Sudan University of Science and Technology
College of Graduate Studies

Solving Quadratic Optimal Control Problems using Legendre Function

حل مسائل الأمثل التربيعية بإستخدام دالة اللجيندر

A Thesis Submitted in Partial Fulfillment for the Requirements for Master Degree in Engineering

Submitted by:

AHMED EZZELDEEN KARAMALLAH

Supervisor:

Dr. EISA BASHIER MOHAMED ELTAYEB

July 2017

الاية

بسم الله الرحمن الرحيم

"وَقُلِ اعْمَلُواْ فَسَيَرَى اللّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ وَسَتُرَدُّونَ إِلَى عَالِمِ الْغَيْبِ وَالشَّهَادَةِ فَيُنَبِّئُكُم بِمَا كُنتُمْ تَعْمَلُونَ "

التوبة: (105)

ACKNOWLEDGEMENT

To Dr. Eisa Bashier who guided me through this thesis and to everyone who helped me to accomplish this work.

ABSTRACT

The proposed methods to solve optimal control problems are classified as direct methods and indirect methods. This thesis is based on solving optimal control problems using direct methods in which an optimal control problem is converted into a mathematical programming problem. The direct methods can be employed by using the parameterization technique which can be applied in three different ways: control parameterization, control-state parameterization and state parameterization. Here, we used control-state parameterization.

This thesis presents numerical methods to solve unconstrained optimal control problems. The solution method is based on using the iteration approach to replace the nonlinear optimal control problem by a sequence of time-varying linear quadratic optimal control problems. Each of these problems is solved by converting it into quadratic programming problem. The control-state parameterization technique is done by using the Legendre polynomials to approximate the system state variables.

The proposed method has been applied on several examples and we find that it gives acceptable results compared with some other methods.

مستخلص

تصنف الطرق المقترحة لحل مشاكل التحكم الأمثل الى طرق مباشرة و غير مباشرة. يستند هذا البحث على حل مشاكل التحكم الأمثل باستخدام الطرق المباشرة التي يتم فيها تحويل مشكلة التحكم الأمثل الى مشكلة برمجة رياضية. الطرق المباشرة يمكن استخدامها عن طريق استخدام تقنية الباراميترايزيشن التي يمكن تطبيقها بثلاث طرق مختلفة: باراميترايزيشن الدخل ، باراميترايزيشن الحالة, و باراميترايزيشن الدخل والحالة. هنا استخدمنا باراميترايزيشن الدخل والحالة.

تقدم هذه الأطروحة الطرق العددية لحل مشاكل التحكم الأمثل غير المقيدة. وتستند طريقة الحل على استخدام نهج التكرار ليحل محل مشكلة التحكم الأمثل غير الخطية سلسلة من مشاكل التحكم الامثل الخطية المتغيرة مع الزمن. كل هذه المشاكل يتم حلها عن طريق تحويلها إلى مشكلة البرمجة التربيعية. و تتم تقنية براميترايزيشن الدخل والحالة باستخدام كثيرات حدود لجندر لتقريب متغيرات حالة النظام. وقد طبقت الطريقة المقترحة على العديد من الأمثلة ونجد أنها تعطي نتائج مقبولة مقارنة مع بعض الطرق الأخرى.

LIST OF FIGURES

LIST OF ABBREVIATIONS

Abbreviation	Meaning
OCP	Optimal Control Problem
LTI	Linear Time Invariant
LTV	Linear Time Varying
HJB	Hamilton-Jacobi-Bellman.
LSF	Legendre Scaling Function

LIST OF TABLES

LIST OF CONTENTS

CHAPTER ONE

INTRODUCTION

1.1 Overview

Classical control system design is generally a trial-and-error process in which various methods of analysis are used iteratively to determine the design parameters of an "acceptable" system. The acceptable performance is generally defined in terms of time and frequency domain criteria such as rise time, settling time, peak overshoot, gain and phase margin, and bandwidth. Radically different performance criteria must be satisfied, however, by the complex, multiple-input, multiple-output systems required to meet the demands of modern technology. For example, the design of a spacecraft attitude control system that minimizes fuel expenditure is not amenable to solution by classical methods. A new and direct approach to the combination of these complex systems, called optimal control theory, has been made feasible by the development of the digital computer.[1]

Optimal control is an important science that deals with nonlinear Optimal Control Problem (OCP) and the main objective of optimal control is to find an optimal control that can be applied to the nonlinear system and to extrmize a certain cost function within the system's physical constraints. The methods of optimal control can be classified as direct and indirect methods. Indirect methods are generally converting OCP into two-point boundary value problem, then solving it by Hamilton-Jacobi-Bellman (HJB) equation or Euler Lagrange technique. The direct methods can be preceded using parameterization or discretization methods.

Parameterization is the method that using polynomials in order the performance index and the constraints with these polynomials and the coefficients that make this representation valid. The indirect methods have many disadvantages such as: complete knowledge of system model is needed; difficulties to obtain exact solution of nonlinear OCPs using Euler Lagrange or Hamilton-Jacobi-Bellman increasing; computational problem in using artificial co-states λ_{is}, also there are many advantages of direct methods such as: direct methods convert dynamic OCP into a static optimization problem; there is no need to use co-states variables λ_{is}. Thus, researchers are encouraged to use direct parameterization methods which are based on orthogonal functions and polynomials. This Thesis will present a technique to solve the nonlinear optimal control problem by using Legendre scaling functions to parameterize the state and control for nonlinear optimal control problem combined with an iterative approach.

1.2 Problem Statement

The nonlinear optimal control problem can be stated as: Find an optimal control $u^*(t)$ on $0 \le t \le t_f$ which minimizes the performance index:

$$J = \int_0^{t_f} (x^T Q x + u^T R u)\, dt \tag{1.1}$$

Subject to:

$$\dot{x} = f(x(t), u(t), t) \qquad x(0) = x_0 \tag{1.2}$$

Where $x \in R^n$, $u \in R^m$, Q is $n \times n$, positive semidefinite matrix and R is $m \times m$ positive definite matrix, f is continuously differentiable with respect to all its arguments.

This problem is very difficult to solve using indirect method so that; we proposed to solve it using control state parameterization via Legendre polynomials

1.3 Objectives

i. Using Legendre function to approximate the state and control variables of OCPS to solve linear time invariant and linear time-varying OCPS.

ii. Solving nonlinear OCPS by converting it into a sequence of time varying quadratic OCPS.

iii. Comparing the proposed method results with exact values and other methods results.

1.4 Methodology

- The proposed method is to convert the nonlinear OCPS into sequence of time varying OCPS using iterative approach, then the optimal problem is converted to quadratic programming problem, which it solved by MATLAB.
- Using Legendre function to approximate the state and control variables of optimal control problems to solve linear time invariant and linear time-varying optimal control problems.

1.5 Thesis Layout

Chapter two the previous studies in optimal control problem will be reviewed, these studies include the methods of solving OCPs. In Chapter three wavelets and Legendre scaling function will be introduced. These introductions include the approximation via Legendre function of linear quadratic OCPs method, state and control variables parameterization via Legendre scaling function are also presented in chapter three. Chapter four describes a method for solving unconstrained nonlinear OCPs by converting nonlinear OCPs into a sequence of linear time-varying OCPs using iterative approach. The conclusions and recommendations are drawn in chapter five.

CHAPTER TWO

OPTIMAL CONTROL PROBLEM

2.1 Introduction

The objective of optimal control is to determine an optimal control signal that forces the system to satisfy the system physical constraints and at the same time minimizes or maximizes a performance criterion.The formulation of an optimal control problem requires:

1. A set of first order deferential equations which represent mathematical model of the system to be controlled

$$\dot{x} = f(x(t), u(t), t) \quad , \quad t \in [t_0, t_f] \tag{2.1}$$

   Where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, f is continuously differentiable with respect to all its arguments.

2. A set of boundary conditions on the state variables which gives the value of the system states at the initial time

$$x(t_0) = x_0 \tag{2.2}$$

   Where x_0 is a known vector of initial conditions.

3. A performance index which describes some desired specifications, it selected by designer. An optimal control is defined as one that minimizes (or maximizes) the performance index. The performance index can be expressed as

$$J = \theta(x(t_f), t_f) + \int_{t_0}^{t_f} \varphi(x(t), u(t), t)\, dt \tag{2.3}$$

Where θ and φ are scalar functions [1].

2.2 Statement of the Optimal Control Problem

The optimal control problem is to find the optimal control $u^*(t)$ that minimizes the following performance index:

$$J = \theta\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} \varphi(x(t), u(t), t)\, dt \tag{2.4}$$

Subject to:

$$\dot{x} = f(x(t), u(t), t) \qquad x(t_0) = x_0 \tag{2.5}$$

Where $t \in [t_0, t_f]$, $x \in R^n$ is the states vector, $u \in R^m$ is the controls vector, f is assumed continuous differentiable function, while θ and φ are scalar functions. This problem, basically, can be solved by one of the following methods:

- Bellman's dynamic programming method.
- Variational method and Pontryagin's minimum principle (Euler-Lagrange equations).
- Direct methods using discretization or parameterization.

These methods will briefly be discussed in the following sections. In general it is not possible to solve the problem (2.4)-(2.5) analytically. However, an analytical solution is possible for a special case of this problem, the linear quadratic optimal control problem, in which the performance index is quadratic and the system state equations are linear. This problem can be stated as follows: Find the optimal control that minimizes:

$$J = x^T(t_f) S x(t_f) + \int_{t_0}^{t_f} \varphi(x^T Q x + u^T R u)\, dt \tag{2.6}$$

Subject to

$$\dot{x} = A(t)x + B(t)u \qquad x(t_0) = x_0 \tag{2.7}$$

Where S and Q are positive semidefinite matrices and R is a positive definite matrix. For this problem the solution can be expressed in feedback form

$$u^*(x,t) = -R^{-1}B^T(t)P(t)x \tag{2.8}$$

Where $P(t)$ is the solution of Riccati equation.

2.3 Dynamic Programming

Dynamic programming is a commonly used method of optimally solving complex problems by breaking them down into simpler problems, the use of the principle of optimality, usually known as dynamic programming, to derive an equation for solving optimal control problem was first proposed by Bellman [2]. The application of this principle on continuous optimal control problem has led to the invention of the famous Hamilton-Jacobi-Bellman (HJB) equation. It is a nonlinear first order hyperbolic partial differential equation which is used for constructing a nonlinear optimal feedback control law. For the optimal control problem (2.4)-(2.5), the HJB equation is given by:

$$\frac{\partial J^*(x(t),t)}{\partial t} = -\min_{u(t)}\left\{\varphi(x(t),u(t),t) + \frac{\partial J^*(x(t),t)}{\partial x} f(x(t),u(t),t)\right\} \tag{2.9}$$

And the boundary condition is:

$$J^*\big(x(t_f),t_f\big) = \phi\big(x(t_f),t_f\big) \tag{2.10}$$

To obtain a solution of Equation (2.9), we proceed in two steps. The first step is to perform the indicated minimization. This leads to a control law of the form:

$$u^* = \psi\left(\frac{\partial J^*}{\partial x}, x, t\right) \tag{2.11}$$

The second step is to substitute (2.11)back into (2.9) and solve the nonlinear, partial differential equation:

$$-\frac{\partial J^*(x,t)}{\partial t} = \varphi(x,\psi,t) + \frac{\partial J^*(x,t)^T}{\partial x} f(x,\psi,t) \tag{2.12}$$

For $J^*(x,t)$, subject to the boundary condition (2.10). Then the gradient of $J^*(x,t)$ with respect to x is computed, and the optimal feedback control law is obtained as follows:

$$u^* = \psi\left(\frac{\partial J^*}{\partial x}, x, t\right) = \Phi(x,t) \tag{2.13}$$

The derivation of HJB equation can be found in any standard optimal control textbook, for example [3]. This equation is a sufficient condition for optimality. The HJB equation is satisfied for all time-state pairs $(x(t),t)$ by the optimal value function $J^*(x(t),t)$. An advantage of using the HJB approach to solve the optimal control problem is that we obtain optimal feedback control law. However, the HJB equation does not in general, possess classical solution, that is, solutions $J^*(x(t),t)$ which are differentiable with respect to t and x. In general it is not possible to solve (2.12) analytically. However, in the case of linear quadratic optimal control problem (2.6)-(2.7), the HJB equation reduces to Riccati differential equation, which is given by:

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q - P(t)B(t)R^{-1}B^T(t)P(t) \tag{2.14}$$

$$P(t_f) = S \tag{2.15}$$

This result can be obtained if the value $J^* = x^T\, P(t)x$ is substituted in the HJB equation.

2.4 Necessary Conditions of Optimality

The Necessary condition of optimality can be divided into two categories as follows:

2.4.1 Euler-Lagrange equations

The necessary conditions can be derived by the methods of calculus of variations which are based on the fact that, at each stationary point, the variation in the cost function should vanish for arbitrary variation in the control [4]. To solve the optimal control problem (2.4)-(2.5), we shall use Lagrange multipliers $\lambda(t) \in R^n$ to adjoin the system state equations (2.5), to the performance index (2.4). Therefore, the augmented performance index is given by:

$$J_A = \theta\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} [\varphi(x,u,t) + \lambda^T(f(x,u,t) - \dot{x})]\, dt \tag{2.16}$$

Introducing the Hamiltonian function H defined by:

$$H(x,u,\lambda,t) = \varphi(x,u,t) + \lambda^T f(x,u,t) \tag{2.17}$$

We can rewrite equation(2.16) in the form:

$$J_A = \theta\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} H(x,u,\lambda,t)dt - \int_{t_0}^{t_f} \lambda^T \dot{x}\, dt \tag{2.18}$$

Applying the integration by parts for the last term in the above equation we get:

$$\int_{t_0}^{t_f} \lambda^T \dot{x}\, dt = \lambda^T(t_f)x(t_f) - \lambda^T(t_0)x(t_0) - \int_{t_0}^{t_f} \dot{\lambda}^T x\, dt \tag{2.19}$$

And therefore equation (2.18) becomes

$$J_A = \theta(x(t_f), t_f) - \lambda^T(t_f)x(t_f) + \lambda^T(t_0)x(t_0) + \int_{t_0}^{t_f} [H(x, u, \lambda, t) + \dot{\lambda}^T x]dt \tag{2.20}$$

The original problem (2.4)-(2.5) has been converted to the problem of minimizing (2.20) without constraints. To achieve the stationary state, the first order effect of control variations on the cost function must be zero for $0 \le t \le t_f$. Assuming that the initial time t_0 and final time t_f are fixed, and then the first variation of J_A due to control variation is:

$$\delta J_A = \left[\left(\frac{\partial \theta}{\partial x} - \lambda^T\right)\delta x\right]_{t=t_f} + \lambda^T \delta x|_{t=t_0} + \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x\right] dt \tag{2.21}$$

Since $\lambda(t)$ is arbitrary so far, we may set it to be:

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial f}{\partial x} - \frac{\partial \varphi}{\partial x} \tag{2.22}$$

With boundary condition:

$$\lambda^T(t_f) = \frac{\partial \theta}{\partial x}\bigg|_{t=t_f} \tag{2.23}$$

Equation (2.22) is called costate equation and the Lagrange multiplier $\lambda(t)$ is called the costate. Since the initial condition $x(t_0)$ is fixed, this implies $\delta x(t_0)$ vanishes. Therefore, equation (2.21) reduced to:

$$\delta J_A = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u} \delta u\right] dt \tag{2.24}$$

For a local minimum, it is necessary that δJ_A vanishes for arbitrary δu, hence it is necessary that:

$$\frac{\partial H}{\partial u} = \left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial \varphi}{\partial u}\right)^T = 0 \quad , \quad t_0 \leq t \leq t_f \tag{2.25}$$

Equations (2.5), (2.22), (2.23) and (2.25) are necessary conditions to be satisfied by optimal solutions of the problem, when the final time is fixed. These equations are called the Euler-Lagrange equations. In summary, to find the optimal control $u^*(t)$ that minimizes the performance index (2.4) subject to the system Equation (2.5), the following equations must be solved

$$\dot{x} = f(x(t), u(t), t) \tag{2.26}$$

$$x(t_0) = x_0 \tag{2.27}$$

$$\dot{\lambda} = -\left(\frac{\partial f}{\partial x}\right)^T \lambda - \left(\frac{\partial \varphi}{\partial x}\right)^T \tag{2.28}$$

$$\lambda(t_f) = \left(\frac{\partial \phi}{\partial x}\right)^T \tag{2.29}$$

Where $u^*(t)$ is determined by:

$$\left(\frac{\partial f}{\partial u}\right)^T \lambda + \left(\frac{\partial \varphi}{\partial u}\right)^T = 0 \tag{2.30}$$

Thus, the solution of the optimal control problem is determined by a two-point boundary value problem, expressed by the state equation (2.26) with the initial condition (2.27) and the costate equation (2.28) with the final condition (2.29).

