Sudan University of Science & Technology

College of Graduate Studies

Evolution of Stars and Universe within the Framework of Generalized Special Relativity Theory

تطور النجوم والكون في إطار النظرية النسبية الخاصة المعممة

A Thesis Submitted for the Fulfillment for the Requirements of degree of doctor of Philosophy Physics

Presented by:

Mohammed Saeed Dawood Aamir Mohammed

Supervisor by:

Prof. Mubarak Dirar Abdallah

Co supervisor:

D. Ahmed Elhassan Alfaki

January 2017

قَالَ تَعَالَىٰ:

﴿ وَجَعَلْنَا ٱلَّيْلَ وَٱلنَّهَارَ ءَايَتَيْنِ ۖ فَمَحَوْنَآ ءَايَةَ ٱلَّيْلِ وَجَعَلْنَآ ءَايَةَ ٱلنَّهَارِ مُبْصِرَةً
لِّتَبْتَغُوا۟ فَضْلًا مِّن رَّبِّكُمْ وَلِتَعْلَمُوا۟ عَدَدَ ٱلسِّنِينَ وَٱلْحِسَابَ ۚ وَكُلَّ شَىْءٍ فَصَّلْنَٰهُ
تَفْصِيلًا ﴾

الإسراء: (12)

Dedication

To my mother & father …

To my wife and To my sons & daughters …

To my brothers and sisters …

To all of my family …

I dedicate this work

Acknowledgement

My thanks and foremost to my GOD as without his will no thing is possible. Thanks my supervisor Prof. Mubarak Dirar Abdallah for his encourage at first and for supervised to this thesis. Thanks to Department of Physics, College of Graduate Studies, Sudan University of Science and Technology.

Abstract

Generalized special relativistic energy expression, beside Fermi momentum and ordinary Newtonian gravity potential were used for stars equilibrium conditions. The radius which makes the energy minimum shows that stability requires the mass to be less than certain critical mass which reflects quantum gravity behavior. This condition was similar to that of general relativity, where the radius should be greater than certain critical value. This critical value was typical to that of general relativity for black hole. The equilibrium condition show that pressure and centrifugal force should counter balance attractive gravity force. It also shows that kinetic energy balances potential energy at equilibrium. This agrees with previous models. The mathematical model was simple compared to general relativity model.

Generalized special relativity energy-momentum relation beside the positivity or negativity of energy were used to construct star evolution model. In the first approach short range repulsive beside long range attractive gravity force were assumed to contribute to the total energy. This shows the existence of finite self energy of matter in the form of string. It shows that the star radius was that of general relativity black hole radius. The minimization of energy with respect to potential, radius and mass shows in all cases the string nature of matter building blocks. The star evolution to become supernova or black hole was shown to be related to the relation of thermal to attractive gravity force in the same sense shown by general relativity.

The conditions of star equilibrium is discussed on the basis of the relation between pressure and gravity forces. The pressure expression was found first by using Gibbs and quantum laws. This leads to an equilibrium radius that depends on particle and mass density. The star explosion requires the energy to be positive. In this case, thermal energy exceeds gravity potential. When the generalized special relativistic energy is negative contraction takes place when gravity energy exceeds the thermal one. Star equilibrium requires the radius to have critical value typical to that of black hole and the critical mass to be less than a certain critical temperature dependent mass.

Using generalized special relativity together with Newton's laws of gravitation and treating particles as quantum strings, a useful expression for self energy was found. The critical radius of a star when particles are created is that of a black hole. The critical radius and mass are dependent on the speed of light and gravitational constant. For mass formation, the radius and mass should be small which agrees with the fact that elementary particles have very small mass and radius. The formation should also takes place at Planck time which also conforms with that proposed by big bang model.

مستخلص

استخدمت صيغة الطاقة للنظرية النسبية الخاصة المعممة بالإضافة لصيغة اندفاع فيرمي وصيغة طاقة الوضع النيوتينيَّة لمعرفة شروط اتزان النجوم. من صيغة القطر الذي يجعل الطاقة أقل ما يمكن وضح أن الاتزان يستدعى أن تكون الكتلة أقل من كتلة حرجة معينة مما يعكس الطبيعة الكمية التثاقلية في هذه الحالة. وهذا الشرط مشابه لشرط النسبية العامة حيث يجب أن يكون القطر أكبر من قيمة حرجة. هذه القيمة الحرجة هي نفس القيمة الحرجة لقطر الثقب الأسود في النسبية العامة. وقد أوضحت شروط الاتزان ضرورة أن الضغط وقوة الطرد المركزية ينبغي أن تتزن مع قوة الجذب التثاقلي. كما أوضحت ضرورة تساوي طاقة الحركة مع طاقة الوضع. وهذا يتفق مع النماذج السابقة. ويتميز النموذج الرياضي بأنه أبسط من نموذج النسبية العامة.

استخدم البحث علاقة الطاقة والاندفاع في النسبية الخاصة المعممة بالإضافة لشروط الطاقة الموجبة والسالبة لصياغة نموذج تطور النجوم. في النموذج الأول تم افتراض وجود جهد تثاقلي تنافري قصير المدى وجهد تجاذب تثاقلي بعيد المدى للمساهمة في صيغة الطاقة، مما بيَّن وجود طاقة ذاتية محدودة للمادة في هيئة وتر. وبيَّن وجود قطر حرج مماثل لذلك الذي للثقب الاسود في النسبية العامة. وبينت شروط أقل طاقة بالنسبة للجهد ونصف القطر والكتلة الطبيعة الوترية للبنات المادة. وبينت صيغ تطور النجوم لتصبح مستعر أو ثقب أسود أن لها علاقة مع الطاقة الحرارية والجهد الجاذبي التثاقلي لنفس طريقة النسبية العامة.

تمت مناقشة شروط الاتزان للنجم استناداً على العلاقة بين الضغط وقوى التثاقل. تم الحصول على صيغة الجهد أولاً باستخدام صيغة جيبس والكميّة. وقاد هذا لصيغة القطر عند الاتزان وهي تعتمد على كثافة الجسيمات والكتلة. ويتطلب انفجار النجم أن تكون الطاقة موجبة. في هذه الحالة تتجاوز الطاقة الحرارية جهد التثاقل. ولكن عندما تصبح طاقة النسبية الخاصة المعممة سالبة فإن التقلص يحدث, عندما تتجاوز طاقة التثاقل الطاقة الحرارية. ويتطلب اتزان النجم أن يكون للقطر قيمة حرجة هي نفس قيمتها للثقب الاسود, وتكون للكتلة قيمة أقلَّ من قيمة حرجة تعتمد على درجة الحرارة.

باستخدام نظرية النسبية الخاصة المعممة وقوانين نيوتن وباعتبار الجسيمات أوتار كميّة تم الحصول على صيغة مفيدة للطاقة الذاتية. وتكون القيمة الحرجة لنصف القطر للنجم عند تخلق الجسيمات هي نفس قيمتها للثقب الاسود. وتعتمد القيمة الحرجة للكتلة ونصف القطر على سرعة الضوء وثابت التثاقل. لتكوّن المادة يشترط أن تكون أنصاف الاقطار والكتلة ذات قيم صغيرة وهذا يتفق مع حقيقة أن كتل وأنصاف اقطار الجسيمات الأولية صغيرة. هذا التخلق يجب أن يحدث في زمن بلانك، وهذا يتسق مع ما تقول به نظرية الانفجار الكبير.

Contents

	الآية	i.
	Dedication	ii.
	Acknowledgement	iii.
	Abstract	iv.
	Abstract in Arabic	v.

Chapter One

Introduction

1.1	Cosmology and Stars	1
1.2	Research Problem	4
1.3	Aim of the Work	4
1.4	Previous Studies	4
1.5	Presentation of the Thesis	6

Chapter Two

Stars Evolution within the Framework of General Relativity

2.1	Introduction	7
2.2	The Equation of the Gravitation Field	7
2.3	The Field Equation for the Universe in A commoving Coordinate System	9
2.4	Generalization of Schwarzschild Metric	15
2.5	The Singularity of Schwarzschild Solution	17
2.6	Differential Equations for Stellar Structure	19
2.7	Newtonian Stars, Polytropes and White Dwarfs	24
2.8	Neutron Stars	30
2.9	Supermassive Stars	36
2.10	Stars of Uniform Density	39
2.11	Time Dependent Spherically Symmetric Field	45
2.12	Commoving Coordinates System	48
2.13	Gravitational Collapse	51

Chapter Three

Equilibrium of Stars By Equations of Motion to Relate Pressure and Gravity

3.1	Introduction	61
3.2	Coordinate System and the Mass Distribution	61
3.3	The Equations of Motion and Hydrostatic Equilibrium	63
3.4	Relativistic Electron Gas in Star	65
3.5	Stars Equilibrium	70

Chapter Four

Literature Review

4.1	Introduction	75
4.2	Gravitational Self Energy Mass and A model Based on Generalized General Relativity	75
4.3	Gravitational Self Energy Mass	76
4.4	Short Range Gravitation Field Equation Solution	80
4.5	Gravitational Waves	83
4.6	Graviton Equation of Motion	84
4.7	Graviton Energy	85
4.8	Spatial and Time Dependent Scalar Curvature and Gravitational Constant Quantization	86

Chapter Five

Stars Evolution, String Self Energy And Nonsingular Black Hole on the Basis Generalized Special Relativity

5.1	Introduction	90
5.2	Equilibrium Conditions	90
5.3	Conditions of Stars Evolution	99
5.4	Evolution of Stars By Kinetic Theory and Quantum Physics	107
5.5	Generation of Elementary Particles Inside Black Holes	118

5.6	Discussion	125
5.7	Conclusion	128
	References	129

Chapter One

Introduction

1.1 Cosmology and Stars:

Einstein's theory of general relativity (EGR) is one of biggest achievements in physics. This theory describes gravitation in a geometrical language by utilizing curved Riemann geometry [1]. EGR is found to be successful in describing a large number of astronomical observations [2]. Despite these successes, EGR suffers from noticeable setbacks. It is the most disastrous, and is the description of the radiation and pre-radiation eras, where elementary particles are dominant [3]. The elementary particles description is done by using quantum theory. Thus one needs a quantum gravitational theory to describe the early universe [4, 5], from the start of the big bang, pre plank and blank quantum era [6], up to radiation era. Quantum gravity theory is also needed to describe the behavior of black holes, neutron stars and pulsars [7]. Many attempts were made to construct a quantum gravity model [8]. Some of them are based on super string or string theory [8]. Others, like the wave function of the universe are proposed by Dwelt [9] and Hawking [10].

These attempts, although they are promising, but they are still done for from giving a complete full consistent quantum gravity theory. This failure stems from the fact that EGR derivation is not inconformity with the conventional method used to derive the field equations. In this conventional method the equation of motion and the energy momentum equation are different. The first one results from the variation of the field variables, while the second one results from the space-time variation [11]. In EGR the equation of motion and the energy-momentum equation are the same. They stem from the replacements of the potential field term by a geometrical term, and the matter term by the energy-momentum term [12]. This situation makes EGR isolated from other field theories, including quantum field theory. This bizarre situation necessitates searching for a new version of EGR, by keeping its beautiful geometrical language and abandoning Newton Poisson equation. This new version is first proposed by Lanczos [13] and then by Ali Eltahir [14]. It is based on the conventional action approach [14]. This generalized EGGR reduces to EGR thus shares with it all their successes.

Moreover, GEGR is proved to be nonsingular [15] and capable of solving the gravity energy-momentum problem, horizon, entropy and flatness problem, beside galaxy formation problem.

Stars play a central role in the universe. It is responsible for delivering energy to the surrounding astromical object [16, 17, 18]. In our solar system the sun which is a star, delivers energy to the earth. This energy is important for human life as well as plants and animal life. Without solar energy life cannot exist, including human life [19, 20]. The energy of the sun, which is now termed solar energy, is also utilized widely in generating electricity by using solar cells, and generating mechanical energy by using wind energy. The solar energy is also the main source of petroleum energy, vital energy and most of the energy sources in the world [21, 22, 23].

Gravitational fields are so weak that the practicing astrophysicist can usually ignore general relativity. This Thesis deals with various sorts of objects in which relativistic effects play an important. Or in some cases a dominant, role. One of these is the neutron star, a cold star composed primarily of neutrons and supported against collapse by neutron degeneracy pressure. Another is the supermassive star, a giant object supported by radiation pressure, in which general relativity effect can tip the balance between stability and instability. Most impressive of all is the black hole, a body caught in an inexorable gravitational collapse. The existence of neutron stars and black holes was suggested in the 1930'*s* on purely theoretical grounds, chiefly through the work of J. Robert Oppenheimer and his collaborators [24].

In the last few years a new species of astronomical exotica was discovered the pulsars, radio sources that pulse at regular frequencies ranging from a few tenths Hz to 30 Hz. The pulsars are often associated with optical and even X-ray sources that pulse at the same rate. There appears now to be a general consensus that pulsars are the neutron stars discovered theoretically in the 1930'*s*, but with a rapid rate of rotation that somehow or other produces the observed pulses [25].

A main-sequence hydrogen-burning star, such as the Sun, is maintained in equilibrium via the balance of the gravitational attraction tending to make it collapse, and the thermal pressure tending to make it expand. of course, the thermal energy of the star is generated by nuclear reactions occurring deep inside its core. Eventually, however, the star will run out of burnable fuel, and, therefore, start to collapse, as it radiates away its remaining thermal energy. What is the ultimate fate of such a star.

As the star collapses, its density increases, so the mean separation between its constituent particles decreases. Eventually, the mean separation becomes of order wavelength of the electrons, and the electron gas becomes degenerate. Note, that the wavelength of the ions is much smaller than that of the electrons, so the ion gas remains non-degenerate. Now, even at zero temperature, a degenerate electron gas exerts a substantial pressure, because the Pauli exclusion principle prevents the mean electron separation from becoming significantly smaller than the typical wavelength of the electrons. Thus, it is possible for a burnt-out star to maintain itself against complete collapse under gravity via the degeneracy pressure of its constituent electrons. Such stars are termed white dwarfs [26].

At stellar densities which greatly exceed white dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white dwarf is effectively transformed into a gas of neutrons. Eventually, the mean separation between the neutrons becomes comparable with their wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star. A star which is maintained against gravity in this manner is called a neutron star.it is found that there is a critical mass and critical radius above which a neutron star cannot be maintained against gravity. This critical radius, which is known as the radius of Schwarzschild. A star whose radius exceeds the radius of Schwarzschild. cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a black hole [26].

General relativity theory is one of the most successful theory that describes the universe. The so called big bang model describes the evolution of the universe [27, 28]. It states that the universe starts with singularity in space-time. It then expands, where matter, i.e. elementary particles is formed at early universe. These particles join together to form light atoms. Later on these particles are assembled in a cloud forming galaxies, stars, planets and all other astronomical objects [29].

The formation of stars is one of most striking features of general relativity. These stars can be come white dwarfs or red giant stars, supernova or black holes [30]. However the evolution of stars suffers from noticeable set backs, for instance the so called black holes results from space-time singularity which means break down of the laws of physics [31]. This draw back can be cured in this Thesis by using generalized special relativity (GSR).

1.2 Research Problem:

The evolution of stars within the framework of general relativity needs to be promoted, especially the formation of black holes which is accompanied by the existence of space time singularity.

1.3 Aims of the Work:

Is to construct theoretical model based on the generalized special relativity to obtain nonsingular model explaining stars evolution and formation of black holes.

1.4 Previous Studies:

Different attempts were made to describe the nature of stars and their evolution [32, 33]. Some of them are concerned with equilibrium of stars [34, 35, 36]. While others describe the behavior of black holes [37, 38, 39, 40, 41, 42]. In this section one caut some of them. In one of these papers:

In the work of M. Dirar, a simple derivation of the generalized field equation with a source term is presented by restricting ourselves to a locally inertial frame. It reduces to Einstein's field equation when the lagrangian is linear. Assuming the metric to be Minkowskian, simple solutions for the scalar curvature show the existence of a short range field and the emission of gravitational waves by objects which have strong field. The fact that the generalized field equation with a source term reduces to general relativity in a weak field limit indicates that it shares with general relativity all its successes in this limit. The solutions of the generalized field equation differ from those of general relativity in many respects. First of all the scalar curvature does not vanish outside the source. Secondly the expression for the potential shows the existence of a short range field or presumably a possible link with the strong nuclear force. On the other hand the travelling wave solution is in conformity with the recently observed declining in the orbit period of the binary pulsars [43, 44, 45, 46].

Also in the work of M. Dirar and others, the mass resulting from self energy is obtained by utilizing the generalized relativity. The expression for the mass which results from the gravitational field is finite. This expression is found by considering the mass first as small tiny string and second as small sphere. A useful equation for the propagation of graviton waves in space indicates that graviton propagates as travelling wave. By treating gravitation waves as wave packets a plank quantum expression for graviton energy dependent on the frequency is also found. The gravitational constant (parameter) is quantized also in this work. The capability of equation generalized general relativity to quantize the gravitational field and gravitational constant indicates that it can secure a good basis for a full quantum gravitational theory. The ability of this model to explain the origin of the mass in relation to the gravitational field, and to be an amenable to quantization, raises a hope of unifying all fundamental forces by bridging the gap between general relativity and quantum mechanics, beside finding a pathway to unify gravity with other forces by using Riemannian geometry as a common language [47].

In the work of Dong Lai, he describes what happens to a neutron star or white dwarf near its maximum mass limit when it is brought into a close binary orbit with a companion. Such situation may occur in the progenitors of Type IA supernovae and in coalescing neutron star binaries. Using an energy variational principle, we show that tidal field reduces the central density of the compact object, making it more stable against radial collapse. For a cold white dwarf, the tidal field increases the maximum stable mass only slightly, but can actually lower the maximum central density by as much as 30%. Thus a white dwarf in a close binary may be more susceptible to general relativistic instability than the instability associated with electron capture and pycronuclear reaction (depending on the white dwarf composition). We analyses the radial stability of neutron star using post-Newtonian approximation with an ideal degenerate neutron gas equation of state. The tidal stabilization effect implies that the neutron star in coalescing neutron star-neutron star or neutron star-black hole binaries does not collapse prior to merger or tidal disruption [48].

In the work of G. Dillon, a definition of a Newtonian black hole is possible which incorporates the mass-energy equivalence from special relativity. However, exploiting a spherical double shell model, it will be shown that the ensuing gravitational self energy and mass renormalization prevent the formation of such an object [49].

In the work of Paolo Christillin, it was shown that space curvature can be disposed of by properly taking into account gravitational self energies. This leads to a parameter free modification of Newton's law, violating Gauss theorem, which accounts for the crucial tests of gravitation in a at space. Strong gravitational fields entail opposing big gravitational self energies. The negative gravitational self energy of a gravitational composite object, which results in a mass defect with respect to the sum of the constituents, thus cancels out the latter at the Schwarzschild radius. Hence a black hole, possible end result of the radiative shrinkage of a star, having zero total energy cannot any longer interact with other objects. Baryon number non conservation may result [50].

Finally the work of Abhay Ashtekar, shows a set of boundary conditions defining a non-rotating isolated horizon are given in Einstein-Maxwell theory. A space-time representing a black hole which itself is in equilibrium but whose exterior contains radiation admits such a horizon. Physically motivated, (quasi) local definitions of the mass and surface gravity of an isolated horizon are introduced. Although these definitions do not refer to infinity, the quantities assume their standard values in Reissner-Nordstrom solutions. Finally, using these definitions, the zeroth and first laws of black hole mechanics are established for isolated horizons [51].

1.5 Presentation of the Thesis:

The thesis consists of five chapters. Chapters one and two are concerned with introduction and stars evolution. Chapters three and four are devoted for stars equilibrium and literature review. The contribution is in chapter five.

Chapter Two

Stars Evolution within the Framework of General Relativity

2.1 Introduction:

Gravitational fields were so weak that the practicing astrophysicist can usually ignore general relativity. This chapter deals with various sorts of objects in which relativistic effects play an important. Or in some cases a dominant, role. One of these is the neutron star, a “cold” star composed primarily of neutrons and supported against collapse by neutron degeneracy pressure. Another is the supermassive star, a giant object supported by radiation pressure, in which general relativity effect can tip the balance between stability and instability. Most impressive of all is the black hole, a body caught in an inexorable gravitational collapse [30].

In preparing this chapter, I have tried to restrict myself to the simplest calculations, which can be carried out analytically without too much trouble. These simple calculations are not very useful for a detailed understanding of astronomical observations, but they provide a valuable insight into the possible roles that general relativity can play in astrophysical phenomena.

2.2 The Equation of the Gravitation Field:

Differently electromagnetic field which does not influence its source the charge and which is determined by linear partial differential equation. The gravitational field does not affect the mass producing it and therefore should be described by nonlinear equation to obtain these equation Einstein started from be life that they must generalized from of Newtonian gravitational equation where the scalar potential φ can be component of the metric tensor by [52].

$$\varphi_g \approx -\frac{1}{2} g_{00} \tag{2.2.1}$$

The corresponding Poisson equation reads [53]:

$$\nabla^2 \varphi = -4\pi G\rho \tag{2.2.2}$$

Where φ the Newtonian gravitation field potential, ρ matter density, G gravitational constant.

We generalize equation (2.2.2):

$$G_{00} = -8\pi G T_{00}$$

Thus by equation (2.2.1) one obtains

$$\nabla^2 g_{00} = -8\pi G\rho = -8\pi G T_{00} \qquad (2.2.3)$$

Where the mass density in this case equals the energy density T_{00} if one extends the right hand side of equation (2.2.3) so that $T_{\infty} \rightarrow T_{\alpha\beta}$. Then by tensor analysis the left hand side should be equal to some second rank spatial tensor $G_{\alpha\beta}$. This means

$$G_{\alpha\beta} = -8\pi G T_{\mu\upsilon} \ , \quad \alpha\,\beta = 1,2,3 \qquad (2.2.4)$$

$G_{\alpha\beta}$ is a linear combination of $g_{\alpha\beta}$ and its first and second derivatives. by the equivalence principle these equation can be further generalized to

$$G_{\mu\upsilon} = -8\pi G\, T_{\mu\upsilon} \qquad (2.2.5)$$

To obtain the equation that govern he behavior of the gravitational field one need to find the form of $G_{\mu\upsilon}$ one therefore sets a number of requirements with regard to the properties of the gravitational field and which should be observed constructing the sought equation. Thus the $G_{\mu\upsilon}$ following requirements should be satisfied by [54]:

1. By definition it is a tensor consisting of the matric and its derivation.
2. This tensor should only contain terms that either quadratic in the first derivatives of the metric tensor or linear in its second derivatives $T_{\mu\upsilon}$.
3. It should be symmetric as.
4. Since $T_{\mu\upsilon}$ is conserved it should be equally so and vice versa.
5. It should be reducible to Newtonian limit.

By the fulfillment of these requirement and employing certain properties of the curvature tensor and its contraction it can be seen that the right hand side of equation (2.5.5) should have the form

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon}\, R = G_{\mu\upsilon} \qquad (2.2.6)$$

This expression is called Einstein tensor. Thus equation (2.5.5) becomes

$$R_{\mu\upsilon} - \frac{1}{2}\, g_{\mu\upsilon}\, R = -8\pi G T_{\mu\upsilon} \qquad (2.2.7)$$

Contracting with $g_{\mu\upsilon}$ yield

$$R = 8\pi G T_{\lambda}^{\lambda} \qquad (2.2.8)$$

Hence

$$R_{\mu\upsilon} = -8\pi\, G\left(T_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} T_{\lambda}^{\lambda}\right) \tag{2.2.9}$$

Equations (2.2.7) and (2.2.9) are the Einstein field equation. That describe the gravitation field and summarize the theory of general relativity. These equation can be alternatively by exploiting the variation action principle. Where $T_{\mu\upsilon}$ is the general energy-momentum tensor, and $G_{\mu\upsilon}$ called the Einstein tensor which is a combination of possible derivative of $g_{\mu\upsilon}$ and their products.

