Sudan University of Sciences & Technology

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Branching Brownian Motion with Absorption in Presence of Selection or Coalescence and the All-Time Minimum with Drift

الحركة البراونية المتفرعة مع الإمتصاص في وجود الإختيار او التقلص وأصغرية كل الأزمان مع الإنجراف

A thesis submitted in partial fulfillment for the degree of M.Sc in Mathematics

By: Asaad Mohamed Ibrahim Mohamed

Supervisor:

Prof. Dr. Shawgy Hussein Abd Alla

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Dedication

To my dear lovely parents..

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Abstract

We show and characterise the critical branching Brownian motion in a strip and with absorption. We determine the large deviations for branching Brownian motion in presence of selection, coalescence and the all-time minimum with drift.

الخلاصة

أوضحنا وشخصنا حركة براونيان المتفرعة الحرجة في الشريط ومع الامتصاص.

حددنا الانحرافات الكبيرة لأجل حركة براونيان المتفرعة في الحضور للاختيار والعزل وكل– أصغرية الزمن مع الانجراف.

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Chapter 1
Branching Brownian Motion in Strip: Survival Near Criticality

We consider branching Brownian motion with linear drift in which particles are killed on exiting the interval $(0, K)$ and study the evolution of the process on the event of survival as the width of the interval shrinks to the critical value at which survival is no longer possible. We combine spine techniques and a backbone decomposition to obtain exact asymptotics for the near-critical survival probability.

Section (1.1): Spine Techniques and Martingale Changes of Measure

We consider branching Brownian motion in which each particle performs a Brownian motion with drift $-\mu$, for $\mu \geq 0$, and is killed on hitting 0 or K. All living particles undergo branching at constant rate β to be replaced by a random number of offspring particles, A, where A is an independent random variable with distribution $\{q_k; k = 0, 1, \ldots\}$, finite mean $m > 1$ and such that $E(A \log^+ A) < \infty$. Once born, offspring particles move off independently from their birth position, repeating the stochastic behavior of their parent.

In other words, the motion of a single particle is governed by the infinitesimal generator

$$L = \frac{1}{2}\frac{d^2}{dx^2} - \mu\frac{d}{dx}, \qquad x \in (0, K), \tag{1}$$

defined for all functions $u \in C^2(0, K)$, the space of twice continuously differentiable functions on $(0, K)$, with $u(0+) = u(K-) = 0$. The branching activity is characterised by the branching mechanism

$$F(s) = \beta(G(s) - s), \qquad s \in [0, 1], \tag{2}$$

where $G(s) = \sum_{k=0}^{\infty} q_k s^k$ is the probability generating function of A.

Let us introduce some notation. Denote by N_t and $|N_t|$ the set of and the number of particles alive at time t respectively. For a particle $u \in N_t$, we write $x_u(t)$ for its spatial position at time t. We define $X_t = \sum_{u \in N_t} \delta_{x_u}(t)$ to be the spatial configuration of particles alive at time t and we set $X = (X_t, t \geq 0)$. Denote by P_ν^K the law of X with $X_0 = \nu$ where $\nu \in \mathcal{M}_a(0, K)$, the space of finite atomic measures on $(0, K)$ of the form $\sum_{i=1}^{n} \delta_{x_i}$ with $x_i \in (0, K)$ and $n \in \mathbb{N}$. If the process is initiated from a single particle at $x \in (0, K)$, then we simply write P_x^K (instead of $P_{\delta_x}^K$). We will sometimes neglect the dependence on the initial configuration and write P^K without a subscript. We call the process X a P^K-branching diffusion.

Further, $(\xi = (\xi_t, t \geq 0), \mathbb{P}_x^K)$ will henceforth denote a Brownian motion with drift $-\mu$ starting from $x \in (0, K)$ which is killed upon exiting the interval $(0, K)$. $\mathbb{P}^K$ is the law of the single particle motion under P^K. For $x \in [0, K]$ we define the survival probability $p_K(x) = P_x^K(\zeta = \infty)$ where $\zeta = \inf\{t > 0: |N_t| = 0\}$ is the time of extinction. As a first result we identify the critical width K_0 below which survival is no longer possible.

As we want to study the evolution of the P^K-branching diffusion on survival, we will develop a decomposition which identifies the particles with infinite genealogical lines of descent, that is, particles which produce
a family of descendants which survives forever. To illustrate this, in a realisation of X, let us colour blue all particles with an infinite line of descent and colour red all remaining particles. Thus, on the event of survival, the resulting picture consists of a blue tree 'dressed' with red trees whereas, on the event of extinction, we see a red tree only.

The branching rates of the branching diffusions corresponding to the blue and red trees can be intuitively derived as follows. For simplicity, consider the dyadic branching case only. A particles dies and is replaced by two offspring, at position x say, at rate β. The probability that one of its offspring has an infinite genealogical line of descent is $p_K(x)$, independent of the other particle. Thus, with probability $p_K(x)^2$ both offspring particles are blue and likewise with probability $\left(1 - p_K(x)\right)^2$ and probability $2p_K(x)\left(1 - p_K(x)\right)$ two red ones, respectively one blue and one red particle are born. Thus, given a particle is blue, it branches into two blue particles at rate $\beta \frac{p_K(\cdot)^2}{p_K(\cdot)} = \beta p_K(\cdot)$ and, given a particle is red, it branches into two red particles at rate $\beta\left(1 - p_K(\cdot)\right)$ while, given a particle is blue, immigration of a red particle occurs at rate $2\beta\left(1 - p_K(\cdot)\right)$. Similar reasoning gives the result for the general branching mechanism case. There we shall also see that particles in branching diffusion corresponding to the red trees, respectively the blue tree, move according to a Brownian motion with drift $-\left(\mu + \frac{p_K'}{1-p_K}\right)$, respectively $-\left(\mu - \frac{p_K'}{p_K}\right)$. This results from h-transforms of L using $h = 1 - p_K$ and $h = p_K$ respectively. In fact we will show that the laws of the branching diffusions corresponding to the red tree, the blue and the dressed blue tree arise from martingale changes of measure which, on the level of infinitesimal generators, correspond to the aforementioned h-transforms.

Suppose we know the branching mechanisms of the branching diffusions corresponding to the blue and the red trees in the general case as well as the immigration rates (we will see that a branching mechanism of the general form induces a second type of immigration at the branching times of the blue tree). Intuitively speaking, the coloured tree starting from $x \in (0, K)$ is then constructed by flipping a coin with probability $1 - p_K(x)$ of 'heads' and if it lands 'heads' we grow a red tree with initial particle at x, while if it lands 'tails' we grow a blue tree at x and dress its branches with red trees. Let us write $\mathbf{P}_x^K$ for the law of the coloured tree which is defined on the filtration $\mathcal{F}_t^c := \sigma\{\mathcal{F}_t, c(u)_{u \in N_t}\}$, where $(\mathcal{F}_t, t \geq 0)$ is the natural filtration of (X, P^K) and $c(u)$ is the colour of particle $u \in N_t$. Note that the colours of all particles are $\mathcal{F}_\infty$- measurable. Then we can state (what will turn out to be a simplified version of) the so-called backbone decomposition.

Theorem (1.1.1) [1]: (Backbone Decomposition)

Let $K > K_0$ and $x \in (0, K)$. On the filtration $(\mathcal{F}_t, t \geq 0)$ (which means we ignore the colouring), $(X, \mathbf{P}_x^K)$ is equal in law to (X, P_x^K). This is, for all $t \in [0, \infty]$ and $A \in \mathcal{F}_t$, we have $\mathbf{P}_x^K(A) = P_x^K(A)$.

We show that the backbone decomposition arises naturally from combining changes of measure which condition (X, P^K) on either the event of survival or the event of extinction. A significant convenience of the backbone decomposition is that conditioning the P^K-branching diffusion on survival is the same as conditioning on there being a dressed blue tree, that is a blue tree 'dressed' with red trees. Thus, instead of studying the quasi-stationary limit $\lim_{K \downarrow K_0} P_x^K(\cdot \mid \zeta = \infty)$ it suffices to study the evolution of the branching diffusion corresponding to a dressed blue tree as $K \downarrow K_0$.

In order to do this, we need to know the asymptotics of the survival probability p_K near criticality. For a first asymptotic result note that $u = 1 - p_K$ solves the differential equation $Lu + F(u) = 0$ on $(0, K)$ with boundary condition $u(0) = u(K) = 1$. Near criticality we may assume that $p_K(x)$ is very small for a fixed x and neglecting all terms of order $\left(p_K(x)\right)^2$ and higher we obtain the linearization $Lp_K + m\beta p_K = 0$. This suggests $p_K(x) \sim C_K \sin\left(\frac{\pi x}{K_0}\right) e^{\mu x}$. In fact we have the following result.

The construction of the backbone via a martingale change of measure allows us to give a very simple proof of this quasi-stationary limit result. Theorem (1.2.10) can be seen as an extension of the spine decomposition we mentioned in

the discussion following Theorem (1.1.5) to the critical width K_0. We emphasize however that the result, as stated, only holds over finite time horizons $[0, T]$.
We demonstrate the robustness of our approach by applying the results for the P^K-branching diffusion to study the evolution of a supercritical super-Brownian motion with absorption at 0 and K near criticality. We outline a backbone decomposition analogous to Theorem (1.1.9) in which we will see that the backbone of the super-Brownian motion with absorption at 0 and K is the same as the backbone of an associated P^K-branching diffusion. This connection allows us to deduce asymptotic results for the survival rate of the super-Brownian motion with absorption on $(0, K)$ directly from the results on the survival probability of the associated P^K-branching diffusion. Further, we can find a quasi-stationary limit result for the super-Brownian motion equivalent to Theorem (1.2.10). We intended to highlight the applicability of the backbone approach and we will only sketch the proofs of the results therein.

We present the proof of Theorem (1.1.5) using spine techniques. We show that the backbone arises from a martingale change of measure which conditions (X, P_ν^K) on survival, and we establish the backbone decomposition. We prove the asymptotic results for the survival probability given in Theorem (1.2.1).
The proof of the quasi-stationary limit result in Theorem (1.2.10) follows sketches the analogous results for the super-Brownian motion on $(0, K)$.

Spine techniques of the type used in the proof of Theorem (1.1.5) were developed in the theory of branching processes.
The results for superprocesses are complemented by the decomposition which considers the $(1+\beta)$-super-process conditioned on survival. This work is of particular interest in the current context since it also presents the equivalent result for the approximating branching particle system. However we should point out that in their case the immigrants are conditioned to become extinct up to a fixed time T whereas, in our setting, we condition on extinction in the strip $(0, K)$. Thus the underlying transformations are time-dependent in contrast to the space-dependent h-transforms we see in our setting.
We also point out that our derivation of the backbone decomposition differs in that we show that the backbone arises from combining changes of measure which condition (X, K_ν^K) on either the event of survival or the event of extinction.

However, it has not been possible so far to give such an explicit expression for the constant analogous to C_K.

A similarly fashioned result to Theorem (1.2.10) was obtained in the aforementioned. Their result extends the Evans immortal particle representation for superprocesses which is the equivalent of the spine representation for branching processes. Again we point out that, in contrast to our setting, extinction is a time-dependent phenomenon. Further, our martingale change 8 of measure approach to the backbone decomposition allows us to give a very simple proof of the quasi-stationary limit result.

Remark (1.1.2) [1]:

The martingale construction above applies more generally to branching diffusions with spatially dependent branching mechanism. Suppose we have a branching diffusion Y on $[0, K]$ with branching mechanism $F(s,x), s \in [0,1],\ x \in (0,K)$ and set $F'(x,1) := \frac{d}{ds}F(x,s)|_{s=1}$. Let $\widehat{\Upsilon} = \left(\widehat{\Upsilon}(t), t \geq 0\right)$ be a unit-mean martingale for the single particle motion and accordingly, for $u \in N_t$, define $\widehat{\Upsilon}_u(t)$ as the same object but with the associated single particle replaced by the particle position $y_u(t)$. Then

$$\hat{Z}(t) = \sum_{u \in N_t} \exp\left\{-\int_0^t F'(y_u(s),1)\, ds\right\} \widehat{\Upsilon}_u(t) \tag{3}$$

defines a martingale with respect to $\sigma(Y_t, t \geq 0)$. Changing measure with the martingale $\hat{Z}$ induces a spine decomposition in which the spine has martingale density $\widehat{\Upsilon}(t)$ with respect to the law of the single particle motion in Y.

Let us continue with the study of the martingale Z^K. Since we assumed $E(A \log^+ A) < \infty$, Proposition (1.1.3) gives a necessary and sufficient condition for the $L^1(P_x^K)$-convergence of Z^K.

Proposition (1.1.3) [1]:

Recall that $\lambda(K) = (m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K^2}$ and let $0 < x < K$.

(i) If $\lambda(K) > 0$ then the martingale Z^K is $L^1(P_x^K)$-convergent and in particular uniformly integrable.

(ii) If $\lambda(K) \leq 0$ then $\lim_{t\to\infty} Z^K(t) = 0$ P_x^K-a.s.

We refrain from giving the proof of Proposition (1.1.3) since it is a straightforward adaptation which presents the L^1-convergence result in the case of a branching Brownian motion with absorption at a space-time barrier.

We will now show that the martingale limit $Z^K(\infty)$ is zero if and only if the process becomes extinct.

Proposition (1.1.4) [1]:

For $x \in (0,K)$, the events $\{Z^K(\infty) = 0\}$ and $\{\zeta < \infty\}$ agree P_x^K-a.s.

An essential idea in the proof of Proposition (1.1.4**)** is to embed the killed branching diffusion in a branching diffusion with killing on a larger strip. Let us introduce this procedure and some notation now as it will be used again later.
Denote by $P_x^{(a,b)}$ the law under which X is our usual branching Brownian motion but with killing upon exiting the interval (a,b), where $-\infty \leq a < b \leq \infty$ (and we simply write P_x^b instead of $P_x^{(0,b)}$ in accordance with our previous notation). We denote by τ_u and σ_u the birth and death time respectively of a particle u and write $v \leq u$ if v is an ancestor of u (u is considered to be an ancestor of itself). For an $\epsilon > 0$, we choose a and b such that $a \leq 0 < \epsilon \leq b$. Under $P_x^{(a,b)}$, we define

$$N_t|_{(0,\epsilon)} = \{u \in N_t : x_v(s) \in (0,\epsilon) \forall v \leq u, \qquad \tau_v \leq s \leq \sigma_v \wedge t\},$$

which is the set of particles $u \in N_t$ whose ancestors (not forgetting u itself) have not exited $(0,\epsilon)$ up to time t. Now we can define the restriction of X to $(0,\epsilon)$ under $P_x^{(a,b)}$ by

$$X_t|_{(0,\epsilon)} = \sum_{u \in N_t|_{(0,\epsilon)}} \delta_{x_u(t)}, \qquad t \geq 0.$$

Then we conclude immediately that, for an initial position in $(0,\epsilon)$, the restricted process $X|_{(0,\epsilon)} = \left(X_t|_{(0,\epsilon)}, t \geq 0\right)$ under $P_x^{(a,b)}$ has the same law as (X, P_x^ϵ).
Proof. Clearly $\{\zeta < \infty\} \subset \{Z^K(\infty) = 0\}$ and it remains to show that survival implies that $Z(\infty)$ is strictly positive. We consider the cases $\lambda(K) \leq 0$ and $\lambda(K) > 0$ separately.
Assume $\lambda(K) \leq 0$. Suppose for a contradiction that $Z^K(\infty) = 0$ on survival. This requires the terms $e^{\mu x_u(t)} \sin\left(\frac{\pi x_u(t)}{K}\right)$ to vanish which can only happen if all particles move towards the killing boundaries 0 and K. That is to say, for any $\epsilon > 0$, all particles leave the interval $(\epsilon, K-\epsilon)$ eventually, and thus we may assume without loss of generality that the process survives in a small strip $(0,\epsilon)$, for any $\epsilon > 0$. We will now lead this argument to a contradiction by showing that, for ϵ small enough, the P_x^ϵ-branching diffusion, $x \in (0,\epsilon)$, will become extinct a.s.
We embed the P^ϵ-branching diffusion in a $P^{(-\delta,\epsilon+\delta)}$-branching diffusion according to the previously described procedure. Now we choose δ and ϵ small enough such that $\lambda(\epsilon+2\delta) := (m-1)\beta - \mu^2/2 - \frac{\pi^2}{2(\epsilon+2\delta)^2} < 0$. Then, under $P(-\delta, \epsilon+\delta)$, the process

$$Z^{(-\delta,\epsilon+\delta)}(t) := \sum_{u \in N_t} \left\{ e^{\mu(x_u(t)+\delta) - \lambda(\epsilon+2\delta)t} \sin\left(\frac{\pi(x_u(t)+\delta)}{(\epsilon+2\delta)}\right) \right\}, \qquad t \geq 0,$$

is a martingale of the form in Proposition (1.1.3). Considering now the contribution coming from the particles in the set $N_t|_{(0,\epsilon)}$ only, we first note that survival of the P^{ϵ}-branching diffusion ensures that this set is non-empty for any time t. Further, for particles $u \in N_t|_{(0,\epsilon)}$, the terms $e^{\mu(x_u(t)+\delta)} \sin\left(\frac{\pi(x_u(t)+\delta)}{(\epsilon+2\delta)}\right)$ are uniformly bounded from below by a constant $c > 0$ and hence, under $P_x^{(-\delta,\epsilon+\delta)}$, we get

$$Z^{(-\delta,\epsilon+\delta)}(t) \geq cN_t|_{(0,\epsilon)} e^{-\lambda(\epsilon+2\delta)t}.$$

Since we have chosen δ and ϵ such that $\lambda(\epsilon + 2\delta) < 0$, we now conclude that $Z^{(-\delta,\epsilon+\delta)}(\infty) = \infty, P_x^{(-\delta,\epsilon+\delta)}$ -a.s. This is a contradiction since $Z^{(-\delta,\epsilon+\delta)}$ is a positive martingale and therefore has a finite limit. Hence, for $\lambda(K) \leq 0$, the martingale limit $Z^K(\infty)$ cannot be zero on survival. Assume now $\lambda(K) > 0$. Suppose for a contradiction that $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ is non empty and work on this event from now on. Now let $z_K(x) = P_x^K(Z^K(\infty) = 0)$, for $x \in (0, K)$. Define $M_\infty := \mathbb{1}_{\{Z^K(\infty)=0\}}$ and set

$$M_t := E_x^K(M_\infty|\mathcal{F}_t) = \prod_{u \in N_t} z_K(x_u(t)),$$

where the second equality follows from the branching Markov property. Then $(M_t, t \geq 0)$ is a uniformly integrable P_x^K-martingale with limit M_∞. Hence its limit on the event $\{\zeta = \infty\} \cap \{Z^K(\infty) = 0\}$ is $1, P_x^K$-a.s. This requires in turn that all particles $x_u(t), u \in N_t$ move towards 0 and K as $t \to \infty$, since we know from Proposition (1.1.3) (i) that $z_K(x) < 1$ for x within $(0, K)$. The previous part of this proof already showed that this leads to a contradiction. Thus, for $\lambda(K) > 0$, the martingale limit cannot be zero on survival. This completes the proof.

Theorem (1.1.5) [1]:
If $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$ where $K_0 := \pi(2(m-1)\beta - \mu^2)^{-1}$, then $p_K(x) > 0$ for all $x \in (0, K)$; otherwise $p_K(x) = 0$ for all $x \in [0, K]$.

The proof of Theorem (1.1.5) uses a spine argument, decomposing X into a Brownian motion conditioned to stay in $(0, K)$ dressed with independent copies of (X, P^K) which immigrate along its path.

Proof . We use classical spine techniques applications in the setting of Branching Brownian motion with absorption at 0.

We will briefly recall the key steps in the spine construction. Recall that we denote by $(\xi, \mathbb{P}_x^K)$ a Brownian motion with drift $-\mu$ initiated from $x \in (0, K)$ which is killed upon exiting $(0, K)$. Then the process

$$\Upsilon^K(t) = sin\left(\frac{\pi\xi_t}{K}\right)e^{\mu\xi_t+\left(\frac{\mu^2}{2}+\frac{\pi^2}{2K^2}\right)t}, \qquad t \geq 0, \qquad (4)$$

is a martingale with respect to $\sigma(\xi_s: s \leq t)$. Define $\mathbb{Q}_x^K$ to be the probability measure which has martingale density $\Upsilon^K(t)$ with respect to $\mathbb{P}_x^K$ on $\sigma(\xi_s: s \leq t)$. Under $\mathbb{Q}_x^K$, ξ is now a Brownian motion conditioned to stay in $(0, K)$.

We can use Υ^K to construct a martingale with respect to $\mathcal{F}_t = \sigma(X_s, s \leq t)$, the filtration generated by the P^K-branching diffusion up to time t. For each $u \in N_t$, write $\Upsilon_u^K(t) = \sin(\pi x_u(t)/K)\, e^{\mu x_u(t)+\left(\frac{\mu^2}{1}+\frac{\pi^2}{2K^2}\right)t}, t \geq 0$ Define the process $Z^K = (Z^K(t), t \geq 0)$ given by

$$Z^K(t) = \sum_{u\in N_t} e^{-(m-1)\beta t}\Upsilon_u(t) = \sum_{u\in N_t} e^{\mu x_u(t)-\lambda(K)t}\sin(\pi x_u(t)/K), \qquad t \geq 0,$$

where we set $\lambda(K) := (m-1)\beta - \frac{\mu^2}{2} - \frac{\pi^2}{2K^2}$. Then Z is a nonnegative $(P_x^K, \mathcal{F}_t)$-martingale. For $x \in (0, K)$, we define a martingale change of measure on the probability space of the P^K-branching diffusion via

$$\left.\frac{dQ_x^K}{dP_x^K}\right|_{\mathcal{F}_t} = \frac{Z^K(t)}{Z^K(0)}. \qquad (5)$$

This change of measure induces the following spine construction for the path of X under Q_x^K. From the initial position x, we run a $\mathbb{Q}_X^K$-diffusion, that is a Brownian motion conditioned to stay in $(0, K)$, and we call it a spine. At times of a Poisson process with rate $m\beta$ we immigrate $\tilde{A}$ independent copies of (X, P^K) rooted at the spatial position of the spine at this time. The number of immigrants $\tilde{A}$ has the size-biased offspring distribution

$$\tilde{q}_k = \frac{1+k}{m}q_{k+1}, \qquad k \geq 0.$$

Let us remark that the change of measure with the martingale Υ^K in (4) is equivalent to a Doob's h-transform on L using $h(x) = \sin(\pi x/K)e^{\mu x}$. The infinitesimal generator L^* of a Brownian motion conditioned to stay in $(0, K)$ is therefore

$$L_K^* = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\pi/K_0}{\tan(\pi x/K_0)}\frac{d}{dx}, \qquad \text{on } (0, K,),$$

with domain $C^2(0, K)$. We then call the Brownian motion conditioned to stay in $(0, K)$ a L_K^* -diffusion and note that it is positive recurrent with invariant density $\frac{2}{K}\sin^2(\pi x/K)$, for $x \in (0, K)$.

Note that $\lambda(K) \geq 0$ if and only if $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$. The result follows now immediately from Proposition (1.1.3) and Propositon (1.1.4).

Remark (1.1.6) [1]:
We can apply the same argument given for the process M in the proof of Proposition (1.1.4) to show that

$$E_x^K\left(\mathbb{1}_{\{\zeta^K<\infty\}}|\mathcal{F}_t\right) = \prod_{u\in N_t}\left(1 - p_K\big(x_u(t)\big)\right), \qquad t \geq 0$$

is a uniformly integrable product martingale which, gives that $1 - p_K(x)$ solves

$$\begin{aligned} Lu + F(u) &= 0 \qquad on\ (0, K) \\ u(0) = u(K) &= 1. \end{aligned} \tag{6}$$

We will show later in Remark (1.1.12) that, if there exists a non trivial solution to (6), then it is unique. This will then again imply that $\{Z^K(\infty) = 0\}$ and $\{\zeta < \infty\}$ agree P_x^K-a.s.

We decompose the P^K-branching diffusion into branching diffusions corresponding to the blue and red trees described in our intuitive picture. The blue tree which consists of all genealogical lines of descent that will never become extinct will be shown to correspond to a branching diffusion which we will henceforth refer to as the backbone. Secondly, the red trees which contain all remaining lines of descent will be shown to correspond to copies of the P^K-branching diffusion conditioned on becoming extinct.
The law of the branching diffusion corresponding to the coloured tree, $\mathbf{P}^K$, is defined by the law of X under P^K and a subsequent deterministic colouring of the particles as described previously. Its natural filtration is $\tilde{\mathcal{F}}_t := \sigma\{\mathcal{F}_t, c(u)_{u\in N_t}\}$ where $c(u)$ is the colour of a particle $u \in N_t$. We say a particle u is blue if it has an infinite genealogical line of descent and we write $c(u) = b$, otherwise we say it is red and write $c(u) = r$. We have, for all $t \geq 0$,

$$\left.\frac{d\mathbf{P}_x^K}{dP_x^K}\right|_{\mathcal{F}_\infty} = \prod_{u\in N_t}\left(\mathbb{1}_{\{c(u)=b\}} + \mathbb{1}_{\{c(u)\}=r}\right) = 1$$

and thus

$$\left.\frac{d\mathbf{P}_x^K}{dP_x^K}\right|_{\mathcal{F}_t} = E_x^K\left(\prod_{u\in N_t}\left(\mathbb{1}_{\{c(u)=b\}} + \mathbb{1}_{\{c(u)\}=r}\right)|\mathcal{F}_t\right)$$

$$= \sum_{(c_u)_{u\in N_t}} \prod_{u\in N_t} P_x^K(c(u) = c_u|\mathcal{F}_t)$$

$$= \sum_{(c_u)_{u\in N_t}} \prod_{u\in N_t} p_K\big(x_u(t)\big) \prod_{u\in N_t, c_u=r} \Big(1 - p_K\big(x_u(t)\big)\Big) = 1, \qquad (7)$$

where $(c_u)_{u\in N_t}$ is the set of all possible colourings of N_t. In particular, for $A \in \mathcal{F}_t$, we get

$$\mathbf{P}_x^K(A; c(u) = c_u\ \forall u \in N_t|\mathcal{F}_t)$$

$$= \mathbb{1}_A \prod_{u\in N_t, c_u=b} p_K\big(x_u(t)\big) \prod_{u\in N_t, c_u=r} \Big(1 - p_K\big(x_u(t)\big)\Big).$$

We can now easily derive the change of measure for the branching diffusion corresponding to the red tree. Let $A \in \mathcal{F}_t$, and write $c(0) = r$ for the event that the initial particle is red and thus

$$\mathbf{P}_x^{R,K}(A) \coloneqq \mathbf{P}_x^K(A|c(0) = r) = \frac{\mathbf{P}_x^K(A; c(u)=r\ \forall u\in N_t)}{\mathbf{P}_x^K(c(0)=r)}$$

$$= \frac{E_x^K\Big(\mathbb{1}_A \prod_{u\in N_t}\big(1-p_K(x_u(t))\big)\Big)}{1-p_K(x)}. \qquad (8)$$

Clearly, conditioning the genealogical line of the initial particle to become extinct agrees with conditioning the whole process to become extinct and therefore the law of X under $\mathbf{P}^{R,K}$ agrees with the law of X conditioned on extinction. The following Proposition characterizes the process under $\mathbf{P}^{R,K}$.

Proposition (1.1.7) [1]:

For $\nu \in \mathcal{M}_a(0, K)$, define $\mathbf{P}_\nu^{R,K}$ via (3.2). Then $\left(X, \mathbf{P}_\nu^{R,K}\right)$ is a branching process with single particle motion characterised by the infinitesimal generator

$$L_K^R = \frac{1}{2}\frac{d^2}{dx^2} - \left(\mu + \frac{p_K'}{1 - p_K}\right)\frac{d}{dx} \qquad on\ (0, K),$$

for $u \in C^2(0, K)$ with $u(0+) = u(K-) = 0$ and the branching activity is governed by the space- dependent branching mechanism

$$F_K^R(s, y) = \frac{1}{1 - p_K(y)}\left(F\left(s\big(1 - p_K(y)\big)\right) - sF\big(1 - p_K(y)\big)\right),$$

for $s \in [0, 1]$ and $y \in (0, K)$.

Proof. The change of measure in (8) preserves the branching property in the following sense. Let $\nu = \sum_{i=1}^{n} \delta_{x_i}$ be an initial configuration in $(0, K)$ and $A \in \mathcal{F}_t$. Then

$$\mathbf{P}_{\nu}^{R,K}(A) := E_x^K\left(\mathbb{1}_A \frac{\prod_{u\in N_t}\left(1 - p_K\left(x_u(t)\right)\right)}{\prod_{i=1}^{n}\left(1 - p_K(x_i)\right)}\right)$$

$$= \prod_{i=1}^{n} E_{x_i}^K\left(\mathbb{1}_A \frac{\prod_{u\in N_t}\left(1 - p_K\left(x_u(t)\right)\right)}{1 - p_K(x_i)}\right) = \left(\bigoplus_{i=1}^{n} \mathbf{P}_{x_i}^{R,K}\right)(A).$$

The process $(X, \boldsymbol{P}^{R,K})$ is therefore completely characterised by its evolution up to the first branching time T. Let $\xi = \{\xi_t, 0 \le t \le T\}$ denote the path of the initial particle up to time T and let H be a positive bounded measurable functional of this path. We begin with considering the case $t < T$. We have

$$\boldsymbol{E}_x^{R,K}(H(\xi_s, s \le t); T > t) = \mathbb{E}_{\boldsymbol{x}}^K\left(H(\xi_s, s \le t)\frac{1 - p_K(\xi_t)}{1 - p_K(x)}; T > t\right)$$

$$= e^{-\beta t}\mathbb{E}_{\boldsymbol{x}}^{R,K}\left(H(\xi_s, s \le t)e^{-\int_0^t \frac{F\left(1-p_K(\xi_s)\right)}{1-p_K(\xi_s)}ds}\right), \qquad (9)$$

where $\mathbb{P}_x^{R,K}$ is defined by the change of measure

$$\left.\frac{d\mathbb{P}_x^{R,K}}{d\mathbb{P}_x^K}\right|_{\mathcal{G}_t} = \frac{1 - p_K(\xi_t)}{1 - p_K(x)}e^{\int_0^t \frac{F\left(1-p_K(\xi_s)\right)}{1-p_K(\xi_s)}ds}, \quad t \ge 0 \qquad (10)$$

and $(\mathcal{G}_t, t \ge 0)$ denotes the natural filtration of $(\xi, \mathbb{P}_x^K)$. Thus the initial particle performs a $\mathbb{P}^{R,K}$ -motion which is governed by the infinitesimal generator L_K^R as given in the statement of the proposition. Taking $H = 1$ above, we see immediately that under $\boldsymbol{P}^{R,K}$ the branching rate changes to

$$\beta^R(y) = \frac{F\left(1 - p_K(y)\right) + \beta\left(1 - p_K(y)\right)}{1 - p_K(y)} = \beta\sum_{k\ge 0} q_k\left(1 - p_K(y)\right)^{k-1}, \quad (11)$$

for $y \in (0, K)$.

It remains to identify the offspring distribution and we therefore study the process at its first branching time T. We get

$$\boldsymbol{E}_x^{R,K}(H(\xi_s, s \le T); N_T = k; T \in dt)$$

$$= E_{\boldsymbol{x}}^K\left(\frac{\left(1 - p_K(\xi_T)\right)^{N_T}}{1 - p_K(x)} H(\xi_s, s \le T); T \in dt; N_T = k\right)$$

$$= E_{\boldsymbol{x}}^K\left(\frac{\left(1 - p_K(\xi_T)\right)^{k}}{1 - p_K(x)} H(\xi_s, s \le T)\beta e^{-\beta T} q_k\right)$$

$$= \mathbb{E}_{\boldsymbol{x}}^{R,K}\left(H(\xi_s, s \le T)\beta^R(\xi_T) e^{-\int_0^T \beta^R(\xi_s)ds} \frac{\beta}{\beta^R(\xi_T)} q_k \left(1 - p_K(\xi_T)\right)^{k-1}\right).$$

We see that, in addition to the change in the motion and the branching rate, the offspring distribution under $\boldsymbol{P}^{R,K}$ becomes $\{q_k^R, k \ge 0\}$ where

$$q_k^R(y) = \beta\left(\beta^R(y)\right)^{-1} q_k \left(1 - p_K(y)\right)^{k-1}, \quad k \ge 0. \tag{12}$$

A simple computation shows that $F_K^R(s, y) = \beta^R(y)(\sum_{k\ge0} q_k^R(y)s^k - s)$ takes the desired form.

Similarly to the above reasoning, we obtain the law of the dressed backbone, that is a backbone with immigration of $\boldsymbol{P}^{R,K}$-branching diffusions, by conditioning on the first particle being blue. Thus

$$\boldsymbol{P}_x^{D,K}(A) \coloneqq \boldsymbol{P}_x^K(A | c(0) = b)$$

$$= \frac{\boldsymbol{P}_x^K(A; c(u) = b \text{ } for \text{ } at \text{ } least \text{ } one \text{ } u \in N_t)}{\boldsymbol{P}_x^K(c(0) = b)}$$

$$= \frac{E_{\boldsymbol{x}}^K\left(\mathbf{1}_A\left(1 - \prod_{u\in N_t}\left(1 - p_K\left(x_u(t)\right)\right)\right)\right)}{p_K(x)} \tag{13}$$

Then $(X, \boldsymbol{P}^{D,K})$ certainly agrees with (X, P^K) conditioned on survival. Let us characterise the evolution under $\boldsymbol{P}_{\boldsymbol{x}}^{D,K}$. We use the previous notation and in addition let $\tau = T \wedge \tau_{(0,K)}$ denote the death time of the initial particle, where $\tau_{(0,K)}$ is the first time this particle exits $(0, K)$. Then

$$\boldsymbol{E}_x^{D,K}(H(\xi_s, s \le t); \tau > t) = e^{-\beta t}\mathbb{E}_{\boldsymbol{x}}^K\left(H(\xi_s, s \le t)\frac{p_K(\xi_t)}{p_K(x)}; \tau_{(0,K)} > t\right)$$

$$= e^{-\beta t}\mathbb{E}_{\boldsymbol{x}}^{B,K}\left(H(\xi_s, s \le t) e^{\int_0^t \frac{F\left(1 - p_K(\xi_s)\right)}{p_K(\xi_s)} ds}\right), \tag{14}$$

where $\mathbb{P}_x^{B,K}$ is defined by the change of measure, for $t \geq 0$,

$$\left.\frac{d\mathbb{P}_x^{B,K}}{d\mathbb{P}_x^K}\right|_{\mathcal{G}_t} = \frac{p_K(\xi_t)}{p_K(x)} \exp\left\{-\int_0^t \frac{F\big(1-p_K(\xi_s)\big)}{p_K(\xi_s)} ds\right\} \mathbf{1}_{\{\tau_{(0,K)}>t\}}, \tag{15}$$

and $(\mathcal{G}_t, t \geq 0)$ is again the natural filtration of $(\xi, \mathbb{P}_x^K)$. Thus, setting

$$\begin{aligned} \beta^D(x) &= -\frac{F\big(1-p_K(x)\big) - \beta_{PK}(x)}{p_K(\xi_s)} \\ &= -\beta \frac{1-\sum_{k=0}^{\infty}\big(1-p_K(x)\big)^k q_k}{p_K(\xi_s)}, \quad \text{for } x \in (0,K), \end{aligned} \tag{16}$$

we see that (14) simplifies to

$$\boldsymbol{E}_x^{D,K}(H(\xi_s, s \leq t); \tau > t) = \mathbb{E}_{\boldsymbol{x}}^{B,K}\left(H(\xi_s, s \leq t) e^{\int_0^t \beta^D(\xi_s)ds}\right). \tag{17}$$

We deduce from this that, under $\boldsymbol{P}^{D,K}$, the motion of the initial particle is given by the change of measure in (15) and it branches at rate $\beta^D(\cdot)$ as in (16).

It remains to specify the offspring distribution. We begin with the expression in (13) and then use (15) and (16) to get

$$\boldsymbol{E}_x^{D,K}(H(\xi_s, s \leq T); T \in dt; N_T = k)$$

$$= E_{\boldsymbol{x}}^K\left(\frac{1-\big(1-p_K(\xi_T)\big)^k}{p_K(x)} H(\xi_s, s \leq T)\beta e^{-\beta T} q_k\right)$$

$$= \mathbb{E}_{\boldsymbol{x}}^{R,K}\left(H(\xi_s, s \leq T)\beta^D e^{-\int_0^T \beta^D(\xi_s)ds} \frac{\beta}{\beta^D(\xi_T)} q_k \frac{1-\big(1-p_K(\xi_T)\big)^k}{p_K(\xi_T)}\right).$$

Again this reveals the evolution of the initial particle as described above and we further see that the offspring distribution of the initial particle under $\boldsymbol{P}^{D,K}$ is given by $\{q_k^D, k \geq 0\}$ where

$$q_k^D(x) \propto q_k \frac{1-\big(1-p_K(\xi_T)\big)^k}{p_K(\xi_T)}, \quad \text{for } x \in (0,K)$$

up to the normalising constant $\beta\left(\beta^D(x)\right)^{-1}$. We note that $q_0(x)=0$ for all $x\in(0,K)$ which we expected to see since $(X,\boldsymbol{P}^{D,K})$ is equal in law to (X,P^K) conditioned on survival. However, we have so far neglected the fact that the initial particle can give birth to particles of the same type, i.e. blue particles (referred to as branching), and red particles which evolve as under $\boldsymbol{P}^{R,K}$ (referred to as immigration). We will split up the rate β^D and the offspring distribution q_K^D into terms corresponding to branching respectively immigration. Firstly, note that we can decompose the rate β^D into

$$\beta^D(y)=\beta\sum_{k\geq 2}\sum_{n\geq k}q_n\binom{n}{k}p_K(y)^{k-1}\left(1-p_K(y)\right)^{n-k}+\beta\sum_{n\geq 1}q_n n\left(1-p_K(y)\right)^{n-1}$$

$$=:\beta^B(y)+\beta^I(y). \tag{18}$$

Then β^I is the rate at which the initial particle gives birth to one blue particle and a random number of (red) immigrants (immigration rate) while β^B is the rate at which the initial particle gives birth to at least two particles of the blue type and a random number of (red) immigrants occur (branching rate of the branching diffusion corresponding to the blue tree). We can now rewrite the offspring distribution q_k^D as

$$q_k^D\propto q_k\frac{1-\left(1-p_K(x)\right)^{N_T}}{p_K(x)}$$

$$=q_k\sum_{i=2}^{k}\binom{k}{i}p_K(x)^{i-1}\left(1-p_K(x)\right)^{k-i}+q_k\left(1-p_K(x)\right)^{k-1},\ \ k\geq 1. \tag{19}$$

Then the term in (18) gives, up to normalisation, the probability that the initial particle branches into i particles of its type and, at the same branching time, $k-i$ particles immigrate. The term in (19) is the probability that $k-1$ immigrants occur, again up to a normalizing constant.

Note that $(X,\boldsymbol{P}^{D,K})$ inherits the branching Markov property from (X,P^K) by (13) in a similar spirit to the case of $(X,\boldsymbol{P}^{R,K})$. Thus the description of the initial particle also characterises the evolution of all particles of the blue type and together with the characterisation of the immigrating $\boldsymbol{P}^{R,K}$-branching diffusions in Proposition (1.1.12) we have completely characterised the evolution of X under $\boldsymbol{P}^{D,K}$. The following result is now an immediate consequence.

Theorem (1.1.8) [1]: (The Dressed Backbone)

Let $K > K_0$ and $x \in (0, K)$. The process $(X, \boldsymbol{P}^{D,K})$ evolves as follows.

(i) From x, we run a $\mathbb{P}_x^{B,K}$ -diffusion which dies at rate β^B,

(ii) at the space-time position of its death, it is replaced by A^B particles where A^B is distributed according to the probabilities,

$$q_k^B(y) = \beta\beta^B(y)^{-1} \sum_{n\geq k} q_n \binom{n}{k} p_K(y)^{k-1}\left(1 - p_K(y)\right)^{n-k}, \qquad (20)$$

for $k \geq 2$ and $y \in (0, K)$.

(iii) Each of the offspring particles repeats its parent's stochastic behaviour.

(iv) Conditionally on the branching diffusion, say X^B, generated by steps **(i) - (iii)**, we have the following.

- (Continuous immigration) Along the trajectories of each particle in X^B, an immigration with $n \geq 1$ immigrants occurs at rate

$$\beta_n^I(y) = \beta q_{n+1}(n+1)\left(1 - p_K(y)\right)^n, \quad y \in (0, K). \qquad (21)$$

- (Branch point immigration) At a branch point of X^B with $k \geq 2$ particles, we see an immigration of $n \geq 0$ immigrants with probability

$$q_n^I(y) = q_{n+k}\binom{n+k}{k} p_K(y)^{k-1}\left(1 - p_K(y)\right)^n, \qquad y \in (0, K). \quad (22)$$

Each immigrant initiates an independent copy of $(X, \boldsymbol{P}^{R,K})$ from the space-time position of its birth.

Theorem (1.1.9) [1]: (Backbone Decomposition)

Let $K > K_0$ and $v \in \mathcal{M}_a(0, K)$ such that $v = \sum_{i=1}^n \delta_{x_i}$ with $x_i \in (0, K)$, $n \geq 1$. For $t \geq 0$, we can define

$$\left.\frac{d\boldsymbol{P}_v^K}{dP_v^K}\right|_{\mathcal{F}_t} = \sum_{k=0}^{n} \sum_{(x_1,\dots,x_k)} \prod_{i=1}^{k} p(x_i) \left.\frac{d\boldsymbol{P}_v^K}{dP_v^K}\right|_{\mathcal{F}_t} \prod_{j=k+1}^{n} \left(1 - p(x_i)\right) \left.\frac{d\boldsymbol{P}_{x_i}^{R,K}}{dP_{x_i}^K}\right|_{\mathcal{F}_t}, \qquad (23)$$

where the second sum above is taken over all k-tuples of $x_1, \dots, x_n$. However note that the right-hand side of (23) is equal to 1 and thus, trivially, on the filtration $(\mathcal{F}_t)_{t\geq 0}$, $(X, \boldsymbol{P}_v^K)$ is Markovian and equal in law to (X, P_v^K).

Proof.

The change of measure is just a restatement of (7) and the result follows from Proposition (1.1.7) and Theorem (1.1.8).

Intuitively speaking, we can describe the evolution under $\boldsymbol{P}_{v}^{D,K}$ and thus also under P_{v}^{K} as follows. Independently for each initial particle x_i, we flip a coin with probability $p(x_i)$ of 'heads'. If it lands 'heads', we initiate a copy of $\left(X,\boldsymbol{P}_{x_i}^{D,K}\right)$ and otherwise we initiate a copy of $\left(X,\boldsymbol{P}_{x_i}^{R,K}\right)$.

Corollary (1.1.10) [1]:

Given the number of particles of (X,P_{v}^{K}) and their positions, say $x_1,\dots,x_n$ for some $n\in\mathbb{N}$, at a fixed time t, the number of particles of X_t^B is the number of successes in a sequence of n independent Bernoulli trials each with success probability $p(x_1),\dots,p(x_n)$.

We refer to the branching diffusion generated by steps **(i)-(iii)** of Theorem (1.1.9) above as the backbone. Its law can be characterised as follows.

Proposition (1.1.11) [1]: (The Backbone)

For $v\in\mathcal{M}_a(0,K)$ such that $v=\sum_{i=1}^{n}\delta_{x_i}$ with $x_i\in(0,K)$, $n\geq 1$, we define the measure $\boldsymbol{P}_{v}^{B,K}$ via the following change of measure. For $t\geq 0$,

$$\left.\frac{d\boldsymbol{P}_{v}^{B,K}}{dP_{v}^{K}}\right|_{\mathcal{F}_t}=\prod_{v\in\boldsymbol{\mathcal{T}}_t}\frac{p_K\big(x_v(\sigma_v\wedge t)\big)}{p_K\big(x_v(\tau_v)\big)}\mathbf{1}_{\left\{t<\tau_{(0,K)}^{v}\right\}}$$

$$\times\exp\left\{\int_{\tau_v}^{\sigma_t\wedge t}F'\left(1-p_K\big(x_v(s)\big)\right)+\beta ds\right\}$$

$$\times\prod_{v\in\boldsymbol{\mathcal{T}}_{t-}}\frac{q_{A_v}^{B}\big(x_v(\sigma_v)\big)}{q_{A_v}\beta\big(x_v(\sigma_v)\big)\left(\beta^B\big(x_v(\sigma_v)\big)\right)^{-1}},$$

where $\boldsymbol{\mathcal{T}}_t$ is the set of all particles $v\in\boldsymbol{\mathcal{T}}$ with $\tau_v<t$ and v is in $\boldsymbol{\mathcal{T}}_t-$ if, in addition, $\sigma_v<t$. As usual, τ_v and σ_v are the birth respectively death times, $\tau_{(0,K)}^{v}$ is the first exit time from $(0,K)$ and A_v is the random number of offspring of a particle $v\in\boldsymbol{\mathcal{T}}_{t-}$.

The branching diffusion $\left(X,\boldsymbol{P}_{v}^{B,K}\right)$ has infinitesimal generator

$$L_K^B = \frac{1}{2}\frac{d^2}{dx^2} - \left(\mu - \frac{p_K'}{p_K}\right)\frac{d}{dx} \quad on \quad (0, K),$$

defined for all $u \in C^2(0, K)$, and space-dependent branching mechanism

$$F_K^B(s, y) = \frac{1}{p_K(y)}\Big(F\big(sp_K(y) + (1 - p_K(y))\big) - (1 - s)F\big(1 - p_K(y)\big)\Big),$$

for $s \in [0,1]$ and $y \in (0, K)$. The process $\left(X, \boldsymbol{P}_x^{B,K}\right)$ evolves according to the steps **(i)-(iii)** of Theorem (1.1.9).

Proof. First note that the motion under $\mathbb{P}^{B,K}$, given by the change of measure in (15), is governed by the infinitesimal generator L_K^B as given in the statement. A simple computation also shows that $F_K^B(s, y) = \beta^B(y)(\sum_{k\geq 2} q_k^B(x)s^k - s)$ with β^B and q_k^B as in (18) and (20) gives the desired form. The result then follows from rewriting the change of measure up to the first branching time T as

$$\left.\frac{d\boldsymbol{P}_x^{B,K}}{dP_x^K}\right|_{\mathcal{F}_T} = \frac{p_K(\xi_T)}{p_K(x)} \exp\left\{-\int_0^T \frac{F\big(1 - p_K(\xi_s)\big)}{p_K(\xi_s)} ds\right\} \mathbf{1}_{\{t<\tau_{(0,K)}\}}$$

$$\times \frac{1}{\beta}\beta^B(\xi_T) \exp\left\{-\int_0^T \beta^B(\xi_s) - \beta ds\right\} \times \frac{q_{N_T}^B(\xi_T)}{q_{N_T}},$$

noting that the first line on the right-hand side accounts for the change of motion, the first term in the second line for the change in the branching rate and the last term in the second line for the change in the offspring distribution.

Remark (1.1.12) [1]:

As promised earlier, with the help of Corollary (1.1.11), we can show that, if (6) has a non-trivial solution, then it is unique. Assume $g_K(x)$ is a non-trivial solution to (2.4). It follows that

$$M^K(t) = \prod_{u\in N_t} g_K\big(x_u(t)\big), \quad t \geq 0,$$

is a P_x^K -product martingale. Since M^K is uniformly integrable, its limit $M^K(\infty)$ exits P_x^K -a.s. On the event of extinction, $M^K(\infty) = 1$. On the event of survival, we have

$$M^K(t) = \prod_{u\in N_t} g_K(x_u(t)) \leq \prod_{u\in N_t^B} g_K(x_u^B(t)), \qquad (24)$$

where N_t^B is the set and $x_u(t)$ are the spatial positions of the particles in X_t^B. Clearly $|N_t|^B \to \infty$ as $t \to \infty$ since each particle in X^B is replaced by at least two offspring and there is no killing. Further particles in X^B perform an ergodic motion and it is therefore not possible that $\liminf g(x_u(t))$ tends to 1. Thus the right-hand side of (24) tends to 0 and we conclude that $M^K(\infty) = \mathbf{1}_{\{\zeta<\infty\}}$. Hence $g_K(x) = E_x^K(M^K(\infty)) = P_x^K(\zeta < \infty)$ which implies uniqueness.

In particular we may conclude that (6) has a non-trivial solution if and only if $\mu < \sqrt{2(m-1)\beta}$ and $K > K_0$.

Section (1.2): Super-Brownian Motion in a Strip

Theorem (1.2.1) [1]:
Uniformly for all $x \in (0, K_0)$, we have

$$p_K(x) \sim C_K \sin\left(\frac{\pi x}{K_0}\right) e^{\mu x}, \qquad as\ K \downarrow K_0, \qquad (23)$$

where C_K is independent of *x* and can explicitly be determined as

$$C_K = (K - K_0)\,\frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0} + 1)}, \quad as\ K \downarrow K_0,$$

and in particular $C_K \downarrow 0$ as $K \downarrow K_0$.

We will prove a first part of Theorem (1.2.1) using spine techniques. It is to be particularly emphasized that we are able to determine C_K here. C_K is a 'non-linear' constant and 'linear' spine techniques fail when trying to identify it. Nevertheless, a careful application of the backbone decomposition given in Theorem (1.1.1) will deliver an explicit expression here since the blue tree captures enough 'non-linear' branching information about the evolution of (X, P^K) on survival. With Theorem (1.1.1) and (1.2.1) in hand we look for a quasi-stationary limit result for the law of the branching diffusion corresponding to a dressed blue tree, which agrees with the law of X conditioned on survival, as we approach criticality. In the dyadic case, the heuristic derivation of the branching rates already suggests that, given the particle positions $x_u(t)$ for $u \in N_t$ in X_t, the number of particles in the blue tree at time t, is the number of successes in a sequence of independent Bernoulli trials each with probability of success $p_K(x_u(t))$, $u \in N_t$ (We will address this thinning argument rigorously in Remark

(1.1.2)). Now, as $K \downarrow K_0$, the probabilities $p_K(\cdot)$ tend to 0 uniformly by Theorem (1.2.1) and thus the blue tree becomes increasingly thinner on $(0, K_0)$. Under conditioning on survival, it cannot vanish completely though since the genealogical line of the initial blue particle cannot become extinct and thus one may believe that, over a fixed time interval $[0, T]$, the blue tree thins down to a single genealogical line at criticality. In the case of a dyadic branching mechanism this conjecture can readily by confirmed by looking at the branching rates. The blue branching rate βp_K drops down to 0 as $K \downarrow K_0$, at the same time the red branching rate $\beta(1 - p_K)$ increases to β and the rate of immigration $2\beta(1 - p_K)$ rises to 2β at criticality.