Remarks:

1- If $\varphi(x,u,t)$ and $f(x,u,t)$ are not functions of time explicitly, then the Hamiltonian is constant during all optimal path.

2- In the case of free end time, the following necessary condition is obtained for optimality with free end time.

$$\left(\frac{\partial\theta}{\partial t} + H\right)_{t=t_f} = 0 \tag{2.31}$$

From this equation, it is clear that if the terminal cost $\theta\left(x\left(t_f\right), t_f\right)$ does not depend on the time explicitly, then

$$H\,|_{t=t_f} = 0 \tag{2.32}$$

Therefore, if H also does not depend explicitly on time, then $H = 0$ for all $0 \le t \le t_f$.

3- It is assumed in the previous derivations that the final state $x\left(t_f\right)$ is free. If the final state is fixed.

$$x\left(t_f\right) = x_f \tag{2.33}$$

Then (2.29) is replaced by (2.33).

2.4.2 Pontryagin minimum principle

In real problems, the control variables are usually bounded; therefore we can't differentiate the Hamiltonian with respect to the control, Equation (2.30). Let the bounded control lie in the subset $U \in R^m$. In this case, the necessary conditions are derived from the Minimum Principle which was developed by Pontryagin.

Pontryagin minimum principle:

Suppose that $u^*(t)$ is the optimal control with corresponding optimal trajectories $x^*(t)$, and let the Hamiltonian be defined by Equation (2.17). In order that $u^*(t)$ and $x^*(t)$ be optimal of the problem (2.4)-(2.5), then there must exist a costate vector $\lambda^*(t)$ such that the following conditions hold:

$$\dot{\lambda} = -\frac{\partial H^T}{\partial x} \tag{2.34}$$

$$\lambda(t_f) = \left(\frac{\partial \theta}{\partial x}\right)^T \tag{2.35}$$

And

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t) \tag{2.36}$$

For any $t \in [t_0, t_f]$ and for all controls $u(t) \in U$, this indicates that the optimal control must minimize the Hamiltonian. Inequality (2.36) is very useful to obtain the optimal control if the control is bounded by inequality constraints. It should be pointed out that Pontryagin's minimum principle is a generalization of the calculus of variations approach. The difference between the calculus of variations approach and the minimum principle is that Equation (2.30) is replaced by (2.36).

From the previous discussion, it is clear that the variational approach and the minimum principle lead to a nonlinear two-point boundary value problem which is very difficult to solve analytically. There are a very large number of methods which have been proposed to obtain numerical solutions of the HJB equation and the nonlinear two-point boundary value problem. These methods are called indirect methods. There is another class of methods to solve the optimal control problem, called direct methods. The direct methods are based on solving the optimal control problem by transforming it into a nonlinear programming problem.

2.5 Indirect Methods

These are the methods that based on solving the optimal control problem using HJB equation or the nonlinear two-point boundary value problem. These methods can be divided into two categories as follows:

2.5.1 Closed loop control methods

Some of the methods which were proposed to obtain the feedback optimal control are summarized as follows:

- The first approach to obtain feedback optimal control is based on using the power series expansion to solve either the HJB equation or the nonlinear two-point boundary value problem successively to obtain an approximate optimal feedback control law. This approach has been applied by Lukes [5], to find an approximate solution of HJB equation of the infinite horizon general nonlinear optimal control problem. The solution of HJB equation is reduced to solving successively systems of linear algebraic equation. Using the same idea, Willemstein [6] extended Lukes' work to handle the finite time nonlinear optimal control problem. The work of Lukes has been applied by Garrard and Jordan [7] to control F-8 fighter aircraft. The power series technique has also been used by Nishikawa, Sannomiya and Itakura [8] to obtain the approximate optimal solution of finite time quadratic performance index subject to the perturbed system given by

$$\dot{x} = A(t)x + \epsilon f(x,t) + B(t)u \quad (2.37)$$

This optimal control problem was solved by expanding the costate by a power series with respect to ϵ, and the solution was reduced to solving a sequence of linear partial differential equations. Also, similar idea was applied by Yoshida and Loparo [9] to solve the finite and the infinite time quadratic performance indices subject to the system:

$$\dot{x} = f(x) + Bu \tag{2.38}$$

In this case, the vector $x^{[k]}$ was used to express the function f(x) in a power series about the origin and also to express the costates by a power series of unknown parameters. The solution of the finite time optimal control problem was reduced to solving a Riccati equation and a sequence of ordinary linear differential equations, while the solution of the infinite time optimal control problem was reduced to solving sequence of algebraic equations.

- The second approach to obtain the optimal feedback control is to obtain the neighboring optimal feedback control which can be obtained either by linearizing the necessary conditions of the optimality around the optimal solution or expanding the performance index up to the second order and the constraints up to the first order around the optimal solution [4].
- The third approach to find the optimal feedback control law is based on writing the nonlinear state equations into a linear form state equation as follows:

$$\dot{x} = f(x,u,t) = A(x,u,t)x + B(x,u,t)u \tag{2.39}$$

And then the quadratic optimal control problem is solved by solving the following Riccati equation:

$$\dot{P}(x,u,t) = P(x,u,t)A(x,u,t) + A^T(x,u,t)P(x,u,t)$$
$$-P(x,u,t)B(x,u,t)R^{-1}B^T(x,u,t)P(x,u,t) + Q \tag{2.40}$$

And the optimal control is given by:

$$u^*(x,t) = -R^{-1}B^T(x,u,t)P(x,u,t)x(t) \tag{2.41}$$

Thus for a given state x the optimal control is found by simultaneously solving Equations (2.40) and (2.41). This method was developed by Burghart [10], Wernli and Cook [11].

- ❖ The fourth approach to use linear quadratic model-Riccati equation method is essentially based on the linear quadratic problem and its closed-loop solution, which requires the integration of the Riccati differential equation, can be found in Nedeljkovic [12].

2.5.2 Open loop control methods

The optimal open loop control methods are based on solving the nonlinear two-point boundary value problem. Some of these methods are: gradient methods, quasilinearization, penalty function methods, and neighboring extremal methods. These are standard methods to solve the optimal control problems, for details of these methods can refer to [4, 1].

2.6 Direct Methods

This is another major class of methods for solving the optimal control problems. These methods offer some advantages when applied to optimal control problems. The first advantage is that the difficult dynamic optimal control problem can be converted into static parameters optimization problem which is easier than the original one; the second advantage is that there are well-developed algorithms to solve the nonlinear programming problems; the third advantage is that it is possible to treat different types of constraints easily.

Due to these attractive features of the direct methods and the drawbacks, mentioned earlier, of the indirect methods, a number of authors have used the direct methods to solve the optimal control problem. The direct methods are based on obtaining the solution through a direct minimization of the performance index, subject to constraints, of the optimal control problem. These methods can be applied by converting the nonlinear optimal control problem into a nonlinear mathematical programming problem [13- 19].

The optimal control problem can be converted into a mathematical programming problem by using either the discretization or the parameterization techniques. The work in this thesis is based on using the parameterization technique to convert the optimal control problem into mathematical programming problem.

2.6.1 Discretization methods

All discretization approaches divide the time interval into n_s segments as:

$t_0 < t_1 < t_2 < \cdots < t_{ns} = t_f$ where the time points are referred to as mesh points, grid points or nodes. One approach to apply this method is to discretize both the state variables and the control variables, therefore we have the following sequence of unknown values of state variables and control variables $z = (x_0, x_1, \dots x_{ns}, u_0, u_1, \dots, u_{ns-1})$.

Then the system state equations are replaced by a set of algebraic equations which are considered as equality constraints. Hence this problem can be solved using any of the nonlinear programming techniques. One of the disadvantages of this approach is the high dimensionality of the vector z. Another approach is to discretize the control variables only $z = (u_0, u_1, \dots, u_{ns-1})$ and then the system state equations have to be integrated to find the state variables as a function of the control variables.

2.6.2 Parameterization methods

The parameterization technique is an essential part of this thesis; therefore we will explain this approach in some details. The parameterization technique can be applied in one of the following three forms

1. Control parameterization: The control parameterization [14, 15] is based on approximating the control variables by choosing an appropriate structure with finitely many unknown parameters as follows

$$u_l(t) = \sum_{i=0}^{N} b_i^{(l)} \Phi_i(t) \qquad l = 1,2,\ldots,m \tag{2.42}$$

   Where b_i's are unknown parameters and $\Phi_i(t)$ denotes an appropriate set of functions forming a basis of a finite dimensional control space. The state variables are obtained as a function of the unknown parameters of the control variables, by integrating the system state equations. And by substituting the approximated control variables and the corresponding state variables into the performance index, the optimal control problem can be converted into a static parameters programming problem, which can be solved easier than the original one. The control parameterization approach is the most widely used parameterization approach.

2. Control-state parameterization

   The control-state parameterization approach [17, 18] is based on approximating both the state variables and the control variables by a sequence of known functions with unknown parameters as follows:

$$x_j(t) = \sum_{i=0}^{N} a_i^{(j)} \Phi_i(t) \qquad j = 1,2,\ldots,n \tag{2.43}$$

$$u_l(t) = \sum_{i=0}^{N} b_i^{(l)} \Phi_i(t) \qquad l = 1,2, \dots, m \tag{2.44}$$

Where a_i, b_i are unknown parameters, and $\Phi_i(t)$ is an appropriate set of functions. Using this method, the optimal control problem can be converted into a nonlinear mathematical programming problem.

3. State parameterization

   The idea of the state parameterization is to approximate only the system state variables by a sequence of given functions with unknown parameters as:

$$x_j(t) = \sum_{i=0}^{N} a_i^{(j)} \Phi_i(t) \qquad j = 1,2, \dots, n \tag{2.45}$$

And then the control variables are obtained from the state equations [16, 20].

CHAPTER THREE

LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

3.1 Introduction

The Legendre function is one of the wavelets, so that we need to Know what is the wavelets, wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. Wavelets were developed independently in the fields of mathematics, quantum physics, electrical engineering, and seismic geology. Interchanges between these fields during the last ten years have led to many new wavelet applications such as image compression, turbulence, human vision, radar, and earthquake prediction [21]. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [22]

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right), \qquad a, b \in \mathbb{R}\,, a \neq 0 \tag{3.1}$$

3.2 Legendre Wavelets

Legendre wavelet $\psi_{n,m} = \psi(k, m, t)$ has four arguments $n = 1,2,3,..,2^{k-1}$, k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0, 1] as:

$$\psi_{n,m}(k,m,t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}} P_m(2^k t - 2n + 1), & for\ \frac{2n-2}{2^k} \le t \le \frac{2n}{2^k} \\ 0, & otherwise \end{cases} \tag{3.2}$$

Where $n = 1,2,3,..,2^{k-1}$, $m = 0,1,\ldots,M-1$, $P_m(t)$ are known Legendre polynomials of order m and defined for the time interval and $t \in [-1,1]$ and satisfy the following formula:

$$\begin{aligned} &P_0(t) = 1 \\ &P_1(t) = t \\ &P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\, P_m(t) - \left(\frac{m}{m+1}\right) t\, P_{m-1}(t),\ m = 1,2,3,\ldots \end{aligned} \tag{3.3}$$

There are many scientists generating function for the Legendre polynomials one of them is Rodrigues, he gives analytical form of Legendre polynomial -called Rodrigues' formula- as [24]:

$$P_m(t) = \frac{1}{2^m m!} \frac{d^m}{dx^m} (t^2 - 1)^m \tag{3.4}$$

Now for $M = 3$, $k = 2 \Longrightarrow n = 2$ using (3.4) we have three Legendre polynomials P_m for $m = 0,1,2$ as shown:

$$\begin{aligned} &P_0(t) = 1 \\ &P_1(t) = t \\ &P_2(t) = 0.5(3t^2 - 1) \end{aligned} \tag{3.5}$$

Now by using (3.2) we can find Legendre function for $M = 3$ and $k = 2$ as shown in below equations:

$$\begin{cases} \psi_{1,0}(t) = \sqrt{2} \\ \psi_{1,1}(t) = \sqrt{6}(4t-1) \\ \psi_{1,2}(t) = \sqrt{2.5}\,[3(4t-1)^2 - 1] \end{cases} \quad 0 \le t < 0.5 \tag{3.6}$$

$$\begin{cases} \psi_{2,0}(t) = \sqrt{2} \\ \psi_{2,1}(t) = \sqrt{6}(4t-3) \\ \psi_{2,2}(t) = \sqrt{2.5}\,[3(4t-3)^2 - 1] \end{cases} \quad 0.5 \le t < 1 \tag{3.7}$$

The below figure show the Legendre wavelet for $M = 3$ and $k = 2$

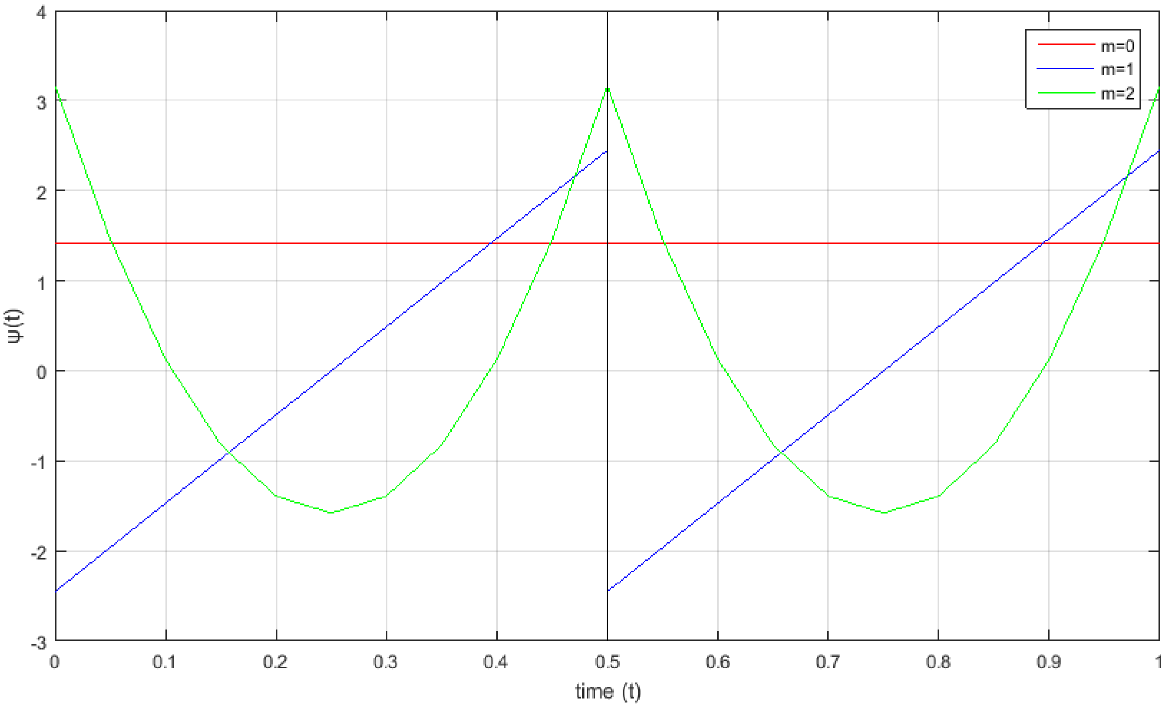

Figure 3.1: Legendre wavelets for $M = 3$, $k = 2$

3.3 Function Approximation

A function $f(t)$ defined over [0, 1] may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m}(t) \tag{3.8}$$

$$C_{n,m} = \big(f(t), \psi_{n,m}(t)\big)$$

If the infinite series in Equation (3.8) is truncated, then it can be written as:

$$f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m} \psi_{n,m}(t) = C^T \Psi(t) \tag{3.9}$$

where C and $\Psi(t)$ are $N \times 1$ $(N = 2^{K-1}M)$ matrices given by:

$$C = \left[C_{1,0}, C_{1,1}, \dots, C_{1,M-1}, C_{2,0}, \dots, C_{2,M-1}, \dots, C_{2^{K-1},M-1}\right]^T \tag{3.10}$$

$$\Psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \dots, \psi_{1,M-1}, \psi_{2,0}, \dots, \psi_{2,M-1}, \dots, \psi_{2^{K-1},M-1}\right]^T$$