By criteria (1) → (5) the required equation yield

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R - \Lambda\, g_{\mu\upsilon} = G_{\mu\upsilon} = 0$$

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R = \Lambda g_{\mu\upsilon} \tag{2.2.10}$$

Where Λ is called the cosmological constant. It should reduce to stationary weak field limit for, $T_{\mu\upsilon} \rightarrow T_{\infty}$. we should set, $\Lambda = 0$. The Einstein's field equation reads

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R = G_{\mu\upsilon} \tag{2.2.11}$$

2.3 The Field Equation for the Universe in A commoving Coordinate System:

The large scale phenomena of universe are strongly affected by the gravitational interaction. Since gravity is the dominating interaction. General relativity should be able to give as it noted before a full description of the universe. Therefore what is needed is to find a model of the universe as a whole, which constitutes a solution of Einstein's equation. The solution of cosmological problem within the framework of general relativity consists of determining a large scale metric of the four-dimensional world and a corresponding large-scale matter distribution satisfying Einstein's equation theoretical cosmologists always make the idealizing assumption that, on sufficiently large scale, matter can be considered to homogeneously and isotropic ally distribute. This means that the energy-momentum tensor of matter in the universe is exemplified by that of a perfect fluid. on a sufficiently large scale the gross features of the universe, such as the mass density, indicate that the universe is homogeneous and isotropic. The modern cosmological theory is built on the cosmological principle, i.e. the hypothesis that the universe is spatially homogeneous and isotropic. The space-time metric of such a universe is given by [55]:

$$d\tau^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) \quad (2.3.1)$$

Where $a^2(t)$ is an unknown function of time called the cosmic scale factor and k is a constant known as the spatial curvature. Which by a suitable choice of units for r can be set to take only three values: 1, 0, or -1 for a closed. Spatially flat and open universe respectively. The spatial polar coordinate r, θ, ϕ. Form co-moving system in the sense that typically galaxies have constant spatial coordinates r, θ, ϕ.
Applying the cosmological principle to energy-momentum tensor that describes cosmic matter shows that it take the same form as for perfect fluid

$$T^{\mu\upsilon} = (\rho + p)U_\mu U_\upsilon + p g_{\mu\upsilon} \quad (2.3.2)$$

Where ρ: mass density, p: pressure.

$$U_t = 0 \quad , \quad U_i = 0$$

$$T_{tt} = \rho(t)\, T_{ii} = g_{ii}\, p \quad , \quad i = r, \theta, \phi \quad (2.3.3)$$

Equation (2.3.1) is suitable matric for describing expanding universe, which is written by Roberson-Walker metric thus Einstein equation (2.2.7) gives

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (2.3.4)$$

$$\frac{\dot{a}}{a} = \frac{-8\pi G}{3}(p + 3\rho) + \frac{\Lambda}{3} \quad (2.3.5)$$

The conservation of the energy also gives

$$\dot{\rho} + 3H(p + \rho) = 0 \quad (2.3.6)$$

Where

$$H = \frac{\dot{a}}{a} \quad (2.3.7)$$

Is called Hubble parameter, to solve these equation a relationship between $\rho \, \Lambda \, p$ is needed. This relation can be written in the form

$$p = \omega\rho \quad (2.3.8)$$

Where ω is constant independent of time. Thus the conservation of energy equation (2.3.7) reads

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \quad (2.3.9)$$

This can be integrated to obtain

$$\rho \propto a^{-3(1+\omega)} \quad (2.3.10)$$

Einstein equation for spatial component gives

$$\frac{\dot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \quad (2.3.11)$$

Which by using equation (2.3.8) takes the following form

$$\frac{\dot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G\rho(1+\omega) \quad (2.3.12)$$

In this case ($\omega = 1$), and the energy density is independent of a since the energy density in matter.

2.3.1 Vacuum Era:

In vacuum matter does not exist. This needs introducing a cosmological constant is equivalent to existence of energy momentum tensor for the vacuum [56] i.e.

$$G_{\mu\upsilon} = 8\pi G T_{\mu\upsilon} + \Lambda g_{\mu\upsilon} \quad (2.3.13)$$

$$T_{\mu\upsilon}^{vac} = \frac{-\Lambda}{8\pi G} g_{\mu\upsilon} \quad (2.3.14)$$

This has the form a perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} \ , \quad T_{\mu\upsilon}^{vac} = p g_{\mu\upsilon} \quad (2.3.15)$$

2.3.2 Radiation Era:

During this era radiation dominates, i.e. most is in the form of radiation. may describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light. Although radiation is perfect fluid and thus the energy momentum tensor. The equation of this state is [57]:

$$p = \frac{1}{3}\rho \quad (2.3.16)$$

This energy density in radiation takes the form

$$\rho = a^{-4} \tag{2.3.17}$$

At the universe expanse it cool down, thus some elementary particles combine to form atoms, which in turn accumulate to matter. The cosmological term Λ in Einstein general relativity can be considered as standing for vacuum energy [58]:

$$\rho_v = \frac{\Lambda}{8\pi G} \tag{2.3.18}$$

Where general relativity takes the form

$$R_{\mu v} - \frac{1}{2}\, g_{\mu v} R = -8\pi G \left(T_{\mu v} - \frac{\Lambda}{8\pi G}\, g_{\mu v} \right) \tag{2.3.19}$$

Experimentally observation indicate that vacuum energy negligible small at present. On the hand, large vacuum energy at the early universe can produce inflation where

$$a = Ae^{\mu t}$$

This inflation can solve some of cosmological problems. The diversity between the present small value of the vacuum energy and the need of large vacuum energy at the early universe is known as cosmological constant problems [59].

2.3.3 Equation of Motion:

In order to find the Einstein equation we approximate the matter, averaged over long distances, as an ideal fluid which is rest in the cosmic coordinate system of the Robertson-Walker metric, i.e. The energy-momentum tensor is given by [60]:

$$T^{\mu v} = (\rho c^2 - p) U^{\mu} U^{v} - p g^{\mu v} \tag{2.3.20}$$

Where U^{μ} is the four-velocity of the matter, the unite tangent vector field of the time like congruence. We have $T^{\mu}_{\mu} = \rho c^2 - 3p = 0$ for scale invariant case like EM radiation, $p = 0$ for matter at rest, like cosmic dust and $T^{\mu v} \sim g^{\mu v}$, $p = -\rho c^2$ in the vacuum.

Notice that $\rho c^2 + 3p \geq 0$ in each case. The metric (2.3.1) is based on the time coordinate r therefore $U^{\mu} = (1,0,0,0)$.

The first two term in the divergence of a tensor $A^{\mu v}$:

$$D_v A^{\mu v} = \partial_v A^{\mu v} + \Gamma^{v}_{\rho v} A^{\mu \rho} + \Gamma^{\mu}_{\rho v} A^{\rho v} \tag{2.3.21}$$

Looks as the covariant divergence of a four-vector which can be written in a simpler manner according

$$D_\upsilon A^{\mu\upsilon} = \frac{1}{-g^{1/2}}\partial_\upsilon\left[-g^{1/2}A^{\mu\upsilon}\right] + \Gamma^{\mu}_{\rho\upsilon}A^{\rho\upsilon} \tag{2.3.22}$$

Thus the expression (2.3.20) leads to the energy-momentum conservation law

$$-\partial_\upsilon p g^{\mu\upsilon} + \frac{1}{-g^{1/2}}\ \partial_\upsilon\left[-g^{1/2}(\rho c^2 + p)U^{\mu}U^{\upsilon}\right] + \Gamma^{\mu}_{\rho\upsilon}(\rho c + p)U^{\rho}U^{\upsilon} = 0 \tag{2.3.23}$$

Where the metric admissibility, $D_g = 0$, was used, $t\,00$. The rest frame condition, $U^{\mu} = (1{,}0{,}0{,}0)$, renders the spatial components, $\mu = 1{,}2{,}3$ of this equation trivial and the temporal part $\mu = 0$ reads as

$$a^3\dot{p} = \frac{d}{dr}\left[a^3(\rho c^2 + p)\right] \tag{2.3.24}$$

Or

$$a\dot{\rho}c^2 + 3\dot{a}(\rho c^2 + p) = 0 \tag{2.3.25}$$

Giving

$$\frac{d}{dr}(a^3\rho c^2) = 0 \tag{2.3.26}$$

$\rho \sim 1/a^3$ for dust. In the case of radiation we write

$$\frac{2}{3}\frac{d}{dr}(a^3\rho c^2) + \frac{1}{3}a^3\dot{\rho}c^2 = \frac{1}{3}\frac{d}{dr}(a^4\rho c^2) = 0 \tag{2.3.27}$$

Resulting in $\rho \sim 1/a^4$. The density drops faster in the latter case during the expansion of the universe (growing a) than for dust. Though the radiation represents a negligible component in the actual universe, it was dominant in an earlier phase.

The Einstein's equation read finally as

$$R_{00} - \frac{1}{2}R - \Lambda = 3\frac{\dot{a}^2 + k}{a^2} - \Lambda = 8\pi G T_{rr} = 8\pi G\rho c^2 \tag{2.3.28}$$

For the component

$$\frac{1}{a^2}\left(R_{rr} + \frac{a^2}{2}R + a^2\Lambda\right) = \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) + \Lambda$$

$$= \frac{-2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} + \Lambda = \frac{8\pi G}{a^2}\ T_{rr} = 8\pi G\rho \tag{2.3.29}$$

We can express the acceleration $\ddot{a}$ by making forming a suitable linear superposition of these two equations

$$\frac{\ddot{a}}{a} = -\frac{\Lambda}{2} - \frac{4}{3}\ \pi G(3p + \rho c^2) \tag{2.3.30}$$

The cosmological constant introduces a pressure in the absence of matter and leads to violation of the Newtonian gravitational law in the slow motion, weak gravitational field limit. We shall set $\Lambda = 0$ in the rest of the discussion for simplicity. The first remark is that there is no static solution, $\ddot{a} < 0$ for $3p + \rho c^2 > 0$. The rate of change of spatial distances

$$v = \frac{dl}{dr} = \frac{l}{a}\dot{a} = Hl \qquad (2.3.31)$$

Where

$$H = \frac{\dot{a}}{a} \qquad (2.3.32)$$

Called Hubble constant, though its value has slow time dependence on astrophysical time scale. The universe is expanding at the present, $\dot{a} > 0$ but in view of $\ddot{a} < 0$ the expansion rate must have been faster in the past. by assuming constant expansion rate $T = 1/H$. Time ago the universe would have been point-like. Due to the slowing expansion rate the big bang, the zero-side universe must have occurred lees time before and the inverse Hubble-constant gives only an order of magnitude estimate of the lifetime of universe. The zero size signals a singularity in the time evolution which prevents us to inquire about the state of the universe. Before the big bang. The so called singularity theorems of general relativity assures that the singularity at the big bang is present even without assuming homogeneity and isotropy. For the flat or open universe, $k = 0$ or $k = -1$, respectively $\dot{a} \neq 0$ according to equation (2.3.28) which can be written as

$$\dot{a}^2 = \frac{8\pi G}{3}a^2\rho c^2 - k \qquad (2.3.33)$$

And the expansion continues forever. In fact, $\rho c^2 = f(a^{-3})$ or $\rho c^2 = f(a^{-4})$ for dust or radiation dominated universe, $\rho c^2 a^2 \to 0$ as $r \to \infty$ and $\dot{a}$ approaches zero from above. For closed universe, $k = 1$, the matter contribution to equation (2.3.32) decreases compared to k during the expansion and there is a maximal value of a, $a \leq a_0$. But the maximal value cannot be approached asymptotically because $\ddot{a}$ dose not tend to zero according to equation (2.3.30) but instead a big crunch occurs at some finite time where $a = 0$ is reached and the universe cease to exists. The component of the Einstein equation (2.3.28) for $\Lambda = 0$ shows that the universe is closed or open if $\rho > \rho_c$ or $\rho < \rho_c$, respectively where

$$\rho_c c^2 = \frac{3H^2}{8\pi G} \tag{2.3.34}$$

The actual observational and theoretical background suggests that the cosmological constant Λ actually plays an important role in determining the age of the universe [60], in particular the choice. $\rho_{matter} \approx 0.27\rho_c$, $\rho_\Lambda \approx 0.73\rho_c$, $\rho_{mer} + \rho_\Lambda \approx \rho_c$ (is preferred).

2.4 Generalization of Schwarzschild Metric:

One will discuss in this section a generalized solution for Schwarzschild metric. This solution is considered as a general solution to describe the weak gravitational field and the strong. Schwarzschild metric is a good approximation to the gravitation field of a slowly rotating body like the Earth or Sun. but in the case of a strong gravitational field the general relativity will fail to describe what is happing inside. We can derive the Schwarzschild metric from my new metric, that is in the case of weak gravitational field where both of them give the same results. But in the case of strong gravitational field my new matric gives an accurate results which are different from the Schwarzschild metric. The Schwarzschild metric is given by [61]:

$$ds^2 = g_{tt}dt^2 - g_{tt}^{-1}dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.4.1}$$

Where

$$g_{tt} = 1 - \frac{2GM}{c^2 r}$$

Now if there is clock located in a gravitational field at distance r from the center of mass M, then the reading of this clock for an observed far away from the gravitational field is given as the equation

$$\Delta t' = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \Delta t \tag{2.4.2}$$

Where, Δt is the reading of the observer from his clock is far away from the gravitational field and $\Delta t'$ is the reading of the same observer from the clock on the gravitational field. From the metric given in (2.4.1) the Schwarzschild radius r_s is given as

$$r_s = \frac{2GM}{c^2 r} \tag{2.4.3}$$

The Schwarzschild solution to Einstein's equations given exact solution in the case of weak gravitational field, but in the case of strong gravitational field the Schwarzschild solution disable to describe what happened accurately, where the laws of physics stop there according to his solution.

General metric which is describing the space-time of the gravitational field (weak or strong) is given as [61]:

$$ds^2 = g_{tt}dt^2 - g_{tt}^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.4.4}$$

Where

$$g_{tt} = \left(1 - \frac{GM}{c^2 r}\right)^2$$

We get

$$g_{tt} = 1 - \frac{2GM}{c^2 r} + \frac{G^2 M^2}{c^4 r^2} \tag{2.4.5}$$

Equation (2.4.5) illustrates the lost term in the Schwarzschild solution where it is G^2M^2/c^4r^2 this term has no effect in the case of the weak gravitational field where it is too small to be perceived. So in this case we can neglect it and the metric will take the Schwarzschild form. But in the case of the strong gravitation field we can't neglect this term. Now according to my solution we can generalize equation (2.4.2) to be [61]:

$$\Delta t' = \left(1 - \frac{GM}{c^2 r}\right)\Delta t \tag{2.4.6}$$

Equation (2.4.6) is a general formula that describes the time dilation in a strong and weak gravitational field. This equation is in agreement with equation (2.4.2). And it is easy to show that (2.4.6) takes the form of equation (2.4.2) in the case of weak gravitational field where the term G^2M^2/c^4r^2 is neglected.

Now if we consider γ^{-1}, then we can compute the radius that the mass should be compressed to be transformed into a black hole. This is known as the Schwarzschild radius. Thus

$$1 - \frac{GM}{c^2 r} = 0$$

Thus

$$r_s = \frac{GM}{c^2} \tag{2.4.7}$$

We see equation (2.4.7) is different from equation (2.4.3) by the factor of equation (2.4.7) is in agreement with equation when deriving the Schwarzschild radius. If we look carefully to my metric we'll see that it takes the form of flat Murkowski space

$$ds^2 = dt^2 - dr^2 - r(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.4.8}$$

That is when

$$\frac{2GM}{c^2 r} = \frac{G^2 M^2}{c^4 r^2} \tag{2.4.9}$$

This happened when

$$r = r_A \tag{2.4.10}$$

Where

$$r_A = \frac{GM}{2c^2} \tag{2.4.11}$$

From that we get

$$r_A = \frac{1}{2} r_s \tag{2.4.12}$$

r_s given in equation (2.4.7) and from equation (2.4.6) where : $r = r_A$ Then $\Delta t' = \Delta t$.

2.5 The Singularity of Schwarzschild Solution:

In this section we consider the irregular behavior of Schwarzschild solution. The Schwarzschild solution can be applied to the gravitational field of the sun, however, the Schwarzschild metric (2.4.1) is singular not only at $r = 2mG$, but also at $r = 0$. The existence of the singularity has been interpreted by Penrose and Hawking [62] as a manifestation of the collapse a massive star, the condition of its occurrence is determined by Chandrasekhar [63].

The distance from the source in the proximity of which the collapsed star may turn to be a neutron star with extremely high density, or otherwise, continue collapsing to infinite density is called singular horizon. Beyond this horizon, the continuous collapse of a certain star results the so called black hole, it implies that the star has disappeared by leaving a hole.

In other words, under the gigantic gravitational force the collapsing star is doomed to further collapse then the gravitational field by passing its Schwarzschild radius will be so large that any particles including photon will be captured inward. In spite of the afore-discussed behavior, the singularity at $r = 2GM$, is not real since it can be removed by a certain transformation of coordinates.

This was done by Kruskal-Szekeres [64, 65]. However, the singularity at $r = 0$ is independent of the choice of the coordinate system, hence it is a real singularity which cannot be removed by any kind of coordinate the transformation. On the other hand, regarding a real star as a point mass in Schwarzschild metric, looks only unreasonable, but also physically inapplicable.

This may imply that despite the successes of Schwarzschild solution in describing weak field gravity, it looks inadequate to describe situations, where it predicts the collapse of a massive star. In other words, with this inherent singularity, general relativity is not capable of describing gravity at distances where the singularity takes place, i.e. when gravity is strong. Thus, this reflects the limitations of general relativity. This means, general relativity by predicting the singular behavior of the metric space it predicts its own inapplicably of that singularity. Thus it contradicts itself.

This limitation of Einstein's model necessitates a search for an alternative model, which would share with general relativity its successes at weak field gravity, and will be hopefully capable of giving a rotational description of the strong gravitational field.

We find that Schwarzschild solution is quite applicable to the gravitational field at finite distances from the sun. This ensures that the solution is the base for all weak field prediction of general relativity. Though, the solution has its limitation in the strong field domain, and hence, it predicts a singular behavior, which entails the existence of black hole due to the gravitational collapse. The Schwarzschild solution of Einstein's equations constitutes the exact solution for the symmetrized form of these equations. The symmetrizedation of the solution doesn't affect the validity of the afore stated predictions of the theory in the case of non-symmetrized space (Birkhoff

Theorem) [66]. It is rather a man of simplification, which allows the obtaining of this solution.

Thus, we conclude that the gravitational equation satisfied by Schwarzschild space time metric gives of the adequate description of gravity at weak field limit where predictions have been well verified by observation. As for the prediction of the theory, at the strong field areas the theory doesn't only contradict itself theoretically, but also has no experimental evidence.

2.6 Differential Equations for Stellar Structure:

We first set up the general relativistic machinery for computing the pressure, density, and gravitational fields within a spherically symmetric static star [30]. The metric

$$g_{rr} = A(r)\,, \quad g_{\theta\theta} = r^2\,, \quad g_{\phi\phi} = r^2\sin^2\theta\,, \qquad g_{tt} = -B(r) \tag{2.6.1}$$

$$g_{\mu v} = 0 \quad \text{for} \quad \mu \neq v$$

The energy-momentum tensor is assumed to be that that for a perfect fluid

$$T_{\mu v} = p g_{\mu v} + (p+\rho)\, U_\mu U_v \tag{2.6.2}$$

With p the proper pressure, ρ the proper total energy density, and U^μ the velocity four-vector, defined so that

$$g^{\mu v} U_\mu U_v = -1 \tag{2.6.3}$$

Since the fluid is at rest, we take

$$U_r = U_\theta = U_\phi = 0 \;\;,\;\; U_t = -(-g^{tt})^{-1/2} \;=\; -\sqrt{B(r)} \tag{2.6.4}$$

Our assumptions of time independence and spherical symmetry imply that p and ρ are functions only of the radial coordinate r. By making use equations (2.6.1)-(2.6.4) and the Ricci tensor components. we find that the Einstein equation (2.2.9) read

$$R_{rr} = \frac{B''}{2B} - \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} = -4\pi G(\rho - p)A \tag{2.6.5}$$

$$R_{\theta\theta} = -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} = -4\pi G(\rho - p)r^2 \tag{2.6.6}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rA} = -4\pi G(\rho + 3p)B \tag{2.6.7}$$

A prime denotes d/dr. (we do not need to write down the equation for $R_{\phi\phi}$, which is identical to that for $R_{\theta\theta}$, or the equations for off-diagonal elements of $R_{\mu v}$, which simply say that zero equals zero) In addition, we may recall the equation for hydrostatic equilibrium

$$\frac{B'}{B} = -\frac{2p'}{p+\rho} \tag{2.6.8}$$

Our first step in solving these equations is to derive an equation for $A(r)$ alone, by forming the quantity

$$\frac{R_{rr}}{2A} + \frac{R_{\theta\theta}}{r^2} + \frac{R_{tt}}{2B} = -\frac{A'}{rA^2} - \frac{1}{r^2} + \frac{1}{Ar^2} = -8\pi G\rho \tag{2.6.9}$$

This equation can by written

$$\left(\frac{r}{A}\right)' = 1 - 8\pi G\rho r^2 \tag{2.6.10}$$

The solution with $A(0)$ finite is

$$A(r) = \left(1 - \frac{2GM(r)}{r}\right)^{-1} \tag{2.6.11}$$

Where

$$M(r) = \int_0^r 4\pi r'^2 \rho(r') dr' \tag{2.612}$$

We can now use equations (2.6.11) and (2.6.8) to eliminate the gravitational fields $A(r)$, $B(r)$ from equation (2.6.6), which becomes

$$-1 + \left(1 - \frac{2GM}{r}\right)\left(1 - \frac{rp'}{p+\rho}\right) + \frac{GM}{r} - 4\pi G\rho r^2 = -4\pi G(\rho - p)r^2$$

We rewrite this as

$$-r^2 p'(r) = GM(r)\rho(r)\left[1 + \frac{p(r)}{\rho(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1} \tag{2.6.13}$$

The reader may recognize this differential equation as the fundamental equation of Newtonian astrophysics (see section 2.7), with general relativity corrections supplied by the last three factors.

We are primarily concerned in this chapter with stars that are isentropic, that is, in which the entropy per nucleon s does not vary throughout the star. This is the case for two very different kinds of star [30]:

1. Stars at absolute zero: When a star exhausts its thermonuclear fuel it can become a white dwarf (section 2.7), or a neutron star (section 2.8), in which the temperature is essentially at absolute zero. According to Nernst's theorem, the entropy per nucleon will then be zero throughout the star.

2. Stars in convective equilibrium: If the most efficient mechanism for entropy transfer within the star is convection, then in equilibrium the entropy per nucleon must be nearly constant throughout the star, because otherwise a small element of fluid containing A nucleons could gain or lose an energy $A\Delta S/T$ when transported from one part of the star to another, and convection would therefore disturb the energy distribution. The supermassive stars discussed in (section 2.9) are generally presumed to be in convective equilibrium. We also assume that the stars we consider have a chemical composition that is constant throughout.

The importance of the preceding assumptions lies in the fact that the pressure p may in general be expressed as a function of the density ρ, the entropy per nucleon s, and the chemical composition. Hence, with s and the chemical composition constant throughout the star, $p(r)$ may be regarded as a function of $\rho(r)$ alone, with no explicit dependence on radius r.

Given $p(r)$ as a function $p\big(\rho(r)\big)$, we now formulate our problem as a pair of first-order differential equations for $\rho(r)$ and $M(r)$. one of these is equation (2.6.13), the other is the derivative of equation (2.6.12):

$$M'(r) = 4\pi r^2 \rho(r) \qquad (2.6.14)$$

In addition, equation (2.6.12) provides an initial condition

$$M(0) = 0 \qquad (2.6.15)$$

Equations (2.6.13), (2.6.14) and (2.6.15), together with an equation of state giving $p(\rho)$, serve to determine $\rho(r)$, $M(r)$, and $p(r)$, and so on, throughout the star, once we specify the other initial condition, that is, the value of $\rho(0)$. The differential equations (2.6.13) and (2.6.14) must be integrated out from the center of the star, until $p\big(\rho(r)\big)$ drops to zero at some point $r = R$, Which we then interpret as radius of the particular star with central density $\rho(r)$.

