}{i}\frac{1}{r}\frac{du}{dt} = Eu \qquad (5.3.21)$$

Rearranging (5.3.21) to separate u from r yield

$$\int \frac{du}{u} = \frac{iE}{c\hbar} r \int d\theta + C_3$$

Thus

$$\ln U = \frac{iE}{c\hbar} r\theta + C_4$$

$$U = e^{C_4} e^{\frac{iE}{c\hbar} r\theta} = U_0 e^{\frac{iE}{c\hbar} r\theta}$$

Then

$$U(r, \theta) = U_0 e^{\frac{iE}{c\hbar} r\theta} \tag{5.3.22}$$

Expression (5.3.22) indicates that the particle behaves as a pure wave in the θ direction.

5.4 The ϕ Dependent Part

If P_θ^θ is the only momentum component, the Hamiltonian and quantum equations become different from the previous ones

The component P_θ^θ is given according to equation.

$$P_\phi^\phi = 2\alpha g^{\phi\phi} R_{;\phi;\phi} + \alpha R^2 \tag{5.4.1}$$

Using equation one gets

$$P_\phi^\phi = \alpha R^2 + \frac{\dot{R}}{rA} \tag{5.4.2}$$

But the Hamiltonian is given according to equation by

$$H = \alpha R^2 + \frac{\alpha \dot{B}}{AB} \dot{R} \tag{5.4.3}$$

In view of (5.4.2) one gets

$$\alpha R^2 = P_\phi^\phi - \frac{\dot{R}}{rA} \tag{5.4.4}$$

Thus in view of equations (5.4.4) and (5.4.3) one gets the Hamiltonian in the from

$$E = H = cP_{\phi}^{\phi} - \frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R} \quad (5.4.5)$$

Multiply both sides by ψ to get

$$E\psi = cP_{\phi}^{\phi}\psi - \frac{\dot{R}}{rA}\psi + \frac{\alpha\dot{B}}{AB}\dot{R}\psi \quad (5.4.6)$$

To obtain the quantum equation one replaces the physical quantities, like energy and momentum by their corresponding operators to get

$$\hat{H}\psi = c\hat{P}_{\phi}^{\phi}\psi - \frac{\dot{R}}{AB}\psi + \frac{\dot{B}\dot{R}}{AB}\psi \quad (5.4.7)$$

To obtain the quantum equation the energy and momentum operates takes the differential form

$$\hat{H} = i\hbar\frac{\partial}{\partial t}\hat{P}_{\phi}^{\phi} = \frac{\hbar}{i}\frac{1}{rsin\theta}\frac{\partial}{\partial\phi} \quad (5.4.8)$$

Inserting (5.4.8) in (5.4.7) yields

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{c\hbar}{i}\frac{1}{rsin\theta}\frac{\partial}{\partial t}\psi - \frac{\dot{R}}{rR}\psi + \frac{\alpha\dot{B}}{AB}\dot{R}\psi \quad (5.4.9)$$

In free space, where no field exists, and for constant (Scalar curvature) in general

$$R = R_0 = constant \qquad \dot{R} = 0 \quad (5.4.10)$$

Thus

$$i\hbar\frac{\partial\psi}{\partial t} = c\frac{\hbar}{i}\frac{1}{rsin\theta}\frac{\partial\psi}{\partial t} \quad (5.4.11)$$

Again by setting

$$\psi = v(t) \quad u(r,\theta,\phi) \quad (5.4.12)$$

One gets

$$i\hbar\frac{udv}{\partial t} = \frac{c\hbar}{i}\frac{1}{rsin\theta}v\frac{\partial u}{\partial\phi}$$

Thus dividing by uv

$$\frac{i\hbar}{v}\frac{\partial v}{\partial t} = \frac{c\hbar}{i}\frac{1}{rsin\theta u}\frac{du}{d\phi} = E \quad (5.4.13)$$