3.4 Some Properties of Legendre Wavelets

The integration of the vector $\Psi(t)$, can be obtained as:

$$\int_0^t \Psi(\tau) d\tau = P\Psi(t) \tag{3.11}$$

Where P is $N \times N$ operational matrix for integration and is given in [25] as:

$$P = \frac{1}{2^K} \begin{bmatrix} L & F & \cdots & F \\ O & L & \cdots & F \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & L \end{bmatrix} \tag{3.12}$$

Where F and L are $M \times M$ matrices given by:

$$F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

`$L =$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3\sqrt{5}} & \cdots & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{5}}{5\sqrt{3}} & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{7}}{7\sqrt{5}} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} & 0 & \frac{\sqrt{2M-3}}{(2M-3)\sqrt{2M-1}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} & 0 \end{bmatrix} \quad (3.13)$$

The integration of multiplication of Legendre scaling function and its transpose in the interval [0, 1] is equal to identity matrix as follows

$$\int_0^1 \Psi(t)\Psi^{\mathrm{T}}(t)dt = I_N \quad (3.14)$$

Where I_N is identity matrix of dimension N . One of the important properties of Legendre wavelet is multiplication of two Legendre function

$$\mathrm{C}^{\mathrm{T}}\Psi(t)\Psi^{\mathrm{T}}(t) = \Psi^{\mathrm{T}}(t)\tilde{\mathrm{C}} \quad (3.15)$$

For $K = 2\,, M = 3$

$$\Psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \psi_{2,0}, \psi_{2,1}, \psi_{2,2}\right]^T \quad (3.16)$$

$$C = \left[C_{1,0}, C_{1,1}, C_{1,2}, C_{2,0}, C_{2,1}, C_{2,2}\right]^T \quad (3.17)$$

Using Equation (3.16) we get:

$$\Psi(t)\Psi^{T}(t)$$

$$= \begin{bmatrix} \psi_{10}\psi_{10} & \psi_{10}\psi_{11} & \psi_{10}\psi_{12} & \psi_{10}\psi_{20} & \psi_{10}\psi_{21} & \psi_{10}\psi_{22} \\ \psi_{11}\psi_{10} & \psi_{11}\psi_{11} & \psi_{11}\psi_{12} & \psi_{11}\psi_{20} & \psi_{11}\psi_{21} & \psi_{11}\psi_{22} \\ \psi_{12}\psi_{10} & \psi_{12}\psi_{11} & \psi_{12}\psi_{12} & \psi_{12}\psi_{20} & \psi_{12}\psi_{21} & \psi_{12}\psi_{22} \\ \psi_{20}\psi_{10} & \psi_{20}\psi_{11} & \psi_{20}\psi_{12} & \psi_{20}\psi_{20} & \psi_{20}\psi_{21} & \psi_{20}\psi_{22} \\ \psi_{21}\psi_{10} & \psi_{21}\psi_{11} & \psi_{21}\psi_{12} & \psi_{21}\psi_{20} & \psi_{21}\psi_{21} & \psi_{21}\psi_{22} \\ \psi_{22}\psi_{10} & \psi_{22}\psi_{11} & \psi_{22}\psi_{12} & \psi_{22}\psi_{20} & \psi_{22}\psi_{21} & \psi_{22}\psi_{22} \end{bmatrix} \tag{3.18}$$

$$\psi_{ij}\psi_{kl} = 0, \quad if\ i \neq k$$

$$\psi_{i0}\psi_{ij} = \sqrt{2}\psi_{ij}$$

$$\psi_{i1}\psi_{i1} = \frac{4}{\sqrt{10}}\psi_{i2} + \sqrt{2}\psi_{i0}$$

We can reduce Equation (3.18) as:

$$\Psi(t)\Psi^{T}(t) =$$

$$\begin{bmatrix} \sqrt{2}\psi_{10} & \sqrt{2}\psi_{11} & \sqrt{2}\psi_{12} & 0 & 0 & 0 \\ \sqrt{2}\psi_{11} & \sqrt{2}\psi_{10} + \frac{4}{\sqrt{10}}\psi_{12} & \frac{4}{\sqrt{10}}\psi_{11} & 0 & 0 & 0 \\ \sqrt{2}\psi_{12} & \frac{4}{\sqrt{10}}\psi_{11} & \sqrt{2}\psi_{10} + \frac{20}{7\sqrt{10}}\psi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}\psi_{20} & \sqrt{2}\psi_{21} & \sqrt{2}\psi_{22} \\ 0 & 0 & 0 & \sqrt{2}\psi_{21} & \sqrt{2}\psi_{20} + \frac{4}{\sqrt{10}}\psi_{22} & \frac{4}{\sqrt{10}}\psi_{21} \\ 0 & 0 & 0 & \sqrt{2}\psi_{22} & \frac{4}{\sqrt{10}}\psi_{21} & \sqrt{2}\psi_{20} + \frac{20}{7\sqrt{10}}\psi_{22} \end{bmatrix} \tag{3.19}$$

By using vector C in Equation (3.17), the 6×6 matrix $\tilde{C}$ can be written as:

$$\tilde{C} = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix} \tag{3.20}$$

Where B_i matrices given by:

$$\begin{bmatrix} \sqrt{2}\psi_{i0} & \sqrt{2}\psi_{i1} & \sqrt{2}\psi_{i2} \\ \sqrt{2}\psi_{i1} & \sqrt{2}\psi_{i0} + \frac{4}{\sqrt{10}}\psi_{i2} & \frac{4}{\sqrt{10}}\psi_{i1} \\ \sqrt{2}\psi_{i2} & \frac{4}{\sqrt{10}}\psi_{i1} & \sqrt{2}\psi_{i0} + \frac{20}{7\sqrt{10}}\psi_{i2} \end{bmatrix} \tag{3.21}$$

3.5 LTI Optimal Control Problem Reformulation

The linear quadratic optimal control problem can be stated as follows:

Find an optimal control $u^*(t)$ that minimizes the following quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u)\, dt \tag{3.22}$$

Subject to:

$$\dot{x} = Ax + Bu \qquad x(0) = x_0 \tag{3.23}$$

Where $x \in R^n$, $u \in R^m$, A, B are $n \times n, n \times m$ real-valued matrices respectively. Q is a positive semidefinite matrix and R is a positive definite matrix, $t \in [0, t_f]$. Because Legendre wavelets are defined on the time interval $t \in [0,1]$ and since our problem is defined on the interval $t \in [0, t_f]$ it is necessary before using Legendre wavelets to transform the time interval of the optimal control problem into the interval $t \in [0,1]$. We can obtained that by using:

$$\tau = \frac{t}{t_f} \tag{3.24}$$

$$dt = t_f d\tau \tag{3.25}$$

Then the optimal control problem became as:

$$J = t_f \int_0^1 (x^T Q x + u^T R u)\, d\tau \tag{3.26}$$

$$\frac{dx}{d\tau} = t_f (Ax + Bu) \tag{3.27}$$

3.5.1 Control state parameterization

The basic idea is to approximate the state and control variables by a finite series of LSF as follows:

$$x_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} a^i{}_{nm}\, \psi_{nm}(t) \qquad i = 1,2,\dots,s \tag{2.28}$$

$$u_i(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} b^i{}_{nm}\, \psi_{nm}(t) \qquad i = 1,2,\dots,r \tag{2.29}$$

We can write these two equations in compact form as:

$$x_i(t) = \left(I_s \otimes \Psi^{\mathrm{T}}(t)\right) a \tag{3.30}$$

$$u_i(t) = \left(I_r \otimes \Psi^{\mathrm{T}}(t)\right) b \tag{3.31}$$

$$a^i = \left[a^i_{1,0}, a^i_{1,1}, \dots, a^i_{1,M-1}, a^i_{2,0}, \dots, a^i_{2,M-1}, \dots, a^i_{2^{K-1},M-1}\right] \quad i = 1,2,\dots,s$$

$$a = [a^1,\ a^2\ ,\ a^3\ ,\dots, a^s]^T \tag{3.32}$$

$$b^i = \left[b^i_{1,0}, b^i_{1,1}, \dots, b^i_{1,M-1}, b^i_{2,0}, \dots, b^i_{2,M-1}, \dots, b^i_{2^{K-1},M-1}\right] \quad i = 1,2,\dots,r$$

$$b = [b^1,\ b^2\ ,\ b^3\ ,\dots, b^r]^T \tag{3.33}$$

Where a and b are $Ns \times 1$, $Nr \times 1$ respectively

$$\Psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \dots, \psi_{1,M-1}, \psi_{2,0}, \dots, \psi_{2,M-1}, \dots, \psi_{2^{K-1},M-1}\right]^T \tag{3.34}$$

To approximate the state equation via LSF Equation (3.23) can be integrated as:

$$x(t) - x_0 = \int_0^t Ax(\tau)\, d\tau + \int_0^t Bu(\tau)\, d\tau \tag{3.35}$$

3.5.2 Initial condition

Here the initial value of state are approximated using LSF

$$x_0 = \frac{\sqrt{2}}{2^{k/2}}\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right)[\alpha_0^1 \ \ \alpha_0^2 \ \ldots \ \alpha_0^s]$$

$$= \delta\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right) g_0 \qquad (3.36)$$

Where $g_0 = [\alpha_0^1 \ \ \alpha_0^2 \ \ldots \ \alpha_0^s], \delta = \frac{\sqrt{2}}{2^{k/2}}$ and $\alpha_0^i =$ $[x_i(0) \ \ 0 \ \ 0 \ldots 0 \ \ x_i(0) \ \ 0 \ \ 0 \ldots 0 \ \ x_i(0) \ \ 0 \ \ 0 \ldots 0]$. By substituting Equations (3.30), (3.31) and (3.36) into (3.35) we get:

$$\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right) a - \delta\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right) g_0$$

$$= \int_0^t A\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right) a \, d\tau + \int_0^t B\left(I_r \otimes \Psi^{\mathrm{T}}(t)\right) b \qquad (3.37)$$

Using properties of Legendre function and Kronecker product Equation (3.37) can reduce to:

$$\left((A \otimes P^T) - I_{Ns}\right)a + (B \otimes P^T)b = -\delta g_0 \qquad (3.38)$$

3.5.3 Performance index approximation

Substitute (3.30), (3.31) into (3.22) we get:

$$J = \int_0^1 \left[a^T\left(\left(I_s \otimes \Psi(t)\right)Q\left(I_s \otimes \Psi^{\mathrm{T}}(t)\right)\right)a\right.$$

$$\left. + b^T\left(\left(I_r \otimes \Psi(t)\right)R\left(I_r \otimes \Psi^{\mathrm{T}}(t)\right)\right)b\right] dt \qquad (3.39)$$

Equation (3.39) can reduce to :

$$J = [a^T \quad b^T]\begin{bmatrix} Q \otimes I_N & 0_{Ns \times Nr} \\ 0_{Nr \times Ns} & R \otimes I_N \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \qquad (3.40)$$

$J^* = Z^T H Z$

Where $Z^T = [a^T \quad b^T]$ and $H = \begin{bmatrix} Q \otimes I_N & 0_{Ns\times Nr} \\ 0_{Nr\times Ns} & R \otimes I_N \end{bmatrix}$

3.5.4 Additional constraints

Science the Legendre function consists from different sections we must add constraints to insure the continuity of these sections. There are $2^{k-1} - 1$ Points at which the continuity of the state and control variables has to ensure.

Theses points are:

$$t_i = \frac{i}{2^{k-1}} \qquad i = 1,2,\dots,2^{k-1} - 1 \tag{3.41}$$

So there are $(2^{k-1} - 1)s$ equality constraints for state given by:

$$\left(I_s \otimes \Psi'(t)\right)a = 0_{(2^{k-1}-1)s\times 1} \tag{3.42}$$

Where

$$\Psi' = \begin{bmatrix} \psi_{1m}(t_1) & -\psi_{2m}(t_1) & 0 & 0 & 0 & 0 \\ 0 & \psi_{2m}(t_2) & -\psi_{3m}(t_2) & 0 & 0 & 0 \\ 0 & 0 & \psi_{3m}(t_3) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \psi_{(2^{k-1}-1)m}\left(t_{2^{k-1}-1}\right) & \psi_{(2^{k-1})m}\left(t_{2^{k-1}-1}\right) \end{bmatrix} \tag{3.43}$$

Ψ' is $(2^{k-1} - 1) \times N$ and $(2^{k-1} - 1)r$ equality constraints for control given by:

$$\left(I_r \otimes \Psi'(t)\right)b = 0_{(2^{k-1}-1)r\times 1} \tag{3.44}$$

By combining (3.38), (3.42) and (3.44) we have

$$\begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ \left(I_s \otimes \Psi'(t)\right) & 0_{(2^{k-1}-1)s \times Nr} \\ 0_{(2^{k-1}-1)r \times Ns} & \left(I_r \otimes \Psi'(t)\right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\delta g_0 \\ 0_{(2^{k-1}-1)s \times 1} \\ 0_{(2^{k-1}-1)r \times 1} \end{bmatrix} \tag{3.45}$$

$$FZ = h$$

Where $F = \begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ \left(I_s \otimes \Psi'(t)\right) & 0_{(2^{k-1}-1)s \times Nr} \\ 0_{(2^{k-1}-1)r \times Ns} & \left(I_r \otimes \Psi'(t)\right) \end{bmatrix}$ and $h = \begin{bmatrix} -\delta g_0 \\ 0_{(2^{k-1}-1)s \times 1} \\ 0_{(2^{k-1}-1)r \times 1} \end{bmatrix}$

3.6 Numerical Example 1 [26]

Find the optimal control $u^*(t)$ that minimizes

$$J = \int_0^1 \left(x^2 + \frac{1}{2}u^2\right) dt$$

Subject to

$$\dot{x} = \frac{1}{2}x + u \quad x(0) = 1$$

With the exact solution for the performance index $J = 0.8641644978$

We solved this problem when $k = 2$ and $M = 3$ so $N = 6$. Then we approximate the state and control variables as

$$x_i(t) = \sum_{n=1}^{2} \sum_{m=0}^{2} a_{nm} \psi_{nm}(t)$$

$$u_i(t) = \sum_{n=1}^{2} \sum_{m=0}^{2} b_{nm} \psi_{nm}(t)$$

For $k = 2$ and $M = 3$ we get:

$$\begin{cases} \psi_{1,0}(t) = \sqrt{2} \\ \psi_{1,1}(t) = \sqrt{6}(4t - 1) \\ \psi_{1,2}(t) = \sqrt{2.5}\,[3(4t - 1)^2 - 1] \end{cases}$$

$$\begin{cases} \psi_{2,0}(t) = \sqrt{2} \\ \psi_{2,1}(t) = \sqrt{6}(4t-3) \\ \psi_{2,2}(t) = \sqrt{2.5}\,[3(4t-3)^2 - 1] \end{cases}$$

$$\Psi(t) = \left[\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \psi_{2,0}, \psi_{2,1}, \psi_{2,2}\right]^T$$

$$a = \left[a_{1,0}, a_{1,1}, a_{1,2}, a_{2,0}, a_{2,1}, a_{2,2}\right]$$

$$b = \left[b_{1,0}, b_{1,1}, b_{1,2}, b_{2,0}, b_{2,1}, b_{2,2}\right]$$

$$\delta = \frac{\sqrt{2}}{2^{k/2}} = \frac{1}{\sqrt{2}}$$

$$g_0 = [1\ , 0\ , 0\ , 1\ , 0\ , 0]$$

There are:

$2^{k-1} - 1 = 1$ points

this $t_1 = \frac{i}{2^{k-1}} = 0.5$

So there are $(2^{k-1} - 1)s = 1$ equality constraints for state given by:

$$\left(I_s \otimes \Psi'(t)\right)a = 0_{(2^{k-1}-1)s\times 1}$$

and $(2^{k-1} - 1)r = 1$ equality constraints for control given by:

$$\left(I_r \otimes \Psi'(t)\right)b = 0_{(2^{k-1}-1)r\times 1}$$

Ψ' is $(2^{k-1} - 1) \times N$ so that Ψ' is 1×6 matrix

$$\Psi' = \left[\psi_{1,0}(0.5), \psi_{1,1}(0.5), \psi_{1,2}(0.5), -\psi_{2,0}(0.5), -\psi_{2,1}(0.5), -\psi_{2,2}(0.5)\right]$$

$$\Psi' = [1.4142 \quad 2.4495 \quad 3.1623 \quad -1.4142 \quad 2.4495 \quad -3.1623]$$

By solving the corresponding quadratic programming problem we obtained the optimal value of performance index $J^* = 0.864198473010324$

$error = 3.397521032433293e - 05$.