Formalising this idea and taking into account the change in the single particle motion, the general results reads as follows.

Proof. We break up Theorem (1.2.1) into two parts which will be proved in the following. We begin with a preliminary result which ensures that the survival probability p_K is right-continuous at K_0.

Lemma (1.2.2) [1]:

Let $x \in (0, K_0)$. Then $\lim\limits_{K \downarrow K_0} p_K(x) = 0$.

Proof. We fix $x \in (0, K_0)$ throughout the proof and consider $p_K(x)$ as a function in K. For a fixed $t > 0$, let us define the probability $p_K(x, t) \coloneqq P_x^K$ (survival in $(0, K)$ up to time t). By monotonicity of measures we have $\lim\limits_{K \downarrow K_0} p_K(x, t) = p_{K_0}(x, t)$. Now we can write $p_K(x) = \inf\limits_{t>0} p_K(x, t)$. Hence $p_K(x)$ is the infimum of a sequence of functions which are continuous at K_0 and thus upper semicontinuous at K_0, that is

$$\limsup_{K \downarrow K_0} p_K(x) \leq p_{K_0}(x).$$

Furthermore, $p_K(x)$ is decreasing as $K \downarrow K_0$ and bounded, so the right limit exists and

$$p_{K_0}(x) \leq \lim_{K \downarrow K_0} p_K(x).$$

Combining the two inequalities above we obtain right-continuity of $p_K(x)$ at K_0. By Theorem (1.1.5), $p_{K_0}(x) = 0$ and so we have $\lim\limits_{K \downarrow K_0} p_K(x) = 0$.

The following lemma is the essential part in the proof of Proposition (1.2.4).

Definition (1.2.3) [5]: (Lebesgue Dominated Convergence Theorem)

Suppose $f_n: \mathbb{R} \to [-\infty, \infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists.

Assume there is an integrable $g: \mathbb{R} \to [0, \infty]$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n \, d\mu = \int_{\mathbb{R}} f d\mu$$

Lemma (1.2.4) [1]:

Let $y \in (0, K_0)$. Then we have

$$\lim_{K\downarrow K_0} \frac{p_K(x)}{p_K(y)} = \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)}. \tag{26}$$

uniformly for all $x \in (0, K_0)$.

Proof. Fix $y \in (0, K_0)$. We begin with showing that the asymptotics hold uniformly for all $x \in (0, y)$.

Let $\boldsymbol{\mathcal{T}}$ denote the set of labels of particles realised in (X, P_x^K). Define T_y as the set containing all particles which are the first ones in their genealogical line to exit the strip $(0, y)$, i.e.

$$T_y = \{u \in \boldsymbol{\mathcal{T}}: \exists s \in [\tau_u, \sigma_u] \text{ s.t. } x_u(s) \notin (0, y)$$

$$\text{and } x_v(\tau) \in (0, y) \text{ for all } v < u, r \in [\tau_v, \sigma_v]\},$$

where $v < u$ means that v is a strict ancestor of u. Further, for $u \in T_y$ denote by T_y^u the first exit time of u from $(0, y)$. The random set T_y is a stopping line.

Since $y \in (0, K_0)$ the width of the strip $(0, y)$ is subcritical and hence, for any initial position $x \in (0, y)$, all particles will exit it eventually which ensures that T_y is a dissecting stopping line. Now let $|T_y|$ be the number of particles which are the first ones in their line of descent to hit y, which can be written as

$$|T_y| = \sum_{u\in T_y} \mathbf{1}_{\{x_u(T_y^u)=y\}}.$$

Recall from Remark (1.1.6) that $\left(\prod_{u\in N_t}\left(1-p_K\left(x_u(t)\right)\right), t\geq 0\right)$ is a P_x^K - martingale. Since T_y is dissecting it follows that we can stop the martingale at T_y and obtain, for $x\in(0,y)$,

$$1-p_K(x)=E_x^K\left(\prod_{u\in T_y}\left(1-p_K\left(x_u(T_y^u)\right)\right)\right)=E_x^K\left(\left(1-p_K(y)\right)^{|T_y|}\right), \quad (27)$$

where we have used that the process started at zero becomes extinct immediately, i.e. $p_K(0)=0$. Further $|T_y|$ has the same distribution under P_x^K and $P_x^{K_0}$ since we consider particles stopped at level y below K_0 and thus we can replace E_x^K by $E_x^{K_0}$ on the right-hand side above.

Now, using first (27) and then the geometric sum $\sum_{j=0}^{n-1}a^j=\frac{1-a^n}{1-a}$, we get

$$\frac{p_K(x)}{p_K(y)}=E_x^{K_0}\left(\frac{1-\left(1-p_K(y)\right)^{|T_y|}}{1-\left(1-p_K(y)\right)}\right)=E_x^{K_0}\left(\sum_{j=0}^{|T_y|-1}\left(1-p_K(y)\right)^j\right). \quad (28)$$

The sum on the right-hand side is dominated by $|T_y|$ which does not depend on K and has finite expectation (which will shortly be shown below). We can therefore apply the Dominated convergence theorem to the right-hand side in (28) and we conclude that

$$\lim_{K\downarrow K_0} E_x^{K_0}\left(\sum_{j=0}^{|T_y|-1}\left(1-p_K(y)\right)^j\right)$$

$$=E_x^{K_0}\left(\sum_{j=0}^{|T_y|-1}\lim_{K\downarrow K_0}\left(1-p_K(y)\right)^j\right)=E_x^{K_0}\left(|T_y|\right), \quad (29)$$

where the convergence holds point-wise in $x\in(0,y)$. In order to get uniform convergence we observe the following.

We set $\varphi(x,K)=E_x^{K_0}\left(\sum_{j=0}^{|T_y|-1}\left(1-p_K(y)\right)^j\right)$, for $x\in[0,y]$ (with the convention that the P^K-branching diffusion becomes extinct immediately for initial position

$x = 0$ respectively stopped for $x = y$) and denote by $\varphi(x) = E_x^{K_0}(|T_y|)$ its point-wise limit. Since $1 - p_K(y) \leq 1 - p_{K'}(y)$, for $K \geq K'$, we have $\varphi(x, K) \leq \varphi(x, K')$ and thus, for any $x \in [0, y]$, the sequence $\varphi(x, K)$ is monotone increasing as $K \downarrow K_0$. Moreover the functions $\varphi(x, K)$ and $\varphi(x)$ are continuous in x, for any K. In conclusion, we have an increasing sequence of continuous functions on a compact set with a continuous point-wise limit and therefore the convergence also holds uniformly in $x \in [0, y]$.

Combining (28) and (29) and the uniformity argument, we arrive at

$$\lim_{K \downarrow K_0} \frac{p_K(x)}{p_K(y)} = E_x^{K_0}(|T_y|), \tag{30}$$

where, for fixed y, the convergence holds uniformly in $x \in (0, y)$. Now let $\tau_\xi := \inf\{t > 0 : \xi_t \in (0, y)\}$ be the first time a Brownian motion ξ with drift $-\mu$ exists the interval $(0, y)$. Since T_y is dissecting it follows that we can apply the Many-to-one Lemma for the stopping line T_y. This gives

$$E_x^{K_0}(|T_y|) = \mathbb{Q}_x^{K_0}\left(\frac{\sin(\pi x/K_0)\, e^{\mu x}}{\sin\left(\pi \xi_{\tau_\xi}/K_0\right) e^{\mu \xi_{\tau_\xi} + \left(\mu^2/2 + \pi^2/2K_0^2\right)\tau_{\xi_y}}} e^{(m-1)\beta \tau_{\xi_y}}, \mathbf{1}_{\left(\xi_{\tau_{\xi_y}} = y\right)}\right)$$

$$= \frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)} e^{\mu(x-y)} \mathbb{Q}_x^{K_0}\left(\xi_{\tau_{\xi_y}} = y\right),$$

where we have used that $(m-1)\beta - \frac{\mu^2}{2} - \pi^2/2K_0^2 = 0$ (and $\mathbb{Q}_x^{K_0}$ is used as an expectation operator). Under $\mathbb{Q}_x^{K_0}, \xi$ will never hit 0 since it is conditioned to stay in $(0, K_0)$. However as ξ is positive recurrent it will eventually cross y and therefore $\mathbb{Q}_x^{K_0}\left(\xi_{\tau_{\xi_y}} = y\right) = 1$. This proves our earlier claim that $|T_y|$ has finite expectation and together with (30) it completes the argument.

For uniformity for all $x \in (0, K)$, it remains to show that (26) also holds uniformly for $x \in (y, K_0)$. Instead of approaching criticality by taking the limit in K we can now fix a $K > K_0$ and consider a (supercritical) strip (z, K) and let $z \uparrow z_0$ where $z_0 := K - K_0$. Denote by $p_{(z,K)}(x + z)$ the probability of survival in the strip (z, K) when starting from $x + z$. We then have

$$\lim_{K\downarrow K_0}\frac{p_K(x)}{p_K(y)}=\lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)}.$$

Hence (26) is equivalent to showing that, uniformly for $x \in (y, K_0)$,

$$\lim_{z\uparrow z_0}\frac{p_{(z,K)}(x+z)}{p_{(z,K)}(y+z)}=\frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)}e^{\mu(x-y)}.$$

Then consider the stopping line containing all particles which exit the strip $(y+z, K)$ and accordingly the set of particles which are the first in their genealogical line to exit $(y+z, K)$ at y. Noting that the latter has the same law under $P_{x+z}^{z,K}$ and $P_{x+z}^{z_0,K}$, we can then repeat the argument in the first part.

Proposition (1.2.4) [1]:

Uniformly for all $x \in (0, K_0)$,

$$p_K(x) \sim c_K \sin(\pi x/K_0)\, e^{\mu x}, \quad as\ K \downarrow K_0,$$

where c_K is independent of x and $c_K \downarrow 0$ as $K \downarrow K_0$.

Proof. Choose a $y \in (0, K_0)$. Then an application of Lemma (1.2.3) gives, as $K \downarrow K_0$,

$$p_K(x)=p_K(y)\frac{p_K(x)}{p_K(y)}\sim p_K(y)\frac{\sin(\pi x/K_0)}{\sin(\pi y/K_0)}e^{\mu(x-y)}=c_K\sin(\pi x/K_0)\,e^{\mu x},$$

uniformly for all $x \in (0, K_0)$, where $c_K := \frac{p_K(y)}{\sin(\pi y/K_0)}e^{-\mu y}$. By Proposition (1.1.3), $c_K \downarrow 0$ as $K \downarrow K_0$ which completes the proof.

Step (i) (The growth rate of the backbone) Consider a process $Y^B = (Y_t^B, t \geq 0)$ performing the single particle motion of the backbone, that is according to the infinitesimal generator L_K^B which is given in Proposition (1.1.11) as

$$L_K^B=\frac{1}{2}\frac{d^2}{dx^2}-\left(\mu-\frac{p_K'}{p_K}\right)\frac{d}{dx} \quad on \quad (0,K),$$

with domain $C^2(0, K)$. Let Π_K^B be the invariant density for L_K^B, i.e. the positive solution of $\tilde{L}_K^B \Pi_K^B = 0$ where $\tilde{L}_K^B$ is the formal adjoint of L_K^B.

Then

$$\Pi_K^B(x) \propto p_K(x)^2 e^{-2\mu x}, \qquad x \in (0,K).$$

For $t \geq 0$, we define $\Gamma(t,A) = \int_0^t \mathbf{1}_{\{Y_s^B \in A\}} ds$, $A \subset [0,K]$A $\subset$ [0,K], to be the occupation time up to t of Y^B in A. Then large deviation theory suggests that the probability that the measure $t^{-1}\Gamma(t,\cdot)$ is 'close' to $\int_0^K \mathbf{1}_{\{\cdot\}}(y) f^2(y) \Pi_K^B(y) dy$ should be roughly

$$\exp\left\{t \int_0^K L_K^B f(y) f(y) \Pi_K^B(y) dy\right\}.$$

Now, as each particle in the backbone moves according to L_K^B, we guess that the expected number of particles at time t with occupation density like $f^2 \Pi_K^B$ is very roughly

$$\exp\left\{t \int_0^K \left(L_K^B + F_K^{B'}(1,y)\right) f(y) f(y) \Pi_K^B(y) dy\right\}, \tag{31}$$

where

$$F_K^{B'}(1,x) \coloneqq \frac{d}{ds} F^B(s,x)|_{s=1} = (m-1)\beta + \frac{F\left(1-p_K(x)\right)}{p_K(x)}, \quad x \in (0,K).$$

The expected growth rate of the blue tree is given by maximising the integral appearing in (31) over all f with $\int_0^K f^2(x) \Pi_K^B(x) dx = 1$. We can compute this optimal function f^* explicitly as the normalised eigenfunction corresponding to the largest eigenvalue λ where

$$\left(L_K^B + F_K^{B'}(1,x)\right) f^*(x) = \lambda f^*(x) \quad \text{ in } (0,K), \tag{32}$$

and we find that, in fact, $\lambda = \lambda(K) = (m-1)\beta - \mu^2/2 - \pi^2/2K^2$ and

$$f^*(x) \propto \frac{\sin(\pi x/K)}{p_K(x)} e^{\mu x}, \quad x \in (0,K), \tag{33}$$

up to a normalising constant. Then we find the 'optimal' occupation density as

$$\Pi_K^{B,*}(x) \coloneqq \left(f^*(x)\right)^2 \Pi_K^B(x) = \frac{2}{K} \sin^2(\pi x/K), \quad x \in (0,K).$$

In summary, we guess that the expected growth rate of the number of particles in the blue tree is $\lambda(K)$ and that

$$\lambda(K) = \int_0^K \left(L_K^B + F_K^{B'}(1, y) \right) f^*(y) f^*(y) \Pi_K^B(y) dy. \qquad (34)$$

We would anticipate that the a.s. growth rate is also $\lambda(K)$ in agreement with the expected growth rate.

Step (ii)(Upper bound on $\boldsymbol{\lambda}(\boldsymbol{K})$) The term $L_K^B f^* f^*$ is non-positive as it represents the cost of spending time like $(f^*)^2 \Pi_K^B(x)$, hence omitting it will give an upper bound for $\lambda(K)$, that is

$$\int_0^K F_K^{B'}(1, y) \Pi_K^B(y) dy \leq \lambda(K)$$

Step (iii)(Lower bound on $\boldsymbol{\lambda}(\boldsymbol{K})$) By taking $f = 1$ in (34), we get a lower bound on $\lambda(K)$ since f^* maximizes the expression in (31). Thus

$$\lambda(K) \leq \int_0^K F_K^{B'}(1, y) \Pi_K^{B,*}(y) dy.$$

Step (iv) (Asymptotics) By Theorem (1.2.1), $p_K(x) \sim c_K \sin(\pi x / K_0)\, e^{\mu x}$, as $K \downarrow K_0$, and we can easily deduce that $\Pi_K^B(x) \sim \Pi_{K_0}^{B,*}(x)$, as $K \downarrow K_0$.

We will make rigorous later that $F_K^{B'}(1, x) \sim (m-1) \beta c_K \sin(\pi x / K_0)\, e^{\mu x}$ as $K \downarrow K_0$. Our conjecture is therefore that

$$\lambda(K) \sim c_K \frac{2\beta}{K_0} \int_0^{K_0} \sin^3(\pi y / K_0)\, e^{\mu y} dy, \quad \text{as } K \downarrow K_0.$$

Since we can calculate the integral explicitly this gives an exact asymptotic for c_K which agrees with the one given in Proposition (1.2.9) and Theorem (1.2.1).

Lemma (1.2.5) [1]:

The function f^* is uniformly bounded in $(0, K)$.

Proof. The function f^* is continuous in $(0, K)$ and it is therefore sufficient to show that $\limsup_{x \downarrow 0} f^*(x)$ and $\limsup_{x \uparrow K} f^*(x)$ are bounded.

An application of L'Hôpital's rule gives

$$\lim_{x\downarrow 0}\frac{\sin(\pi x/K)\,e^{\mu x}}{\frac{\pi}{-2K\mu}(e^{-2\mu x}-1)}=1 \tag{35}$$

To conclude that $\limsup_{x\downarrow 0} f^*(x)<\infty$, it therefore suffices to show that there exists a constant $c>0$ such that

$$c\,\tfrac{1}{-2\mu}(1-e^{2\mu x})\leq p_K(x),\ \text{ for all } x \text{ sufficiently close to zero.}$$

By Remark (1.1.6), $\left(\prod_{u\in N_t}\left(1-p_K\left(x_u(t)\right)\right),t\geq 0\right)$ is a $P_{\boldsymbol{x}}^K$-martingale and it follows then by a standard Feynman-Kac argument that $1-p_K(x)$ satisfies

$$1-p_K(x)=1+\mathbb{E}_{\boldsymbol{x}}^K\int_0^{\tau_{(0,K)}}F\big(1-p_K(\xi_s)\big)ds\,,\quad x\in(0,K),$$

where $\tau_{(0,K)}$ is the first time ξ exists the interval $(0,K)$. To compute the expectation above we use the potential density of ξ, and we get

$$\begin{aligned}-p_K(x)&=\mathbb{E}_{\boldsymbol{x}}^K\int_0^{\tau_{(0,K)}}F\big(1-p_K(\xi_s)\big)ds\\&=\frac{1}{-\mu}(e^{-2\mu x}-1)\int_0^K F\big(1-p_K(y)\big)\frac{\left(e^{-2\mu(K-y)}-1\right)}{(e^{-2\mu K}-1)}dy\\&\quad+\frac{1}{\mu}\int_0^K F\big(1-p_K(y)\big)\left(e^{-2\mu(x-y)}-1\right)dy.\end{aligned} \tag{36}$$

Since $F(s)<0$ for $0<s<1$, the first integral in the last equality on the right-hand side of (36) is strictly negative. Regarding boundedness of this integral it is clear that the integrand is bounded for y near K. By an application of L'Hôpital's rule it follows that the integrand is also bounded near 0. Hence we can set

$$c:=-\int_0^K F\big(1-p_K(y)\big)\frac{\left(e^{-2\mu(K-y)}-1\right)}{(e^{-2\mu K}-1)}dy>0.$$

With the second integral in the last equality on the right-hand side of (36) being non-positive, for x close to 0, we get

$$p_K(x)\geq 2c\,\tfrac{1}{-2\mu}(e^{-2\mu x}-1),\ \text{ for all } x \text{ sufficiently close to zero.}$$

which, by (35), gives the desired result. To establish boundedness as x approaches K, we observe that $p_K(x) = \bar{p}_K(K-x)$, where $\bar{p}_K$ denotes the survival probability for a branching diffusion which evolves as under $P_{\boldsymbol{x}}^K$ but with positive drift μ. Similarly to the previous argument we can then show that there exists a constant $c > 0$ such that $c\bar{p}_K(K-x) \geq \sin(\pi x/K)\, e^{\mu x}$, for x sufficiently close to K, which finishes the proof.

Proposition (1.2.6) [1]:

For $x \in (0, K)$,

$$\lim_{t\to\infty} \frac{1}{t} \log|N_t| = \lambda(K), \qquad \boldsymbol{P}_{\boldsymbol{x}}^{B,K} - \text{a.s.}$$

Step (i) of the heuristic suggests that the growth rate of the backbone is $\lambda(K)$ and, moreover, that it can be expressed as (34). The idea of the rigorous proof of Proposition **1.7** is now to construct a martingale of the form in (3) with $\widehat{\Upsilon}$ built from f^* in (33). We then find upper and lower bounds for this martingale which will, in turn, give bounds on the growth of the number of particles. Before we do this we prove an auxiliary result on the boundedness of f^*.

Proof. We proof the upper bound by contradiction. Recall the embedding procedure described. Choose ε small enough such that $\lambda(K+2\varepsilon) > 0$. Choose a $\delta > 0$ and suppose that there exists an increasing (random) sequence $t_n, n = 1,2,\ldots$, which tends to infinity, such that $\log N_{t_n}$ is bigger than $(\lambda(K+2\varepsilon)+\delta)t_n$, for any $n \in \mathbb{N}$ under P^K. Then, under $P^{(\varepsilon,K+\varepsilon)}$,

$$Z^{(\varepsilon,K+\varepsilon)} := \sum_{u\in N_t} \sin\big(\pi(x_u(t)+\varepsilon)/(K+2\varepsilon)\big)\, e^{-\lambda(K+2\varepsilon)t} e^{\mu(x_u(t)+\varepsilon)}$$

$$\geq \big|N_t|_{(0,K)}\big| e^{-\lambda(K+2\varepsilon)t} \times \sin\big(\pi\varepsilon/(K+2\varepsilon)\big).$$

Since we assumed that, along the sequence $t_n, n = 1,2,\ldots$, the number of particles $\left|N_{t_n}|_{(0,K)}\right|$ is bounded from below by $\exp\{(\lambda(K+2\varepsilon)+\delta)t_n\}$, the right-hand side above tends to infinity along this sequence as $n \to \infty$ which contradicts Proposition (1.1.3). As we can take ε and δ arbitrary small we obtain $\limsup_{t\to\infty}(\lambda(K)t)^{-1}\log|N_t| \leq 1$, under P^K. By the thinning argument in Corollary (1.1.10), we immediately get that $\lambda(K)$ is also an upper bound for the growth rate of $|N_t|$ under $\boldsymbol{P}^{B,K}$.

For the lower bound, as alluded to above, we begin with constructing a $\boldsymbol{P}^{B,K}$-martingale of the form (3). Since f^* satisfies $\left(L_K^B + F_K^{B'}(1) - \lambda(K)\right) f^* = 0$, it follows by an application of Itô's formula that

$$f^*(\xi_t) e^{\int_0^t \left(F'(\xi_s,1) - \lambda(K)\right) ds}, \qquad t \geq 0$$

is a martingale with respect to $\sigma(\xi_t, t \geq 0)$, where $(\xi, \mathbb{P}^{B,K})$ is an L_K^B-diffusion. Appealing to the discussion in Remark (1.1.2) we then see that

$$M_{f^*}(t) = \sum_{u \in N_t} f^*\big(x_u(t)\big) e^{-\lambda(K)t}, \qquad t \geq 0,$$

is a $\boldsymbol{P}_x^{B,K}$ -martingale.

The proof of $L^1\left(\boldsymbol{P}_x^{B,K}\right)$ -convergence of M_{f^*} follows by a classical spine decomposition argument, and is therefore omitted.

$L^1\left(\boldsymbol{P}_x^{B,K}\right)$ -convergence implies then that $\boldsymbol{P}_x^{B,K}\left(M_{f^*}(\infty) > 0\right) > 0$. Now set $g(x) \coloneqq \boldsymbol{P}_x^{B,K}\left(M_{f^*}(\infty) = 0\right)$, for $x \in (0, K)$. Then the product

$$\pi^g(t) = \prod_{u \in N_t} g\big(x_u(t)\big), \quad t \geq 0,$$

is a $\boldsymbol{P}_x^{B,K}$ -martingale with almost sure $\lim \mathbf{1}_{\left\{M_{f^*}(\infty)=0\right\}}$ (cf. proof of Proposition (1.1.3)). Therefore we have

$$g(x) = \boldsymbol{E}_x^{B,K}\left(\pi^g(t)\right) \leq \mathbb{E}_{\boldsymbol{x}}^{B,K}\big(g(\xi_t)\big), \text{ for all } x \in (0, K).$$

Hence we conclude that the process $(g(\xi_t), t \geq 0)$ is a $[0,1]$ -valued $\mathbb{P}_{\boldsymbol{x}}^{B,K}$-submartingale and it converges $\mathbb{P}_{\boldsymbol{x}}^{B,K}$-a.s. to a limit g_∞. However ξ is positive recurrent under $\mathbb{P}_{\boldsymbol{x}}^{B,K}$ and thus $g(\xi_t)$ can only converge if it is constant, hence $g(x) = g_\infty$ for all $x \in (0, K)$. Since $0 \leq g \leq 1$ we then have $g_\infty \in [0,1]$. Assume now that $g_\infty \in [0,1)$. Note that, under $\boldsymbol{P}_x^{B,K}$, $|N_t|$ tends to infinity as $t \to \infty$ since each particle in $\left(X, \boldsymbol{P}_x^{B,K}\right)$ is replaced by at least two offspring when it dies and there is no killing. Thus we get $\pi^g(t) \to 0$, $\boldsymbol{P}_x^{B,K}$-a.s. and therefore $g(x) = \boldsymbol{E}_x^{B,K}\left(\pi^g(\infty)\right) = 0$. In conclusion, g is identical to either 0 or 1. But we already know that the martingale limit $M_{f^*}(\infty)$ is strictly positive with positive probability and consequently, $g(x) = \boldsymbol{P}_x^{B,K}\left(M_{f^*}(\infty) = 0\right) = 0$, for all $x \in (0, K)$.

We may now conclude that $\lim\inf_{t\to\infty} \log M_{f^*}(t)/\lambda(K)t \geq 0$ $\boldsymbol{P}_x^{B,K}$-a.s. We look for an upper bound on $M_{f^*}(t)$ which will, in turn, provide a lower bound on $|N_t|$ under $\boldsymbol{P}_x^{B,K}$. By Lemma (1.2.5), f^* is bounded by a constant $c > 0$ in $(0, K)$. Thus, under $\boldsymbol{P}_x^{B,K}$, for $t \geq 0$,

$$M_{f^*}(t) \leq c|N_t|e^{-\lambda(K)t},$$

and we see that, $\boldsymbol{P}_x^{B,K}$-a.s.,

$$\lim_{t\to\infty}\inf \frac{\log|N_t|}{\lambda(K)t} \geq \lim_{t\to\infty}\inf \frac{\log M_{f^*}(t) - \log c + \lambda(K)t}{\lambda(K)t} \geq 1,$$

which completes the proof.

In **step (ii)** of the heuristic we claimed that $(L_K^B f^*)f^*$ is non-positive to get an upper bound on $\lambda(K)$ which is essentially what we will now do. Recall that the invariant density for the infinitesimal generator L_K^B respectively the optimal occupation density of $\left(X, \boldsymbol{P}_x^{B,K}\right)$ are given by

$$\Pi_K^B(y) = \frac{p_K(y)^2 e^{-2\mu y}}{\int_0^K p_K(z)^2 e^{-2\mu z} dz}$$

and

$$\Pi_K^{B,*}(y) = \left(f^*(y)\right)^2 \Pi_K^B(y) = \frac{2}{K}\sin^2(\pi y/K). \tag{37}$$

Lemma (1.2.7) [1]:

For $K > K_0$, we have

$$\lambda(K) \leq \int_0^K F_K^{B'}(1, y)\Pi_K^{B,*}(y)dy.$$

Proof. We have $\int_0^K \left(f^*(y)\right)^2 \Pi_K^B(y)dy = \int_0^K 2/K \sin^2(\pi x/K)\, dx = 1$. Then multiplying by $\lambda(K)$ gives

$$\lambda(K) = \int_0^K \lambda(K) f^*(y) f^*(y) \Pi_K^B(y) dy$$

Recall that f^* is given by (33) and satisfies the ODE in (32). Thus we can replace the term $\lambda(K)f^*$ above by $\left(L_K^B + F_K^{B'}(1,y)\right) f^*(y)$. Therefore

$$\lambda(K) = \int_0^K \left(\frac{1}{2}\left(f^*(y)\right)'' - \left(\mu - \frac{p_K(y)'}{p_K(y)}\right)\left(f^*(y)\right)'\right) f^*(y)\Pi_K^B(y)dy$$
$$+ \int_0^K F_K^{B'}(1,y)\left(f^*(y)\right)^2 \Pi_K^B(y)dy. \qquad (38)$$

Noting that $\Pi_K^{B,*}(y) = \left(f^*(y)\right)^2 \Pi_K^B(y)$, the result then follows if we can show that the first integral in (38) is non-positive.

We use integration by parts for the first term in the first integral in (38) to get

$$\int_0^K \frac{1}{2}\left(f^*(y)\right)'' f^*(y)\Pi_K^B(y)dy$$
$$= \frac{1}{2}\left(\left[\left(f^*(y)\right)' f^*(y)\Pi_K^B(y)\right]_0^K - \int_0^K \left(f^*(y)\right)' \left(\left(f^*(y)\right)'\Pi_K^B(y)dy + f^*(y)\left(\Pi_K^B(y)\right)' dy\right)\right). \quad (39)$$

We want to show that the first term on the right-hand side above is zero. By Lemma (1.2.5), f^* takes a finite value at 0 and K and hence it suffices to show that $(f^*)'\Pi_K^B$ evaluated at 0 and K is zero. By simply differentiating f^* and recalling that $\Pi_K^B(y) \propto p_k(y)^2 e^{-2\mu y}$ we get

$$\left(f^*(y)\right)'\Pi_K^B(y)$$
$$\propto e^{-\mu y}\left(\left(\mu \sin(\pi y/K) + \frac{\pi}{K}\cos(\pi y/K)\right) p_K(y) - \sin(\pi y/K)\, p_K'(y)\right).$$

Differentiating both sides of equation (1.12) with respect to x, it is easily seen that $p_K'(y)$ is bounded for all $x \in [0, K]$. Therefore $\left(f^*(y)\right)'\Pi_K^B(y)$ is equal to 0 at 0 and K and thus the first term on the right-hand side of (39) vanishes.

The first integral in (38) now becomes

$$\int_0^K \left(\frac{1}{2}\big(f^*(y)\big)'' - \left(\mu - \frac{p_K'(y)}{p_K(y)}\right)\big(f^*(y)\big)'\right) f^*(y)\Pi_K^B(y)dy$$

$$= -\frac{1}{2}\int_0^K \left(\left(\big(f^*(y)\big)'\right)^2 - \big(f^*(y)\big)' f^*(y)\big(\Pi_K^B(y)\big)'\right)\Pi_K^B(y)dy$$

$$- \int_0^K \left(\mu - \frac{p_K'(y)}{p_K(y)}\right) f^*(y)\big(f^*(y)\big)'\Pi_K^B(y)dy. \quad (40)$$

Differentiating Π_K^B gives $(\Pi_K^B)' = -2\left(\mu - \frac{p_K'}{p_K}\right)\Pi_K^B$. Thus, in the righthand side of (40), the second term in the first integral cancels with the second integral and we arrive at

$$\int_0^K \left(\frac{1}{2}\big(f^*(y)\big)'' - \left(\mu - \frac{p_K'(y)}{p_K(y)}\right)\big(f^*(y)\big)'\right) f^*(y)\Pi_K^B(y)dy$$

$$= -\frac{1}{2}\int_0^K \left(\big(f^*(y)\big)'\right)^2 \Pi_K^B(y)dy,$$

which is less than or equal to zero and the proof is complete.

Step (iii) of the heuristic claims that we can lower bound $\lambda(K)$ by replacing f^* in (33) with $f = 1$. This suggests to modify the martingale argument in the proof of Proposition (1.2.6) using a martingale of the form (3) with $\widehat{\Upsilon}$ built from the constant function **1**.

Lemma (1.2.8) [1]:

For $x \in (0, K)$,

$$\lambda(K) \geq \int_0^K F_K^{B'}(1, y)\Pi_K^{B,*}(y)dy.$$

Proof. The constant process $\hat{\Upsilon} = \big(\hat{\Upsilon}(t) = 1, t \geq 0\big)$ is a trivial martingale with respect to $\sigma(\xi_t, t \geq 0)$, where $(\xi, \boldsymbol{P}^{B,K})$ is an L_K^B-diffusion.

Thus according to Remark (1.1.2), the process $M_1 = (M_1(t) = 1, t \geq 0)$ defined by

$$M_1(t) = \sum_{u \in N_t^B} e^{-\int_0^t F_K^{B'}(1, x_u(s))ds}, \qquad t \geq 0,$$

is a $\boldsymbol{P}_x^{B,K}$-martingale. $L^1(\boldsymbol{P}_x^{B,K})$-convergence and uniform integrability of M_1 again follow by a spine decompisition argument in the manner of the proof of Theorem (1.1.5). $L^1(\boldsymbol{P}_x^{B,K})$-convergence implies that $\boldsymbol{P}_x^{B,K}(M_1(\infty) > 0) > 0$ and repeating the argument in the proof of Proposition (1.2.6) we immediately see that $\boldsymbol{P}_x^{B,K}(M_1(\infty) > 0) = 1$.

Therefore, we conclude that $\liminf_{t\to\infty} \log M_1(t)/\lambda(K)t \geq 0$, $\boldsymbol{P}_x^{B,K}$-a.s. On the filtration $\mathcal{F}_t = \sigma(X_s, s \leq t)$ we define a change of measure by

$$\left.\frac{dP_x^{M_1,K}}{d\boldsymbol{P}_x^{B,K}}\right|_{\mathcal{F}_t} = \frac{M_1(t)}{M_1(0)}, \qquad t \geq 0,$$

Since the martingale M_1 is of the form in (3), it induces as spine decomposition which, according to Remark (1.1.2), reads as follows. Under $P_x^{M_1,K}$, the spine ξ is an L_K^B-diffusion and along its path we immigrate independent copies of the $\boldsymbol{P}^{B,K}$-branching diffusion (we do not need to specify the rate of immigration and the distribution of the number of immigrants since they will not be relevant).

Next, fix an $\varepsilon > 0$ and define, for $t \geq 0$ and each $u \in N_t^B$, the set

$$A_t^u = \left\{\left|\frac{1}{t}\int_0^t F_K^{B'}(1, x_u(s))ds - \int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy\right| < \varepsilon\right\}$$

and consider the process we obtain from M_1 by considering the particles in A_t^u only, that is

$$\widetilde{M}_1(t) = \sum_{u\in N_t^B} \mathbf{1}_{A_t^{u,\varepsilon}} e^{-\int_0^t F_K^{B'}(1,x_u(s))ds}, \qquad t \geq 0.$$

Let A_t^ξ be the event we get if we simply replace $x_u(t)$ by the spine process ξ_t in the definition of A_t^u. Since $(\xi, P^{M_1,K})$ has invariant density Π_K^B we have $\mathbf{1}_{A_t^\xi} \to 1$ $P_x^{M_1,K}$-a.s. $\widetilde{M}_1$ therefore has the same limit as M_1 under $P_x^{M_1,K}$, and moreover, since M_1 is

uniformly integrable, this also holds true under $\boldsymbol{P}_x^{B,K}$. In particular we have

$$\liminf_{t\to\infty}\frac{\log \widetilde{M}_1(t)}{\lambda(K)t} = \liminf_{t\to\infty}\frac{\log M_1(t)}{\lambda(K)t} \geq 0, \qquad \boldsymbol{P}_x^{B,K}\text{-a.s.}$$

For $t \geq 0$, we now get an upper bound for $\widetilde{M}_1(t)$ under $\boldsymbol{P}_x^{B,K}$ by

$$\widetilde{M}_1(t) \leq |N_t| e^{-t\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy - \varepsilon}.$$

Consequently, $\boldsymbol{P}_x^{B,K}$-a.s.,

$$\liminf_{t\to\infty} \frac{\log|N_t|}{t\left(\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy\right)}$$
$$\geq \liminf_{t\to\infty} \frac{\log \widetilde{M}_1(t) + t\left(\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy - \varepsilon\right)}{t\left(\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy\right)}$$
$$\geq \frac{\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy - \varepsilon}{\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy},$$

and taking $\varepsilon \downarrow 0$ gives the result.

Proposition (1.2.9) [1]:

The constant c_K in Proposition (1.2.4) satisfies

$$c_K \sim (K - K_0)\frac{(K_0^2\mu^2 + \pi^2)(K_0^2\mu^2 + 9\pi^2)}{12(m-1)\beta\pi K_0^3(e^{\mu K_0} + 1)} \quad as\ K \downarrow K_0, \qquad (41)$$

Theorem (1.2.1) then follows by defining C_K to be the expression on the left-hand side in (41).

We will provide entirely probabilistic proofs of the results above. We remark that, although it would take some effort to make rigorous, it is also possible to recover the asymptotics of p_K and the explicit constant C_K in an analytic approach using a careful asymptotic expansion of the non-linear ODE $Lu + F(u) = 0$ with $u(0) = u(K) = 1$.

Proof. We will present the proof which gives an explicit expression for the constant c_K appearing in the asymptotics for the survival probability in Proposition (1.2.8). We outline a heuristic derivation of the explicit constant c_K in Proposition (1.2.9) which will give the intuition for the rigorous proofs presented subsequently.

Let us now come to the rigorous proof. Recall that the backbone $(X, \boldsymbol{P}^{B,K})$ is the process constructed in **steps (i)-(iii)** in Theorem (1.1.9), which was further characterised in Proposition (1.1.11). First, we need to confirm the conjecture that the number of particles in $\left(X, \boldsymbol{P}_{\boldsymbol{x}}^{B,K}\right)$ grows at rate $\lambda(K)$.

By Lemma (1.2.7) and (1.2.8), we get the following bounds on $\lambda(K)$

$$\int_0^K F_K^{B'}(1,y)\Pi_K^B(y)dy \leq \lambda(K) \leq \int_0^K F_K^{B'}(1,y)\Pi_K^{B,*}(y)dy, \tag{42}$$

where Π_K^B and $\Pi_K^{B,*}$ were defined in (37). By Proposition (1.1.8), we have, as $K \downarrow K_0$,

$$\Pi_K^B(y) = \frac{p_K(y)^2 e^{-2\mu y}}{\int_0^K p_K(z)^2 e^{-2\mu z} dz} \sim \frac{2}{K_0}\sin^2(\pi y/K) = \Pi_{K_0}^{B,*}(y), \tag{43}$$

where we have used that the asymptotics in Proposition (1.1.8) hold uniformly to deal with the integral in the denominator of the second term in (43). The uniformity in Proposition (1.1.8) also ensures that (43) holds uniformly for all $y \in (0, K_0)$. Further

$$\lim_{s\uparrow 1}\frac{F(s)}{s(s-1)} = \lim_{s\uparrow 1}\frac{\beta\left(\sum_{n\geq 2} q_n s^n - 1\right)}{s-1} = \lim_{s\uparrow 1}\beta\sum_{n\geq 2} q_n n s^{n-1} = (m-1)\beta,$$

where we applied L'Hôpitals rule in the second equality above. Then, together with Proposition (1.1.8), as $K \downarrow K_0$,

$$F_K^{B'}(1,y) = (m-1)\beta + \frac{F\left(1-p_K(y)\right)}{p_K(y)}$$

$$\sim (m-1)\beta p_K(y) \sim (m-1)\beta c_K \sin(\pi y/K_0)\, e^{\mu y}. \tag{44}$$

Note that $\frac{F\left(1-p_K(y)\right)}{p_K(y)} = -\frac{F(1)-F\left(1-p_K(y)\right)}{1-\left(1-p_K(y)\right)}$. Convexity of F yields then that $\left|\frac{F\left(1-p_K(y)\right)}{p_K(y)}\right|$ is bounded by $(m-1)\beta$. Thus $\left|F_K^{B'}(1,y)\right| \leq 2(m-1)\beta$ and we can appeal to bounded convergence as we take the limit in (42). With (43) and (44) we get

$$\lambda(K) \sim c_K \frac{2(m-1)\beta}{K_0}\int_0^{K_0}\sin^3(\pi y/K_0)\, e^{\mu y} dy, \qquad \text{as } K \downarrow K_0.$$

Evaluating the integral gives

$$\lambda(K) \sim c_K \frac{12(m-1)\beta\pi^3(e^{\mu K_0}+1)}{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}, \qquad \text{as } K \downarrow K_0.$$

Finally, $\lambda(K) \sim \pi^2(K-K_0)K_0^{-3}$ as $K \downarrow K_0$ which follows from the linearization

$$\lambda(K) = (m-1)\beta - \mu^2/2 - \pi^2/2K^2 = \frac{\pi^2}{2K_0^2} - \frac{\pi^2}{2K^2}$$

$$= \frac{\pi^2(K-K_0+K_0)^2}{2K_0^2K^2} - \frac{\pi^2K_0^2}{2K_0^2K^2} = \frac{\pi^2(K-K_0)}{2K_0K^2} - \frac{\pi^2(K-K_0)^2}{2K_0^2K^2}$$

and noting that the first term in the last line is the leading order term as $K \downarrow K_0$. This completes the proof.

Theorem (1.2.10) [1]:
Let $x \in (0, K_0)$. Consider a process $X^* = (X_t^*, t \geq 0)$ which evolves as follows. X^* is initiated from a single particle at x performing a Brownian motion conditioned to stay in $(0, K_0)$, i.e. a strong Markov process with infinitesimal generator

$$L_{K_0}^* = \frac{1}{2}\frac{d^2}{dx^2} + \frac{\pi/K_0}{\tan(\pi x/K_0)}\frac{d}{dx}, \tag{45}$$

defined for all $u \in C^2(0, K_0)$. Along its path we immigrate $\tilde{A}$ independent copies of (X, P^K) at rate $m\beta$ where $\tilde{A}$ has the size-biased offspring distribution $(\tilde{q}_k, k = 0, 1, \ldots)$ with

$$\tilde{q}_k = q_{k+1}\frac{k+1}{m}, \qquad k \geq 0.$$

Denote the law of X^* by P_x^*. Then, for any fixed time $T > 0$, the law of $(X_t, 0 \leq t \leq T)$ under the measure $\lim_{K \downarrow K_0} P_x^K(\cdot \mid \zeta = \infty)$ is equal to $(X_t^*, 0 \leq t \leq T)$ under P_x^*.

Proof. Recall that $(X, \boldsymbol{P}^{D,K})$ was defined as the process (X, P^K) conditioned on the event of survival and characterized via the change of measure in (13) and Theorem (1.1.8).

Fix a $K' > K_0$ and further denote by $N_t|_{(0,K)}$ the set of particles whose ancestors (including themselves) have not exited $(0, K)$ up to time t.

Then, for $K \leq K'$, and for $x \in (0, K_0)$ and $A \in \mathcal{F}_t$,

$$\lim_{K \downarrow K_0} \boldsymbol{P}_x^{D,K}(A) = E_x^{K'}\left(\mathbf{1}_A \lim_{K \downarrow K_0} \frac{1 - \prod_{u \in N_t|_{(0,K)}}\left(1 - p_K\left(x_u(t)\right)\right)}{p_K(x)}\right),$$

since $N_t|_{(0,K)}$ has the same law under P^K and $P^{K'}$. Suppose the particles in $N_t|_{(0,K)}$ are ordered, for instance according to their spatial positions, and we write $u_1, \ldots, u_{N_t|_{(0,K)}}$. We can now expand the term within the expectation on the right-hand side as

$$\frac{1 - \prod_{u \in N_t|_{(0,K)}} \left(1 - p_K\left(x_u(t)\right)\right)}{p_K(x)}$$

$$= \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K\left(x_{u_i}(t)\right)}{p_K(x)} \prod_{j<i} \left(1 - p_K\left(x_{u_j}(t)\right)\right) \quad (46)$$

which is bounded from above by $\left|N_t|_{(0,K)}\right| \left(p_K(x)\right)^{-1}$. Recall the asymptotics for p_K in Theorem (1.2.1) and in particular Lemma (1.2.3), noting that these results hold uniformly in $(0, K_0)$. Since $\left|N_t|_{(0,K)}\right|$ has finite expectation, we can apply the Dominated convergence theorem to the expression in (46), and we get

$$E^{K'}\left(\mathbf{1}_A \lim_{K \downarrow K_0} \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{p_K\left(x_{u_i}(t)\right)}{p_K(x)} \prod_{j<i} \left(1 - p_K\left(x_{u_j}(t)\right)\right)\right)$$

$$= E^{K_0}\left(\mathbf{1}_A \sum_{i=1}^{|N_t|_{(0,K)}|} \frac{\sin\left(\pi x_{u_i}(t)/K_0\right) e^{\mu x_{u_i}(t)}}{\sin(\pi x/K_0)\, e^{\mu x}}\right).$$

Hence, for $A \in \mathcal{F}_t$, we arrive at

$$\lim_{K \downarrow K_0} E_x^{D,K}(A) = E_x^{K_0}\left(\mathbf{1}_A \lim_{K \downarrow K_0} \frac{\sum_{u \in N_t} \sin\left(\frac{\pi x_u(t)}{K_0}\right) e^{\mu x_u(t)}}{\sin\left(\frac{\pi x}{K_0}\right) e^{\mu x}}\right)$$

$$= E_x^{K_0}\left(\mathbf{1}_A \lim_{K \downarrow K_0} \frac{Z^{K_0}(t)}{Z^{K_0}(0)}\right),$$

where Z^{K_0} is the martingale used in the change of measure in (5). The evolution under this change of measure is described in the paragraph following (5) and agrees with that of (X^*, P_x^*) as defined in Theorem (1.2.10).

Recall from (1) that the infinitesimal generator L is given as $L = \frac{1}{2}\frac{d^2}{dx^2} - \mu\frac{d}{dx}$, $x \in (0, K_0)$, defined for all functions $u \in C^2(0,K)$ with $u(0+) = u(K-) = 0$. Changing the domain to $u \in C^2(0,K)$ with $u''(0+) = u''(K-) = 0$, then L corresponds to Brownian motion with absorption (instead of killing) at 0 and K. For technical reason, we will assume from now on that $\boldsymbol{\mathcal{P}}^K = \{\boldsymbol{\mathcal{P}}_t^K, t \geq 0\}$ is the corresponding diffusion semi-group of Brownian motion with absorption and is therefore conservative. Note that all the results presented for branching Brownian motion with killing at 0 and K also hold in the setting of absorption at 0 and K when we restrict the process with absorption to particles within $(0,K)$, in particular when defining N_t as the number of particles who are alive and have not been absorbed at time t.

Suppose $Y = \{Y_t, t \geq 0\}$ is a Super-Brownian motion with associated semi-group $\boldsymbol{\mathcal{P}}^K$ and branching mechanism ψ of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_0^\infty \left(e^{-\lambda y} - 1 + \lambda y\right)\Pi(dy), \quad \lambda \geq 0,$$

where $\alpha = -\psi'(0+) \in (0,\infty)$, $\beta \geq 0$ and Π is a measure concentrated on $(0,\infty)$ satisfying $\int_{(0,\infty)}(x \wedge x^2)\Pi(dx) < \infty$. For an initial configuration $\eta \in \mathcal{M}_f(0,K)$, the space of finite measures supported on $(0,K)$, we denote the law of Y by $\tilde{P}_\eta^K$.

Since $\alpha = -\psi'(0+) > 0$, the function ψ is the branching mechanism of a supercritcal continuous-state branching process (CSBP), say Z. We assume henceforth that ψ satisfies the non-explosion condition $\int_{0+}|\psi(s)|^{-1}ds = \infty$ and further that $\psi(\infty) = \infty$. The last condition, together with $\psi'(0+) < 0$, ensures that ψ has a unique positive root λ^*.

The parameter λ^* is the survival rate of Z in the sense that the probability of the event $\left\{\lim_{t\to\infty} Z_t = 0\right\}$ given $Z_0 = x$ is $e^{-\lambda^* x}$, which is strictly positive. We further assume from now on that $\int^{+\infty}\left(\psi(s)\right)^{-1}ds < \infty$, which guarantees that the event $\left\{\lim_{t\to\infty} Z_t = 0\right\}$ agrees with the event of extinction, that is $\{\exists t > 0 \colon Z_t = 0\}$ a.s. This implies in turn that, for the Super-Brownian motion Y, the event of becoming extinguished and the event of extinction agree $\tilde{P}^K$-a.s. We denote the event of extinction of Y by $\varepsilon = \{\exists t > 0 \colon Y_t(0,K) = 0\}$, where $Y_t(0,K)$ is the total mass within $(0,K)$ at time t. We can characterise the $\tilde{P}_\eta^K$-superdiffusion via its Laplace functional.

Lemma (1.2.11) [1]:

For all $f \in B_+(0,K)$,

$$\tilde{E}_x^K\left(e^{-\langle f, Y_t\rangle}\right) = e^{-\langle \tilde{u}_f^K(\cdot,t),\eta\rangle}, \qquad \eta \in \mathcal{M}_f(0,K), \quad t \geq 0,$$

where $\tilde{u}_f^K(x,t)$ is the unique non-negative solution to the semi-group equation

$$\tilde{u}_f^K(x,t) = \mathcal{P}_t^K[f(\cdot)](x) - \int_0^t \mathcal{P}_{t-s}^K\left[\psi\left(\tilde{u}_f^K(\cdot,s)\right)\right](x)ds. \tag{47}$$

We call the function $\tilde{u}_f^K(x,t)$ the Laplace functional of $\left(Y, \tilde{P}_\eta^K\right)$. We have used the notation $\langle f,\eta\rangle = \int_0^K f(x)\eta(dx)$, for $\eta \in \mathcal{M}_f[0,K]$. We define the survival rate w_K of the $\tilde{P}^K$-superdiffusion as the function satisfying

$$\tilde{P}_x^K(\varepsilon) = \exp\{-w_K(x)\}, \qquad \text{for } \eta \in \mathcal{M}_f[0,K] \tag{48}$$

and, taking $f \equiv \theta$ constant in Lemma **6**, we can deduce that

$$-\log \tilde{P}_x^K(\varepsilon) = \lim_{t\to\infty}\lim_{\theta\to\infty}\langle \tilde{u}_\theta(\cdot),\eta\rangle = \langle w_K,\eta\rangle.$$

It can be derived, again by Lemma (1.2.11), that w_K is a solution to

$$Lu - \psi(u) = 0 \quad \text{with} \quad u(0) = u(K) = 0 \tag{49}$$

Analogous to Theorem (1.1.5), it is possible to give a necessary and sufficient condition for a positive survival rate which follows from a spine change of measure argument in the spirit, now using the $\tilde{P}_x^K$ -martingale

$$\tilde{Z}^K(t) = \int_0^K \sin(\pi x/K)\, e^{\mu x - \lambda(K)t} Y_t(dx), \quad t \geq 0, \tag{50}$$

where here $\lambda(K) = -\psi'(0+) - \mu^2/2 - \pi^2/2K^2$. Assuming henceforth in addition that $\int_0^\infty x \log x \prod(dx) < \infty$, one can then show that $\tilde{Z}^K$ is an $L^1\left(\tilde{P}_x^K\right)$ martingale if and only if $\lambda(K) > 0$. It can thus be concluded that w_K is positive if $\lambda(K) > 0$.