Here

$$i\hbar\frac{dv}{\partial t} = Edt \tag{5.4.14}$$

Thus the time dependent part is given according to equation (5.4.13) by

$$i\hbar\int\frac{dv}{v} = \int Edt$$

$$i\hbar \ln v = Et + C_5$$

$$\ln v = \frac{E}{i\hbar}t + \hbar C_6$$

$$v = v_0 e^{\frac{E}{i\hbar}t} \tag{5.4.15}$$

Similarly the spatial part can be obtained from equation (5.4.13) to be

$$\frac{c\hbar}{i}\frac{du}{u} = Ersin\theta d\phi$$

$$c\int\frac{h}{i}\frac{du}{u} = \int Er\ sin\theta d\phi \ \Rightarrow \frac{h}{i}\int\frac{du}{u} = Er\int sin\theta d\phi$$

$$\frac{h}{i}\ln u = Er\int sin\theta d\phi \Rightarrow \ \ln u = \frac{i}{\hbar}Er\int sin\theta d\phi$$

$$U = U_0 e^{\frac{i}{\hbar}Er\int sin\theta d\phi}$$

Thus

$$U = U_0 e^{-\frac{i}{\hbar}E(rsin\theta)\phi} \tag{5.4.16}$$

5.5 The Full Spherical Quantum Gravity Equation

In the system of units where $c \neq 1$ the (quantum general equation) becomes

$$\hat{E}\psi = c\hat{P}\psi + f(r)\psi \tag{5.5.1}$$

This equation can be rewritten by taking in to account the fact that in (classical mechanic), the energy is given by:

$$E = \int F.dr = \int m\frac{dv}{dt}.\underline{dr} = m\int d\underline{v}.\frac{dr}{dt}$$

$$E = m\int \underline{v}.d\underline{v} = \frac{mv^2}{2} \quad (5.5.2)$$

But if the system is oscillating the velocity can thus give by

$$v(t) = v_m sin\omega t \quad (5.5.3)$$

Where the effective velocity is given by:

$$v = \frac{v_{max}}{\sqrt{2}} = \frac{v_m}{\sqrt{2}}\ ,\ \ V^2 = \frac{v_m^2}{2} \quad (5.5.4)$$

If one rewrite (5.5.2) to be

$$E = \frac{mv_m^2}{2} \quad (5.5.5)$$

It follows that

$$E = mv^2 \quad (5.5.6)$$

Alternatively for harmonic oscillator

$$E = \frac{1}{2}m\omega^2A^2 = \frac{1}{2}mv_m^2 = m(\frac{V_m}{\sqrt{2}})^2 = mv_e^2 = mv^2 \quad (5.5.7)$$

Thus

$$E = mv^2 = (mv).v = P.v \quad (5.5.8)$$

$$E = P_xV_x + P_yV_y + P_zV_z \quad (5.5.9)$$

For light

$$V_x = V_y = V_z = c \quad (5.5.10)$$

Thus

$$E = cP_x + cP_y + cP_z \quad (5.5.11)$$

In spherical coordinate

$$E = cP_r + cP_\theta + cP_\phi \quad (5.5.12)$$

In view of (5.5.12) equation (5.5.1) can be written as

$$E = cP_r + cP_\theta + cP_\phi + f(r) \quad (5.5.13)$$

Multiply both sides by ψ to E get

$$E\psi = cP_r\psi + cP_\theta\psi + cP_\phi\psi + f(r)\psi \qquad (5.5.14)$$

Replacing $E \ and \ P$ by their corresponding operates one gets

$$\hat{E}\psi = c\hat{P}_r\psi + c\hat{P}_\theta\psi + c\hat{P}_\phi\psi + f(r) \qquad (5.5.15)$$

But the energy and momentum operators take the form

$$\hat{E} = i\frac{\partial}{\partial t}$$

$$\hat{P}_r = \frac{\hbar}{i}\frac{\partial}{\partial t}$$

$$\hat{P}_\theta = -\frac{\hbar}{ir}\frac{\partial}{\partial\theta}\hat{P}_\phi = \frac{\hbar}{i}\frac{1}{rsin\theta}\frac{\partial}{\partial\phi} \qquad (5.5.16)$$

Thus the full quantum equation becomes

$$\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\left[c\frac{\partial\psi}{\partial r} + \frac{c}{r}\frac{\partial\psi}{\partial\theta} + \frac{c}{rsin\theta}\frac{\partial\psi}{\partial\phi} + f(r)\psi\right]$$

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar}{i}\left[c\frac{\partial\psi}{\partial r} + \frac{c}{r}\frac{\partial\psi}{\partial\theta} + \frac{c}{rsin\theta}\frac{\partial\psi}{\partial\phi} + f(r)\psi\right] \qquad (5.5.17)$$

Using separation of variables

$$\psi(r,\theta,\phi,t) = \tilde{R}(r) + \theta\Phi(\phi)\ v(t) \qquad (5.5.18)$$