The Figures 3.2 - 3.5 show the Example 1 control and state trajectories for different values of and M .

Table 3.1 shows the values of performance index and error according to different values of K and M. The comparison of results between the method used in this thesis and other method are drawn in table 3.2.

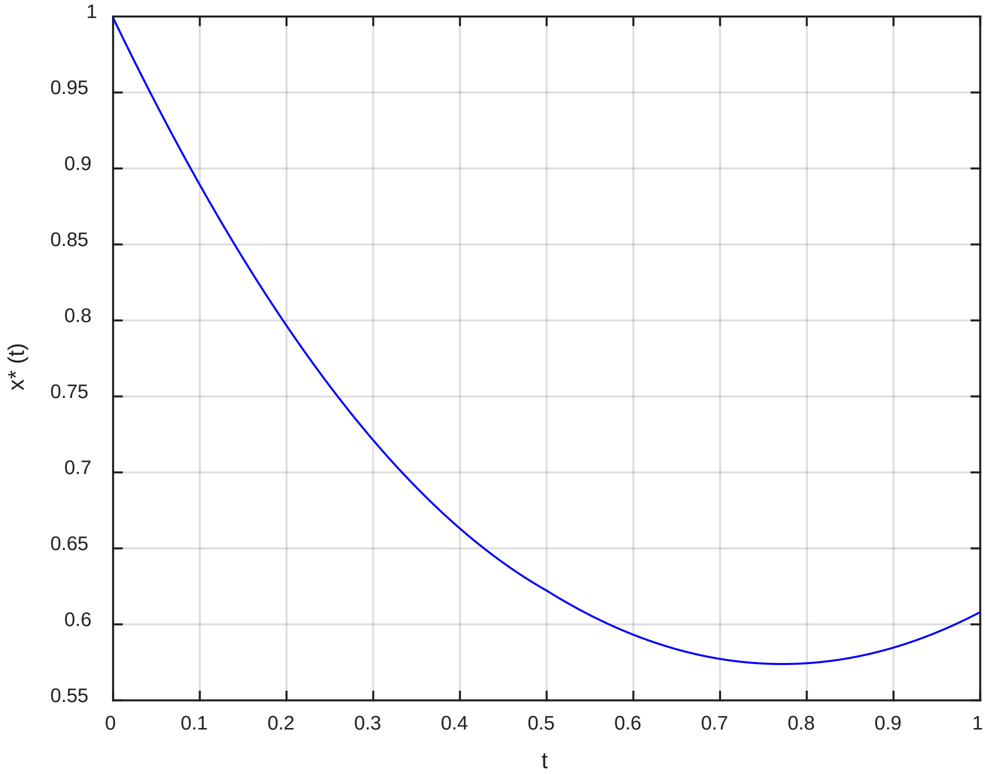

Figure 3.2**:** Example1 optimal state trajectory for $= 2$, $M = 3$

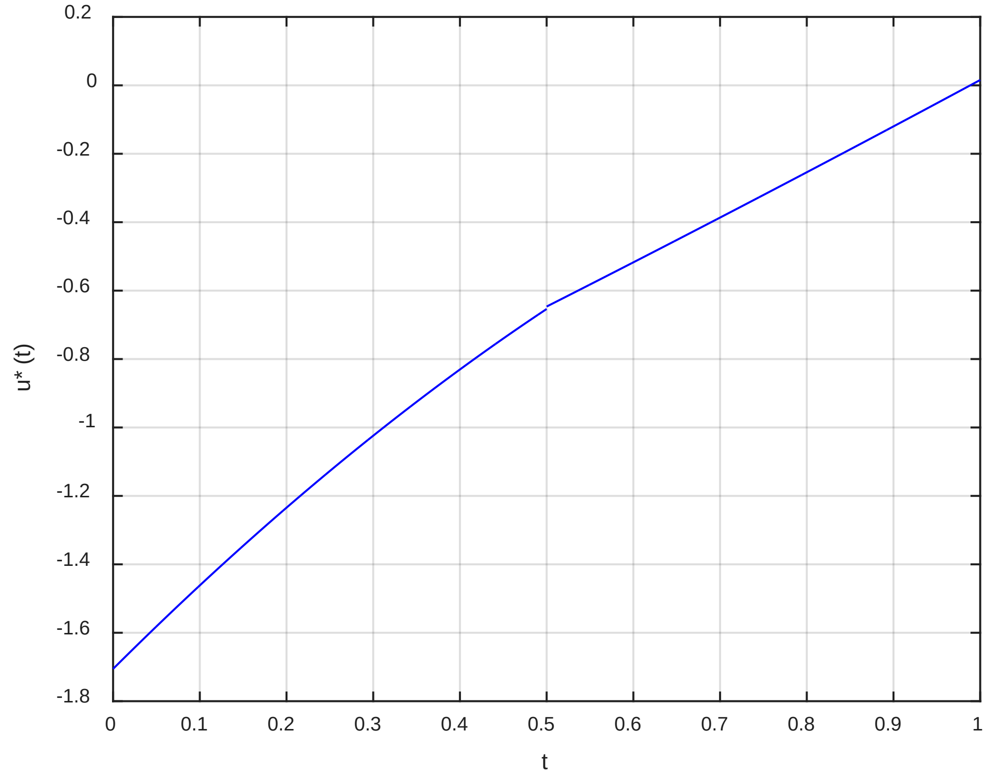

Figure 3.3**:** Example 1 optimal control trajectory for $= 2$, $M = 3$

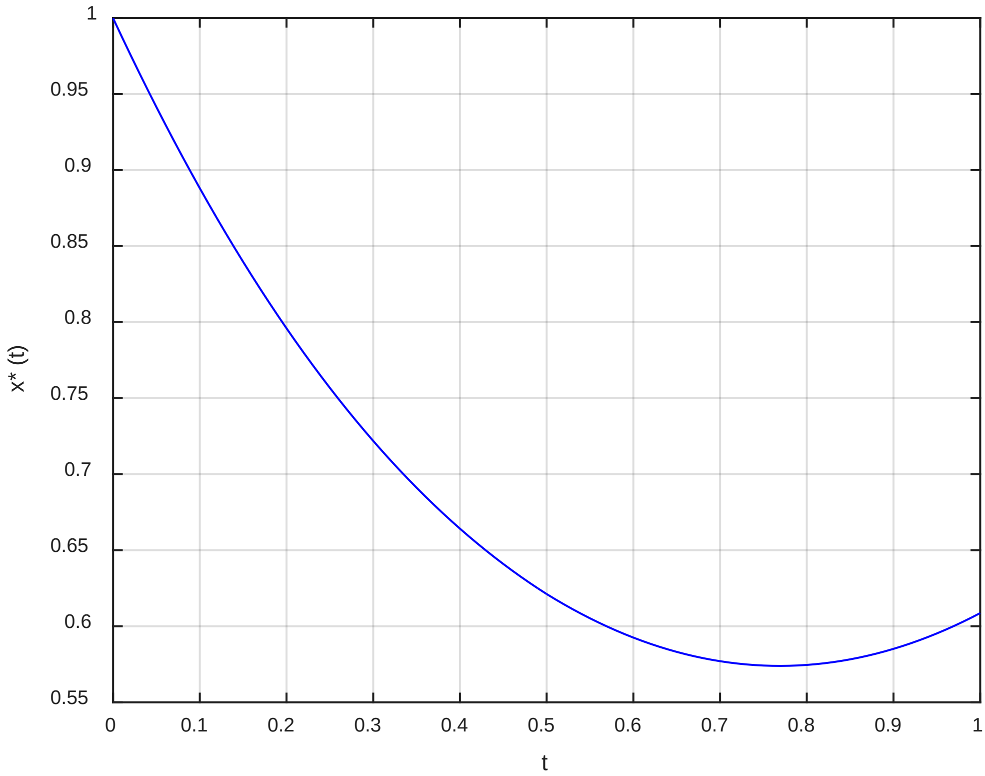

Figure 3.4**:** Example 1 optimal state trajectory for $= 2$, $M = 4$

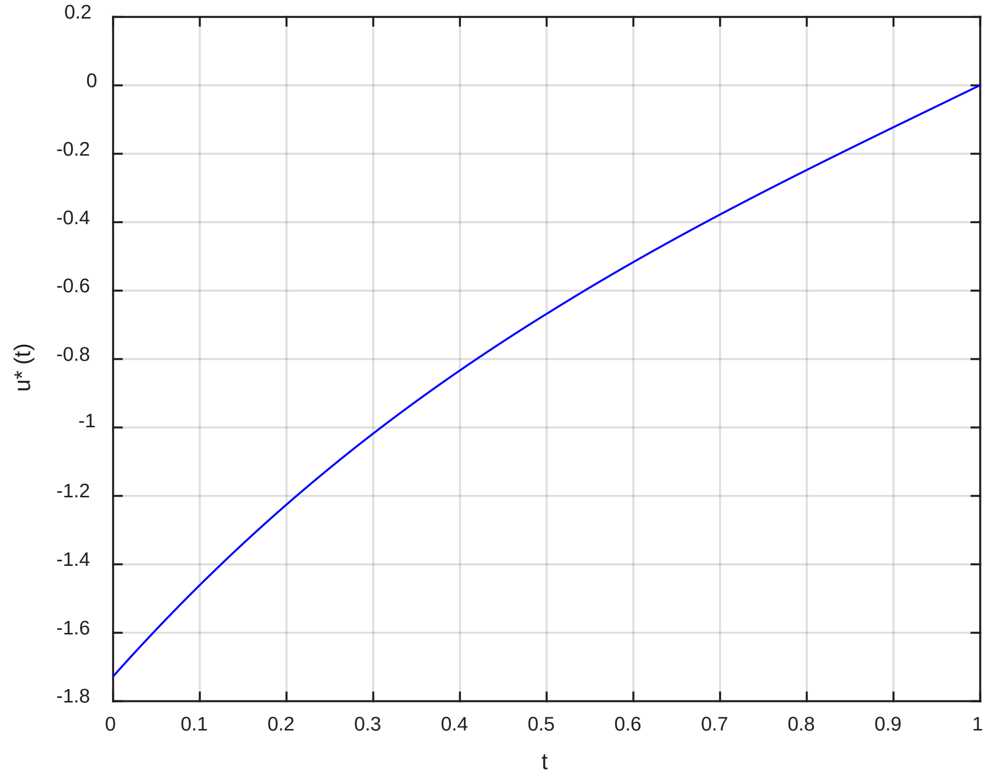

Figure 3.5**:** Example 1 optimal and control trajectory for $= 2$, $M = 4$

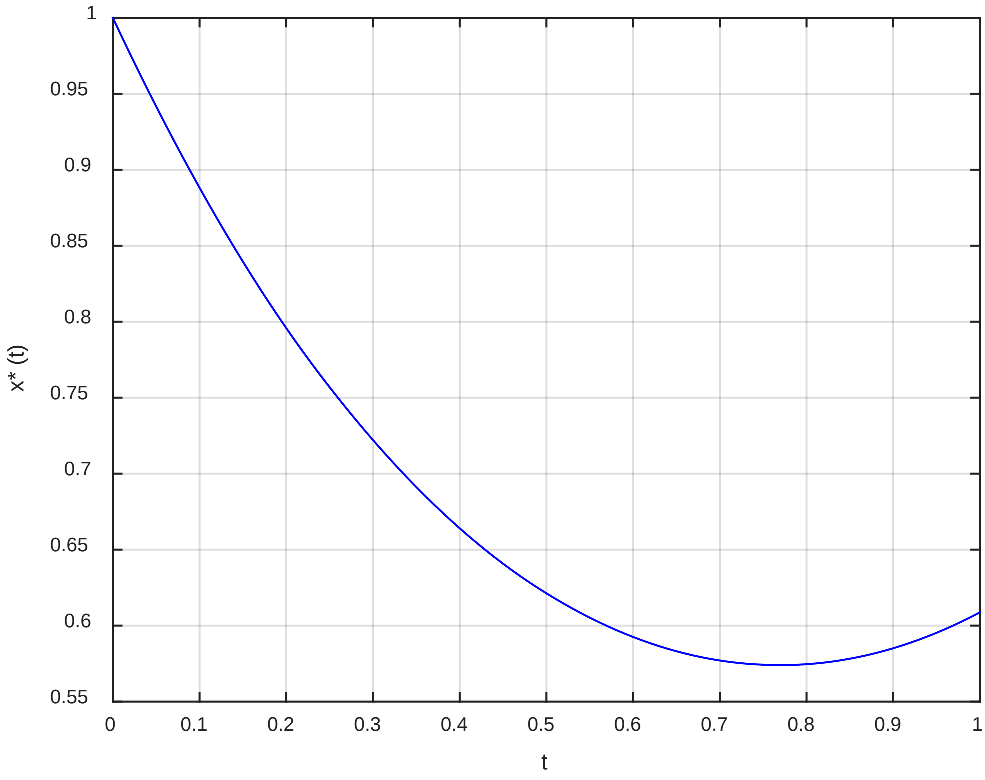

Figure 3.6**:** Example 1 optimal state trajectory for $= 3$, $M = 3$

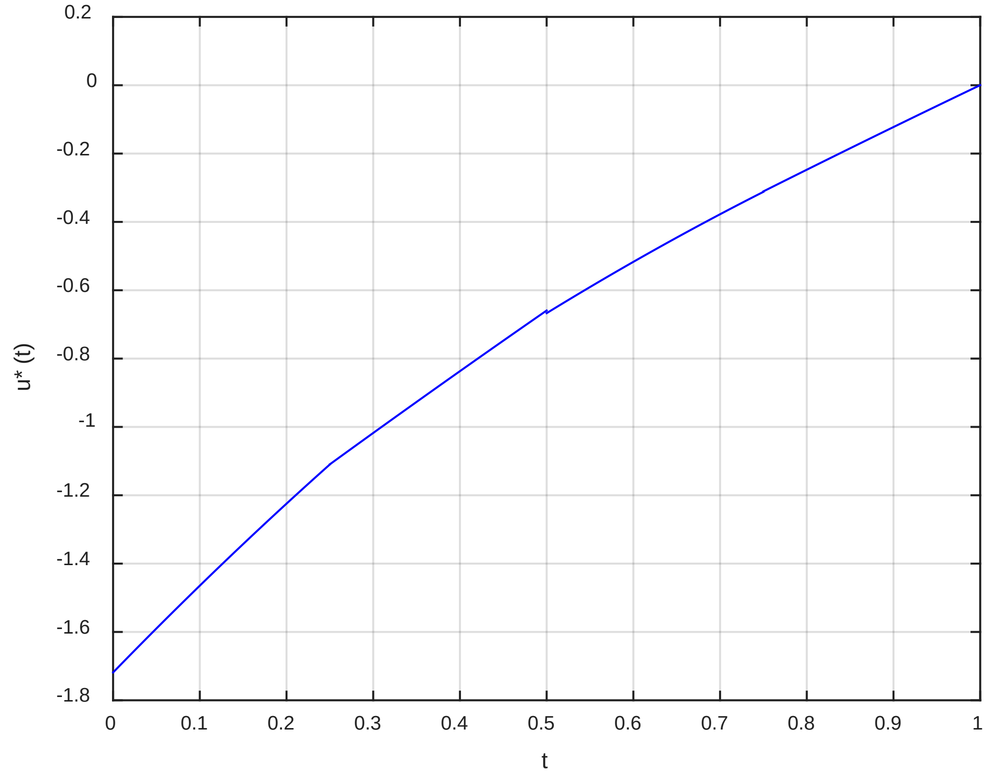

Figure 3.7**:** Example 1 optimal control trajectory for $= 3$, $M = 3$

Table 3.1: Optimal Values of performance index for Example (1)

k, M	J	Error
$k = 2, M = 3$	0.864198473010324	$3.397521032433293e-05$
$k = 2, M = 4$	0.864164513248718	$1.544871797154457e-08$
$k = 3, M = 3$	0.864168451296065	$3.953496064523776e-06$