Let us now establish the connection between the $\tilde{P}^K$-superdiffusion and a P^K-branching diffusion via the following relations. Set

$$F(s) = \frac{1}{\lambda^*}\psi\big(\lambda^*(1-s)\big), \quad s \in (0,1), \tag{51}$$

$$\overline{w}_K(x) = \lambda^* p_K(x), \qquad x \in (0,K), \tag{52}$$

where p_K is the survival probability of the P^K-branching diffusion.

We can show that (51) is the branching mechanism of a Galton-Watson process and they identify the Galton-Watson process with branching mechanism F of (51) as the backbone of the CSBP with branching mechanism ψ. If we can show that $\overline{w}_K$ in (52) is indeed the survival rate w_K then the following Theorem is a direct consequence of Theorem (1.1.5) and Theorem (1.2.1).

Theorem (1.2.12) [1]:

(i) If $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$ where $K_0 := \pi\big(\sqrt{-2\psi'(0+)}\big)^{-1}$, then $w_K(x) > 0$ for all $x \in (0,K)$; otherwise $w_K(x) = 0$ for all $x \in (0,K)$.

(ii) Uniformly for $x \in (0,K_0)$, as $K \downarrow K_0$,

$$w_K(x) \sim \lambda^*(K-K_0)\frac{(K_0^2\mu^2+\pi^2)(K_0^2\mu^2+9\pi^2)}{12\psi'(0+)\pi K_0^3(e^{\mu K_0}+1)}\sin(\pi x/K_0)\, e^{\mu x}. \tag{53}$$

Proof. The relation in (51) gives $(m-1)\beta = F'(1-) = -\psi'(0+)$ and hence the K_0 in Theorem (1.2.12) is the same as the one in Theorem (1.1.5) and the $\lambda(K)$ defined earlier agrees with $\lambda(K)$ as in Proposition (1.1.3). In particular, $\lambda(K) > 0$ if and only if $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$.

Suppose $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$. By Remark (1.1.12), p_K is the unique non-trivial solution to $L(u) - F(1-u) = 0$ on $(0,K)$ with $u(0) = u(K) = 0$. Using (51) it follows then that $\overline{w}_K$ given by (52) solves (49) and the uniqueness carries over. That is, for $\mu < \sqrt{-2\psi'(0+)}$ and $K > K_0$, $\overline{w}_K$ is the unique non-trivial solution to (49). On the other hand, we know that w_K solves (49) and, by the spine argument we mentioned after (50), we know that $w_K$is positive within $(0,K)$. By uniqueness, we have $\overline{w}_K = w_K$.

Suppose $\mu \geq \sqrt{-2\psi'(0+)}$ or $K \leq K_0$. Then p_K is identically zero and (6) does not have a non-trivial solution. By the transformation in (51), the same holds true for (49) and since w_K is always a solution to (49) it must be equal to zero. Thus (52) holds true again.

The result is now a direct consequence of Theorems (1.1.5) and (1.2.1).

We will now outline the backbone decomposition for the $\tilde{P}_\eta^K$–superdiffusion which consists of a copy of $(Y, \tilde{P}_\eta^K)$ conditioned on becoming extinct and further independent copies of $(Y, \tilde{P}_\eta^K)$ conditioned on becoming extinct which immigrate along a $\boldsymbol{P}^{B,K}$-branching diffusion.

Recall that the $\boldsymbol{P}^{B,K}$-branching diffusion is also the backbone of the P^K-branching diffusion (Theorem (1.1.9)).

Let us begin by studying the process $(Y, \tilde{P}_\eta^K)$ conditioned on becoming extinct.

Proposition (1.2.13) [1]:

Define for $\eta \in \mathcal{M}_f[0, K]$ and $t \geq 0$,

$$\left.\frac{d\widetilde{\boldsymbol{P}}_\eta^{R,K}}{d\tilde{P}_\eta^K}\right|_{\tilde{\mathcal{F}}_t} = \frac{e^{-\langle w_K, Y_t\rangle}}{e^{-\langle w_K, \eta\rangle}},$$

where $(\tilde{\mathcal{F}}_t, t \geq 0)$ is the natural filtration generated by $(Y, \tilde{P}_\eta^K)$. Then $(Y, \widetilde{\boldsymbol{P}}_\eta^{R,K})$ is equal in law to $\left(Y, \tilde{P}_\eta^K(\cdot\,|\varepsilon)\right)$. Further $(Y, \widetilde{\boldsymbol{P}}_\eta^{R,K})$ has spatially dependent branching mechanism

$$\psi^{R,K}(s, x) = \psi\big(s + w_K(x)\big) - \psi\big(w_K(x)\big), \quad s \geq 0 \text{ and } x \in [0, K],$$

and diffusion semigroup $\mathcal{P}^K$.

Proof.

We point out that the motion of the $\widetilde{\boldsymbol{P}}^{R,K}$- superdiffusion remains unchanged and it is therefore different from the motion of the $\boldsymbol{P}^{R,K}$-branching diffusion in Proposition (1.1.7). However, set $\tilde{u}_f^R(x,t) = \lambda^*\big(1 - p_K(x)\big)\left(1 - u_f^R(x,t)\right)$, where u_f^R is the Laplace functional of the $\widetilde{\boldsymbol{P}}^{R,K}$-branching diffusion (Laplace functionals for branching diffusions are defined in a similar fashion to Lemma (1.2.11). Then together with the relations (51) and (52) we can find that $\tilde{u}_f^R$ is the Laplace functional of the $\widetilde{\boldsymbol{P}}^{R,K}$-superdiffusion. Thus the $\widetilde{\boldsymbol{P}}^{R,K}$-superdiffusion can in this way be seen as the analogue of the $\boldsymbol{P}^{R,K}$-branching diffusion in the superdiffusion setting.

We need to introduce some more notation before we can establish the backbone decomposition. Associated to the laws $\{\widetilde{\boldsymbol{P}}^{R,K}_{\delta_x}, x \in [0,K]\}$ is the family of the so-called excursion measures $\{\mathbb{N}^{R,K}_x, x \in [0,K]\}$, defined on the same measurable space, which satisfy

$$\mathbb{N}^{R,K}_x(1 - \exp\{-\langle f, Y_t \rangle\}) = -\log \widetilde{\boldsymbol{E}}^{R,K}_{\delta_x}(\exp\{-\langle f, Y_t \rangle\}), \qquad (54)$$

for any $f \in B_+[0,K]$ and $t \geq 0$. Intuitively speaking, the branching property implies that $\widetilde{\boldsymbol{P}}^{R,K}_x$ is an infinitely divisible measure on the path space of Y and (54) is a 'Lévy–Khintchine' formula in which $\mathbb{N}^{R,K}_x$ plays the role of the Lévy measure. In this sense, $\mathbb{N}^{R,K}_x$ can be considered as the rate at which $\widetilde{\boldsymbol{P}}^{R,K}$-superdiffusions with infinitesimally small initial mass contribute to a unit mass at position x. Further we define, for $n \geq 2$, $x \in (0,K)$,

$$\rho_n(dy, x) = \frac{\beta w_K(x)^2 \delta_0(dy) \mathbf{1}_{\{n=2\}} + w_K(x)^n \frac{y^n}{n!} e^{w_K(x) y} \prod(dy)}{q^{B,K}_n(x) w_K(x) \beta^{B,K}(x)},$$

which will turn out to be the distribution of the initial mass of the immigrating $\widetilde{\boldsymbol{P}}^{R,K}$-superdiffusions at branch points of the backbone on the event of n offspring.

Definition (1.2.14) [1]:

Let $K > K_0$ and $\nu \in \mathcal{M}_a(0,K)$. Let $X^B = (X^B_t, t \geq 0)$ be a $\boldsymbol{P}^{R,K}$- branching diffusion with initial configuration ν. Suppose $I^{\widetilde{\boldsymbol{P}}^{R,K}} = \left(I^{\widetilde{\boldsymbol{P}}^{R,K}}_t, t \geq 0\right)$, $I^{\mathbb{N}^{R,K}} = \left(I^{\mathbb{N}^{R,K}}_t, t \geq 0\right)$ and $I^{\rho} = \left(I^{\rho}_t, t \geq 0\right)$ are three immigration processes (defined below) which are, conditionally on X^B, independent of each other. Then we define the process $Y^D = (Y^D_t, t \geq 0)$ by

$$Y^D_t = I^{\mathbb{N}^{R,K}}_t + I^{\widetilde{\boldsymbol{P}}^{R,K}} + I^{\rho}, \qquad t \geq 0$$

and denote its law by $\widetilde{\boldsymbol{P}}^{D,K}_{\nu}$.

The immigration processes are constructed as follows:

(i) Continuous immigration: The process $I^{\mathbb{N}^{R,K}} = \left(I^{\mathbb{N}^{R,K}}_t, t \geq 0\right)$ is defined as

$$I_t^{\mathbb{N}^{R,K}} = \sum_{u\in\tau^B} \sum_{u\in\tau_u^B\wedge t\leq s\leq\sigma_u^B\wedge t} Y_{t-s}^{(1,u,s)}, \qquad t \geq 0,$$

where, given X^B, independently for each $u \in \tau^B$ such that $\tau_u^B < t$, the processes $Y^{(1,u,s)}$ are countable in number and correspond to Poissonian immigration along the space-time trajectory $\{(x_u^B(s), s): s \in (\tau_u^B, \sigma_u^B]\}$ with rate $2\beta ds \times d\mathbb{N}_{x_u^B(s)}^R$.

(ii) Discontinuous immigration: The process $I^{\widetilde{\boldsymbol{P}}^{R,K}} = \left(I_t^{\widetilde{\boldsymbol{P}}^{R,K}}, t \geq 0\right)$ is defined as

$$I_t^{\widetilde{\boldsymbol{P}}^{R,K}} = \sum_{u\in\tau^B} \sum_{u\in\tau_u^B\wedge t\leq s\leq\sigma_u^B\wedge t} Y_{t-s}^{(2,u,s)}, \qquad t \geq 0,$$

where, given X^B, independently for each $u \in \tau^B$ such that $\tau_u^B < t$ the processes $Y^{(2,u,s)}$ are countable in number and correspond to Poissonian immigration along the space-time trajectory $\{(x_u^B(s), s): s \in (\tau_u^B, \sigma_u^B]\}$ with rate $ds \times \int_0^\infty y \exp\{-w_K(x_u^B(s))y\} \prod(dy) \times dP_{y\delta_{x_u^B(s)}}^{R,K}$.

(iii) Immigration at branch points: The process $I^\rho = \left(I_t^\rho, t \geq 0\right)$ is defined as

$$I_t^\rho = \sum_{u\in\tau^B} 1_{\{\sigma_u^B\leq t\}} Y_{t-\sigma_u^B}^{(3,u)}, \qquad t \geq 0,$$

where, given X^B, independently for each $u \in \tau^B$ such that $\sigma_u^B \leq t$ the processes $Y^{(3,u)}$ is an independent copy of $\left(Y, \widetilde{\boldsymbol{P}}_{Y_u\delta_{x_u^B(\sigma_u)}}^{R,K}\right)$ issued at space-time position $(x_u^B(\sigma_u), \sigma_u)$. At a branch point of u with $n \geq 2$ offspring the initial mass Y_u is distributed according to $\rho_n\left(dy, x_u^B(\sigma_u)\right)$.

Theorem (1.2.15) [1]: (Backbone decomposition)

For $K > K_0$ and $\eta \in \mathcal{M}_f[0, K]$. Let $Y^R = (Y_t^R, t \geq 0)$ be an independent copy of $\left(Y, \widetilde{\boldsymbol{P}}_\eta^{R,K}\right)$.

Suppose that ν is a Poisson random measure on $(0, K)$ with intensity $w_K(x)\eta(dx)$. Let $\left(Y^D, \widetilde{\boldsymbol{P}}_\nu^{D,K}\right)$ be the process constructed in Definition

1. Define the process $\tilde{Y} = \left(\tilde{Y}_t, t \geq 0\right)$ by

$$\tilde{Y}_t = Y_t^R + Y_t^D, \qquad t \geq 0, \tag{55}$$

and denote its law by $\widetilde{\boldsymbol{P}}_\eta^K$. Then the process $\left(\tilde{Y}, \widetilde{\boldsymbol{P}}_\eta^K\right)$ is Markovian and equal in law to $\left(Y, \widetilde{\boldsymbol{P}}_\eta^K\right)$.

Proof. In principle it should be possible to get the proof by using that $\left(Y, \widetilde{\boldsymbol{P}}_\eta^K\right)$ conditioned on non-extinction arises from the martingale change of measure

$$\left.\frac{d\widetilde{\boldsymbol{Q}}_\nu^K}{d\tilde{P}_\nu^K}\right|_{\tilde{\mathcal{F}}_t} = \frac{1-e^{-\langle w_K, Y_t\rangle}}{1-e^{-\langle w_K, \nu\rangle}}, \quad t \geq 0,$$

and showing that $\left(Y, \widetilde{\boldsymbol{Q}}_\nu^K\right)$ agrees in law with the process $\left(Y^D, \widetilde{\boldsymbol{P}}_\nu^{D,K}\right)$ of Definition (1.2.14).

The analogy between the P^K-branching diffusion and the $\tilde{P}^K$-superdiffusion indicates that there is a quasi-stationary limit result equivalent to Theorem (1.2.10). Let us begin with constructing the analogue of the process (X^*, P^*) in Theorem (1.2.10) for the superdiffusion setting. We introduce the family of $\mathbb{N}$-measures now associated with the laws $\left\{\tilde{P}_{\delta_x}^{K_0}, x \in [0, K_0]\right\}$. Consider the family $\left\{\mathbb{N}_x^{K_0}, x \in [0, K_0]\right\}$ satisfying

$$\mathbb{N}_x^{K_0}(1-\exp\{-\langle f, Y_t\rangle\}) = -\log \widetilde{\boldsymbol{E}}_{\delta_x}^{K_0}\left(e^{-\langle f, Y_t\rangle}\right),$$

for $f \in B^+[0, K]$, $t \geq 0$.

Let $\eta \in \mathcal{M}_f(0, K)$. Suppose $\xi^* = (\xi_t^*, t \geq 0)$ is a Brownian motion conditioned

to stay in $(0, K_0)$ with initial position x distributed according to

$$\frac{\sin(\pi x/K_0)\, e^{\mu x}}{\int_{(0,K_0)} \sin(\pi z/K_0)\, e^{\mu z} \eta(dz)} \eta(dx), \quad x \in (0, K_0). \tag{56}$$

Let $I^{\mathbb{N}^{K_0}} = \left(I_t^{\mathbb{N}^{K_0}}, t \geq 0\right)$ and $I^{\widetilde{\boldsymbol{P}}^{K_0}} = \left(I_t^{\widetilde{\boldsymbol{P}}^{K_0}}, t \geq 0\right)$ be two immigration processes (defined below) which, conditionally on ξ^*, are independent of each other. Then we define the process $Y^s = (Y_t^s, t \geq 0)$ by

$$Y_t^s = I_t^{\mathbb{N}^{K_0}} + I_t^{\widetilde{\boldsymbol{P}}^{K_0}}, \qquad t \geq 0. \tag{57}$$

The immigration processes $I^{\mathbb{N}^{K_0}}$ and $I^{\widetilde{\boldsymbol{P}}^{K_0}}$ are defined pathwise as follows.

(i) Continuous immigration: The process $I^{\mathbb{N}^{K_0}} = \left(I_t^{\mathbb{N}^{K_0}}, t \geq 0\right)$ is defined as

$$I_t^{\mathbb{N}^{K_0}} = \sum_{s\le t} Y_{t-s}^{(\mathbb{N},s)}, \qquad t \ge 0,$$

where, given ξ^*, the processes $Y^{(\mathbb{N},s)}$ are countable in number and correspond to Poissonian immigration along the space-time trajectory $\{(\xi_s^*, s): s \ge 0\}$ with rate $2\beta ds \times d\mathbb{N}_{\xi_s^*}^{K_0}$;

(ii) Discontinuous immigration: The process $I^{\widetilde{\boldsymbol{P}}^{K_0}} = \left(I_t^{\widetilde{\boldsymbol{P}}^{K_0}}, t \ge 0\right)$ is defined as

$$I_t^{\widetilde{\boldsymbol{P}}^{K_0}} = \sum_{s\le t} Y_{t-s}^{(\widetilde{\boldsymbol{P}}^{K_0},s)}, \qquad t \ge 0,$$

where, given ξ^*, the processes $Y^{(\widetilde{\boldsymbol{P}}^{K_0},s)}$ are countable in number and correspond to Poissonian immigration along the space-time trajectory $\{(\xi_s^*, s): s \ge 0\}$ with $ds \times \int_0^\infty y \prod(dy) \times \tilde{P}_{y\delta_{\xi_s^*}}^{K_0}$.

Then define the process $Y^* = (Y_t^*, t \ge 0)$ by setting

$$Y_t^* = Y_t' + Y_t^s, \qquad t \ge 0, \tag{58}$$

where Y' is an independent copy of $\left(Y, \tilde{P}_\eta^{K_0}\right)$. We denote the law of Y^* by $\tilde{P}_\eta^*$. The evolution of Y^* under $\tilde{P}^*$ can thus be seen as a path-wise description of Evans' immortal particle picture for the critical width K_0; for a similar construction of Evans' immortal particle picture.

Further, we note that $\left(Y^*, \tilde{P}_\eta^{K_0}\right)$ has the same law as Y under the measure which has martingale density $\tilde{Z}^{K_0}(t)$ of (50) with respect to $\tilde{P}_\eta^{K_0}$.

Theorem (1.2.16) [1]:

Let $K > K_0$ and $\eta \in \mathcal{M}_f[0, K]$. For a fixed time $t \ge 0$, the law of Y_t under the measure $\lim_{K\downarrow K_0} \tilde{P}_\eta^K \left(\cdot \mid \lim_{t\to\infty} \|Y_t\| > 0\right)$ is equal to Y_t^* under $\tilde{P}_\eta^*$.

Proof. By Theorem (1.2.15), $\left(Y, \tilde{P}_\eta^K\right)$ is equal in law to $\left(\tilde{Y}, \widetilde{\boldsymbol{P}}_\eta^K\right)$. The latter is equal in law to $\left(Y, \widetilde{\boldsymbol{Q}}_\eta^K\right)$ where

$$\left.\frac{d\widetilde{\boldsymbol{Q}}_\eta^K}{d\tilde{P}_\eta^K}\right|_{\tilde{\mathcal{F}}_t} = \frac{1 - e^{-\langle w_K, Y_t\rangle}}{1 - e^{-\langle w_K, \eta\rangle}}, \quad t \ge 0.$$

The uniform asymptotics for w_K in Theorem (1.2.12) let us conclude that

$$\lim_{K\downarrow K_0}\frac{1-e^{-\langle w_K,Y_t\rangle}}{1-e^{-\langle w_K,\eta\rangle}}=\lim_{K\downarrow K_0}\frac{\langle w_K,Y_t\rangle}{\langle w_K,\eta\rangle}=\frac{\int_0^{K_0}\sin(\pi x/K_0)\,e^{\mu x}Y_t(dx)}{\int_0^{K_0}\sin(\pi x/K_0)\,e^{\mu x}\mu(dx)}=\frac{\tilde{Z}^{K_0}(t)}{\tilde{Z}^{K_0}(0)},$$

where $\tilde{Z}^{K_0}$ is the martingale in (50). As mentioned before, the law of Y under a change of measure with $\tilde{Z}^{K_0}$ is equal to $\left(Y^*,\tilde{P}_\eta^{K_0}\right)$.

Chapter 2
Critical Branching Brownian Motion with Absorption: Particle Configurations

We consider critical branching Brownian motion with absorption, in which there is initially a single particle at $x > 0$, particles move according to independent one-dimensional Brownian motions with the critical drift, and particles are absorbed when they reach zero. We obtain asymptotic results concerning the behavior of the process before the extinction time. We estimate the number of particles in the system at a given time.

Section (2.1): Preliminary Estimates

We consider branching Brownian motion with absorption. At time zero, there is a single particle at $x > 0$. Each particle moves independently according to one-dimensional Brownian motion with a drift of $-\mu$, and each particle independently splits into two at rate 1. Particles are absorbed when they reach the origin. With positive probability there are particles alive at all times if $\mu < \sqrt{2}$, but all particles are eventually absorbed almost surely if $\mu \geq \sqrt{2}$.

There has been a surge of renewed interest in this process. Some of this interest has been driven by connections between branching Brownian motion with absorption and the FKPP equation. We used branching Brownian motion with absorption to establish existence and uniqueness results for the FKPP traveling-wave equation. In other work, branching Brownian motion with absorption or a very similar process has been used to model a population undergoing selection. In this setting, particles represent individuals in a population, branching events correspond to births, the positions of the particles are the fitnesses of the individuals, and absorption at zero models the death of individuals whose fitness becomes too low.

In this chapter, we consider branching Brownian motion with absorption in the critical case with $\mu = \sqrt{2}$. This process is known to die out with probability one, but we are able to use techniques developed to obtain some new and rather precise results about the behavior of the process before the extinction time. We focus on asymptotic results about the number of particles, the position of the right-most particle, and the configuration of particles as the position x of the initial particle tends to infinity. Let $N(s)$ be the number of particles at time s, and let $X_1(s) \geq X_2(s) \geq \cdots \geq X_{N(s)}(s)$ denote the positions of the particles at time s. Let

$$Y(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)}. \tag{1}$$

Throughout the chapter, we will use the constants

$$\tau = \frac{2\sqrt{2}}{3\pi^2}, \qquad c = \tau^{-1/3} = \left(\frac{3\pi^2}{2\sqrt{2}}\right)^{1/3}. \tag{2}$$

Let $t = \tau x^3$, which is approximately the extinction time of the process when x is large. More precisely, it was shown that for all $\epsilon > 0$, there is a positive constant β such that for sufficiently large x, the extinction time is between $t - \beta x^2$ and $t + \beta x^2$ with probability at least $1 - \epsilon$.
Our first result shows how the number of particles evolves over time. For times s between Bx^2 and $(1-\delta)t$, where B is a large constant and δ is a small constant, with high probability this result estimates the number of particles at time s to within a constant factor.

We now explain some of the contexts in which these result might be applied, and some related open problems that are raised by them.

Yaglom limit laws. Let $x > 0$ be fixed, and consider a branching Brownian motion with critical drift started from one particle at x. Then with high probability, the process dies out in finite time. But conditional on survival up to a large time t, what does the process look like?
More precisely, what is the empirical distribution of particles at times $0 \le s \le t$? This problem is known as the Yaglom conditional limit law, in the case of ordinary branching processes.

An interesting related question is the following: conditional upon survival up to time t, what is the actual number of particles at that time? Note that the results give sharp estimates, up to constants, for the probability of survival up to time t.

Fleming-Viot processes. The process studied here (critical branching Brownian motion with absorption) shares several features with the Fleming-Viot process.
In the case where the underlying motion is simple random walk with negative drift and absorption at 0. For that process the main question concerns the limiting behaviour of the empirical distribution of particles, which under fairly general conditions is believed to be the minimal quasi-stationary distribution of the underlying motion. For a recent verification of this in the case where the underlying motion is that of a subcritical branching process. We point out that the function $e^{-\sqrt{2}x} \sin{(\pi x/L)}$ is precisely a Dirichlet eigen function of

$$\frac{1}{2}\frac{d^2}{dx^2} + \sqrt{2}\frac{d}{dx},$$

and corresponds to a quasi-stationary distribution of Brownian motion with drift $-\sqrt{2}$ in $(0, \infty)$. It is in fact the minimal such distribution.

Extreme configurations. Theorem (2.3.5) gives us information about the position of the rightmost particle at time s, and localises it to within $O(1)$. A natural open problem is to get a convergence in distribution for the position of the rightmost particle. More generally, one can ask about the distribution of the particle configuration as seen from the rightmost particle, or from the median of the rightmost particle. We point out that these questions also make sense in the nearly-critical case, and that the proof of Theorem (2.3.5) can be adapted to that setting. Getting information about the extremal configurations particles is of interest, among other things, because of the role of these particles in the spin glass interpretation of these branching diffusions.

In the following, we obtain or recall some preliminary estimates concerning branching Brownian motion in which particles are killed not only at the origin but also when they travel sufficiently far to the right. We will consider two cases. One is when the Brownian particles are killed at some level $L > 0$. The other is when particles are killed when they reach $L(s) = c(t - s)^{1/3}$ for some s.

As before, let $N(s)$ be the number of particles at time s, and denote the positions of the particles at time s by $X_1(s) \geq X_2(s) \geq \cdots \geq X_{N(s)}(s)$. Define $Y(s)$ as in (1). Let $(\mathcal{F}_s, s \geq 0)$ denote the natural filtration associated with the branching Brownian motion. Let $q_s(x, y)$ denote the density of the branching Brownian motion, meaning that if initially there is a single particle at x and A is a Borel subset of $(0, \infty)$, then the expected number of particles in A at time s is

$$\int_A q_s(x, y)\, dy.$$

Let $L > 0$, we consider here the case in which particles are killed upon reaching either 0 or L. The following result is Lemma (2.1.1).

Lemma (2.1.1) [2]:

For $s > 0$ and $, y \in (0, L)$), let

$$p_s(x, y) = \frac{2}{L} e^{-\pi^2 s/2L^2} e^{\sqrt{2}x} \sin\left(\frac{\pi x}{L}\right) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L}\right).$$

and define $D_s(x, y)$ so that $q_s(x, y) = p_s(x, y)\big(1 + D_s(x, y)\big)$.

Then for all $x, y \in (0, L)$, we have

$$|D_s(x, y)| \leq \sum_{n=2}^{\infty} n^2 e^{-\pi^2(n^2-1)s/2L^2} \tag{3}$$

Lemma (2.1.1) allows us to approximate $q_s(x,y)$ by $p_s(x,y)$ when s is sufficiently large. Lemma (2.1.2) below collects some further results about the density $q_s(x,y)$.

Lemma (2.1.2) [2]:

Fix a positive constant $b > 0$. There exists a constant C (depending on b) such that for all s such that $s \geq bL^2$, we have

$$q_s(x,y) \leq Cp_s(x,y)\,, \ \ \forall x,y \in [0,L] \tag{4}$$

and for all s such that $s \geq bL^2$, we have

$$q_s(x,y) \leq \frac{CL^3}{s^{3/2}} p_s(x,y)\,, \ \ \forall x,y \in [0,L] \tag{5}$$

The following inequalities hold in general (for all $s > 0$ and $x,y \in [0,L]$):

$$q_s(x,y) \leq \frac{Ce^{\sqrt{2}(x-y)}e^{-(x-y)^2/2s}}{s^{1/2}} \tag{6}$$

$$\int_0^L q_s(x,y)\,dy \leq e^s \tag{7}$$

$$\int_0^\infty q_s(x,y)\,ds \leq \frac{2e^{\sqrt{2}(x-y)}x(L-y)}{L} \tag{8}$$

$$\int_0^L e^{\sqrt{2}y} q_s(x,y)\,dy \leq e^{\sqrt{2}x} \min\left\{1, \frac{L-x}{s^{1/2}}\right\} \tag{9}$$

Proof. Equation (4) holds because the right-hand side of (3) is bounded by a constant when $s/L^2 \geq b$. The result (5) is established by breaking the sum on the right-hand side of (3) into blocks of size approximately $L/\sqrt{s}$. Equation (6) is obtained by comparing $q_s(x,y)$ to the density of standard Brownian motion at time s. Equation (7) follows from the fact that the expected number of particles at time s is at most e^s because branching occurs at rate 1. Equation (8) is proved using Green's function estimates for Brownian motion in a strip.

Finally, to prove (9), let $v_s(x,y)$ be the density of Brownian motion killed at 0 and L, meaning that if A is a Borel subset of $(0,L)$, then the probability that a

Brownian motion started at x is in A at time s and has not hit 0 or L before time s is $\int_A \; v_s(x,y)\,dy$. We have

$$q_s(x,y) = e^{\sqrt{2}(x-y)} v_s(x,y). \tag{10}$$

Let $(B(t), t \geq 0)$ be standard Brownian motion with $B(0) = x$. Then, by the Reflection Principle,

$$\int_0^L v_s(x,y)\,dy = \mathbb{P}(B(t) \in (0,L) \text{ for all } t \in [0,s])$$

$$\leq \mathbb{P}\left(\max_{0 \leq t \leq s} B(t) \leq L\right)$$

$$= 2\int_0^{L-x} \tfrac{1}{\sqrt{2\pi s}} e^{-y^2/2s} dy$$

$$\leq \min\left\{1, \tfrac{L-x}{s^{1/2}}\right\}, \tag{11}$$

and (9) follows from (10) and (11).

Let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin\left(\tfrac{\pi X_i(s)}{L}\right).$$

Lemma (2.1.3) [2]:

For all initial configurations of particles at time zero, we have

$$\mathbb{E}[Z(s)] = e^{-\pi^2 s/2L^2} Z(0) \tag{12}$$

and

$$\mathbb{E}[Y(s)] = \tfrac{4}{\pi} e^{-\pi^2 s/2L^2} Z(0)\big(1 + D(s)\big), \tag{13}$$

where $|D(s)|$ is bounded above by the right-hand side of (3).

Lemma (2.1.4) [2]:

Fix a constant $b > 0$. Suppose initially there is a single particle at x. Then there exists a positive constant C, depending on b but not on L or x, such that for all $s \geq bL^2$,

$$\mathbb{E}[Z(s)^2] \leq \frac{Ce^{\sqrt{2}x}e^{\sqrt{2}L}\,s}{L^4}.$$

Lemma (2.1.5) [2]:

Suppose $f: (0, L) \to [0, \infty)$ is a bounded measurable function. Suppose initially there is a single particle at x. Then

$$\mathbb{E}\left[\sum_{i=1}^{N(s)} f(X_i(s))\right] = \int_0^L f(y)q_s(x, y)\,dy$$

and

$$\mathbb{E}\left[\left(\sum_{i=1}^{N(s)} f(X_i(s))\right)^2\right]$$

$$= \int_0^L f(y)^2 q_s(x, y)\,dy + 2\int_0^s \int_0^L q_u(x, z)\left(\int_0^L f(y)q_{s-u}(z, y)\,dy\right)^2 dz\,du.$$

Fix any time $t > 0$. for $s \in [0, t]$, let

$$L(s) = c(t - s)^{1/3},$$

where c was defined in (2). Consider branching Brownian motion with drift $-\sqrt{2}$ in which particles are killed if they reach zero, or if they reach $L(s)$ at time s. Note that all particles must be killed by time t because $L(t) = 0$. We recall here some results, where they were proved. Let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin\left(\frac{\pi X_i(s)}{L(s)}\right),$$

a quantity of crucial importance in what follows. The next result, we provide a precise estimate of $\mathbb{E}[Z(s)]$.

Lemma (2.1.6) [2]:

For $0 < r < s < t$, let

$$G_r(s) = \exp\left(-(3\pi^2)^{1/3}((t-r)^{1/3} - (t-s)^{1/3})\right)\left(\frac{t-s}{t-r}\right)^{1/6}. \qquad (14)$$

There exist positive constants C_3 and C_4 such that if $0 < s < t$, then

$$Z(0)G_0(s)\exp\left(-C_3(t-s)^{1/3}\right) \leq \mathbb{E}[Z(s)] \leq Z(0)G_0(s)\exp\left(C_4(t-s)^{1/3}\right)$$

and, more generally, if $0 < r < s < t$, then

$$Z(r)G_r(s)\exp\left(-C_3(t-s)^{1/3}\right) \leq \mathbb{E}[Z(s)|\mathcal{F}_r] \leq Z(r)G_r(s)\exp\left(C_4(t-s)^{1/3}\right).$$

The following result, which is the $r = 0$ case, establishes bounds on the density up to a constant factor.

Lemma (2.1.7) [2]:

For $x, y > 0$ and $0 < s < t$, let

$$\psi_s(x,y) = \frac{1}{L(s)} e^{-(3\pi^2)^{1/3}(t^{1/3}-(t-s)^{1/3})}\left(\frac{t-s}{t}\right)^{1/6} e^{\sqrt{2}x}\sin\left(\frac{\pi x}{L(0)}\right) e^{-\sqrt{2}y}\sin\left(\frac{\pi y}{L(s)}\right).$$

Fix a positive constant b. There exists a constant $A > 0$ and positive constants C' and C'', with C'' depending on b, such that if $L(0)^2 \leq s \leq t - A$, then

$$q_s(x,y) \geq C'\psi_s(x,y)$$

and if $bL(0)^2 \leq s \leq t - A$, then $q_s(x,y) \leq C''\psi_s(x,y)$.

We will also require estimates on the number of particles killed at the right boundary. The result below is the $s = 0$ case.

Lemma (2.1.8) [2]:

Suppose there is initially a single particle at x, where $0 < x < L(0)$. Let R be the number of particles killed at $L(s)$ for some $s \in [0, t]$. Then there are positive constants C' and C'' such that

$$C'h(x) \leq \mathbb{E}[R] \leq C''\left(h(x) + j(x)\right),$$

where

$$h(x) = e^{\sqrt{2}x}\sin\left(\frac{\pi x}{ct^{1/3}}\right) t^{1/3}\exp(-(3\pi^2 t)^{1/3})$$

and $$j(x) = xe^{\sqrt{2}x}t^{1/3}\exp(-(3\pi^2 t)^{1/3}).$$

Finally, we will need the following bound on the second moment of $Z(s)$.

Proposition (2.1.9) [2]:

Fix $\kappa > 0$ and $\delta > 0$. Then there exists a positive constant C, depending on κ and δ but not on t, such that for all $t \geq 1$ and all s satisfying $\kappa t^{2/3} \leq s \leq (1-\delta)t$,

$$\operatorname{Var}\big(Z(s)\big) \leq C\mathbb{E}[Z(s)]^2\left(\frac{e^{\sqrt{2}L(0)}}{L(0)Z(0)} + \frac{e^{\sqrt{2}L(0)}Y(0)}{L(0)^2 Z(0)^2}\right).$$

Proof.

Choose times $0 = s_0 < s_1 < \cdots < s_K = s$ such that $\kappa t^{2/3} \leq s_{i+1} - s_i \leq 2\kappa t^{2/3}$ for $i = 0,1,\dots,K-1$. Note that $K \leq Ct^{1/3}$.

By Lemma (2.1.6), for $i = 0,1,\dots,K-1$,

$$\mathbb{E}\big[Z(s_{i+1})|\mathcal{F}_{s_i}\big]$$

$$= \exp\big(-(3\pi^2)^{1/3}((t-s_i)^{1/3} - (t-s_{i+1})^{1/3})\big)\left(\tfrac{t-s_{i+1}}{t-s_i}\right)^{1/6} Z(s_i)D_i, \quad (15)$$

where

$$\exp\big(-C_3\delta^{-1/3}t^{-1/3}\big) \leq D_i \leq \exp\big(C_4\delta^{-1/3}t^{-1/3}\big). \quad (16)$$

Because the particles alive at time s_{i+1} are a subset of the particles that would be alive at time s_{i+1} if particles were killed at $L(s_i)$, rather than $L(s)$, for $s \in [s_i, s_{i+1}]$, and the right-hand side of (3) is bounded by a constant when $s \geq \kappa t^{2/3}$ and $l \leq Ct^{1/3}$, it follows from (13) that

$$\mathbb{E}\big[Y(s_{i+1})|\mathcal{F}_{s_i}\big] \leq CZ(s_i) \quad (17)$$

for $i = 0,1,\dots,K-1$, let

$$Z'(s_{i+1}) = \sum_{i=1}^{N(s_{i+1})} e^{\sqrt{2}X_i(s_{i+1})}\sin\left(\tfrac{\pi X_i(s_{i+1})}{L(s_i)}\right),$$

which is the same as $Z(s_{i+1})$ except $L(s_i)$ rather than $L(s_{i+1})$ appears in the denominator. Because $\sin(\pi x/L(s_{i+1})) \leq C\sin(\pi x/L(s_i))$ for all $x \in [0, L(s_{i+1})]$, we have $Z(s_{i+1}) \leq CZ'(s_{i+1})$.

By Lemma (2.1.4), if there is a single particle at x at time s_i, then

$$\mathrm{Var}\big(Z(s_{i+1})|\mathcal{F}_{s_i}\big) \le \mathbb{E}\big[Z(s_{i+1})^2|\mathcal{F}_{s_i}\big] \le C\mathbb{E}\big[Z'(s_{i+1})^2|\mathcal{F}_{s_i}\big] \le \frac{Ce^{\sqrt{2}x}e^{\sqrt{2}L(s_i)}(s_{i+1}-s_i)}{L(s_i)^4}.$$

Because particles move and branch independently, it follows by summing over the particles at time s_i that

$$\mathrm{Var}\big(Z(s_{i+1})|\mathcal{F}_{s_i}\big) \le \frac{CY(s_i)e^{\sqrt{2}L(s_i)}(s_{i+1}-s_i)}{L(s_i)^4} \le Ct^{-2/3}Y(s_i)e^{\sqrt{2}L(s_i)}. \qquad (18)$$

Using the conditional variance formula, equations (15) and (18), and the fact that $s < (1-\delta)t$,

$$\mathrm{Var}\big(Z(s_{i+1})\big) = \mathbb{E}\big[\mathrm{Var}\big(Z(s_{i+1})|\mathcal{F}_{s_i}\big)\big] + \mathrm{Var}\big(\mathbb{E}\big[Z(s_{i+1})|\mathcal{F}_{s_i}\big]\big)$$

$$\le Ct^{-2/3}e^{\sqrt{2}L(s_i)}\mathbb{E}[Y(s_i)] + D_i^2 e^{-2(3\pi^2)^{1/3}\left((t-s_i)^{1/3}-(t-s_{i+1})^{1/3}\right)}\left(\frac{t-s_{i+1}}{t-s_i}\right)^{1/3}\mathrm{Var}\big(Z(s_i)\big)$$

$$\le Ct^{-2/3}e^{\sqrt{2}L(s_i)}\mathbb{E}[Y(s_i)] + D_i^2 e^{-2(3\pi^2)^{1/3}\left((t-s_i)^{1/3}-(t-s_{i+1})^{1/3}\right)}\mathrm{Var}\big(Z(s_i)\big).$$

Therefore, by induction,

$$\mathrm{Var}\big(Z(s)\big) \le Ct^{-2/3}\sum_{i=0}^{K-1} e^{\sqrt{2}L(s_i)}\left(\prod_{j=i+1}^{K-1} D_i^2 e^{-2(3\pi^2)^{1/3}\left((t-s_i)^{1/3}-(t-s_{i+1})^{1/3}\right)}\right)\mathbb{E}[Y(s_i)]$$

$$\le Ct^{-2/3}\sum_{i=0}^{K-1} e^{\sqrt{2}L(s_i)}\left(\prod_{j=i+1}^{K-1} D_i^2\right)e^{-2(3\pi^2)^{1/3}\left((t-s_i)^{1/3}-(t-s_{i+1})^{1/3}\right)}\mathbb{E}[Y(s_i)].$$

By (2.19), for $i = 0,1,\dots,K-1$, we have

$$\mathbb{E}[Y(s_i)] = \mathbb{E}\left[\mathbb{E}\big[Y(s_i)|\mathcal{F}_{s_{i-1}}\big]\right] \le C\mathbb{E}[Z(s_{i-1})].$$

By (2.18) and the fact that $K \le Ct^{1/3}$, for $i = 0,1,\dots,K-1$ we have

$$\prod_{j=i+1}^{K-1} D_i^2 \le \prod_{j=i+1}^{K-1} \exp\big(2C_4\delta^{-1/3}t^{-1/3}\big) \le \exp\left(\prod_{j=i+1}^{K-1} 2C_4\delta^{-1/3}t^{-1/3}\right) \le C.$$

It follows that

$$\mathrm{Var}\big(Z(s)\big) \le Ct^{-2/3} \sum_{i=1}^{K-1} e^{\sqrt{2}L(s_i)} e^{-2(3\pi^2)^{1/3}\left((t-s_{i+1})^{1/3}-(t-s)^{1/3}\right)}\, \mathbb{E}[Z(s_{i-1})]$$

$$+Ct^{-2/3}e^{\sqrt{2}L(0)}e^{-2(3\pi^2)^{1/3}\left((t-s_1)^{1/3}-(t-s)^{1/3}\right)}Y(0). \qquad (19)$$

Denote the two terms on the right-hand side of (19) by T_1 and T_2.

Because $[(t-s)/t]^{1/6}$ is bounded above and below by positive constants when $0 \le s \le (1-\delta)t$, it follows from Lemma (2.1.6) that there are constants C' and C'', depending on δ, such that for $i = 0,1,\dots,K$,

$$C'Z(0)\exp\left(-(3\pi^2)^{1/3}\big(t^{1/3}-(t-s_i)^{1/3}\big)\right) \le \mathbb{E}[Z(s_i)]$$

$$\le C'Z(0)\exp\left(-(3\pi^2)^{1/3}\big(t^{1/3}-(t-s_i)^{1/3}\big)\right).$$

Therefore, using that $\sqrt{2}C = (3\pi^2)^{1/3}$,

$$T_1 \le Ct^{-2/3}\sum_{i=1}^{K-1}\exp\Bigg(\sqrt{2}L(s_i) - 2(3\pi^2)^{\frac{1}{3}}\left((t-s_{i+1})^{\frac{1}{3}} - (t-s)^{\frac{1}{3}}\right)$$
$$- 2(3\pi^2)^{1/3}\big(t^{1/3}-(t-s_{i-1})^{1/3}\big)\Bigg)Z(0)$$

$$= Ct^{-2/3}\sum_{i=1}^{K-1}\exp\Big((3\pi^2)^{1/3}(t-s_i)^{1/3} - 2(3\pi^2)^{1/3}\big((t-s_{i+1})^{1/3}-(t-s)^{1/3}\big)$$
$$- (3\pi^2)^{1/3}\big(t^{1/3}-(t-s_{i-1})^{1/3}\big)\Big)Z(0)$$

$$= Ct^{-2/3}\exp\big(2(3\pi^2)^{1/3}(t-s)^{1/3} - (3\pi^2)^{1/3}t^{1/3}\big)Z(0)$$

$$\times \sum_{i=1}^{K-1}\exp\Big((3\pi^2)^{1/3}\big((t-s_i)^{1/3} - 2(t-s_{i+1})^{1/3} + (t-s_{i-1})^{1/3}\big)\Big). \quad (20)$$

For $i = 0,1,\dots,K-1$, we have $t - s_{i+1} \ge \delta$t, and so $(t-s_i)^{1/3} - (t-s_{i+1})^{1/3} \le C$. Therefore, the sum on the right-hand side of (20) is bounded by $C(K-1) \le Ct^{1/3}$. Thus, using $t > 1$ and Lemma (2.1.6) again,

$$T_1 \le Ct^{-1/3}\exp\big((3\pi^2)^{1/3}t^{1/3}\big)\exp\big(2(3\pi^2)^{1/3}(t-s)^{1/3} - t^{1/3}\big)\frac{Z(0)^2}{Z(0)}$$

$$\le Ct^{-1/3}\exp\big((3\pi^2)^{1/3}t^{1/3}\big)\frac{\mathbb{E}[Z(s)]^2}{Z(0)}$$

$$\le \frac{Ce^{\sqrt{2}L(0)}\mathbb{E}[Z(s)]^2}{L(0)Z(0)}. \qquad (21)$$

Also, using that $t^{1/3}-(t-s_1)^{1/3}\le C$,

$$T_2 \le Ct^{-2/3}e^{\sqrt{2}L(0)}\exp\left(-2(3\pi^2)^{1/3}\big((t-s_1)^{1/3}-(t-s)^{1/3}\big)\right)Y(0)$$

$$\le Ct^{-2/3}e^{\sqrt{2}L(0)}\exp\left(-2(3\pi^2)^{1/3}\big(t^{1/3}-(t-s)^{1/3}\big)\right)Y(0)$$

$$\le \frac{Ce^{\sqrt{2}L(0)}Y(0)\mathbb{E}[Z(s)]^2}{L(0)^2Z(0)^2}. \qquad (22)$$

The result now follows from (21), (23), and (24).

Section (2.2): Number and Configuration of Particles

We return to the model presented, in which there is initially a single particle at x and we are concerned with the asymptotic behavior of the process as $x\to\infty$. We consider how the branching Brownian motion evolves during the initial period between time 0 and time κx^2, where $\kappa>0$ is an arbitrary positive constant. We will use the following result.

Lemma (2.2.1) [2]:

Consider branching Brownian motion with drift $-\sqrt{2}$ and no absorption, started with a single particle at the origin. For each $y\ge 0$, let $K(y)$ be the number of particles that reach $-y$ in a modified process in which particles are killed upon reaching $-y$. Then there exists a random variable W, with $P(0<W<\infty)=1$ and $E(W)=\infty$, such that

$$\lim_{y\to\infty} ye^{-\sqrt{2}y}K(y)=W \quad \text{a.s.}$$

For our process which begins with a single particle at x, let $K(y)$ be the number of particles that would reach $x-y$, if particles were killed upon reaching $x-y$. Note that $K(y)<\infty$ almost surely. The critical branching Brownian motion with absorption dies out. If y is sufficiently large, then $ye^{-\sqrt{2}y}K(y)$ will have approximately the same distribution as the random variable W in Lemma (2.2.1). Our strategy for studying the branching Brownian motion between time 0 and time κx^2 will be to choose a sufficiently large constant y, wait for $K(y)$ particles to reach $x-y$, and then consider $K(y)$ independent branching Brownian motions started from $x-y$.

Let $\alpha\in\mathbb{R}$, and let

$$Z_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \sin\left(\frac{\pi X_i(\kappa x^2)}{x+\alpha}\right) \mathbf{1}_{\{X_i(\kappa x^2) \le x+\alpha\}}. \tag{23}$$

The following result describes the behavior of the configuration of particles at time κx^2.

Definition (2.2.2) [6]: (Martingale)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a filtration is a collection of σ-algebras $\{\mathcal{F}_\alpha : \alpha \in I = \mathbb{N}_0\}$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ and $\bigcup_{n=0}^{\infty} \mathcal{F}_n \subset \mathcal{F}$.

Suppose now $\{X_n : n = 0,1,2,\ldots\}$ is a stochastic process defined on the same probability space as $\{\mathcal{F}_n\}$. Then $\{X_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$ if for all $n = 0,1,2,\ldots,$

- $\mathbb{E}[|X_n|] < \infty$,
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$.

Lemma (2.2.3) [2]:

For all $\epsilon > 0$, there exists a positive constant C_5, depending on κ and ε but not on x, such that for sufficiently large x,

$$\mathbb{P}\left(Y(\kappa x^2) \le C_5 x^{-1} e^{\sqrt{2}x}\right) \ge 1 - \varepsilon. \tag{24}$$

Also, there exist positive constants C_6 and C_7, depending on κ and ε but not on x or α, such that for sufficiently large x,

$$\mathbb{P}\left(C_6 x^{-1} e^{\sqrt{2}x} \le Z_\alpha \le C_7 x^{-1} e^{\sqrt{2}x}\right) \ge 1 - \varepsilon. \tag{25}$$

Furthermore,

$$\lim_{x\to\infty} \mathbb{P}(X_1(\kappa x^2) \le x + \alpha) = 1. \tag{26}$$

Proof. Choose $\eta > 0$ sufficiently small and $B > 0$ sufficiently large such that the random variable W in Lemma (2.2.1) satisfies $\mathbb{P}(W \le 2\eta) \le \varepsilon/8$ and $\mathbb{P}(W \ge B - \eta) < \varepsilon/8$. By Lemma (2.2.1), we can choose $y > 0$ large enough that, for some random variable W having the same distribution as the random variable W in Lemma (2.2.1),

$$\mathbb{P}\left(\left|ye^{-\sqrt{2}y}K(y) - W\right| \ge \eta\right) < \frac{\varepsilon}{8}.$$

These conditions imply that

$$\mathbb{P}\left(ye^{-\sqrt{2}y}K(y) \leq \eta\right) < \frac{\varepsilon}{4} \tag{27}$$

and

$$\mathbb{P}\left(ye^{-\sqrt{2}y}K(y) \geq B\right) < \frac{\varepsilon}{4}. \tag{28}$$

We can also choose y to be large enough that $y \geq 2|\alpha|$ and $\frac{Be^{-\sqrt{2}\alpha}}{y} < \varepsilon/8$.

For $1 \leq i \leq N(\kappa x^2)$ and $0 \leq s \leq \kappa x^2$, let $x_i(s)$ be the position of the particle at time s that is the ancestor of the particle at the location $X_i(\kappa x^2)$ at time κx^2. Let $v_i = \inf\{s: x_i(s) = x - y\}$. Let $0 < u_1 < \cdots < u_{K(y)}$ denote the times at which particles would hit $x - y$, if particles were killed upon reaching $x - y$. Note that $\left\{v_1, \ldots, v_{N(\kappa x^2)}\right\} \subset \{u_1, \ldots, u_{K(y)}\}$. Let $\mathcal{G}$ denote the σ-field generated by the set of times $\{u_1, \ldots, u_{K(y)}\}$. We can choose a positive number $\rho > 0$, depending on y but not on x, such that

$$\mathbb{P}\left(u_{K(y)} \leq \rho\right) > 1 - \frac{\varepsilon}{8}. \tag{29}$$

Throughout the proof, we will assume that x is large enough that $x \geq y$, so that particles are not killed at the origin before reaching $x - y$, and that $\kappa x^2/2 \geq \rho$, so that with high probability all particles will have reached $x - y$ well before time κx^2. Let

$$M(s) = \sum_{i=1}^{N(s)} X_i(s) e^{\sqrt{2}X_i(s)}. \tag{30}$$

It is well-known that the process $(M(s), s \geq 0)$ is a martingale. If there is initially a single particle at $x - y$, then by the Optional Sampling Theorem, the probability that some particle eventually reaches $x + \alpha$ is at most

$$\frac{(x-y)e^{\sqrt{2}(x-y)}}{(x+\alpha)e^{\sqrt{2}(x+\alpha)}}.$$

Therefore, conditional on $\mathcal{G}$, the probability that some descendant of a particle that reaches $x - y$ eventually reaches $x + \alpha$ is at most

$$\frac{K(y)(x-y)e^{\sqrt{2}(x-y)}}{(x+\alpha)e^{\sqrt{2}(x+\alpha)}} \leq \frac{e^{-\sqrt{2}\alpha}}{y}. ye^{-\sqrt{2}y}K(y).$$

Thus, the unconditional probability that some descendant of a particle that reaches $x - y$ eventually reaches $x + \alpha$ is at most

$$\mathbb{P}\left(ye^{-\sqrt{2}y}K(y) > B\right) + \frac{Be^{-\sqrt{2}\alpha}}{y} < \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{3\varepsilon}{8}.$$

In particular, $\mathbb{P}(X_1(\kappa x^2) > x + \alpha) \leq \mathbb{P}\left(u_{K(y)} > \rho\right) + 3\varepsilon/8 \leq \frac{\varepsilon}{2}$ for sufficiently large x, which by letting $\varepsilon \to 0$ implies (26).