Let us return to the problem of calculating the metric. Once we compute $\rho(r)$, $M(r)$, and $p(r)$ we can immediately obtain $A(r)$ from equation (2.6.11); to find $B(r)$ we use equation (2.6.13) to rewrite (2.6.8) as

$$\frac{B'}{B} = \frac{2G}{r^2}[M + 4\pi r^3 p]\left[1 - \frac{2GM}{r}\right]^{-1}$$

The solution with $B(\infty) = 1$ is

$$B(r) = \exp\left(-\int_r^\infty \frac{2G}{r^2}[M(r') + 4\pi r'^3 p(r')]\left[1 - \frac{2GM(r')}{r'}\right]^{-1} dr'\right) \qquad (2.6.16)$$

Our solution is now complete. (incidentally, we did not need to use equations (2.6.5) and (2.6.7) for R_{rr} and R_{tt} separately, because these equations follow from (2.6.6), (2.6.8), and (2.6.9), which were used in our calculation. This should not be surprising, because equation (2.6.8), which is really just the equation for momentum conservation, follows from the Einstein equations (2.6.5), (2.6.6) and (2.6.7) via the Bianchi identities). Outside the star, $p(r)$ and $\rho(r)$ vanish, and $M(r)$ is the constant $M(R)$, so equations (2.6.11) and (2.6.16) give

$$B(r) = A^{-1}(r) = 1 - \frac{2GM(R)}{r} \qquad for \qquad r \geq R \qquad (2.6.17)$$

The gravitational field (2.6.17) must equal the mass M of the star, defined as the total energy of the star and its gravitational field, that is

$$M = M(R) \equiv \int_0^R 4\pi r^2 \rho(r) dr \qquad (2.6.18)$$

Thus (2.6.17) is just the familiar exterior Schwarzschild solution. It may appear paradoxical that M, which must include the energy of the gravitational field, is given in (2.6.18) as the integral of the energy density $\rho(r)$ of matter (including radiation) alone. The resolution is that (2.6.18) does not say that M is the total energy of the matter. The total material energy is not really well defined, but it might be computed by splitting up the star into small volume elements and adding up the energies of each element as measured in a locally inertial reference frame; this would give the material energy as

$$M_{matter} \equiv \int \sqrt{g}\,\rho\; dr\; d\theta\; d\phi = \int_0^R 4\pi r^2 \sqrt{A(r)\,B(r)}\;\rho(r)\; dr \qquad (2.6.19)$$

The difference between equations (2.6.18) and (2.6.19) can be regarded as the energy of the gravitational field. However, this decomposition is not particularly useful, and will not be employed here. It is more informative to compare (2.6.18) with the energy M_0 that the matter of the star would have if dispersed to infinity. This is simply

$$M_0 = m_N N \tag{2.6.20}$$

Where $m_N = 1.66 \times 10^{-24}\ g$ is the rest-mass of a nucleon and N is the number of nucleons in the star. The nucleon number is given by

$$N = \int \sqrt{g} J_{N^0}\, dr\, d\theta\, d\phi = \int_0^R 4\pi r^2 \sqrt{A(r)B(r)}\ j_{N^0}\,(r)\, dr \tag{2.6.21}$$

Where j_{N^μ} is the conserved nucleon number current. It is convenient to express j_{N^0} in terms of the proper nucleon number density n, that is, the nucleon number density measured in a locally inertial reference frame at rest in the star, which is

$$n = -U_\mu J_{N^\mu} = \sqrt{B}\, j_{N^0} \tag{2.6.22}$$

(see equation (2.6.4), and recall that in a locally inertial coordinate frame $U_0 = -1$) equation (2.6.21) then becomes

$$N = \int_0^R 4\pi r^2 \sqrt{A(r)}\, n(r)\, dr\ = \int_0^R 4\pi r^2 \left(1 - \frac{2GM(r)}{r}\right)^{-1/2} n(r)\, dr \tag{2.6.23}$$

The proper number density $n(r)$ is in general a function of the proper density $\rho(r)$, the chemical composition, and the entropy per nucleon s, so $n(r)$ and N are fixed for a star with a given constant s and chemical composition, once we choose $\rho(0)$. The internal energy of the star is now given by

$$E \equiv M - m_N N \tag{2.6.24}$$

We can also define a proper internal material energy density as

$$e(r) \equiv \rho(r) - m_N n(r) \tag{2.6.25}$$

And write (2.6.24) as

$$E = T + V \tag{2.6.26}$$

Where T and V are the thermal and gravitational energies, respectively, of the star

$$T \equiv \int_0^R 4\pi r^2 \left(1 - \frac{2GM(r)}{r}\right)^{-1/2} e(r)\, dr \tag{2.6.27}$$

$$V \equiv \int_0^R 4\pi r^2 \left[1 - \left(1 - \frac{2GM(r)}{r}\right)^{-1/2}\right] \rho(r)\, dr \qquad (2.6.28)$$

Expanding the square roots gives

$$T = \int_0^R 4\pi r^2 \left(1 + \frac{GM(r)}{r} + \cdots\right) e(r) dr \qquad (2.6.29)$$

$$V = -\int_0^R 4\pi r^2 \left(\frac{GM(r)}{r} + \frac{3G^2M^2(r)}{2r^2} + \cdots\right) \rho(r)\, dr \qquad (2.6.30)$$

The first terms in T and V are recognizable as the Newtonian values for the thermal and gravitational energies of the star; in particular, note that the first term in V may be written

$$-G \int_0^R 4\pi r M(r) \rho(r) dr = -\frac{G}{2} \int_0^R \frac{1}{r} d\big(M^2(r)\big) = -\frac{GM^2}{2R} - \frac{G}{2} \int_0^R \frac{M^2(r)}{r^2}\, dr$$

$$= \frac{\phi(R) M(R)}{2} - \frac{1}{2} \int_0^R M(r) d\phi(r) = \frac{1}{2} \int_0^R \phi(r)\, dM(r) \qquad (2.6.31)$$

Where ϕ is the Newtonian potential, given inside the star by

$$\phi(r) = -\frac{GM}{R} - G \int_r^R \frac{M(r')}{r'^2}\, dr' \qquad (2.6.32)$$

The higher terms in T and V are discussed in section (2.9). To repeat out main conclusion: Once we specify that a star has a definite uniform entropy per nucleon and chemical composition, all properties of the star, including $\rho(r)$, $p(r)$, $n(r)$, $e(r)$, M, N, and E, are determined as function of the central density $\rho(0)$. This is not the case for ordinary stars like the sun, in which the entropy distribution is not uniform and has to be determined from the equations of radiative equilibrium. However, the considerations of this section do provide an adequate basis for the study of the exotic structures discussed in this chapter.

2.7 Newtonian Stars, Polytropes and White Dwarfs:

Most of the stars in the sky are adequately described by Newtonian physics, without taking account of general relativity. Such Newtonian stars deserve some attention here, both because they serve us limiting cases for the more exotic objects that interest the general relativity, and because they can guide us in understanding the qualitative properties of these objects. In Newtonian astrophysics the internal energy and pressure are very much less than the rest-mass density

$$e \ll nm_N \quad , \qquad p \ll nm_N \tag{2.7.1}$$

So that total density is dominated by the density of rest mass

$$\rho \simeq nm_N \tag{2.7.2}$$

And also

$$p \ll \rho \quad , \qquad 4\pi r^3 p \ll M$$

In addition, the gravitational potential is everywhere small, so

$$\frac{2GM}{r} \ll 1 \tag{2.7.3}$$

The fundamental equation (2.6.13) thus simplifies to

$$-r^2 p'(r) = GM(r)\rho(r) \tag{2.7.4}$$

With $M(r)$ still defined by

$$M(r) \equiv \int_0^r 4\pi {r'}^2 \rho(r')\, dr' \tag{2.7.5}$$

Dividing (2.7.4) by $\rho(r)$ and differentiating allows us to combine both (2.7.4) and (2.7.5) in a single second-order differential equation:

$$\frac{d}{dr}\frac{r^2}{\rho(r)}\frac{dp(r)}{dr} = -4\pi G r^2 \rho(r) \tag{2.7.6}$$

In order that $\rho(0)$ be finite, it is necessary that $p'(0)$ vanish. Thus, given an equation of state $p = p(\rho)$ (with $dp/d\rho \neq 0$), we can obtain $\rho(r)$ by solving equation (2.7.6) with the initial conditions that $\rho(0)$ have some given value and that

$$\rho'(0) = 0 \tag{2.7.7}$$

(equation (2.7.7) also follows from the requirement that $\rho(r)$ be an analytic function of x, y, and z at $x = y = z = 0$). We still need to prescribe an equation of state. It is often the case that the internal energy density is proportional to the pressure, that is

$$e \equiv \rho - m_N n = (\gamma - 1)^{-1} p \tag{2.7.8}$$

(here $(\gamma - 1)^{-1}$ is jest a constant proportionality coefficient, γ will not be the ratio of specific heats unless e and p are proportional to the temperature). The condition of uniform entropy per nucleon then reads

$$0 = \frac{d}{dr}\left(\frac{\rho}{n}\right) + p\frac{d}{dr}\left(\frac{1}{n}\right) = \frac{d}{dr}\left(\frac{e}{n}\right) + p\frac{d}{dr}\left(\frac{1}{n}\right)$$
$$= \frac{1}{\gamma - 1}\left[\gamma p\frac{d}{dr}\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)\frac{dp}{dr}\right]$$

And therefore

$$p \propto n^{\gamma}$$

Or, since

$$\rho \simeq m_N\, n$$
$$p = K\rho^{\gamma} \tag{2.7.9}$$

The proportionality constant K depends on the entropy per nucleon and chemical composition, but it does not depend on r or on $\rho(0)$. Any star for which the equation of state takes the form equation (2.7.9) is called a polytrope. For Newtonian stars, M is dominated by the total rest-mass Nm_N, so the nucleon number of the star is given to a good approximation by

$$N \simeq \frac{M}{m_N} \tag{2.7.10}$$

We also want to know the internal energy $E \equiv M - Nm_N$. For general Newtonian stars this is given by equations (2.6. 26), (2.6.29) and (2.6.30) as

$$E = T + V \tag{2.7.11}$$

With the thermal energy T and the gravitational energy V given by

$$T = \int_0^R 4\pi r^2 e(r)\, dr \tag{2.7.12}$$

$$V = -\int_0^R 4\pi r G M(r)\rho(r)\, dr \tag{2.7.13}$$

We now show that for polytropes, T and V are given by the remarkably simple formulas [67]:

$$T = \frac{1}{(5\gamma - 6)}\frac{GM^2}{R} \tag{2.7.14}$$

$$V = -\frac{3(\gamma - 1)}{(5\gamma - 6)}\frac{GM^2}{R} \tag{2.7.15}$$

So the total internal energy is

$$E = -\frac{(3\gamma - 4)}{(5\gamma - 6)} \frac{G\,M^2}{R} \tag{2.7.16}$$

To prove the formula for V, we use equation (2.7.4) to rewrite (2.7.13) as

$$V = 4\pi \int_0^R r^3 \frac{dp(r)}{dr} dr = -12\pi \int_0^R r^2 p(r)\, dr \tag{2.7.17}$$

Multiplying and diving in the integrand by $\rho(r)$, we have

$$V = -3 \int_0^R \frac{p(r)}{\rho(r)}\, dM(r) = 3 \int_0^R M(r)\, d\left(\frac{p(r)}{\rho(r)}\right)$$

(we assume here that $\gamma > 1$, so that p/ρ vanishes at R) This can be evaluated by using the equation of state to calculate

$$\frac{d}{dr}\left(\frac{p(r)}{\rho(r)}\right) = \left(\frac{\gamma - 1}{\gamma}\right)\frac{p'(r)}{\rho(r)} = -\left(\frac{\gamma - 1}{\gamma}\right)\frac{GM(r)}{r^2}$$

So

$$V = -3\left(\frac{\gamma - 1}{\gamma}\right)\int_0^R \frac{GM^2(r)}{r^2} dr \tag{2.7.18}$$

Sine $dr/r^2 = -d(1/r)$, we can integrate by parts once again, and find

$$V = 3\left[\frac{\gamma - 1}{\gamma}\right]\left[\frac{G\,M^2}{R} - 2\int_0^R 4\pi r G M(r)\rho(r) dr\right] = 3\left[\frac{\gamma - 1}{\gamma}\right]\left[\frac{GM^2}{R} + 2V\right]$$

Solving for V then gives the desired result (2.7.15), to calculate T we use equation (2.7.8) in (2.7.17), which gives

$$V = -3(\gamma - 1)T \tag{2.7.19}$$

Equations (2.7.15) and (2.7.19) then give the desired result (2.7.14). From equations (2.7.10), (2.7.12), (2.7.13) and (2.7.8) give

$$N = \frac{4\pi}{3m_N}\rho R^3 \tag{2.7.20}$$

$$T = \frac{4\pi}{3}(\gamma - 1)^{-1}\, K\rho^\gamma R^3 \tag{2.7.21}$$

$$V = -\frac{16\pi^2}{15} G\rho^2 R^5 \tag{2.7.22}$$

So, eliminating R

$$E = T + V = a\,\rho^{\gamma - 1} - b\rho^{1/3} \tag{2.7.23}$$

Where

$$a = \frac{KM}{(\gamma - 1)} \tag{2.7.24}$$

$$b = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/3} GM^{5/3} \tag{2.7.25}$$

For $\gamma > 4/3$, E has a minimum at

$$\rho = \left(\frac{b}{3a(\gamma - 1)}\right)^{1/(\gamma-4/3)} = \left(\frac{M^{2/3}G(4\pi/3)^{1/3}}{5K}\right)^{1/(\gamma-4/3)} \tag{2.7.26}$$

Corresponding to a configuration of stable equilibrium. for $\gamma = 4/3$, E is stationary with respect to ρ only if it vanishes everywhere, which requires that $a = b$, or

$$M = \left(\frac{5K}{G}\right)^{3/2} \left(\frac{4\pi}{3}\right)^{-1/2} \tag{2.7.27}$$

For $\gamma < 4/3$, E has a maximum at the point (2.7.26), corresponding to a state of unstable equilibrium. Incidentally, equation (2.7.26) gives an estimate for the mass

$$M \simeq \frac{4\pi}{3}\rho^{(3\gamma-4)/2}\left(\frac{15K}{4\pi G}\right)^{3/2}$$

The variational approach also provides a simple method for estimating the oscillation frequency for dilation and contraction of the star. Equations (2.7.20), (2.7.21) and (2.7.22) show that for fixed N,

$$T \propto R^{3(1-\gamma)} \quad , \quad V \propto R^{-1}$$

We can be use equations (2.7.14) and (2.7.15) to fix the correct values of T and V at the equilibrium radius (which we shall now write as R_{eq}, to distinguish it from the instantaneous radius R of an oscillating configuration). This gives then

$$E = \frac{1}{(5\gamma - 6)}\frac{GM^2}{R_{eq}^{(4-3\gamma)}}R^{3(1-\gamma)} - \frac{3(\gamma - 1)}{5\gamma - 6}GM^2R^{-1}$$

For $\gamma > 4/3$, this has a minimum at $R = R_{eq}$, as it should . for R near R_{eq}, E behaves like

$$E \to E_{eq} + \frac{3(\gamma - 1)(3\gamma - 4)}{2(5\gamma - 6)}\,\frac{GM^2}{R_{eq}^3}\left(R - R_{eq}\right)^2$$

The uniform dilation of a sphere with uniform density will give it a kinetic energy

$$U = \frac{3}{10}M\dot{R}^2$$

So the condition of energy conservation, that $U + E$, be constant, leads to modes with

$$R - R_{eq} \propto \sin \omega_0 t$$

$$\omega_0 \approx \left[\frac{5(\gamma - 1)(3\gamma - 4)}{5\gamma - 6} \frac{GM^2}{R_{eq}^3}\right]^{1/2} \tag{2.7.28}$$

Finally, we note that a uniform sphere rotating with angular velocity Ω will have kinetic energy

$$U = \frac{1}{5} M R_{eq}^2 \, \Omega^2$$

This must be less than the binding energy $-E$, so the maximum angular velocity with which a star can rotate is of order

$$\Omega_{max} \approx \left[\frac{5(3\gamma - 4)}{(5\gamma - 6)} \frac{GM^2}{R_{eq}^3}\right]^{1/2} \approx \frac{\omega_0}{\sqrt{\gamma - 1}} \tag{2.7.29}$$

Of course a star rotating this fast will no longer be a sphere, and (2.7.29) only gives an order of magnitude estimate of the actual maximum rotation frequency.

Now let us apply what we have learned to the stars known as white dwarfs. Imagine an aged star that exhausts its nucleon fuel and begins to cool and contract . when the temperature is sufficiently low (see below for just how low), the electrons will be frozen into the lowest available energy levels. The Pauli principle tells us that there will be tow electrons in each level (because of the tow spin states available) an there are $4\pi k^2 (2\pi\hbar)^{-3} \, dk$ levels per unit volume with momenta between k and $k + dk$, so the number of electrons per unit volume will be related to the maximum momentum k_F by

$$n = \frac{8\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 \, dk = \frac{k_F^3}{3\pi^2 \hbar^3} \tag{2.7.30}$$

The mass density is

$$\rho = n m_N \mu \tag{2.7.31}$$

Where μ is the number nucleons per electron, $\mu \simeq 2$ for stars that have used up their hydrogen. This gives

$$k_F = \hbar \left(\frac{3\pi^2 \rho}{m_N \mu}\right)^{1/3} \tag{2.7.32}$$

The condition that the temperature is negligible is

$$kT \ll [k_F^2 + m_e^2]^{1/2} - m_e$$

The kinetic energy density and pressure of these electrons are

$$e = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}\left[(k^2+m_e^2)^{1/2} - m_e\right]k^2 dk \tag{2.7.33}$$

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}\frac{k^2}{(k^2+m_e^2)^{1/2}}k^2 dk \tag{2.7.34}$$

The equation of state here is not simple, but it reduces to a polytrope in two extreme cases, distinguished by the criteria $\rho \ll \rho_c$ or $\rho \gg \rho_c$ where ρ_c is the critical density at which k_F becomes equal to m_e (in e.g.s. units)

$$\rho_c = \frac{m_N\mu\, m_e^3\, c^3}{3\pi^2\hbar^3} = 0.97\times 10^6\mu \ \ \mathrm{g}/cm^3 \tag{2.7.35}$$

(A) $\rho \ll \rho_c$ in this case $k_F \ll m_e$, so equations (2.7.33) and (2.7.34) give

$$e = \frac{3}{2}p$$

$$p = \frac{8\pi k_F^5}{15m_e(2\pi\hbar)^3} = \frac{\hbar^2}{15m_e\pi^2}\left(\frac{3\pi^2\rho}{m_N\mu}\right)^{5/3}$$

This is a polytrope, with

$$\gamma = \frac{5}{3} \quad , \quad K = \frac{\hbar^2}{15m_e\pi^2}\left(\frac{3\pi^2}{m_N\mu}\right)^{5/3} \tag{2.7.36}$$

(B) $\rho \gg \rho_c$ in this case $k_F \gg m_e$, so equations (2.7.33) and (2.7.34) give

$$e = 3p$$

$$p = \frac{8\pi k_F^4}{12(2\pi\hbar)^3} = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2\rho}{m_N\mu}\right)^{4/3}$$

This is a polytrope, with

$$\gamma = \frac{4}{3} \quad , \quad K = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2}{m_N\mu}\right)^{4/3} \tag{2.7.37}$$

2.8 Neutron Stars:

We saw in the last section that a white star supported by the pressure of cold degenerate electrons cannot be in equilibrium if its mass is greater than the

Chandrasekhar limit, about $\hbar^{3/2}/m_N^2 G^{3/2}$. Also, the gravitational potential at the surface of such a star cannot be greater than about m_e/m_N, so general relativity plays no role in its structure.

Continuing our search for astrophysical application of general relativity, let us ask what happens when a star whose mass is above the Chandrasekhar limit reaches the end of its thermonuclear evolution and grows cold. Its internal pressure then fails to the support it, and it collapses. On possibility is that the star will simply go on collapsing forever, in which case general relativity will certainly come into play. Another possibility is that the star will become so heated during its collapse that it will explode, becoming a supernova. It might then blow off enough matter so that its mass drops below the Chandrasekhar limit [68]. It is believed that in this case the highly compressed remnant does not find its quietus as a white dwarf, but rather becomes a super dense neutron star [68]. A neutron star is like a white dwarf, except that it consists almost entirely of "cold" degenerate neutrons, all electrons and protons having been converted into neutrons through the reaction $p + \bar{e} \rightarrow n + \upsilon$. The neutrinos escaping the star. Enough electrons and protons must remain so that the Pauli principle prevents neutron beta decay, $n \rightarrow p + \bar{e} + \bar{\upsilon}$, this sets a lower limit on the mass of stable neutron stars, to be evaluated below. Neutron stars of low mass are much like white dwarfs of the same mass, except that neutron degeneracy pressure replaces electron degeneracy pressure, and thus m_e should be replaced in all formulas with m_n. Thus, by noting how m_e enters in formulas equations (2.7.35), (2.7.36) and (2.7.37) for small white dwarfs, we can immediately conclude that a neutron star of small mass will have a central density higher than that of a white dwarf with the same mass and $(\mu = 2)$ by a factor $\frac{1}{2}(m_n/m_e)^3 = 3.1 \times 10^9$, and will have a radius smaller by a factor, $m_n/2m_e = 920$.

Another difference that is even more interesting is that, whereas a white dwarf whose electrons are moderately relativistic will have a surface gravitational potential GM/R of order m_e/m_n , a neutron star of equal mass will have a surface potential roughly of order unity. Thus general relativity will necessarily play a role in the theory of the more massive neutron stars. In order to formulate the quantitative theory of neutron star, we begin by writing down expressions for the total energy density and pressure of an ideal Fermi gas of neutrons with maximum momentum k_F:

$$\rho = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}(k^2+m_n^2)^{1/2}k^2\,dk = 3\rho_c\int_0^{k_F/m_n}(u^2+1)^{1/2}u^2\,du \quad (2.8.1)$$

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_F}\frac{k^2}{(k^2+m_n^2)^{1/2}}k^2dk = \rho_c\int_0^{k_F/m_n}(u^2+1)^{-1/2}u^4\,du \quad (2.8.2)$$

Where now (in c.g.s units)

$$\rho_c \equiv \frac{8\pi m_n^4c^3}{3(2\pi\hbar)^3} = 6.11\times 10^{15}\,gm/cm^3 \quad (2.8.3)$$

By eliminating k_F/m_n equations (2.8.1) and (2.8.2), we obtain the equation of state in the form

$$\frac{p}{\rho_c} = f\left(\frac{\rho}{\rho_c}\right) \quad (2.8.4)$$

With f a definite transcendental function. The structure of a neutron star with given central density $\rho(0)$ is to be calculated by solving (2.6.13) with p given as a function of ρ by equation (2.8.4). Since the only dimensional quantities in these equations are $\rho(0)$, ρ_c, and G, the solution must give the mass and radius as functions of $\rho(0)$ of the form

$$M = M_0 f\left(\frac{\rho(0)}{\rho_c}\right) \quad (2.8.5)$$

$$R = R_0\,g\left(\frac{\rho(0)}{\rho_c}\right) \quad (2.8.6)$$

Where (in c.g.s units)

$$R_0 \equiv c(8\pi G\rho_c)^{-1/2} = 3Km \quad (2.8.7)$$

$$M_0 \equiv \frac{c^2R_0}{G} = 2M_s \quad (2.8.8)$$

And f and g are unknown dimensionless functions. This problem, like that of the white dwarfs, is analytically tractable only for very large and very small central densities. With ρ_c now given by equation (2.8.3). For $\rho(0) \gg \rho_c$, the neutrons near the center of the star have $k_F \gg m_n$, so (2.8.1) and (2.8.2) give

$$\rho = \frac{3\rho_c}{4}\left(\frac{k_F}{m_n}\right)^5 \quad , \quad p = \frac{\rho_c}{4}\left(\frac{k_F}{m_n}\right)^5$$

And therefore

$$p = \frac{\rho}{3} \tag{2.8.9}$$

As would be expected for a gas of highly relativistic particles. Using this equation of state in the fundamental differential equation (2.6.13) gives

$$-r^2\rho'(r) = 4GM(r)\rho(r)\left[1 + \frac{4\pi r^3\rho(r)}{3M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1} \tag{2.8.10}$$

Amazingly, we can find an exact solution of this equation [69]:

$$\rho(r) = \frac{3}{56\pi G r^2} \tag{2.8.11}$$

Corresponding to the limit $\rho(0) \to \infty$. However, even in the limit of infinite central density, this $\rho(r)$ will drop below ρ_0 at a radius r of order R_0, so that the equation of state (2.8.9) is not valid for the outer layers of any neutron star. To deal with the crust of nonrelativistic neutron, it is necessary to solve the full equation (2.6.13) using the equation of state (2.8.4); the condition of infinite central density is imposed by (2.8.11) for $r \ll R_0$. We shall not do this here; the important points are that the solution has a finite radius R where ρ. vanishes, and that the mass M within this radius is finite, because the singularity in equation (2.8.11) is integrable at $r = 0$. Thus the mass and radius of a neutron star approach finite limits as $\rho(0) \to \infty$. Numerical solution of the fundamental equation (2.6.13) gives these limits as [68].

$$M_\infty = 0.171\, M_0 \qquad R_\infty = 1.06\, R_0 \tag{2.8.12}$$

This expectation is confirmed by detailed calculation [68] using equations (2.6.13) and (2.8.1)- (2.8.3). The mass M of a pure ideal-gas neutron star reaches a maximum

$$M_m = 0.36\, M_0 = 0.7\, M_s \tag{2.8.13}$$

At a radius

$$R_m = 3.2\, R_0 = 9.6\, km \tag{2.8.14}$$

Since this a point where $\partial M / \partial \rho(0)$ vanishes, we expect a transition here from stability to instability with respect to radial oscillation. Thus equations (2.8.13) and (2.8.14) characterize a neutron star with the greatest mass and central density allowed by the requirement that the star be stable. The mass (2.8.13) is known as the Oppenheimer-Volkoff limit. Note that the fractional red shift of a spectral line emitted from the surface of such a neutron star is

$$z \equiv \frac{\Delta\lambda}{\lambda} = B^{-1/2}(R_m) - 1 = \left(1 - \frac{2M_m G}{R_m}\right)^{-1/2} - 1 = 0.13 \qquad (2.8.15)$$