Table 3.2: Comparison the Optimal Value of Example (1) with other method

	Method	J	Error
Exact value		0.8641644978	0
Kafash and Delavarkhalafi [26]	Restarted State Parameterization	0.8643546452	$1.9\,e-04$
This research	Legendre	0.864164513248718	$1.5e-08$

3.7 Numerical Example 2 [17]

$$minmize\ J = \frac{1}{2}\int_0^1 \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0.01u^2 \right\} dt$$

Subject to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

With the exact solution for the performance index $J = 0.06936094$

Figures 3.8-3.13 show Example 2 optimal control and state trajectories according to different values of K and M. Table 3.3 shows the values of performance index and error according to different values of K and M. The comparison of results between the method used in this thesis and other method are drawn in table 3.4.

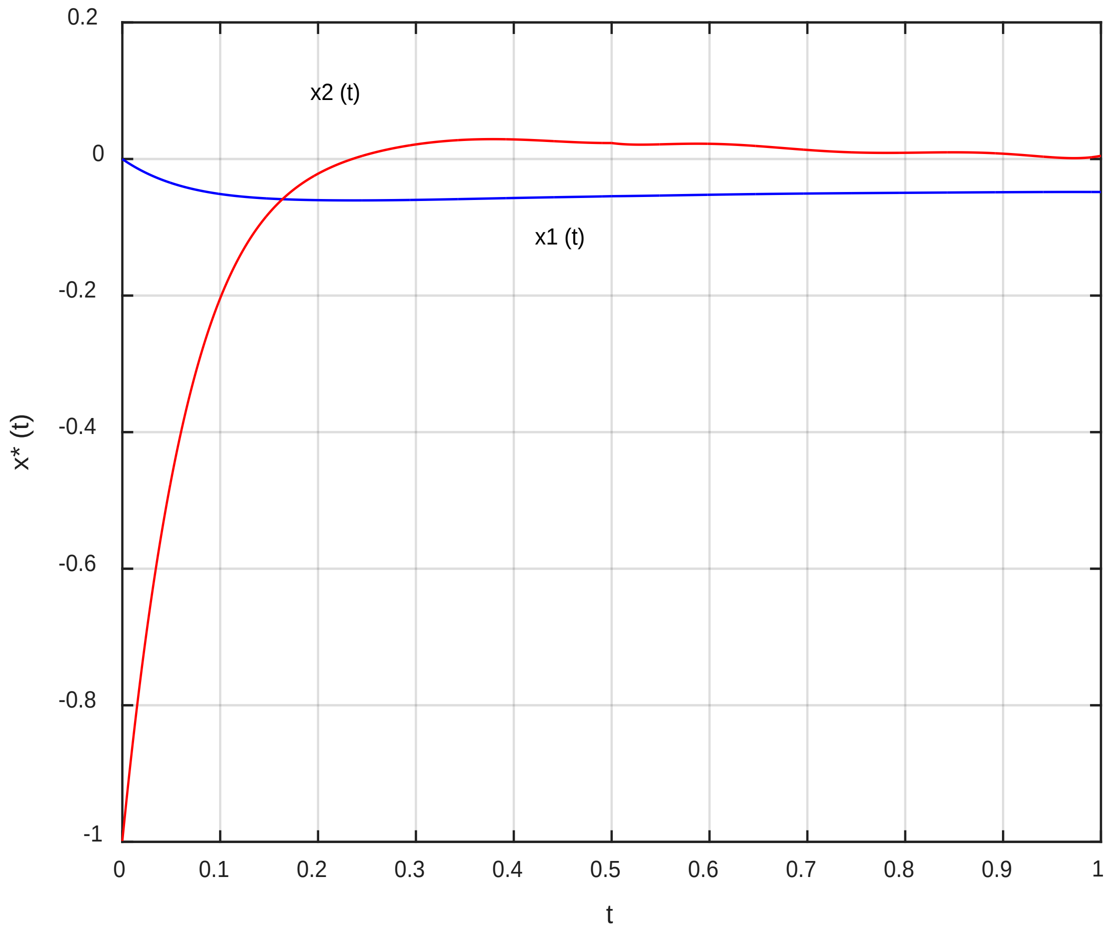

Figure 3.8: Example 2 optimal state trajectories for $M = 7\ , k = 2$

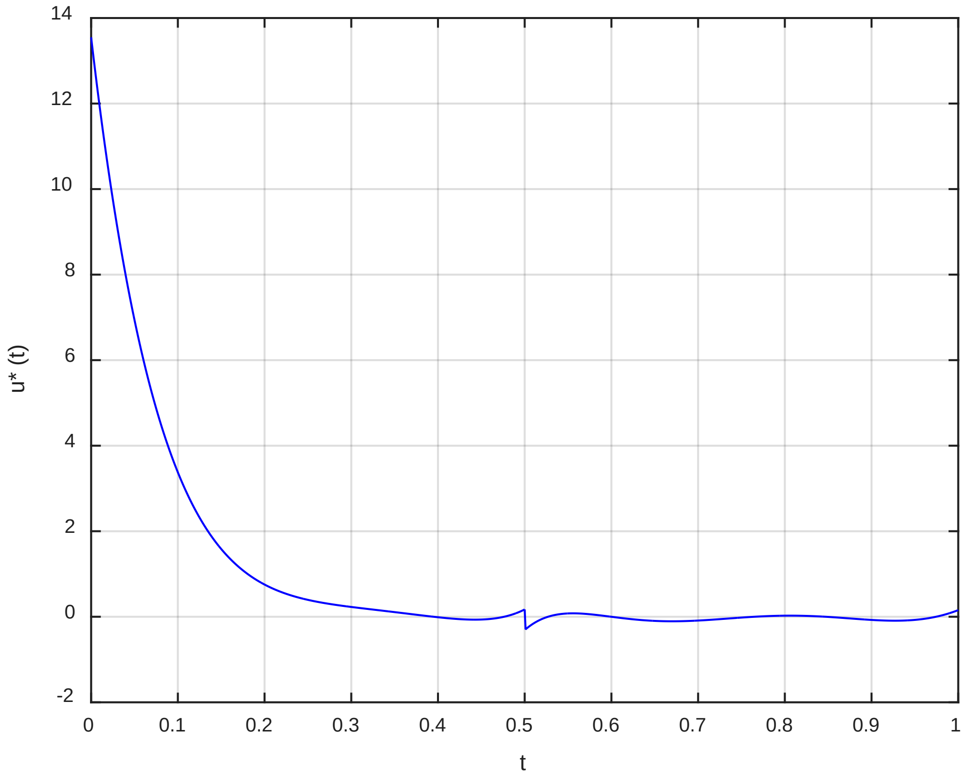

Figure 3.9: Example 2 optimal control trajectory for $M = 7, k = 2$

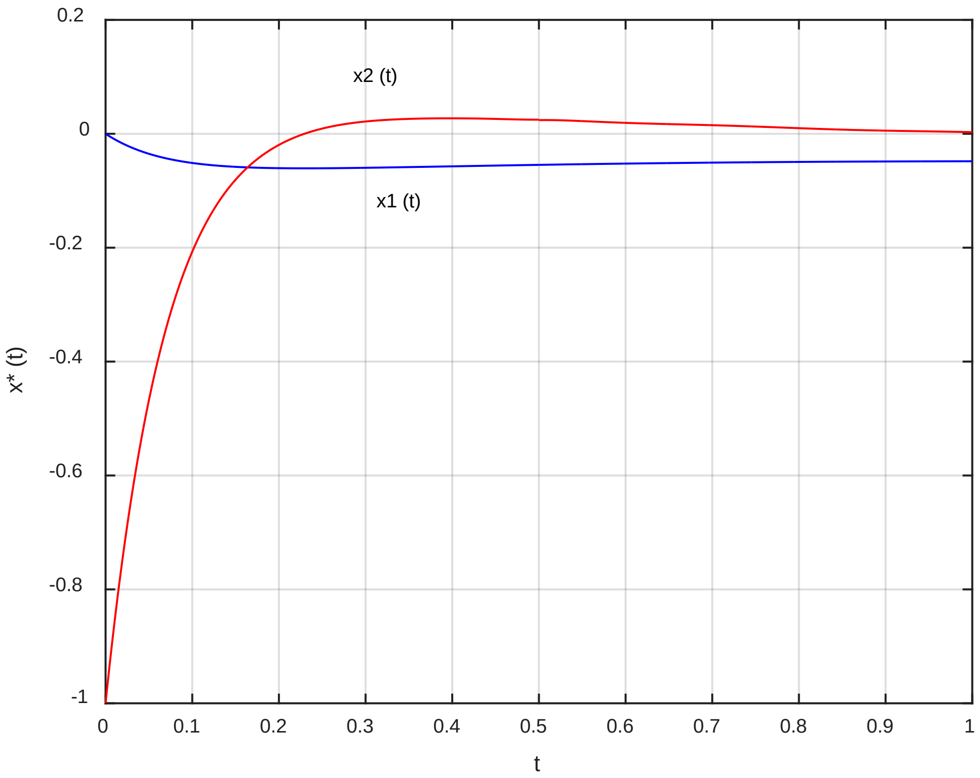

Figure 3.10: Example 2 optimal state trajectories for $M = 8, k = 2$

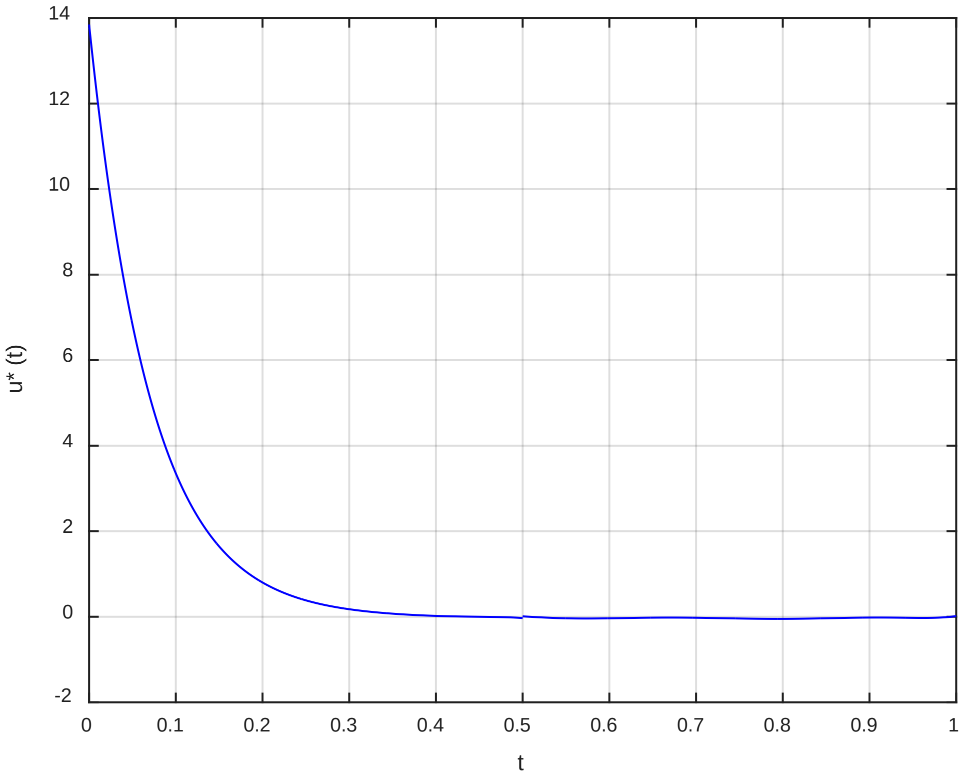

Figure 3.11: Example 2 optimal control trajectory for $M = 8, k = 2$

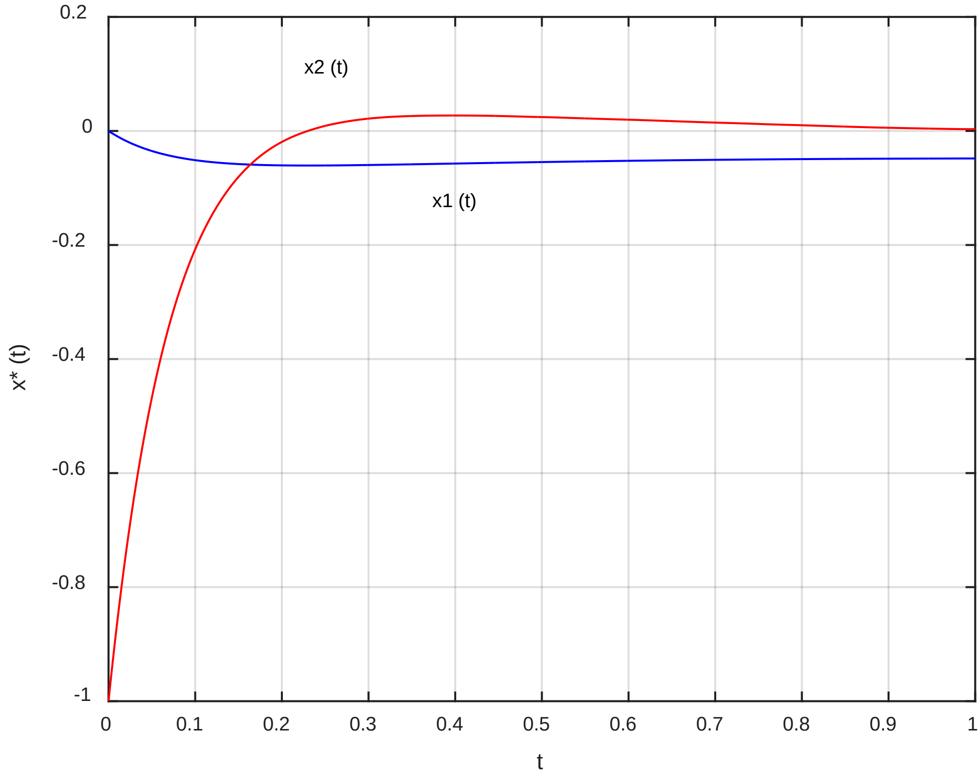

Figure 3.12: Example 2 optimal state trajectories for $k = 3, M = 7$

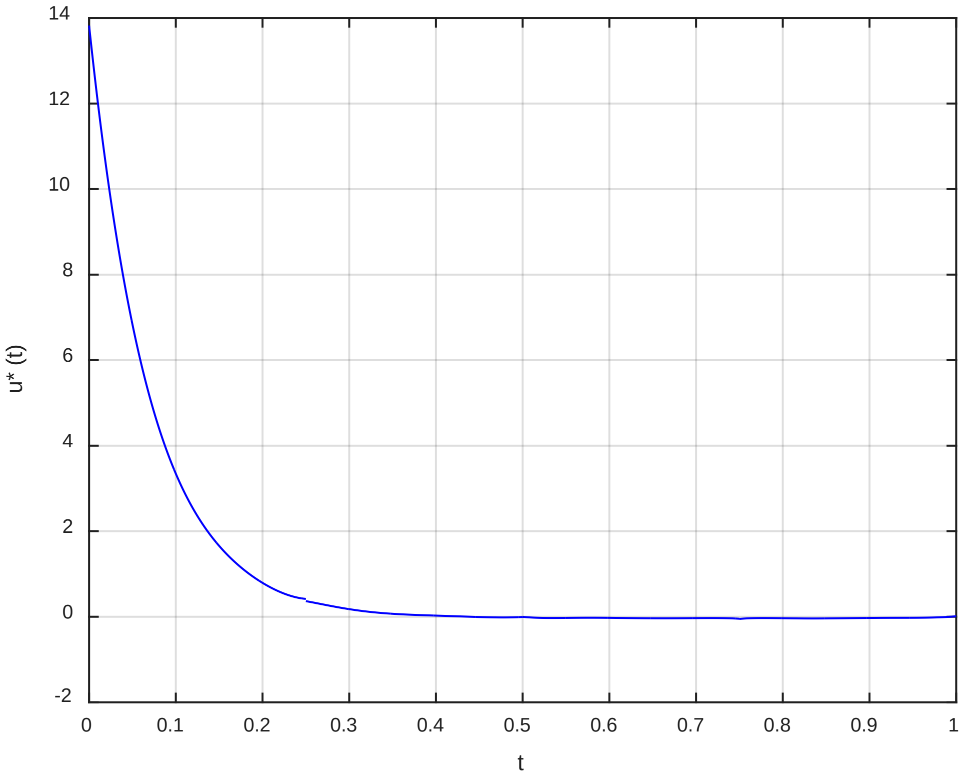

Figure 3.13: Example 2 optimal control trajectory for $k = 3, M = 7$

Table 3.3: Optimal Values of performance index for Example (2)

k, M	J	Error
$M = 7, k = 2$	0.069384847631541	$2.390763154117015e - 05$
$M = 8, k = 2$	0.069361250298555	$3.102985551561854e - 07$
$M = 7, k = 3$	0.069361200683422	$2.606834219182064e - 07$

Table 3.4: Comparison the Optimal Value of Example (2) with other method

	Method	J	Error
Exact value		0.06936094	0
C. P. NEUMAN and A. SEN	Cubic Splines [17]	0.06989	$5.3e-04$
This research	Legendre	0.069361200683422	$2.6e-07$

In this section, we proposed a numerical method for solving linear time invariant quadratic optimal control problems. In this method we used Legendre wavelet to approximate controls and states of the system using a finite length of Legendre function. Then we solved two examples, the first example contains one state and the second example contains two states, compared with other researches, our research gives better or comparable results with other researches.

As we saw in this chapter we converted the difficult linear quadratic optimal control problem into a quadratic programming problem which was easy to solve, and solved it by MATLAB program.

We conclude from Tables 3.1 and 3.3 that when we increase k or M we can obtain the results of performance index J more closed to the exact value. .

3.8 LTV Optimal Control Problem Reformulation

The linear time varying quadratic optimal control problem can be stated as follows:

Find an optimal control $u^*(t)$ that minimizes Equation (3.22)

Subject to

$$\dot{x} = A(t)x(t) + B(t)u(t) \qquad x(0) = x_0 \tag{3.46}$$

The different between LTI and LTV that A , B are function of t in LTV but A and B are constant in LTI, so that we use the most of equations in LTI.

3.8.1 Time varying approximation

Since the time varying elements $A(t)\ and\ B(t)$ are known matrices we can approximate them using Legendre function as:

$$A(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} A_{nm}\,\psi_{nm}(t) \tag{2.47}$$

$$B(t) = \sum_{n=1}^{2^{K-1}} \sum_{m=0}^{M-1} B_{nm}\,\psi_{nm}(t) \tag{2.48}$$

Where $A_{nm}\ and\ B_{nm}$ are:

$$A_{nm} = \int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}} A(t)\psi_{nm}(t)\,dt \tag{2.49}$$

$$B_{nm} = \int_{\frac{2n-2}{2^k}}^{\frac{2n}{2^k}} B(t)\psi_{nm}(t)\,dt$$

$$\mathrm{A} = [A_{10} \quad A_{11} \quad \cdots \quad A_{1,M-1} \quad \cdots \quad A_{2,M-1} \quad \cdots \quad A_{2^{K-1},M-1}]$$

$$\mathrm{B} = [B_{10} \quad B_{11} \quad \cdots \quad B_{1,M-1} \quad \cdots \quad B_{2,M-1} \quad \cdots \quad B_{2^{K-1},M-1}]$$

$$A(t) = A\Psi \tag{3.50}$$

$$B(t) = B\Psi \tag{3.51}$$

3.8.2 Control state parameterization

The basic idea is to approximate the state and control variables using Equations (2.28) and (2.29)

To approximate the state equation via LSF Equation (3.46) can be integrated as:

$$x(t) - x_0 = \int_0^t A(\tau)x(\tau)\,d\tau + \int_0^t B(\tau)u(\tau)\,d\tau \tag{3.52}$$

by substituting Equations $(3.50), (3.51), (3.30)$and (3.31) into (3.52) we get:

$$\Psi^{T} a - \delta \Psi^{T} g_0 = \int_0^t A \Psi \Psi^{T} a \, d\tau + \int_0^t B \Psi \Psi^{T} b \, d\tau \qquad (3.53)$$

Using Legendre properties Equation (3.53) can reduce to:

$$\left((P^T \otimes I_s) \widetilde{A} - I_{Ns} \right) a + (P^T \otimes I_s) \widetilde{B} b = -\delta g_0 \qquad (3.54)$$

By combining (3.54) and the additional constraints, we have:

$$\begin{bmatrix} (P^T \otimes I_s)\widetilde{A} - I_{Ns} & (P^T \otimes I_s)\widetilde{B} \\ (\Psi'(t) \otimes I_s) & 0_{(2^{k-1}-1)s \times Nr} \\ 0_{(2^{k-1}-1)r \times Ns} & \left(I_r \otimes \Psi'(t) \right) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\delta g_0 \\ 0_{(2^{k-1}-1)s \times 1} \\ 0_{(2^{k-1}-1)r \times 1} \end{bmatrix} \qquad (3.55)$$

$$FZ = h$$

Where $F = \begin{bmatrix} (P^T \otimes I_s)\widetilde{A} - I_{Ns} & (P^T \otimes I_s)\widetilde{B} \\ \left(\Psi'(t) \otimes I_s\right) & 0_{(2^{k-1}-1)s \times Nr} \\ 0_{(2^{k-1}-1)r \times Ns} & \left(I_r \otimes \Psi'(t) \right) \end{bmatrix}$ and $h = \begin{bmatrix} -\delta g_0 \\ 0_{(2^{k-1}-1)s \times 1} \\ 0_{(2^{k-1}-1)r \times 1} \end{bmatrix}$

3.9 Numerical Example 3 [27]

$$J = \frac{1}{2} \int_0^1 \left(x^2(t) + u^2(t) \right) dt$$

$$\dot{x} = tx + u \ , \qquad x(0) = 1$$

We solve this problem for:

$$k = 2, M = 3$$

$$J = 0.484290813333797$$

Figures 3.14-3.21 shows the optimal trajectories for several values of k and M.