Let $S(\alpha) = \{i: x_i(s) < x + \alpha \text{ for all s} \in [v_i, \kappa x^2]\}$. Then let

$$Y'_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \mathbf{1}_{\{i \in S(\alpha)\}}$$

and

$$Z'_\alpha = \sum_{i=1}^{N(\kappa x^2)} e^{\sqrt{2}X_i(\kappa x^2)} \sin\left(\frac{\pi X_i(\kappa x^2)}{x+\alpha}\right) \mathbf{1}_{\{i \in S(\alpha)\}}.$$

The argument in the previous paragraph implies that

$$\mathbb{P}(Y'_\alpha = Y(\kappa x^2) \text{ and } Z'_\alpha = Z_\alpha) \geq 1 - \frac{\varepsilon}{2}. \tag{31}$$

By the Strong Markov Property, the configuration of particles at time κx^2 has the same distribution as the configuration that we would get by starting with $K(y)$ particles at $x - y$ and stopping their descendants at the times $\kappa x^2 - u_i$. Furthermore, restricting to particles in $S(\alpha)$ is equivalent to killing particles when they reach $x + \alpha$. Therefore, the tools with $L = x + \alpha$, can be used to estimate the first and second moments of Y'_α and Z'_α.

We first apply (13) with $s = \kappa x^2 - u_i$, which when $u_{K(y)} \leq \rho$u is at least $\kappa x^2/2$. Because the right-hand side of (3) is bounded by a constant when s is of the order L^2, it follows from (13) that there is a constant C, depending on κ, such that on the event $\{u_{K(y)} \leq \rho\}$,

$$\mathbb{E}[Y'_\alpha | \mathcal{G}] \leq CK(y)e^{\sqrt{2}(x-y)} \sin\left(\frac{\pi(x-y)}{x+\alpha}\right).$$

Using $\sim$ to denote that the ratio of the two sides tends to one as $x \to \infty$, we have

$$\sin\left(\frac{\pi(x-y)}{x+\alpha}\right) \sim \frac{\pi(y+\alpha)}{x+\alpha} \sim \frac{\pi(y+\alpha)}{x}. \tag{32}$$

Because $y \geq 2|\alpha|$, it follows that there exists a constant C_8 such that on the event $\{u_{K(y)} \leq \rho\}$, for sufficiently large x,

$$\mathbb{E}[Y'_\alpha|\mathcal{G}] \leq C_8 x^{-1} e^{\sqrt{2}x}. ye^{-\sqrt{2}y} K(y).$$

Therefore, choosing $C_5 = 8C_8B/\varepsilon$ and using (28), (29), and the conditional Markov's inequality,

$$\mathbb{P}\Big(Y'_\alpha \geq C_5 x^{-1} e^{\sqrt{2}x}\Big) \leq \mathbb{P}\big(u_{K(y)} > \rho\big) + \mathbb{P}\Big(ye^{-\sqrt{2}y} K(y) \geq B\Big) + \mathbb{P}\left(Y'_\alpha \geq \tfrac{8\mathbb{E}[Y'_\alpha|\mathcal{G}]}{\varepsilon}\right)$$

$$\leq \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} = \frac{\varepsilon}{2}. \qquad (33)$$

The result (24) now follows from (33) and (32).

By (12), on the event $\{u_{K(y)} \leq \rho\}$, we have

$$e^{-\pi^2 \kappa x^2/2(x+\alpha)^2} K(y) e^{\sqrt{2}(x-y)} \sin\left(\tfrac{\pi(x-y)}{x+\alpha}\right)$$

$$\leq \mathbb{E}[Z_\alpha|\mathcal{G}] \leq e^{-\pi^2(\kappa x^2-\rho)/2(x+\alpha)^2} K(y) e^{\sqrt{2}(x-y)} \sin\left(\tfrac{\pi(x-y)}{x+\alpha}\right).$$

Because (32) holds and $e^{-\pi^2 \kappa x^2/2(x+\alpha)^2} \sim e^{-\pi^2\kappa/2} \sim e^{-\pi^2(\kappa x^2-\rho)/2(x+\alpha)^2}$, there are constants C_9 and C_{10}, depending on κ, such that

$$C_9 x^{-1} e^{\sqrt{2}x}. ye^{-\sqrt{2}y} K(y) \leq \mathbb{E}[Z'_\alpha|\mathcal{G}] \leq C_{10} x^{-1} e^{\sqrt{2}x}. ye^{-\sqrt{2}y} K(y) \qquad (34)$$

when $u_{K(y)} \leq \rho$ for sufficiently large x. Furthermore, by applying Lemma (2.1.4) to the configuration with a single particle at $x - y$ at time zero and then summing over the particles, we get

$$\mathrm{Var}(Z'_\alpha|\mathcal{G}) \leq \tfrac{CK(y)e^{\sqrt{2}(x-y)}e^{\sqrt{2}(x+\alpha)}\kappa x^2}{2(x+\alpha)^4} \leq \frac{Ce^{\sqrt{2}\alpha}. ye^{-\sqrt{2}y}K(y)}{y}\left(x^{-1}e^{\sqrt{2}x}\right)^2$$

for sufficiently large x. By the conditional Chebyshev's Inequality, on the event $\{u_{K(y)} \leq \rho\}$,

$$\mathbb{P}\left(|Z'_\alpha - \mathbb{E}(Z'_\alpha|\mathcal{G})| > \tfrac{1}{2}\mathbb{E}(Z'_\alpha|\mathcal{G})\Big|\,\mathcal{G}\right) \leq \frac{4\mathrm{Var}(Z'_\alpha|\mathcal{G})}{\big(\mathbb{E}(Z'_\alpha|\mathcal{G})\big)^2} \leq \frac{Ce^{\sqrt{2}\alpha}}{y. ye^{-\sqrt{2}y}K(y)}.$$

In view of (27), it follows that for y large enough that $Ce^{\sqrt{2}\alpha}/\eta y < \varepsilon/8$ and sufficiently large x,

$$\mathbb{P}\left(|Z'_\alpha - \mathbb{E}(Z'_\alpha|\mathcal{G})| > \tfrac{1}{2}\mathbb{E}(Z'_\alpha|\mathcal{G})\middle|\mathcal{G}\right) \le \mathbb{P}\big(u_{K(y)} > \rho\big) + \mathbb{P}\Big(ye^{-\sqrt{2}y}K(y) \le \eta\Big) + \frac{Ce^{\sqrt{2}\alpha}}{\eta y} < \frac{\varepsilon}{2}.$$

Combining this result with (34), we get that for sufficiently large x, the event

$$\tfrac{C_9}{2}.\,x^{-1}e^{\sqrt{2}x}.\,ye^{-\sqrt{2}y}K(y) \le Z'_\alpha \le \tfrac{3C_{10}}{2}.\,x^{-1}e^{\sqrt{2}x}.\,ye^{-\sqrt{2}y}K(y)$$

holds with probability at least $1-\varepsilon/2$. Thus, using (27) and (28), for sufficiently large x we have

$$\tfrac{C_9\eta}{2}.\,x^{-1}e^{\sqrt{2}x} \le Z'_\alpha \le \tfrac{3BC_{10}}{2}.\,x^{-1}e^{\sqrt{2}x}$$

with probability at least $1-\varepsilon$. The result (25) now follows by setting $C_6 = C_9\eta/2$ and $C_7 = 3BC_{10}\eta/2$ and invoking (31).

Let $t = \tau x^3 = 2\sqrt{2}x^3/(3\pi^2)$. For $0 < s < t$, recall that

$$L(s) = x\left(1 - \tfrac{3\pi^2 s}{2\sqrt{2}x^3}\right)^{1/3} = c(t-s)^{1/3}$$

and let

$$Z(s) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)} \sin\left(\tfrac{\pi X_i(s)}{L(s)}\right)\mathbf{1}_{\{X_i(s)\le L(s)\}}.$$

Our goal is to find a lower bound for $Z(s)$. Such a bound will be provided by Proposition (2.2.3) below.

To prove this result, we will consider the following new process, which will also be useful. Fix $\alpha \in \mathbb{R}$, and let $t_\alpha = \tau(x+\alpha)^3$, so that $ct_\alpha^{1/3} = x+\alpha$, where c is defined in (2). For $0 < s < t_\alpha$, let $L_\alpha(s) = c(t_\alpha - s)^{1/3}$. Note that $L_0(s) = L(s)$. Now suppose that, in addition to being killed at the origin, particles to the right of $x+\alpha$ are killed at time κx^2, and for $\kappa x^2 < s < t_\alpha + \kappa x^2$, particles are killed at time s if they reach $L_\alpha(s-\kappa x^2)$. Let $N_\alpha(s)$ be the number of particles alive at time s, and let $X_{1,\alpha}(s) \ge X_2(s) \ge \cdots \ge X_{N_\alpha(s),\alpha}(s)$ denote the positions of these particles at time s. Let

$$Z_\alpha(s) = \sum_{i=1}^{N_\alpha(s)} e^{\sqrt{2}X_{i,\alpha}(s)} \sin\left(\tfrac{\pi X_{i,\alpha}(s)}{L_\alpha(s-\kappa x^2)}\right).$$

Note that $Z_\alpha(\kappa x^2)$ is the same as Z_α defined in (23). Also, let

$$Y_\alpha(s) = \sum_{i=1}^{N_\alpha(s)} e^{\sqrt{2}X_{i,\alpha}(s)}. \quad (36)$$

Proposition (2.2.3) [2]:

For all $\varepsilon > 0$, there exists a constant $C > 0$, depending on κ,δ and ε such

that for sufficiently large x,

$$\mathbb{P}\left(Z(s) \geq Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) > 1-\varepsilon$$

for all $s \in [2\kappa x^2, (1-\delta)t]$.

Proof. We consider the process defined above. Recall that $(\mathcal{F}_u)_{u\geq 0}$ is the natural filtration associated with the branching Brownian motion. By Markov property, there exist positive constants C' and C'', depending on κ and δ, such that for all $s \in [2\kappa x^2, (1-\delta/2)t_\alpha]$

$$C'Z_\alpha G_0(s-\kappa x^2) \leq \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}] \leq C''Z_\alpha G_0(s-\kappa x^2).$$

Because $(t_\alpha - (s-\kappa x^2))^{1/3} - (t_\alpha - s)^{1/3}$ is bounded by a constant, it follows from (14) that

$$C'Z_\alpha \exp\left(-(3\pi^2)^{1/3}\left(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}\right)\right) \leq \mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]$$

$$\leq C''Z_\alpha \exp\left(-(3\pi^2)^{1/3}\left(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}\right)\right). \quad (36)$$

Likewise, by Proposition (2.1.9),

$$\mathrm{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) \leq C\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 \left(\frac{e^{\sqrt{2}L_\alpha(0)}}{L_\alpha(0)Z_\alpha} + \frac{e^{\sqrt{2}L_\alpha(0)}Y(\kappa x^2)}{L_\alpha(0)^2 Z_\alpha^2}\right)$$

$$= C\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 \left(\frac{e^{\sqrt{2}x}e^{\sqrt{2}\alpha}}{(x+\alpha)Z_\alpha} + \frac{e^{\sqrt{2}x}e^{\sqrt{2}\alpha}Y(\kappa x^2)}{(x+\alpha)^2 Z_\alpha^2}\right).$$

Let A be the event that $Y(\kappa x^2) \leq C_5 x^{-1}e^{\sqrt{2}x}$ and $Z_\alpha \geq C_6 x^{-1}e^{\sqrt{2}x}$, where C_5 and C_6 are the constants from Lemma (2.2.2) applied with $\varepsilon/8$ in place of ε. Lemma (2.2.2) then gives $\mathbb{P}(A) > 1-\varepsilon/4$ for sufficiently large x. On A, we have

$$\mathrm{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) \le C\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2 e^{\sqrt{2}\alpha}\left(\frac{x}{C_6(x+\alpha)} + \frac{C_5 x}{C_6^2(x+\alpha)^2}\right).$$

Therefore, if α is chosen to be a large enough negative number that $Ce^{\sqrt{2}\alpha}C_6 < \varepsilon/8$, then $\mathrm{Var}(Z_\alpha(s)|\mathcal{F}_{\kappa x^2}) \le (\varepsilon/8)\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]^2$ on A for sufficiently large x. It follows from the conditional Chebyshev's Inequality that for sufficiently large x,

$$\mathbb{P}\left(Z_\alpha(s) < \tfrac{1}{2}\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}]\right) \le \mathbb{P}(A^c) + \frac{4\varepsilon}{8} < \frac{3\varepsilon}{4}. \qquad (37)$$

By (36), on A we have

$$\mathbb{E}[Z_\alpha(s)|\mathcal{F}_{\kappa x^2}] \ge Cx^{-1}\exp\left(\sqrt{2}x - (3\pi^2)^{1/3}\left(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}\right)\right).$$

Thus, using (37) and the fact that $\mathbb{P}(A^c) < \varepsilon/4$, there is a positive constant C such that for all $s \in [2\kappa x^2, (1-\delta/2)t_\alpha]$,

$$\mathbb{P}\left(Z_\alpha(s) > Cx^{-1}\exp\left(\sqrt{2}x - (3\pi^2)^{1/3}\left(t_\alpha^{1/3} - (t_\alpha - s)^{1/3}\right)\right)\right) \ge 1 - \varepsilon$$

for sufficiently large x. Note that $\left|t_\alpha^{1/3} - t^{1/3}\right|$ is bounded by a constant which depends on α, and thus on ε. Likewise,

$$\sup_{\kappa x^2 \le s \le (1-\delta/2)t_\alpha} \left|(t_\alpha - s)^{1/3} - (t-s)^{1/3}\right| \qquad (38)$$

is bounded by a constant which depends on α and δ. Furthermore, we have $\sqrt{2}x = (3\pi^2)^{1/3}t^{1/3}$. Because $(1-\delta/2)t_\alpha \le (1-\delta)t$ for sufficiently large x, we obtain the result of the proposition with $Z_\alpha(s)$ in place of $Z(s)$, provided that α is a sufficiently large negative number.

To complete the proof, recall that $L(s) = c(t-s)^{1/3}$ and $L_\alpha(s - \kappa x^2) = c(t_\alpha - s + \kappa x^2)^{1/3}$, where $t = \tau x^3$ and $t_\alpha = \tau(x+\alpha)^3$. Therefore, there is a constant $\alpha_0 < 0$ such that if $\alpha < \alpha_0$, then $L_\alpha(s - \kappa x^2) < L(s)$ for sufficiently large x. Also, $L(s)/2 < L_\alpha(s - \kappa x^2)$ for sufficiently large x. Thus, if $\alpha < \alpha_0$, there exists a constant C such that for sufficiently large x,

$$\sin\left(\tfrac{\pi z}{L_\alpha(s-\kappa x^2)}\right) \le C\sin\left(\tfrac{\pi z}{L(s)}\right)$$

for all $z \in [0, L_\alpha(s-\kappa x^2)]$. Because killing particles at a right boundary can only reduce the number of particles in the system, it follows that if $\alpha < \alpha_0$, then $Z_\alpha(s) \le CZ(s)$ for sufficiently large x. The result follows.

Recall that $t = \tau x^3$. The next lemma shows that it is unlikely for any particle ever to get far to the right of $L(s)$ for $s \in [2\kappa x^2, (1-\delta)t]$.

Lemma (2.2.4) [2]:

Let $\varepsilon > 0$. For all $\alpha > 0$, let $t_\alpha = \tau(x+\alpha)^3$, and let $L_\alpha(s) = c(t_\alpha - s)^{1/3}$ for $0 \le s \le t_\alpha$. Then there exists a positive constant C_{11}, depending on κ,δ and ε but not on α or x, such that for sufficiently large x,

$$\mathbb{P}(X_1(s) \le L_\alpha(s-\kappa x^2) \; for\; all\; s \in [\kappa x^2, (1-\delta)t]) \ge 1 - \varepsilon - C_{11}e^{-\sqrt{2}\alpha}.$$

Proof. Suppose there is a particle at the location $z \le ct_\alpha^{1/3} = x + \alpha$ at time κx^2. By Lemma (2.1.8) with $t = t_\alpha$, the probability that a descendant of this particle reaches $L_\alpha(s-\kappa x^2)$ for some $s \in [\kappa x^2, (1-\delta/2)t_\alpha]$ is at most

$$Ce^{-(3\pi^2 t_\alpha)^{1/3}}\left(e^{\sqrt{2}z}\sin\left(\frac{\pi z}{L_\alpha(0)}\right)t_\alpha^{1/3} + ze^{\sqrt{2}z}t_\alpha^{-1/3}\right).$$

Therefore, using the bound $zt_\alpha^{-1/3} \le c$ and applying the Markov property, we get that the conditional probability, given $\mathcal{F}_{\kappa x^2}$, on the event $X_1(\kappa x^2) < x + \alpha$, that a particle reaches $L_\alpha(s-\kappa x^2)$ for $\kappa x^2 \le s \le (1-\delta/2)t_\alpha$ is at most

$$Ce^{-\sqrt{2}(x+\alpha)}\left(t_\alpha^{1/3}Z_\alpha(\kappa x^2) + Y(\kappa x^2)\right). \tag{39}$$

Let A be the event that $X_1(\kappa x^2) < x + \alpha$, $Y(\kappa x^2) \le C_5 x^{-1}e^{\sqrt{2}x}$ and $Z_\alpha(\kappa x^2) \le C_7 x^{-1}e^{\sqrt{2}x}$, where C_5 and C_7 are the constants from Lemma (2.2.2) with $\varepsilon/3$ in place of ε. On A, for sufficiently large x, the expression in (39) is at most

$$Ct_\alpha^{1/3}x^{-1}e^{-\sqrt{2}\alpha} + Cx^{-1}e^{-\sqrt{2}\alpha} \le C_{11}e^{-\sqrt{2}\alpha}.$$

Because $\mathbb{P}(A) > 1 - \varepsilon$ for sufficiently large x by Lemma (2.2.2) and the fact that $(1-\delta/2)t_\alpha \ge (1-\delta)t$ for sufficiently large x, the result follows.

The next lemma shows that at any fixed time $s \in [2\kappa x^2, (1-\delta)t]$, it is unlikely that there is any particle near or to the right of $L(s)$.

Lemma (2.2.5) [2]:

Let $a > 0$ be a positive constant. Let $\varepsilon > 0$. Then for sufficiently large x, we have

$$\mathbb{P}(X_1(s) > L(s) - a) < \varepsilon$$

for all $s \in [2\kappa x^2, (1-\delta)t]$.

Proof. We consider the process defined in which at time κx^2, particles to the right of $x + \alpha$ are killed, and for $\kappa x^2 < s < t_\alpha + \kappa x^2$, particles are killed at time s if they reach $L_\alpha(s - \kappa x^2)$. By (26), for sufficiently large x, the probability that some particle is killed at time κx^2 is at most $\varepsilon/4$. By applying Lemma (2.2.4) with $\varepsilon/4$ in place of ε and choosing $\alpha > 0$ large enough that $C_{11}e^{-\sqrt{2}\alpha} < \varepsilon/4$, we get that the probability that a particle is killed between times κx^2 and $(1-\delta)t$ is at most $\varepsilon/2$. Thus, with probability at least $1 - 3\varepsilon/4$, no particle is killed until at least time $(1-\delta)t$.

Suppose $s \in [2\kappa x^2, (1-\delta)t]$. Let $K_\alpha(s)$ be the number of particles at time s between $L(s) - a$ and $L_\alpha(s - \kappa x^2)$. By Lemma (2.1.7) with t_α in place of t, we have

$$\mathbb{E}[K_\alpha(s)|\mathcal{F}_{\kappa x^2}] \le C t_\alpha^{-1/3} e^{-(3\pi^2)^{1/3}\left(t_\alpha^{1/3} - (t_\alpha - s + \kappa x^2)^{1/3}\right)} Z_\alpha$$
$$\times \int_{L(s)-a}^{L_\alpha(s-\kappa x^2)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L_\alpha(s-\kappa x^2)}\right) dy.$$

For sufficiently large x, the expression

$$L_\alpha(s - \kappa x^2) - (L(s) - a) = c(t_\alpha - s + \kappa x^2)^{1/3} - c(t-s)^{1/3} + a$$

is bounded above by a constant depending on α and a, and thus

$$\int_{L(s)-a}^{L_\alpha(s-\kappa x^2)} e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L_\alpha(s-\kappa x^2)}\right) dy \le \frac{Ce^{-\sqrt{2}L_\alpha(s-\kappa x^2)}}{L_\alpha(s-\kappa x^2)} \le C t_\alpha^{-1/3} e^{-\sqrt{2}L_\alpha(s-\kappa x^2)}$$

Therefore, on the event that $Z_\alpha \le C_7 x^{-1} e^{\sqrt{2}x}$, where C_7 is the constant from Lemma (2.2.2) with $\varepsilon/8$ in place of ε, for sufficiently large x,

$\mathbb{E}[K_\alpha(s)|\mathcal{F}_{\kappa x^2}]$

$$\le Ct_\alpha^{-2/3}x^{-1}exp\left(\sqrt{2}x-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s+\kappa x^2)^{1/3}\right)-\sqrt{2}L_\alpha(s-\kappa x^2)\right)$$

$$\le Cx^{-3}\exp\left(\sqrt{2}x-\sqrt{2}(x+\alpha)+(3\pi^2)^{1/3}(t_\alpha-s+\kappa x^2)^{1/3}-(3\pi^2)^{1/3}(t_\alpha-s+\kappa x^2)^{1/3}\right)$$

$$\le Cx^{-3}$$

because the exponential is a constant which depends on α. Therefore, by the conditional Markov's Inequality and Lemma (2.2.2), for sufficiently large x,

$$\mathbb{P}(K_\alpha(s)>0)\le\mathbb{P}\left(Z_\alpha>C_7x^{-1}e^{\sqrt{2}x}\right)+Cx^{-3}<\frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\frac{\varepsilon}{4}.$$

Because with probability at least $1-3\varepsilon/4$, no particle is killed until at least time $(1-\delta)t$, it follows that for sufficiently large x, we have $\mathbb{P}(X_1(s)>L(s)-a)<\varepsilon$ for all $s\in[2\kappa x^2,(1-\delta)t]$.

Proposition (2.2.6) [2]:

For all $\varepsilon>0$, there exists a constant $C>0$ depending on κ,δ and ε such that for sufficiently large x,

$$\mathbb{P}\left(Z(s)\le Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right)>1-\varepsilon$$

for all $s\in[2\kappa x^2,(1-\delta)t]$.

Proof. We again work with the process defined earlier. By (36) and the conditional Markov's Inequality, there is a constant C depending on κ,δ and ε such that for all $s\in[2\kappa x^2,(1-\delta)t]$,

$$\mathbb{P}\left(Z_\alpha(s)\le CZ_\alpha\exp\left(-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s)^{1/3}\right)\right)\right)>1-\frac{\varepsilon}{4}.$$

Therefore, by (2.27), for all $s\in[2\kappa x^2,(1-\delta)t]$,

$$\mathbb{P}\left(Z_\alpha(s)\le Cx^{-1}\exp\left(\sqrt{2}x-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s)^{1/3}\right)\right)\right)>1-\frac{\varepsilon}{2}$$

for sufficiently large x. Because $\left[t_\alpha^{1/3}-t^{1/3}\right]$ and the expression in (38) are bounded by constants depending on α and $\sqrt{2}x=(3\pi^2t)^{1/3}$, it follows that

$$\mathbb{P}\left(Z_\alpha(s)\le Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right)>1-\frac{\varepsilon}{2}\qquad(40)$$

for sufficiently large x.

From Lemma (2.2.4) with $\varepsilon/8$ in place of ε, we see that with probability at least $1-\varepsilon/8-C_{11}e^{-\sqrt{2}\alpha}$, no particles are killed between times κx^2 and $(1-\delta)t$. Therefore, if α is chosen large enough that $C_{11}e^{-\sqrt{2}\alpha}<\varepsilon/8$, then with probability at least $1-\varepsilon/4$, we have $N_\alpha(s)=N(s)$ and $X_i(s)=X_{i,\alpha}(s)$ for $i=1,\dots,N(s)$. Furthermore, provided α is also large enough that $L_\alpha(s-\kappa x^2)\geq L(s)$, for sufficiently large x it holds that for $0\leq x\leq L(s)$, we have

$$\sin\left(\frac{\pi x}{L_\alpha(s-\kappa x^2)}\right)\geq C\sin\left(\frac{\pi x}{L(s)}\right)$$

for some positive constant C. By Lemma (2.2.5), for sufficiently large x the probability that $X_1(s)>L(s)$ is less than $\varepsilon/4$. It follows that for sufficiently large x, we have $Z_\alpha(s)\geq CZ(s)$ with probability at least $1-\varepsilon/2$. Combining this observation with (40) yields the result.

Proposition (2.2.7) [2]:

For all $\varepsilon>0$, there exists a constant $C>0$ depending on κ,δ and ε such that for sufficiently large x,

$$\mathbb{P}\left(Y(s)\leq Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right)>1-\varepsilon$$

for all $s\in[2\kappa x^2,(1-\delta)t]$.

Proof. We again work with the process defined earlier. Recall the definition of $Y_\alpha(s)$ from (35). By Lemma (2.2.4), we can choose $\alpha>0$ sufficiently large that with probability at least $1-\varepsilon/2$, we have $X_1(s)\leq c(t_\alpha-s+\kappa x^2)^{1/3}$ for all $s\in[\kappa x^2,(1-\delta)t_\alpha]$. Therefore, for all $s\in[2\kappa x^2,(1-\delta)t]$, we have $\mathbb{P}\big(Y_\alpha(s)=Y(s)\big)>1-\varepsilon/2$.

By Lemma (2.1.7) with t_α in place of t, for all $s\in[2\kappa x^2,(1-\delta)t]$,

$$\mathbb{E}[Y_\alpha(s)|\mathcal{F}_{\kappa x^2}]$$

$$\leq\frac{C}{L_\alpha(s-\kappa x^2)}e^{-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s+\kappa x^2)^{1/3}\right)}Z_\alpha\times\int_0^{L_\alpha(s-\kappa x^2)}\sin\left(\frac{\pi y}{L_\alpha(s-\kappa x^2)}\right)dy$$

$$\leq Ce^{-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s+\kappa x^2)^{1/3}\right)}Z_\alpha.$$

By combining this result with the conditional Markov's inequality and (25), we get that there is a constant C such that for sufficiently large x,

$$\mathbb{P}\left(Y_\alpha(s) \leq Cx^{-1}e^{\sqrt{2}x}e^{-(3\pi^2)^{1/3}\left(t_\alpha^{1/3}-(t_\alpha-s+\kappa x^2)^{1/3}\right)}\right) > 1-\frac{\varepsilon}{2}$$

for all $s \in [2\kappa x^2, (1-\delta)t]$. Because $\left|(t_\alpha - s - \kappa x^2)^{1/3} - (t-s)^{1/3}\right|$ is bounded by a constant which depends on α, and $(3\pi^2)^{1/3}t_\alpha^{1/3} = \sqrt{2}(x-\alpha)$, there is a constant C depending on α such that

$$\mathbb{P}\left(Y_\alpha(s) \leq Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) > 1-\frac{\varepsilon}{2}$$

for all $s \in [2\kappa x^2, (1-\delta)t]$.

The result follows because $\mathbb{P}(Y_\alpha(s) = Y(s)) > 1-\varepsilon/2$.

Suppose $\kappa > 0$ and $\delta > 0$. Let $\varepsilon > 0$. Choose a constant $B > 0$ sufficiently large that if $s = BL^2$, the right-hand side of (3) is at most ε. Now fix a time s such that

$$(B+3\kappa)x^2 \leq s \leq (1-\delta)t.$$

Let $f\colon [0,\infty) \to \mathbb{R}$ and $\phi\colon [0,1] \to \mathbb{R}$ be bounded continuous functions. Let $\|f\| = \sup_{x\geq 0}|f(x)|$ and $\|\phi\| = \sup_{0\leq x\leq 1}|\phi(x)|$. We are interested here in the quantities

$$\sum_{i=1}^{N(s)} f\left(X_i(s)\right). \tag{41}$$

and

$$\sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)}\phi\left(\frac{X_i(s)}{L(s)}\right)\mathbf{1}_{\{X_i(s)\leq L(s)\}}. \tag{42}$$

Let $r = s - Bx^2$. Let A be the event that $X_1(u) \leq L(s)$ for all $u \in [r,s]$. By Proposition (2.2.7), there is a positive constant C such that

$$\mathbb{P}\left(Y(r) \leq Cx^{-1}\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right)\right) > 1-\varepsilon \tag{43}$$

for sufficiently large x. Because $L(r) - L(s)$ is bounded above by a constant, Lemma (2.2.5) implies that

$$\mathbb{P}(X_1(r) \leq L(s)) > 1-\varepsilon \tag{44}$$

for sufficiently large x. Because $M(r)$, as defined in (30), is bounded by $X_1(r)Y(r)$, we have

$$M(r) \leq CL(s)x^{-1}\exp\big((3\pi^2)^{1/3}(t-r)^{1/3}\big)$$

when the events in (43) and (44) both occur. By the Optional Sampling Theorem, the probability, conditional on $\mathcal{F}_r$, that some particle reaches $L(s)$ between times r and s is at most $M(r)/\Big(L(s)e^{\sqrt{2}L(s)}\Big)$. Therefore,

$$\mathbb{P}(A^c) \leq 2\varepsilon + Cx^{-1}\exp\big((3\pi^2)^{1/3}(t-r)^{1/3} - \sqrt{2}L(s)\big). \qquad (45)$$

Because $\sqrt{2}L(s) = (3\pi^2)^{1/3}(t-s)^{1/3}$, the exponential on the right-hand side of (45) is bounded by a constant. Therefore, the second term on the right-hand side of (45) tends to zero as $x \to \infty$, and thus $\mathbb{P}(A^c) < 3\varepsilon$ for sufficiently large x.

Let S be the set of all $i \in \{1, \dots, N(s)\}$ such that for all $u \in [r, s]$, the particle at time u that is the ancestor of the particle at $X_i(s)$ at time s is positioned to the left of $L(s)$. We will work with the quantities

$$X(f) = \sum_{\mathrm{i}=1}^{\mathrm{N(s)}} f\left(\mathrm{X_i(s)}\right)1_{\{i \in s\}}$$

and

$$X'(\phi) = \sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)}\phi\left(\tfrac{X_i(s)}{L(s)}\right)\mathbf{1}_{\{i\in s\}}.$$

Note that $X(f)$ and $X'(\phi)$ equal the sums in (41) and (42) respectively on the event A, so we have the following result.

Lemma (2.2.8) [2]:

Suppose ε, B, r, and s are as defined above. Then for sufficiently large x, with probability greater than $1 - 3\varepsilon$, the quantity $X(f)$ equals the sum in (41) and $X'(\phi)$ equals the sum in (42) for all bounded continuous functions $f: [0, \infty) \to \mathbb{R}$ and $\phi: [0,1] \to \mathbb{R}$.

Because $X(f)$ and $X'(\phi)$ are the sums that would be obtained if particles were killed at $L(s)$ between times r and s, we can compute conditional moments of

$X(f)$ and $X'(\phi)$ by applying Lemma (2.1.5) with Bx^2 in place of s and $L(s)$ in place of L. We define $q_u(x, y)$ as in Lemma (2.1.1) with $L(s)$ in place of L.

Define

$$\hat{Z} = \sum_{i=1}^{N(r)} e^{\sqrt{2}X_i(r)} \sin\left(\frac{\pi X_i(r)}{L(s)}\right) \mathbf{1}_{\{X_i(r) \le L(s)\}}. \tag{46}$$

Note that $\hat{Z}$ is defined in the same way as $Z(r)$, except that $L(s)$ is used instead of $L(r)$ in the denominator of the sine function and in the indicator. Lemma (2.2.5) implies that with probability tending to one as $x \to \infty$, we have $X_1(r) \le L(r) - 2(L(r) - L(s))$. Therefore, there are positive constants C' and C'' such that for sufficiently large x,

$$\mathbb{P}\left(C'Z(r) \le \hat{Z} \le C''Z(r)\right) > 1 - \varepsilon. \tag{47}$$

Lemma (2.2.9) [2]:

For sufficiently large x, we have

$$\left| \mathbb{E}[X(f)|\mathcal{F}_r] - \hat{Z}\frac{\pi}{L(s)^2} e^{-\pi^2 Bx^2/2L(s)^2} \int_0^\infty f(y)g(y)dy \right| < \frac{2\pi\|f\|\varepsilon}{L(s)^2} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z},$$

where $g(y) = 2ye^{-\sqrt{2}y}$ as in Theorem (2.2.15).

Proof. Because the right-hand side of (3) is at most ε when $s = Bx^2$, it follows from Lemma (2.1.5) and Lemma (2.1.1) that

$$\mathbb{E}[X(f)|\mathcal{F}_r] = \sum_{i=1}^{N(r)} \int_0^{L(s)} f(y) q_{Bx^2}(X_i(r), y) dy$$

$$= \frac{2(1 + D)}{L(s)} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^{L(s)} f(y) e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right) dy,$$

where $|D| < \varepsilon$. Note that

$$\lim_{x \to \infty} L(s) \int_0^{L(s)} f(y) e^{-\sqrt{2}y} \left| \frac{\pi y}{L(s)} - \sin\left(\frac{\pi y}{L(s)}\right) \right| dy = 0$$

and

$$\lim_{x\to\infty}\int_{L(s)}^{\infty} f(y)e^{-\sqrt{2}y}.\pi y\,dy = 0.$$

It follows that

$$L(s)\int_0^{L(s)} f(y)e^{-\sqrt{2}y}\sin\left(\tfrac{\pi y}{L(s)}\right)dy = \int_0^{\infty} f(y)e^{-\sqrt{2}y}.\pi y\,dy + \gamma(x),$$

where $\gamma(x) \to 0$ as $x \to \infty$. Therefore,

$$\mathbb{E}[X(f)|\mathcal{F}_r] = \hat{Z}\,\frac{\pi(1+D)}{L(s)^2}\,e^{-\pi^2 Bx^2/2L(s)^2}\left(\int_0^{\infty} f(y)g(y)dy + \tfrac{2\gamma(x)}{\pi}\right). \quad (48)$$

To obtain the result from (48), first note that the error term involving $\gamma(x)$ is bounded by $2(1+\varepsilon)L(s)^{-2}e^{-\pi^2 Bx^2/2L(s)^2}\hat{Z}\gamma(x)$, and then bound the remaining error term involving D by $\pi\varepsilon L(s)^{-2}e^{-\pi^2 Bx^2/2L(s)^2}\|f\|\hat{Z}$.

Lemma (2.2.10) [2]:

There is a constant C such that for sufficiently large x,

$$\mathrm{Var}(X(f)|\mathcal{F}_r) \le \frac{C\,Y(r)e^{\sqrt{2}L(s)}}{x^{11/2}}.$$

Proof. By summing over the contributions of the particles at time r and applying Lemma (2.1.5), we get

$$\mathrm{Var}(X(f)|\mathcal{F}_r) \le \sum_{i=1}^{N(r)}\int_0^{L(s)} f(y)^2 q_{Bx^2}(X_i(r),y)dy$$

$$+2\sum_{i=1}^{N(r)}\int_0^{Bx^2}\int_0^{L(s)} q_u(X_i(r),z)\left(\int_0^{L(s)} f(y)q_{Bx^2-u}(z,y)dy\right)^2 dz\,du. \quad (49)$$

The first term on the right-hand side of (49) is bounded by $\|f\|^2\mathbb{E}[X(1)|\mathcal{F}_r]$, where $X(1)$ denotes the value of $X(f)$ when $f(x)=1$ for all x. Consequently, by Lemma (2.2.9), this term is bounded above by

$$C\hat{Z}x^{-2} \le CY(r)x^{-2} \le CY(r)e^{\sqrt{2}L(s)}/x^{11/2},$$

It remains to bound the second term. The strategy involves splitting the outer integral into four pieces. Suppose $0 < w < L(s)$. Using Lemma (2.1.1) and equations (4) and (8),

$$\int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} f(y)q_{Bx^2-u}(z,y)dy\right)^2 dz\,du$$

$$\leq \int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} \frac{C}{L(s)}e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)e^{-\sqrt{2}y}\sin\left(\tfrac{\pi y}{L(s)}\right)dy\right)^2 dz\,du$$

$$\leq \frac{C}{L(s)^2}\int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)e^{2\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^2\left(\int_0^{L(s)} e^{-\sqrt{2}y}\sin\left(\tfrac{\pi y}{L(s)}\right)dy\right)^2 dz\,du$$

$$\leq \frac{C}{L(s)^4}\int_0^{L(s)} e^{2\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^2\left(\int_0^{Bx^2/2} q_u(w,z)du\right)dz.$$

$$\leq \frac{Ce^{\sqrt{2}w}}{L(s)^4}\int_0^{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^2\frac{w(L(s)-z)}{L(s)}dz$$

$$\leq \frac{Ce^{\sqrt{2}w}e^{\sqrt{2}L(s)}}{L(s)^6}. \tag{50}$$

Using Lemma (2.1.1) and (5),

$$\int_{Bx^2/2}^{Bx^2-L(s)^{7/4}}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} f(y)q_{Bx^2-u}(z,y)dy\right)^2 dz\,du$$

$$\leq \int_{Bx^2/2}^{Bx^2-L(s)^{7/4}}\int_0^{L(s)} \frac{C}{L(s)}e^{\sqrt{2}w}\sin\left(\tfrac{\pi w}{L(s)}\right)e^{-\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)$$

$$\times\left(\int_0^{L(s)} \frac{C}{L(s)}e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)e^{-\sqrt{2}y}\sin\left(\tfrac{\pi y}{L(s)}\right).\tfrac{CL(s)^3}{(Bx^2-u)^{3/2}}dy\right)^2 dz\,du$$

$$\leq CL(s)^3 e^{\sqrt{2}w}\sin\left(\tfrac{\pi w}{L(s)}\right)\left(\int_{Bx^2/2}^{Bx^2-L(s)^{7/4}} \tfrac{1}{(Bx^2-u)^3}du\right)$$

$$\times\left(\int_0^{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^3 dz\right)\left(\int_0^{L(s)} e^{-\sqrt{2}y}\sin\left(\tfrac{\pi y}{L(s)}\right)dy\right)^2$$

$$\le CL(s)^3 e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right) \cdot \frac{1}{L(s)^{7/2}} \cdot \frac{e^{\sqrt{2}L(s)}}{L(s)^3} \cdot \frac{1}{L(s)^2}$$

$$= \frac{Ce^{\sqrt{2}w}e^{\sqrt{2}L(s)}}{L(s)^{11/2}} \sin\left(\tfrac{\pi w}{L(s)}\right). \tag{51}$$

Using (6), we get

$$\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_0^{2L(s)/3} q_u(w,z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z,y) dy \right)^2 dz\, du$$

$$\le \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_0^{2L(s)/3} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right)$$

$$\times e^{-\sqrt{2}z} \sin\left(\tfrac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} \frac{Ce^{\sqrt{2}(z-y)}}{(Bx^2-u)^{1/2}} dy \right)^2 dz\, du$$

$$\le \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right) \left(\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \frac{1}{Bx^2-u} du \right) \left(\int_0^{2L(s)/3} e^{\sqrt{2}z} \sin\left(\tfrac{\pi z}{L(s)}\right) dz \right)$$

$$\le \frac{Ce^{\sqrt{2}w} e^{2\sqrt{2}L(s)/3} \log L(s)}{L(s)} \sin\left(\tfrac{\pi w}{L(s)}\right). \tag{52}$$

and

$$\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_{2L(s)/3}^{L(s)} q_u(w,z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z,y) dy \right)^2 dz\, du$$

$$\le \int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \int_{2L(s)/3}^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right)$$

$$\times e^{-\sqrt{2}z} \sin\left(\tfrac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} \frac{Ce^{\sqrt{2}(z-y)} e^{-(z-y)^2/2(Bx^2-u)}}{(Bx^2-u)^{1/2}} dy \right)^2 dz\, du$$

$$\le \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right) \left(\int_{Bx^2-L(s)^{7/4}}^{Bx^2-1} \frac{1}{Bx^2-u} du \right)$$

$$\times \int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} \sin\left(\tfrac{\pi z}{L(s)}\right) \left(\int_0^{L(s)} e^{-\sqrt{2}y} e^{-(z-y)^2/2L(s)^{7/4}} dy \right)^2 dz$$

$$\leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right)$$

$$\times \int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} \left(\int_0^{L(s)/3} e^{-\sqrt{2}y} e^{-(L(s)/3)^2/2L(s)^{7/4}} dy + \int_{L(s)/3}^{L(s)} e^{-\sqrt{2}y} \, dy \right)^2 dz$$

$$\leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right) \left(\int_{2L(s)/3}^{L(s)} e^{\sqrt{2}z} dz \right) \left(e^{-L(s)^{1/4}/18} + e^{-\sqrt{2}L(s)/3} \right)^2$$

$$\leq \frac{C \log L(s)}{L(s)} e^{\sqrt{2}w} e^{\sqrt{2}L(s)} e^{-L(s)^{1/4}/9} \sin\left(\tfrac{\pi w}{L(s)}\right). \quad (53)$$

Finally, using (7),

$$\int_{Bx^2-1}^{Bx^2} \int_0^{L(s)} q_u(w,z) \left(\int_0^{L(s)} f(y) q_{Bx^2-u}(z,y) dy \right)^2 dz \, du$$

$$\leq \int_{Bx^2-1}^{Bx^2} \int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w} \sin\left(\tfrac{\pi w}{L(s)}\right) e^{-\sqrt{2}z} \sin\left(\tfrac{\pi z}{L(s)}\right) (\|f\| e)^2 dz \, du$$

$$\leq \frac{C e^{\sqrt{2}w}}{L(s)^2} \sin\left(\tfrac{\pi w}{L(s)}\right). \quad (54)$$

The expressions in (50), (51), (52), (53), and (54) are all bounded by $Ce^{\sqrt{2}w} e^{\sqrt{2}L(s)}/L(s)^{11/2}$. Because $L(s)$ and x are the same to within a constant factor, we get after summing over the positions of the particles at time r that the second term on the right-hand side of (49) is bounded by $CY(r)e^{\sqrt{2}L(s)}/x^{11/2}$. The result follows.

Lemma (2.2.11) [2]:

For sufficiently large x, we have

$$\left| \mathbb{E}[X'(\phi)|\mathcal{F}_r] - \tfrac{4\hat{Z}}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \int_0^1 \phi(y) h(y) dy \right| < \tfrac{4\|\phi\|\varepsilon}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z},$$

where $h(y) = \frac{\pi}{2} \sin(\pi y)$ as in Theorem (2.2.17).

Proof. Because the right-hand side of (3) is at most ε when $s = Bx^2$, it follows from Lemma (2.1.5) and Lemma (2.1.1) that

$$\mathbb{E}[X'(\phi)|\mathcal{F}_r] = \sum_{i=1}^{N(r)} \int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\tfrac{y}{L(s)}\right) q_{Bx^2}(X_i(r), y)dy$$

$$= \frac{2(1+D)}{L(s)} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^{L(s)} \phi\left(\tfrac{y}{L(s)}\right) \sin\left(\tfrac{\pi y}{L(s)}\right) dy$$

$$= \frac{4(1+D)}{\pi} e^{-\pi^2 Bx^2/2L(s)^2} \hat{Z} \int_0^1 \phi(y)h(y)dy$$

where $|D| < \varepsilon$. Because h is a probability density, the error term involving D is bounded by $(4\|\phi\|\varepsilon/\pi)e^{-\pi^2 Bx^2/2L(s)^2}\hat{Z}$, as claimed.

Lemma (2.2.12) [2]:

There is a constant C such that for sufficiently large x,

$$\operatorname{Var}(X'(\phi)|\mathcal{F}_r) \leq \frac{CY(r)e^{\sqrt{2}L(s)} \log x}{x^2}.$$

Proof. By summing over the contributions of the particles at time r and applying Lemma (2.1.5), we get

$$\operatorname{Var}(X'(\phi)|\mathcal{F}_r) \leq \sum_{i=1}^{N(r)} \int_0^{L(s)} e^{2\sqrt{2}y} \phi\left(\tfrac{y}{L(s)}\right)^2 q_{Bx^2}(X_i(r), y)dy$$

$$+2\sum_{i=1}^{N(r)} \int_0^{Bx^2} q_u(X_i(r), z)\left(\int_0^{L(s)} e^{\sqrt{2}y} \phi\left(\tfrac{y}{L(s)}\right) q_{Bx^2-u}(z, y)dy\right)^2 dz\, du. \quad (55)$$

To bound the first term on the right-hand side of (55), note that if $0 < w < L(s)$, then, by Lemma (2.1.1) and (4),

$$\int_0^{L(s)} e^{2\sqrt{2}y} \phi\left(\tfrac{y}{L(s)}\right)^2 q_{Bx^2}(w, y)dy \leq \frac{Ce^{\sqrt{2}w}}{L(s)} \sin\left(\tfrac{\pi w}{L(s)}\right) \int_0^{L(s)} e^{\sqrt{2}y} \sin\left(\tfrac{\pi y}{L(s)}\right) dw$$

$$\leq \frac{Ce^{\sqrt{2}w} e^{\sqrt{2}L(s)}}{L(s)^2} \sin\left(\tfrac{\pi w}{L(s)}\right). \quad (56)$$

We bound the second term on the right-hand side of (55) by breaking the outer integral into two pieces. Using (8), if $0 < w < L(s)$, then

$$\int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} e^{\sqrt{2}y}\phi\left(\tfrac{y}{L(s)}\right) q_{Bx^2-u}(z,y)dy\right)^2 dz\,du$$

$$\leq \int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)\sin\left(\tfrac{\pi y}{L(s)}\right)dy\right)^2 dz\,du$$

$$\leq C\int_0^{Bx^2/2}\int_0^{L(s)} q_u(w,z)e^{2\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^2 dz\,du$$

$$\leq C\int_0^{L(s)} e^{\sqrt{2}w}e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)^2 \frac{w(L(s)-z)}{L(s)}dz$$

$$\leq \frac{Ce^{\sqrt{2}w}e^{\sqrt{2}L(s)}}{L(s)^2}. \tag{57}$$

Furthermore, by using (9) in the third line, making the substitution $v = Bx^2 - u$ in the fourth line, and breaking the inner integral into the piece from 0 to 1 and the piece from 1 to $Bx^2/2$ in the fifth line, we get

$$\int_{Bx^2/2}^{Bx^2}\int_0^{L(s)} q_u(w,z)\left(\int_0^{L(s)} e^{\sqrt{2}y}\phi\left(\tfrac{y}{L(s)}\right) q_{Bx^2-u}(z,y)dy\right)^2 dz\,du$$

$$\leq \int_{Bx^2/2}^{Bx^2}\int_0^{L(s)} \frac{C}{L(s)} e^{\sqrt{2}w}\sin\left(\tfrac{\pi w}{L(s)}\right)e^{-\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)\left(\int_0^{L(s)} e^{\sqrt{2}y}q_{Bx^2-u}(z,y)dy\right)^2 dz\,du$$

$$\leq \frac{Ce^{\sqrt{2}w}}{L(s)}\sin\left(\tfrac{\pi w}{L(s)}\right)\int_{Bx^2/2}^{Bx^2}\int_0^{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)\min\left\{1,\tfrac{(L(s)-z)^2}{Bx^2-u}\right\}dz\,du$$

$$\leq \frac{Ce^{\sqrt{2}w}}{L(s)}\sin\left(\tfrac{\pi w}{L(s)}\right)\int_0^{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)\left(\int_0^{Bx^2/2}\min\left\{1,\tfrac{(L(s)-z)^2}{v}\right\}dv\right)dz$$

$$\leq \frac{Ce^{\sqrt{2}w}}{L(s)}\sin\left(\tfrac{\pi w}{L(s)}\right)\int_0^{L(s)} e^{\sqrt{2}z}\sin\left(\tfrac{\pi z}{L(s)}\right)(1+(L(s)-z)^2\log x)dz$$

$$\leq \frac{Ce^{\sqrt{2}w}e^{\sqrt{2}L(s)}\log x}{L(s)^2}\sin\left(\tfrac{\pi w}{L(s)}\right). \tag{58}$$

The expressions in (56), (57), and (58) are all bounded by $\left(Ce^{\sqrt{2}w}e^{\sqrt{2}L(s)}\log x\right)/L(s)^2$. By summing over the positions of the particles at time r, we get that the right-hand side of (55) is bounded by $\left(CY(r)e^{\sqrt{2}L(s)}\log x\right)/x^2$, which implies the result.

Theorem (2.2.13) [2]:

Fix $\varepsilon > 0$ and $\delta > 0$. Then there exists a positive constant B depending on ϵ and positive constants C_1 and C_2 depending on B, δ, and ϵ such that for sufficiently large x, we have

$$\mathbf{P}\left(\frac{C_1}{x^3}e^{\sqrt{2}(1-s/t)^{1/3}x} \le N(s) \le \frac{C_2}{x^3}e^{\sqrt{2}(1-s/t)^{1/3}x}\right) > 1-\varepsilon$$

for all $s \in [Bx^2, (1-\delta)t]$.