(see equations (2.6.1) and (2.6.17)) evidently general relativity is just beginning to be important for the most massive stable neutron stars. of course, a neutron star cannot consist purely of neutrons, if only because we need a Fermi sea of electrons so that the Pauli exclusion principle can black the neutrons beta decay. In order to get a first taste of the chemical composition in a neutron star, let us consider the equilibrium among neutrons, protons and electrons. The energy density and number density of each one of these three Fermi gases are given (For $i = n, p, e$) by

$$\rho_i = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_{F,i}} \sqrt{k^2 + m_i^2 k^2 dk} \qquad (2.8.16)$$

$$n_i = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_{F,i}} k^2 dk = \frac{k_{F\prime i}^3}{3\pi^2\hbar^3} \qquad (2.8.17)$$

At any given point in the star, the reactions $n \to p + e + \bar{v}$ and $p + e \to n + v$ can convert neutrons into protons and vice versa. (the neutrons escape) these reactions preserve the total number density of baryons

$$n_n + n_p = n_B \qquad (2.8.18)$$

And preserve charge neutrality

$$n_p + n_e = 0 \qquad (2.8.19)$$

But with n_B fixed, the total energy density may be expressed in terms of n_n alone

$$\rho \equiv \rho_n + \rho_e + \rho_p$$

$$\rho = 3C^{-3}\int_0^{Cn_n^{1/3}} \sqrt{k^2 + m_n^2}\, k^2\, dk + \int_0^{C\,[n_B - n_n]^{1/3}} \sqrt{k^2 + m_p^2}\; k^2\, dk$$

$$+ \int_0^{C\,[n_B - n_n]^{1/3}} \sqrt{k^2 + m_e^2}\, k^2\, dk \qquad (2.8.20)$$

Where $C \equiv (3\pi^2\hbar^3)^{1/3}$

Chemical equilibrium is reached when this function is minimum, that is, as

$$\frac{d\rho}{dn_e} = \left(C^2 n_n^{2/3} + m_n^2\right)^{1/3} - \left(C^2 [n_B - n_n]^{2/3} + m_p^2\right)^{1/2}$$

$$-\left(C^2 [n_B - n_n]^{2/3} + m_e^2\right)^{1/2} = 0$$

We can solve for $n_p = n_B - n_n$ as a function of n_n, and find

$$\frac{n_p}{n_n} = \frac{1}{8}\left[\frac{1 + \frac{2\left(m_n^2 - m_p^2 - m_e^2\right)}{C^2 n_n^{2/3}} + \frac{\left(m_n^2 - m_p^2\right)^2 - 2m_e^2\left(m_n^2 + m_p^2\right) + m_e^4}{C^4 n_n^{4/3}}}{1 + \frac{m_n^2}{C^2 n_n^{2/3}}}\right]$$

The nucleon mass difference $Q = m_n - m_p$ and the electron mass me are of comparable magnitude and very much less than m_n, so this result can be written more simply as

$$\frac{n_p}{n_n} = \frac{1}{8}\left[\frac{1 + \frac{4Q}{m_n}\left(\frac{\rho_c}{m_n n_n}\right)^{2/3} + \frac{4(Q^2 - m_e^2)}{m_n^2}\left(\frac{\rho_c}{m_n n_n}\right)^{4/3}}{1 + \left(\frac{\rho_c}{m_n n_n}\right)^{2/3}}\right] \qquad (2.8.21)$$

Where $\rho_c = m_n^4/C^3$is the critical density previously defined in equation (2.8..3). The condition for the neutrons to be stable against beta decay is that the electrons Fermi sea should be filled up to momentum greater than maximum momentum k_{max} of electron emitted in neutron beta decay.

$$k_{F,e} > k_{max} \qquad (2.8.22)$$

Where

$$k_{max} = \frac{\left[\left(m_n^2 - m_p^2\right)^2 - 2m_e^2\left(m_n^2 - m_p^2\right) + m_e^4\right]^{1/2}}{2m_n}$$

$$\simeq [Q^2 - m_e^2]^{1/2} = 1.19\ MeV \qquad (2.8.23)$$

The electron Fermi momentum is given by equations (2.8.17) and (2.8.19) as

$$k_{F,e}^2 = C^2 n_e^{1/3} = C^2 n_p^{2/3} = m_n^2 \left(\frac{m_n n_n}{\rho_n}\right)^{2/3} \left(\frac{n_p}{n_n}\right)^{2/3}$$

$$= \frac{\frac{m_n^2}{4}\left(\frac{m_n n_n}{\rho_c}\right)^{4/3} + Q m_n \left(\frac{m_n n_n}{\rho_c}\right)^{2/3} + Q^2 - m_e^2}{\left(\frac{m_n n_n}{\rho_c}\right)^{2/3} + 1} \qquad (2.8.24)$$

This is smallest at $n_n = 0$, where $k_{F,e}$ barely equals the value k_{max}. Hence the condition (2.8.22) for neutron beta stability is indeed satisfied for any positive neutron density. The proton ratio (2.8.21) is large and decreasing for $m_n n_n$ equal to the transition density

$$\rho_r \simeq \rho_c \left(\frac{4(Q^2 - m_e^2)}{m_n^2}\right)^{3/4} = 1.28 \times 10^{-4}\, \rho_c \tag{2.8.25}$$

Where

$$\left(\frac{n_p}{n_n}\right)_{min} \simeq \left(\frac{Q^2 + \frac{1}{2}(Q^2 - m_e^2)^{1/2}}{m_n}\right)^{3/2} - 0.002 \tag{2.8.26}$$

2.9 Supermassive Stars:

We now turn to a different kind of “star” [70] in which general relativity enters in quite a different way. Let us consider a Newtonian star that is supported by the pressure of radiation rather than of matter, the conditions under which this occurs will be discovered as we go along. Let us also assume that the star is in convective equilibrium (see section 2.6) and has uniform chemical composition. Radiation has an energy density $e = 3p$, so this star will be a polytrope with $\gamma = 4/3$, that is

$$p = K\rho^{4/3} \tag{2.9.1}$$

This radiation pressure is given by the Stefan-Boltzmann law

$$p_r = \frac{\pi^2 (KT)^4}{45\hbar^3} \tag{2.9.2}$$

So with $p \simeq p_r$, the temperature is given by

$$KT = \left(\frac{45\hbar^3 K}{\pi^2}\right)^{1/4} \rho^{1/3} \tag{2.9.3}$$

The pressure of matter here is given by the ideal-gas law

$$p_m = \rho \frac{KT}{\bar{m}} \tag{2.9.4}$$

Where $\bar{m}$ is the mean mass of the gas particles. Thus the ratio of matter to radiation pressure is

$$\beta = \frac{p_m}{p_r} = \frac{45\hbar^3}{\pi^2 \bar{m}} \frac{\rho}{(KT)^3} = \frac{1}{\bar{m}} \left(\frac{45\hbar^3}{\pi^2 K^3} \right)^{1/4} \tag{2.9.5}$$

This is a constant throughout the star, so we can use β instead of K (or the entropy per nucleon, on which they both depend) to define the equation of state, writing

$$K = \left(\frac{45\hbar^3}{\bar{m}^4 \pi^2 \beta^4} \right)^{1/3} \tag{2.9.6}$$

The value of $\rho(0)$ for which the internal energy E is stationary. To calculate E, we use equations (2.6.29), (2.6.30) and (2.6.31), which to first order in GM/R give

$$E \simeq \int_0^R 4\pi r^2 e(r) dr + \int_0^R 4\pi G r M(r) e(r)\, dr$$
$$- \int_0^R 4\pi G r M(r) dr - \int_0^R 6\pi G^2 M^2(r) \rho(r)\, dr \tag{2.9.7}$$

The internal energy density e is

$$e = \frac{\pi^2}{15} \frac{(KT)^4}{\hbar^3} + \frac{1}{\Gamma - 1} \frac{\rho k T}{\bar{m}} = 3p_r \left[1 + \frac{\beta}{3(\Gamma - 1)} \right]$$

Where, Γ is the specific heat ratio of the matter. (for ionized hydrogen, $\Gamma = 5/3$) the total pressure is

$$p = p_r + p_m = p_r(1 + \beta)$$

Therefore, to first order in the small parameter β, the ratio of energy density to pressure is given by

$$e \simeq 3p \left[1 - \frac{(3\Gamma - 4)}{3(\Gamma - 1)} \beta + O(\beta^2) \right] \tag{2.9.8}$$

The small correction of order β can be ignored in the second term in (2.9.7), which is already smaller than the first term by a factor of order GM/R, but it must be kept in the large first term, and therefore

$$E \simeq \left[1 - \frac{(3\Gamma - 4)}{3(\Gamma - 1)} \beta \right] \int_0^R 12\pi r^2 p(r) dr + \int_0^R 12\pi G r M(r) p(r)\, dr$$
$$- \int_0^R 4\pi G r M(r) dr - \int_0^R 6\pi G^2 M^2(r) \rho(r)\, dr - \cdots \tag{2.9.9}$$

The first integral can be rewritten by integrating by parts:

$$\int_0^R 12\pi r^2 p(r)\, dr = \int_0^R p(r) d(4\pi r^3) = -\int_0^R 4\pi r^3 p'(r)\, dr$$

To calculate $p'(r)$, we expand the fundamental equation (2.6.13) to first order in GM/R:

$$-r^2 p'(r) \simeq GM(r)\rho(r)\left[1 + \frac{p(r)}{\rho(r)} + \frac{4\pi r^3 p(r)}{M(r)} + \frac{2GM(r)}{r}\right]$$

So

$$\int_0^R 12\pi r^2 p(r) dr \simeq \int_0^R 4\pi G r M(r)\rho(r) dr + \int_0^R 4\pi G r M(r) p(r)\, dr$$

$$+ \int_0^R 16\pi^2 G r^4 \rho(r) p(r) dr + \int_0^R 8\pi G^2 \rho(r) M^2(r)\, dr$$

The β-correction needs to be kept in only the first term, which is larger than the others by a factor of order GM/R, so to first order in β and GM/R, equation (2.9.9) reads

$$E \simeq -\frac{(3\Gamma - 4)}{3(\Gamma - 1)}\beta \int_0^R 4\pi G r M(r)\rho(r) dr + \int_0^R 16\pi G r M(r) p(r)\, dr$$

$$+ \int_0^R 16\pi^2 G r^4 \rho(r) p(r) dr + \int_0^R 2\pi G^2 M^2(r) \rho(r)\, dr \qquad (2.9.10)$$

Now every term is small, so they can all be evaluated using for ρ, p and M the values obtained by solving the Newtonian equation

$$-r^2 p'(r) \simeq GM(r)\rho(r)$$

For a Newtonian polytrope with $\gamma = 4/3$. In particular, the first integral in (2.9.10) is given by setting $\gamma = 4/3$ in equation (2.7.15)

$$\int_0^R 4\pi G r M(r)\rho(r) dr = -V = \frac{3GM^2}{2R}$$

Whereas an integration by parts lets us write the third term as

$$\int_0^R 16\pi^2 G r^4 \rho(r) p(r)\, dr = \int_0^R 4\pi r^2\, p(r)\, dM(r)$$

$$= -\int_0^R 4\pi G r^2\, p'(r) M(r)\, dr - \int_0^R 8\pi G r p(r) M(r)\, dr$$

$$= \int_0^R 4\pi G^2 M^2(r) \rho(r)\, dr - \int_0^R 8\pi G r p(r) M(r)\, dr$$

Equation (2.9.10) now reads

$$E \simeq -\frac{(3\Gamma-4)}{2(\Gamma-1)}\beta\frac{GM^2}{R} + \int_0^R 8\pi GrM(r)p(r)\,dr + \int_0^R 6\pi G^2M^2(r)\rho(r)\,dr$$

So, putting this all together, we have at last

$$E \simeq -\frac{(3\Gamma-4)}{2(\Gamma-1)}\beta\frac{GM^2}{R} + 5.1\frac{G^2M^2}{R^2} \qquad (2.9.11)$$

The star is certainly stable when R is so large that general relativity can be neglected, for then the star behaves like a Newtonian polytrope with

$$\gamma \equiv 1 + \frac{p}{e} \simeq \frac{4}{3} + \frac{(3\Gamma-4)}{9(\Gamma-1)}\beta > \frac{4}{3}$$

(see equation (2.9.8)). The transition from stability will occur when R decreases to a value where

$$\frac{\partial E}{\partial R} = \frac{\partial E}{\partial\rho(0)}\frac{\partial\rho(0)}{\partial R} = 0$$

The derivative must be taken with constant entropy per nucleon, and hence in this case with β fixed and M fixed. (see equation (2.9.6)). Thus the minimum radius for stability is

$$R_{min} = \frac{20.4(\Gamma-1)}{(3\Gamma-4)}\,\frac{GM}{\beta} \qquad (2.9.12)$$

The maximum energy that can be released by letting the star shrink slowly (through radiation at its surface) to this minimum stable radius is

$$-E(R_{min}) = \frac{(3\Gamma-4)^2\beta^2M}{81.6\,(\Gamma-1)^2} \qquad (2.9.13)$$

For instance, a star with $\beta = 0.1$ will have $M \simeq 7200\,M_s$ if $\Gamma = 5/3$ then the minimum radius is $1.45\times10^6\,km$, and the fraction of its rest-mass that can be released by assembling the star is 0.03%. The maximum value of the surface potential GM/R for $\Gamma = 5/3$.

2.10 Stars of Uniform Density:

General relativity finds an interesting application to one other class of stable stars, those consisting of incompressible fluids, with equation of state constant

$$\rho = constant \qquad (2.10.1)$$

These stars are of interest, not because they actually exist (they don't), but because they are simple enough to allow an exact solution of Einstein's equation [71] and because they set an upper limit to the gravitational red shift of spectral lines from the surface of any star [72]. With ρ constant, the fundamental equation (2.6.13) may be writing

$$\frac{-p'(r)}{[\rho + p(r)]\left[\frac{\rho}{3} + p(r)\right]} = 4\pi Gr\left[1 - \frac{8\pi G\rho r^2}{3}\right]^{-1} \tag{2.10.2}$$

The pressure must now be determined by integrating inward from the surface where $p = 0$, rather than outward, as for more realistic models. This gives

$$\frac{p(r) + \rho}{3p(r) + \rho} = \left[\frac{\left(1 - \frac{8\pi G\rho R^2}{3}\right)}{\left(1 - \frac{8\pi G\rho r^2}{3}\right)}\right]^{1/2}$$

Solving for $p(r)$, and expressing ρ in terms of the stellar mass

$$\rho = \frac{3M}{4\pi R^3} \quad for \quad r < R \tag{2.10.3}$$

We find

$$p(r) = \frac{3M}{4\pi R^3}\left[\frac{\left(1 - \frac{2MG}{R}\right)^{1/2} - \left(1 - \frac{2MGr^2}{R^3}\right)^{1/2}}{\left(1 - \frac{2MGr^2}{R^3}\right)^{1/2} - 3\left(1 - \frac{2MG}{R}\right)^{1/2}}\right] \tag{2.10.4}$$

The metric component $A(r)$ is immediately given by equation (2.6.11)

$$A(r) = \left[\frac{1 - 2MGr^2}{R^3}\right]^{-1} \tag{2.10.5}$$

Whereas $B(r)$ can be calculated by using (2.10.4) in the integral (2.6.16):

$$B(r) = \frac{1}{4}\left[3\left(1 - \frac{2MG}{R}\right)^{1/2} - \left(1 - \frac{2MGr^2}{R^3}\right)^{1/2}\right]^2 \tag{2.10.6}$$

The most interesting feature of this solution is that it does not make sense for all values of M and R. The pressure given by equation (2.10.4) will become infinite at a point r_∞ where

$$r_\infty^2 = 9R^2 - \frac{4R^3}{MG} \tag{2.10.7}$$

(Also, the metric becomes singular at r_∞ because $B(r_\infty)$ vanishes.) But the pressure is a scalar, and so an infinity in $p(r)$ cannot be blamed on an injudicious choice of coordinate system. We must see to it that $p(r)$ is not singular for any real r, and the only way to accomplish this is to have r_∞^2 negative, or

$$\frac{MG}{R} < \frac{4}{9} \tag{2.10.8}$$

Note that the Schwarzschild radius $2MG$ is then less than 8/9 the actual radius R, so there is no singularity in either the exterior solution (2.10.17) or the interior solutions (2.10.5) and (2.10.6).

This is not the first time that we have discovered an upper bound on the absolute value GM/R of the gravitational potential of a star. We learned in Section 2.8 that for a stable ideal-gas neutron star, GM/R is never greater than 0.36/3.2, or 0.11(see equations (2.8.13) and (2.8.14)). Is there then an absolute upper limit to GM/R imposed by the structure of the Einstein equations, irrespective of the equation of state. To frame this question as a mathematical problem, we consider ρ as an arbitrary finite positive function, subject only to these general requirements

(A) The radius R is fixed, with

$$\rho(r) = 0 \qquad \text{for} \qquad r > R \tag{2.10.9}$$

(B) The mass M is fixed, with

$$\int_0^R 4\pi\pi^2 \rho(r)\, dr = M \tag{2.10.10}$$

(C) The metric coefficient $A(r)$ given by (2.6.11) must not be singular, so

$$M(r) < \frac{r}{2G} \tag{2.10.11}$$

Where

$$M(r) \equiv \int_0^r 4\pi r'^2 \rho(r')\, dr'$$

(D) The density $\rho(r)$ must not increase outward

$$\rho'(r) \leq 0 \tag{2.10.12}$$

(It is difficult to imagine that a fluid sphere with a larger density near the surface than near the center could be stable). Given any function $\rho(r)$, satisfying these conditions, we can calculate $A(r)$ from equation (2.6.11); we can determine $p(r)$ by integrating equation (2.6.13) inward from the surface (with the boundary condition that $p(R) = 0$); and we can then calculate $B(r)$ from equation (2.6.16). Equation (2.10.11) guarantees that $A(r)$ is well behaved, and as long as $p(r)$ is finite, equation (2.6.13) will give $p(r) \geq 0$, and equation (2.6.16) will give a finite positive-definite $B(r)$. Thus any absolute limitations on the input function $\rho(r)$ (such as an upper bound on GM/R) can only come from the condition that equation (2.6.13) must yield a finite solution for the pressure $p(r)$. We shall exploit this condition rather indirectly, by concentrating on the metric coefficient $B(r)$ rather than on $p(r)$ itself. We first derive an equation that allows $B(r)$ to be calculated for a given density function $\rho(r)$, without having to solve for $p(r)$; from equations (2.6.5) and (2.6.7), we have

$$3R_{rr}B + R_{tt}A = B'' - \frac{B'}{2}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{3BA'}{rA} - \frac{B'}{r} = -16\pi G\rho AB$$

Or

$$B'' - \frac{B'}{2}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{2}{r}\right) = \frac{B}{rA}\left[3A' - 16\pi G\rho r A^2\right]$$

This equation can be linearized by defining

$$B = \xi^2 \tag{2.10.13}$$

Introducing equation (2.6.11) for $A(r)$, and rearranging a bit, we find

$$\frac{d}{dr}\left[\frac{1}{r}\left(1 - \frac{2GM(r)}{r}\right)^{1/2}\frac{d\xi(r)}{dr}\right] = G\left[1 - \frac{2GM(r)}{r}\right]^{-1/2}\left[\frac{M(r)}{r^3}\right]'\xi(r) \tag{2.10.14}$$

The initial conditions at $r = R$ can be determined directly from equation (2.6.16), or from the condition that $B(r)$ fit smoothly to the exterior solution (2.6.17), either way, we find that

$$\xi(R) = \left[1 - \frac{2MG}{R}\right]^{1/2} \tag{2.10.15}$$

$$\xi'(R) = \frac{MG}{R^2}\left[1 - \frac{2MG}{R}\right]^{-1/2} \tag{2.10.16}$$

The solution for $\xi(r)$ must be positive, because $\xi(r)$ can become negative only if it passes through the value zero, at which point B would vanish, and, according to equation (2.6.16), B can vanish only if the pressure $p(r)$ has a singularity.

We next proceed to derive an upper bound for $\xi(0)$. If ξ is positive, then the right-hand side of (2.10.14) is negative, because $3M(r)/4\pi r^3$is the mean density within the radius r, and the mean density cannot increase with r if the density does not. Thus equation (2.10.14) gives

$$\frac{d}{dr}\left[\frac{1}{r}\left(1-\frac{2GM(r)}{r}\right)^{1/2}\frac{d\xi(r)}{dr}\right]\leq 0$$

The equality being attained only for uniform density. Integrating this inequality from r to R and using (2.10.16), we have

$$\xi'(r)\geq\frac{MGr}{R^3}\left[1-\frac{2GM(r)}{r}\right]^{-1/2}$$

Integrating again from 0 to R and using (2.10.15) gives

$$\xi(0)\leq\left(1-\frac{2GM(r)}{r}\right)^{1/2}-\frac{MG}{R^3}\int_0^R\frac{r\,dr}{[1-(2GM(r)/r)]^{1/2}}$$

The right-hand side is largest when $M(r)$ is as small as possible. For a given mass M and radius R, the density distribution with $\rho'(r)\leq 0$ that gives an $M(r)$ that is everywhere as small as possible has $\rho(r)$ constant, in which case

$$M(r)=\frac{Mr^3}{R^3}$$

Using this in the integral, our inequality is

$$\xi(0)\leq\frac{3}{2}\left(1-\frac{2MG}{R}\right)^{1/2}-\frac{1}{2}\tag{2.10.17}$$

We have already noted that $\xi(r)$ must be positive-definite; hence (2.10.17) implies that

$$\frac{MG}{R}<\frac{4}{9}\tag{2.10.18}$$

This is just the upper limit found earlier for stars of uniform density, but now we know that (2.10.18) holds for all stars, uniform or not.

It can also be proved that for a given mass and radius, the stable stars with smallest central pressure are those with uniform density. Hence the central pressure of any star is not less than the value obtained by setting $r = 0$, in equation (2.10.4) that is

$$p(0) \geq \frac{3M}{4\pi R^3}\left[\frac{\left(1-\frac{2MG}{R}\right)^{1/2}-1}{1-3\left(1-\frac{2MG}{R}\right)^{1/2}}\right] \qquad (2.10.19)$$

This again shows that GM/R can never equal the forbidden value $4/9$. Our result can be immediately translated into a statement about the red shift of spectral lines from the surface of any star. According to equations (2.6.1) and (2.6.17), this is

$$z \equiv \frac{\Delta\lambda}{\lambda} = B^{-1/2}(R) - 1 = \left(1-\frac{2MG}{R}\right)^{-1/2} - 1$$

Equation (2.10.18) imposes on z the upper bound

$$z < 2 \qquad (2.10.20)$$

However, we should not jump to the conclusion that these red shifts are necessarily due to strong gravitational field, for red shifts near $z < 2$ require the star to be composed of a nearly incompressible fluid, with $\partial\rho/\partial p$ very small. This would seem unphysical, since we do not want the speed of sound $(\partial\rho/\partial p)^{1/2}$ to become larger than the speed light [73] Bondi [74] has shown that for a stable star with $\partial\rho/\partial p < 1$ and with $p/\rho \leq 1/3$.

However, there is no theorem that limits the red shifts of light signals from the interior of static spherically symmetric bodies [75, 76]. For instance, a light signal from the center of a transparent uniform star would have a red shift given by equations (2.6.1) and (2.10.6):

$$1 + z = B^{-1/2}(0) = \frac{2}{3\left(1-\frac{2MG}{R}\right)^{1/2} - 1}$$

As GM/R approaches the maximum value 4/9, this red shift becomes infinite. Hoyle and Fowler [76] have suggested that a quasi-star object can consist of a cluster of small dense stars, with the red shifts arising from emission and absorption in a hot cloud of gas trapped near the cluster center.

2.11 Time-Dependent Spherically Symmetric Field:

We now turn to the problems of star dynamics and begin by writing down the metric and Ricci tensor for a spherically symmetric but time-dependent system. spherical symmetry requires the proper time interval $d\tau^2$ to depend only on the rotational invariants

$$t, dt, r, x\,.\,dx = rdr, dx^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\; d\phi^2)$$

So it can be written

$$d\tau^2 \;= C(r,t)\; dt^2 - D(r,t)\; dr^2 - 2E(r,t)\; dr\; dt - F(r,t)\; r^2(d\theta^2 + \sin^2\theta\; d\phi^2)$$

The function F can be removed by defining a new radial variable

$$r' \equiv r\; F^{1/2}(r,t)$$

The metric will then be of the same from, but with new functions C', D', E' in place of C, D, E, and of course with r' in place of r and no factor F. Dropping primes, we have then

$$d\tau^2 = C(r,t)\; dt^2 - D(r,t)\; dr^2 - 2E(r,t)\; dr\; dt - \; r^2(d\theta^2 + \sin^2\theta\; d\phi^2)$$

We next remove E, by defining a new time

$$dt' = \; \eta(r,t)[C(r,t)\; dt - E(r,t)\; dr]$$

Where η is an integrating factor defined to make the right-hand side a perfect differential, that is, so that

$$\frac{\partial}{\partial}[\eta(r,t)\; C(r,t)] = -\frac{\partial}{\partial t}\; [\eta(r,t)E(r,t)]$$

(this equation can be solved by treating it as an initial value problem; given $\eta(r, t_0)$ for all r, we can solve for $\partial\eta(r,t)/\partial t$ at $t = \; t_0$ and thus determine $\eta(r, t_0 + dt)$for all r) the proper time is then

$$d\tau^2 = \; \eta^{-2}\; C^{-1}\; dt'^2 - (D + C^{-1}E^2)\; dr^2 - r^2(d\theta^2 + \sin^2\theta\; d\phi^2)$$

Or, introducing new functions A and B in place of $D + \; C^{-1}E^2$ and $\eta^{-2}\; C^{-1}$ and dropping the prime on t.

$$d\tau^2 = B(r,t)\; dt^2 - A(r,t)\; dr^2 - \; r^2(d\theta^2 + \sin^2\theta\; d\phi^2) \qquad (2.11.1)$$

Thus we can use the metric in its familiar "standard" from, the only new feature being that A and B now depend on t as well as r.