Equation (5.5.17) becomes

$$iR\phi\Theta\frac{dv}{dt} = \frac{\hbar c}{i\times i}vH\Theta\phi\frac{\partial\ddot{R}}{\partial r} + \frac{\hbar c}{i\times i}v\tilde{R}\phi\frac{\partial\Theta}{\partial\theta} + \frac{c\hbar}{irsin\theta}\tilde{R}\Theta\frac{\partial\phi}{\partial\phi} + f(r)R\Theta\phi v$$

Divide by $R\ \Theta\phi v$

$$\frac{i\hbar}{v}\frac{dv}{\partial t} = \frac{c\hbar}{i}\frac{1}{\tilde{R}}\frac{\partial\tilde{R}}{\partial r} + \frac{c\hbar}{i}\frac{1}{r\Theta}\frac{\partial\Theta}{\partial\theta} + f(r) + \frac{c\hbar}{i}\frac{1}{r\phi sin\theta}\frac{\partial\phi}{\partial\phi} = C_0 = E$$

Thus

$$\frac{i\hbar}{v}\frac{\partial v}{\partial t} = E \qquad (5.5.19)$$

$$f(r) + \frac{c\hbar}{i}\frac{1}{\tilde{R}}\frac{d\tilde{R}}{\partial r} + \frac{ch}{i}\frac{1}{r\Theta}\frac{\partial\Theta}{\partial\theta} + \frac{ch}{i\phi(crsin\theta)}\frac{\partial\phi}{\partial\phi} = E$$

Multiply by r

$$\frac{c\hbar}{i}\frac{d\tilde{R}}{dr} + f(r)r + \frac{c\hbar}{i\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\phi(isin\theta)}\frac{d\Phi}{d\phi} = Er$$

$$\frac{c\hbar}{i}\frac{r}{\tilde{R}} + (f(r) - E)r = \frac{ch}{i}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{1}{sin\theta}\frac{d\Phi}{d\phi} = C_1$$

Thus

$$\frac{ch}{i}\frac{r}{\tilde{R}}\frac{d\tilde{R}}{dr} + [f(r) - E]r = C_1 \tag{5.5.20}$$

$$\frac{c\hbar}{i\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{1}{sin\theta}\frac{d\Phi}{d\phi} = C_1 \tag{5.5.21}$$

Multiply by $(sin\theta)$

$$\frac{c\hbar}{i}\frac{sin\theta}{\Theta}\frac{d\Theta}{d\theta} + \frac{ch}{\Phi i}\frac{d\Phi}{d\phi} = C_1 sin\theta \tag{5.5.22}$$

$$c\hbar\frac{sin\theta}{i\Theta}\frac{d\Theta}{d\theta} - C_1 sin\theta = -\frac{ch}{\Phi i}\frac{d\Phi}{d\phi} = C_1 \tag{5.5.23}$$

Thus

$$\frac{c\hbar}{i}\frac{sin\theta}{\Theta}\frac{d\Theta}{d\theta} - C_1 sin\theta = C_2 \tag{5.5.24}$$

The function $f(r)$can be found from equations (5.2.3), (5.2.4), (5.3.3) and (4.4.1), where the GGR Hamiltonian is given by

$$H = \alpha R^2 + \alpha\frac{\dot{B}}{AB}\dot{R} = \frac{1}{3}\alpha R^2 + \frac{1}{3}\alpha R^2 + \frac{1}{3}\alpha R^2 + \alpha\frac{\dot{B}}{AB}\dot{R} \tag{5.5.25}$$

The corresponding momentum components in spherical coordinates is given by

$$\frac{1}{3}P_r = \frac{1}{3}\alpha R^2 + \frac{2}{3}\frac{\alpha\ddot{R}}{A} - \frac{\alpha}{3}\frac{\dot{A}\dot{R}}{R^2}$$

$$\frac{1}{3}P_\theta = \frac{1}{3}\alpha R^2 + \frac{\dot{R}}{3rA}\frac{1}{3}P_\phi = \frac{1}{3}\alpha R^2 + \frac{\dot{R}}{3rA} \tag{5.5.26}$$

Thus inserting equation (5.5.26) in equation (5.5.25) yields

$$H = \frac{1}{3}\left[P_r + P_\theta + P_\phi\right] - \frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R} \quad (5.5.27)$$

In the system of units where $c \neq 1$

The Hamiltonian becomes

$$H = \frac{c}{3}\left[P_r + P_\theta + P_\phi\right] + f(r) \quad (5.5.28)$$

Or by following Dirac relativistic quantum equation approach

$$H = c\alpha_r P_r + \alpha_\theta P_\theta + \alpha_\phi P_\phi + f(r) \quad (5.5.29)$$

Where

$$f(r) = -\frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R}$$

$$f(r) = -\frac{2}{3}\frac{\alpha\ddot{R}}{A} + \frac{\alpha}{3}\frac{\dot{A}}{A^2}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} + \frac{\alpha\dot{B}}{AB}\dot{R} \quad (5.5.30)$$

5.6 Solution of the Radial Part

Since most of astronomical objects have spherical shape. Therefore it is suitable to use spherical coordinates.