For $0 \le s \le t$, define

$$L(s) = x(1-\tfrac{s}{t})^{1/3} = c(t-s)^{1/3}. \quad (59)$$

The next result shows that at time s, the right-most particle is usually slightly to the left of $L(s)$.

Proof. Let $\kappa = 1$. Choose B as at before. Choose $s \in [(B+3\kappa)x^2, (1-\delta)t]$, and let $r = s - Bx^2$. Throughout the proof, the constants C, C', and C'' will be allowed to depend on B, δ and ε. Recall that $X(1)$ denotes the value of $X(f)$ when $f(x) = 1$ for all x. By Lemma (2.2.8),

$$\mathbb{P}(X(1) = N(s)) > 1 - 3\varepsilon \quad (60)$$

for sufficiently large x. By Lemma (2.2.9),

$$(1-2\varepsilon)\hat{Z}\frac{\pi}{L(s)^2}e^{-\pi^2Bx^2/2L(s)^2} \le \mathbb{E}[X(1)|\mathcal{F}_r] \le (1+2\varepsilon)\hat{Z}\frac{\pi}{L(s)^2}e^{-\frac{\pi^2Bx^2}{2L(s)^2}}. \quad (61)$$

Using Lemma (2.2.10) and the conditional Chebyshev's Inequality,

$$\mathbb{P}\left(|X(1) - \mathbb{E}[X(1)|\mathcal{F}_r]| > \tfrac{1}{2}\mathbb{E}[X(1)|\mathcal{F}_r]\,\middle|\,\mathcal{F}_r\right) \le \frac{CY(r)e^{\sqrt{2}L(s)}}{x^{11/2}\mathbb{E}[X(1)|\mathcal{F}_r]^2} \le \frac{CY(r)e^{\sqrt{2}L(s)}}{x^{3/2}\hat{Z}^2}. \quad (62)$$

By (47), Proposition (2.2.3), Proposition (2.2.6), and Proposition (2.2.7), there are constants C, C', and C'' such that with probability at least $1-4\varepsilon$, we have

$$C'x^{-1}\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right) \le \hat{Z} \le C''x^{-1}\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right) \quad (63)$$

and

$$Y(r) \le Cx^{-1}\exp\big((3\pi^2)^{1/3}(t-r)^{1/3}\big). \tag{64}$$

Thus, on an event of probability at least $1-4\varepsilon$, the quantity on the right-hand side of (62) is bounded above by

$$\begin{aligned} &Cx^{-1/2}\exp\big(\sqrt{2}L(s)-(3\pi^2)^{1/3}(t-r)^{1/3}\big) \\ &\qquad = Cx^{-1/2}\exp\big((3\pi^2)^{1/3}(t-s)^{1/3}-(3\pi^2)^{1/3}(t-r)^{1/3}\big), \end{aligned}$$

which tends to zero as $x\to\infty$ because the exponential term is bounded by a constant. By (61), on this same event of probability $1-4\varepsilon$, there are constants C' and C'' such that

$$\begin{aligned} &C'x^{-3}\exp\big((3\pi^2)^{1/3}(t-s)^{1/3}\big) \le \tfrac{1}{2}\mathbb{E}[X(1)|\mathcal{F}_r] \\ &\qquad \le \tfrac{3}{2}\mathbb{E}[X(1)|\mathcal{F}_r] \le C''x^{-3}\exp\big((3\pi^2)^{1/3}(t-s)^{1/3}\big). \end{aligned}$$

Combining these results with (60), we get

$$\begin{aligned} &\mathbb{P}\left(C'x^{-3}\exp\big((3\pi^2)^{1/3}(t-s)^{1/3}\big) \le N(s) \le C''x^{-3}\exp\big((3\pi^2)^{1/3}(t-s)^{1/3}\big)\right) \\ &\qquad > 1-7\varepsilon \end{aligned}$$

for sufficiently large x. Because the constants C' and C'' do not depend on s and

$$(3\pi^2)^{1/3}(t-s)^{1/3} = \sqrt{2}\left(1-\tfrac{s}{\tau x^3}\right)^{1/3}x,$$

the result follows.

The following proposition implies Theorem (2.2.15). Here $\sim$ means that the ratio of the two sides tends to one as $n\to\infty$.

Proposition (2.2.14) [2]:

Suppose $0<u<\tau$. Consider a sequence of times $(s_n)_{n=1}^{\infty}$ such that $s_n \sim u x_n^3$.

Let

$$\chi_n(u) = \frac{1}{N(s_n)}\sum_{i=1}^{N(s_n)} \delta_{X_i(s_n)}$$

Let μ be the probability measure on $(0, \infty)$ with density $g(y) = 2ye^{-\sqrt{2}y}$. Then $\chi_n(u) \Longrightarrow \mu$ as $n \to \infty$.

Proof. To show that $\chi_n(u) \Longrightarrow \mu$ as $n \to \infty$, it suffices to show that for all bounded continuous functions $f: [0, \infty) \to \mathbb{R}$, we have

$$\frac{1}{N(s_n)} \sum_{i=1}^{N(s_n)} f(X_i(s_n)) \to_p \int_0^\infty g(y) f(y) dy, \tag{65}$$

where $g(y) = 2ye^{-\sqrt{2}y}$ for $y \geq 0$ and $\to_p$ denotes convergence in probability as $n \to \infty$.

Fix a bounded continuous function $f: [0, \infty) \to \mathbb{R}$. Let $\varepsilon > 0$, and choose B as before. Let $r_n = s_n - Bx^2$. By Lemma (2.2.8), for sufficiently large n,

$$\mathbb{P}\left(\frac{1}{N(s_n)} \sum_{i=1}^{N(s_n)} f(X_i(s_n)) = \frac{X(f)}{X(1)} \right) > 1 - 3\varepsilon. \tag{66}$$

By Lemma (2.2.10) and the conditional Chebyshev's Inequality,

$$\begin{aligned} \mathbb{P}\left(\left| X(f) - \mathbb{E}\left[X(f) | \mathcal{F}_{r_n} \right] \right| > x_n^{-19/6} e^{\sqrt{2}L(s_n)} \middle| \mathcal{F}_{r_n} \right) &\leq \frac{CY(r_n) e^{\sqrt{2}L(s_n)}}{x_n^{-11/2}} \cdot \frac{x_n^{-19/3}}{e^{2\sqrt{2}L(s_n)}} \\ &\leq \frac{CY(r_n) x_n^{5/6}}{e^{\sqrt{2}L(s_n)}}. \end{aligned} \tag{67}$$

Both (63) and (64) hold, with r_n in place of r, with probability at least $1 - 4\varepsilon$ for sufficiently large n. Because $(t - r_n)^{1/3} - (t - s_n)^{1/3}$ is bounded by a constant, the expression obtained by replacing $Y(r_n)$ on the right-hand side of (67) by the upper bound from (64) tends to zero as $n \to \infty$, and thus is less than ε for sufficiently large n. The same convergence holds with $X(1)$ in place of $X(f)$ on the left-hand side of (67). Thus, for sufficiently large n, on an event of probability at least $1 - 5\varepsilon$, we have

$$\frac{\mathbb{E}\left[X(f) | \mathcal{F}_{r_n} \right] - x_n^{-19/6} e^{\sqrt{2}L(s_n)}}{\mathbb{E}\left[X(1) | \mathcal{F}_{r_n} \right] + x_n^{-19/6} e^{\sqrt{2}L(s_n)}} \leq \frac{X(f)}{X(1)} \leq \frac{\mathbb{E}\left[X(f) | \mathcal{F}_{r_n} \right] + x_n^{-19/6} e^{\sqrt{2}L(s_n)}}{\mathbb{E}\left[X(1) | \mathcal{F}_{r_n} \right] - x_n^{-19/6} e^{\sqrt{2}L(s_n)}}.$$

This inequality, when combined with Lemma (2.2.9), becomes

$$\frac{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/L(s_n)^2}\left(\int_0^\infty f(y)g(y)dy - 2\|f\|\varepsilon\right) - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\hat{Z}\pi L(s_n)^{-2}e^{-\frac{\pi^2 Bx_n^2}{L(s_n)^2}}(1+2\varepsilon) + x_n^{-\frac{19}{6}}e^{\sqrt{2}L(s_n)}} \le \frac{X(f)}{X(1)}$$

$$\le \frac{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/L(s_n)^2}\left(\int_0^\infty f(y)g(y)dy + 2\|f\|\varepsilon\right) + x_n^{-19/6}e^{\sqrt{2}L(s_n)}}{\hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/L(s_n)^2}(1-2\varepsilon) - x_n^{-19/6}e^{\sqrt{2}L(s_n)}}.$$

When (63) holds, we have $x_n^{-3}e^{\sqrt{2}L(s_n)} \le C\hat{Z}L(s_n)^{-2}$, and thus for sufficiently large n,

$$x_n^{-19/6}e^{\sqrt{2}L(s_n)} \le \hat{Z}\pi L(s_n)^{-2}e^{-\pi^2 Bx_n^2/L(s_n)^2}\varepsilon.$$

Therefore, for sufficiently large n,

$$\frac{1}{1+3\varepsilon}\left(\int_0^\infty f(y)g(y)dy - 2\|f\|\varepsilon - \varepsilon\right) \le \frac{X(f)}{X(1)}$$
$$\le \frac{1}{1-3\varepsilon}\left(\int_0^\infty f(y)g(y)dy + 2\|f\|\varepsilon + \varepsilon\right)$$

with probability at least $1-5\varepsilon$. In view of (66), we can let $\varepsilon \to 0$ to obtain (65).

Theorem (2.2.15) [2]:

Suppose $0 < u < \tau$, and let $s = ux^3$. Define the probability measure

$$\chi(u) = \frac{1}{N(s)}\sum_{i=1}^{N(s)} \delta_{X_i(s)}.$$

Define μ as in Proposition (2.2.14).Then $\mathcal{X}(u) \Longrightarrow \mu$ as $x \to \infty$.

Proposition (2.2.16) [2]:

Suppose $0 < u < \tau$. Consider a sequence of times $(s_n)_{n=1}^\infty$ such that $s_n \sim ux_n^3$ as $n \to \infty$. Let

$$\eta_n(u) = \frac{1}{Y(s_n)}\sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)}\delta_{X_i(s_n)/L(s_n)}.$$

Let v be the probability measure on $(0,1)$ with density $h(y) = \frac{\pi}{2}\sin(\pi y)$. Then $\eta_n(u) \Longrightarrow v$ as $n \to \infty$.

Proof. The proof is very similar to the proof of Theorem (2.2.15). It suffices to show that we have $\mathbb{P}(X_1(s_n) < L(s_n)) \to 1$ as $n \to \infty$, and that for all bounded continuous functions $\phi: [0,1] \to \mathbb{R}$,

$$\frac{1}{Y(s_n)} \sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)} \phi\left(\frac{X_i(s_n)}{L(s_n)}\right) \to_p \int_0^1 \phi(y)h(y)dy, \tag{68}$$

That $\mathbb{P}(X_1(s_n) < L(s_n)) \to 1$ as $n \to \infty$ follows immediately from Lemma (2.2.5) with $a = 0$.

Fix a bounded continuous function $\phi: [0,1] \to \mathbb{R}$. Let $\varepsilon > 0$, and choose B as before. Let $r_n = s_n - Bx_n^2$. Let $X'(1)$ denote the value of $X'(\phi)$ when $\phi(x) = 1$ for all $x \in [0,1]$. By Lemma (2.2.8), for sufficiently large x,

$$\mathbb{P}\left(\frac{1}{Y(s_n)} \sum_{i=1}^{N(s_n)} e^{\sqrt{2}X_i(s_n)} \phi\left(\frac{X_i(s_n)}{L(s_n)}\right) = \frac{X'(\phi)}{X'(1)}\right) > 1 - 3\varepsilon. \tag{69}$$

By Lemma (2.2.12) and the conditional Chebyshev's Inequality,

$$\begin{aligned}\mathbb{P}\left(\left|X'(\phi) - \mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}]\right| > x_n^{-4/3} e^{\sqrt{2}L(s_n)} \middle| \mathcal{F}_{r_n}\right) \\ &\le \frac{CY(r_n)e^{\sqrt{2}L(s_n)} \log x_n}{x_n^2} \cdot \frac{x_n^{8/3}}{e^{2\sqrt{2}L(s_n)}} \\ &\le \frac{CY(r_n)x_n^{2/3} \log x_n}{e^{\sqrt{2}L(s_n)}}. \end{aligned} \tag{70}$$

Recall that (63) and (64) both hold with probability at least $1 - 4\varepsilon$ for sufficiently large n. The expression obtained by replacing $Y(r_n)$ with the right-hand side of (64) on the right-hand side of (70) tends to zero as $x_n \to \infty$, and the same result holds when $X'(\phi)$ is replaced by $X'(1)$ on the left-hand side. Thus, for sufficiently large n, on an event of probability at least $1 - 5\varepsilon$, we have

$$\frac{\mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}] - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X'(1)|\mathcal{F}_{r_n}] + x_n^{-4/3} e^{\sqrt{2}L(s_n)}} \le \frac{X'(\phi)}{X'(1)} \le \frac{\mathbb{E}[X'(\phi)|\mathcal{F}_{r_n}] + x_n^{-4/3} e^{\sqrt{2}L(s_n)}}{\mathbb{E}[X'(1)|\mathcal{F}_{r_n}] - x_n^{-4/3} e^{\sqrt{2}L(s_n)}}.$$

Combining this inequality with Lemma (2.2.11) gives

$$\frac{4\pi^{-1}\hat{Z}\pi e^{-\pi^2 Bx_n^2/L(s_n)^2}\left(\int_0^1 \phi(y)h(y)dy - \|\phi\|\varepsilon\right) - x_n^{-4/3}e^{\sqrt{2}L(s_n)}}{4\pi^{-1}\hat{Z}\pi e^{-\pi^2 Bx_n^2/L(s_n)^2}(1+\varepsilon) + x_n^{-4/3}e^{\sqrt{2}L(s_n)}} \le \frac{X'(\phi)}{X'(1)}$$

$$\le \frac{4\pi^{-1}\hat{Z}\pi e^{-\pi^2 Bx_n^2/L(s_n)^2}\left(\int_0^1 \phi(y)h(y)dy + \|\phi\|\varepsilon\right) + x_n^{-4/3}e^{\sqrt{2}L(s_n)}}{4\pi^{-1}\hat{Z}\pi e^{-\pi^2 Bx_n^2/L(s_n)^2}(1-\varepsilon) - x_n^{-4/3}e^{\sqrt{2}L(s_n)}}.$$

Because $x_n^{-1}e^{\sqrt{2}L(s_n)} \le C\hat{Z}$ when (63) holds, we have

$$x_n^{-4/3}e^{\sqrt{2}L(s_n)} \le 4\pi^{-1}\hat{Z}\pi e^{-\pi^2 Bx_n^2/L(s_n)^2}\varepsilon$$

for sufficiently large n when (63) holds. Therefore, for sufficiently large n,

$$\frac{1}{1+2\varepsilon}\left(\int_0^1 \phi(y)h(y)dy - \|\phi\|\varepsilon - \varepsilon\right) \le \frac{X'(\phi)}{X'(1)}$$
$$\le \frac{1}{1-2\varepsilon}\left(\int_0^1 \phi(y)h(y)dy + \|\phi\|\varepsilon + \varepsilon\right)$$

with probability at least $1-5\varepsilon$. In view of (69), we can let $\varepsilon \to 0$ to obtain (68).

Theorem (2.2.17) [2]:

Suppose $0 < u < \tau$, and let $s = ux^3$. Define the probability measure

$$\eta(u) = \frac{1}{Y(s)}\sum_{i=1}^{N(s)} e^{\sqrt{2}X_i(s)}\delta_{X_i(s)/L(s)}.$$

Let v be defined as in Proposition (2.2.16). Then $\eta(u) \Longrightarrow v$ as $x \to \infty$.

Section (2.3): Position of the Right-Most Particle

Consider branching Brownian motion without killing and with a drift of $-\sqrt{2}$. Let $u(t,w)$ be the probability that if at time zero there is a single particle at the origin, then the position of the right-most particle at time t will be greater than or equal to w. Define $m(t) = \inf\{w: u(t,w) \ge 1/2\}$. There exist positive constants T, C', C'', and C_{12} such that if $t \ge T$, then

$$u(t,w) \le C''e^t \int_{-1}^{0} \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}}\left(1 - e^{-2(z+1)(w-m(t))/t}\right)dz \quad (71)$$

and

$$u(t,w) \ge C'e^t \int_{-1}^{0} \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi t}}\left(1 - e^{-2(z+1)(w-m(t))/t}\right)dz \quad (72)$$

for all $w \geq m(t) + 1$, where

$$\left| m(t) + \frac{3}{2\sqrt{2}} \log t \right| \leq C_{12}. \tag{73}$$

Lemma (2.3.1) [2]:

Suppose $0 < \gamma \leq 1$. Suppose that $t = \gamma x^2$ and that $w = -(3/2\sqrt{2}) \log t + y$, where $1 + C_{12} \leq y \leq C_{13} x$ for some positive constant C_{13}. Then there exists $x_0 > 0$, depending on γ, such that for $x \geq x_0$,

$$C' y e^{\sqrt{2} y} e^{-y^2/2t} \leq u(t, w) \leq C'' y e^{\sqrt{2} y} e^{-y^2/2t},$$

where $C'$and C'' are positive constants that do not depend on γ.

Remark (2.3.2) [2]:

We note that similar bounds on u may be obtained directly by PDE methods, and these have in fact been used to reprove Bramson's logarithmic correction result and to extend it to the setup of periodic branching rates.

Proof. We may assume that x is large enough that $t \geq \max\{1, T\}$. If $-1 \leq z \leq 0$, then using (73),

$$\frac{2(z+1)(w - m(t))}{t} \leq \frac{2(y - (3/2\sqrt{2}) \log t - m(t))}{t} \leq \frac{2(y + C_{12})}{t} \leq \frac{4y}{t}. \tag{74}$$

It follows that

$$1 - e^{-2(z+1)(w - m(t))/t} \leq \frac{4y}{t}. \tag{75}$$

Because $y \leq C_{13} x$ and $t = \gamma x^2$, the expression in (74) tends to zero as $x \to \infty$. Therefore, if $-1/2 \leq z \leq 0$, we have, for sufficiently large x,

$$1 - e^{-2(z+1)(w - m(t))/t} \geq \frac{1}{2} \cdot \frac{2(z+1)(w - m(t))}{t} \geq \frac{y - (3/2\sqrt{2}) \log t - m(t)}{2t}$$
$$\geq \frac{y - C_{12}}{2t} \geq \frac{Cy}{t}. \tag{76}$$

Next, observe that

$$e^{-(w + \sqrt{2} t - z)^2/2t} = e^{-(y-z)^2/2t} e^{(3/2\sqrt{2})(y-z) \log t / t} e^{-9(\log t)^2/16t} e^{-\sqrt{2}(y-z)} e^{-t} t^{3/2}.$$

If $-1 \leq z \leq 0$, then $e^{-\sqrt{2}} \leq e^{\sqrt{2}z} \leq 1$. Also, $e^{-9(\log t)^2/16t}$ tends to one as $x \to \infty$. Furthermore, because $t = \gamma x^2$ and $y \leq C_{13}x$, we have $e^{(3/2\sqrt{2})(y-z)\log t/t} \to 1$ and $e^{-(y-z)^2/2t}/e^{-y^2/2t} \to 1$ as $x \to \infty$. It follows that there exists $x_0 > 0$, depending on γ, and positive constants $C'$and C''such that if $x \geq x_0$, then

$$C'e^{-y^2/2t}e^{-\sqrt{2}y}e^{-t}t^{3/2} \leq e^{-(w+\sqrt{2}t-z)^2/2t} \leq C''e^{-y^2/2t}e^{-\sqrt{2}y}e^{-t}t^{3/2}. \quad (77)$$

Combining (71), (75), and (77), we get that for sufficiently large x,

$$\begin{aligned} u(t,w) &\leq Ce^t \int_{-1}^{0} \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi}t} \left(1 - e^{-2(z+1)(w-m(t))/t}\right) dz \\ &\leq Ce^t \int_{-1}^{0} \frac{e^{-y^2/2t}e^{-\sqrt{2}y}e^{-t}t^{3/2}}{\sqrt{2\pi}t} \frac{y}{t} dz \\ &\leq Cye^{-\sqrt{2}y}e^{-y^2/2t}. \end{aligned} \quad (78)$$

By similar reasoning using (71), (76), and (77), we get that for sufficiently large x,

$$\begin{aligned} u(t,w) &\geq Ce^t \int_{-1/2}^{0} \frac{e^{-(w+\sqrt{2}t-z)^2/2t}}{\sqrt{2\pi}t} \left(1 - e^{-2(z+1)(w-m(t))/t}\right) dz \\ &\geq Ce^t \int_{-1/2}^{0} \frac{e^{-y^2/2t}e^{-\sqrt{2}y}e^{-t}t^{3/2}}{\sqrt{2\pi}t} \frac{y}{t} dz \\ &\geq Cye^{-\sqrt{2}y}e^{-y^2/2t}. \end{aligned} \quad (79)$$

The result follows from (78) and (79).

Lemma (2.3.3) [2]:

Suppose $0 < \gamma \leq 1$. Suppose $t \leq \gamma x^2$ and $w \geq C_{14}x$ for some positive constant C_{14}. Then there exists $x_0 > 0$, depending on γ, such that for $x \geq x_0$,

$$u(t,w) \leq C\gamma^{-3/2}x^{-3}we^{-\sqrt{2}w}e^{-C_{15}/\gamma} \quad (80)$$

for some positive constants C and C_{15} that do not depend on γ.

Proof. If $-1 \leq z \leq 0$, then

$$1 - e^{-2(z+1)(w-m(t))/t} \leq \frac{2(z+1)(w-m(t))}{t} \leq \frac{Cw}{t}. \quad (81)$$

Also, for sufficiently large x,

$$e^{-(w+\sqrt{2}t-z)^2/2t} = e^{-t}e^{-\sqrt{2}(w-z)}e^{-(w-z)^2/2t} \leq Ce^{-t}e^{-\sqrt{2}w}e^{-C_{14}^2x^2/t}. \quad (82)$$

By (71), (81), and (82), we get that when $T \leq t \leq \gamma x^2$,

$$u(t,w) \leq Cwe^{-\sqrt{2}w}t^{-3/2}e^{-C_{14}^2x^2/t}.$$

The function $t \mapsto t^{-3/2}e^{-C_{14}^2x^2/t}$ is increasing when $t \leq (2C_{14}^2x^2)/3$ which means that for $\gamma \leq 2C_{14}^2/3$, we have

$$u(t,w) \leq C\gamma^{-3/2}x^{-3}we^{-\sqrt{2}w}e^{-C_{14}^2/2\gamma}$$

whenever $T \leq t \leq \gamma x^2$. This is enough to imply (80) except in the case when $t < T$. However, when $t < T$, by the Many-to-One Lemma and Markov's Inequality, $u(t,w)$ is bounded above by e^t times the probability that an individual Brownian particle started at the origin is to the right of w by time t. For the purpose of obtaining an upper bound on $u(t,w)$, we may ignore the drift of $-\sqrt{2}$. Therefore, using that

$$\int_z^\infty e^{-x^2/2}dx \leq z^{-1}\, e^{-z^2/2},$$

we have

$$u(t,w) \leq e^t \int_{w/\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dz \leq \frac{e^t\sqrt{t}}{\sqrt{2\pi}w}e^{-w^2/2t} \leq \frac{e^T T}{\sqrt{2\pi}w}e^{-w^2/2T}.$$

Because $w \geq C_{14}x$, this expression is bounded above by the right-hand side of (80) for $x \geq x_0$, where x_0 depends on γ.

We now return to the setting of Theorem (2.3.5), in which there is initially a particle at x and particles are killed when they reach the origin.

Lemma (2.3.4) [2]:

Let $\varepsilon > 0$. Let $0 < u < \tau$, and let $s = ux^3$. Let $\gamma > 0$. Let D be the number of particles that are killed at the origin between times $s - \gamma x^2$ and s. Then there exists a positive constant C, depending on u and ε but not on γ, such that for sufficiently large x,

$$\mathbb{P}\left(D > C\gamma x^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right)\right) \leq 6\varepsilon.$$

Proof. Let $A = 2\gamma$, and let $r = s - Ax^2$. For $u \in [s - \gamma x^2, s]$ define $X_u(1)$ in the same way as $X(1)$, but with u playing the role of s. That is, $X_u(1)$ consists of the number of particles at time u whose ancestor was positioned to the left of $L(u)$ at time v for all $v \in [r, u]$. By the argument leading to Lemma (2.2.8),

$$\mathbb{P}(N(u) = X_u(1) \text{ forall } u \in [s - \gamma x^2, s]) > 1 - 3\epsilon \qquad (83)$$

for sufficiently large x. By Lemma (2.2.9), there is a positive constant C such that $\mathbb{E}[X_u(1)|\mathcal{F}_r] \leq Cx^{-2}\hat{Z}$ for sufficiently large x, where $\hat{Z}$ is defined as in (46) but with u in place of s. The argument leading to (47) implies that on an event with probability greater than $1 - \varepsilon$, we have $\mathbb{E}[X_u(1)|\mathcal{F}_r] \leq Cx^{-2}Z(r)$ for all $u \in [s - \gamma x^2, s]$ for sufficiently large x, where C is some other positive constant.

Define times $s - \gamma x^2 = u_0 < u_1 < \cdots < u_j = s$, where the u_i are chosen such that $1/2 \leq u_i - u_{i-1} \leq 1$ for $i = 1,2, \dots, j$. For $i = 0,1, \dots, j-1$, let D_i be the number of particles that are killed at the origin between times u_i and u_{i+1}. Let D_i' be the number of such particles that are descended from particles at time u_i that are counted in $X_{u_i}(1)$, meaning that their ancestor was positioned to the left of $L(u_i)$ throughout the time period $[r, u_i]$. Even in the absence of killing between times u_i and u_{i+1}, the expected number of descendants at time u_{i+1} produced by a given particle at time u_i is at most $e^{u_{i+1} - u_i} \leq e$. It follows that for sufficiently large x,

$$\mathbb{E}[D_i'|\mathcal{F}_r] \leq e\mathbb{E}\left[X_{u_i}(1)|\mathcal{F}_r\right] \leq Cx^{-2}Z(r)$$

for all i on an event of probability at least $1 - \varepsilon$, and therefore,

$$\mathbb{E}\left[\sum_{i=0}^{j-1} D_i' \middle| \mathcal{F}_r\right] \leq C\gamma Z(r)$$

on an event of probability at least $1 - \varepsilon$. In view of Proposition (2.2.6), there is a positive constant C such that for sufficiently large x,

$$\mathbb{E}\left[\sum_{i=0}^{j-1} D_i' \middle| \mathcal{F}_r\right] \leq C\gamma x^{-1}\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right)$$

on an event of probability at least $1 - 2\varepsilon$. By Markov's Inequality, there is a positive constant C such that for sufficiently large x,

$$\mathbb{P}\left(\sum_{i=0}^{j-1} D_i' > C\gamma x^{-1}\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right)\right) \leq 3\varepsilon.$$

Because $\mathbb{P}\left(D = \sum_{i=0}^{j-1} D_i'\right) > 1 - 3\varepsilon$ by (83) and

$$\exp\left((3\pi^2)^{1/3}(t-r)^{1/3}\right) \leq C\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right),$$

the result follows.

Theorem (2.3.5) [2]:

Suppose $0 < u < \tau$, and let $s = ux^3$. Let $\varepsilon > 0$. Then there exist $d_1 > 0$ and $d_2 > 0$, depending on u and ϵ, such that for sufficiently large x,

$$\mathbf{P}\left(L(s) - \frac{3}{\sqrt{2}}\log x - d_1 \leq X_1(s) \leq L(s) - \frac{3}{\sqrt{2}}\log x + d_2\right) > 1 - \varepsilon.$$

We are also able to obtain results about the entire configuration of particles. The key idea is that at time s, the density of particles near $y \in (0, L(s))$ will be roughly proportional to

$$e^{-\sqrt{2}y}\sin\left(\frac{\pi y}{L(s)}\right). \tag{84}$$

Establishing a rigorous version of this statement requires proving two theorems. In Theorem (2.2.15), we consider the probability measure in which a mass of $1/N(s)$ is placed at the position of each particle at time s. Because most particles are close to the origin and $\sin(\pi y/L(s)) \approx \pi y/L(s)$ for small y, in the limit this probability measure has a density proportional to $ye^{-\sqrt{2}y}$. In Theorem (2.2.17), we consider the probability measure in which a particle at position z is assigned a mass proportional to $e^{\sqrt{2}z}$. In this case, particles over the entire interval from 0 to $L(s)$ contribute significantly even in the limit, and the sinusoidal shape is observed.

For these results, we use $\Longrightarrow$ to denote convergence in distribution for random elements in the Polish space of locally finite measures on $(0, \infty)$, endowed with the vague topology. We also use δ_y to denote a unit mass at y.

Proof. Fix $d \in \mathbb{R}$. Let $\gamma \in (0,1]$. Let $r = s - \gamma x^2$. Let

$$p_i = u\left(\gamma x^2, L(s) - \frac{3}{\sqrt{2}}\log x + d - X_i(r)\right).$$

Let $R(s)$ be the position of the right-most particle at time s for a modified process in which particles that reach the origin between times r and s are not killed. Then

$$\mathbb{P}\left(R(s) \geq L(s) - \frac{3}{\sqrt{2}}\log x + d \,\middle|\, \mathcal{F}_r\right) = 1 - \prod_{i=1}^{N(r)}(1 - p_i).$$

Therefore,

$$1 - \exp\left(1 - \sum_{i=1}^{N(r)} p_i\right) \leq \mathbb{P}\left(R(s) \geq L(s) - \frac{3}{\sqrt{2}}\log x + d \,\middle|\, \mathcal{F}_r\right) \leq \sum_{i=1}^{N(r)} p_i\,. \tag{85}$$

Consequently, the key to the proof will be obtaining a precise estimate of $\sum_{i=1}^{N(r)} p_i$.

Note that

$$p_i = u\left(\gamma x^2, L(s) - \frac{3}{2\sqrt{2}}\log \gamma x^2 + \frac{3}{2\sqrt{2}}\log \gamma + d - X_i(r)\right).$$

Because $L(r) - L(s)$ is bounded above by a constant depending on u, it follows from Lemma (2.2.5) that with probability tending to one as $x \to \infty$, we have

$$X_1(r) \leq L(s) + \frac{3}{2\sqrt{2}}\log \gamma + d - 1 - C_{12}, \tag{86}$$

where C_{12} is the constant from (73). By Lemma (2.3.1), on this event for sufficiently large x we have

$$C' R_i S_i T_i \leq p_i \leq C'' R_i S_i T_i \tag{87}$$

for all i, where

$$R_i = L(s) + \frac{3}{2\sqrt{2}}\log \gamma + d - X_i(r),$$

$$S_i = \exp\left(-\sqrt{2}\left(L(s) + 2/2\sqrt{2}\log \gamma + d - X_i(r)\right)\right),$$

$$T_i = \exp\left(-\frac{\left(L(s) + 2/2\sqrt{2}\log \gamma + d - X_i(r)\right)^2}{2\gamma x^2}\right).$$

Let

$$a = L(s) - L(r) + \frac{3}{2\sqrt{2}}\log \gamma + d.$$

Then

$$R_i = L(r)\left(1 - \frac{X_i(r)}{L(r)} + \frac{a}{L(r)}\right). \tag{88}$$

Also,

$$S_i = \gamma^{-3/2} e^{-\sqrt{2}d} e^{-\sqrt{2}L(s)} e^{\sqrt{2}X_i(r)}. \tag{89}$$

Finally, because

$$\frac{L(s)^2}{2\gamma x^2} = \frac{c^2(t-s)^{2/3}}{2\gamma c^2 t^{2/3}} = \frac{1}{2\gamma}\left(1 - \frac{s}{t}\right)^{2/3} = \frac{1}{2\gamma}\left(1 - \frac{u}{\tau}\right)^{2/3},$$

we have

$$T_i = \exp\left(-\frac{1}{2\gamma x^2}\left(\big(L(r) - X_i(r)\big)^2 + 2a\big(L(r) - X_i(r)\big) + a^2\right)\right)$$

$$= \exp\left(-\frac{L(s)^2(L(s)^2 - L(r)^2)}{2\gamma x^2}\left(1 - \frac{X_i(r)}{L(r)}\right)^2 - \frac{2a\big(L(r) - X_i(r)\big) + a^2}{2\gamma x^2}\right)$$

$$= \exp\left(-\frac{1}{2\gamma}\left(1 - \frac{u}{\tau}\right)^{2/3}\left(1 - \frac{X_i(r)}{L(r)}\right)^2\right) U_i, \tag{90}$$

where $U_i \to 1$ as $x \to \infty$ uniformly in i because $a/x \to 0$ and $(L(s)^2 - L(r)^2)/x^2 \to 0$ as $x \to \infty$. Therefore, by (88), (89), and (90),

$$\sum_{i=1}^{N(r)} R_i S_i T_i = \gamma^{-3/2} e^{-\sqrt{2}d} e^{-\sqrt{2}L(s)} L(r) \sum_{i=1}^{N(r)} U_i\, e^{\sqrt{2}X_i(r)}\left(1 - \frac{X_i(r)}{L(r)} + \frac{a}{L(r)}\right)$$

$$\times \exp\left(-\frac{1}{2\gamma}\left(1 - \frac{u}{\tau}\right)^{2/3}\left(1 - \frac{X_i(r)}{L(r)}\right)^2\right). \tag{91}$$

Consider the function $\phi\colon [0,1] \to \mathbb{R}$ defined by

$$\phi(z) = (1-z)\exp\left(-\frac{1}{2\gamma}\left(1 - \frac{u}{\tau}\right)^{2/3}(1-z)^2\right).$$

By (68), applied with $s_n = ux_n^3 - \gamma x_n^2$, where $(x_n)_{n=1}^{\infty}$ is a sequence tending to infinity, we have,

$$\frac{1}{Y(r)} \sum_{i=1}^{N(r)} e^{\sqrt{2} X_i(r)} \phi\left(\frac{X_i(r)}{L(r)}\right) \to_p$$

$$\frac{\pi}{2} \int_0^1 (1-z) \exp\left(-\frac{1}{2\gamma}\left(1-\frac{u}{\tau}\right)^{2/3} (1-z)^2\right) \sin(\pi z)\, dz\,, \qquad (92)$$

Now let $\alpha = (2\gamma)^{-1/2}(1-u/\tau)^{1/3}$ and make the substitution $y = \alpha(1-z)$ to get that the right-hand side of (92) is

$$\frac{\pi}{2} \int_0^\alpha \frac{y}{\alpha} e^{-y^2} \sin\left(\frac{\pi y}{\alpha}\right) \cdot \frac{1}{\alpha}\, dy \asymp \frac{1}{\alpha^3} \asymp \gamma^{3/2}, \qquad (93)$$

where $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants.

Furthermore, $\sum_{i=1}^{N(r)} e^{\sqrt{2} X_i(r)} = Y(r)$ and $a/L(r)$ tends to zero as $x \to \infty$. It thus follows from (91), (92), and (93) that on the event (86), we have

$$\sum_{i=1}^{N(r)} R_i S_i T_i = e^{-\sqrt{2} d} e^{-\sqrt{2} L(s)} L(r) Y(r) H(u, x, \gamma), \qquad (94)$$

where $H(u, x, \gamma)$ converges in probability as $x \to \infty$ to some number which is bounded between two positive constants that do not depend on γ. Note that $e^{-\sqrt{2} L(s)} = e^{-(3\pi^2)^{1/3}(t-s)^{1/3}}$. Therefore, because $Z(r) \leq Y(r)$, we can use Propositions (2.2.3) and (2.2.7) to conclude that with probability at least $1 - 2\varepsilon$, we have $C' \leq e^{-\sqrt{2} L(s)} L(r) Y(r) \leq C''$ for sufficiently large x. Combining this result with (87) and (94), we get that there are constants C_{16} and C_{17}, not depending on γ, such that for sufficiently large x,

$$\mathbb{P}\left(C_{16} e^{-\sqrt{2} d} \leq \sum_{i=1}^{N(r)} p_i \leq C_{17} e^{-\sqrt{2} d}\right) > 1 - 3\varepsilon. \qquad (95)$$

Now choose $d_2 > 0$ large enough that $C_{17} e^{-\sqrt{2} d_2} < \varepsilon$. By (85) and (95),

$$\begin{aligned} \mathbb{P}\left(X_1(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d_2\right) &\leq \mathbb{P}\left(R(s) \geq L(s) - \frac{3}{\sqrt{2}} \log x + d_2\right) \\ &\leq C_{17} e^{-\sqrt{2} d_2} + 3\varepsilon \\ &\leq 4\varepsilon. \qquad (96) \end{aligned}$$

Likewise, we can choose $d_1 > 0$ large enough that $\exp\left(-C_{16} e^{-\sqrt{2} d_1}\right) \leq \varepsilon$. By (85) and (95),

$$\mathbb{P}\left(R(s) \leq L(s) - \frac{3}{\sqrt{2}}\log x - d_1\right) \leq \exp\left(-C_{16}e^{-\sqrt{2}d_1}\right) + 3\varepsilon \leq 4\varepsilon. \quad (97)$$

It remains to bound the probability that $R(s) > L(s) - \left(3/\sqrt{2}\right)\log x - d_1$ but $X_1(s) \leq L(s) - \left(3/\sqrt{2}\right)\log x - d_1$. This could only happen if some particle reaches 0 between times r and s and then, for the modified process in which killing is suppressed during this time, some descendant particle is to the right of $L(s) - \left(3/\sqrt{2}\right)\log x - d_1$ at time s. However, by Lemma (2.3.4), with probability at least $1 - 6\varepsilon$, at most $C\gamma x^{-1}\exp\left((3\pi^2)^{1/3}(t-s)^{1/3}\right) = C\gamma x^{-1}e^{\sqrt{2}L(s)}$ particles reach the origin between times r and s. Conditional on this event, by Lemma (2.3.3), the expected number of these particles with a descendant to the right of $L(s) - (3/\sqrt{2})\log x + y$ at time s is at most

$$C\gamma x^{-1}e^{\sqrt{2}L(s)} \cdot \gamma^{-3/2}x^{-3}L(s)e^{-\sqrt{2}\left(L(s)-\left(3/\sqrt{2}\right)\log x - d_1\right)}e^{-C_{15}/\gamma}$$

$$\leq C_{18}\gamma^{-1/2}e^{-\sqrt{2}d_1}e^{-C_{15}/\gamma}.$$

Combining this result with (2.97) and Markov's Inequality, and choosing γ small enough that $C_{18}\gamma^{-1/2}e^{-\sqrt{2}d_1}e^{-C_{15}/\gamma} < \varepsilon$, we get, for sufficiently large x,

$$\mathbb{P}\left(X_1(s) \leq L(s) - \frac{3}{\sqrt{2}}\log x - d_1\right) \leq 4\varepsilon + 6\varepsilon + C_{18}\gamma^{-1/2}e^{-\sqrt{2}d_1}e^{-C_{15}/\gamma} \leq 11\varepsilon. \quad (98)$$

The result follows from (96) and (98).

Chapter 3
Large Deviations for the Branching Brownian Motion in Presence of Selection or Coalescence

We estimate the large deviation function of the position of the rightmost particle for several such generalizations: the L-BBM, the N -BBM, and the CBRW (coalescing branching random walk) which is closely related to the noisy FKPP equation. Our approach allows us to obtain only upper bounds on these large deviation functions. One noticeable feature of our results is their non analytic dependence on the parameters (such as the coalescence rate in the CBRW).

Section (3.1): The Physical Picture

Branching Brownian motions (BBM) and branching random walks (BRW) are among the simplest stochastic models of a growing population in space and time. They describe particles which perform Brownian motions or random walks and branch independently at random times. If one starts with a single particle, the size of the region of space occupied by the particles grows linearly with time. Since the mid seventies, one has a precise understanding of the fluctuations of the size of this region. For example, in the one dimensional case one knows that the probability distribution of the position of the rightmost particle of a BBM can be obtained by solving an FKPP (Fisher-Kolmogorov-Petrovskii-Piskounov) equation: for a BBM starting at the origin, where particles diffuse according to

$$\langle [X(t)-X(0)]^2\rangle = \sigma^2 t$$

and branch at rate 1, one can show that, at time t, the probability $P(x,t)$ that the rightmost particle is on the right of x is the solution of the FKPP equation

$$\frac{\partial P(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 P(x,t)}{\partial x^2} + P(x,t) - P^2(x,t) \tag{1}$$

with a step initial condition $P(x,0) = 1-\theta(x)$ (where $\theta(x)$ is the Heaviside function). In the long time limit, it is known that the probability $-\partial P(x,t)/\partial x$ that the position of the rightmost particle $X_{\max}(t) = x$ is concentrated around $X_t \simeq \sqrt{2}\sigma t - \frac{3\sigma}{2\sqrt{2}}\ln t$.

One can also show from (1) that the large deviation function ψ_{BBM} of the position $X_{\max}(t)$ of the rightmost particle for $v > \sqrt{2}\sigma$

$$\mathbf{P}(X_{\max}(t) > vt) \sim \exp[-t\,\psi_{BBM}(v)] \tag{2}$$

is given by

$$\psi_{BBM}(v) = \frac{v^2}{2\sigma^2} - 1. \tag{3}$$

In (2) and everywhere below the symbol $\sim$ means that

$$\lim_{t\to\infty} \frac{\ln \mathbf{P}(X_{\max}(t)>vt)}{t} = -\psi_{BBM}(v). \tag{4}$$

Over the last decade a number of generalizations of the branching Brownian motion have been considered where, due to some selection or coalescence mechanism, the density of particles generated by the BBM saturates. These extensions of the BBM are expected to be described by noisy versions of the FKPP equation. In these noisy versions, the main effect of the noise is to shift the velocity of the front and to make its position fluctuate. A phenomenological approach has been proposed which gives a prediction for the cumulates of this position. Our goal here is to understand the large *positive* deviations of this position. The case of large *negative* deviations (for branching random walks with coalescence) would require a rather different approach and will not be discussed in this chapter except for some comments in the conclusion; in particular the large deviation function may depend on the number of particles one starts with.

Now we study how (3) is modified by these selection or coalescence mechanisms. We discuss three models:

In the L-BBM, one starts at time t = 0 with a single particle at the origin. This particle branches and diffuses like a usual branching Brownian motion. The only difference with the usual BBM is that whenever a particle gets at a distance larger than L from the rightmost particle, it is eliminated. Therefore at any given time t the system consists of a random number $\mathcal{N}(t) \geq 1$ of particles at positions $X_1(t), X_2(t), \dots, X_{\mathcal{N}}(t)$ which all satisfy $X_{\max}(t) - L \leq X_i(t) \leq X_{\max}(t)$ where $X_{\max}(t) = \max_{1\leq i\leq \mathcal{N}(t)} X_i(t)$.

This number of particles $\mathcal{N}(t)$ fluctuates but one can show that the evolution of the L-BBM leads to a steady state where the event $\mathcal{N}(t) = 1$ is recurrent.

For large t one can also show that the probability distribution of the position $X_{\max}(t)$ of the rightmost particle has a large deviation from

$$\mathbf{P}_{\mathbf{LBBM}}(X_{\max}(t) > vt) \sim \exp[-t\, \psi_{LBBM}(v)]. \tag{5}$$

One of our results is the following upper bound for $v > \sqrt{2}\sigma$ and large L

$$0 \leq \psi_{LBBM}(v) - \psi_{BBM}(v) \lesssim e^{-\alpha(v)L/\sigma} \tag{6}$$

with

$$\alpha(v) = \begin{cases} \frac{2\sqrt{2}(v-v_c)}{v_c} & \text{for } v_c < v < \frac{3}{2}v_c \\ \frac{v+\sqrt{v^2-2v_c^2}}{\sqrt{2}v_c} & \text{for } v > \frac{3}{2}v_c \end{cases} \tag{7}$$

where

$$v_c = \sqrt{2}\sigma. \tag{8}$$

In (6) and everywhere else in this chapter, the symbol $\lesssim$ " means that

$$\lim_{L\to\infty} \frac{\ln\left(\psi_{LBBM}(v) - \psi_{BBM}(v)\right)}{L/\sigma} \leq -\alpha(v).$$

In the N-BBM one starts as above with a single particle at $t = 0$ which diffuses and branches but the size of the population cannot exceed a fixed value N. As long as the number of particles $\mathcal{N}(t)$ is less than N the evolution is exactly the same as for the BBM. However, when $\mathcal{N}(t) = N$, as soon as a new branching event occurs, the left-most particle is eliminated so that the total number of particles remains subsequently equal to N.

For the N-BBM we will obtain for the large deviation function

$$\mathbf{P_{NBBM}}(X_{\max}(t) > vt) \sim \exp[-t\,\psi_{NBBM}(v)]. \tag{9}$$

an upper bound

$$0 \le \psi_{NBBM}(v) - \psi_{BBM}(v) \lesssim N^{-\beta(v)} \tag{10}$$

where

$$\beta(v) = \begin{cases} \frac{v^2}{v_c^2} - 1 & \text{for } v_c < v \le \sqrt{2}v_c \\ \frac{v^2}{2v_c^2} & \text{for } v \ge \sqrt{2}v_c \end{cases}, \tag{11}$$

where v_c is given by (8). In fact, as discussed in the conclusion, we believe that $\beta(v) = \frac{v^2}{v_c^2}$ remains valid even for $v > \sqrt{2}v_c$.

An important motivation in the study of the CBRW is its dual relation with the noised FKPP equation.

To explain how the CBRW is defined let us first consider a branching random walk BRW on a one dimensional lattice with lattice spacing σ: a particle on site x jumps to site $x + \sigma$ at rate $1/2$, to site $x - \sigma$ at rate $1/2$ and branches at rate r to give rise to two new particles on the same site.

The trajectory of each particle is a random walk and in the long time limit the probability that such a random walk reaches a position $x = vt$ is of the form

$$\mathbf{P_{RW}}(x = vt) \sim e^{-tf(v)} \tag{12}$$

where

$$f(v) = 1 - \sqrt{1 + \frac{v^2}{\sigma^2}} + \frac{v}{\sigma}\ln\left(\frac{v}{\sigma} + \sqrt{1 + \frac{v^2}{\sigma^2}}\right). \tag{13}$$

Using the fact that $\langle e^{\lambda x}\rangle = e^{tg(\lambda)}$ with

$$g(\lambda) = \cosh(\lambda\sigma) - 1, \tag{14}$$

the large deviation function (13) can be easily obtained from the parametric form as

$$f(v) = -g(\lambda) + \lambda g'(\lambda); \quad v = g'(\lambda) \tag{15}$$

As the particles branch at rate r, the distribution of the position $X_{\max}(t)$ of the right- most particle of this BRW, (in absence of coalescence), is of the form

$$\mathbf{P_{BRW}}(X_{\max}(t) > vt) \sim \exp[-t\,\psi_{BRW}(v)]. \tag{16}$$

with

$$\psi_{BRW}(v) = f(v) - r. \tag{17}$$

Now in the coalescing branching random walk (CBRW), in addition to the diffusion and the branching, we let each pair of particles on the same site coalesce at rate μ. We will show

$$\mathbf{P_{CBRW}}(X_{\max}(t) > vt) \sim \exp[-t\,\psi_{CBRW}(v)]. \tag{18}$$

and that for $\mu \to 0$

$$0 \leq \psi_{CBRW}(v) - \psi_{BRW}(v) \lesssim \mu^{-\gamma(v)} \quad (19)$$

where

$$\gamma(v) = \begin{cases} \frac{f'(v)}{f'(y)} - 1 & \text{for } v_c < v < v_1 \\ 1 & \text{for } v > v_1 \end{cases} \quad (20)$$

and, where for each v, y is solution of

$$\frac{f(y)-r}{f'(y)} - y = \frac{f(v)-r}{f'(v)} - v \quad (21)$$

with v_c and v_1 given by

$$\psi_{BRW}(v_c) = 0 \quad ; \quad \gamma(v_1) = 1, \quad (22)$$

(i.e. v_1 is the value of v such that $f'(v) = 2f'(y)$).
The general expression (20) simplifies when $r \ll 1$. One then has $v_c \simeq \sqrt{2r}\sigma$ and in the whole range $v_c < v \ll \sigma$

$$f(v) \simeq \frac{v^2}{2\sigma^2}$$

instead of (14). All the other steps remain the same with $y = \frac{2\sigma^2}{v}, v_1 = \sqrt{2}v_c$ and therefore

$$\psi_{BRW}(v) = \frac{v^2}{v_c^2} - 1\,; \quad \gamma(v) = \begin{cases} \frac{v^2}{v_c^2} - 1 & \text{for } v_c < v < \sqrt{2}v_c \\ 1 & \text{for } v > \sqrt{2}v_c \end{cases}, \quad (23)$$

If one would consider more general branching random walks, characterized by the rate $\rho(y)$ at which a particle jumps a distance y from the site it occupies, $g(\lambda)$ would be given by

$$g(\lambda) = \sum_y \rho(y)(e^{\lambda y} - 1) \quad (24)$$

and all the rest (15-22) would remain unchanged with only (14) replaced by (24).

Consider first all the possible trees of a BBM which, starting with a single particle at the origin, contain at least one particle which reaches, at time t, a position on the right of vt at time t.

Here we focus on velocities $v > v_c$ (for the BBM one knows that $v_c = \sqrt{2}\sigma$. The probability that the tree has at time t at least one particle on the right of vt is (2,3) for $v > v_c$

$$P \sim \exp\left[t\left(1 - \frac{v^2}{2\sigma^2}\right)\right] = \exp\left[t\left(1 - \frac{v^2}{v_c^2}\right)\right]. \quad (25)$$

For each such tree event, we will call red particles all the particles which end up on the right of vt. Given its position at time t, the trajectory of a red particle is, up to a shift (linear in time), a Brownian bridge (in fact it is more like a Brownian excursion but this has no incidence on the discussion below).