The non-vanishing elements of the metric tensor and its inverse are

$$\left.\begin{array}{llll} g_{rr} = A, & g_{\theta\theta} = r^2, & g_{\phi\phi} = r^2 \sin^2\theta, & g_{tt} = -B \\ g^{rr} = A^{-1}, & g^{\theta\theta} = r^{-2}, & g^{\phi\phi} = r^{-2}(\sin^2\theta)^{-2}, & g^{tt} = -B^{-1} \end{array}\right\} \tag{2.11.2}$$

It follows that the non-vanishing elements of the affine connection are

$$\left.\begin{array}{lll} \Gamma^r_{rr} = \dfrac{A'}{2A} & \Gamma^r_{\theta\theta} = -\dfrac{r}{A} & \Gamma^r_{\phi\phi} = -\dfrac{r\sin^2\theta}{A} \\ \Gamma^r_{tt} = \dfrac{B'}{2A} & \Gamma^r_{rt} = \Gamma^r_{tr} = \dfrac{\dot{A}}{2A} & \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \dfrac{1}{r} \\ \Gamma^\theta_{\phi\phi} = -\sin\theta\ \cos\theta & \Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \dfrac{1}{r} & \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \\ \Gamma^t_{rr} = \dfrac{A}{2B} & \Gamma^t_{tt} = \dfrac{\dot{B}}{2B} & \Gamma^t_{tr} = \Gamma^t_{rt} = \dfrac{B'}{2B} \end{array}\right\} \tag{2.11.3}$$

We obtain the independent nonzero components of the Ricci tensor:

$$R_{rr} = \frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{A'B'}{4AB} - \frac{A'}{Ar} - \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} + \frac{\dot{A}^2}{4AB} \tag{2.11.4}$$

$$R_{\theta\theta} = -1 + \frac{1}{A} - \frac{rA'}{2A^2} + \frac{rB'}{2AB} \tag{2.11.5}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'A'}{4A^2} - \frac{B'}{Ar} + \frac{B'^2}{4AB} + \frac{\ddot{A}}{2A} - \frac{\dot{A}^2}{4A^2} - \frac{\dot{B}\dot{A}}{4AB} \tag{2.11.6}$$

$$R_{tr} = -\frac{\dot{A}}{Ar} \tag{2.11.7}$$

Also, it follows from the spherical symmetry of the metric that

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta} \tag{2.11.8}$$

$$R_{r\theta} = R_{r\phi} = R_{\theta\phi} = R_{\theta t} = R_{\phi t} = 0 \tag{2.11.9}$$

As a simple but important application of these results, let us consider a spherically asymmetric but not necessarily static field in empty space, where the field equations read $R_{\mu v} = 0$. According to (2.11.7) the field equation $R_{tr} = 0$ just tells us that A is time-independent

$$\dot{A} = 0$$

Inspection of (2.11.4) – (2.11.6) then shows that all time derivatives drop out of the field equations, and they become identical with the equations for a static isotropic gravitational field in empty space. The vanishing of R_{rr} and R_{tt} gives

$$(AB)' = 0$$

And the vanishing of $R_{\theta\theta}$ gives

$$\left(\frac{r}{A}\right)' = 1$$

Since A is time-independent, the general solution is

$$A = \left(1 - \frac{2M}{r}\right)^{-1} \quad , \quad B = f(t)\left(1 - \frac{2MG}{r}\right)$$

With GM a time-independent integration constant, and $f(t)$ an unknown function of t. The function $f(t)$ can be made to equal unity by defining a new time coordinate

$$t' = \int_0^r f^{1/2}(t)\, dt$$

The metric is now entirely time-independent, and agrees with the Schwarzschild solution. We have thus proved the Birkhoff theorem [30], that a spherically symmetric gravitational field in empty space must be static, with a metric given by the Schwarzschild solution.

The Birkhoff theorem is analogous to the result proved by Newton in his theory of the lunar motion, that the gravitational field outside a spherically symmetric body behaves as if the whole mass of the body were concentrated at the center. It is a little surprising that this result should apply in general relativity as well as in Newton's theory, for in general relativity a non-static body will usually radiate gravitational waves. the Birkhoff theorem tells us that, although a pulsating spherically symmetric body can of course produce non-static gravitational fields within its mass, no gravitational radiation can escape into empty space. in this sense, the Birkhoff theorem is analogous to the well-known result of atomic theory, that a photon cannot be emitted in a quantum transition between two states of zero spin.

The Birkhoff theorem may be applied, not only to the gravitational field outside a body, but also to the field inside an empty spherical cavity at the center of a spherically symmetric (but not necessarily static) body. In this case the metric is again given by the Schwarzschild solution, but since the point $r = 0$ is here in empty space, there can be no singularity, so the integration constant MG must vanish. The Birkhoff theorem thus has the corollary that the metric inside an empty spherical cavity at the center of a spherically symmetric system must be equivalent to the flat-space Minkowskian metric $\eta_{\mu\upsilon}$. This corollary is analogous to another famous result of Newtonian theory, that the gravitational field of a spherical shell vanishes inside the shell. Stars do not usually have holes at their centers, so this corollary will not be of much use to us in this chapter. Its importance arises from the fact that the Birkhoff theorem is a local theorem, not depending on any conditions on the metric for $r \to \infty$ (aside from spherical symmetry), so that space must be flat in a spherical cavity at the center of a spherically symmetric system, even if the system is infinite even, in fact, if the system is the whole universe. That the corollary to Birkhoff theorem can be used to justify a limited use of Newtonian mechanics in cosmological problems.

2.12 Commoving Coordinates System:

The metric $g_{\mu\upsilon}$ in commoving coordinates is characterized by certain specially simple features. First, we note that the clocks are in free fall and therefore tell proper time , so the proper time interval between two points (x, t) and $(x, t + dt)$ on a given particle's trajectory is just dt, that is

$$d\tau^2 = -g_{\mu\upsilon} dx^\mu \, dx^\upsilon = -g_{tt} dt^2$$

And therefore

$$g_{tt} = -1 \tag{2.12.1}$$

Also, we note that the particle trajectory x = constant, $t = \tau$ satisfies the equation of free fall, so

$$\Gamma_{tt}^i = \frac{d^2 x^i}{d\tau^2} + \Gamma_{\mu\upsilon}^i \frac{dx^\mu}{d\tau} \frac{dx^\upsilon}{d\tau} = 0$$

Using equation (2.12.1) this gives

$$g^{ij} \frac{\partial g_{jt}}{\partial t} = 0$$

Or, since g^{ij} is generally a nonsingular matrix

$$\frac{\partial g_{jt}}{\partial t} = 0 \tag{2.12.2}$$

We have kept open the option of setting the clocks attached to the different particles in an arbitrary fashion. Suppose that we reset these clocks by a transformation.

$$t' = t + f(x) \qquad x' = x \tag{2.12.3}$$

The new metric will have the elements

$$g'_{tt} = -1 \tag{2.12.4}$$

$$g'_{ti} = g_{ti} + \frac{\partial f}{\partial x'} \tag{2.12.5}$$

$$g'_{ij} = g_{ij} - g_{ti}\frac{\partial f}{\partial x^j} - g_{tj}\frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \tag{2.12.6}$$

It would be a great simplification if the function f could be chosen so that the to two terms in equation (2.12.5) cancel, giving $g'_{it} = 0$. there are two important cases where this is possible:

(A) Suppose that we can reset all clocks so that all particles are at rest at a time $t = 0$. This assumption can be given an absolute physical significance by interpreting it to mean that for each particle p at $t = 0$, it is possible to find a locally inertial coordinate system $\tilde{x}^\mu$ in which the separation between p and neighboring particles is purely spatial

$$\left(\frac{\partial \tilde{x}^0}{\partial x^i}\right)_{t=0\,,x=x_p} = 0$$

And in which the movement of p in a time interval dt is purely temporal

$$\left[\frac{\partial \tilde{x}^i}{\partial t}\right]_{t=0\,,\;\; x=x_p} = 0$$

The metric in this locally inertial system is the Minkowskian metric $\eta_{\mu v}$, so the space-time components of the metric in the commoving system at $t = 0$ are

$$g_{ti}(x_p, 0) = \left[\eta_{\mu v}\frac{\partial \tilde{x}^\mu}{\partial x^i}\frac{\partial \tilde{x}^v}{\partial t}\right]_{t=0,x=x_p} = 0$$

With (2.12.2), it follows that g_{ti} vanishes everywhere, so the metric is given by

$$d\tau^2 = dt^2 - g_{ij}(x,t)\,dx^i\,dx^j \tag{2.12.7}$$

(B) If the metric is manifestly spherically symmetric, then the line element must have the general form with which we started in the last section, that is

$$d\tau^2 = C(r,t)dt^2 - D(r,t)dr^2 - 2E(r,t)\,dr\,dt - F(r,t)r^2(d\theta^2 + \sin^2\theta\,d\phi^2)$$

The only non-vanishing time-space component g_{tj} is $g_{tr} = 2E$, and equation (2.12.2) then tells us that E is time-independent, so

$$g_{tr} = 2E(r)$$

$$g_{t\theta} = g_{t\phi} = 0$$

We can therefore eliminate the components $g_{tj}$by resetting the clocks as in equation (2.12.3), with

$$f = -2\int_0^r E(r)\,dr$$

Using equation (2.12.4) and dropping primes, the metric is now of the from

$$d\tau^2 = dt^2 - U(r,t)dr^2 - V(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (2.12.8)$$

With U and V new unknown functions that replace D and F.

It is of course possible to construct coordinate systems of this sort even if the cloud of freely falling particles is purely imaginary. In differential geometry, coordinate systems satisfying (2.12.1) and (2.12.2) are called Gaussian, and if g_{ti} vanishes, so that the line element takes the form equation (2.12.7), then we call the coordinates Gaussian normal. However, these coordinate systems find their most important applications to system That actually do consist of a freely falling fluid. In this case the fluid velocity four-vector by definition has zero space component

$$U^i = 0 \qquad (2.12.9)$$

And since U^μ is normalized so that

$$g_{\mu\upsilon}U^\mu U^\upsilon = -1 \qquad (2.12.10)$$

The time component of U^μ must be

$$U^t = (-g_{tt})^{-1/2} = 1 \qquad (2.12.11)$$

We shall be working only with spherically symmetric commoving coordinate systems, with line element (2.12.8). The non-vanishing elements of the metric tensor are

$$g_{rr} = U \quad , \quad g_{\theta\theta} = V \quad , \quad g_{\phi\phi} = V\sin^2\theta \quad , \quad g_{tt} = -1$$

$$g^{rr} = U^{-1} \quad , \quad g^{\theta\theta} = V^{-1} \quad , \quad g^{\phi\phi} = (V\sin^2\theta)^{-1} \quad , \quad g^{tt} = -1 \qquad (2.12.12)$$

The non-vanishing elements of the affine connection are readily calculated as

$$\Gamma^r_{rr} = \frac{U'}{2U} \quad , \quad \Gamma^r_{\theta\theta} = -\frac{V'}{2U} \quad , \quad \Gamma^r_{\phi\phi} = -\frac{V'}{2U}\sin^2\theta \quad , \quad \Gamma^r_{rt} = \Gamma^r_{tr} = \frac{\dot{U}}{2U}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{V'}{2V} \quad , \quad \Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} - \frac{\dot{V}}{2V} \quad , \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{V'}{2V} \quad , \quad \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \frac{\dot{V}}{2V} \quad , \quad \Gamma^\phi_{\theta\phi} = \cot\theta$$

$$\Gamma^t_{rr} = \frac{\dot{U}}{2} \quad , \quad \Gamma^t_{\theta\theta} = \frac{\dot{V}}{2} \quad , \quad \Gamma^t_{\phi\phi} = \frac{\dot{V}}{2}\sin^2\theta \tag{2.12.13}$$

The Ricci tensor:

$$R_{rr} = \frac{V''}{V} - \frac{V'^2}{2V^2} - \frac{U'V'}{2UV} - \frac{\ddot{V}}{2} + \frac{\dot{U}^2}{4U} - \frac{\dot{U}\dot{V}}{2V} \tag{2.12.14}$$

$$R_{\theta\theta} = -1 + \frac{V''}{2U} - \frac{U'V'}{4U^2} - \frac{\ddot{V}}{2} - \frac{\dot{V}\dot{U}}{4U} \tag{2.12.15}$$

$$R_{tt} = \frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{\dot{U}^2}{4U^2} - \frac{\dot{V}^2}{2V^2} \tag{2.12.14}$$

$$R_{tr} = \frac{\dot{V}'}{V} - \frac{V'^{\dot{V}}}{2V^2} - \frac{\dot{U}V'}{2UV} \tag{2.12.17}$$

Also, it again follows from the spherical symmetry of the metric that

$$R_{\phi\phi} = R_{\theta\theta}\sin^2\theta \tag{2.12.18}$$

$$R_{r\theta} = R_{r\phi} = R_{\theta\phi} = R_{\theta t} = R_{\phi t} = 0 \tag{2.12.19}$$

2.13 Gravitational Collapse:

We saw in sections (2.7) and (2.8) that a cooling star of mass greater than a few solar masses cannot reach equilibrium as white dwarf or a neutron star. It may be that a massive star will always eject enough matter by the time it reaches the end of its thermonuclear evolution so that its mass drops below the Chandrasekhar or the Oppenheimer-Volkoff limits. If not, then it will collapse.

A proper treatment of gravitational collapse would be prohibitively complicated for this Research. In order to get some feeling for what can happen during collapse, we consider only the simples case, [76] the spherically symmetric collapse of “dust” with negligible pressure. Since the dust particles are acted on by purely gravitational forces, they fall freely, and we can use them as the physical basis of a commoving coordinate system of the sort discussed in the last section. The metric then given by equation (2.12.8):

$$d\tau^2 = dt^2 - U(r,t)dr^2 - V(r,t)(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.13.1}$$

The energy-momentum tensor for a fluid of negligible pressure is given

$$T^{\mu\upsilon} = \rho U^\mu U^\upsilon \tag{2.13.2}$$

Where $\rho(r,t)$ is the proper energy density and U^μ is the velocity four-vector, given for a commoving coordinate system by equations (2.12.9) and (2.12.11):

$$U^r = U^\theta = U^\phi = 0 \quad , \quad U^t = 1 \tag{2.13.3}$$

The equations of momentum conservation $\left(T_i^\mu\right)_{;\mu} = 0$, are automatically satisfied, and the equation for energy conservation reads

$$\left(T_t^\mu\right)_{;\mu} = -\frac{\partial\rho}{\partial t} - \rho\Gamma_{\lambda t}^{\lambda} = -\frac{\partial\rho}{\partial t} - \rho\left(\frac{\dot{U}}{2U} + \frac{\dot{V}}{V}\right) = 0$$

Or in other words

$$\frac{\partial}{\partial t}\left(\rho V\sqrt{U}\right) = 0 \tag{2.13.4}$$

The Einstein field equations can be written

$$R_{\mu v} = -8\pi G S_{\mu v} \tag{2.13.5}$$

Where

$$S_{\mu v} = T_{\mu v} - \frac{1}{2}g_{\mu v}T_{\lambda}^{\lambda} = \rho\left[\frac{1}{2}g_{\mu v} + U_\mu U_v\right] \tag{2.13.6}$$

This may be evaluated with the aid of equation (2.13.1) and (2.13.3); we find that the only non-vanishing components of $S_{\mu v}$ are

$$S_{rr} = \rho\frac{U}{2} \quad , \quad S_{\theta\theta} = \rho\frac{V}{2} \quad , \quad S_{\phi\phi} = S_{\theta\theta}\sin^2\theta \quad , \quad S_{tt} = \frac{\rho}{2} \tag{2.13.7}$$

In particular

$$s_{tr} = 0 \tag{2.13.8}$$

Using equations (2.13.7), (2.13.8) and (2.12.14) - (2.12.17) in (2.13.5) yields four field equations

$$\frac{1}{U}\left[\frac{V''}{V} - \frac{V'^2}{2V^2} - \frac{U'V'}{2UV}\right] - \frac{\ddot{U}}{2U} + \frac{\dot{U}^2}{4U^2} - \frac{\dot{U}\dot{V}}{2UV} = -4\pi G\rho \tag{2.13.9}$$

$$-\frac{1}{V} + \frac{1}{U}\left[\frac{V''}{2V} - \frac{U'V'}{4UV}\right] - \frac{\ddot{V}}{2V} - \frac{\dot{V}\dot{U}}{4VU} = -4\pi G\rho \tag{2.13.10}$$

$$\frac{\ddot{U}}{2U} + \frac{\ddot{V}}{V} - \frac{\dot{U}^2}{4U^2} - \frac{\dot{V}^2}{2V} = -4\pi G\rho \tag{2.13.11}$$

$$\frac{\dot{V}'}{V} - \frac{V'\dot{V}}{2V^2} - \frac{\dot{U}V'}{2UV} = 0 \tag{2.13.12}$$

Let us simplify our model even further, and assume that ρ is independent of position [32]. We can now seek a separable solution, with

$$U = R^2(t)f(r) \quad , \quad V = S^2(t)g(r)$$

Then (2.13.12) requires that $\frac{\dot{S}}{S}$ equal $\frac{\dot{R}}{R}$, so we can normalize f and g so that

$$S(t) = R(t)$$

Also, we are still free to redefine the radial coordinate as an arbitrary function $\bar{r}$ of r, and in particular we can choose $r = \sqrt{g(r)}$, so f and g are replaced with

$$\bar{f} = \frac{fg'^2}{4g} \quad , \quad \bar{g} = \bar{r}^2$$

Dropping the tildes, we have then

$$U = R^2(t)f(r) \quad , \quad V = R^2(t)r^2 \tag{2.13.13}$$

Equations (2.13.9) and (2.13.10) then read

$$-\frac{f'^{(r)}}{rf^2(r)} - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G R^2(t)\rho(t) \tag{2.13.14}$$

$$\left[-\frac{1}{r^2} + \frac{1}{rf^2(r)} - \frac{f'(r)}{2rf^2(r)} -\right] - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G R^2(t)\rho(t) \tag{2.13.15}$$

The first terms in (2.13.14) and (2.13.15) must evidently be equal constants, which we shall call-$2k$:

$$-\frac{f'(r)}{rf^2(r)} = -\frac{1}{r^2} + \frac{1}{r^2 f(r)} - \frac{f'(r)}{2rf^2(r)} = -2k$$

The unique solution is

$$f(r) = [1 - kr^2]^{-1}$$

So the metric takes the form

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right] \tag{2.13.16}$$

Our remaining problem is to calculate the functions $\rho(t)$ and $R(t)$. Using equations (2.13.13) and (2.13.14) in the energy-conservation equation (2.13.4), we find that $\rho(t)R^3(t)$ is constant. We normalize the radial coordinate r so that

$$R(0) = 1 \tag{2.13.17}$$

And therefore

$$\rho(t) = \rho(0)R^{-3}(t) \tag{2.13.18}$$

The field equations (2.13.14) or (2.13.15) and (2.13.11) are now ordinary differential equations

$$-2k - \ddot{R}(t)R(t) - 2\dot{R}^2(t) = -4\pi G\rho(0)R^{-1}(t) \tag{2.13.19}$$

$$\ddot{R}(t)R(t) = -\frac{4\pi G}{3}\rho(0)R^{-1}(t) \tag{2.13.20}$$

We can eliminate $\ddot{R}(t)$ by adding these two equations, and find

$$\dot{R}^2(t) = -k + \frac{8\pi G}{3}\rho(0)R^{-1}(t) \tag{2.13.21}$$

Equations (2.13.19) and (2.13.20) can be recovered from (2.13.21) and its time derivative, so we can forget about them and simply use (2.13.21) to calculate $R(t)$. We shall now assume that the fluid is at rest (in standard coordinates) at $t = 0$, so

$$\dot{R}(0) = 0 \tag{2.13.22}$$

And therefore (2.13.21) and (2.13.17) give

$$k = \frac{8\pi G}{3}\rho(0) \tag{2.13.23}$$

Thus equation (2.13.21) can be written

$$\dot{R}^2(t) = k[R^{-1}(t) - 1] \tag{2.13.24}$$

The solution is given by the parametric equation of a cycloid

$$t = \left(\frac{\Psi + \sin\Psi}{2\sqrt{k}}\right)$$

$$R = \frac{1}{2}(1 + \cos\Psi) \tag{2.13.25}$$

Note that $R(t)$ vanishes when $\Psi = \pi$, and hence when $t = T$, where

$$T = \frac{\pi}{2\sqrt{k}} = \frac{\pi}{2}\left(\frac{3}{8\pi G\rho(0)}\right)^{1/2} \tag{2.13.26}$$

Thus a fluid sphere of initial density $\rho(0)$ and zero pressure will collapse from rest to a state of infinite proper energy density in the finite time T. Although the collapse is complete at a finite coordinate time $t = T$, any light signal coming to us from the sphere's surface will be delayed by its gravitational field, so we on earth will not see the star suddenly vanish. To make this more specific, we have to complete our calculation by finding the metric outside the star.

The Birkhoff theorem proved in (section (2.11)) shows that it is always possible to find a "standard" coordinate system $\bar{r}, \bar{\theta}, \bar{\phi}$,$\bar{t}$ in which the metric outside the sphere takes the form

$$d\tau^2 = \left(1 - \frac{2MG}{\bar{r}}\right) d\bar{t}^2 - \left(1 - \frac{2MG}{\bar{r}}\right)^{-1} d\bar{r}^2 - \bar{r}^2 d\bar{\theta}^2 - \bar{r}^2 \sin^2\bar{\theta}\, d\bar{\phi}^2 \qquad (2.13.27)$$

But this metric is not in the Gaussian normal form (2.13.1), so in order to match solution at the surface we either have to convert the interior solution (2.13.16) into standard coordinates, or the exterior solution (2.13.27) into Gaussian normal coordinates. We choose the former course [77].

Inspection of equation(2.13.16) shows immediately that the standard spatial coordinate $\bar{r}, \bar{\theta}, \bar{\phi}$ must be chosen as

$$\bar{r} = r\, R(r) \ , \quad \bar{\theta} = \theta \quad , \quad \bar{\phi} = 0 \qquad (2.13.28)$$

In order to define a standard time coordinate such that $d\tau^2$ does not contain a cross-term $d\bar{r}\, d\bar{t}$, we employ the "integrating factor" technique described in section (2.11), which gives

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{S(r,t)}^{1} \frac{dR}{\left(1 - \frac{ka^2}{R}\right)} \left(\frac{R}{1 - R}\right)^{1/2} \qquad (2.13.29)$$

Where

$$S(r,t) = 1 - \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} \big(1 - R(t)\big) \qquad (2.13.30)$$

The constant a is arbitrary, but may conveniently be chosen as the radius of the sphere in commoving coordinates. It is straightforward to check that the metric in the coordinate system $\bar{r}, \bar{\theta}, \bar{\phi}, \bar{t}$ takes the standard form

$$d\tau^2 = B(\bar{r}, \bar{t})\, d\bar{t}^2 - A(\bar{r}, \bar{t})\, d\bar{r}^2 - \bar{r}^2 (\, d\bar{\theta}^2 + \ \sin^2\bar{\theta}\, d\bar{\phi}^2\,)$$

With

$$B = \frac{R}{S} \left(\frac{1 - kr^2}{1 - ka^2}\right)^{1/2} \frac{\left(1 - \frac{ka^2}{S}\right)^2}{\left(1 - \frac{kr^2}{R}\right)} \qquad (2.13.31)$$

$$A = \left(1 - \frac{kr^2}{R}\right)^{-1} \tag{2.13.32}$$

It now being understood that S is a function of $\tilde{t}$ defined by equation (2.13.29) and that r and $R(t)$ are function of $\bar{r}$ and S, or $\bar{r}$ and $\bar{t}$, defined by solving equations (2.13.28) and (2.13.30). This is a mass, but at the radius $r = a$ of the star (a constant, since r is a commoving coordinate) we have

$$\bar{r} = \bar{a}(t) \equiv aR(t) \tag{2.13.33}$$

$$\bar{t} = \left(\frac{1 - ka^2}{k}\right)^{1/2} \int_{R(t)}^{1} \frac{dR}{\left(1 - \frac{ka^2}{R}\right)} \left(\frac{R}{1 - R}\right)^{1/2} \tag{2.13.34}$$

$$B(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right) \tag{2.13.35}$$

$$A(\bar{a}, \bar{t}) = \left(1 - \frac{ka^2}{R(t)}\right)^{-1} \tag{2.13.36}$$

We see that the interior and exterior solutions fit continuously at $\bar{r} = aR(t)$ if

$$k = \frac{2MG}{a^3} \tag{2.13.37}$$

With (2.13.23), this just says that

$$M = \frac{4\pi}{3}\rho(0)a^3 \tag{2.13.38}$$

not a surprising result.