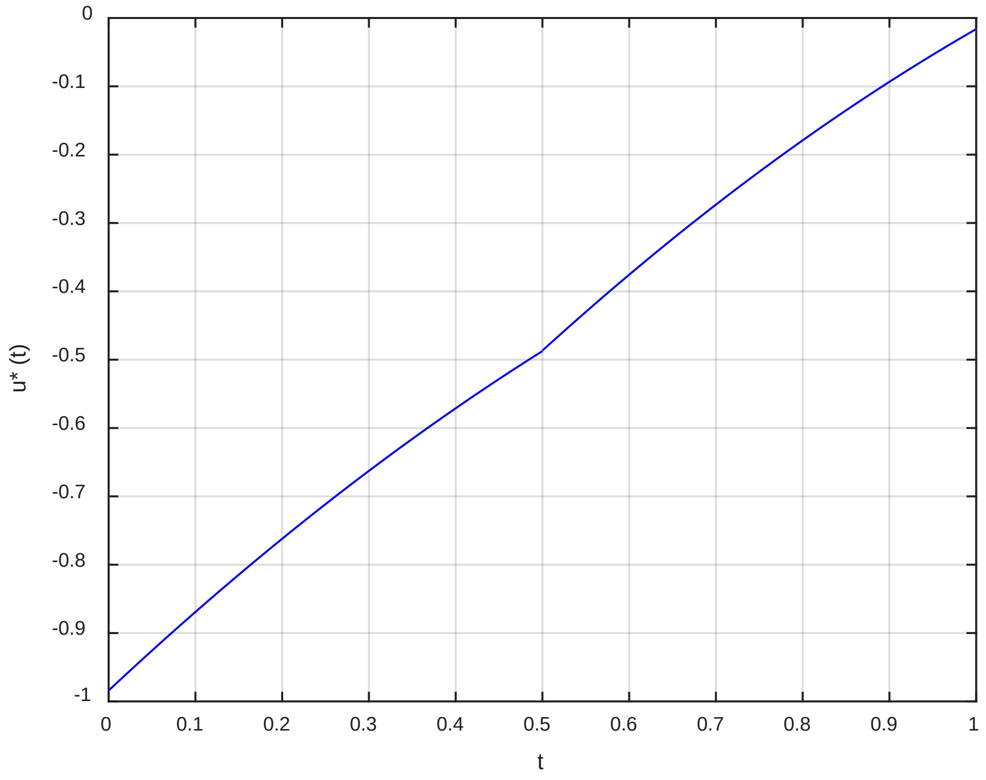

Figure 3.14: Example 3 optimal control trajectory for $k = 2, M = 3$

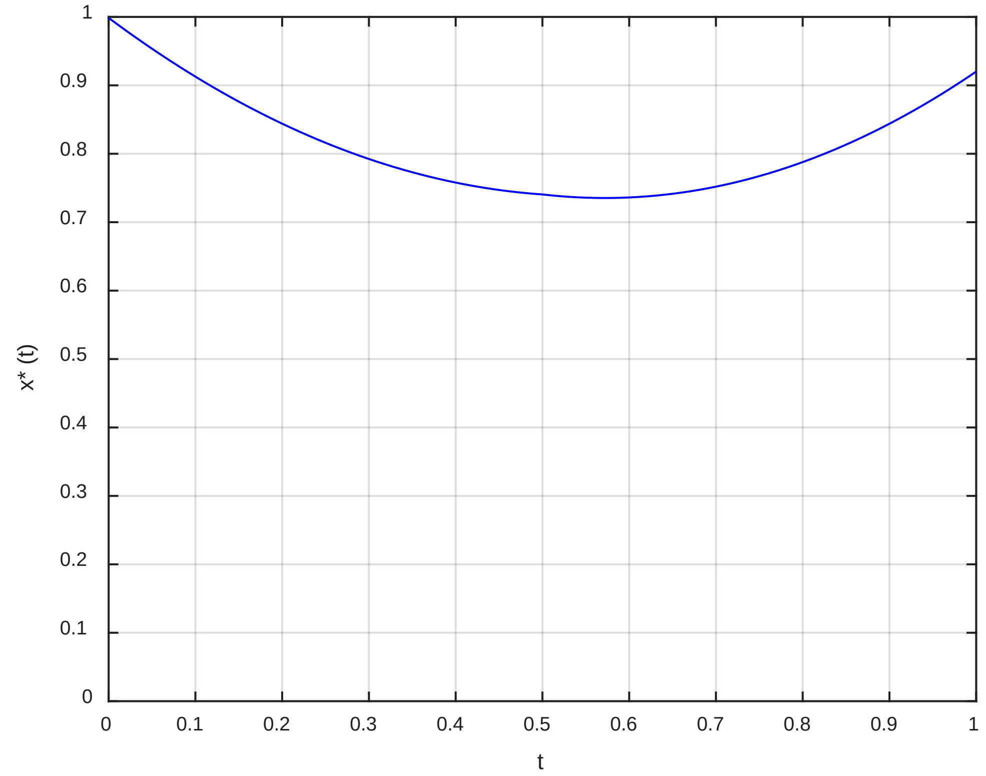

Figure 3.15: Example 3 optimal state trajectory for $k = 2, M = 3$

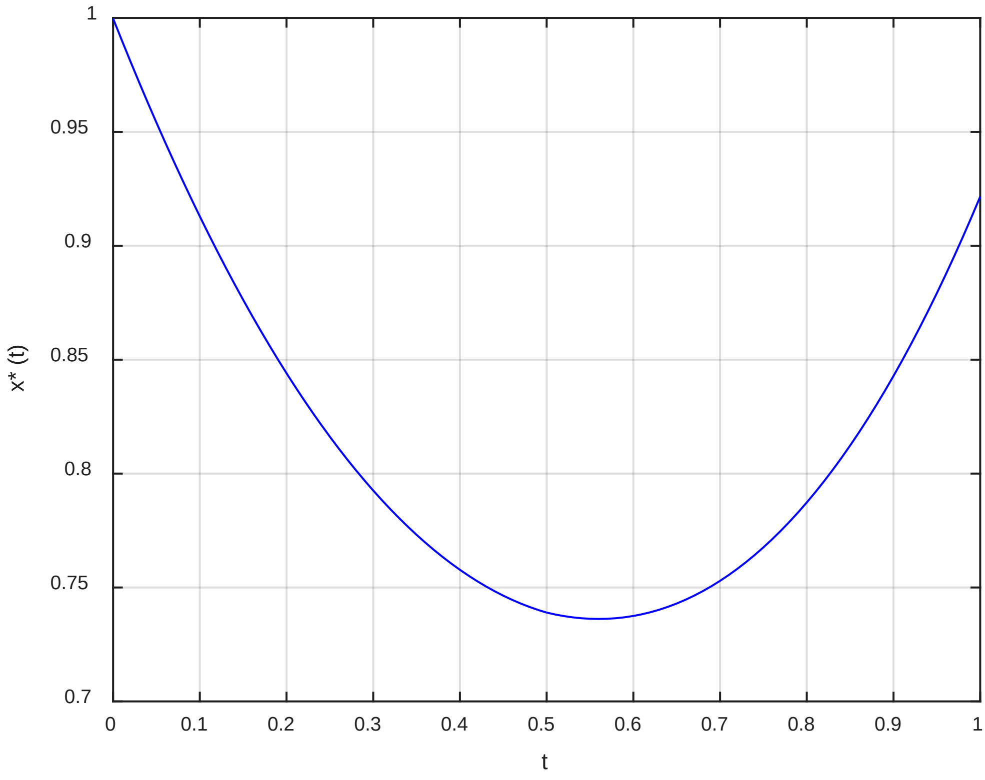

Figure 3.16: Example 3 optimal state trajectory for $k = 2, M = 4$

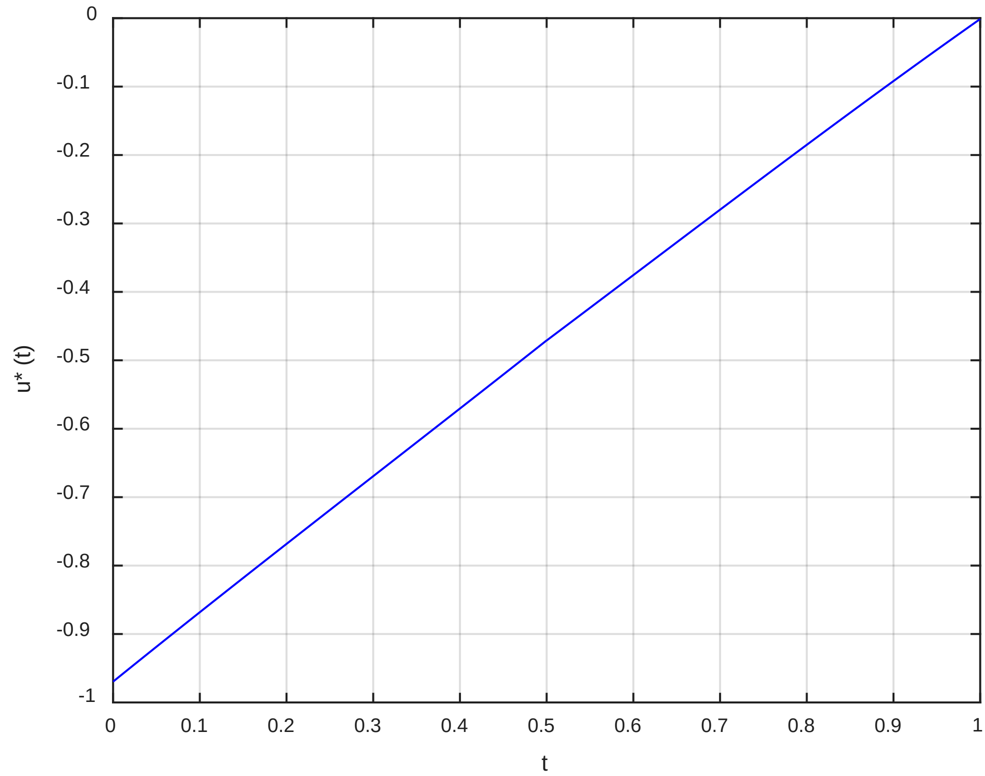

Figure 3.17: Example 3 optimal control trajectory for $k = 2, M = 4$

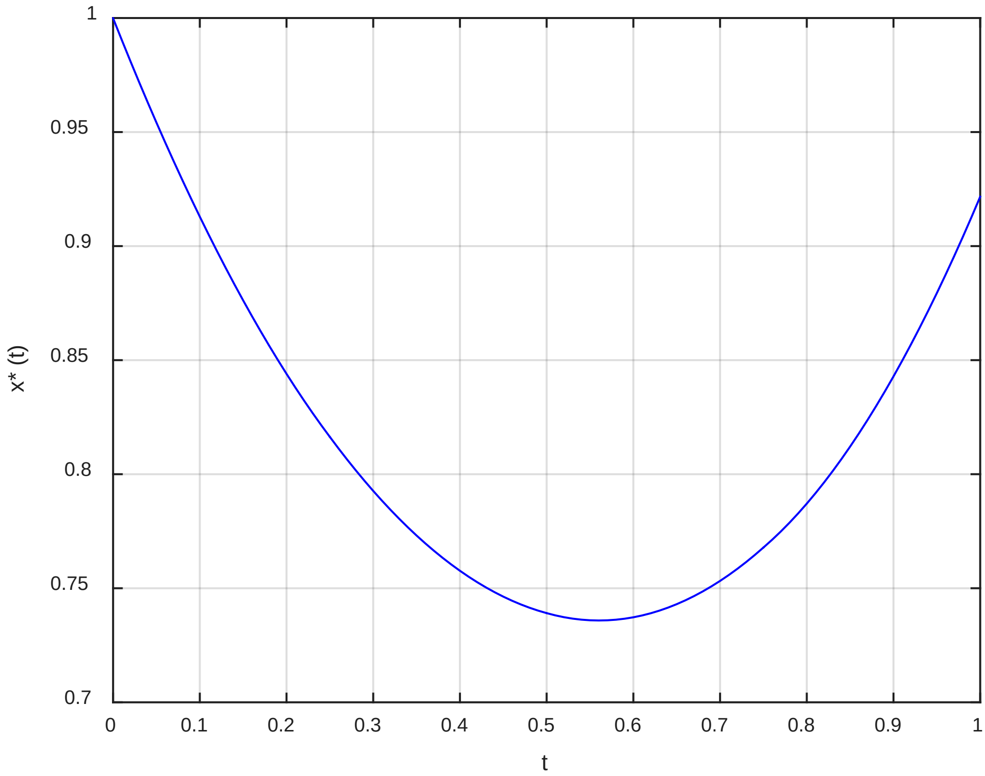

Figure 3.18: Example 3 optimal state trajectories for $k = 3, M = 3$

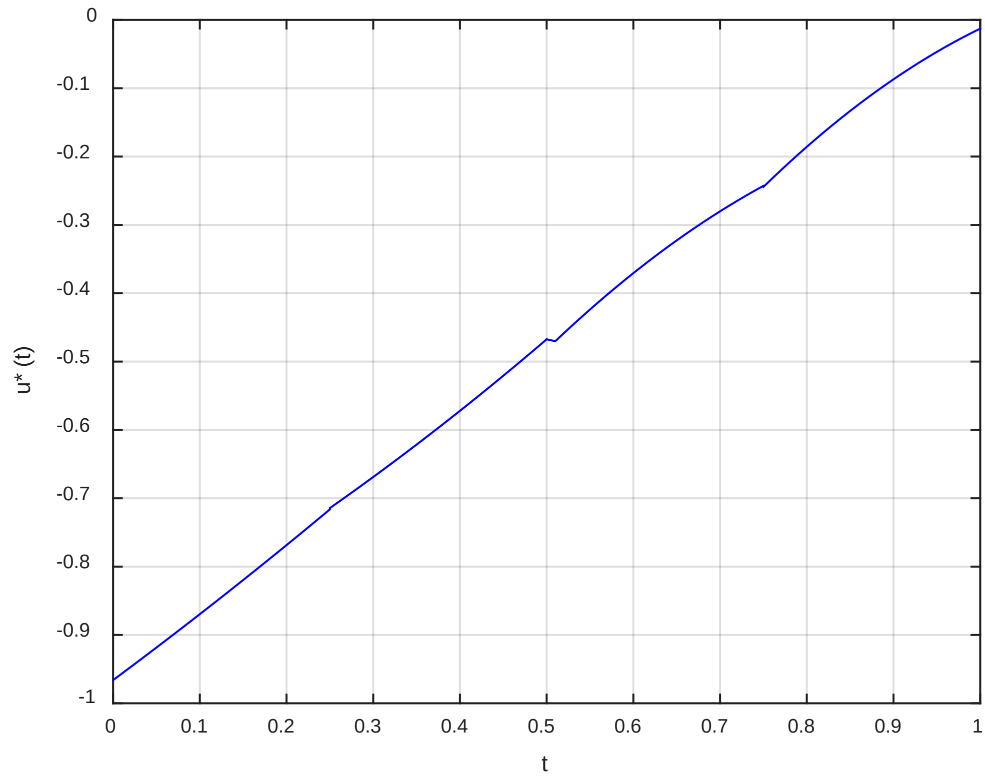

Figure 3.19: Example 3 optimal control trajectory for $k = 3, M = 3$

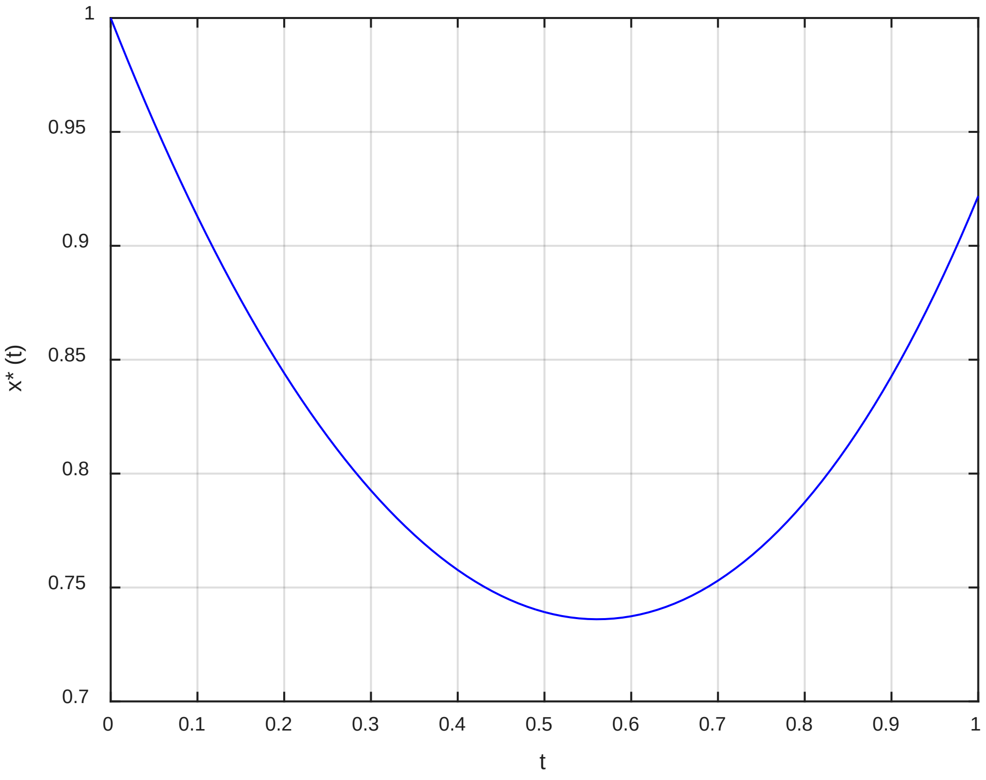

Figure 3.20: Example 3 optimal state trajectory for $k = 2\,, M = 5$

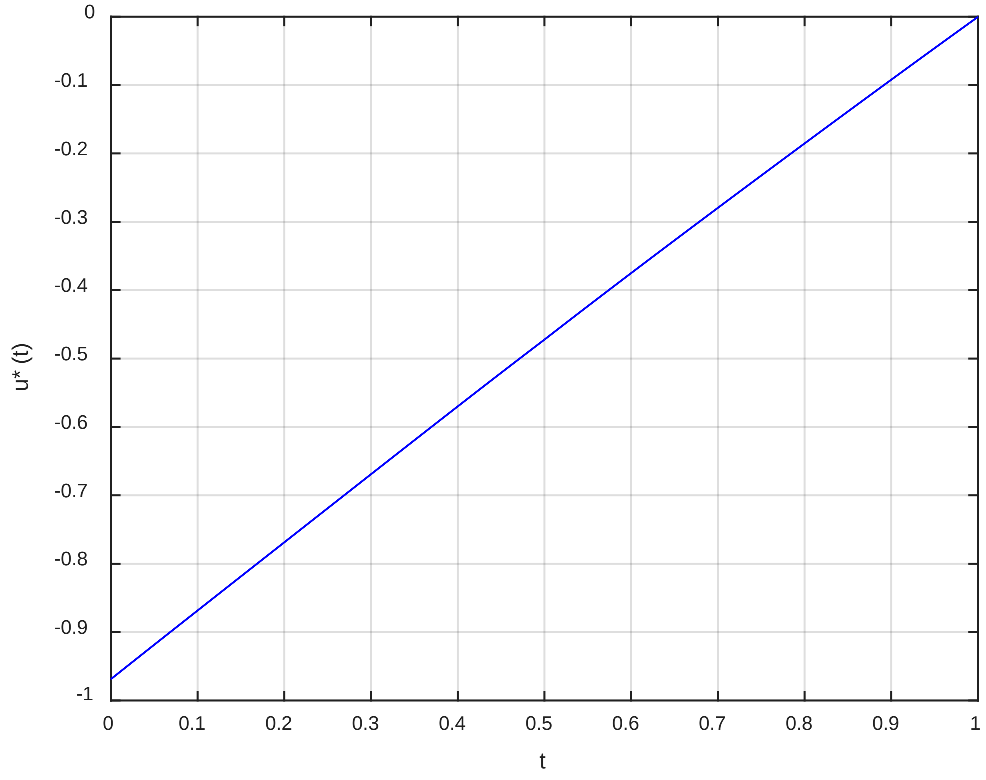

Figure 3.21: Example 3 optimal control trajectory for $k = 2\,, M = 5$

Table 3.5 shows the optimal performance index of Example 3 for different values of k and M, the comparison of our method and another method is shown in table 3.6.

Table 3.5: Optimal Values of performance index for Example (3)

k, M	J
$M = 3, k = 2$	0.484290813333797
$M = 3, k = 3$	0.484274888299967
$M = 4, k = 2$	0.484267796666342
$M = 5, k = 2$	0.484267700378673

Table (3.6): Comparison the Optimal Value of Example (3) with other method

	Method	J
Gamal N. Elnagar **[27]**	Spectral Chebyshev	0.48426764
This research	Legendre	0.484267700378673

Table 3.6 shows the comparison between our research and other researches to solve the previous problem, from the table we notice that our method is good compared with other methods. In this section we solved the OCP time-varying systems using Legendre scaling function; we applied this method at a numerical example to see the effectiveness of the method and compared with other methods. Solving time-varying optimal control problem is very

important because we must need it to solve the nonlinear optimal control problem in the next chapter.

CHAPTER FOUR

NONLINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

4.1 Introduction

In this chapter, we extend the method described in the previous chapter to solve nonlinear optimal control problems; one of the methods to solve the unconstrained nonlinear optimal control problem is to convert it into a nonlinear programming problem by using the direct methods. The nonlinear mathematical programming problem, in its turn, can be solved using different methods, in particular the sequential quadratic programming method, which replaces the nonlinear mathematical programming problem by a sequence of quadratic programming problems.

In this thesis, the nonlinear optimal control problem is converted directly into a sequence of quadratic programming problems, without converting it into nonlinear programming problem. This approximation can be achieved by an iterative approach [28].

Using the iterative approach, the nonlinear optimal control problem is replaced by a sequence of time-varying linear quadratic optimal control problems and then each of these problems is converted into a quadratic programming problem by using the control state parameterization via Legendre polynomials. Since the obtained quadratic programming problem is subject to equality constraints only, it can be solved in one iteration by matrix vector multiplication.

4.2 Iteration Approach

The iteration approach is based on the replacement of the original nonlinear system by a sequence of linear time-varying systems, whose solutions will converge to the solution of the nonlinear problem.

Consider a nonlinear system of the form:

$$\dot{x}(t) = A[x(t)]x(t) + B[x(t)]u(t); \;\; x(0) = x0 \qquad (4.3)$$

$$\dot{x}^{(1)}(t) = A[x_0]x^{(1)}(t) + B[x_0]u^{(1)}(t) \tag{4.4}$$

$$\dot{x}^{(2)}(t) = A\left[x^{(1)}(t)\right]x^{(2)}(t) + B\left[x^{(1)}(t)\right]u^{(2)}(t)$$

$$\vdots$$

$$\dot{x}^{(i)}(t) = A\left[x^{(i-1)}(t)\right]x^{(i)}(t) + B\left[x^{(i-1)}(t)\right]u^{(i)}(t)$$

with initial conditions $x^{(1)}(0) = x^{(2)}(0) = \cdots = x^{(i)}(0) = x_0$ at each iteration.