When one goes from the BBM to the L-BBM, a red particle will survive if between time 0 and time t no other particle of the BBM overtakes it by a distance L. Any tree of the BBM for which a red particle survives contributes to the event that the the rightmost particle of L-BBM is on the right of vt. So the probability that a tree of the BBM reaches position vt and that at least one red particle is never overtaken by any other particle of the BBM by a distance L is a lower bound for the probability that a L-BBM reaches position vt. This is why in the following, by estimating the survival probability of a red particle of a BBM, we will get an upper bound on the large deviation function (5) of the L-BBM.

As a red particle is moving on average faster than v_c the only possibility for it to be killed is that for a relatively short time interval s, i.e. a time $s \ll t$, either this red particle moves slower than v, or one of the other particles of the tree moves sufficiently fast to overtake it by a distance L or both.

So the picture is the following. A red particle moves at velocity v. Along its trajectory, branching events occur which give rise to subtrees. This red particle is then killed if, shortly after one of these branching events, the red particle slows down and one of the particles of the subtree overtakes it by a distance L.

Let us now be quantitative. The discussion below will hold for more general random walks, where the probability (25) would be replaced by

$$P \sim e^{t(1-f(v))} \tag{26}$$

where $f(v)$ is the large deviation function of the position of the random walk. In this general case v_c is given by

$$f(v_c) = 1. \tag{27}$$

The case of the branching Brownian motion will then be recovered by taking

$$f(v) = \frac{v^2}{2\sigma^2} = \frac{v^2}{v_c^2}. \tag{28}$$

One can show that, conditioned on the fact that a red particle moves at velocity v, the probability $P(x,s)$ that during a relatively short time interval $(\tau, \tau + s)$ (here $1 \ll s \ll t$) it moves a distance x is

$$P(x,s) \sim \exp\left[-s\left(f\left(\frac{x}{s}\right) - f(v) - \left(\frac{x}{s} - v\right)f'(v)\right)\right]. \tag{29}$$

Now the probability $Q(x,s)$ that at least one particle of the subtree created at time τ moves a distance $x + L$ during the time interval s is given by

$$Q(x,s) \lesssim \min\left\{1, \exp\left[s\left(1 - f\left(\frac{x+L}{s}\right)\right)\right]\right\}. \tag{30}$$

Therefore the probability p that such a subtree will kill the red particle is

$$p \lesssim \max_{s,x}\{P(x,s)Q(x,s)\}. \tag{31}$$

– If 1 dominates in (30) this means that the particle of the subtree moves at velocity v_c. In this case x and s are related by

$$x + L = v_c s \tag{32}$$

because for $x < L - v_c s$, $Q(x,s)$ would remain ≤ 1 but $P(x,s)$ would get smaller.

One can then see that the value of s which maximizes (31) is solution of

$$f\left(v_c - \frac{L}{s}\right) - f(v) - \left(v_c - \frac{L}{s} - v\right) f'(v) + \frac{L}{s} f'\left(v_c - \frac{L}{s}\right) - \frac{L}{s} f'(v) = 0.$$

This condition takes the form

$$f(y) - yf'(y) + v_c f'(y) = f(v) - vf'(v) + v_c f'(v) \quad (33)$$

where $y = v_c - L/s$ and this gives (31)

$$p \sim e^{-L(f'(v) - f'(y))} \quad (34)$$

Very much like in the remark at the end of the introduction, assuming as above that $g(\lambda) \to \infty$ as $\lambda \to \pm\infty$, one can show that (33) has always a solution.

As the number B_t of branching events along the red trajectory is of order t (for a rigorous justification, the survival probability of the red particle is

$$(1-p)^{B_t} \sim e^{-B_t p}$$

Therefore

$$\mathbf{P_{LBBM}}(X_{\max}(t) > vt) \gtrsim e^{t(1-f(v)) - B_t p}$$

and this implies that

$$\psi_{LBBM} - \psi_{BBM} \lesssim p \sim e^{-L(f'(v) - f'(y))} \quad (35)$$

In the particular case where $f(v) = v^2/(\sqrt{2}\sigma)$ the solution of (33) is $y = 2v_c - v$ and this leads to the announced result (30, 31).

When the second alternative dominates in (30) one needs to find the maximum over s and x of

$$s\left[-f\left(\frac{x}{s}\right) + f(v) + \left(\frac{x}{s} - v\right) f'(v) + 1 - f\left(\frac{x+L}{s}\right)\right].$$

This implies that $y = x/s$ and s are solutions of

$$f'(v) = f'(y) + f'\left(y + \frac{L}{s}\right)$$

$$-f(y) + f(v) + (y - v)f'(v) + 1 - f\left(y + \frac{L}{s}\right) + \frac{L}{s} f'\left(y + \frac{L}{s}\right) = 0. \quad (36)$$

After some algebra which uses (36) one ends up with the same expression (35), the only difference being that y is now solution of (36) instead of (33).

As v_c is solution of (3), one can check that the solution y of (36) reduces to the solution of (33) when $y + L/s \to v_c$, meaning that the rightmost particle of the subtree moves at the velocity v_c.

In the particular case where $f(v) = v^2/(\sqrt{2}\sigma)$ the solution of (36) is $y = (v - \sqrt{v^2 - 2v_c^2})/2$ (where $v_c = 2\sigma$) and this leads to the second line of (31).

In the N-BBM, the picture is rather similar and one has to estimate the probability p that a subtree will kill a red particle. To do so one needs the red particle to slow down so that the subtree produces N particles ahead of the red particle to eliminate it.

The probability that the red particle moves a distance x during time s is still given by (29). We now need to estimate the probability $Q(x, s)$ that the subtree produces, at time s, N particles on the right of position x. We do not have an expression for $Q(x,s)$ (see the discussion in the conclusion for a conjecture). One can however obtain an easy upper bound (using the Markov inequality)

$$Q(x,s) < \frac{\langle N(x,s)\rangle}{N}$$

where $N(x,s)$ is the number of particles of a subtree (of age s) on the right of position x. One has

$$\langle N(x,s)\rangle \sim \exp\left[s - \frac{x^2}{2\sigma^2 s}\right]$$

so that

$$Q(x,s) \lesssim \min\left[1, e^{s-\ln N - \frac{x^2}{2\sigma^2 s}}\right] \quad (37)$$

which, as for the L-BBM, we can write for more generality

$$Q(x,s) \lesssim \min\left[1, e^{s-\ln N - sf\left(\frac{x}{s}\right)}\right] \quad (38)$$

to treat the case of an arbitrary N-BBM.

Now we need to find a bound for p given by (27) and the discussion is very similar to what we did for the L-BBM:

If 1 dominates in (38), then $x = ys$ where s and y are related by

$$s - \ln N - sf(y) = 0. \quad (39)$$

The optimization of (31) under the constraint (39) leads to

$$p \sim e^{s[-f(y)+f(v)+(y-v)f'(v)]}$$

where y is solution of

$$\frac{1-f(y)+yf'(y)}{f'(y)} = \frac{1-f(v)+vf'(v)}{f'(v)} \quad (40)$$

[A solution $y \neq v$ exists for $v > v_c$ for the same reason as in (21).] One gets after some algebra

$$p \sim N^{-\frac{f'(v)-f'(y)}{f'(v)}}. \quad (41)$$

For the N-BBM, one has $f(v) = v^2/(\sqrt{2}\sigma)$ the solution of (40) is $y = 2\sigma^2/v$; so

$$\frac{f'(v)-f'(y)}{f'(v)} = \frac{v^2}{2\sigma^2} - 1,$$

and

$$p \sim N^{-\left(\frac{v^2}{2\sigma^2}-1\right)}. \quad (42)$$

This agrees with the first line of (11).

In the second alternative of (38)

$$p = \max_{s,y}\left(\exp\left[s\left(1-f(y)\right) - \ln N + s\left(-f(y)+f(v)+(y-v)f'(v)\right)\right]\right) \quad (43)$$

given that $s - \ln N + sf(y) \leq 0$. There is also the natural condition $\ln N < s$ (because it is highly unlikely to have more than e^s particles in a time $(1-\epsilon)s$, $\forall \epsilon > 0$) so that

$$s - sf(y) \leq \ln N \leq s. \quad (44)$$

The expression in the exponential (34) being linear in s, the maximum in s is achieved at one of the two boundaries in (44).

If the maximum is realized by the condition $s - sf(y) = \ln N$, one recovers the results (41) and (42). On the other hand, if the maximum is realized by $s = \ln N$, the optimal value of y in (43) is solution of

$$2f'(y) = f'(v) \tag{45}$$

and this leads to

$$p \sim N^{f(v) - vf'(v) - 2f(y) + 2yf'(y)}. \tag{46}$$

One can check that the range of validity of (41) is $v_c < v < v^*$ and for (46) is $v > v^*$ where v^* is the value of v where (40) and (45) have a common solution y. It is remarkable to notice that for $v = v^*$, both (41) and (46) coincide to give $p \sim N^{-1}$. For $f(v) = v^2/(2\sigma^2)$ the solution of (45) is $y = v/2$, which leads to

$$p \sim N^{-\frac{v^2}{4\sigma^2}}; \tag{47}$$

comparing (42) with (47), one can check that (42) holds for $v_c < v < v^* = \sqrt{2} v_c = 2\sigma$, while (47) is valid for $v > \sqrt{2} v_c$, as announced in (3.11).

For a branching random walk on a lattice, the probability that a red particle reaches the position vt with $v > v_c$ at time t is of the form $e^{t(r - f(v))}$.

For example, if the random walk is characterized by the probability $\rho(y)$ that the walker jumps a distance y from the site it occupies, $f(v)$ is given in a parametric form as

$$f(v) = -g(\lambda) + \lambda g'(\lambda); \quad v = g'(\lambda) \tag{48}$$

with $g(\lambda)$ given by (24). Given that the red particle moves on average at velocity v during time t, the probability $P(x, s)$ that it moves a distance x during a time interval $1 \ll s \ll t$ is as before (29) by

$$P(x,s) \sim \exp\left[s\left(-f\left(\frac{x}{s}\right) + f(v) + \left(\frac{x}{s} - v\right) f'(v)\right)\right].$$

On the other hand the number of particles produced by the subtree at position x at time s is $\lesssim e^{s(r - f(x/s))}$. Therefore the probability that the red particle is killed by a subtree of age s is

$$Q(x,s) \lesssim \min\left[1, \mu e^{s\left(r - f\left(\frac{x}{s}\right)\right)}\right]. \tag{49}$$

As for the L-BBM, one needs to distinguish two cases:

– If 1 dominates in (49) this means that x/s satisfies the relation

$$s\left(r - f\left(\frac{x}{s}\right)\right) + \ln \mu = 0 \tag{50}$$

then one has to maximize $P(x,s)$ given by (29) over s and x *given* the constraint(50).

This leads to the fact that $x = sy$ where y is solution of

$$\frac{f(y) - r}{f'(y)} - y = \frac{f(v) - r}{f'(v)} - v \tag{51}$$

and after some algebra to $p \sim \mu^{\frac{f'(v)}{f'(y)} - 1}$. This leads to (21).

– The other case, when (49) is dominated by $\mu e^{s(r - f(x/s))}$, is much easier. The optimum over s gives $s = 0$ and therefore $p \sim \mu$.

Section (3.2): Existence and Bounds for the Large Deviation Function

In this section we prove the existence of the large deviation functions (5,9,18). We first establish two elementary properties of the L-BBM if one starts at time 0 with N surviving particles. In view of the statement, we can assume $N \geq 2$.
For any $s \geq 0$, let $\mathcal{N}(s)$ be the number of surviving particles of the L-BBM at time s (so that $\mathcal{N}(0) = N$).

Lemma (3.2.1) [3]:

Let

$$\tau = \frac{aL^2}{2\sigma^2 \ln N}. \quad (52)$$

Then

$$\mathbf{P}\left[\exists s \in (0,\tau]: \mathcal{N}(s) < N^{\lambda}\right] > 1 - \frac{3}{N^{\mu}} \quad (53)$$

where a, b, λ and μ are constants which satisfy some conditions (56). For example, $a = 36$, $b = 3$, $\lambda = 17/18$ and $\mu = 1/18$ will work.

Proof. It suffices to establish the following upper bound

$$\mathbf{P}\left[\forall s \in (0,\tau]: \mathcal{N}(s) \geq N^{\lambda}\right] < \frac{3}{N^{\mu}}. \quad (54)$$

Let us write

$$M = N^{\lambda}.$$

Without loss of generality, one can choose the origin to be the position of the rightmost particle of the L-BBM at time 0. So all the initial positions are in $[-L, 0]$.

If we assume that $\mathcal{N}(s) \geq M$ at all times $s < \tau$, we want to follow the trajectories $x_1(s) \dots x_M(s)$ of M surviving particles between time $s = 0$ and time τ. At time $s = 0$ we choose any set of M different particles among the N present at time 0. Let $x_1(0) \dots x_M(0)$ be their positions at time 0. These particles move, branch and can get killed according to the rule of the L-BBM (they get killed as soon as their distance to the leading particle of the full L-BBM exceeds L). When one of these M particles gets killed, one replaces it immediately by any of the remaining $\mathcal{N}(s) - (M - 1)$. On the other hand, when one of them branches, one just keeps one of the two branches in our list of M particles and ignore the other branch. We obtain this way M trajectories. Let us denote $x_1(s) \dots x_M(s)$ the positions of these particles. These M trajectories are those of Brownian particles, except that whenever one of these particles gets killed, it is replaced by one of the surviving $\mathcal{N}(s) - M + 1$ particles of the L-BBM (i.e. the corresponding trajectory makes a jump to its right).

Let us consider also M regular Brownian motions which start at time $s = 0$ at the same positions as the above M particles of the L-BBM. We denote by $y_1(s) \dots y_M(s)$ the positions of these M Brownian particles at time s.

By a simple coupling argument it is clear that at any time $0 < s < \tau$ and for $1 \leq i \leq M$, one has $y_i(s) \leq x_i(s)$ so that $\max_{1 \leq i \leq M} y_i(s) \leq \max_{1 \leq i \leq M} x_i(s)$.

Therefore the probability Q that there exists at least one surviving particle of the full L-BBM on the right of some fixed position bL is bound from below by

$$Q \geq \mathbf{P}\left[\max_{1\leq i\leq M} x_i(s) > bL\right]$$
$$\geq \mathbf{P}\left[\max_{1\leq i\leq M} y_i(s) > bL\right] \geq 1 - \left[\int_{-\infty}^{(b+1)L/\sqrt{2\tau\sigma^2}} \frac{e^{-u^2}du}{\sqrt{\pi}}\right]^M.$$

Using the fact that for $x > 2$

$$\int_{-\infty}^{x} \frac{e^{-u^2}du}{\sqrt{\pi}} < 1 - e^{-2x^2} < \exp\left[-e^{-2x^2}\right]$$

and that for $y > 0$

$$e^{-y} < \frac{1}{y}$$

one gets that

$$Q > 1 - N^{\frac{2(b+1)^2}{a}-\lambda}. \qquad (55)$$

To complete the proof of (54), we now show that there is a small probability that the number $\widehat{\mathcal{N}}$ of particles of the L-BBM on the right of position $(b-1)L$ at time τ exceeds M. To do so, we first notice that

$$\mathbf{P}[\mathcal{N} > M] < \mathbf{P}\left[\widetilde{\mathcal{N}} > M\right]$$

where $\widetilde{\mathcal{N}}$ is the number of particles on the right of $(b-1)L$ at time τ generated by N independent BBM's (with no selection) starting all at time 0 at position L. One can calculate the expectation $\widetilde{\mathcal{N}}$

$$E\left[\widetilde{\mathcal{N}}\right] = Ne^{\tau}\int_{\frac{(b-1)L}{\sqrt{2\sigma^2\tau}}}^{\infty} \frac{e^{-u^2}du}{\sqrt{\pi}} < 2\, N^{1-\frac{(b-1)^2}{a}}$$

where we have used that for $x > 0$

$$\int_{x}^{\infty} \frac{e^{-u^2}du}{\sqrt{\pi}} < e^{-x^2}.$$

Therefore by the Markov inequality one gets

$$\mathbf{P}\left[\widehat{\mathcal{N}} > M\right] < 2N^{1-\lambda-\frac{(b-1)^2}{a}}.$$

Now we know that, at time τ, there is a probability Q close to 1 that there is at least one particle on the right of bL and a probability also close to 1 that $\widehat{\mathcal{N}} < M$. Therefore, because when there is at least one particle on the left of bL and no more than M particles on the right of $(b-1)L$, one knows that the total number of surviving particles of the L-BBM does not exceed M. Consequently,

$$\mathbf{P}[\mathcal{N}(\tau) > M] < 1 - Q + \mathbf{P}\left[\widehat{\mathcal{N}} > M\right] < 3N^{-\mu}$$

if we choose

$$\mu = -1 + \lambda + \frac{(b-1)^2}{a} = \lambda - \frac{2(b-1)^2}{a}. \qquad (56)$$

This completes the proof of (53).

Lemma (3.2.2) [3]:

There exist constants $c_1 > 0$ and $c_2 > 0$, depending only on (L, σ), such that

$$\mathbf{P}[\exists s \in (0, c_1]: \mathcal{N}(s) = 1] \geq c_2.$$

In words, Lemma (3.2.1) says that with a probability close to 1 when N is large, the number of surviving particles $\mathcal{N}(\tau)$ will be greatly reduced within a very short time τ (defined in (52)), whereas Lemma (3.2.2) ensures that no matter how large N is, within a time independent of N (but which may depend on L for example L^2), the total number of surviving particles will have become 1, at least once. In Lemma (3.2.2), it is possible to get moment estimates of the first time when the system has exactly a single particle.

Proof. Let $C > 0$ be a large constant independent of N. It suffices to prove that if one starts with an arbitrary number N of particles of the L-BBM, there is, uniformly in N, a positive probability $\tilde{Q}$ that the number of particles will be less than or equal to C at least once before a time of order 1.

To prove this statement, we use $k = k(C, N)$ times the result (53): the number k of steps needed is such that

$$N^{\lambda^k} < C \leq N^{\lambda^{k-1}}.$$

According to (53), one has

$$\tilde{Q} > \left(1 - \tfrac{3}{N^{\mu}}\right) \dots \left(1 - \tfrac{3}{N^{\mu\lambda^{k-1}}}\right) \geq \prod_{n=0}^{\infty} \left(1 - \tfrac{3}{C^{\mu\lambda^{-n}}}\right) > 0,$$

if the constant C is chosen sufficiently large such that $\frac{3}{N^{\mu\lambda^{k-1}}} < 1$; on the other hand, the time needed (52) for this to happen will be less than

$$\frac{aL^2}{2\sigma^2} \sum_{n\geq 0} \frac{\lambda^n}{\ln C} = \frac{aL^2}{2\sigma^2(1-\lambda)\ln C}.$$

This proves Lemma (3.2.2).

Now that we have proved Lemmas (3.2.1) and (3.2.2), it is quite easy to deduce the existence of the large deviation function for the L-BBM. Let $v \in (-\infty, \infty)$, and let

$$E_t := \{\exists \text{ particle in the } L - \text{BBM whose position at time } t \text{ is in } [vt, \infty)\}.$$

[Clearly, E_t depends on v, t and N.] The existence of the large deviation function we need to prove means the existence of $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t)$. We prove this by considering

$$E_t^{(1)} := \{\mathcal{N}(t) = 1, \text{and the unique particle at time } t \text{ lies in } [vt, \infty)\},$$

where $\mathcal{N}(t)$ denotes as before the number of particles in the L-BBM at time t. Clearly,

$$\mathbf{P}\left(E_{t+t'}^{(1)}\right) \geq \mathbf{P}\left(E_t^{(1)}\right)\mathbf{P}\left(E_{t'}^{(1)}\right), \quad \forall t \geq 0, t' \geq 0.$$

As such, the function $t \mapsto \ln \mathbf{P}(E_t^{(1)})$ is superadditive on $(0, \infty)$, and as t goes to infinity, $\frac{1}{t} \ln \mathbf{P}(E_t^{(1)}) \to \sup_{s>0} \frac{1}{s} \ln \mathbf{P}(E_s^{(1)}) \in (-\infty, 0]$.

The existence of $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t^{(1)})$ implies the existence of $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t)$; indeed, we trivially have

$$\mathbf{P}(E_t) \geq \mathbf{P}\left(E_t^{(1)}\right), \quad \forall t > 0,$$

because $E_t \supset E_t^{(1)}$. Conversely, by Lemma (3.2.2),

$$\mathbf{P}\left(E_{t+c_1}^{(1)}\right) \geq c_2\mathbf{P}(E_t), \quad \forall t > 0,$$

The last two inequalities together yield the existence of $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t)$, which equals $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t^{(1)})$.

For the N-BBM, we start with two simple but useful monotonicity properties. We include the elementary proof for the sake of self-containedness. We say that $(u_i)_{1\leq i\leq M}$ dominates $(v_i)_{1\leq i\leq N}$ if $\sum_{i=1}^{M} 1_{\{u_i\geq a\}} \geq \sum_{i=1}^{N} 1_{\{v_i\geq a\}}$ for all $a \in (-\infty, \infty)$ (so in particular, $M \geq N$).

Lemma (3.2.3) [3] (First Monotonicity Property for the N-BBM)

Let $x_1 \geq \cdots \geq x_N$ and $y_1 \geq \cdots \geq y_N$ be such that $x_i \geq y_i$ for all $1 \leq i \leq N$. There exists a coupling for two N-BBM systems on a same probability space, starting at positions $(x_i)_{1\leq i\leq N}$ and $(y_i)_{1\leq i\leq N}$ respectively, such that the first system dominates the second at all time.

Proof. Consider two N-BBM systems, the first starting at positions $(x_i)_{1\leq i\leq N}$, and the second at $(y_i)_{1\leq i\leq N}$. We attach the same Brownian motion to particles starting at x_i and y_i (for $1 \leq i \leq N$) respectively in the two systems, and also attach the same Poisson process which determines the branching times along the paths. As such, the first branching time is identical in the two systems, and before this time, the x-system obviously dominates the y-system. It is also easy to check that right after the first branching time, the x-system still dominates the y-system. Then by attaching as before the same Brownian motions and the same Poissonian clocks to the x- and the y-particles, the x-system will continue to dominate the y-system. And so on. The procedure leads to the desired coupling.

Lemma (3.2.4) [3] (Second Monotonicity Property for the N-BBM)

Let $N' \geq N$. Let $x_1 \geq \cdots \geq x_{N'}$ and $y_1 \geq \cdots \geq y_N$ be such that $x_i \geq y_i$ for all $1 \leq i \leq N$. There exists a coupling for an N'-BBM and an N-BBM on a same probability space, with initial positions $(x_i)_{1\leq i\leq N'}$ and $(y_i)_{1\leq i\leq N}$ respectively, such that the N'-BBM dominates the N-BBM all time.

Proof. If $N' = N$, this amounts to the previous lemma. So let us assume $N' > N$.

Then, as in the proof of the previous lemma, if initially the N rightmost particles of the system with N' particles dominates the other system, this remains true subsequently. The remaining $N' - N$ particles can only reinforce this domination.

Let us now turn to the proof of the existence of the large deviation function for the N-BBM. Let $v \in R$. Consider the following event for the N-BBM:

$$E_t := \{\exists \text{ particle whose position at time } t \text{ lies in } [vt, \infty)\}.$$

To prove the existence of the large deviation function, we need to show that the limit $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t)$ exists. We prove this by an argument of superadditivity. By removing all particles at time t except the rightmost one, the second monotonicity property stated in Lemma (3.2.4) tells us that

$$\mathbf{P}(E_{t+t'}) \geq \mathbf{P}(E_t)\mathbf{P}(E_{t'}), \quad \forall t \geq 0\,, t' \geq 0.$$

So the function $t \mapsto \ln \mathbf{P}(E_t)$ is superadditive on $(0, \infty)$. In particular,

$$\lim_{t\to\infty} \tfrac{1}{t} \ln \mathbf{P}(E_t) = \sup_{t>0} \frac{1}{t} \ln \mathbf{P}(E_t) \in (-\infty, 0],$$

exists.

The existence of the large deviation function of the CBRW is very similar. As in Lemma (3.2.4) for the N-BBM, the probability of the large deviation event increases with the number of initial particles. Consequently, by removing all particles except the rightmost one at time t, on sees that if E_t denotes the event that in the CBRW, there exists a particle lying in $[vt, \infty)$ at time t,

$$\mathbf{P}(E_{t+t'}) \geq \mathbf{P}(E_t)\mathbf{P}(E_{t'}), \quad \forall t \geq 0\,, t' \geq 0,$$

from which the existence of $\lim_{t\to\infty} \frac{1}{t} \ln \mathbf{P}(E_t)$ follows immediately.

We describe the strategy for the L-BBM. The strategy for the N-BBM will be along similar lines, with a few appropriate modifications indicated below.

Let E_t^{LBBM} denote as before the event that there exists at least one particle in the L-BBM whose position at time t lies in $[vt, \infty)$. To bound from below $\mathbf{P}(E_t^{LBBM})$, we consider the following event of the BBM (without selection):

$$\tilde{E}_t^{LBBM} := \bigcup_{i=1}^{\mathcal{N}(t)} \{\text{the particle } i \text{ lies in } [vt, \infty) \text{at time } t, \\ \text{not } L-\text{dominated, and leans to the left}\}. \tag{57}$$

Here, $\mathcal{N}(t)$ denotes, as before, the number of particles at time t. Leaning to the left means that the path of the particle lies in $(-\infty, t'v + t^{2/3}]$ for all $t' \in [0, t]^2$. We say that a particle with trajectory $[X_{t'}, t' \in [0, t])$ is L-dominated if at some time $t' \in [0, t]$ there is a particle lying in $[X_{t'} + L, \infty)$.

Clearly, if $\tilde{E}_t^{LBBM}$ is realized, then one can construct an L-BBM such that the large deviation event E_t^{LBBM} is realized. Therefore,

$$\mathbf{P}\big(\tilde{E}_t^{LBBM}\big) \leq \mathbf{P}(E_t^{LBBM}).$$

We estimate $\mathbf{P}\big(\tilde{E}_t^{LBBM}\big)$ which will serve as a lower bound for $\mathbf{P}(E_t^{LBBM})$. To bound $\mathbf{P}\big(\tilde{E}_t^{LBBM}\big)$ from below, let us write

$$\#\tilde{E}_t^{LBBM} := \sum_{i=1}^{\mathcal{N}(t)} 1_{\{\text{the particle } i \text{ lies in } [vt,\infty) \text{ at time } t, \text{not } L-\text{dominated,and leans to the left}\}}\,. \tag{58}$$

By the Cauchy–Schwarz inequality, we have

$$\mathbf{P}\big(\tilde{E}_t^{LBBM}\big) \geq \frac{\left[\mathbf{E}\big(\#\tilde{E}_t^{LBBM}\big)\right]^2}{\mathbf{E}\left[\big(\#\tilde{E}_t^{LBBM}\big)^2\right]}.$$

Therefore

$$\mathbf{P}(E_t^{LBBM}) \geq \frac{\left[\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right)\right]^2}{\mathbf{E}\left[\left(\#\tilde{E}_t^{LBBM}\right)^2\right]}. \tag{59}$$

We need to bound $\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right)$ from below, and bound $\mathbf{E}\left[\left(\#\tilde{E}_t^{LBBM}\right)^2\right]$ from above. The main estimates for the L-BBM which we obtain below are as follows:

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) \gtrsim \exp\left[-\left(\frac{v^2}{2\sigma^2} - 1 + e^{-[\alpha(v)+o_L(1)]L}\right)t\right], \tag{60}$$

$$\mathbf{E}\left[\left(\#\tilde{E}_t^{LBBM}\right)^2\right] \gtrsim \exp\left[-\left(\frac{v^2}{2\sigma^2} - 1\right)t\right], \tag{61}$$

with the value of $\alpha(v)$ given in (7). As before, the notation $a(t) \gtrsim b(t)$ or $b(t) \lesssim a(t)$ means that $\liminf_{t\to\infty} \frac{1}{t}\ln\left(\frac{a(t)}{b(t)}\right)$, whereas $o_L(1)$ denotes a term not depending on t, such that $\lim_{L\to\infty} o_{L(1)} = 0$. In view of the Cauchy–Schwarz inequality (59), it is clear that (60) and (61) together will imply the upper bound stated in (6) for the large deviation function ψ_{LBBM} of the L-BBM.
The next subsection is devoted to the proof of (60). The proof of (61), which is identical for all the three models.

We write $X = (X_u, u \in [0,t])$ for the trajectory of the particle i in the definition of $\#\tilde{E}_t^{LBBM}$, and write

$$A_t := \left\{X_u \leq uv + t^{\frac{2}{3}}, \forall u \in [0,t]\right\}, \tag{62}$$

which stands for the event that the particle i leans to the left. Then

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) = \int_{tv}^{\infty} \frac{e^{t-\frac{y^2}{2\sigma^2 t}}}{(2\pi\sigma^2 t)^{1/2}} \mathbf{E}\left(1_{A_t} \prod_{j:\tau_j \leq t} 1_{D_t^{LBBM}(\tau_j)} \middle| X_t = y\right) dy, \tag{63}$$

where, for all $u \in [0,t]$, $D_t^{LBBM}(u)$ stands for the event that the subtree of BBM branched at time u on the path of X does not produce any descendant going beyond X by distance $\geq L$ at any time during $[u,t]$. Here, $(\tau_j, j \geq 1)$ is a rate-2 Poisson process. The identity above, which is intuitively clear (except, maybe, for the rate being 2 instead of 1 which is a property of the Poisson process; we mention that the rate of the Poisson process plays no role in the final result).

It is easily guessed that the essential contribution to the integral $\int_{tv}^{\infty} \ldots dy$ on the right-hand side comes from the neighbourhood of $y = vt$. In any case, we can limit ourselves to the neighbourhood of $y = vt$ to pretend that it only gives a lower bound:

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) \gtrsim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t} \mathbf{E}\left(1_{A_t} \prod_{j:\tau_j \leq t} 1_{D_t^{LBBM}(\tau_j)} \middle| X_t = vt\right).$$

By conditioning upon $X := (X_u, u \in [0,t])$ and $\tau := (\tau_j, j \geq 1)$, we have

$$\mathbf{E}\left(1_{A_t}\prod_{j:\tau_j\le t}1_{D_t^{LBBM}(\tau_j)}\middle|X,\tau\right)=1_{A_t}\prod_{j:\tau_j\le t}\mathbf{P}_{\mathrm{X}}\big(D_t^{LBBM}(\tau_j)\big|\tau\big),$$

where $\mathbf{P}_{\mathrm{X}}(\cdot)\coloneqq\mathbf{P}(\cdot\,|X)$ denotes conditional probability given X. As such, writing $\mathbf{E}_{\mathrm{X}}$for expectation with respect to $\mathbf{P}_{\mathrm{X}}$, we have

$$\mathbf{E}_{\mathrm{X}}\left(1_{A_t}\prod_{j:\tau_j\le t}1_{D_t^{LBBM}(\tau_j)}\right)=1_{A_t}\mathbf{E}_{\mathrm{X}}\left(\prod_{j:\tau_j\le t}\mathbf{P}_{\mathrm{X}}\big(D_t^{LBBM}(\tau_j)\big|\tau\big)\right)$$
$$=1_{A_t}e^{-2\int_0^t\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du},$$

the second identity being a consequence of the fact that $(\tau_j,j\ge1)$ is a rate-2 Poisson process. Accordingly,

$$\mathbf{E}\big(\#\tilde{E}_t^{LBBM}\big)\gtrsim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t}\mathbf{E}\left\{1_{A_t}e^{-2\int_0^t\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du}\middle|X_t=vt\right\}.$$

Given $X_t=vt$, the process $(X_u,u\in[0,t])$ is a Brownian bridge of length t; it can be realized as $X_u=vu+\sigma(W_u-\frac{u}{t}W_t)$, where W is a standard Brownian motion (of variance 1). Thus

$$\mathbf{E}\big(\#\tilde{E}_t^{LBBM}\big)\gtrsim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t}\mathbf{E}\left\{1_{A_t^{(W)}}e^{-2\int_0^t\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du}\right\}.\qquad(64)$$

where

$$A_t^{(W)}:=\left\{W_u-\frac{u}{t}W_t\le\frac{t^{\frac{2}{3}}}{\sigma},\forall u\in[0,t]\right\}.$$

We will see that the indicator $1_{A_t^{(W)}}$ brings no significant difference to the expectation. Writing the conditional probability

$$\mathbf{P}^t(\cdot)\coloneqq\mathbf{P}\left(\cdot\,\middle|A_t^{(W)}\right),$$

and $\mathbf{E}^t(\cdot)$ for the associated expectation, we obtain:

$$\mathbf{E}\big(\#\tilde{E}_t^{LBBM}\big)\gtrsim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t}\mathbf{P}\left(A_t^{(W)}\right)\mathbf{E}^t\left(e^{-2\int_0^t\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du}\right).$$

By scaling, $\mathbf{P}\left(A_t^{(W)}\right)=\mathbf{P}\left\{W_r-rW_1\le\frac{t^{\frac{1}{6}}}{\sigma},\forall r\in[0,1]\right\}$, which converges to 1 when $t\to\infty$. So in our notation for "$\gtrsim$", we have

$$\mathbf{E}\big(\#\tilde{E}_t^{LBBM}\big)\gtrsim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t}\mathbf{E}^t\left(e^{-2\int_0^t\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du}\right)$$
$$\ge\exp\left\{-\left(\frac{v^2}{2\sigma^2}-1\right)t-2\int_0^t\mathbf{E}^t\left[1-\mathbf{P}_{\mathrm{X}}\big(D_t^{LBBM}(u)\big)\right]du\right\},$$

the last line following from Jensen's inequality. By definition,

$$\mathbf{E}^t\left[1-\mathbf{P}_{\mathrm{X}}\big(D_t^{LBBM}(u)\big)\right]=\frac{\mathbf{E}\left\{\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]1_{A_t^{(W)}}\right\}}{\mathbf{P}\left(A_t^{(W)}\right)}$$
$$\le\frac{\mathbf{E}\left[1-\mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]}{\mathbf{P}\left(A_t^{(W)}\right)}.$$

We have already seen that $\mathbf{P}\left(A_t^{(W)}\right) \to 1$, $t \to \infty$. So for all sufficiently large t (which will be taken for granted from now on), we have

$$\frac{2}{\mathbf{P}\left(A_t^{(W)}\right)} \leq 3.$$

As such,

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) \gtrsim \exp\left\{-\left(\frac{v^2}{2\sigma^2} - 1\right)t - 3\int_0^t \mathbf{E}\left[1 - \mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right)\right]du\right\}. \quad (65)$$

[So the presence of the indicator function $1_{A_t^{(W)}}$in (64) indeed has no significant influence.]

For all $s > 0$, let us write $M(s)$ for the maximal position at time s of a BBM independent of X. [This was denoted by $X_{\max}(s)$ in the introduction.] By definition of $D_t^{LBBM}(u)$,

$$\begin{aligned}1 - \mathbf{P}_{\mathrm{X}}\left(D_t^{LBBM}(u)\right) &= \mathbf{P}_{\mathrm{X}}(\exists s \in (0, t-u]: M(s) \geq L + X_{s+u} - X_u) \\ &\leq \int_0^{t-u} \mathbf{P}_{\mathrm{X}}(M(s) \geq L + X_{s+u} - X_u)ds. \quad (66)\end{aligned}$$

[The inequality in (66) is heuristic; it would be trivially true if s were an integer (in which case we would have a sum over s instead of an integral on the right-hand side). However, we can easily make it rigorous by arguing that

$$\begin{aligned}&\mathbf{P}_{\mathrm{X}}(\exists s \in (0, t-u]: M(s) \geq L + X_{s+u} - X_u) \\ &\leq \sum_{i=1}^{[t-u]+1} \mathbf{P}_{\mathrm{X}}\left(\sup_{s\in[i-1,i]} M(s) \geq L + \inf_{s\in[i-1,i]}(X_{s+u} - X_u)\right).\end{aligned}$$

The rest of the argument will go through, by noting that the tail probability of $\sup_{s\in[i-1,i]} M(s)$ behaves like the tail probability of $M(i)$ (in the sens of "$\lesssim$"), and that in the estimates of $j_t^{(1)}(u,s)$ and $j_t^{(2)}(u,s)$, instead of using the exact Gaussian distribution of $W_{s+u} - W_u - \frac{s}{t}W_t$, we can use the fact that the negative tail distribution of $\inf_{s\in[i-1,i]}\left(W_{s+u} - W_u - \frac{s}{t}W_t\right)$ is bounded by the Gaussian tail. The same argument applies to the N-BBM. For the CBRW, the situation is slightly different due to the fact that the space is discrete, but some obvious modifications to the argument readily make it rigorous.]

By the Markov inequality, $\boldsymbol{P}_X\{M(s) \geq L + X_{s+u} - X_u\}$ is bounded by the $\boldsymbol{P}_X$-expectation of the number of particles located beyond $L + X_{s+u} - X_u$ at time s; this $\boldsymbol{P}_X$-expectation is bounded by $\exp\left(s - \frac{(L+X_{s+u}-X_u)^2}{2\sigma^2 s}\right)$. Of course, this bound is interesting only when $L + X_{s+u} - X_u \geq (2\sigma^2)^{1/2}s$; otherwise, we use the trivial inequality $\boldsymbol{P}_X\{M(s) \geq L + X_{s+u} - X_u\} \leq 1$. As a consequence,

$$\begin{aligned}&1 - \boldsymbol{P}_X\left(D_t^{LBBM}(u)\right) \\ &\leq \int_0^{t-u}\left[1_{\left\{L+X_{s+u}-X_u<(2\sigma^2)^{\frac{1}{2}}s\right\}} + 1_{\{L+X_{s+u}-X_u\geq(2\sigma^2)^{1/2}s\}}\exp\left(s - \frac{(L+X_{s+u}-X_u)^2}{2\sigma^2 s}\right)\right]du.\end{aligned}$$

With the notation $X_u = vu + \sigma\left(W_u - \frac{s}{t}W_t\right)$, we have $L + X_{s+u} - X_u = L + vs + \sigma\left(W_{s+u} - W_u - \frac{s}{t}W_t\right)$. Assembling these pieces yields that

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) \gtrsim \exp\left\{-\left(\frac{v^2}{2\sigma^2} - 1\right)t - 3\int_0^t \left(\int_0^{t-u} \left[j_t^{(1)}(u,s) + j_t^{(2)}(u,s)\right] ds\right) du\right\}.$$

where

$$j_t^{(1)}(u,s) \coloneqq \mathbf{P}\left(L + vs + \sigma\left(W_{s+u} - W_u - \frac{s}{t}W_t\right) < (2\sigma^2)^{1/2}s\right),$$

$$j_t^{(2)}(u,s) \coloneqq \mathbf{E}\left(1_{\{L+vs+\sigma(W_{s+u}-W_u-\frac{s}{t}W_t)\geq(2\sigma^2)^{1/2}s\}} \times \exp\left(s - \frac{\left[L+vs+\sigma\left(W_{s+u}-W_u-\frac{s}{t}W_t\right)\right]^2}{2\sigma^2 s}\right)\right).$$

The random variable $W_{s+u} - W_u - \frac{s}{t}W_t$ has the Gaussian $\mathcal{N}\left(0, s\left(1 - \frac{s}{t}\right)\right)$ law. Some elementary but tedious computations lead to the following conclusion: in case $v > \left(\frac{9\sigma^2}{2}\right)^{1/2}$, the subtrees move forward faster than the usual speed $(2\sigma^2)^{1/2}$ (i.e., the integral of $j_t^{(2)}(u,s)$ dominates), whereas if $(2\sigma^2)^{1/2} < v \leq \left(\frac{9\sigma^2}{2}\right)^{1/2}$, these subtrees make no particular effort: they only need, in this case, to wait for the occasions when the red particle makes some fluctuations toward the left (which happens with some frequency). Letting $t \to \infty$ and then $L \to \infty$ (in this order), we obtain:

$$\mathbf{E}\left(\#\tilde{E}_t^{LBBM}\right) \geq \exp\left[-(1 + o(1))\left(\frac{v^2}{2\sigma^2} - 1 + e^{-(1+o_L(1))\alpha(v)L}\right)t\right],$$

where $\alpha(v)$ is given in (7). This is the desired lower bound (60).

The proof for the N-BBM is similar to the proof for the L-BBM, so we present only an outline, indicating the places where modifications are needed. We fix $0 < \epsilon < 1$, and write $M = M(\epsilon) \coloneqq [N^{1-\epsilon}]$. Consider

$$\tilde{E}_t^{NBBM} \coloneqq \bigcup_{i=1}^{\mathcal{N}(t)} \{\text{the particle } i \text{ lies in } [vt, \infty), \text{leans to the left,}$$
$$\text{does not split much, is not } M-\text{dominated}\}.$$

Let us explain the definition of $\tilde{E}_t^{NBBM}$. The meaning of "leans to the left" is as for the L-BBM: the path of the particle lies in $(-\infty, t'v + t^{2/3}]$ for all $t' \in [0,t]$. By "does not split much", we mean that the number of branchings (from the path of the particle i) at each of the time intervals $[(k-1)(\ln N)^2, k(\ln N)^2]$, for $1 \leq k \leq \frac{t}{(\ln N)^2}$, is bounded by $(\ln N)^3$. By "M-dominated", we mean the existence of a time $u \in [0,t]$ such that either there are at least M particles branching at time u from the path of the particle i lying in $[X_{t'}, \infty)$ at some time $t' \in [u, u + (\ln N)^2]$, or there is a particle branching at time u from the path of the particle i lying in $[X_{t'}, \infty)$ at some time $t' \in [u + (\ln N)^2, t]$ (if the interval is not empty).

The event $\tilde{E}_t^{NBBM}$ is the analogue, for the N-BBM, of the event $\tilde{E}_t^{LBBM}$ in (57). The probability $\mathbf{P}(\tilde{E}_t^{NBBM})$ will serve as a lower bound for the probability of the large deviation event for the N-BBM, because by definition, $\tilde{E}_t^{NBBM}$ implies the large deviation event for the N-BBM.

Write as before $\#\tilde{E}_t^{NBBM}$ for the number of i satisfying the conditions in $\tilde{E}_t^{NBBM}$. The main estimates for the N-BBM we are going to prove are:

$$\mathbf{E}\left(\#\tilde{E}_t^{NBBM}\right) \gtrsim \exp\left[-\left(\frac{v^2}{2\sigma^2} - 1 + M^{-\beta(v)+o_L(1)}\right)t\right], \qquad (67)$$

$$\mathbf{E}\left[\left(\#\tilde{E}_t^{NBBM}\right)^2\right] \lesssim \exp\left[-\left(\frac{v^2}{2\sigma^2} - 1\right)t\right], \qquad (68)$$

where $\beta(v)$ is defined in (11), and $o_L(1)$ stands for a term not depending on t such that $\lim_{L\to\infty} o_L(1) = 0$. Since ϵ can be as small as possible, (67) and (68) together with the Cauchy–Schwarz inequality will yield the upper bound stated in (10) for the large deviation function for the N-BBM.

The proof of (68), which is identical for all the three models. The rest of this discussion is devoted to the proof of (67).

Writing $X = (X_u, u \in [0,t])$ again for the trajectory of the red particle i, and $A_t := \left\{X_u \le uv + t^{\frac{2}{3}}, \forall u \in [0,t]\right\}$ as in (62), we have

$$\mathbf{E}\left(\#\tilde{E}_t^{NBBM}\right) = \int_{tv}^{\infty} \frac{e^{t-\frac{y^2}{2\sigma^2 t}}}{(2\pi\sigma^2 t)^{1/2}} \mathbf{E}\left(1_{A_t}\left(\prod_{k=1}^{t/(\ln N)^2} 1_{G_k}\right)\prod_{j:\tau_j\le t} 1_{D_t^{NBBM}(\tau_j)}\middle| X_t = y\right)dy,$$

where, for all $u \in [0,t]$, $D_t^{NBBM}(u)$ stands for the event that the subtree of BBM branched at time u on the path of X does not produce M descendants going beyond X at any time during $[u, u+(\ln N)^2]$ and does not produce any descendant going beyond X at any time during $[u+(\ln N)^2, t]$ (if the interval is non empty). Here, $(\tau_j, j \ge 1)$ is as before the atoms of a rate-2 Poisson process, and for each k, G_k is the event that the number of atoms $(\tau_j, j \ge 1)$ lying in $[(k-1)(\ln N)^2, k(\ln N)^2]$ is bounded by $(\ln N)^3$.

Once again, the essential contribution to the integral $\int_{tv}^{\infty} \ldots dy$ on the right-hand side comes from the neighbourhood of $y = vt$; we write

$$\mathbf{E}\left(\#\tilde{E}_t^{NBBM}\right) = e^{-\left(\frac{v^2}{2\sigma^2 t}-1\right)t}\mathbf{E}\left(1_{A_t}\left(\prod_{k=1}^{t/(\ln N)^2} 1_{G_k}\right)\prod_{j:\tau_j\le t} 1_{D_t^{NBBM}(\tau_j)}\middle| X_t = vt\right).$$

Compared to the discussions for the L-BBM in the previous subsection, we have a new factor $\prod_{k=1}^{t/(\ln N)^2} 1_{G_k}$; conditionally on the path of X, the probability of $\bigcap_{k=1}^{t/(\ln N)^2} G_k$ is at least $(1 - e^{-c_3(\ln N)^2})^{t/(\ln N)^2}$ (for some constant $c_3 > 0$), which is greater than or equal to $e^{-te^{-tc_4(\ln N)^2}}$ (for some constant $c_4 > 0$). As such, using again $\boldsymbol{P}_X$ to denote the conditional probability given X, we have

$$\mathbf{E}_X\left(\left(\prod_{k=1}^{t/(\ln N)^2} 1_{G_k}\right)\prod_{j:\tau_j\le t} 1_{D_t^{NBBM}(\tau_j)}\right)$$

$$\ge e^{-te^{-tc_4(\ln N)^2}}\mathbf{E}_X\left(\prod_{j:\tau_j\le t} \mathbf{P}_X\left(D_t^{NBBM}(\tau_j)\middle|\tau\right)\middle| \bigcap_{k=1}^{t/(\ln N)^2} G_k\right).$$

We have

$$\mathbf{E}_X\left(\prod_{j:\tau_j\le t}\mathbf{P}_X\big(D_t^{NBBM}(\tau_j)|\tau\big)\middle|\bigcap_{k=1}^{t/(\ln N)^2}G_k\right)\ge \mathbf{E}_X\left(\prod_{j:\tau_j\le t}\mathbf{P}_X\big(D_t^{NBBM}(\tau_j)|\tau\big)\right),$$

which equals $\exp\left\{-2\int_0^t[1-\mathbf{P}_X(D_t^{NBBM}(u))]du\right\}$.We can now carry out the same computations as in the case of the L-BBM, to see that

$$\mathbf{E}\big(\#\tilde{E}_t^{NBBM}\big)\gtrsim \exp\left\{-\left(\frac{v^2}{2\sigma^2}-1+e^{-c_4(\ln N)^2}\right)t-3\int_0^t\mathbf{E}\big[1-\mathbf{P}_X\big(D_t^{NBBM}(u)\big)\big]du\right\}.$$

[This is the analogue for the N-BBM, of the inequality in (65).]

For all $s>0$ and $x\in(-\infty,\infty)$, let us write $\mathcal{N}(x,s)$ for the number of particles lying in $[x,\infty)$ at time s in an BBM independent of X, and $M(s)$ the maximal position at time s of the BBM. By definition of $D_t^{NBBM}(u)$,

$$1-\mathbf{P}_X\big(D_t^{NBBM}(u)\big)\le \mathbf{P}_X(\exists s\in[(\ln N)^2,t-u]:M(s)\ge X_{s+u}-X_u)$$
$$+\mathbf{P}_X(\exists s\in[0,t-u]:\mathcal{N}(X_{s+u}-X_u,s)\ge M).$$

We argue that this implies

$$1-\mathbf{P}_X\big(D_t^{NBBM}(u)\big)\le\int_{(\ln N)^2}^{t-u}\mathbf{P}_X(M(s)\ge X_{s+u}-X_u)ds$$
$$+\int_0^{t-u}\mathbf{P}_X(\mathcal{N}(X_{s+u}-X_u,s)\ge M)ds,$$

even though the rigorous meaning of the inequality should be formulated as in the paragraph following (66).

The first probability expression on the right-hand side $\mathbf{P}_X(M(s)\ge X_{s+u}-X_u)$ is bounded by $\min\left[1,e^{s-\frac{(X_{s+u}-X_u)^2}{2\sigma^2 s}}\right]$. The probability $\mathbf{P}_X(\mathcal{N}(X_{s+u}-X_u,s)\ge M)$ was denoted by $Q(X_{s+u}-X_u,s)$ (with M in place of N), and we have seen in (38) that

$$\mathbf{P}_X(\mathcal{N}(X_{s+u}-X_u,s)\ge M)\le\min\left[1,e^{s-\ln M-\frac{(X_{s+u}-X_u)^2}{2\sigma^2 s}}\right].$$

As such,

$$1-\mathbf{P}_X\big(D_t^{NBBM}(u)\big)$$
$$\le\int_{(\ln N)^2}^{t-u}\left[1_{\left\{X_{s+u}-X_u<(2\sigma^2s^2)^{\frac12}\right\}}+1_{\left\{X_{s+u}-X_u\ge(2\sigma^2s^2)^{\frac12}\right\}}\exp\left(s-\frac{(X_{s+u}-X_u)^2}{2\sigma^2 s}\right)\right]du$$

$$+\int_{\ln M}^{t-u}\left[1_{\left\{X_{s+u}-X_u<[2\sigma^2s)(s-\ln M)]^{\frac12}\right\}}+1_{\left\{X_{s+u}-X_u\ge[2\sigma^2s)(s-\ln M)]^{\frac12}\right\}}\exp\left(s-\frac{(X_{s+u}-X_u)^2}{2\sigma^2 s}\right)\right]du.$$

With the notation $X_u=vu+\sigma\big(W_u-\frac{s}{t}W_t\big)$ (where W denotes again a standard Brownian motion with variance 1, we have $X_{s+u}-X_u=vs+\sigma\big(W_{s+u}-W_u-\frac{s}{t}W_t\big)$. The random variable $W_{s+u}-W_u-\frac{s}{t}W_t$ has the Gaussian $\mathcal{N}\left(0,s\big(1-\frac{s}{t}\big)\right)$ law. As for the L-BBM, some elementary computations yield that, in case $v>(4\sigma^2)^{1/2}$, the subtrees move forward faster than the usual speed $(2\sigma^2)^{1/2}$, whereas if $(2\sigma^2)^{1/2}<v\le(4\sigma^2)^{1/2}$, these subtrees make no particular effort, and wait only for the occasions

when the red particle makes some fluctuations toward the left. Letting $t \to \infty$ and then $N \to \infty$, we obtain:

$$\mathbf{E}\left(\#\tilde{E}_t^{NBBM}\right) \gtrsim \exp\left[-\left(\frac{v^2}{2\sigma^2} - 1 + e^{-c_4(\ln N)^2} + M^{-\beta(v)+o_N(1)}\right)t\right],$$

where $\beta(v)$ is defined in (11), and $o_N(1)$ stands for a term not depending on t such that $\lim_{N\to\infty} o_N(1) = 0$. Note that $e^{-c_4(\ln N)^2}$ is negligible compared to $M^{-\beta(v)+o_N(1)}$. This yields the desired lower bound (67).