Now we return to the problem of calculating the behavior of light signals emitted from the surface of the collapsing sphere. A light signal emitted in a radial direction at a standard time $\bar{t}$ will have $d\bar{r}/d\bar{t}$ given by equation (2.13.27) and the condition $d\tau = 0$, so it will arrive at a distant point $\bar{r}$ at a time

$$\bar{t}' = \bar{t} + \int_{aR(t)}^{\bar{r}'} \left(1 - \frac{2MG}{r}\right)^{-1} dr \tag{2.13.39}$$

The most striking consequence of equations (2.13.39) and (2.13.34) is that both $\bar{t}$ and $\overline{t'}$ approach infinity when the radius (2.13.33) of the sphere approaches the Schwarzschild radius $2MG$, that is, when

$$R(r) \rightarrow \frac{2GM}{a} = ka^2 \tag{2.13.40}$$

The collapse to the Schwarzschild radius therefore appears to an outside observer to take an infinite time, and the collapse to $R = 0$ is utterly unobservable from outside.

Although the collapsing sphere does not suddenly disappear, it does fade out of sight, because light from its surface is subject to an increasing red shift. The proper time for a light source on the sphere's surface is just the commoving time t, so the commoving time interval between emission of wave crests at the surface equals the natural wavelength λ_0 that would be emitted by the source in the absence of gravitation. The standard time interval $d\bar{t}'$ between arrivals of wave crests at $\bar{r}'$ is the observed wavelength λ' thus the fractional change of wavelength is

$$z \equiv \frac{\lambda' - \lambda_0}{\lambda_0} = \frac{d\bar{t}'}{dt} - 1 = \frac{d\bar{t}}{dt} - a\dot{R}(t)\left(1 - \frac{2MG}{aR(t)}\right)^{-1} - 1$$

$$= -\dot{R}(t)\left[1 - \frac{ka^2}{R(t)}\right]^{-1}\left[\left(\frac{1 - ka^2}{k}\right)^{1/2}\left(\frac{R(t)}{1 - R(t)}\right)^{1/2} + a\right] - 1$$

Using equation (2.13.24) to determine $\dot{R}(t)$, this is

$$z = \left[1 - \frac{ka^2}{R(t)}\right]^{-1}\left[(1 - ka^2)^{1/2} + a\sqrt{k}\left(\frac{1 - R(t)}{R(t)}\right)^{1/2}\right] - 1 \qquad (2.13.41)$$

In order to see how the red shift z varies with $\bar{t}$, let us assume that the sphere is initially very much larger than its Schwarzschild radius

$$ka^2 = \frac{2GM}{a} \ll 1 \qquad (2.13.42)$$

And distinguish two periods in the history of the collapse:

(A) Until t gets close to T, we have

$$\frac{ka^2}{R(t)} \ll 1 \qquad (2.13.43)$$

Using equations (2.13.42) and (2.13.43) in (2.13.34), (2.13.39) and (2.13.41) gives (with $\bar{r}' \gg a$)

$$\bar{t} \simeq t$$

$$\bar{t}' \simeq \bar{t} + \bar{r}' - aR(t) \simeq t + \bar{r}' - aR(t) \simeq t + \bar{r}'$$

$$z \simeq a\sqrt{k}\left(\frac{1 - R(t)}{R(t)}\right)^{1/2} \simeq a\sqrt{k}\left(\frac{1 - R(\bar{t}' - \bar{r}')}{R(\bar{t}' - \bar{r}')}\right)^{1/2} \qquad (2.13.44)$$

(B) Eventually we have

$$\frac{ka^2}{R(t)} \to 1$$

at a time t_1 given by equation (2.13.25) as

$$t_1 \simeq \frac{1}{2\sqrt{k}}\left[\pi - \frac{4}{3}(ka^2)^{3/2}\right] \tag{2.13.45}$$

Now (2.13.34), (2.13.39), and (2.13.41) give

$$\bar{t} \simeq -ka^3\, Ln\left[1 - \frac{ka^2}{R(t)}\right] + \text{constant}$$

$$\overline{t'} \simeq \bar{t} - ka^3\, Ln\left[1 - \frac{ka^2}{R(t)}\right] + \text{constant}$$

$$\simeq \ -2ka^3\, Ln\left[1 - \frac{ka^2}{R(t)}\right] + \text{constant}$$

$$z \simeq 2\left(1 - \frac{ka^2}{R(t)}\right)^{-1} \propto \ \exp\left(\frac{t'}{2ka^3}\right) \tag{2.13.46}$$

Putting (A) and (B) together, we conclude that the red shift z seen by an observer at $\bar{r}'$ is zero when the collapse is observed to begin, increases gradually but remains of order $a\sqrt{k} \ll 1$ until a time very close to $T = \pi/2\sqrt{k}$ has passed, and then grows exponentially with a rate $1/2ka^3$. for example; a collapsing sphere with a mass $M = 10^8 M_s$ and radius $a = 100$ light years will have a red shift z of order 10^{-3} for a period of order 10^5 years, after which the red shift suddenly begins growing exponentially with an e-folding time of order $1min$. For practical purposes, the collapsing sphere is suddenly and completely cut off from communication with the rest of the universe.

Completely cut off. Even if a collapsing body does fade out of sight, it still has a gravitational field, and, as shown in (section (2.10) and (2.11)), the measurement of this field at great distances can be used to determine the energy, momentum, and angular momentum of the body. If the body has a net electric charge, then measurement of the electric field at great distances will, via Gauss's theorem, also tell us the charge. It is interesting to ask whether measurements of the gravitational and/or electromagnetic fields outside a collapsing body can yield any information about the body beyond the energy, momentum, angular momentum, and charge.

In the case of a spherically symmetric electrically neutral body, which we have been considering in this chapter the answer is provided by Birkhoff theorem: the gravitational field outside a spherically symmetric body must be of the Schwarzschild form, so all we can ever learn about the body is its mass. (spherical symmetry, of course, implies zero momentum and zero angular momentum.)

Carter [78] has shown that when the gravitational field of an axially symmetric collapsing body settles down to a stationary state, its exterior metric belongs to a two-parameter family of solutions, such as the Kerr metrics (see section (2.11)) that are completely specified by the total mass and angular momentum. It is widely believed that the gravitational field of any electrically neutral collapsing body will eventually approach the Kerr form.

An mentioned in the introduction to this chapter, interest in the phenomenon of gravitational collapse was rekindled in the last decade by the discovery of quasi-stellar sources, which appear to require some powerful new source of energy. The maximum energy available from fusion of hydrogen into the most stable nuclei, say iron, is only 8MeV per nucleon, or less than 1% of the rest-mass. Matter-antimatter annihilation could have 100% efficiency (apart from neutrino energy losses), but this process can be important only if there is some abundant natural source of antinucleons. Otherwise the only likely mechanism for conversion of mass into energy with high efficiency is gravitational collapse [79].

A cloud of dust that is collapsing as in the Oppenheimer-Snyder model will obviously release no energy to the outside world. To extract the growing kinetic energy of the falling particles, something must slow them on the way down either a macroscopic "bounce" of the whole system, or particle-particle collisions that heat the collapsing gas. Detailed calculations reveal a discouragingly low efficiency for conversion of mass into available energy in the gravitational collapse of an isolated body [80]. However, particles falling into a Kerr metric can reemerge with a higher energy, acquired at the expense of the rotational energy of the collapsing body [81].

Whether or not gravitational collapse has anything to do with quasi-stellar sources, the question remains: what happens to a real cooling star whose mass is above the

Chandrasekhar and Oppenheimer-Volkoff limits. In recent years topological methods have been used by Penrose and Hawking to prove a number of powerful theorems [82], to the effect that under reasonable conditions (validity of general relativity, positivity of energy, ubiquity of matter, causality) collapse becomes inevitable once a trapped surface forms. A trapped surface is a closed space like two dimensional surface for which both the outgoing and the ingoing families of future-directed null geodesics orthogonal to the surface are converging. (for the Schwarzschild metric, the spheres with r and t constant are trapped surfaces for r within the Schwarzschild radius $2MG$) However, it is not known whether a real massive star will actually develop a trapped surface, or merely explode into fragments with small enough mass to form stable neutron stars or white dwarfs.

If gravitational collapse is indeed the inevitable fate of massive bodies, then we must expect that the universe is full of black hole collapsing bodies whose presence is betrayed only through their gravitational fields or through the energy released when matter is drawn in [83]. Our best hope of observing gravitational collapse would be to find a binary star , one member an ordinary visible star, and the other member a black hole [84, 85].

Chapter Three

Equilibrium of Stars By Equation of Motion to Relate Pressure and Gravity

3.1 Introduction:

One will discuss in this chapter the natural behavior of the stars through the stages of building them, and the terms of their equilibrium, based on the pressure and gravity, which keeps these stars in equilibrium through the gravitational equivalent attraction, which may lead to undermine it, and thermal stress may make them explode, it is possible to remain the star without happen to him undermine the final (an explosion in the opposite direction of time) if a condition related mass and radius. We shall see that stars are generally in a state of almost complete mechanical equilibrium. The same method can understand the behavior of the universe and the condition of its equilibrium across the different stages of his life.

3.2 Coordinate Systems and the Mass Distribution:

The assumption of spherical symmetry implies that all interior physical quantities (such as density ρ, pressure p, temperature T, etc)depend only on one radial coordinate. The obvious coordinate to use in a Eulerian coordinate system is the radius of a spherical shell, r $(\in\ 0 \dots R)$. In an evolving star, all quantities also depend on time t. When constructing the differential equations for star structure one should thus generally consider partial derivatives of physical quantities with respect to radius and time, $\partial/\partial r$ and $\partial/\partial t$, taken at constant t and r, respectively. The principle of mass conservation applied to the mass dm of a spherical shell of thickness dr at radius r gives [86].

$$dm(r,t) = 4\pi r^2 \rho dr - 4\pi r^2 \rho v dt \tag{3.2.1}$$

Where υ is the radial velocity of the mass shell. Therefore one has

$$\frac{\partial m}{\partial r} = 4\pi r^2 \rho \quad , \qquad \frac{\partial m}{\partial t} = -4\pi r^2 \rho v \tag{3.2.2}$$

The first of these partial differential equations relates the radial mass distribution in the star to the local density, it constitutes the first fundamental equation of star structure.

Note that $\rho = \rho(r, t)$ is not known a priori, and must follow from other conditions and equations. The second equation of (3.2.2) represents the change of mass inside a sphere of radius r due to the motion of matter through its surface; at the stellar surface this gives the ass-loss rate (if there is a star wind with $v > 0$) or mass-accretion rate (if there is inflow with $v < 0$). In a static situation, where the velocity is zero, the first equation of (3.2.2) becomes an ordinary differential equation

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{3.2.3}$$

This is almost always a good approximation for star interiors, as we shall see. Integration yields the mass $m(r)$ inside a spherical shell of radius r

$$m(r) = \int_0^r 4\pi r'^2 \, \rho \, dr'$$

Since $m(r)$ increases monotonically outward, we can also use $m(r)$ as our radial coordinate, instead of r. This mass coordinate, often denoted as m_r or simply m, is a Lagrangian coordinate that moves with the mass shells

$$m = m_r = \int_0^r 4\pi r'^2 \, \rho dr' \qquad (m \in 0 \ldots M) \tag{3.2.4}$$

It is often more convenient to use a Lagrangian coordinate instead of a Eulerian coordinate. The mass coordinate is defined on a fixed interval, $m \in 0 \ldots M$, as long as the star does not lose mass. On the other hand r depends on the time-varying star radius r. Furthermore the mass coordinate follows the mass elements in the star, which simplifies many of the time derivatives that appear in the star evolution equations (e.g. equations for the composition). We can thus write all quantities as functions of m, i.e.

$$r = r(m) \quad , \quad \rho = \rho(m) \quad , \quad p = p(m) \quad \text{etc.}$$

Using the coordinate transformation $r \to m$, i.e.

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r} \frac{\partial r}{\partial m} \tag{3.2.5}$$

The first equation of star structure becomes in terms of the coordinate m

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{3.2.6}$$

Which again becomes an ordinary differential equation in a static situation.

3.3 The Equations of Motion and Hydrostatic Equilibrium:

We next consider conservation of momentum inside a star, i.e. Newton's second law of mechanics. The net acceleration on a gas element is determined by the sum of all forces acting on it. In addition to the gravitational force considered above, forces result from the pressure exerted by the gas surrounding the element. Due to spherical symmetry, the pressure forces acting horizontally (perpendicular to the radial direction) balance each other and only the pressure forces acting along the radial direction need to be considered. By assumption we ignore other forces that might act inside a star. Hence the net acceleration of a cylindrical gas element with mass [86]:

$$dm = \rho dr ds \tag{3.3.1}$$

Where dr is its radial extent and ds is its horizontal surface area, is given by, Pressure (net force due to difference in pressure between upper and lower faces)

$$F_p = p(r)ds - p(r+dr)ds$$

$$= p(r)ds - \left[p(r) + \frac{dp}{dr}dr\right]ds = -\frac{dp}{dr}drds$$

We can write

$$p(r+dr) = p(r) + \left(\frac{\partial p}{\partial r}\right)dr$$

Mass of element $dm = \rho dr ds$ Applying Newton's second law

$$\ddot{r}dm = F_g + F_p$$

$$\ddot{r}dm = -g\, dm + p(r)ds - p(r+dr)ds \tag{3.3.2}$$

Acceleration equal zero everywhere if star static, setting acceleration to zero, and substituting for dm

$$-\frac{Gm\rho\, dr\, ds}{r^2} - \frac{dp}{dr}drds = 0 \tag{3.3.3}$$

Equation of hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{Gm}{r^2}\rho \tag{3.3.4}$$

Hence after substituting equations (3.2.1) and (3.3.1) we obtain the equation of motion for a gas element inside the star

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3.3.5}$$

This is a simplified from of the Navier-Stokes equation of hydrodynamics, applied to spherical symmetry. Writing the pressure gradient $\partial p/\partial r$ in terms of the mass coordinate m by substituting equation (3.2.6), the equation of motion is

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial p}{\partial m} \tag{3.3.6}$$

Hydrostatic equilibrium The great majority of stars are obviously in such long-lived phases of evolution that no change can be observed over human lifetimes. This means there is no noticeable acceleration, and all forces acting on a gas element inside the star almost exactly balance each other. Thus most stars are in a state of mechanical equilibrium which is more commonly called hydrostatic equilibrium. The state of hydrostatic equilibrium, setting Acceleration equal zero in equation (3.3.5), yields the second differential equation of star structure

$$\frac{dp}{dr} = -\frac{Gm}{r^2}\rho \tag{3.3.7}$$

Or with equation (3.2.6):

$$\frac{dp}{dm} = \frac{dp}{dr}\frac{dr}{dm} = -\frac{Gm}{r^2}\rho\ \frac{1}{4\pi r^2 \rho} = -\frac{Gm}{4\pi r^4} \tag{3.3.8}$$

Alternate form of hydrostatic equilibrium equation.

A direct consequence is that inside a star in hydrostatic equilibrium, the pressure always decreases outwards. Equations (3.2.6) and (3.3.8) together determine the mechanical structure of a star in hydrostatic equilibrium. These are two equations for three unknown functions of $m(r,\ p$ and $\rho)$, so they cannot be solved without a third condition. This condition is usually a relation between p and ρ called the equation of state. We integrate this equation

$$\frac{dp}{dr} = -\frac{GM(r)\rho(r)}{r^2} \tag{3.3.9}$$

Assuming the star has constant density ρ_0

$$M(r) = \frac{4}{3}\pi r^3 \rho_0$$

$$\frac{dp}{dr} = -\frac{G}{r^2}\left(\frac{4}{3}\pi r^3 \rho_0\right)\rho_0 = -\frac{4\pi G \rho_0^2 r}{3}$$

$$\int_{P_c}^{0} dp_c = \int_0^R -\frac{4\pi G {\rho_0}^2}{3} r\, dr = -\frac{4\pi G {\rho_0}^2}{3}\int_0^R r\, dr$$

$$p_c = \frac{4\pi {\rho_0}^2 G}{3}\,\frac{R^2}{2} = \left(\frac{\rho_0 G}{2R}\right)\left(\frac{4}{3}\pi R^3 \rho_0\right) = \frac{M\rho_0 G}{2R} \qquad (3.3.10)$$

3.4 Relativistic Electron Gas in Star:

White dwarf stars, a main-sequence hydrogen burning star, such as the Sun, is maintained in equilibrium via the balance of the gravitational attraction tending to make it collapse, and the thermal pressure tending to make it expand of course, the thermal energy of the star is generated by nuclear reactions occurring deep inside its core. Eventually, however, the star will run out of burnable fuel, and, therefore, start to collapse, as it radiates away its remaining thermal energy. What is the ultimate fate of such a star. A burnt-out star is basically a gas of electrons and ions. As the star collapses, its density increases, so the mean separation between its constituent particles decreases. Eventually, the mean separation becomes of order the de Broglie wavelength of the electrons, and the electron gas becomes degenerate. Note, that the de Broglie wavelength of the ions is much smaller than that of the electrons, so the ion gas remains non-degenerate. Now, even at zero temperature, a degenerate electron gas exerts a substantial pressure, because the Pauli exclusion principle prevents the mean electron separation from becoming significantly smaller than the typical de Broglie wavelength. Thus, it is possible for a burnt-out star to maintain itself against complete collapse under gravity via the degeneracy pressure of its constituent electrons. Such stars are termed white dwarfs. Let us investigate the physics of white dwarfs in more detail [87, 88].

The total energy of a white dwarf star can be written

$$E = K + U \qquad (3.4.1)$$

Where K is the total kinetic energy of the degenerate electrons (the kinetic energy of the ion is negligible) U is the gravitational potential energy. Let us assume, for the sake of simplicity, that the density of the star is uniform. In this case, the gravitational potential energy takes the form

$$U = -\int_0^R 4\pi r^2 \rho \frac{Gm(r)}{r}\, dr$$

Where

$$m(r) = \frac{4}{3}\pi r^3 \rho$$

Hence

$$U = -\frac{16}{3}\pi^2 \rho^2 G \int_0^R r^4\, dr = -\frac{16}{15}\pi^2 \rho^2 G R^5$$

$$\rho = \frac{3m}{4\pi R^3}$$

gravitational potential energy

$$U = -\frac{3}{5}\frac{GM^2}{R} \tag{3.4.2}$$

Where, G is the gravitational constant, M is the stellar mass, R is the stellar radius.

we employed classical mechanics to deal with the translational degrees of freedom of the constituent particles, and quantum mechanics to deal with the non-translational degrees of freedom. Let us now discuss ideal gases from a purely quantum mechanical standpoint. It turns out that this approach is necessary to deal with either low temperature or high density gases [89].

Consider a gas consisting of N identical, non-interacting, structure less particles enclosed within a container of volume V. Cell volume in a vacuum phase:

$$\int dx\, dy\, dz \int dp_{F_x} dp_{F_y} dp_{F_z} = V \int dp_{F_x} dp_{F_y} dp_{F_z}$$

$$\text{The number of cells} = \frac{\text{the number of electrons (N)}}{2} = \frac{\text{the total volume of cells}}{\text{one cell volume}}$$

$$\text{The number of cells} = \frac{V \int dp_{F_x} dp_{F_y} dp_{F_z}}{h^3} = \frac{N}{2}$$

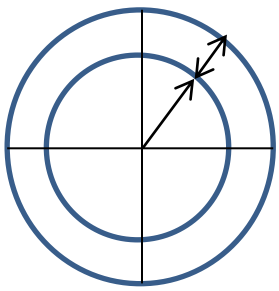

Figure (3.1): p-space of the spherical shell between radii $p + dp$

Let us assume that the electron gas is highly degenerate, which is equivalent to taking the limit $T \to 0$. In this case, we know, from the previous section, that the Fermi momentum can be written

$$\int dp_{F_x} dp_{F_y} dp_{F_z} = h^3 \frac{N}{2V} = \frac{h^3}{2}\left(\frac{N}{V}\right)$$

$$\frac{4}{3}\pi p_F^3 = \frac{h^3}{2}\left(\frac{N}{V}\right)$$

$$p_F^3 = \frac{3h^3}{8\pi}\left(\frac{N}{V}\right) = 3\pi^2 \frac{h^3}{8\pi^3}\left(\frac{N}{V}\right) = 3\pi^2 \hbar^3 \left(\frac{N}{V}\right)$$

$$p_F = (3\pi^2)^{1/3} \hbar \left(\frac{N}{V}\right)^{1/3} = \Lambda \left(\frac{N}{V}\right)^{1/3} \tag{3.4.3}$$

Where N the total number of electrons in the star.

$$\Lambda = (3\pi^2)^{1/3} \hbar \tag{3.4.4}$$

The volume of the star is equal to:

$$V = \frac{4}{3}\pi r^3 \tag{3.4.5}$$

is the star volume, and N is the total number of electrons contained in the star. Furthermore, the number of electron states contained in an annular radius of p-space lying between radii p and $p + dp$ is

$$dN = 2 \times the\ number\ of\ cells = \frac{2V(4\pi p_F^2\, dp)}{h^3}$$

$$dN = \frac{8\pi^3 V\, p_F^2\, dp}{\pi^2 h^3} = \frac{V p_F^2\, dp}{\pi^2 \hbar^3}$$

$$dN = \frac{3V}{\Lambda^3} p_F^2\, dp \tag{3.4.6}$$

Where

$$\Lambda^3 = 3\pi^2 \hbar^3$$

The density of modes per unit volume when the magnitude of k lies in the range k to $k + dk$ is given by multiplying the density of modes per unit volume by the "volume" in k-space of the spherical shell between radii k and $+dk$. It follows that the number of allowed values of k (i.e., the number of allowed modes) when k_x lies in the range

k_x to $k_x + dk_x$, k_y lies in the range k_y to $k_y + dk_y$, and k_z lies in the range k_z to $k_z + dk_z$, is

$$\rho d^3k = \left(\frac{L_x}{2\pi}dk_x\right)\left(\frac{L_y}{2\pi}dk_y\right)\left(\frac{L_z}{2\pi}dk_z\right) = \frac{V}{(2\pi)^3}dk_x dk_y dk_z \tag{3.4.7}$$

Let us calculate the Fermi energy of a Fermi Dirac gas at $T = 0$. The energy of each electron is related to its momentum $p = \hbar k$ via

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{3.4.8}$$

Where k is the de Broglie wave-vector. At $T = 0$ all quantum states whose energy is less than the Fermi energy E_F are filled. The Fermi energy corresponds to a Fermi momentum $p_F = \hbar k_F$ is thus given by

$$E_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} \tag{3.4.9}$$

Thus, at $T = 0$ all quantum states with $k < k_F$ are filled, and all those with $k > k_F$ are empty [86]. Now, we know, by analogy with equation (3.4.7), that there are $(2\pi)^{-3}V$ allowable translational states per unit volume of k-space. The volume of the sphere of radius k_F in k-space is $(4/3)\pi k_F^3$. It follows that the Fermi sphere of radius k_F contains $(4/3)\pi k_F^3(2\pi)^{-3}V$ translational states. The number of quantum states inside the sphere is twice this, because electrons possess two possible spin states for every possible translational state. Since the total number of occupied states (i.e. the total number of quantum states inside the Fermi sphere) must equal the total number of particles in the gas, it follows that

$$2\frac{V}{(2\pi)^3}\left(\frac{4}{3}\pi k_F^3\right) = N \tag{3.4.10}$$

The above expression can be rearranged to give

$$k_F = (3\pi^2 n)^{1/3} = \frac{\Lambda}{\hbar}\left(\frac{N}{V}\right)^{1/3} \tag{3.4.11}$$

Hence

$$\lambda_F = \frac{2\pi}{(3\pi^2 n)^{1/3}} = \frac{2\pi\hbar}{\Lambda}\left(\frac{V}{N}\right)^{1/3} \tag{3.4.12}$$

Which implies that the de Broglie wavelength λ_F corresponding to the Fermi energy is of order the mean separation between particles $(V/N)^{1/3}$, all quantum states with de Broglie wavelengths $\lambda > \lambda_F$ are occupied at $T = 0$, where as all those with $\lambda < \lambda_F$ are empty.