In view of equation (4.5.20) the radial part is given by

$$\frac{c\hbar}{i}\frac{r}{R}\frac{d\tilde{R}}{dr} + [f(r) - E]r = C_1 \quad (5.6.1)$$

Thus, separation of R and r dependent parts yield

$$\int \frac{d\tilde{R}}{\tilde{R}} = \frac{iC_1}{c\hbar}\int \frac{dr}{r} + \int \frac{i}{c\hbar}\big(E - f(r)\big)dr + C_4$$

$$\ln\tilde{R} = \frac{iC_1}{c\hbar}\ln r + \frac{i}{c\hbar}\left[Er - \int f(r)dr\right] + C_4 \ln\tilde{R} - \ln r^{ic_3}$$

$$= \frac{i}{c\hbar}\left[Er - \int f(r)dr\right] + C_4$$

$$\tilde{R} = C_5 r^{ic_3} e^{\frac{i}{c\hbar}[Er - \int f(r)dr]} \quad (5.6.2)$$

Where

$$C_3 = \frac{C_1}{c\hbar} C_5 = e^{c_4} \qquad (5.6.3)$$

The radial wave function can be rewritten in the form

$$\tilde{R} = e^{ic_3 \ln r} e^{\frac{i}{\hbar}[Er - \int f(r)dr]} \qquad (5.6.4)$$

Equation (5.2.3) shows that the gravity energy density is constant when

$$R = R_0 = constant \qquad (5.6.5)$$

Where

$$H = \alpha R_0^2 = constant \qquad (5.6.6)$$

But since the energy density is equation to graviton energy multiplied by the number of them, therefore

$$H = \hbar\omega|\psi|^2 = \hbar\omega\left|\tilde{R}\right|^2 = constant \qquad (5.6.7)$$

Therefore the probability or the number of gravitons is also constant, I.e.

$$\left|\tilde{R}\right|^2 = C_6 \qquad \tilde{R} = C_7 = constant \qquad (5.6.8)$$

For simplicity let

$$C_1 = 0 \qquad C_3 = 0 \qquad (5.6.9)$$

But since R is constant, thus according to equation (5.5.30)

$$f(r) = 0 \qquad (5.6.10)$$

In view of equation (5.6.4) and (5.6.8)

$$e^{\frac{i}{c\hbar}Er_0} = c_7 \qquad (5.6.11)$$

But, since the number of gravitons are constant, and from (5.6.6) which hews that $R$is real as for as the Hamiltonian (energy) is real thus

$$\mathrm{R} = \cos\frac{Er_0}{c\hbar} + i\sin\frac{Er_0}{c\hbar} = c_7\,, \cos\frac{Er_0}{c\hbar} = c_7 \sin\frac{Er_0}{c\hbar} = 0 \qquad (5.6.12)$$

Hence

$$\frac{Er_0}{c\hbar} = 2n\pi \quad , \quad E = \frac{2n\pi c\hbar}{r_0} \tag{5.6.13}$$

$$E = \frac{nc\hbar}{r_0} \tag{5.6.14}$$

According to Bohr model the minimum Bohr radian corresponds to

$$2\pi r_0 = \lambda$$

Thus

$$E = \frac{nc\hbar(2\pi)}{\lambda}$$

$$E = 2\pi nrf \tag{5.6.15}$$

5.7 Solution of Constant Scalar Curvature

There are many useful solutions which can be found when the mass density is constant. According to GR it implies constant R.using equation (5.5.30)

$$f(r) = -\frac{2}{3}\alpha\frac{\ddot{R}}{A} + \frac{\alpha\dot{A}}{3A^2}\dot{R} + \frac{\alpha\dot{B}}{AB}\dot{R} - \frac{2}{3}\frac{\dot{R}}{rA} \tag{5.7.1}$$

For constant R

$$R = R_0 \qquad \dot{R} = 0 \qquad \ddot{R} = 0 \tag{5.7.2}$$

Thus (5.7.1) given (5.7.2)

$$f(r) = Zero$$

Substituting in (5.5.17) and consider ψ to depend on r only for afield generated by a spherical body.