The proof for the CBRW is along the lines of the proof for the L-BBM and for the N-BBM. Let

$$\tilde{E}_t^{CBRW} := \bigcup_{i=1}^{\mathcal{N}(t)} \{\text{the particle } i \text{ lies in } [vt,\infty), \text{leans to the left, does not coalesce}\}.$$

The meaning of "leans to the left" is as before: the path of the particle lies in $(-\infty, t'v + t^{2/3}]$ for all $t' \in [0,t]$. By "does not coalesce", we mean that at no time during $[0,t]$ does the particle coalesce with any other particle.

Let $\#\tilde{E}_t^{CBRW}$ denote the number of i satisfying the conditions in $\tilde{E}_t^{CBRW}$. The main estimates for the CBRW are:

$$\mathbf{E}\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right) \gtrsim \exp\left[-\left(f(v) - r + \mu^{\gamma(v)+o_\mu(1)}\right)t\right], \tag{69}$$

$$\mathbf{E}\left[\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right)^2\right] \gtrsim \exp[-(f(v) - r)t], \tag{70}$$

where $\gamma(v)$ and $f(v)$ are defined in (20) and (15) respectively, and $o_\mu(1)$ stands for a term not depending on t such that $\lim_{\mu\to\infty} o_\mu(1) = 0$. Equations (69) and (70) together with the Cauchy–Schwarz inequality will yield the upper bound stated in (19) for the large deviation function for the CBRW.

The proof of (70), which is identical for all the three models. The rest of this subsection is devoted to the proof of (69).

Writing $X = (X_u, u \in [0,t])$ again for the trajectory of the red particle i, and $A_t := \left\{X_u \le uv + t^{\frac{2}{3}}, \forall u \in [0,t]\right\}$ as in (62), we have

$$\mathbf{E}\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right) = \sum_{k: vt \le k\sigma \le vt + t^{2/3}} e^{rt} P(k\sigma;t) \mathbf{E}\left(1_{A_t} \prod_{j:\tau_j \le t} 1_{D_t^{\mathrm{CBRW}}(\tau_j)} \middle| X_t = k\sigma\right),$$

where $P(k\sigma;t)$ is the probability that a random walk is at position $k\sigma$ at time t, and for all $u \in [0,t]$, $D_t^{\mathrm{CBRW}}(u)$ stands for the event that none of the particles in the subtree of BBM branched at time u on the path of X coalesces with the red particle. Here, $(\tau_j, j \ge 1)$ is as before the atoms of a rate-2 Poisson process.

For $t \to \infty$, $P(k\sigma;t) \sim e^{-tf(k\sigma/t)}$ where f is as in (12), and the essential contribution to the sum on the right-hand side comes from $k \approx \frac{vt}{\sigma}$; we treat $\frac{vt}{\sigma}$ as an integer, and write

$$\mathbf{E}\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right) \gtrsim e^{-tf(k\sigma/t)} \mathbf{E}\left(1_{A_t} \prod_{j:\tau_j \le t} 1_{D_t^{\mathrm{CBRW}}(\tau_j)} \middle| X_t = vt\right).$$

The same computations as for the L-BBM (see (65)) give that

$$\mathbf{E}\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right) \gtrsim \exp\left\{t(r - f(v)) - 3\int_0^t \mathbf{E}\left[1 - \mathbf{P}_{\mathrm{X}}\left(D_t^{CBRW}(u)\right)\right] du\right\}.$$

As for the L-BBM, we argue that

$$1 - \mathbf{P}_{\mathrm{X}}\left(D_t^{CBRW}(u)\right) \leq \int_0^{t-u} \mathbf{P}_{\mathrm{X}}\left(B_{u,s}\right) ds,$$

where $B_{u,s}$ denotes the event that there exists a particle branched at time u that coalesces with the red particle at time $u + s$. [For a rigorous meaning of this inequality, see the paragraph following (66).] By the Markov inequality, $\mathbf{P}_{\mathrm{X}}\left(B_{u,s}\right)$ is bounded by the $\mathbf{P}_{\mathrm{X}}$-expected number of particles branched at time u that coalesce with the red particle at time $u + s$, and this $\mathbf{P}_{\mathrm{X}}$-expected number is approximately $\mu \exp\left[s\left(r - f\left(\frac{X_{s+u}-X_u}{s}\right)\right)\right]$.

On the other hand, $\mathbf{P}_{\mathrm{X}}\left(B_{u,s}\right) \leq 1$. So

$$\mathbf{P}_{\mathrm{X}}\left(B_{u,s}\right) \leq \min\left[1, \mu \exp\left(s\left(r - f\left(\frac{X_{s+u}-X_u}{s}\right)\right)\right)\right].$$

Taking expectation with respect to the law of the red particle, we arrive that

$$\mathbf{E}\left(\#\tilde{E}_t^{\mathrm{CBRW}}\right) \gtrsim \exp\left\{t\left(r - f\left(\frac{X_{s+u}-X_u}{s}\right)\right)\right.$$

$$\left. - 3\int_0^t du \int_0^{t-u} ds\, \mathbf{E}\min\left[1, \mu \exp\left(s\left(r - f\left(\frac{X_{s+u}-X_u}{s}\right)\right)\right)\right]\right\}.$$

From here, we can use the computations presented which leading to (50) and (51). This yields (69).

We use a common proof for (61) and (68), for the L-BBM and the N-BBM, respectively. The proof of (70), for the CBRW, is along similar lines, and is omitted.

It suffices to prove that

$$\mathbf{E}(\Lambda_t^2) \lesssim e^{-\left(\frac{v^2}{2\sigma^2 t}-1\right)t},$$

if $\Lambda_t = \Lambda_t(v)$ denotes the number of particles in the BBM (without selection) at time t lying in $[vt, \infty)$ and leaning on the left (i.e., whose trajectories are in $(-\infty, t'v + t^{2/3}]$ for all $t' \in [0, t]$).

By definition,

$$\mathbf{E}(\Lambda_t^2) \leq \mathbf{E}(\Lambda_t) + \int_0^t d\tau \int_{-\infty}^{v\tau+t^{2/3}} dy \frac{e^{\tau - \frac{y^2}{2\sigma^2\tau}}}{(2\pi\sigma^2\tau)^{1/2}} \left(\int_{v\tau}^{\infty} dz \frac{e^{(t-\tau)-\frac{(z-y)^2}{2\sigma^2(t-\tau)}}}{(2\pi\sigma^2(t-\tau))^{1/2}}\right)^2.$$

[It is an inequality because the trajectories are not required to lean on the left, but only lie in $(-\infty, v\tau + t^{2/3}]$ at time τ, when they split.] We have $\mathbf{E}(\Lambda_t) \leq e^{-\left(\frac{v^2}{2\sigma^2 t}-1\right)t}$.

It is convenient to split $\int_{-\infty}^{v\tau+t^{2/3}} dy$ into the sum of $\int_{-\infty}^{v\tau} dy$ and $\int_{v\tau}^{v\tau+t^{2/3}} dy$.

Since $y \longmapsto \frac{y^2}{2\sigma^2\tau} + \frac{(z_1-y)^2}{2\sigma^2(t-\tau)} + \frac{(z_2-y)^2}{2\sigma^2(t-\tau)}$ is non-decreasing on $[0, v\tau]$ (for all $z_1 \geq v\tau$ and $z_2 \geq v\tau$), it follows for the first integral that

$$\int_0^t d\tau \int_{-\infty}^{v\tau} dy \frac{e^{\tau - \frac{y^2}{2\sigma^2\tau}}}{(2\pi\sigma^2\tau)^{\frac{1}{2}}} \left(\int_{v\tau}^{\infty} dz \frac{e^{(t-\tau) - \frac{(z-y)^2}{2\sigma^2(t-\tau)}}}{\left(2\pi\sigma^2(t-\tau)\right)^{\frac{1}{2}}} \right)^2$$

$$\leq \int_0^t d\tau e^{\tau - \frac{v^2\tau}{2\sigma^2}} \left(e^{(t-\tau) - \frac{v^2(t-\tau)}{2\sigma^2}} \right)^2$$

$$\leq \int_0^t d\tau e^{-\left(\frac{v^2}{2\sigma^2}-1\right)(2t-\tau)}$$

$$\lesssim e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t},$$

using again our notation $a(t) \lesssim b(t)$ means that $\limsup_{t\to\infty} \frac{\ln[a(t)/b(t)]}{\ln t} \leq 0$.

A few more lines of elementary computations show that the extra integral $\int_{v\tau}^{v\tau+t^{2/3}} dy$ leads to an upper bound $e^{-\left(\frac{v^2}{2\sigma^2}-1\right)t+o(t)}$. Therefore, we get the claimed upper bound for $\mathbf{E}(\Lambda_t^2)$.

Chapter 4
Branching Brownian Motion with Absorption and the All-Time Minimum of Branching Brownian Motion with Drift

We study a dyadic branching Brownian motion on the real line with absorption at 0, drift $\mu \in \mathbb{R}$ and started from a single particle at position $x > 0$. When μ is large enough so that the process has a positive probability of survival, we consider $K(t)$, the number of individuals absorbed at 0 by time t and for $s \geq 0$ the functions $w_s(x) := \mathbb{E}^x\left[s^{K(\infty)}\right]$. We show that $w_s < \infty$ if and only of $s \in [0, s_0]$for some $s_0 > 1$ and we study the properties of these functions.

We give three descriptions of the family $w_s,\ s \in [0, s_0]$ through a single pair of functions, as the two extremal solutions of the Kolmogorov-Petrovskii-Piskunov (KPP) traveling wave equation on the half-line, through a martingale representation and as an explicit series expansion.

Section (4.1): The Tail Behaviour of $K(\infty)$ and the All-Time Minimum in a Branching Brownian Motion

Consider a branching Brownian motion in which particles move according to a Brownian motion with drift $\mu \in \mathbb{R}$ and split into two particles at rate β independently one from another. Call $\mathcal{N}_{all}(t)$ the population of all particles at time t and call $X_u(t)$ the position of a given particle $u \in \mathcal{N}_{all}(t)$. When we start with a single particle at position x we write $\mathbb{P}^x$ for the law of this process.
We considered the branching Brownian motion with absorption, i.e. the model just described with the additional property that particles entering the negative half-line $(-\infty, 0]$ are immediately absorbed and removed. We write $\mathcal{N}_{live}(t)$ for the set of particles alive (not absorbed) in the branching Brownian motion with absorption and $K(t)$ the number of particles that have been absorbed up to time t. The system with absorption is said to become extinct if $\exists t \geq 0 : \mathcal{N}_{live}(t) = \emptyset$ and to survive otherwise. We let $K(\infty) := \lim_{t \to \infty} K(t) \in \mathbb{R} \cup \{\infty\}$.

Depending on the value of μ one has the following behaviours:
Regime A: if $\mu \leq -\sqrt{2\beta}$, the drift towards origin is so large that the system goes extinct almost surely. $K(\infty)$ is finite and non-zero
Regime B: if $-\sqrt{2\beta} \leq \mu \leq \sqrt{2\beta}$ there is a non-zero probability of survival. On survival, there will always be particles near 0 and $K(\infty) = \infty$ almost surely.
Regime C: if $\mu \geq \sqrt{2\beta}$ there is still a non-zero probability of survival, but the system is drifting so fast away from 0 that, on survival, $min_{u \in \mathcal{N}_{all}(t)} X_u(t)$ drifts to $+\infty$ almost surely as $t \to \infty; K(\infty)$ is thus almost surely finite. Furthermore, there is a non-zero probability that $K(\infty) = 0$.

The behaviour of $K(\infty)$ in regime **A** ($\mu \leq -\sqrt{2\beta}$) has been the subject of very active research recently, including a conjecture by Aldous which was recently settled by P. Surprisingly, relatively little was known concerning the regimes **B** and **C**. Our main results at present concern the study of $K(\infty)$ and of certain related KPP-type equations.

In regime **A** ($\mu \leq -\sqrt{2\beta}$), the variable $K(\infty)$ has a very fat tail. More precisely we shows that, as $z \to \infty$, there exists two constants c, c' which depend on *x* such that

$$\mathbb{P}^x[K(\infty) > z] \sim \begin{cases} \dfrac{c}{z\log(z)^2} & \text{for } \mu = -\sqrt{2\beta}, \\ c'z^{-a(\mu)} & \text{for } \mu < -\sqrt{2\beta} \text{ where } a(\mu) = \dfrac{\mu + \sqrt{\mu^2 - 2\beta}}{\mu - \sqrt{\mu^2 - 2\beta}}. \end{cases}$$

In regime **B** ($-\sqrt{2\beta} < \mu < \sqrt{2\beta}$) it is clear that $K(\infty) = \infty$ on survival so one would essentially condition on extinction to study the tail behaviour of $K(\infty)$.

In regime **C** ($\mu \geq \sqrt{2\beta}$), however, $K(\infty)$ is almost surely finite. We introduce for $s \geq 0$ and $x \geq 0$,

$$w_s(x) := \mathbb{E}^x\left[s^{K(\infty)}\right], \quad w_s(0) = 0. \tag{1}$$

When $s \in [0,1]$ this quantity is the generating function of $K(\infty)$. We show that $w_s(x)$ is finite for some values of s larger than 1.

The probability that $K(\infty) = 0$ for a system started from x, is also the probability that the all-time minimum of a full branching Brownian motion with drift μ started from zero does not go below $-x$:

$$w(x) := w_0(x) = \mathbb{P}^x[K(\infty) = 0] = \mathbb{P}^0\left[\min_{t\geq 0} \min_{u \in \mathcal{N}_{all}(t)} X_u(t) > -x\right]. \tag{2}$$

This quantity, of course, is not trivial only in regime **C** ($\mu \geq \sqrt{2\beta}$). Then, since

$$\lim_{t\to\infty} \min_{u \in \mathcal{N}_{all}(t)} X_u(t) = +\infty$$

almost surely, we see that there is a well defined all-time minimum for the branching Brownian motion and we conclude that $\lim_{x\to\infty} X_u(t) = 1$.

It is not hard to see by standard arguments that w must satisfy a KPP-type differential equation with boundary conditions:

$$\begin{cases} 0 = \frac{1}{2}w'' + \mu w' + \beta(w^2 - w), & x \geq 0, \\ w(0) = 0, & w(\infty) = 1. \end{cases} \tag{3}$$

In fact, $w_s(x)$ introduced in (1), if finite, is solution to the same equation with the boundary condition $w(0) = 0$ replaced by $w_s(0) = 0$:

$$\begin{cases} 0 = \frac{1}{2} w_s'' + \mu w_s' + \beta(w_s^2 - w_s), & x \geq 0, \\ w_s(0) = 0, \quad w_s(\infty) = 1. \end{cases} \tag{4}$$

This is an example of the deep connection between branching Brownian motion and the KPP equation which noticed that one can represent solutions of the KPP equation as expectations of functionals of branching Brownian motions.

Until now this is very classical, however there is one unexpected difficulty here: both (3) and (4) admit infinitely many solutions and are not sufficient to characterize $w(x)$.

In this Chapter, we present three largely independent ways to characterize $w(x)$ which are laid out in the three following subsections. The first approach relies on partial differential equations, the second gives $w(x)$ as the expectation of a certain martingale and the third one gives $w(x)$ as a power series. One salient property of w is that it converges to 1 rather quickly.

A first way to characterize $w(x)$ is to track the probability that no particle got absorbed up to time t. Define

$$u(t,x) := \mathbb{P}^x[K(t) = 0]. \tag{5}$$

The function $u: \mathbb{R}_+^2 \mapsto [0,1]$ is increasing in x and decreasing in t and, clearly, for each x, $u(t,x) \mapsto w(x)$ as $t \to \infty$. Furthermore, u satisfies the KPP equation with boundary conditions

$$\begin{cases} \partial_t u = \frac{1}{2} \partial_{xx} u + \mu \partial_x u + \beta(u^2 - u), \\ u(t,0) = 0 \ (\forall t \geq 0), \ \ u(0,x) = 1 \ (\forall x > 0), \end{cases} \tag{6}$$

which, by Cauchy's Theorem has only one solution.
Therefore, to obtain $w(x)$, one can in principle solve (6) and take the large time limit.

There is an explicit probabilistic representation of the maximum standing wave w in regime **C** ($\mu \geq \sqrt{2\beta}$). Recall that $\mathcal{N}_{all}(t)$ is the population of all the particles in the branching Brownian motion with no absorption and $\mathcal{N}_{live}(t)$ is the population of particles alive at time t when we kill at 0. We now define on the same probability space a third process based on the branching Brownian motion in which particles that hit 0 are stopped but not removed from the system (they neither move nor branch). We denote by $\mathcal{N}_{live+abs.}(t)$ the set of particles alive at time t in this model. With a slight abuse of notations we continue to write $X_u(t)$ for the positions of particles when $u \in \mathcal{N}_{live+abs.}(t)$.
Let us define the following two processes:

$$Z_{live+abs.}(t) := \sum_{u \in \mathcal{N}_{live+abs.}(t)} e^{-rX_u(t)}, Z_{all}(t) := \sum_{u \in \mathcal{N}_{all}(t)} e^{-rX_u(t)} \tag{7}$$

where r is the asymptotic decay of $w(x)$. Rewriting $X_u(t) = Y_u(t) + \mu t$ it is clear that $\{Y_u(t), u \in \mathcal{N}_{all}(t)\}$ is simply a standard branching Brownian motion with no drift.
Therefore Z_{all} is the usual exponential martingale with parameter r associated with the branching Brownian motion Y. The process $Z_{live+abs.}$ is the martingale Z_{all} stopped on the line $t \wedge T_0$ (i.e. particles are stopped at time t or when they hit 0 for the first time). It is therefore also a martingale.

Lemma (4.1.1) [4]:

In regime **C** ($\mu \geq \sqrt{2\beta}$) the martingale $(Z_{live+abs.}(t), t \geq 0)$ converges almost surely and in L^1 to $K(\infty)$ and therefore $\mathbb{E}^x[K(\infty)] = Z_{live+abs.}(0) = e^{-rx}$.

We introduce a new probability measure $\mathbb{Q}^x$

$$\frac{d\mathbb{Q}^x}{d\mathbb{P}^x} = e^{rx} K(\infty).$$

Note that since $Z_{live+abs.}$ is a closed martingale we have that $\mathbb{E}^x[K(\infty)|\mathcal{F}_t] = Z_{live+abs.}(t)$. Thus

$$\left.\frac{d\mathbb{Q}^x}{d\mathbb{P}^x}\right|_{\mathcal{F}_t} = \frac{Z_{live+abs.}(t)}{Z_{live+abs.}(0)}.$$

Under this tilted probability measure, the law of the process is the same as the original $\mathbb{P}^x$ law except for the movement and branching rate of a distinguished particle (the spine particle ξ). The spine moves according to a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$, branches at an accelerated rate of 2β and stops (i.e. sticks and stops reproducing) upon hitting 0.

Proof. Recall that by (7)

$$Z_{live+abs.}(t) := \sum_{u \in \mathcal{N}_{live+abs.}(t)} e^{-rX_u(t)}, Z_{all}(t) := \sum_{u \in \mathcal{N}_{all}(t)} e^{-rX_u(t)} \qquad (8)$$

are positive martingales which therefore converge $\mathbb{P}$-almost surely to their respective limits $Z_{live+abs.}$ and Z_{all}. Furthermore, as $Z_{all}(t)$ is the usual additive martingale with parameter $r \geq \sqrt{2\beta}$, one has $Z_{all} = 0$. As the bounds

$$K(t) \leq Z_{live+abs.}(t) \leq K(t) + Z_{all}(t)$$

always hold, it is clear that $Z_{live+abs.} = K(\infty)$. The only thing left is to show that the convergence also holds in L^1.

We start by recalling the description of the measure $\mathbb{Q}^x|_{\mathcal{F}_t}$ which is defined by

$$\left.\frac{d\mathbb{Q}^x}{d\mathbb{P}^x}\right|_{\mathcal{F}_t} = \frac{Z_{live+abs.}(t)}{Z_{live+abs.}(0)}.$$

Standard arguments allow us to conclude that under $\mathbb{Q}^x$ the process behaves as follows: for $t \geq 0$, there is a distinguished line of descent (the *spine*) denoted $\xi(t) \in \mathcal{N}_{live+abs.}(t)$. Under $\mathbb{Q}^x$ the particle ξ moves according to a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$ and therefore almost surely hits 0 in finite time; we call $\tau_\xi = \inf\{t \geq 0: X_{\xi(t)}(t) = 0\}$ the time at which it reaches 0. For $t < \tau_\xi$, the spine branches at rate 2β creating non-spine particles which start new independent branching Brownian motion behaving according to the usual $\mathbb{P}$ law. After τ_ξ, the spine particle is frozen at zero (no motion, no branching). Observe that $\mathbb{Q}^x$ is actually the projection of the measure just described since under $\mathbb{Q}^x$ we do not know which is the spine particle ξ.

To prove the L^1 convergence of $Z_{live+abs.}(t)$ towards its limit $Z_{live+abs.} := \lim_t Z_{live+abs.}(t)$, it is sufficient to show that

$$\mathbb{Q}^x(Z_{live+abs.} < \infty) = 1.$$

As the time τ_ξ at which the spine is absorbed at 0 is $\mathbb{Q}$-almost surely finite, there are only finitely many branching events from the spine $\mathbb{Q}$-almost surely as well. At each of these events, a non-spine particle u starts its own independent $\mathbb{P}$ branching Brownian motion and we call $K_u(\infty)$ the total number of particles frozen at 0 that are descended from u. Let us also call $Z^{(u)}_{live+abs.}$ the analogue of the limit $Z_{live+abs.}$ (but we sum only on particles descended from u) and $Z^{(u)}_{all}$ is the same as $Z^{(u)}_{live+abs.}$but without any absorption or freezing at 0. It is clear as above that

$$K_u(\infty) \leq Z^{(u)}_{live+abs.} \leq K_u(\infty) + Z^{(u)}_{all}.$$

and that $Z^{(u)}_{all} = 0$. We conclude that $Z^{(u)}_{live+abs.} = K_u(\infty) < \infty$, $\mathbb{Q}$-almost surely, and finally

$$Z_{live+abs.} = K(\infty) < \infty \quad \mathbb{Q}\text{-almost surely.}$$

Observe that since $K(\infty) \geq 1$ almost surely under $\mathbb{Q}$, we have $\mathbb{Q} \sim \mathbb{P}$. Thus we know that under $\mathbb{P}$ $Z_{live+abs.}(t) \to Z_{live+abs.} = K(\infty)$ in L^1. Hence, $\mathbb{E}^x[K(\infty)] = Z_{live+abs.}(0) = e^{-rx}$.

Lemma (4.1.2) [4]:

Recall $w(x) = \mathbb{P}^x(K(\infty) = 0)$ and $w_s(x) := \mathbb{E}^x\left[s^{K(\infty)}\right]$ for $0 < s \leq s_0$ as usual. Then

$$1 - w(x) = \mathbb{Q}^x\left(\frac{1}{K(\infty)}\right)e^{-rx}$$

and

$$1 - w_s(x) = \mathbb{Q}^x\left(\frac{1 - s^{K(\infty)}}{K(\infty)}\right)e^{-rx}.$$

Proof. As $K(\infty) > 0$ $\mathbb{Q}^x$-almost surely, it is sufficient to prove the second assertion. Using that $K(\infty) = Z_{live+abs.}$ $\mathbb{P}^x$-almost surely,

$$1 - w_s(x) = \mathbb{P}^x\left[1 - s^{K(\infty)}\right] = \mathbb{P}^x\left[\left(1 - s^{K(\infty)}\right)\mathbb{I}_{\{Z_{live+abs.}>0\}}\right]$$

$$= \mathbb{P}^x\left[\left(1 - s^{K(\infty)}\right)\frac{Z_{live+abs.}(0)}{Z_{live+abs.}}\frac{Z_{live+abs.}}{Z_{live+abs.}(0)}\mathbb{I}_{\{Z_{live+abs.}>0\}}\right]$$

$$= \mathbb{Q}^x\left[\frac{Z_{live+abs.}(0)}{Z_{live+abs.}}\left(1 - s^{K(\infty)}\right)\mathbb{I}_{\{Z_{live+abs.}>0\}}\right]$$

$$= \mathbb{Q}^x\left(\frac{1-s^{K(\infty)}}{K(\infty)}\right)e^{-rx}.$$

Since we already know that $(1 - w_s(x))e^{rx}$ tends to a constant $B > 0$, it is now clear that the $\mathbb{Q}^x$ expectations in Lemma (4.1.2) also converge to B as $x \to \infty$. However, we are now going to define $\mathbb{Q}^\infty$ as the law of the process under which we can couple all the $\mathbb{Q}^x$ together and interpret the limit constant B as the expectation of a limit variable under $\mathbb{Q}^\infty$. Loosely speaking, we want $\mathbb{Q}^\infty$ to be the law of the process where the spine particle starts at $x = +\infty$ before drifting to 0. In fact it is easier to reverse time and have the spine start at 0 and drift to $+\infty$.

The function $w(x)$ can be understood in terms of series expansion. Let $\{a_n\}_{n\geq 1}$ be the sequence defined by

$$a_1 = 1, \quad a_n = \frac{\beta}{\frac{1}{2}n^2r^2 - n\mu r + \beta}\sum_{j=1}^{n-1} a_j a_{n-j}$$

$$= \frac{1}{(n-1)(\frac{r^2}{2\beta}n - 1)\frac{1}{2}n^2r^2 - n\mu r + \beta}\sum_{j=1}^{n-1} a_j a_{n-j} \;, n \geq 2 \qquad (9)$$

(recall that r was defined in (8), that $\frac{1}{2}r^2 - \mu r + \beta = 0$ and that $r \geq \mu \geq \sqrt{2\beta}$) and Φ the function defined by the series

$$\Phi(z) = \sum_{n\geq 1} a_n z^n \qquad (10)$$

The KPP partial differential equation,

$$\partial_t h = \frac{1}{2}\partial_{xx}h + \mu\partial_x u + \beta(h^2 - h), \qquad (11)$$

where $0 \leq h(t,x) \leq 1, h(t,-\infty) = 0$ and $h(t,+\infty) = 1$, describes how a stable phase ($h = 0$ on the left) invades an unstable phase ($h = 0$ on the right). It is well known that it admits travelling wave solutions of the form

$$h(t,x) = h_\mu(x - \mu t), \qquad 0 < h_\mu < 1, \quad h_\mu(-\infty) = 0, \quad h_\mu(+\infty) = 1,$$

for any velocity μ greater or equal to $\sqrt{2\beta}$. The travelling wave $x \mapsto h_\mu(x)$ is then solution to

$$\frac{1}{2}h_\mu'' + \mu h_\mu' + \beta\left(h_\mu^2 - h_\mu\right) = 0, \quad h_\mu(-\infty) = 0, \quad h_\mu(+\infty) = 1. \qquad (12)$$

The solution to (12) is unique up to translation.

Equation (12) for h_μ is very similar to equation (3) for $w(x)$, but h_μ, surprisingly, does not have the same asymptotic behaviour as w for large x, in the region where h_μ and w are close to 1. Indeed, linearising (12) around 1, one gets

$$\frac{1}{2}(1-\tilde{h}_\mu)'' + \mu(1-\tilde{h}_\mu)' + \beta(1-\tilde{h}_\mu) = 0, \text{ [linearized]} \qquad (13)$$

(a term of order $(1-\tilde{h}_\mu)^2$ has been neglected) and the general solution to (13) is, for some constants *A* and *B*,

$$1-\tilde{h}_\mu(x) = \begin{cases} Ae^{-\left(\mu-\sqrt{\mu^2-2\beta}\right)x} + Be^{-\left(\mu+\sqrt{\mu^2-2\beta}\right)x}, & for\ \ \mu > \sqrt{2\beta} \\ (Ax+B)e^{-\sqrt{2\beta}x} & , for\ \mu = \sqrt{2\beta} \end{cases} \text{ [linearized]} (14)$$

For x large, $\tilde{h}_\mu$ is close to h_μ with the meaning that for some constant *A* and *B*, $1-\tilde{h}_\mu \sim 1-h_\mu$. Of course, if $A \neq 0$, the term in factor of *B* is negligible compared to the term in factor of *A* and the *A* term alone is an equivalent to $1-h_\mu$. When solving (12), it turns out that the solution has a non-zero *A* term and that, therefore,

$$1-\tilde{h}_\mu(x) = \begin{cases} Ae^{-\left(\mu-\sqrt{\mu^2-2\beta}\right)x} & , for\ \ \mu > \sqrt{2\beta} \\ Axe^{-\sqrt{2\beta}x} & , for\ \mu = \sqrt{2\beta} \end{cases} \text{ [linearized]} \qquad (15)$$

where *A* depends on μ.

We now consider equation (3) for $w(x)$. Of course, the boundary condition of (3) is not sufficient to determine a unique solution, and for a range of values of *c* there exists a solution to

$$0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v), v(-\infty) = 0, v(+\infty) = 1, v'(0) = c. \qquad (16)$$

(The difference with equation (3) for $w(x)$ is the added condition $v'(0) = c$. One can then do, as above, a large *x* analysis of *v* and, the partial differential equation being the same, one finds again that $1-v \sim 1-\tilde{h}_\mu$ (*x* large) as given in (14) for some *c* and μ dependent values of *A* and *B*. Generically, *A* is non-zero and $1-v$ decays as $1-h_\mu$ in (15) (up to a multiplicative constant; the *A* is usually different and can even be negative).

However, for a well chosen value of c (depending on μ), one has $A = 0$, and the asymptotic decay of $1-v$ is given by the *B* term (that is: it decays much faster). We will show that $w(x)$ is precisely that very special solution to (3) that decays unlike all the other ones and unlike the travelling wave h_μ.

It is also interesting to remark that an equation very similar to (3) appears in the study of the the extinction probability of a branching Brownian motion with absorption and supercritical drift $\mu > -\sqrt{2\beta}$ (regimes **B** and **C**):

let $\theta(x) = \mathbb{P}^x[\mathcal{N}_{live}(\infty) = \emptyset]$ be the extinction probability when the system is started from *x*; then one has

$$\begin{cases} 0 = \frac{1}{2}\theta'' + \mu\theta' + \beta(\theta^2 - \theta)\,, & x \geq 0, \\ \theta(0) = 1, \quad \theta(\infty) = 0. \end{cases} \tag{17}$$

Equations (3) and (17) differ only by their boundary conditions; however (17) has a unique solution, whereas (3) has many.

A possible way to understand the difference is that an asymptotic analysis of $\theta(x)$ for large x similar to (14) yields only one possible exponential decay: $\theta(x) \sim A\exp[-(\mu + \sqrt{\mu^2 + 2\beta})x]$ for some constant A, which means that, up to translations, there is only one solution which does converge to zero at infinity, whereas for $w(x)$ there were two possible exponential decays and infinitely many solutions. Otherwise said, if one were to impose $\theta(0) = 1$ and $\theta'^{(0)} = -c$, there would only be one value of c for which θ would converge to zero at infinity.

In regimes **A** and **B** ($\mu < \sqrt{2\beta}$) one has $w(x) = 0$ because $K(\infty) > 0$ almost surely, which is not very interesting. What is more interesting is the way that $u(t,x)$, defined in (5) as $\mathbb{P}^x[K(t) = 0]$, converges to zero: it does so by assuming the shape of the critical travelling wave of the KPP equation. Let us recall quickly the well known facts on this critical travelling wave. Consider the KPP equation (11) without drift on the whole line with Heaviside initial conditions:

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_{xx} h + \beta(u^2 - h), \\ h(0,x) = 0\ (\forall x < 0),\ \ h(0,x) = 1\ (\forall x > 0). \end{cases} \tag{18}$$

It is well known that $h(t,x)$ is the probability that the leftmost particle at time t of a branching Brownian motion started at x is to the right of zero. Furthermore, this probability converges to the critical travelling wave in the following sense:

$$h(t, m_t + x) \to h_*(x) \text{ uniformly in } x \text{ as } t \to \infty$$

with (Bramson's displacement m_t [8])

$$m_t := \sqrt{2\beta}t - \frac{3}{2\sqrt{2\beta}}\log t + Cste \tag{19}$$

and where $h_* := h_{\sqrt{2\beta}}$ is the travelling wave moving at the minimal possible velocity $\sqrt{2\beta}$, see (12). To fix the invariance by translation, we impose the further condition $h_*(0) = \frac{1}{2}$:

$$\begin{cases} 0 = \frac{1}{2}h_*'' + \sqrt{2\beta}h_*' + \beta(h_*^2 - h_*), \\ h_s(-\infty) = 0,\ h_*(0) = \frac{1}{2},\ h_*(+\infty) = 1. \end{cases} \tag{20}$$

and the solution to (20) is now unique.

Adding a drift $+\mu\partial_x h$ to (18) would only shift the solutions by $-\mu t$ and would make (18) very similar to (6): the only difference would be that u is defined on R^+ and h on $\mathbb{R}$, but as both equations converge quickly to zero around the origin in regimes **A** and **B** ($\mu < \sqrt{2\beta}$), this difference turns out to be minimal.

We consider exclusively regime **C** ($\mu \geq \sqrt{2\beta}$) and we focus on the problem of the exponential moments of $K(\infty)$. We first establish some properties of $w_s(x) = \mathbb{E}^x\left[s^{K(\infty)}\right]$ as defined in (1) and proceed to prove Theorems (4.1.5) and Proposition (4.1.12). We then prove the asymptotic behaviour to complete the proof of Theorem (4.1.15).

The first property we need is that for a given s, the quantity $w_s(x)$ is either finite for all $x > 0$ or infinite for all $x > 0$:

Lemma (4.1.3) [4]:

For a given s, $(\exists x > 0 : w_s(x) < +\infty) \Leftrightarrow (\forall x > 0 : w_s(x) < +\infty)$.

Proof. Fix $s > 0, x > 0$ and $y > 0$. There is a positive probability, which we note $\epsilon(x, y)$, that the initial particle starting from x reaches position y before any branching or killing happens. Then

$$w_s(x) = \mathbb{E}^x\left[s^{K(\infty)}\right] \geq \epsilon(x, y)\mathbb{E}^y\left[s^{K(\infty)}\right] = \epsilon(x, y) w_s(y).$$

Therefore, if $w_s(x)$ is finite, then $w_s(y)$ is also finite.

We write $w_s < \infty$ when the conditions of the lemma (4.1.3) are met. Clearly, this is the case when $s \leq 1$ Furthermore, as $s \mapsto w_s(x)$ is obviously increasing, if $w_{s_0} < \infty$ for some s_0, then $w_s < \infty$ for all $s < s_0$.

When $w_s < \infty$, it is clear by standard arguments that $w_s(x)$ is solution to

$$\begin{cases} 0 = \frac{1}{2} w_s'' + \mu w_s' + \beta(w_s^2 - w_s), \\ w_s(0) = s. \end{cases} \tag{21}$$

Let us now prove Theorem (4.1.5). A slightly more general result is given by the following Lemma:

Lemma (4.1.4) [4]:

If $w_s < \infty$ then, for any $x \geq 0$ and $h \geq 0$,

$$w_s(x + h) = w_{w_s(h)}(x).$$

Setting $s = 0$ and renaming $w_0(h)$ as s gives the first line of the Theorem. Once we have proved that s_0 exists, setting $s = s_0$ and renaming $w_{s_0}(h)$ as s gives the second line of the Theorem.

Proof. Instead of starting our branching process at position x and killing particles at 0, it is here more convenient to think of the process as started at 0 and particles being absorbed at $-x$.

This allows to couple different values of the killing position. In particular, if $\mathcal{H}_x$ designates the particles stopped when they first hit $-x$ and $K_x(\infty)$ is the number of particles in $\mathcal{H}_x$, we have that

$$w_s(x+h) = \mathbb{E}\left[\prod_{u\in\mathcal{H}_x} s^{K_h^{(u)}(\infty)}\right] = \mathbb{E}\left[\prod_{u\in\mathcal{H}_x} w_s(h)\right] = \mathbb{E}\left[w_s(h)^{K_x(\infty)}\right] = w_{w_s(h)}(x),$$

where $K_h^{(u)}(\infty)$ is the total number of descendent of the particle u which are killed at $-x-h$ (which by translation invariance of the branching Brownian motion and the branching property is an independent copy of $K_h(\infty)$).

We now have a monotonicity result:

Theorem (4.1.5) [4]:

In regime **C** ($\mu \geq \sqrt{2\beta}$),

i. For each $s \in [0,1)$, $\quad w_s(x) = w_0\left(x + w_0^{-1}(s)\right)$,
ii. For each $s \in (1, s_0]$, $\quad w_s(x) = w_{s_0}\left(x + w_{s_0}^{-1}(s)\right)$.

We do not have an explicit expression for s_0 as a function of $\frac{\mu}{\sqrt{\beta}}$, but we can evaluate it numerically with a good precision. In the critical case $\mu = \sqrt{2\beta}$, we obtain $s_0 = 1.3486$ ...
The prove follow from Lemma (4.1.4).

Lemma (4.1.6) [4]:
i. If $s < 1, x \mapsto w_s(x)$ is increasing function converging to 1.
ii. If $s > 1$ and $w_s < \infty$ $x \mapsto w_s(x)$ is decreasing function converging to 1.

Proof. Once the increasing/decreasing part is proved, the fact that the limit is 1 is obvious: from its definition, it is clear that $w_s < 1$ if $s < 1$ and $w_s > 1$ if $s > 1$. Assuming w_s is increasing or decreasing (depending on s), it must have a limit, and from (21) that limit must be 1.
From its interpretation as the distribution of the all-time minimum of a branching Brownian motion, see (2), it is furthermore clear that $w_0 = w$ is an increasing function. Then, the coupling provided by Lemma (4.1.4) (or more simply Theorem (4.1.5)) implies that w_s is an increasing function for all $s < 1$.
Therefore, it only remains to prove that for $s > 1, w_s$ is decreasing when it is finite.
Assume $s > 1$ and $w_s < \infty$. We first show that w_s is monotonous by considering two cases:

- If $w_s'(0) > 0$ then, for all $h > 0$ small enough, $w_s(h) > s$. But, for x fixed, $s \mapsto w_s(x)$ is a strictly increasing function so $w_{w_s(h)}(x) > w_s$. Then by Lemma (4.1.4), $w_s(x+h) > w_s(x)$ for all x and all $h > 0$ small enough: w_s is increasing.
- If $w_s'(0) \leq 0$ then, for all $h > 0$ small enough, $w_s(h) < s$ because in the limit case $w_s'(0) = 0$, one has $w_s''(0) < 0$ from (21). Then, as in previous case,

$w_s(x+h) < w_{w_s(h)}(x) < w_s(x)$ for all x and all $h > 0$ small enough: w_s is decreasing.

It now remains to rule out the possibility that w_s is increasing for $s > 1$. Imagine that $s > 1$ and w_s increases. Then, from (21), $w_s''(x) \le -2\beta\big(w_s^2(x) - w_s(x)\big) \le -2\beta(s^2 - s)$ and $w_s'(x) \le w_s'(0) - 2\beta(s^2 - s)x$, which becomes negative for x large enough, in contradiction with the fact that w_s increases. So w_s must decrease for $s > 1$.

We need now to characterize the values of s for which $w_s < \infty$.

Lemma (4.1.7) [4]:

Assume $s > 1$. If there exists a function v which solves

$$\begin{cases} 0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v)\,, \\ v(0) = s, \qquad v(x) \ge 1 (\forall x > 0), \end{cases} \tag{22}$$

then $w_s < \infty$. Define

$$s_0 = \sup\{s \ge 1 : w_s < \infty\} = \sup\{s \ge 1 : \text{a solution to (22)exists}\} \tag{23}$$

and because $s \mapsto w_s(x)$ increases, one has $w_s < \infty$ for all $s < s_0$.

Proof. We present two proofs: one probabilistic and one analytical.

Choose $s > 1$ such that (22) has a solution v. We introduce the process

$$M_t \coloneqq \prod_{u \in \mathcal{N}_{live+abs.}(t)} v\big(X_u(t)\big),$$

where we recall that $\mathcal{N}_{live+abs.}(t)$ is the set of particles in the branching Brownian motion where particles are frozen at the origin.

M_t is a positive local martingale and therefore a positive super-martingale which thus converges almost surely to M_∞. Observe that under $\mathbb{P}^x$

$$v(x) = M_0 \ge \mathbb{E}^x(M_t) \ge \mathbb{E}^x(M_\infty).$$

But since for all $t \ge 0$ one has

$$M_t \ge v(0)^{K(t)} = s^{K(t)},$$

we see that $M_\infty \ge s^{K(\infty)}$ and therefore

$$w_s(x) = \mathbb{E}^x\big[s^{K(\infty)}\big] \le v(x) < \infty.$$

The same result can be proved analytically through the maximum principle. Let us introduce

$$u_s(t,x) \coloneqq \mathbb{E}^x\big[s^{K(\infty)}\big], \tag{24}$$

which is clearly solution to

$$\begin{cases} \partial_t u_s = \frac{1}{2}\partial_{xx} u_s + \mu \partial_x u_s + \beta({u_s}^2 - u_s), \quad x \ge 0, \\ u_s(t,0) = s \ (\forall t \ge 0), \ \ u_s(0,x) = 1 \ (\forall x > 0). \end{cases} \tag{25}$$

(Compare to (5).) With v as above, one clearly has $\forall x \ge 0, u_s(0,x) \le v(x)$. Therefore, by the maximum principle we have that

$$u_s(t,x) \le v(x), \ \ \forall t \ge 0, x \ge 0$$

and as $u_s(t,x) \nearrow w_s(x)$ as $t \to \infty$ we see that $w_s(x) \le v(x) < \infty$.

It is obvious that s_0 defined in (23) depends only on the ratio $\frac{\bar{\mu}}{\sqrt{\bar{\beta}}}$ by a simple scaling argument: the branching Brownian motion with drift μ and branching rate β is transformed, when time is scaled by λ and space by $\sqrt{\lambda}$, into a branching Brownian motion with drift $\mu\sqrt{\lambda}$ and branching rate $\beta\lambda$.

In particular, $w_{s,\beta,\mu}(x) = w_{s,\beta\lambda,\mu\sqrt{\lambda}}(\sqrt{\lambda}x) = w_{s,1,\mu/\sqrt{\beta}}(x/\sqrt{\beta})$ with the obvious new notation. What remains to be shown are the following properties of w_s: s_0 is finite, w_{s_0} is finite, $w'_{s_0}(0) = 0$ and $s_0 > 1$.

Lemma (4.1.8) [4]:

$s_0 < \infty$, that is: $K(\infty)$ does not have exponential moments of all orders.

Proof. For the system started from $x > 0$, consider the following family of events for $n \in \mathbb{N}$:

$$\mathcal{A}_n = \begin{cases} K(\infty) = n, \text{and} \\ \text{for all integers } i \le n, K(i) = i, \text{and} \\ \text{for all integers } i \le n, \text{there is at time } i \text{ only one particle alive and it sits in } [x, x+1]. \end{cases}$$

In words, for each $i \in \{0,1,\dots,n-1\}$ there is one particle alive at time i, it sits in $[x, x+1]$, and during a time interval one, this particle splits exactly once, one the offspring gets absorbed and the other is again in $[x, x+1]$ at time $i+1$. The one particle alive at time n generates a tree drifting to infinity with no more absorbed particles.

Let $\epsilon_{y,z}\mathrm{d}z$ be the probability that a particle sitting at y has, during a time interval one, exactly one splitting event with one offspring being absorbed and the other one ending up in $\mathrm{d}z$.

Define furthermore

$$q = \min_{y \in [x,x+1]} \int_x^{x+1} \epsilon_{y,z}\mathrm{d}z.$$

q is the minimal probability for a particle sitting somewhere in $[x, x+1]$ to have, during a time interval one, exactly one splitting event with one offspring being absorbed and the other one ending up in $[x, x+1]$. It is clear that $q > 0$ and that, furthermore,

$$\mathbb{P}^x[K(\infty) = n] > \mathbb{P}^x(\mathcal{A}_n) > q^n \mathbb{P}^x(K(\infty) = 0).$$

This implies that $\mathbb{E}^x\left[q^{-K(\infty)}\right] = \infty$ and that $s_0 \le {}^1\!/_q < \infty$.

Lemma (4.1.9) [4]:

$$w_{s_0} < \infty.$$

Proof. If $s_0 = 1$, this is trivial as $w_1 = 1$. Assume now $s_0 > 1$ and let us fix $x > 0$. For any $1 < s < s_0$, as w_s is decreasing, one has $w_s(x) < s < s_0$. This implies that (using the monotone convergence Theorem) $w_{s_0}(x) = \lim_{s \nearrow s_0} w_s(x)$ is finite which entails the result.

Lemma (4.1.10) [4]:

$$w'_{s_0}(0) = 0.$$

Proof. We already know that $w'_{s_0}(0) \leq 0$. Assuming $w'_{s_0}(0) < 0$, one could continue the function w_{s_0} to negative arguments using (22) and one could find a $x_0 < 0$ such that $w_{s_0}(x_0) > s_0$; then the function $x \mapsto w_{s_0}(x_0 + x)$ satisfies (22) with $s > s_0$, which is a contradiction.
The proof that $s_0 > 1$ for all $\mu \geq \sqrt{2\beta}$ is divided into two steps. First, we show that $s_0 > 1$ in the critical case $\mu = \sqrt{2\beta}$. Then we conclude by proving Proposition (4.1.12), which states that s_0 is an increasing function of $\mu/\sqrt{\beta}$.

Lemma (4.1.11) [4]:

In the critical case $\mu = \sqrt{2\beta}$, for $s > 1$ small enough, there exists solutions to (22), that is $s_0 > 1$.
Proof. Assume $\mu = \sqrt{2\beta}$. After the change of variables $\ell(x) := e^{\mu x}(v(x) - 1)$, (22) reads

$$\frac{1}{2}\ell'' + \beta e^{-\mu x}\ell^2 = 0 \tag{26}$$

Let us consider the solution to (26) with $\ell(0) = \ell'(0) = \epsilon$ for some $\epsilon > 0$. We want to prove that $\forall x, \ell(x) > 0$ if ϵ is small enough. Assume otherwise and call $x_0 = \inf\{x \geq 0 : \ell(x) = 0\}$. Then, as $\ell''(x) \leq 0$, we have $\ell(x \leq \epsilon + x\epsilon)$ and thus on $[0, x_0]$ (where $\ell(x) \geq 0$),

$$\ell''(x) \geq -2\beta\epsilon^2(1 + x)^2 e^{-\mu x}.$$

We conclude that

$$\ell'(x_0) \geq \epsilon - 2\beta\epsilon^2 \int_0^{x_0} (1 + x)^2 e^{-\mu x} \mathrm{d}x \geq \epsilon - 2\beta\epsilon^2 \int_0^{\infty} (1 + x)^2 e^{-\mu x} \mathrm{d}x,$$

which is strictly positive for s small enough. This contradicts the definition of x_0 and thus we have found a solution of (26) such that $\ell(x) > 0$ for all $x \geq 0$. Then $v(x) = 1 + \ell(x)e^{-\mu x}$ is a solution to (22) started from $s = 1 + \epsilon$; in other words $s_0 \geq 1 + \epsilon$ in the $\mu = \sqrt{2\beta}$ case.
Proposition (4.1.12) [4]:
s_0 is an increasing function of $\frac{\mu}{\sqrt{\beta}}$ and furthermore $s_0 \sim \frac{c\mu^2}{\beta}$ for some constant c as $\frac{\mu}{\sqrt{\beta}} \to \infty$.

Proof. Let us fix $\mu \geq \sqrt{2\beta_1} > \sqrt{2\beta_2}$. One can easily construct two branching Brownian motions with parameters (μ, β_1) and (μ, β_1) on the same probability space to realize a coupling so that the particles of the second one are a subset of the particles of the first one. It is then clear that for any $s > 1$ one has

$$w_{s,\beta_1,\mu}(x) \geq w_{s,\beta_2,\mu}(x) \tag{27}$$

(with the obvious extension of notation) so that

$$s_0\left(\mu/\sqrt{\beta_1}\right) \leq s_0\left(\mu/\sqrt{\beta_2}\right) \tag{28}$$

This already gives non-strict monotonicity and concludes the proof that $s_0 > 1$ for all μ, β with $\mu \geq \sqrt{2\beta}$.
We can now prove that the inequality (28) is strict. Assume otherwise; one would have $w_{s_0,\beta_1,\mu}(0) = w_{s_0,\beta_2,\mu}(0) = s_0$ (where $s_0 > 1$ would be the common value), $w'_{s_0,\beta_1,\mu}(0) = w'_{s_0,\beta_2,\mu}(0) = 0$ (from Lemma (4.1.10)) and, from (21), $w''_{s_0,\beta_1,\mu}(0) = -\beta_1(s_0^2 - s_0) < w''_{s_0,\beta_2,\mu}(0) = -\beta_2(s_0^2 - s_0)$, which would imply by Taylor expansion that for $x > 0$ small enough $w_{s_0,\beta_1,\mu}(x) < w_{s_0,\beta_2,\mu}(x)$ in contradiction with (27).