According to equation (3.4.9), the Fermi energy at $T = 0$ takes the form

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} = \frac{\Lambda^2}{2m}\left(\frac{N}{V}\right)^{2/3} \qquad (3.4.13)$$

It is easily demonstrated that $E_F \gg KT$ for conventional metals at room temperature. Hence, the total kinetic energy of the electron gas can be written

$$K = \int \frac{p_F^2}{2m}\, dN$$

$$K = \frac{3V}{\Lambda^3}\int_0^{p_F} \frac{p_F^2}{2m} p_F^2\, dp = \frac{3}{5}\frac{V}{\Lambda^3}\frac{p_F^5}{2m} \qquad (3.4.14)$$

Where m electron mass. Using two relations (3.4.14) and (3.4.13) becomes kinetic energy as follows

$$K = \frac{3}{5}N\frac{\Lambda^2}{2m}\left(\frac{N}{V}\right)^{2/3} \qquad (3.4.15)$$

The interior of a white dwarf star is composed of atoms like C^{12} and O^{16} Which contain equal numbers of protons, neutrons, and electrons. Thus,

$$M = Nm_n + Nm_p = 2Nm_p \qquad (3.4.16)$$

Where m_p proton mass, is equal to the mass of the proton and neutron mass m_n, and N represents the total number of protons.

Equations (3.4.1), (3.4.2), (3.4.4), (3.4.5), (3.4.15), and (3.4.16) can be combined to give model

$$E = \frac{A}{R^2} - \frac{B}{R} \qquad (3.4.17)$$

Where R radius of star.

Where

$$A = \frac{3}{20}\left(\frac{9\pi}{8}\right)^{2/3}\frac{\hbar^2}{m}\left(\frac{M}{m_p}\right)^{5/3} \qquad (3.4.18)$$

$$B = \frac{3}{5} GM^2 \tag{3.4.19}$$

The radius of the star is a balance R_{eq} that dimension which reduces the total energy E and his can be found by using the minimization of energy with respect to radius

$$\frac{dE}{dR} = 0$$

$$-\frac{2A}{R^3} + \frac{B}{R^2} = -2A + BR = 0$$

The equilibrium radius of the star R_{eq} is that which minimizes the total energy E. In fact, it is easily demonstrated that

$$R_{eq} = \frac{2A}{B} \tag{3.4.20}$$

which yields

$$R_{eq} = \frac{(9\pi)^{2/3}}{8} \frac{\hbar^2}{m} \frac{1}{Gm_p^{5/3}\ M^{1/3}} \tag{3.4.21}$$

The above formula can also be written

$$\frac{R_{eq}}{R_s} = 0.01 \left(\frac{M_s}{M}\right)^{1/3} \tag{3.4.22}$$

Where $R_s = 7 \times 10^5\ Km$ is the solar radius , $M_s = 2 \times 10^{30}\ Kg$ is the solar mass. It follows that the radius of a typical solar mass white dwarf is about 7000 *Km*: i.e., about the same as the radius of the Earth. The first white dwarf to be discovered (In 1862) was the companion of Sirius. Nowadays, thousands of white-dwarfs have been observed, all with properties similar to those described above [87, 88].

3.5 Stars Equilibrium:

One curious feature of white dwarf stars is that their radius decreases as their mass increases (see equation (3.4.22)). It follows, from equation (3.4.14), that the mean energy of the degenerate electrons inside the star increases strongly as the star mass increases: in fact, $K\alpha M^{4/3}$. Hence, if M becomes sufficiently large the electrons become relativistic, and the above analysis needs to be modified. Strictly speaking, the non-relativistic analysis described in the previous section is only valid in the low mass limit $M \ll M_s$. Let us, for the sake of simplicity, consider the ultra-relativistic limit in which $p \gg mc$.

The relativistic relation between the total energy E, momentum p and rest mass m :

$$E^2 = p^2c^2 + m_0^2c^4$$

Which for

$$pc \ll m_0c^2$$

You can also write the total electron energy using special relativity formula and using the relation (3.4.6) as follows

$$K = \int E\, dN = \int (p^2c^2 + m^2c^4)^{1/2}\, dN \tag{3.5.1}$$

The total electron energy (Including the rest mass energy) can be written

$$K = \frac{3V}{\Lambda^3}\int_0^{p_F} (p^2c^2 + m^2c^4)^{1/2}\, p^2 dp \tag{3.5.2}$$

You can use the formula

$$(1 + x)^n = 1 + nx$$

When $x \ll 1$, to simplify energy relativistic formula to get

$$E = (p^2c^2 + m_0^2c^4)^{1/2}$$

$$E = pc\left(1 + \frac{m^2c^2}{p^2}\right)^{1/2} = pc\left(1 + \frac{m^2c^2}{2p^2}\right)$$

Thus, electron energy as follows, by analogy with equation (3.4.14) Thus,

$$K = \frac{3Vc}{\Lambda^3}\int_0^{p_F} \left(p^3 + \frac{m^2c^2}{2}p + \cdots\right) dp \tag{3.5.2}$$

Giving

$$K = \frac{3}{4}\frac{Vc}{\Lambda^3}(p_F^4 + m^2c^2p_F^2 + \cdots) \tag{3.5.3}$$

Using the two relations (3.4.3) and (3.4.5) that produces

$$p_F = \Lambda\left(\frac{N}{V}\right)^{1/3} = \Lambda\left(\frac{3N}{4\pi R^3}\right)^{1/3}\frac{1}{R} \tag{3.5.4}$$

And by reference to the equations (3.4.5) and (3.5.4) it is clear that

$$K = \frac{3c}{4\Lambda^3}\left[\frac{4\pi R^3}{3}\right]\left[\Lambda^4\left(\frac{3N}{4\pi}\right)^{4/3}\frac{1}{R^4} + m^2c^2\Lambda^2\left(\frac{3N}{4\pi}\right)^{2/3}\frac{1}{R^2} + \cdots\right]$$

$$K = \frac{A}{R} + CR \qquad (3.5.5)$$

It follows, from the above, that the total energy of an ultra-relativistic white dwarf star can be written, using equations (3.4.1), (3.4.2) and (3.4.19) can be combined to give model

$$E = \frac{A - B}{R} + CR \qquad (3.5.6)$$

Where

$$A = \frac{3}{8}\left(\frac{9\pi}{8}\right)^{1/3} \hbar c \left(\frac{M}{m_p}\right)^{4/3} \qquad (3.5.7)$$

$$B = \frac{3}{5} G M^2 \qquad (3.5.8)$$

$$C = \frac{3}{4} \frac{1}{(9\pi)^{1/3}} \frac{m^2 c^3}{\hbar} \left(\frac{M}{m_p}\right)^{2/3} \qquad (3.5.9)$$

As already, the radius of the balance R_{eq} is that dimension which reduces the total energy E of the star, of the equation (3.5.6) can be found radius equilibrium by makes the energy minimum E, that is

$$\frac{dE}{dR} = -\frac{(A - B)}{R^2} + C = 0$$

The permission

$$R_{eq} = \left(\frac{A - B}{C}\right)^{1/2} \qquad (3.5.10)$$

As before, the equilibrium radius R_{eq} is that which minimizes the total energy E. However, in the ultra-relativistic case, a non-zero value of R_{eq} only exists for $A - B > 0$ When $A - B < 0$ the energy decreases monotonically with decreasing star radius: in other words, the degeneracy pressure of the electrons is incapable of halting the collapse of the star under gravity. The criterion which must be satisfied for a relativistic white-dwarf star to be maintained against gravity is that

$$\frac{A}{B} > 1 \qquad (3.5.11)$$

In any case, in the case benchtop relativism, it is no non-zero value for the amount of R_{eq}, in order $A - B > 0$. When you are $A - B < 0$. decreasing energy with a decrease of the radius of the star. or unable to pressure the electrons from the dissolved stop undermining the star under the pressure of gravity. The condition that must be achieved in order to dwarf relativistic to remain stable under the pressure of gravity its attractiveness is determined by the relation (3.5.11).

This criterion can be rewritten

$$M < M_c \tag{3.5.12}$$

Where

$$M_c = \frac{15}{64}(5\pi)^{1/2} \frac{(\hbar c/G)^{1/2}}{m_p^2} = 1.72 M_s \tag{3.5.13}$$

(The Chandrasekhar limit)

Is known as the Chandrasekhar limit, after Chandrasekhar who first derived it in 1931. A more realistic calculation, which does not assume constant density, yields

$$M_c = 1.4\, M_s \tag{3.5.14}$$

Thus, if the star mass exceeds the Chandrasekhar limit then the star in question cannot become a white dwarf when its nuclear fuel is exhausted, but, instead, must continue to collapse.what is ultimate fate of such a star [90, 91, 92].

It follows that the radius of a typical solar mass white dwarf is about 7000 *Km*, i.e., about the same as the radius of Earth. At stellar densities which greatly exceed white dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white-dwarf is effectively transformed into a gas of neutrons. Eventually, the mean separation between the neutrons becomes comparable with their de Broglie wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star. A star which is maintained against gravity in this manner is called a neutron star [93, 94, 95, 96].

We see that the white dwarf has the maximum mass $M = M_c$, which it cannot exceed without collapsing to a neutron star or a black hole. This is called the Chandrasekhar limit, and even more detailed calculations give $M_c = 1.4 M_s$.

Neutrons stars can be analyzed in a very similar manner to white-dwarf stars .In fact, the previous analysis can be simply modified by letting $m_p \rightarrow m_p/2$ and $m \rightarrow m_p$. Thus, we conclude that non-relativistic neutrons stars satisfy the mass-radius law [87]:

$$\frac{R_{eq}}{R_s} = 0.000011 \left(\frac{M_s}{M}\right)^{1/3} \tag{3.5.15}$$

It follows that the radius of a typical solar mass neutron star is a mere $10Km$. In 1967 Antony Hewish and Jocelyn Bell discovered a class of com pact radio sources, called pulsars, which emit extremely regular pulses of radio waves. Pulsars have subsequently been identified as rotating neutron stars [97, 98, 99, 100]. To date, many hundreds of these objects have been observed.

When relativistic effects are taken into account, it is found that there is a critical mass above which a neutron star cannot be maintained against gravity. According to our analysis, this critical mass, which is known as the Oppenheimer-Volkoff limit, is given by

$$M_{ov} = 4M_c = 6.9M_s \tag{3.5.16}$$

A more realistic calculation, which does not assume constant density, does not treat the neutrons as point particles, and takes general relativity is taken into account, gives a somewhat lower value of

$$M_{ov} = 1.5 \;—\; 2.5\, M_s \tag{3.5.17}$$

A star whose mass exceeds the Oppenheimer-Volkoff limit cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a black hole [101, 102, 103].

Chapter Four

Literature Review

4.1 Introduction:

General relativity is of one of the big achievement that describes cosmological and astronomical phenomena, successfully. However it fails in describing the behavior of some exotic objects like black holes and neutrons stars [104, 105, 106, 107, 108]. Different attempts were made to cure the defects of general relativity [109, 110]. In this chapter some of these attempts were presented here.

4.2 Gravitational Self Energy Mass and A model Based on Generalized General Relativity:

Was constructed by M. Dirar and others [47]. The attempt is based on a more generalized field equation which generalizes EGR. This generalized general relativity (GGR) was first obtained by Lanczos [13] and then Ali El-Tahir [14]. In the later derivation the principle of least action is utilized by taking the field variables to be the metric tensor $g_{\mu\upsilon}$. This conventional approach leads to the generalized general relativity in the form:

$$L'''\left[g_{\mu\upsilon}g^{\rho\sigma}R_{;\rho}R_{;\sigma}-R_{;\mu}R_{;\upsilon}\right]+L''\left[g_{\mu\upsilon}\Box^{2}\,R-R_{;\mu;\upsilon}\right]-L'R_{\mu\upsilon}+\frac{1}{2}g_{\mu\upsilon}L=0 \quad (4.2.1)$$

$$L'=\frac{\partial L}{\partial R}$$

Where the lagrangian L depends on R.This equation reduces to EGR by considering the linear Lagrangian

$$L=\beta R+\gamma \qquad , \qquad \beta=\frac{1}{16\pi G} \quad (4.2.2)$$

Equation (4.2.1) then reduces to

$$R_{\mu\upsilon}-\frac{1}{2}g_{\mu\upsilon}R=\frac{g_{\mu\upsilon}\gamma}{2\beta}=8\pi G g_{\mu\upsilon}\gamma \quad (4.2.3)$$

One can set

$$g_{\mu\upsilon}\gamma=T_{\mu\upsilon}(m)$$

Where, $T_{\mu\upsilon}(m)$, stands for the matter energy momentum tensor. In this case equation (4.2.1) reduces to general relativity where

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R = -8\pi g T_{\mu\upsilon}(m) \tag{4.2.4}$$

The fact that EGGR reduces to EGR indicates that this new Einstein's version shares with GGR all its successes [111]. Motivated by the quadratic Lagrangian of the electro-magnetic field, a Lagrangian quadratic in R was utilized by some authors to construct a useful gravitational equation. This equation is used for static field to obtain nonsingular solution, and a solution reduced to Schwarzschild solution [15]. The EGGR cosmological model is also constructed and found to share with EGR all its successes. This model is free from the singularity, flatness, entropy and horizon problems [111]. Moreover, this model can also solve the galaxy formation problem [112]. Recently EGGR is utilized to express a quantum model using conventional quantum mechanics. This model is capable to predicting the universe expanding at its early stage [113]. The EGGR model shows that Einstein's general relativity can still be capable of rearranging and refurnishing itself to describe physical phenomena.

4.3 Gravitational Self Energy Mass:

Since the mass of anybody generates gravitational field thus one expects the inverse process to take place i.e. the gravitational field frozen out to generate masses. This resembles what happens in the pair production, where a photon generates a pair of a particle and the anti-particle annihilate to form a photon. To see how the gravitational field generates masses field generates masses one can utilize the contracted form of the generalized general relativity, i.e. equation (4.2.1) to get

$$\Box^2 R = \frac{\beta}{6\alpha} R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha} \tag{4.3.1}$$

Where the Lagrangian takes the form

$$L = \alpha R^2 + \beta R + \gamma$$

$$\Box^2 R = g^{\mu\upsilon} R_{;\mu;\upsilon} = g^{\mu\upsilon} \partial_\upsilon R_{;\mu} - g^{\mu\upsilon} \Gamma^{\lambda}_{\mu\upsilon} R_{;\lambda} \tag{4.3.2}$$

$$R_\mu = \partial_\mu R = \frac{\partial R}{\partial x^\mu} \tag{4.3.3}$$

Using the coordinate condition

$$\Gamma^{\lambda} = g^{\mu\upsilon}\Gamma^{\lambda}_{\mu\upsilon} = 0 \tag{4.3.4}$$

Equation (4.3.1) reduces to

$$g^{\mu\upsilon}\partial_{\mu\upsilon}R = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha} \tag{4.3.5}$$

To describe the behavior of a certain star it is suitable to use static isotropic metric:

$$g_{rr} = A(r), \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = g_{\theta\theta}\sin^2\theta, \quad g_{tt} = -B(r) \tag{4.3.6}$$

The scalar curvature R is a function of r only in this case, i.e.

$$R = R(r) \tag{4.3.7}$$

Thus the only no vanishing terms in this case are the $r - r$ components. Thus equation (4.3.5) reduces to

$$g^{rr}\partial_{rr}R = \frac{\beta}{6\alpha}R + \frac{\gamma}{3\alpha}$$

$$\frac{1}{A(r)}\frac{d^2R}{dr^2} = \frac{1}{\sqrt{A}}\frac{d}{dr}\frac{dR}{\sqrt{A}dr} = \frac{\beta}{6\alpha}R + \frac{\gamma}{3\alpha} = \frac{\beta}{6\alpha}\left(R + \frac{2\gamma}{\beta\alpha}\right) \tag{4.3.8}$$

To simplify this equation, it is convenient to define the variables

$$dx = \sqrt{A}dr \qquad , \qquad f = R + \frac{2\gamma}{\beta} \tag{4.3.9}$$

To get

$$\frac{d^2f}{dx^2} = \frac{\beta}{3\alpha}f \tag{4.3.10}$$

One of the possible solutions of this equation is in the form

$$f = c_0 e^{\sqrt{\frac{\beta}{6\alpha}}x} \tag{4.3.11}$$

In view of equation (4.3.9):

$$R + \frac{2\gamma}{\beta} = c_0 e^{\sqrt{\frac{\beta}{6\alpha}}\int\sqrt{A}dr} \tag{4.3.12}$$

$$R = c_0 e^{\sqrt{\frac{\beta}{6\alpha}}\int\sqrt{A}dr} - \frac{2\gamma}{\beta} \tag{4.3.13}$$

The relation between the scalar curvature and the matter density ρ can be found by using the contracted form of general relativity, equation (4.2.4) to get

$$R = 8\pi G T^{\lambda}_{\lambda} = -8\pi G\rho \tag{4.3.14}$$

Thus (4.3.12) reads

$$8\pi G\rho = c_0 e^{\sqrt{\frac{\beta}{6\alpha}}\int \sqrt{A}dr} - \frac{2\gamma}{\beta} \qquad (4.3.15)$$

When the mass is considered to be generated by gravitational field only, the contribution of no gravitational field via the term γ is ignored i.e. $\gamma = 0$ Thus, equation (4.3.15) becomes

$$\rho = \frac{c_0}{8\pi g}\ e^{\sqrt{\frac{\beta}{6\alpha}}\int \sqrt{A}dr} \qquad (4.3.16)$$

To simplify treatments one can consider flat space or nearly flat space metric, where

$$A(r) \approx 1 \qquad (4.3.17)$$

In this case equation (4.3.16) becomes

$$\rho = \frac{c_0}{8\pi G} e^{\sqrt{\frac{\beta}{6\alpha}}r} = \frac{c_0}{8\pi G} e^{\gamma_0 r} \quad ; \quad \gamma_0 = \sqrt{\frac{\beta}{6\alpha}} \qquad (4.3.18)$$

The physical meaning of the γ can be understood by resolving equation (4.3.10) i.e.

$$f'' = r^2 f \qquad (4.3.19)$$

In the form

$$f = c_1 \sin(\gamma_0 r) \qquad (4.3.20)$$

This represents a wave number

$$k = \frac{2\pi}{\lambda} = \gamma_0 \qquad (4.3.21)$$

Consider now a particle in the form of a small tiny string of the length r_0. The mass of this string can be found from equation (4.3.18) to be

$$m = \frac{c_0}{8\pi G}\int_0^{r_0} e^{kr} dr = \frac{c_0}{8\pi G}\left[\frac{1}{k}\right][e^{kr_0} - 1] \qquad (4.3.22)$$

For very small r_0:

$$e^{kr_0} = 1 - kr_0 \quad , \quad m = \frac{c_0 r_0}{8\pi G} \qquad (4.3.23)$$

If the particle is in the form of sphere of volume

$$V = \frac{4\pi}{3}r^3 \quad , \quad dV = 4\pi r^2 dr$$

The mass then taken form

$$m = \int_0^{r_0} \rho dV = \frac{4\pi c_0}{8\pi G} \int_0^{r_0} r^2 \, e^{\gamma r} dr$$

$$m = \frac{c_0}{2G} \left[\frac{r^2}{\gamma} e^{\gamma r} \right]_0^{r_0} - \frac{2c_0}{2G\gamma} \int_0^{r_0} r^2 \, e^{\gamma r} dr$$

$$= \frac{c_0}{2G} \left[\frac{r_0^2}{\gamma} e^{\gamma r_0} \right] - \frac{c_0}{G\gamma} \left[\frac{r e^{\gamma r}}{\gamma} \right]_0^{r_0} + \frac{c_0}{G\gamma^2} \int_0^{r_0} e^{\gamma r} \, dr$$

$$= \frac{c_0 r_0^2}{2G\gamma} e^{\gamma r_0} - \frac{c_0}{G\gamma^2} [r_0 e^{\gamma r_0}] + \frac{c_0}{G\gamma^3} [e^{\gamma r}]_0^{r_0}$$

$$m = \frac{c_0 r_0^2}{2\gamma G} e^{\gamma r_0} - \frac{c_0 r_0 e^{\gamma r_0}}{G\gamma^2} + \frac{c_0}{G\gamma^3} [e^{\gamma r_0} - 1]$$

If r_0 is very small such that $\gamma \, r_0 \ll 1, \; c_0 r_0 \ll 1$ In this case, $e^{\gamma r_0} = 1 + \gamma \, r_0$. Hence the mass can be given to be

$$m = c_0 r_0^2 [1 + \gamma r_0] - \frac{c_0 r_0}{G\gamma^2} [1 + \gamma r_0] + \frac{c_0}{G\gamma^3} [r r_0]$$

$$m = \frac{c_0 r_0^3}{2G} - \frac{c_0 r_0^2}{2G\gamma} = - \frac{c_0 r_0^2}{2G\gamma} \qquad (4.3.24)$$

Where, one neglects the terms r compared to r_0.Thus the mass is dependent on the radius r, gravitational coupling constant G, the wave number γ beside the free parameter c_0. The mass of the string equation (4.3.23) is finite for very small radius which is true also for equation (4.3.24) when the particle considered as a sphere. If c_0 is positive then one expects the string to represent the particle while the sphere representing an anti-particle. The mass in both versions is directly proportional to the radius, thus increasing with the radius. This in conformity with experiments where the larger the radius the large the mass as in case of protons and neutrons compared to electrons. The mass is also dependent on the gravitational constant G which is also quite natural as for as the mass general gravitational field and is also affected by it. However, the mass of spherical bodies depends on the wave number γ unlike the string mass which is free from this term. The free parameter c_0 provides as with freedom to adjust its value to give as the value of all elementary particles.

4.4 Short Range Gravitation Field Equation Solution:

Let us now see how the field of a certain star looks like within the framework of the generalized field equation [114]. Contracting the generalized field equation (4.2.1) by $g_{\mu\upsilon}$ yields

$$\square^2 R = \frac{L'R - 2L - L'''\, R_{;\rho} R_{;\sigma}}{3L''} + \frac{1}{6L''} T^{\mu}_{\mu(m)} - \frac{2}{3L''}\gamma_{\upsilon} \tag{4.4.1}$$

The model is different from general relativity if we add to the lagrangian terms of higher order. The simplest lagrangian is the one which consists of a quadratic term beside the linear term

$$L = -\alpha R^2 + \beta R + \gamma$$

The contracted equation (4.4.1) thus becomes

$$\square^2 R = \frac{\beta}{6\alpha} R + \frac{\rho - 3P}{12\alpha} + \frac{\gamma_{\upsilon}}{3\alpha} \tag{4.4.2}$$

The field of any isolated star can be described by a static isotropic metric. We now use the static isotropic spherical coordinate (t, r, θ, ϕ) in the invariant proper time interval [115].

$$ds^2 = -g_{\mu\upsilon} dx^{\mu} dx^{\upsilon}$$

$$ds^2 = -(A(r)dr^2 + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2 - B(r)dt^2) \tag{4.4.3}$$

Where $\quad g_{\mu\upsilon} = 0 \;;\quad \mu = \upsilon$

$$g_{rr} = A(r)\;,\qquad g_{\theta\theta} = r^2\;,\qquad g_{\phi\phi} = g_{\theta\theta}\sin^2\theta\;,\qquad g_{tt} = -B(r) \tag{4.4.4}$$

$$g_{\mu\upsilon} = 0 \quad for \quad \mu \neq \upsilon$$

Then by using the relationship (4.4.4) in (4.4.2) we get

$$\square^2 R = \frac{1}{A}\left[\ddot{R} - \dot{R}\left(\frac{\dot{A}}{2A} - \frac{\dot{B}}{2B} - \frac{2}{r}\right)\right] = \frac{\beta}{6\alpha} R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma_{\upsilon}}{3\alpha} \tag{4.4.5}$$

The scalar curvature R is a function of r only in this case i.e. $R = R(r)$. Thus the only no vanishing terms in this case are $r - r$ components. In the case when $R = R_0 = constant$, equation (4.4.5) yields

$$R_0 = -\left(\frac{\rho + P}{2\beta} + \frac{2\gamma_{\upsilon}}{\beta}\right) \tag{4.4.6}$$

This means that (R) feed the existence of both matter and vacuum, and we can consider matter and vacuum here as frozen gravitational field in other words. The constant background gravitational field manifests itself in the form matter and vacuum [116].