$$\psi = \psi(r) \tag{5.7.3}$$

$$\frac{\partial\psi}{\partial\theta} = Zero \qquad\qquad \frac{\partial\psi}{\partial\phi} = Zero \tag{5.7.4}$$

Then

$$i\hbar = \frac{\partial\psi}{\partial t} = \frac{c\hbar}{i}\frac{\partial\psi}{\partial r} + Zero + Zero$$

$$i\hbar = \frac{\partial \psi}{\partial t} = \frac{c\hbar}{i}\frac{\partial \psi}{\partial r} \tag{5.7.5}$$

Substituting

$$\psi = A\sin(kr - \omega t) \tag{5.7.6}$$

$$\frac{\partial \psi}{\partial t} = -\omega A\cos(kr - \omega t)\,\frac{\partial \psi}{\partial r} = kA\cos((kr - \omega t) \tag{5.7.7}$$

Substituting in

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{hc}{i}\frac{\partial \psi}{\partial r} \tag{5.7.8}$$

$$-i\hbar\omega A\cos((kr - \omega t)\,,\frac{hc}{i}kA\cos((kr - \omega t)$$

$$-i\hbar\omega = ck\frac{\hbar}{i}$$

Multiply by i and divide by $\hbar$

$$W = CK \tag{5.7.9}$$

This indicates that the solution is consistent with the ordinary relation between angular frequency and wave number. Thus

$$\psi = A\sin((kr - \omega t) \tag{5.7.10}$$

Is a solution which indicates that gravitons are travelling waves moving with speed of light.

Using the relation

$$cos^2 + sin^2 = 1$$

The probability of existence of particles is

$$|\psi|^2 A^2 cos^2\theta \tag{5.7.11}$$

But

$$\cos 2\theta = cos^2\theta - sin^2\theta = -2cos^2\theta - 1$$

$$cos^2\theta = \frac{1}{2}[cos2\theta + 1]$$

$$h = |\psi|^2 \frac{A^2}{2}[1 + cos2\theta(kr - \omega t)] \quad (5.7.12)$$

This means that a gravitational wave can be propagated with wave function

$$\psi = Acos(kr - \omega t) \quad (5.7.13)$$

The intensity of waves is

$$n = \frac{A^2}{2}[1 + \cos 2(kr - \omega)]$$

$$n = n_0 + \frac{A^2}{2}\cos 2(kr - \omega t) \quad (5.7.14)$$

Where

$$n_0 = \frac{A^2}{2} \quad (5.7.15)$$

The constant form $n_0$can be considered as. a back ground constituting vacuum energy.

5.8 Gravitational Field Quantization

The gravitational field for spherically symmetric body satisfies the equation

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\frac{\partial\psi}{\partial r} \quad (5.8.1)$$

Separating the variables in to time and radial function

$$\psi = uv = u(t)v(r) \quad (5.8.2)$$

Thus

$$i\hbar\frac{v\partial u}{\partial t} = \frac{\hbar c}{i}\frac{u\partial v}{\partial r} \quad (5.8.3)$$

Dividing both sides by uv yield

$$\frac{i\hbar}{u}\frac{\partial u}{\partial t} = \frac{\hbar c}{i}\frac{1}{v}\frac{\partial v}{\partial r} = C_0 = E \quad (5.8.4)$$

Thus

$$i\hbar\frac{\partial u}{\partial t} = Eu \quad (5.8.5)$$

Which is the energy Eigen function consider now the solution of equation in the form

$$u = e^{-i\alpha t} \tag{5.8.6}$$

Substituting equation in equation yield

$$-i^2\hbar\alpha u = Eu \tag{5.8.7}$$

Thus

$$E = \hbar\alpha \tag{5.8.8}$$

The periodicity condition requires

$$u(t + T) = u(t) \tag{5.8.9}$$

$$e^{-i\alpha(t+T)} = e^{i\alpha t} \tag{5.8.10}$$

$$e^{-i\alpha T} = \cos\alpha T - i\sin\alpha T = 1 \tag{5.8.11}$$

Thus

$$\cos\alpha T = 1 \qquad \sin\alpha T = 0 \tag{5.8.12}$$

Hence

$$\alpha T = 2n\pi \tag{5.8.13}$$

$$\alpha = \frac{2n\pi}{T} = 2n\pi f = n\omega \tag{5.8.14}$$

In view of equations the energy is thus given by

$$E = \hbar\alpha = n\hbar\omega \tag{5.8.15}$$

This means that the energy of gravity field is quantized.