The sequence of solutions $x(t)$ converges uniformly on any compact time interval to the nonlinear solution $x(t)$

4.3 Problem Reformulation

To solve the nonlinear optimal control problem (4.1)- (4.2) using the proposed algorithm, the first step is to apply the iteration method, by expanding the state Equations (4.2) up to the first order around nominal trajectories $x(t)^{(k)}, u(t)^{(k)}$, and by expanding the performance. Then the optimal control problem is reduced to the following sequence of problems:

Minimize

$$J^{(i)} = \int_0^{t_f} \left(x^{(i)^T} Q x^{(i)} + u^{(i)^T} R u^{(i)}\right) dt$$

$$\dot{x}^{(i)} = A\big(x(i-1)\big)x^{(i)} + B\big(x(i-1)\big)u^{(i)} \qquad x^{(i)}(0) = x_0$$

We can solve this problem using previous technique

In the first step $i = 1$

$$J^{(1)} = \int_0^{t_f} \left(x^{(1)^T} Q x^{(1)} + u^{(1)^T} R u^{(1)}\right) dt$$

$$\dot{x}^{(1)} = A(x_0)x^{(1)} + B(x_0)u^{(1)} \qquad x^{(1)}(0) = x_0$$

This is LTI optimal control problem it can easily solved and obtain the output:

$x^{(1)}(t), u^{(1)}(t) and\ J^{*(1)}$

$i = 2$

$$J^{(2)} = \int_0^{t_f} \left(x^{(2)^T} Q x^{(2)} + u^{(2)^T} R u^{(2)}\right) dt$$

$$\dot{x}^{(2)} = A\left(x^{(1)}(t)\right)x^{(2)} + B\left(x^{(1)}(t)\right)u^{(2)} \quad x^{(2)}(0) = x_0$$

$x^{(1)}(t), u^{(1)}(t)$ are given from previous step, the problem became LTV optimal control problem, it can solved as in chapter three, after solving the problem the output is $x^{(2)}, u^{(2)} and\, J^{*(2)}$,all the next problems are LTV optimal control problem

$i = 3$

$$J^{(3)} = \int_0^{t_f} \left(x^{(3)^T} Q x^{(3)} + u^{(3)^T} R u^{(3)}\right) dt$$

$$\dot{x}^{(3)} = A\left(x^{(2)}\right)x^{(3)} + B\left(x^{(2)}\right)u^{(3)} \quad x^{(3)}(0) = x_0$$

$x^{(2)}, u^{(2)}$ are given from previous step after solving the problem the output is $x^{(3)}, u^{(3)} and\, J^{*(3)}$. We continue increasing the number of iteration i until $\left| J^{*[i]} - J^{*[i-1]} \right|$ became very small.

4.4 Numerical Example 4 [29]

Find optimal control law $u(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 [x^2(t) + u^2(t)]\, dt$$

Such that the bilinear system dynamics

$$\dot{x}(t) = -2x(t) + x(t)u(t) + 3u(t)\,, \quad x(0) = 5$$

We solve this problem for $k = 2, M = 5$. Firstly we rewrite the problem as:

$$J^{(i)} = \frac{1}{2}\int_0^1 \left(x^{(i)}\right)^2(t) + \left(u^{(i)}\right)^2(t)\, dt$$

$$\dot{x}^{(i)}(t) = -2x^{(i)}(t) + \left(x^{(i-1)} + 3\right)u^{(i)}(t)$$

$i = 1$, we get:

$$x^{(0)} = x(0) = 5$$

$$J^{(1)} = \frac{1}{2}\int_0^1 \left(x^{(1)}\right)^2(t) + \left(u^{(1)}\right)^2(t)\, dt$$

$$\dot{x}^{(1)}(t) = -2x^{(1)}(t) + 8u^{(1)} \quad x^{(1)}(0) = 5$$

By solving the above LTI OCPs we get the optimal value $J = 1.220274788157110$. For $= 2$, we have:

$$J^{(2)} = \frac{1}{2}\int_0^1 \left(x^{(2)}\right)^2(t) + \left(u^{(2)}\right)^2(t)\, dt$$

$$\dot{x}^{(1)}(t) = -2x^{(2)}(t) + \left(x^{(1)} + 3\right)u^{(2)}(t) \qquad x^{(2)}(0) = 5$$

By solving the above LTV OCPs we get the optimal value $J = 1.520044796167813$. Continue increasing i and listed J as shown in table 4.1.Table 4.2 shows comparison between the optimal value of example 4 with other method.

Table 4.1: Values of example (4) performance index for each iteration

Iteration	J
1	1.220274788157110
2	1.520044796167813
3	1.488960969923553
4	1.489546563552286

Table 4.2: Comparison the Optimal Value of Example (4) with other method

	Method	J
B. M. Mohan & Sanjeeb Kumar Kar [29]	Block Pulse Functions and Legendre Polynomials	1.4851
This research	Legendre	1.4895

Figure 4.1 and Figure 4.2 show the optimal state trajectory and the optimal control trajectory of Example 4.

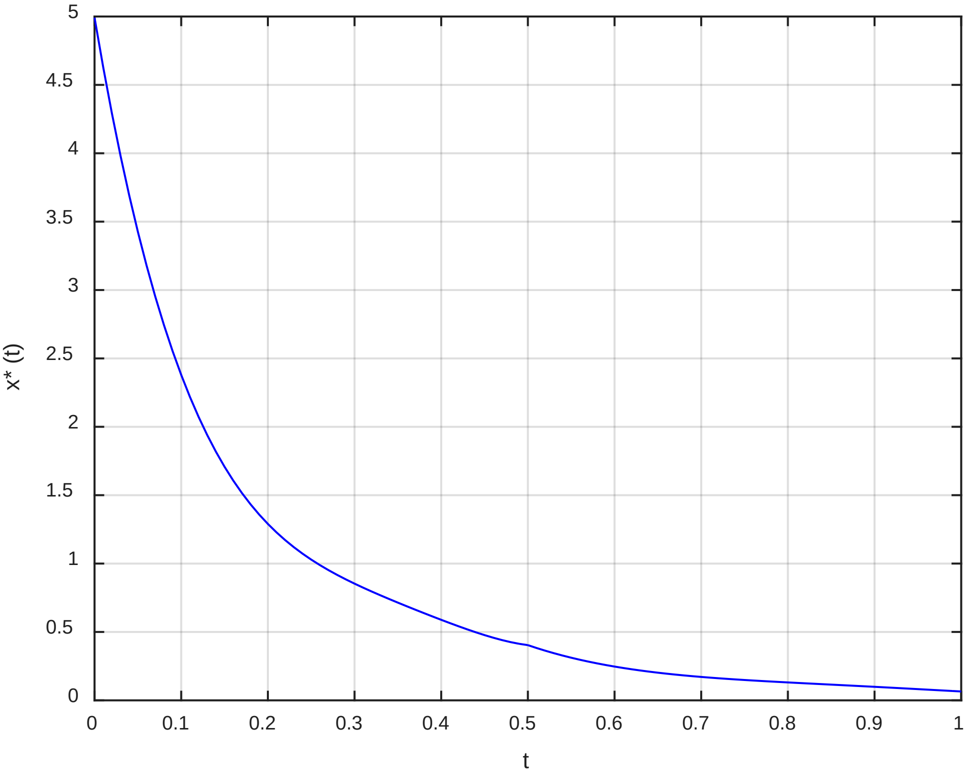

Figure 4.1: Example 4 optimal state trajectory

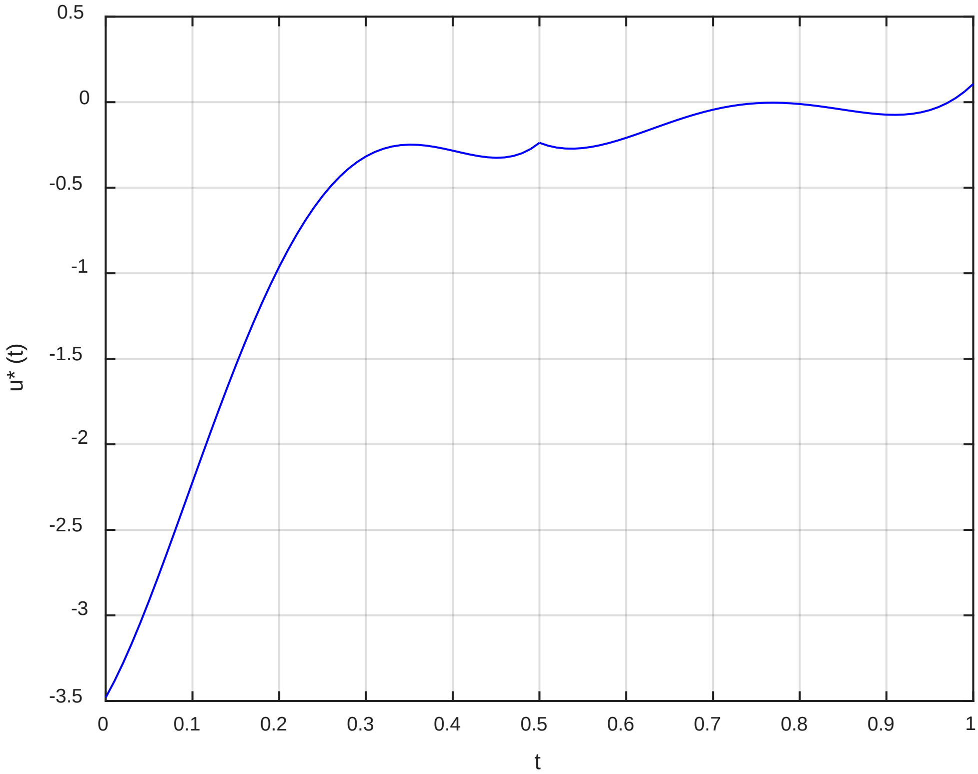

Figure 4.2: Example 4 optimal control trajectory

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this thesis, we proposed numerical methods to solve several types of optimal control problems. These methods are based on using the iteration approach and on parameterizing the system control, state variables using Legendre polynomials. Applying the proposed methods, convert the linear optimal control problem into quadratic programming problem and convert the nonlinear optimal control problem into sequence of quadratic optimal control problems. This method has several advantages such as: easy approximation; no integration of the state equations or co-state equations is needed; explicit formula is derived to approximate the quadratic performance index; small quadratic programming problems are to be solved. When the method is applied on several test examples which were solved by other researchers using different methods, the computational results of the proposed algorithms give better or same results in comparison with some other methods.

5.2 Recommendations

The work of this thesis can be extended in two ways:

- In control-state parameterization with large number of unknown parameters which have to be determined a_i and b_i; the system state equations have to be replaced by a large number of equality constraints; so control parameterization can be used instead of control-state parameterization.
- Using Laguerre or Hermite functions instead of Legendre function and solve the same examples which solved with Legendre Functions and compare the results with my results.

REVERENCE

[1] Donald E. Kirk, “Optimal Control Theory - An Introduction”, Dover Publications, Inc. Mineola, New York, 1970.

[2] Richard E. Bellman and Stuart E. Dreyfus, “Applied Dynamic Programming”, Princeton University Press, 1962.

[3] Brian D. O. Anderson John B. Moore,” Optimal Control - Linear Quadratic Methods”, Prentice Hall, Inc**,** 1989.

[4] Arthur E. Bryson, Jr. and Yu-Chi Ho, “Applied Optimal Control”, Taylor and Francis Group, 1975.

[5] D. L. Lukes, “Optimal Regulation of Nonlinear Dynamical System”, Siam J. Control, Vol. 7, No. 1, 1969.

[6] A. P. Willemstein,” Optimal Regulation of Nonlinear Dynamical Systems On A Finite Interval”, Siam J. Control and Optimization, Vol. 15, No. 6, 1977.

[7] William L. Garrard and John M. Jordan, “Design of Nonlinear Automatic Flight Control Systems”, Automatica, Vol. 13, pp. 497-505, Pergamon Press, 1977.

[8] Y. Nishikawa, N. Sannomiya and H. Itakura, “A Method for Suboptimal Design of Nonlinear Feedback Systems”, Automatica, Vol. 7, pp. 703-712. Pergamon Press, 1971.

[9] Taketoshi Yoshida and Kenneth A. Loparo, “Quadratic Regulatory Theory for Analytic Non-Linear System with Additive Controls”, Automatica. Vol. No. 4. pp. 531-544, 1989

[10] James H. Burghart, “A Technique for Suboptimal Feedback Control of Nonlinear Systems”, IEEE Transactions on Automatic Control, 1969.

[11] Andreas Wernli and Gerald Cook, “Suboptimal Control for the Nonlinear Quadratic Regulator Problem”, Automatica, Vol. 11, pp. 75-84. Pergamon Press, 1975.

[12] Nikola B. Nedeljkovic, “New Algorithms for Unconstrained Nonlinear Optimal Control Problems”, IEEE Transactions on Automatic Control, VOL. AC-26, NO. 4, 1981.

[13] Daniel Tabak, “Applications of Mathematical Programming Techniques in Optimal Control: A Survey”, IEEE Transactions on Automatic Control, 1970.

[14] K L Teo, C J Goh and K H Wong, “A Unified Computational Approach to Optimal Control Problems”, Longman Scientific & Technical, England, 1991.

[15] H. R. Sirisena, “Computation of Optimal Controls Using A Piecewise Polynomial Parameterization”, IEEE Transactions on Automatic Control, 1973.

[16] H. R. Sirisena and F. S. Chou, “State Parameterization Approach to the Solution of Optimal Control Problems”, Optimal Control Applications & Methods, Vol. 2, pp. 289-298, 1981.

[17] C. P. Neuman and A. Sen, “A Suboptimal Control Algorithm for Constrained Problems Using Cubic Splines”, Automatica, vol. 9, pp. 601-613, Pergamon Press, 1973.

[18] Kenneth A. Fegley, Joseph O. Bergholm, Steven Blum, Anthony J. Calise, J. Marowitz, Giacomo Porcelli And Lakshman P. Sinha, “Stochastic and Deterministic Design and Control via Linear and Quadratic Programming”, IEEE Transactions on Automatic Control, vol. AC-16, No. 6, 1971.

[19] G. B Ashein and Mark Enns, “Computation of Optimal Control by A Method Combining Quasi-Linearization and Quadratic Programming”, International Journal of Control,Vol. 16,No 1, pp. 177-187, 1972.

[20] V. Yen And M. Nagurka, “Optimal Control of Linearly Constrained Linear Systems via State Parameterization”, Optimal Control Applications & Methods, Vol. 13, pp. 155-167, 1992.

[21] Amara Graps, “An Introduction to Wavelets”, IEEE Computational Science and Engineering, Vol. 2, No. 2, 1995.

[22] Mohsen Razzaghi and Sohrabali Yousefi,” Legendre Wavelets Method for Constrained Optimal Control Problems”, Math Meth. Appl. Sci.Vol. 25, pp. 529-539, 2002.

[23] Cheng-Chiian Liu A and Yen-Ping Shih, “Analysis and Optimal Control of Time-Varying Systems via Chebyshev Polynomials”, Taylor & Francis, INT. J. Control, Vol. 38, No.5, pp. 1003-1012, 1983.

[24] W. W. Bell, “Special Functions for Scientists and Engineers”, D. Van Nostrand Company Ltd, 1968.

[25] M. Razzaghi and S. Yousefi, “The Legendre Wavelets Operational Matrix of Integration”, International Journal of Systems Science, Vol. 32,No. 4, pp. 495-502, 2001.

[26] B. Kafash and A. Delavarkhalafi, “Restarted State Parameterization Method for Optimal Control Problems”, Journal of Mathematics and Computer Science, pp. 151 – 161,2015.

[27] Gamal N. Elnagar, “State-control spectral Chebyshev parameterization for linearly constrained quadratic optimal control problems”, Journal of Computational and Applied Mathematics, vol. 79, pp.19-40, 1997.

[28] M. Tomas-Rodriguez, C. Navarro-Hernandez, S.P. Banks, “Parametric Approach to Optimal Nonlinear Control Problem Using Orthogonal Expansions”, IFAC, 2005.

[29] B. M. Mohan and Sanjeeb Kumar Kar, “Optimal Control of Nonlinear Systems via Block Pulse Functions and Legendre Polynomials”, Differ Equ Dyn Syst, pp. 149–159, 2012.