Finally, the only remaining point to complete the proof of Theorem (4.1.15) is the asymptotic behaviour (40), i.e. the assertion that for $x > 0$ fixed

$$p_n(x) := \mathbb{P}^x[K(\infty) = n] \sim \frac{-w'_{s_0}(x)}{2s_0^n n^{\frac{3}{2}}\sqrt{\pi\beta(s_0 - 1)}} \text{ as } n \to \infty. \tag{29}$$

Write $D(z, r)$ for the open disc of the complex plane with center $z \in \mathbb{C}$ and radius r. We extend the definition of $s \mapsto w_s(x)$ to $s \in \mathbb{C}$:

$$w_s(x) = \mathbb{E}^x\left[s^{K(\infty)}\right] = \sum_{n\geq 0} s^n p_n, \qquad \forall s \in \mathbb{C} \tag{30}$$

This quantity is analytical on $D(0, s_0)$ because the p_n in (30) are positive and the first singularity on the real axis is at s_0. Furthermore, it is finite on $\overline{D(0, s_0)}$ by uniform convergence because it is finite at s_0.
The key argument is an application of relying on the analysis of generating functions near their singular points. We need to show that

Lemma (4.1.13) [4]:

Fix $x > 0$. There exists $r_x > 0$ such that $s \mapsto w_s(x)$ is analytical in $V = D(s_0, r_x)\backslash[s_0, \infty)$, and

$$\partial_s w_s(x) \sim \frac{-w'_{s_0}(x)}{2\sqrt{\beta(s_0 - s)(s_0^2 - s)}} \text{ as } s \to s_0, s \in V, \tag{31}$$

and that

Proof. We know that $w'_{s_0}(0) = 0$ and $w''_{s_0}(0) = -2\beta(s_0^2 - s_0) < 0$. Since w_{s_0}solves the KPP traveling wave differential equation, for each $x \geq 0$ we can extend $s \longmapsto w_{s_0}(x+z)$ analytically on a neighborhood of zero in $\mathbb{C}$. In particular for $x = 0$ we have the following expansion:

$$w_{s_0}(z) = s_0 + \frac{w''_{s_0}(0)}{2} z^2 + o(z^2) \quad \text{as } z \to 0 \tag{32}$$

The function w_{s_0} is analytic and zero is a zero of order two of $w_{s_0}(z) - s_0$, there exists $r_1 > 0$ and a function ψ analytic and invertible on $D(0, r_1)$ such that

$$w_{s_0}(z) = s_0 + \frac{w''_{s_0}(0)}{2} \psi(z^2) \tag{33}$$

This means that

$$z = \psi^{-1}\left(\sqrt{\frac{w_{s_0}(z) - s_0}{w''_{s_0}(0)/2}}\right) = \psi^{-1}\left(\sqrt{\frac{s_0 - w_{s_0}(z)}{\beta(s_0^2 - s_0)}}\right) \tag{34}$$

for any z in $D(0, r_1)$ such that $w_{s_0}(z) \notin (s_0, \infty)$ (so that the right-hand side is well defined and is analytic on this domain when using the standard definition of the complex square root).
Recall from Lemma (4.1.4) that for any non-negative real x and z one has

$$w_{w_{s_0}(z)}(x) = w_{s_0}(z + x) \tag{35}$$

Replace the z in the right-hand side by its expression (34) and write $w_{s_0}(z)$ as s to obtain

$$w_s(x) = w_{s_0}\left(\psi^{-1}\left(\sqrt{\frac{s - s_0}{\beta(s_0^2 - s_0)}}\right) + x\right) \tag{36}$$

for $s \in w_{s_0}([0, r_1)) = (s_0 - r_2, s_0]$ for some $r_2 > 0$. But (4.42) is an equality between analytical functions as long as $s \in D(s_0, r_x) \backslash [s_0, \infty)$ for some $r_x > 0$ small enough (one must have $D(s_0, r_x) \subset w_{s_0}(D(0, r_1))$for ψ^{-1} to be analytical, which is possible by the open mapping Theorem, and one must have $\psi^{-1}(\ldots)$ small enough for w_{s_0} to be also analytical). From the analytical continuation principle, (36) must hold on the whole $D(s_0, r_x) \backslash [s_0, \infty)$ domain. Now differentiate with respect to s to get

$$\partial_s w_s(x) = -\frac{(\psi^{-1})'\left(\sqrt{\frac{s - s_0}{\beta(s_0^2 - s_0)}}\right)}{2\sqrt{\beta(s_0 - s)(s_0^2 - s)}} w'_{s_0}\left(\psi^{-1}\left(\sqrt{\frac{s - s_0}{\beta(s_0^2 - s_0)}}\right) + x\right), \tag{37}$$

yielding

$$\partial_s w_s(x) \sim -\frac{(\psi^{-1})'(0)}{2\sqrt{\beta(s_0-s)(s_0^2-s)}} w'_{s_0}(x) \quad \text{as } s \to s_0 \text{ in } D(s_0, r_x)\backslash[s_0, \infty). \quad (38)$$

A straightforward computation shows that $(\psi^{-1})'(0) = 1$, which concludes the proof.

Lemma (4.1.14) [4]:

Fix $x > 0$. There exists $\epsilon > 0$ such that $s \mapsto w_s(x)$ is analytical on $D(0, s_0 + \epsilon)\backslash[s_0, \infty)$.

Then, as $s \longmapsto \partial_s w_s(x)$, Lemma (4.1.12) leads to

$$(n+1)p_{n+1}(x) \sim \frac{-w'_{s_0}(x)}{2 s_0^{n+1} n^{\frac{1}{2}} \sqrt{\pi\beta(s_0-1)}} \text{ as } n \to \infty, \quad (39)$$

which obviously implies (29).

Theorem (4.1.15) [4]:

In regime **C** ($\mu \geq \sqrt{2\beta}$), there exists a finite $s_0 > 1$ depending only on $\frac{\mu}{\sqrt{\beta}}$ such that

i. For $s \leq s_0$, $w_s(x)$ is finite for all $x \geq 0$,
ii. For $s > s_0$, $w_s(x)$ is finite for all $x > 0$.

The functions $x \mapsto w_s(x)$ are increasing for any $s \in [0,1)$ and decreasing for any $s \in (1, s_0]$, converging to 1 when $x \to \infty$, and one has $w'_{s_0}(0) = 0$.
Furthermore, one has for n large

$$\mathbb{P}^x[K(\infty) = n] \sim \frac{-w'_{s_0}(x)}{2 s_0^{n} n^{\frac{3}{2}} \sqrt{\pi\beta(s_0-1)}}. \quad (40)$$

Using the branching structure and a simple coupling allows to relate the $w_s(x)$ with each others.

Proof. $w_s(x)$ is already analytical on $D(0, s_0)$. To prove the Lemma it is sufficient to show that it can be analytically extended around any point $s \in \partial D(0, s_0)\backslash\{s_0\}$. Indeed, by the finite covering property of compacts one can then show analycity on an open containing the compact $\partial D(0, s_0)\backslash D(s_0, r_x/2)$ with r_x defined in Lemma (4.1.12), and then we conclude with the help of Lemma (4.1.12).

So it now remains to see why $s \longmapsto w_s(x)$ can be analytically extended to neighborhoods of any $s \neq s_0$ with $|s| = s_0$. As we define

$$a(s) \coloneqq w'_s(0), \quad (41)$$

where we recall that the prime is a derivatice with respect to *x*. We first show analycity of $a(s)$ on $D(0, s_0)$ by writing an integral representation of $a(s)$: multiply (4) by $\exp[(\mu - \sqrt{\mu^2 + 2\beta})x]$ and integrate on $x \in [0, \infty)$. Integrate several times by part to get rid of the derivatives of w_s; one is left with

$$a(s) = \left(\mu - \sqrt{\mu^2 + 2\beta}\right) s + 2\beta \int_0^\infty \mathrm{d}x\, w_s(x)^2 e^{\left(\mu - \sqrt{\mu^2+2\beta}\right)x}. \qquad (42)$$

For any $x \geq 0$ and $s \in \overline{D(0, s_0)}$ one has $|w_s(x)| \leq w_{s_0} \leq s_0$. This implies that the convergence for x close to infinity of the integral in (42) is uniform on the disk $s \in \overline{D(0, s_0)}$. As $s \longmapsto w_s(x)$ is analytical on $D(0, s_0)$, this is sufficient to ensure that $s \longmapsto a(s)$ is also analytical on $D(0, s_0)$. Furthermore, notice that the series (4.35) defining $w_s(x)$ converges uniformly on $s \in \overline{D(0, s_0)}$ because we know it converges absolutely (all the $p_n(x)$ are non-negative) at $s = s_0$. This implies that $s \longmapsto w_s(x)$ is continuous on $\overline{D(0, s_0)}$ and, from the expression (42), so is $s \longmapsto a(s)$ (by dominated convergence Theorem since $|w_s(x)| \leq s_0$ on the closed disc).

We proceed to show that $a(s)$ can be extended analytically around any point $s \neq s_0$ with $|s| = s_0$ and show that the property extends to $w_s(x)$.
In Lemma (4.1.4) we showed for any $s \in [0, s_0]$ and any $x \geq 0$and $h \geq 0$ one had

$$w_s(x + h) = w_{w_s(h)}(x). \qquad (43)$$

One can check that the proof of Lemma (4.1.4) extends to complex s so that (43) remains valid for any $s \in C$ such that $w_s(x)$ is finite.

For fixed (complex) s, by deriving (4.47) with respect to h and then setting $h = 0$, one gets

$$w_s'(x) = a(s)\partial_s w_s(x). \qquad (45)$$

Derive again with respect to x, and then set $x = 0$:

$$w_s''(0) = a(s)\partial_s a(s), \qquad (46)$$

so that the differential equation (21) on $x \longmapsto w_s(x)$ applied at $x = 0$ gives

$$0 = \tfrac{1}{2} a(s)\partial_s a(s) + \mu a(s) + \beta(s^2 - s), \qquad (47)$$

This equation is valid for all $s \in D(0, s_0)$.

We now use the **Fact (*)** which we produce here with some notation for clarity:

Fact (*). Let H be a region in $\mathbb{C}$ and $s \longmapsto \phi(s)$ analytic in H. Let G be a region in $\mathbb{C}^2$ such that $(\phi(s), s) \in G$ for each $s \in H$ and suppose that there exists an analytic function $f: G \to \mathbb{C}$ such that

$$\phi'(s) = f(\phi(s), s), \quad \forall s \in H.$$

Let $s^* \in \partial H$. Suppose $\phi(s)$ is continuous at s^* and that $(\phi(s^*), s^*) \in G$. Then s^* is a regular point of $\phi(s)$ i.e. $\phi(s)$ admits an analytic continuation at s^*.

We apply to our case with $\phi = a, H = D(0, s_0)\backslash\{1\}$ and $G = \mathbb{C}^* \times \mathbb{C}$. From (47), the only candidate values of s in $D(0, s_0)$ such that $a(s) = 0$ are 0 and 1, and we know that $a(0) > 0$, so the condition "$(a(s), s) \in G$ for each $s \in H$" is verified.

The function $f(a, s)$ is obtained from (4.50): $f(a, s) = -2\mu + 2(s - s^2)/a$, and is obviously analytical on G; we have already shown that $a(s)$ is continuous on $\overline{D(0, s_0)}$. Therefore, for any point $s^* \in \partial D(0, s_0)$ such that $a(s^*) \neq 0$ (because we want $(a(s^*), s^*) \in G$), the function $a(s)$ admits an analytic continuation at s^*. We know that $a(s_0) = 0$, and we prove now that one has $a(s^*) \neq 0$ for any $s^* \in \partial D(0, s_0)\backslash\{s_0\}$, which will conclude the proof that $a(s)$ can be analytically continued around any point in $\partial D(0, s_0)\backslash\{s_0\}$. From (30) one can write $a(s)$ as a series:

$$a(s) = \sum_n p_n'(0) s^n. \tag{48}$$

We know that $p_1'(0) \leq 0$ (because $p_1(0) = 1$ and $p_1(x > 0) < 1$) and that for $n \neq 1$, $p_n'(0) \geq 0$ (because $p_n(0) = 0$ and $p_n(x > 0) > 0$). Since $a(s_0) = 0$, we write

$$\sum_{n \neq 1} p_n'(0) s_0^n = -p_1'(0) s_0. \tag{49}$$

All the terms on the left hand side are non-negative and infinitely many of them are non-zero since (47) does not have polynomial solutions. Thus, for any $s^* \in \partial D(0, s_0)\backslash\{s_0\}$ one has

$$\left|\sum_{n \neq 1} p_n'(0) (s^*)^n\right| < \sum_{n \neq 1} p_n'(0) s_0^n. \tag{50}$$

In particular, $a(s^*) \neq 0$ because it is the sum of two terms (the $\sum_{n=1}$ and the term $n = 1$) with different moduli and $s \longmapsto a(s)$ can be extended analytically around s^*.
We now show how the analycity of $a(s)$ translates into analycity of $w_s(x)$. First derive (43) again but this time with respect to x, then set $x = 0$, and rename h into x to obtain

$$w_s' = a[w_s(x)] = a(s)\partial_s w_s(x), \tag{51}$$

where we used (45) for the second equality.
For each given $s^* \in \partial D(0, s_0)\backslash\{s_0\}$ we consider a neighborhood V of s^* where $a(s)$ is analytical and we apply again Fact **(*)** to prove that $s \longmapsto w_s(x)$ is also analytical around s^*. This time, ~~we take~~ $\phi(s) = w_s(x)$ and $f(w, s) = a(w)/a(s)$ from (51). We pick $H = \partial D(0, s_0)\backslash\{1\}$ and $G = D(0, s_0) \times (H \cup V)$. For any $s \in \overline{D(0, s_0)}\backslash\{s_0\}$ one has $|w_s(x)| < w_{s_0} \leq s_0$ so that the condition $(\phi(s), s) \in G$ for each $s \in H$" is satisfied. We have already shown that $s \longmapsto w_s(x)$ is continuous on $\overline{D(0, s_0)}$, so we conclude that s^* is a regular point of $s \longmapsto w_s(x)$.

Theorem (4.1.16) [1]:

In regime **C** ($\mu \geq \sqrt{2\beta}$), the function $w(x)$ defined by (2) is the maximal solution of (3) such that $w(x) < 1$ for all $x \geq 0$.
More generally, $w_s(x)$ for $s < 1$ is the maximal solution of (4) that stays below 1 and $w_s(x)$ for $s \in (1, s_0]$ is the minimal solution of (4) that stays above 1.

Proof. We need to prove that w is the maximal solution remaining below 1 of the differential equation (3). This is an elementary application of the maximum principle again. Suppose that v is any solution of (3) which stays below 1. Since v is a standing wave solution of (6), that is $\tilde{u}(t,x) = v(x)$ for all $t \geq 0$ is a solution of

$$\partial_t \tilde{u} = \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x\tilde{u} + \beta(\tilde{u}^2 - \tilde{u}),$$

and since $v(x) < 1$ for all $x \geq 0$ we have that

$$v(x) \leq u(t,x), \quad \forall t \geq 0, \forall x \geq 0$$

where u is the solution of (6). As for each x we know that $t \longmapsto u(t,x) \searrow w(x)$ we conclude that $w(x) \geq v(x)$ and therefore w is the maximal solution of (3) bounded by 1. The same argument is easily generalized to the case of an arbitrary value of $s \in [0, s_0]$.

Theorem (4.1.17) [4]:

In regime **C** ($\mu \geq \sqrt{2\beta}$),

$$1 - w(x) = \mathbb{Q}^x\left(\frac{1}{K(\infty)}\right)e^{-rx}$$

Furthermore, $\mathbb{Q}^x\left(\frac{1}{K(\infty)}\right)$ converges to a finite constant $B > 0$ when $x \to \infty$ and thus, as $x \to \infty$,

$$1 - w(x) \sim Be^{-rx}.$$

More generally, for any $s \in [0, s_0]$, one has

$$1 - w_s(x) = \mathbb{Q}^x\left(\frac{1 - s^{K(\infty)}}{K(\infty)}\right)e^{-rx}$$

and the expectation $\mathbb{Q}^x(\cdot)$ converges to a finite positive constant as $x \to \infty$.
We will see in the proof that we can give an explicit representation of the constant B which appears as the expectation of $K(\infty)^{-1}$ under the measure $\mathbb{Q}^\infty$ (similar to $\mathbb{Q}^x$ but with the spine particle "started at infinity").

Proof. We start by proving Lemma (4.1.1), i.e. that the martingale $(Z_{live+abs.}(t), t \geq 0)$ converges $\mathbb{P}$-almost surely and in L^1 to $K(\infty)$ and therefore that

$$\mathbb{E}^x[K(\infty)] = e^{-rx}.$$

Lemma (4.1.18) [4]:

Let $(Y(t), t \geq 0)$ be a Brownian motion with drift $+\sqrt{\mu^2 - 2\beta}$, started from 0 conditioned to never hit 0. Otherwise said, Y is solution of the following stochastic differential equation

$$\begin{cases} \mathrm{d}Y(t) = \mathrm{d}B_t + \sqrt{\mu^2 - 2\beta}\coth\left(\sqrt{\mu^2 - 2\beta}Y(t)\right)\mathrm{d}t & \text{if } \mu > \sqrt{2\beta} \\ \mathrm{d}Y(t) = \mathrm{d}B_t + \frac{1}{Y(t)}\mathrm{d}t & \text{if } \mu = \sqrt{2\beta}. \end{cases}$$

Let (t_i) be a Poisson point process on $\mathbb{R}^+$ with intensity 2β. For each $i \geq 0$ start a branching Brownian motion with law $\mathbb{P}^{Y(t_i)}$ and call $\widetilde{K}_i$ the total number of absorbed particles at 0 for this process.

Fix $x > 0$, then the distribution of the variable $K(\infty)$ under $\mathbb{Q}^x$ is the same as that of

$$K^x(\infty) \coloneqq 1 + \sum_{i: t_i \leq \tau_x} \widetilde{K}_i$$

under $\mathbb{Q}^\infty$ where $\tau_x \coloneqq \sup_{t \geq 0}\{Y(t) = x\}$.

This result should be clear once it is realized that the process Y is the reversed path of the spine ξ.

Proof. We only treat the case $\mu > \sqrt{2\beta}$ since the zero-drift case is similar. The only thing we need to prove here is that if $(\xi(t), t \leq \tau_\xi)$ is a Brownian motion with drift $-\sqrt{\mu^2 - 2\beta}$ started form x and stopped at time $\tau_\xi \coloneqq \inf\{t: \xi(t) = 0\}$, then

$$\{(\xi(t), t \leq \tau_\xi), \tau_\xi\} \overset{\mathcal{L}}{=} \{(Y(\tau_x - t), t \leq \tau_x), \tau_x\}.$$

The upshot of Lemma (4.1.18) is that we can now construct the variables $K(\infty)$ under $\mathbb{Q}^x$ for all values of x simultaneously. We write $\mathbb{Q}^\infty$ for the joint law of the variables $(Y(t), t \geq 0), (t_i)_{i \in \mathbb{N}}, \widetilde{K}_i)$ described above. Then under $\mathbb{Q}^\infty$, clearly $(K^x(\infty), x \geq 0)$ is an increasing process in x. We call $K^\infty(\infty)$ its limit which is also the total number of particles absorbed at 0 under $\mathbb{Q}^\infty$.

Definition (4.1.19) [7]: (Borel-Cantelli Lemma)

If the sum of the probabilities of the $\mathbb{E}_n$ is finite

$$\sum_{n=1}^{\infty} Pr(\mathbb{E}_n) < \infty,$$

then the probability that infinitely many of them occur is 0, that is,

$$Pr\left(\limsup_{n\infty\to} \mathbb{E}_n\right) = 0.$$

Lemma (4.1.20) [4]:

We have that $K^\infty(\infty)$ $\mathbb{Q}^\infty$-almost surely.

Proof. We start with the $\mu > \sqrt{2\beta}$ case. First observe that for $\epsilon > 0$ fixed, there exists almost surely a random $i_0 \in \mathbb{N}$ such that

$$\forall i \geq i_0, \quad Y(t_i) \geq \left(\frac{\sqrt{\mu^2-2\beta}}{2\beta} - \epsilon\right) i.$$

This simply comes from the fact that $Y(t)/t \to \sqrt{\mu^2 - 2\beta}$ and $t_i/i \to (2\beta)^{-1}$ almost surely. Now, $1 - w_s(x) \leq e^{-(\mu+\sqrt{\mu^2-2\beta})x}$ because under $\mathbb{Q}^x$ we have that $Z_{live+abs.} \geq 1$ almost surely and therefore $\mathbb{Q}^x(1/Z_{live+abs.}) \leq 1$. Hence, for any $i \geq i_0$ we have that

$$\mathbb{Q}^\infty(\widetilde{K}_i > 0) \leq 1 - w\left(\left[\frac{\sqrt{\mu^2-2\beta}}{2\beta} - \epsilon\right] i\right) \leq e^{-ci}$$

for some positive constant c. Thus a straightforward application of Borel-Cantelli Lemma shows that almost surely, there exists $j_0 \in \mathbb{N}$ such that $\forall i \geq j_0, \widetilde{K}_i = 0$, which yields the desired result. The zero-drift case is similar. One just needs to start the argument by observing that for $\epsilon > 0$ fixed, there exists almost surely a random $i_0 \in \mathbb{N}$ such that

$$\forall i \geq i_0, \quad Y(t_i) \geq c i^{1/2-\epsilon}$$

where c is a constant. The proof then follows as before.

Lemma (4.1.20) [4]:

For $s \in [0, s_0]$ small enough, we have that

$$\mathbb{Q}^x\left(\frac{s^{K(\infty)}}{K(\infty)}\right) \to \mathbb{Q}^\infty\left(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}\right). \tag{52}$$

Proof. The monotone convergence Theorem applies when $s \leq 1$ so we suppose $1 < s \leq s_0$. Observe that the map $t \mapsto s^t/t$ is decreasing on $[1, 1/\log s]$ and increasing on $[1/\log s, \infty)$. Thus we write

$$\begin{aligned}
\mathbb{Q}^x\left(\frac{s^{K(\infty)}}{K(\infty)}\right) &= \mathbb{Q}^\infty\left(\frac{s^{K^x(\infty)}}{K^x(\infty)}\right) \\
&= \mathbb{Q}^\infty\left(\frac{s^{K^x(\infty)}}{K^x(\infty)}; K^x(\infty) \leq \frac{1}{\log s}\right) + \mathbb{Q}^\infty\left(\frac{s^{K^x(\infty)}}{K^x(\infty)}; K^x(\infty) \geq \frac{1}{\log s}\right) \\
&\to \mathbb{Q}^\infty\left(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}; K^\infty(\infty) \leq \frac{1}{\log s}\right) + \mathbb{Q}^\infty\left(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}; K^\infty(\infty) \geq \frac{1}{\log s}\right) \\
&= \mathbb{Q}^\infty\left(\frac{s^{K^\infty(\infty)}}{K^\infty(\infty)}\right)
\end{aligned}$$

where the first convergence comes from the dominated convergence Theorem and the second from the monotone convergence Theorem.

Proposition (4.1.21) [4]:

There exists $B > 0$ such that

$$1 - w(x) \sim B e^{-(\mu+\sqrt{\mu^2-2\beta})x} \quad \text{for large } x. \tag{53}$$

Furthermore, $w(x)$ is the only solution of (3) which remains in [0,1) and converges that fast to 1.

Similarly, for any $s \in [0, s_0)$, there exists $B_s \in \mathbb{R}$ such that

$$1 - w_s(x) \sim B_s e^{-(\mu+\sqrt{\mu^2-2\beta})x} \quad \text{for large } x. \tag{54}$$

where $B_1 = 0,\ B_s > 0$ when $s < 1$ and $B_s < 0$ when $s > 1$.

To simplify notation, we call r the exponential decay rate of $1 - w(x)$ as given in (53):

$$r := \mu + \sqrt{\mu^2 - 2\beta}. \tag{55}$$

It is the largest solution of $\frac{1}{2}r^2 - \mu r + \beta = 0$.

Proof. We consider here all the solutions $x \longmapsto v(x)$ to (3) that remains in [0,1). By Cauchy's theorem, a solution to (3) is entirely determined once the derivative at the origin is given.

Let r and $R < r$ be the two roots of the polynomial $\frac{1}{2}X^2 - \mu X + \beta$:

$$r = \mu + \sqrt{\mu^2 - 2\beta}\,, \qquad R = \mu - \sqrt{\mu^2 - 2\beta}.$$

(See also (55).) From the general theory of differential equations, one has:

Lemma (4.1.22) [4]:

Let v be a solution to

$$0 = \frac{1}{2}v'' + \mu v' + \beta(v^2 - v) \tag{56}$$

such that $v(x)$ converges to 1 as $x \to \infty$. Then, for some non-zero constant A or B,

- if $\mu > \sqrt{2\beta}$ one has either $1 - v(x) \sim Ae^{-Rx}$ or $1 - v(x) \sim Be^{-rx}$ as $x \to \infty$.
- If $\mu = \sqrt{2\beta} = r = R$ one has either $1 - v(x) \sim Axe^{-\mu x}$ or $1 - v(x) \sim Be^{-\mu x}$ as $x \to \infty$.

Furthermore, up to invariance by translation, there are exactly two solutions which converges to 1 in the fast way (as Be^{-rx}); one of them approaching 1 by above and the other from below.

This lemma simply tells that the solutions to the non-linear equation (56) behave around $v = 1$ as the solutions to the equation linearised around 1.

Proof. This follows from a result that shows that if

$$\dot{X} = \Gamma X + F(X) \tag{57}$$

is a non-linear differential system of dimension 2 with Γ a hyperbolic (eigen values have a non-zero real part) matrix and F is C^1 with $F(x) = o(|x|)$ as $x \to 0$ (so 0 is a critical point), then there exists a C^1 diffeomorphism ϕ with derivative the identity at the origin such that $U(t) = \phi(X(t))$ solves

$$\dot{U} = \Gamma U. \tag{58}$$

Otherwise said the solutions of the linearized system and the solutions of the non-linear system are (locally around 0) in one-to-one correspondence through ϕ. We apply this result to the following system where $v = 1 - u$ is a solution of (56)

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}; \dot{X} = \begin{pmatrix} u'(t) \\ u''(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ -2\mu y(t) - 2\beta[x(t) - x^2(t)] \end{pmatrix}, \quad (59)$$

which has a critical point at $(x, y) = (0,0)$. In this case

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -2\beta & -2\mu \end{pmatrix} \quad (60)$$

with eigenvalues $-r = -\mu - \sqrt{\mu^2 - 2\beta}$ and $-R = -\mu + \sqrt{\mu^2 - 2\beta}$ for simplicity we only consider the case $r \neq R$ here) and corresponding eigenvectors $\begin{pmatrix} 1 \\ -r \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -R \end{pmatrix}$. The solutions of $\dot{U} = \Gamma U$ are of the form

$$U(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = Be^{-rt}\begin{pmatrix} 1 \\ -r \end{pmatrix} + Ae^{-Rt}\begin{pmatrix} 1 \\ -R \end{pmatrix}. \quad (61)$$

Thus the only solutions such that $U(t) \sim ce^{-rt}$ for some constant c are those such that $A = 0$. If $B > 0$ then u_1 approaches by above, if $B < 0$ then u_1 approaches by below. Hartman's Theorem tells us that there exists

$$\phi(X) = X + f(X),\ f(X) = o(|x|) \text{ when } x \to 0 \quad (62)$$

such that the solutions $X(t)$ of the non-linear system are locally

$$X(t) = \phi^{-1}(U(t)). \quad (63)$$

Thus, (after a shift in the argument, replacing x by $x + \ln|B|/r$) there is exactly one solution X to the non linear system such that $|X(t)|e^{rt}$ has a non degenerate limit and such that $x(t)$, the first coordinate of $X(t)$, is eventually positive (resp. eventually negative).

Let $s < 1$ be such that there is a solution v of (56) with $v(0) = s, 1 - v(x) \sim ce^{-rt}$, and $v(x) < 1$ for all $x \geq 0$ (we now know that such s exists). Then $w_s(x)$ being the maximal solution of (3) that starts from s and stays below 1, we must have $w_s(x) \geq v(x), \forall x \geq 0$.

Since we also know that the only two possibilities for the asymptotic behavior of w_s are that either $w_s(x)e^{rx} \to c$ or $w_s(x)e^{Rx} \to c$ we conclude that it is the former that holds. The same argument apply for $w_s(x)$ for any $s \leq s_0$ and in the critical case. This concludes the proof of Proposition (4.1.21) for $w(x)$

Proposition (4.1.23) [4]:

The radius of convergence R of Φ is non-zero and there exists $B \in (0, R)$ such That

$$w(x) = 1 - \Phi(Be^{-rx}).$$

More generally, for any $0 \leq s \leq s_0$, there exists a number B_s such that

$$w_s(x) = 1 - \Phi(B_s e^{-rx}). \quad for\ all\ x \geq 0\ such\ that\ |B_s|e^{-rx} < R. \quad (64)$$

$s \longmapsto B_s$ is decreasing, positive for $s < 1$, zero for $s = 1$, and negative for $s > 1$. In particular, for $s \leq 1$, the condition $|B_s|e^{-rx} < R$ is automatically fulfilled.

Numerically, it seems that R is large enough that $|B_s|e^{-rx} < R$ for all $s \in [0, s_0]$ and all $x \geq 0$, but we haven't proved that point. The representation (64) makes it very easy to compute numerically w_s by first computing $\Phi(z)$, the value s_0 is then obtained as 1 minus the first minimum of Φ for negative arguments. This follows easily from the facts that $w'_{s_0} = 0, w''_{s_0} < 0$ and that $w'_{s_0} < 0$ for all $s \in (1, s_0)$.

Proof. We consider the series $\Phi(z) = \sum_{n\geq 1} a_n z^n$ defined in (10) with the coefficients a_n defined in (9). The function $z \mapsto \Phi(z)$ is a well defined object because, by induction on (9) one has easily $0 < a_n \leq 1$ and, therefore, $\mathcal{R} \geq 1$. It is then very easy to check by direct substitution that for any $B \in \mathbb{R}$, the function

$$x \mapsto v(x) = 1 - \Phi(Be^{-rx}) \text{ for } x \text{ such that } |B|e^{-rx} < R, \quad (65)$$

is solution to the partial differential equation $\frac{1}{2}v'' + \mu v' + \beta(v^2 - v)$ which appears in (3) (when discussing w) and in (21) (when discussing w_s). Recall that $r = \mu + \sqrt{\mu^2 - 2\beta}$ is the larger root of $\frac{1}{2}X^2 + \mu X + \beta$. As the coefficients a_n are positive, $\Phi(z)$ is non-negative and increasing for $z \geq 0$. As $a_1 = 1$ and $a_2 > 0$, it is easy to find a $0 < z_0 < 1 \leq \mathcal{R}$ such that $\Phi(z_0) > a_1 z_0 + a_2 z_0^2 > 1$. This implies that there must exists a $B_0 \in (0, \mathcal{R})$ (smaller than z_0) such that $\Phi(B_0) = 1$. With $B = B_0$, the function $v(x)$ in (65) is smaller than 1 and converges to 1 for large x as e^{-rx}. Using Proposition (4.1.21), this implies that $v(x) = w(x) = 1 - \Phi(B_0 e^{-rx})$.

Recall by Theorem (4.1.5) that w_s for $s < 1$ is simply equal to w correctly shifted to have $w_s(0) = s$. This implies that, for $s < 1$, $w_s(x) = 1 - \Phi(B_s e^{-rx})$ where $B_S \in (0, B_0]$ is such that $\Phi(B_s) = 1 - s$.

The case s $= 1$ is trivial, we now turn to $s > 1$. As for $s = 0$, we have the following points:

- for $s > 1$, w_s is the smallest solution to (21) that remains above 1(Theorem (4.1.16)).
- By Lemma (4.1.22), there is exactly one solution to (21) which remains above 1 and decays to 1 as e^{-rx}. Because of the previous point, this solution must be w_s.

Now consider $\Phi(z)$ for negative arguments. Because $\Phi(0) = 0$ and $\Phi'^{(0)} = 1$, there must exists $B \in (-\mathcal{R}, 0)$ such that Φ is negative on $(B, 0]$. Then, the function $x \mapsto 1 - \Phi(Be^{-rx})$ is solution to (21) for $s = 1 - \Phi(B) > 1$, remains above 1 for $x \geq 0$ and converges to 1 as e^{-rx}. Therefore, it must be w_s for that particular s.

But all the functions w_s for≤1 $< s < s_0$ are related through Theorem (4.1.5): they are all shifted versions of w_{s_0}. Therefore, for any $s \in (1, s_0]$, one has

$w_s(x) = 1 - \Phi(B_s e^{-rx})$ for a well chosen negative B_s (which represents the shift), at least for values of x sufficiently large to have $|B_s|e^{-rx} < R$.

Lemma (4.1.24) [4]:

$$\|v^T(\cdot,\cdot) - u(\cdot,\cdot)\|_\infty \to 0 \ \ as\ T \to \infty. \tag{66}$$

In addition, there exists $C \in \mathbb{R}$ such that $C_T \to C\ as\ T \to \infty$.

Indeed, assuming that Lemma (4.1.24) holds, we can conclude:

Theorem (4.1.25) [4]:

In regimes **A** and **B** ($\mu < \sqrt{2\beta}$), there exists a constant C depending on μ and β such that

$$u(t, x + m_t - \mu t + C) \to h_*(x) \text{ uniformly in } x \text{ as } t \to \infty$$

where m_t is given by (19) and h_* is the solution to (20).

It is interesting to compare this result about the behaviour of $u(t,x) = \mathbb{P}^x[K(t) = 0]$ when $\mu < \sqrt{2\beta}$ to the behaviour of the extinction probability $\tilde{u}(t,x) = \mathbb{P}^x[\mathcal{N}_{live}(t) = \emptyset]$ when $\mu \le -\sqrt{2\beta}$. It is not hard to see that $\tilde{u}$ satisfies the same equation (6) as u with different boundary conditions, which is 1 minus the boundary condition in (6); namely $\tilde{u}$ solves

$$\begin{cases} \partial_t \tilde{u} = \dfrac{1}{2}\partial_{xx}\tilde{u} + \beta(\tilde{u}^2 - \tilde{u}), \\ \tilde{u}(t,0) = 1\ (\forall t \ge 0),\ \ \tilde{u}(0,x) = 0\ (\forall x > 0). \end{cases} \tag{67}$$

What is particularly striking is that in the critical case $\mu = -\sqrt{2\beta}$ it is known to survive up to time t one must start with an initial particle at position $x = ct^{1/3}$. This means that if $\tilde{u}(t, x + \tilde{m}_t)$ converges to some limit front shape then the centering term giving the position of the front $\tilde{m}_t$ has to be of order $ct^{1/3}$. However the convergence of the solution of (67) to a travelling wave is at present an open problem.

Proof. We assume to be in regime $\overline{\mathbf{A}}$ or **B** ($\mu < \sqrt{2\beta}$) and we want to show how $u(t,x) = \mathbb{P}^x(K(t) = 0)$ converges to a KPP travelling wave.

The proof is essentially analytic and relies on the maximum principle. The key step is to compare $u(t,x)$ to a new function $v^T: [T, +\infty) \times \mathbb{R} \to \mathbb{R}$ (where $T \ge 0$ is a parameter) where $v^T(t,x)$ is defined as the probability, in the standard branching Brownian motion (without absorption nor stopping) with drift μ starting from x, that no particles are present in the negative half-line between times $t - T$ and t. In symbols

$$v^T(t,x) := \mathbb{P}^x(\forall r \in [t-T, t], \forall u \in \mathcal{N}_{all}(r): X_u(r) > 0), \tag{68}$$

where we recall that $\mathcal{N}_{all}(s)$ is the population of particles at time s in a branching Brownian motion without absorption or stopping. The advantage of v^T is that since it is defined on $\mathcal{N}_{all}$ it satisfies a KPP equation on the whole real line:

$$\begin{cases} \partial_x v^T = \frac{1}{2}\partial_{xx} v^T + \mu \partial_x v^T + \beta((v^T)^2 - v^T), & (t,x) \in [T, +\infty) \times \mathbb{R} \\ v^T(T,x) = u(T,x), & \text{for } x \geq 0, \\ v^T(T,x) = 0, & \text{for } x < 0. \end{cases}$$

Otherwise said the function $\tilde{v}^T(t,x) = v^T(T+t,x)$ solves the KPP equation on the whole line with initial condition $\tilde{v}^T(0,x) = u(T,x)\mathbb{I}_{\{x>0\}}$. Since for $T > 0$ fixed, $1 - u(T,x)$ goes to 0 as $x \to \infty$ with a super exponential decay, ensures that there exists a constant $\tilde{C}_T \in \mathbb{R}$ such that we have

$$\left\| \tilde{v}^T\left(t, \cdot + m_t - \mu t + \tilde{C}_T\right) - h_*(\cdot) \right\|_\infty \to 0 \ \ as\, t \to \infty, \quad (69)$$

where Bramson's displacement m_t is given in (19). The value $\tilde{C}_T$ depends on T because for different T we plug different initial conditions in the KPP equation.
Since $m_t - m_{t-T} \to \sqrt{2\beta}T$ when $t \to \infty$, one obtains taking $C_T = \tilde{C}_T - \sqrt{2\beta}T + \mu T$:

$$\| \tilde{v}^T(t, \cdot + m_t - \mu t + C_T) - h_*(\cdot) \|_\infty \to 0 \ \ as\, t \to \infty. \quad (70)$$

Lemma (4.1.24) show that for t large enough, $u(t,x)$ is close to $v^T(t,x)$.

Fix $s > 0$. Using (70) and (66), choose T large enough that $\|v^T(\cdot,\cdot) - u(\cdot,\cdot)\|_\infty < \epsilon$ and then choose t large enough so that $\|v^T(t, \cdot + m_t - \mu t + C_T) - h_*(\cdot)\|_\infty < \epsilon$. Then, we have that

$$\begin{aligned} \|u(t, \cdot + m_t - \mu t) - h_*(\cdot - C)\|_\infty &\leq \|u(t, \cdot + m_t - \mu t) - v^T(t, \cdot + m_t - \mu t)\|_\infty \\ &\quad + \|v^T(t, \cdot + m_t - \mu t) - h_*(\cdot - C_T)\|_\infty \\ &\quad + \|h_*(\cdot - C_T) - h_*(\cdot - C)\|_\infty \\ &\leq 2\epsilon + c|C_T - C| \end{aligned}$$

where $c = \max_{x \in \mathbb{R}} h'_*(x)$. As $C_T \to C$, for T large enough independently of x this can be made smaller than 3ϵ.Thus $\|u(t, \cdot + m_t - \mu t) - h_*(\cdot - C)\|_\infty \to 0$ as $t \to \infty$, which is the Theorem.

Lemma (4.1.26) [4]:

For any $\epsilon > 0$ there exists T_ϵ such that for all $t \geq T \geq T_\epsilon$ one has $v^T(t,0) \leq \epsilon/(1+\epsilon)$.
(The $1+\epsilon$ in the denominator makes the following easier.)

Proof. We use the representation (68). Let $t \geq T$; obviously

$$v^T(t,0) \leq \mathbb{P}^0\left(\min_{u \in \mathcal{N}_{all}(\mathrm{t})} X_u(t) > 0\right) = h(t, -\mu t),$$

where h is the solution of (18). $h(t, -\mu t)$ is by definition the probability that the leftmost particle at time t of a driftless branching Brownian motion is to the right of $-\mu t$; it is also the probability that the leftmost particle at time t of a branching

Brownian motion with drift μ is to the right of zero. For $\mu < \sqrt{2\beta}$ (regimes **A** and **B**), this probability is known to tend to zero when $t \to \infty$.

Lemma (4.1.27) [4]:

For any $\epsilon > 0$ and any $T > T_\epsilon$ one has

$$(1+\epsilon)v^T(t,x) - \epsilon \le u(t,x) \le v^T(t,x), \quad (t,x) \in [T,\infty) \times \mathbb{R}_+.$$

(The T_ϵ in Lemma (4.1.27) is the same as in Lemma (4.1.26).)

Proof. $u \le v^T$ follows immediately from their definitions as probabilities. Let us introduce $\tilde{u}(t,x) := \frac{u(t,x)+\epsilon}{1+\epsilon}$

We have that

$$(1+\epsilon)\partial_t \tilde{u} = (1+\epsilon)\frac{1}{2}\partial_{xx}\tilde{u} + (1+\epsilon)\mu\partial_x \tilde{u} + \beta\left[\left((1+\epsilon)\tilde{u} - \epsilon\right)^2 - ((1+\epsilon)\tilde{u} - \epsilon)\right]$$

Performing simple calculations we arrive at

$$\partial_t \tilde{u} = \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x \tilde{u} + \beta(\tilde{u}-1)(\tilde{u} - \epsilon + \epsilon\tilde{u})$$
$$\ge \frac{1}{2}\partial_{xx}\tilde{u} + \mu\partial_x \tilde{u} + \beta(\tilde{u}-1)\tilde{u}$$

Since $\tilde{u} \le 1$ and $\epsilon > 0$.

Now, for any $T > T_\epsilon$, we have with Lemma (4.1.26)

$$v^T(t,0) \le \frac{\epsilon}{(1+\epsilon)} = \tilde{u}(t,0), \quad t \ge T.$$

Moreover one checks directly that

$$v^T(T,x) = u(T,x) \le \tilde{u}(T,x), \quad x \ge 0.$$

By the parabolic maximum principle (and the unicity of solutions) we get that for any $T > T_\epsilon$

$$v^T(t,x) \le \tilde{u}(t,x), \quad \forall (t,x) \in [T,\infty) \times \mathbb{R}_+.$$

This proves the first inequality and thus concludes the proof of the lemma.

Lemma (4.1.27) implies that $|u(t,x) - v^T(t,x)| \le \epsilon\left(1 - v^T(t,x)\right) \le \epsilon$ for each $x \in \mathbb{R}_+$ and each t and T with $t \ge T \ge T(\epsilon)$, which is the first assertion of Lemma (4.1.24).

The last step is then to prove that C_T has a limit C for large T.

As $u(t,\cdot)$ is strictly increasing and continuous, $u(t,0) = 0$ and $\lim_{x\to\infty} u(t,x) = 1$, we may define $m_{\frac{1}{2}}: (0,+\infty) \times \mathbb{R}_+$ by $u(t, m_{\frac{1}{2}}(t)) = 1/2$.

Fix $\epsilon > 0$. We have that

$$\left|\frac{1}{2} - h_*\left(m_{\frac{1}{2}}(t) - m_t - \mu t - C_T\right)\right| \le \left|u\left(t, m_{\frac{1}{2}}(t)\right) - v^T\left(t, m_{\frac{1}{2}}(t)\right)\right|$$
$$+ \left|v^T\left(t, m_{\frac{1}{2}}(t)\right) - h_*\left(m_{\frac{1}{2}}(t) - m_t - \mu t - C_T\right)\right| \le 2\epsilon,$$

as long as T and t are large enough by (70) and (66). From this we deduce

$$m_{\frac{1}{2}}(t) - m_t - \mu t - C_T \in \left[h_*^{-1}\left(\frac{1}{2} - 2\epsilon\right), h_*^{-1}\left(\frac{1}{2} + 2\epsilon\right)\right]. \quad (70)$$

Consequently

$$\lim_{t\to+\infty}\sup\left[m_{\frac{1}{2}}(t) - m_t - \mu t\right] - \lim_{t\to+\infty}\inf\left[m_{\frac{1}{2}}(t) - m_t - \mu t\right]$$
$$\leq h_*^{-1}\left(\frac{1}{2} + 2\epsilon\right) - h_*^{-1}\left(\frac{1}{2} - 2\epsilon\right).$$

Since ϵ can be chosen arbitrarily small we have that $\lim_{t\to+\infty}\left[m_{\frac{1}{2}}(t) - m_t - \mu t\right] = C$, for some constant $C \in \mathbb{R}$. This and (70) immediately yields that

$$\lim_{T\to+\infty} C_T = C,$$

where we used that $h_*^{-1}\left(\frac{1}{2}\right)$. This concludes the proof of Lemma (4.1.24).

Section (4.2): Radius of Convergence and Asymptotic Behavior of s_0

We related the $w_s(x)$ to a function $x \mapsto \Phi(z)$defined as a series of which the coefficients a_n follows the recursive equation (16). We write here the same property in a slightly different but equivalent way. Let $p \in (0,1]$ be defined by

$$p := \frac{2\beta}{r^2},$$

and introduce $\Psi^{(p)}(z) = p\Phi(z/p)$ and $b_n^{(p)} = a_n/p^{n-1}$. These quantities satisfy the relation

$$\Psi^{(p)}(z) = \sum_{n\geq 1} b_n^{(p)} z^p, b_1^{(p)} = 1, b_n^{(p)} = \frac{1}{(n-1)(n-p)} \sum_{j=1}^{n-1} b_j^{(p)} b_{n-j}^{(p)}, n \geq 2. \quad (71)$$

Let $\mathcal{R}^{(p)}$ be the radius of convergence of $\Psi^{(p)}$. We know that there exists a B_{s_0} relating $\Psi^{(p)}$ and w_{s_0} through

$$w_{s_0}(x) = 1 - \frac{1}{p}\Psi^{(p)}\left(pB_{s_0}e^{-rx}\right).$$

The following observation will be useful. Since $w_{s_0}'(0) = 0$ and $w_{s_0}''(0) < 0$, the function w_{s_0} (defined on a domain containing zero) has a local maximum in zero. This implies that for $p > 0$ the function $\Psi^{(p)}$ has a local minimum in $m^{(p)} := pB_{s_0} < 0$. In fact $m^{(p)}$ is the first local minimum (and indeed the first point where the first derivative cancels) one encounters left of zero for $\Psi^{(p)}$.

The steps of the proof are the following:

1. We show that $\mathcal{R}^{(p)} \geq 4$ for small enough p (including $p = 0$).
2. We prove that there exists $m^{(0)} \in (-3,0]$ which is the first minimum one encounters left of zero for $\Psi^{(0)}$ and that
$$\left[\Psi^{(0)}\right]'(x)<0, x\in[a,m^{(0)}), \quad \text{and} \quad \left[\Psi^{(0)}\right]'(x) > 0, x \in \left(m^{(0)}, 0\right], \quad (72)$$
for some $a\in[-3, m^{(0)}]$.

3. We show that $\left[\Psi^{(p)}\right]'$ converges to $\left[\Psi^{(0)}\right]'$ uniformly on $(-3,0]$. This implies that
$$\lim_{p\searrow 0} m^{(p)} = m^{(0)} \in (-3,0). \tag{73}$$
4. Since $\left|pB_{s_0}\right| \longrightarrow \left|m^{(0)}\right| < 4$ we conclude that B_{s_0} is within the radius of convergence of Φ for p small enough. The identity
$$s_0 = w_{s_0}(0) = 1 - \Phi\left(B_{s_0}\right) = 1 - \Psi^{(p)}(m^{(p)})/p.$$
shows that
$$\lim_{p\searrow 0} p s_0(p) = \Psi^{(0)}\left(m^{(0)}\right),$$
where we made the dependence of s_0 on $p = 2\beta/r^2$ explicit.

We now prove these points.

1. The key remark is that if for a real $\alpha > 0$ and an integer n_0, one has $b_n^{(p)} \leq (n_0 - p)\alpha^{-n}$ for all $n \in \{1, \dots, n_0 - 1\}$ then, as can be shown by a very simple recursion, the property $b_n^{(p)} \leq (n_0 - p)\alpha^{-n}$ holds for all $n \geq 1$.
   Computing the first values of $b_n^{(0)}$, one checks easily that the maximum of $4^n b_n^{(0)}$ for $n \in \{1, \dots, 14\}$ is around 14.14. For p small enough, by continuity of $p \mapsto b_n^{(p)}$, the maximum of $b_n^{(p)}$ for $n \in \{1, \dots, 14\}$ will be no more than $15 - p$ and hence one has
   $$b_n^{(p)} \leq 15 \times 4^{-n}, \quad \text{(for } p \text{ small enough)} \tag{74}$$
   As a consequence, $\mathcal{R}^{(p)} \geq 4$ for p small enough (including $p = 0$).
2. The bound (74) applies for $p = 0$. Thus, for any $z \in [-3,3]$ using only the the fifty first terms of the expansion leads to an error of at most $\sum_{n\geq 51} 15 \times (3/4)^n$. In that way we computed $\Psi^{(0)}(-3) \approx -0.8528$ and $\Psi^{(0)}(-2.5) \approx -0.8575$. Therefore $\Psi^{(0)}(-2.5)$ is smaller than both $\Psi^{(0)}(-3)$ and $\Psi^{(0)}(0) = 0$, and the function $\Psi^{(0)}$ must have a minimum in $(-3,0)$. In other words we proved $m^{(0)} \in (-3,0)$. It is easy to check that $\sum_{n=1}^{50} n(n-1)a_n x^{n-2} \geq 0.7$ for $x \in [-3,0]$. Estimating an error by $\sum_{n\geq 51} 15n(n-1)(3/4)^n < 0.074$ we conclude that $\left[\Psi^{(0)}\right]''(x) > 0$ for $x \in [-3,0]$. In this way we get (72).
3. By (74) there exist $p_0 > 0$ and $C > 0$ such that the functions $\left[\Psi^{(p)}\right]'$ are analytic in $[-3,0]$ and $\sup_{p\in[0,p_0], x\in[-3,0]}\left|\Psi^{(p)}(x)\right| < C$. By continuity of $p \mapsto b_n^{(p)}$, for any $x \in [-3,0]$ we have $\left[\Psi^{(p)}\right]'(x) \to \left[\Psi^{(0)}\right]'(x)$. The Vitali-Proter theorem strengthen this to uniform convergence. This together with (72) implies easily (73).

List of Symbols

Symbols		Page No
inf	Infimum	2
$\downarrow$	Converges monotonically decreasing to	3
L^1	Hilbert space	5
lim sup	Limit Supremum	20
$\uparrow$	Converges monotonically increasing to	23
max	Maximum	50
Min	Minimum	50
$\to_p$	Convergence in probability	80
$\Longrightarrow$	Convergence in distribution	89
exp	Exponential	93
$\mathcal{N}_{all}(t)$	The population of all particles at time *t*	115
$\mathcal{N}_{live}(t)$	The set of particles alive (not absorbed) in BBM	115
$\mathbb{C}^2$	The space of functions with second continuous derivatives	132
$\mathbb{R}_+$	Set of positive-real numbers	142

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