5.9 Angular momentum Quantization

Equation (5.5.23) can be used to find the relation

$$-\frac{c\hbar}{i\phi}\frac{d\phi}{\varphi} = C_2 \tag{5.9.1}$$

$$-\frac{\hbar}{c}\frac{d\phi}{\varphi} = \frac{C_2}{c}\phi \tag{5.9.2}$$

$$\hat{L}_z\phi = L_z \tag{5.9.3}$$

Where the L_z operator is given by

$$\hat{L}_z = -\frac{\hbar}{i}\frac{\partial}{\partial\varphi} \tag{5.9.4}$$

$$\int \frac{d\phi}{\phi} = -\frac{i}{\hbar}\frac{C_2}{c}\int d\varphi \tag{5.9.5}$$

$$\ln\phi = -\frac{iC_2}{c\hbar} = -iC_0\varphi + C_3 \tag{5.9.6}$$

$$\phi = e^{c_3}e^{-ic_0\varphi} = \phi_0 e^{-ic_0} \tag{5.9.7}$$

$$\phi(\varphi + 2\pi) = \phi(\varphi) \tag{5.9.8}$$

$$e^{-ic_0(\varphi+2\pi)} = e^{ic_0\varphi} \tag{5.9.9}$$

$$e^{-2\pi c_0 i} = \cos 2\pi c_0 + i\sin 2\pi c_0 = 1 \tag{5.9.10}$$

Thus

$$\cos 2\pi c_0 = 1 \qquad \sin 2\pi c_0 = 0 \tag{5.9.11}$$

$$2\pi C_0 = 2n\pi \tag{5.9.12}$$

$$n = o, 1, 2, 3, \dots \dots$$

$$C_0 = n$$

$$L_z = \frac{C_2}{c} = \frac{\hbar c}{c}C_0 = \hbar n \tag{5.9.13}$$

5.10 Discussion

The Gravity quantum relativistic equation is found by relating GGR Hamiltonian to the momentum as shawnn by equations (5.2.11). Then this expression is multiplied byψ. The energy and momentum terms in equation (5.2.140), is replaced by the corresponding energy and momentum operators in equation (5.2.12) and (5.2.13). Thus the full quantum's GGR equation for radical part is obtained as shown by equation (5.2.12). using the separation of variables time and radial equations were found [see equations (5.2.19) and (5.2.20)]. The solution of the radial port for constant scalar curvature predicts travelling wave solution (5.3.37). This equation shows the existence of gravitational waves, with quantum energy typical to that of plank as shown by equation (5.2.40). The graviton moves with the speed of light. The prediction of graviton agrees with that concerning the behavior of binary pulsars which are assumed to emit gravitational waves. Using the periodicity condition for graviton or particles moving in a circular orbit, the energy is shown to be quantized [see equation (5.2.45)]. This quantization rests on the solution (5.2.28). using the same procedures as in section (2), the GGR quantum equation for the angular part θ is also obtained as shown by equation(5.3.10), (5.3.16) and (5.3.18). For constant scalar curvature, one obtains standing wave solution (5.3.22). The uniqueness of the wave function to have the same value at specific point is used as a physical constraint. This constraint shows that the energy of particles moving in a circular orbit or in a closed loop is quantized, as equation (5.3.25) indicates.

A full GGR quantum equation for spherically symmetric motion is obtained in equation (5.5.17). The separation of variables (5.5.18) leads to 4 independent (r, θ, ϕ, t) equations. The time dependent and radial parts [see equation (5.7.5)] predict again the existence of gravitational wave. Applying time periodicity of this wave on equation (5.8.15). Again this energy is no thing but quantum plank ordinary energy. The angular.

The equation of ϕ part of Schrödinger equation is given by equation (5.9.1). Its solution is given by equation (5.9.7). using the fact that the wave function has only one unique value, the angular momentum component L_z is shown to be quantized.

5.11 Conclusion

The quantum model based on GGR appears to be Succeful. This is since it predicts energy and angular momentum quantization. It also predicts existence of gravitational wave and gravitons.

5.12 Recommendation for Future Work

1. The GGR quantum model needs to be used for black hole and quasars.

2. The Classical GGR can be solved for strong gravity to obtain potential which can be used in GGR quantum equation.

3. The super nova and stars evolution can also be described by GGRQ model.

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