Equation (4.4.5) is very complex and highly non-liner, but it can be simplified by assuming the metric to be flat [117]. i.e.

$$A \to 1 \quad , \qquad B \to 1$$

Therefore, in the region out ide the source. equation (4.4.5) reduces to

$$\ddot{R} + \frac{2}{r}\dot{R} = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma_v}{3\alpha} \tag{4.4.7}$$

If we are outside the source $\gamma = 0$, and by setting $\beta = 0$, then equation (4.4.5) reduces to

$$\frac{d\dot{R}}{dr} + \frac{2}{r}\dot{R} = 0$$

Therefore

$$\frac{d\dot{R}}{\dot{R}} = \frac{-2}{r}dr$$

Integrating both sides yields

$$\dot{R} = \frac{c}{r^2}$$

And hence

$$R = -\frac{c}{r} + c_1$$

When we are far away from the source the space is flat, i.e. $R \to 0$ as $r \to \infty$ and as a result, $c_1 = 0$. The scalar curvature is thus given by

$$R = -\frac{c}{r}$$

Using equations

$$\frac{d\dot{R}}{dr} + \frac{2}{r}\dot{R} = 0$$

And

$$L = -\alpha R^2 + \beta R + \gamma$$

And the expression for A in a weak field, i.e.

$$A = \left(1 - \frac{2MG}{r}\right)^{-1}$$

To solve this equation consider a solution of the form

$$R = \frac{c_1}{r}\ e^{c_2 r} + R_o \tag{4.4.8}$$

Putting equation (4.4.8) in (4.4.7) yield

$$\frac{c_1 c_2^2}{r}\ e^{c_2 r} = R_0 + \frac{\beta c_1}{6\alpha r}\ e^{c_2 r} + \frac{\gamma}{3\alpha} \tag{4.4.9}$$

$$c_2 = \pm\left(\frac{\beta}{6\alpha}\right)^{1/2}, \qquad R_0 = -\frac{\gamma_v}{3\alpha}\ , \qquad R = \frac{c_1}{r} e^{-r\left(\frac{\beta}{6\alpha}\right)^{1/2}} - \frac{\gamma_v}{3\alpha}$$

Where c_2 with the plus sign is excluded by the condition $R \to 0$ as $r \to \infty$. To express R in terms of the potential φ we use quasi Minkowskian approximation [118] where

$$g^{ii} \to \eta^{i}$$

$$g^{00} = \eta^{00}\eta^{00} g_{00} = -(1 + 2\varphi)$$

$$R = g^{00} g^{ii} R_{;o;o} = g^{00} g^{ii} \frac{1}{2}\nabla^2 g_{00}$$

$$R_{;o;o} = \frac{1}{2}\nabla^2 g_{00} = -4\pi G\rho$$

$$R = 8\pi G\rho\varphi + 4\pi G\rho \tag{4.4.10}$$

In view of equations (4.4.9) and (4.4.10) the poetical gravitational we get

$$\frac{\gamma}{3\alpha} = -4\pi G\rho \quad , \quad \varphi = \frac{c_1}{8\pi G\rho r}\ e^{-\sqrt{\frac{\beta}{6\alpha}}r} \tag{4.4.11}$$

This indicates again the existence of a short range force or a possible link with strong nuclear force.

If we set

$$c_1 = 8\pi G\rho$$

Then the red shift becomes

$$z = (B)^{-\frac{1}{2}} - 1 = \left(1 + \frac{2}{r}\ e^{-\sqrt{\frac{\beta}{6\alpha}}\ r}\right)^{-1/2} - 1 \approx \left(\frac{r}{2}\right)^{1/2} e^{\frac{1}{2}\sqrt{\frac{\beta}{6\alpha}}\ r} - 1 \tag{4.4.12}$$

When we are just outside the star $\rho = 0$ and one of the possible ways to do this is to set $1/a \to 0$ and for $r = 32$ [119]. $z \approx 3$. Thus the origin of a large red shift of quasars can be explained.

This expression for the potential resembles the Yukawa potential and therefore shows the existence of a short range gravitational field or a possible link between gravitational and strong nuclear force [120].

4.5 Gravitational Waves:

General relativity predicts that large stars that move under the influence of gravitational waves broadcast. These gravitational waves, like electromagnetic waves, carrying energy away from the stars broadcast, but the energy loss rate is usually very little, difficult to be seen. But in 1975 he discovered Contact Hals and Joseph Taylor bilateral pulsating (P SR 1913 +16), a system consisting of two stars rotating neutron compression about each other, and between them a distance of at maximum radius of the Sun. According to the general relativity, the rapid movement means that the session of the system time should be decreasing scale much shorter time frame, due to broadcast a strong signal of gravitational waves. And consistent change predicted by general relativity with careful monitoring conducted by the Contact Hals and Joseph Taylor [121, 122, 123]. Let us now see the possibility of emitting gravitational waves by a certain source in empty space [31]. When we are in free space, equation (4.4.2) becomes

$$\Box^2 R = \frac{\beta}{6\alpha} R \tag{4.5.1}$$

Taking into account the coordinate condition and if we are in empty space then equation (4.5.1) reads

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial z^2}\right) R = \frac{\beta}{6\alpha} R \tag{4.5.2}$$

And one of the possible solutions is

$$R = R_m \sin(\omega t - kr) \tag{4.5.3}$$

Which leads to [31]:

$$\omega = c\left(\frac{\beta}{12\alpha}\right)^{1/2} \tag{4.5.4}$$

The non-vanishing frequency mode shows the possibility of emitting gravitational waves by a certain star where the strong field presumably dominates via the contribution of the quadratic lagrangian. From equations (4.4.10) and (4.5.3) the potential is given by

$$\varphi = \frac{R}{8\pi G\rho} - \frac{1}{2} \approx \varphi_m \sin(\omega t - kr) \tag{4.5.5}$$

The travelling wave solution agrees with the recently observed decline in the orbit period of binary pulsars which was assumed to occur because the system emitted gravitational waves. The fact that the generalized field equation with a source term reduces to general relativity in a weak field limit indicates that it shares with general relativity all its successes [32] in this limit. The solutions of the generalized field equation differ from those of general relativity in many respects. First of all the scalar curvature does not vanish outside the source. Secondly the expression for the potential shows the existence of a short range field or presumably a possible link with the strong nuclear force. On the other hand the travelling wave solution is in conformity with the recently observed declining in the orbit period of the binary pulsars.

4.6 Graviton Equation of Motion:

The graviton is the energy quantum which results from gravity quantization, which should be in the form of gravitational waves. Thus, one need to prove that the gravitational field can be propagated in the form of travelling waves. To do this consider the metric and scalar curvature to be dependent on r and t, i.e.

$$R = R(r,t) \quad , \quad g_{rr} = A(r,t) \quad , \quad g_{tt} = B(r,t) \tag{4.6.1}$$

In view of equation (4.4.6) one gets

$$g^{rr}\partial_{rr}R = g^{tt}\partial_{tt}R = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha} \tag{4.6.2}$$

$$\frac{1}{A}\frac{d^2R}{dr^2} + \frac{1}{B}\frac{d^2R}{dt^2} = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha}$$

$$\frac{1}{\sqrt{A}}\frac{d}{dr}\frac{dR}{\sqrt{A}dr} + \frac{d}{\sqrt{B}dt}\,\frac{d}{\sqrt{B}dt}R = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha}$$

Define α and t to satisfy

$$dx = \sqrt{A}\,dr \;;\; x = \int \sqrt{A}\,dr \quad , \quad d\tau = \sqrt{B}\,dt \;;\; \tau = \int \sqrt{B}\;dt \tag{4.6.3}$$

Thus, equation (4.2.10) reduces to

$$\partial_{xx}R + \partial_{\tau\tau}R = \frac{\beta}{6\alpha}R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma}{3\alpha}$$

$$= \frac{\beta}{6\alpha}\left(R + \frac{\rho - 3p}{2\beta} + \frac{2\gamma}{\beta\alpha}\right) \tag{4.6.4}$$

Outside a given star, one expects the matter density γ to vanish, hence one can consider the solution

$$R = R_0 \sin(\omega t - kx)\,, \qquad \partial_{xx}R = -k^2 R\ , \quad \partial_{rr}R = \omega^2 R\ , \ \gamma = 0 \tag{4.6.5}$$

A direct substitution of equations (4.6.3) in (4.6.4) yields

$$-(k^2 - \omega^2)R = \frac{\beta}{6\alpha}R\,, \qquad -(k^2 - \omega^2) = \frac{\beta}{6\alpha}$$

In the r, t space the travelling wave equation becomes

$$R = R_m \sin(\omega_0 t - k_0 r) \tag{4.6.7}$$

In view of equations (4.6.7), (4.6.5) and (4.6.3), the angular frequency ω_0, and the wave number k_0 are given according to the relations

$$\omega_0 t = \omega\tau = \omega \int \sqrt{B}\ dt \quad , \quad k_0 r = kx = k \int \sqrt{A}\ dr \tag{4.6.8}$$

It is clear that the frequency and the wave are dependent on the gravitational field via the metric components B and A. equation (4.6.7) also indicates that the gravitational field can be propagated in the form of a travelling wave.

4.7 Graviton Energy:

In quantum mechanics the particle is thought to be associated with a wave packet (wave group). The graviton can be treated equally as a wave packet by integrating (4.3.18) in the k-space to get

$$m = \frac{c_0}{8\pi G}\int_0^k e^{kr}\,dk = \frac{c_0}{8\pi G r}(e^{kr} - 1) \approx \frac{c_0}{8\pi G r}e^{kr} \tag{4.7.1}$$

Where, one can be neglected in equation (4.7.1) compare with exponential term. Therefore, equation (4.3.21) in which, $\gamma = k$, is used. The radius at which, m is minimum can be found by differentiating m with respect to r to get

$$\frac{dm}{dr} = -\frac{c_0}{8\pi r^2}e^{kr} + \frac{c_0 k}{8\pi G r}e^{kr} = 0 \tag{4.7.2}$$

$$r = \frac{1}{k} = \frac{\lambda}{2\pi} \quad , \quad \lambda = 2\pi r$$

Hence the circular circumference is occupied by one complete wave. This relation resembles the Bohr radius. Thus, according to equations (4.7.1) and (4.7.2), the mass and the energy of the graviton are given to be

$$m = \frac{c_0}{8\pi G} k e^{kr} \tag{4.7.3}$$

$$E = mc^2 = \frac{c_0 c^2}{8\pi G}\left(\frac{2\pi}{\lambda}\right) e^{kr}$$

$$E = \frac{c_0 c^2}{4G\lambda} e^{kr} = \frac{c_0 c}{4G} f e^{kr} = h_g f \tag{4.7.4}$$

This expression resembles plank quantum photon energy this expression can also be obtained directly from expression (4.3.18) for very small radius $r \to 0$ where the exponential term can be expanded by using Taylor series to get

$$E = \rho c^2 = \frac{c_0 c^2}{8\pi G} e^{kr_0} = \frac{c_0 c^2}{8\pi G}(kr_0 - 1)$$

$$E = \frac{c_0 c^2}{8\pi G} kr_0 + \frac{c_0 c^2}{8\pi G} = \frac{c_0 c (f\lambda)}{8\pi G} r_0 \left(\frac{2\pi}{\lambda}\right) + \frac{c_0 c^2}{8\pi G}$$

$$E = \frac{c_0 c^2}{2G} r_0 f + \frac{c_0 c^2}{8\pi G} = h_g f + E_0 \tag{4.7.5}$$

The energy of the graviton is determined in equation (4.7.5) by treating the graviton as a wave packet. It is interesting to find that the graviton mass is a function of its frequency in complete agreement with the Plank energy for the photon but with new (Plank) gravitational constant, the graviton expression for the energy can also be obtained by utilizing the expression for the mass density, when the radius r_0 is very small (see equation (4.7.5)). Again one gets Plank quantum gravity expression but with a background zero energy. This means that the vacuum E_0 is a media in which graviton transmit it. It is also amazing to find in equation (4.7.5) and that the gravity plank constant h_g is dependent on the gravitational constant G.

4.8 Spatial and Time Dependent Scalar Curvature and Gravitational Constant Quantization:

In the previous sections one tries to find R with direct suggestion of solution for R as a function of r and t directly. In this section one needs to use the separation of variables to solve (4.6.4) without the source term i.e. $\gamma = 0$ to get

$$\partial_{xx}R + \partial_{tt}R = \frac{\beta}{6\alpha}R \quad (4.8.1)$$

Using the method of separation of variables now let R to be a produce of tow function, T which depends on time t, beside X which depends on x.

$$T\partial_{xx} X + X\partial_{tt} T = \frac{\beta}{6\alpha}TX$$

Dividing both sides by TX yields

$$\frac{1}{X}\partial_{xx} X + \frac{1}{T}\partial_{tt} T = \frac{\beta}{6\alpha} \quad (4.8.2)$$

This means that the first term and the second term on the left hand side are constants. Hence one can set the time dependent part to be

$$\frac{1}{T}\partial_{tt} T = \mu^2 \quad (4.8.3)$$

This equation can be solved by suggesting T to be

$$T = c_1 e^{i\omega t}$$

Where

$$-\omega^2 = -\mu^2 \quad \text{and} \quad \omega^2 = \mu^2 \quad (4.8.4)$$

$$\frac{1}{X}\partial_{xx} X = \frac{\beta}{6\alpha} = \mu^2 \quad (4.8.5)$$

$$X = c_2 \sin kx \quad (4.8.6)$$

$$-k^2 = \frac{\beta}{6\alpha} + \mu^2$$

$$k = \sqrt{\frac{-\beta}{6\alpha} - \mu^2} = \sqrt{\frac{-\beta}{6\alpha} - \omega^2} \quad (4.8.7)$$

The term α is found by some researchers to be negative strictly speaking in the Ph.D. work of M. Dirar [111] α was found to be

$$\alpha = -\frac{\sqrt{\beta}}{\sqrt{24}} = -\frac{\sqrt{\beta}}{2\sqrt{6}} \quad , \qquad -\frac{\beta}{6\alpha} = \frac{\sqrt{\beta}\sqrt{6}}{\sqrt{3}} = \frac{\sqrt{6}}{3} \times \frac{1}{\sqrt{6}\sqrt{3\pi}\sqrt{G}} = \frac{1}{3\sqrt{3G}}$$

Hence

$$k = \sqrt{\frac{1}{3\sqrt{3}\sqrt{G}} - \mu^2} \quad (4.8.8)$$

To take the source term γ inconsideration one can define f as in (4.3.9) and substitute it in (4.6.4) to get

$$\partial_{xx} f = \partial_{tt} f = \frac{\beta}{6\alpha} f \tag{4.8.9}$$

By setting

$$f = GT \tag{4.8.10}$$

And following the same procedures done for R in equation (4.8.1) up to (4.8.8) after comparing (4.8.9) with (4.8.1) to get

$$T = c_3 e^{i\omega t} \quad , \quad G = c_4 \sin kx \tag{4.8.11}$$

$$k = \sqrt{\frac{1}{3\sqrt{3}\sqrt{G}} - \omega^2} \tag{4.8.12}$$

Recalling equation (4.3.9) yield

$$f = R + \frac{2\gamma}{\beta} = GT = c_3 c_4 e^{i\omega t} \sin kx$$

$$R = c_3 c_4 e^{i\omega t} \sin kx - \frac{2\gamma}{\beta} \tag{4.8.13}$$

In this expression for, k is real when

$$\frac{1}{3\sqrt{3}\sqrt{G}} > \omega^2 \tag{4.8.14}$$

One can quantize the gravitational field by bearing in mind that outside the universe both gravity R and matter γ vanishes outside the universe near the boundaries, i.e.

$$R = 0 \quad , \quad \gamma = 0 \quad \text{at} \quad x = x_0 \tag{4.8.15}$$

Hence equation (4.8.13) becomes

$$c_3 c_4 e^{i\omega t} \sin kx_0 = 0$$

This can be satisfied if

$$kx_0 = n\pi \quad , \quad n = 0, 1, 2, 3, \ldots$$

Hence

$$k = \frac{n\pi}{x} \sqrt{\frac{1}{3\sqrt{3}\sqrt{G}} - \omega^2} = \frac{n\pi}{x_0}$$

$$\frac{1}{3\sqrt{3}\sqrt{G}} = \frac{n^2x^2}{x_0^2} + \omega^2 \qquad , \qquad 27G = \frac{1}{\left(\frac{n^2\pi^2}{x_0^2} + \omega^2\right)^2}$$

$$G = \frac{1}{\left(\frac{n^2\pi^2}{x_0^2} + \omega^2\right)^2} \tag{4.8.16}$$

Thus the gravitational coupling is constant is quantized. At the early universe x_0 is small thus quantized takes place. However at present $x_0 \to \infty$, hence, $G = 1/27\omega^2$. Thus no quantization is observed.

Equation (4.8.16) shows that the gravitational constant G is quantized. at the early universe x_0 is small and quantized term including the discrete number n dominates. Thus, the gravitational parameter "constant" G in equation (4.8.16) is quantized and is no longer a constant. But at present $x_0 \to \infty$ and the quantized term is smeared out and G is a constant parameter and is no longer quantized.

Chapter Five

Stars Evolution, String Self Energy And Nonsingular Black Hole on the Basis of Generalized Special Relativity

5.1 Introduction:

The theory of general relativity is a model of gravity, the prediction of which leads to a remarkable change in the concepts of nature. it is now understood that in spite of the successes of this theory, it suffers from main defects reflected in its singular behavior at strong field limit .they are flawed in two ways: firstly, abnormal behavior in the strong field makes it conducive to the demolition of the same law that the adoption of it, where predict a so called gravitational collapse, and the attendant emergence of black holes. Secondly, which causes its being isolated from other physical laws, and forbid its being neither quantization nor unified with the rest of physics.

General relativity theory is one of the most successful theory that describes the universe. The big bang model describes the evolution of the universe. It states that the universe starts with singularity in space-time and then expands, where matter, i.e. elementary particles is formed at early universe. These particles join together to form light atoms. Later on these particles are assembled in a cloud forming galaxies, stars, planets and all other astronomical objects.

The formation of stars is one of most striking features of general relativity. These stars can be come white dwarfs or red giant stars, supernova or black holes. However the evolution of stars suffers from noticeable set backs, for instance the so called black holes results from space-time singularity which means break down of the laws of physics. This draw back can be cured in this chapter by using generalized special relativity.

5.2 Equilibrium Conditions:

A main-sequence hydrogen-burning star, such as the Sun, is maintained in equilibrium via the balance of the gravitational attraction tending to make it collapse, and the thermal pressure tending to make it expand. of course, the thermal energy of the star is generated by nuclear reactions occurring deep inside its core. Eventually, however, the star will run out of burnable fuel, and, therefore, start to collapse, as it radiates away its remaining thermal energy [124, 125, 126]. What is the ultimate fate of such a star.

A burnt-out star is basically a gas of electrons and ions. As the star collapses, its density increases, so the mean separation between its constituent particles decreases. Eventually, the mean separation becomes of order wavelength of the electrons, and the electron gas becomes degenerate. Note, that the wavelength of the ions is much smaller than that of the electrons, so the ion gas remains non-degenerate. Now, even at zero temperature, a degenerate electron gas exerts a substantial pressure, because the Pauli exclusion principle prevents the mean electron separation from becoming significantly smaller than the typical wavelength of the electrons. Thus, it is possible for a burnt-out star to maintain itself against complete collapse under gravity via the degeneracy pressure of its constituent electrons. Such stars are termed white-dwarfs.

At stellar densities which greatly exceed white-dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons [127, 128]. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white-dwarf is effectively transformed into a gas of neutrons. Eventually, the mean separation between the neutrons becomes comparable with their wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star. A star which is maintained against gravity in this manner is called a neutron star.it is found that there is a critical mass and critical radius above which a neutron star cannot be maintained against gravity. this critical radius, which is known as the radius of Schwarzschild [129]. A star whose radius exceeds the radius of Schwarzschild. cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a black hole. One will discuss in this section the equilibrium of stars by pressure and gravity forces within the framework of generalized special relativity.

Generalized special relativistic energy expression, beside Fermi momentum and ordinary Newtonian gravity potential are used for stars equilibrium conditions.

Consider first the generalized special relativity (GSR) energy E equilibrium condition by minimizing E w.r.t r

$$E = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{5.2.1}$$

$$\varphi = -\frac{GM}{r} \quad , \quad m_0 = M \tag{5.2.2}$$

$$\frac{v^2}{c^2} = \frac{m^2 v^2}{m^2 c^2} = \frac{p^2}{m^2 c^2} = \frac{p^2}{M^2 c^2} \tag{5.2.3}$$

For simplicity consider the average momentum p is equal to the maximum momentum p_F, by ignoring $\sqrt{2}$, where

$$p = \frac{p_F}{\sqrt{2}}$$

Thus

$$p = p_F = \Lambda \left(\frac{N}{V}\right)^{1/3} = \Lambda n_0$$

Where

$$\Lambda = (3\pi^2)^{1/3} \hbar \tag{5.2.4}$$

Therefore, with the aid of equations (5.2.2) – (5.2.4), equation (5.2.1) reads

$$E = E_F = Mc^2 \left(1 - \frac{2MG}{rc^2}\right)\left(1 - \frac{2MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right)^{-1/2} \tag{5.2.5}$$

The radius r which makes the energy E minimum is given when

$$\frac{dE_r}{dr} = \frac{Mc^2 \left(\frac{2MG}{r^2 c^2}\right)}{\left(1 - \frac{2MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right)^{1/2}} + \frac{Mc^2 \left(1 - \frac{2MG}{rc^2}\right)\left(-\frac{1}{2}\right)\left(\frac{2MG}{r^2 c^2}\right)}{\left(1 - \frac{2MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right)^{3/2}} = 0$$

$$\frac{Mc^2 \left[\left(\frac{2MG}{r^2 c^2}\right)\left(1 - \frac{2MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right) - \left(\frac{MG}{r^2 c^2}\right)\left(1 - \frac{2MG}{rc^2}\right)\right]}{\left(1 - \frac{2MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right)^{3/2}} = 0$$

$$\frac{M^2 c^2 G}{r^2 c^2}\left(-1 + 2 + \frac{4MG}{rc^2} - \frac{p_F^2}{M^2 c^2}\right) = 0 \tag{5.2.6}$$

This is satisfied when

$$\frac{4MG}{rc^2} = \frac{p_F^2}{M^2 c^2} - 1 \tag{5.2.7}$$

Thus the minimum radius is given by

$$r = \frac{4M^3G}{p_F^2 - M^2c^2} \tag{5.2.8}$$

Where

$$p_F = (3\pi^2)^{1/3}\hbar n^{1/3} = \Lambda \left(\frac{N}{V}\right)^{1/3} = \left(\frac{9\pi}{4}\right)^{1/3} \frac{N^{1/3}}{r_F}\hbar \tag{5.2.9}$$

The equilibrium takes place when r is non negative, i.e. when

$$p_F^2 > M^2c^2$$

$$p_F > Mc \tag{5.2.10}$$

The critical mass is given by

$$M_c = \frac{p_F}{c} \tag{5.2.11}$$

Thus for star to be at equilibrium one requires

$$\begin{gathered} \frac{p_F}{c} > M \\ M_c > M \\ M < M_c \end{gathered} \tag{5.2.12}$$

Thus the maximum mass for stable star is

$$M_c = \frac{p_F}{c} = \frac{(3\pi^2)^{1/3}\hbar}{c}\left(\frac{N}{V}\right)^{1/3} \tag{5.2.13}$$

This condition resembles Chandrasekhar limit for stable white dwarf. i.e. the star mass need to be less than the critical value in equation (5.2.11). The equilibrium condition can also be found by using generalized special relativity energy momentum relation

$$g_{00}\, E^2 - p^2c^2 = m_0^2c^4g_{00}^2$$

$$E^2 = g_{00}^{-1}\, p^2c^2 + g_{00}\, m_0^2c^4 \tag{5.2.14}$$

One can rewrite equation (5.2.14) to be

$$E = (a_1 - a_2p^2)^{1/2} \tag{5.2.15}$$

Where

$$a_1 = g_{00}m_0^2c^4 = \left(1 - \frac{2MG}{rc^2}\right)m_0^2c^4 \quad , \quad a_2 = g_{00}^{-1}c^2$$

$$a_2p^2 = a_1\cos^2\theta \tag{5.2.16}$$

$$E = \int_0^{p_F} (a_1 - a_1\cos^2\theta)^{1/2}dp \tag{5.2.17}$$

Where

$$-dp = \sqrt{\frac{a_1}{a_2}}\ \sin\theta d\theta \tag{5.2.18}$$

$$E = \sqrt{a_1}\int (1-\cos^2\theta)^{1/2}\left(-\sqrt{\frac{a_1}{a_2}}\right)\sin\theta\, d\theta \tag{5.2.19}$$

$$= \sqrt{a_1}\left(-\sqrt{\frac{a_1}{a_2}}\right)\int \sin^2\theta\, d\theta \tag{5.2.20}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$E = \frac{\sqrt{a_1}}{2}\left(-\sqrt{\frac{a_1}{a_2}}\right)\left(\theta - \frac{\sin 2\theta}{2}\right) \tag{5.2.21}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \quad \cos\theta = \sqrt{\frac{a_2}{a_1}}\ p$$

$$\sin\theta = (1-\cos^2\theta)^{1/2} = \left[1 - \frac{a_2}{a_1}p^2\right]^{1/2}$$

$$E = \sqrt{a_1}\sqrt{\frac{a_1}{a_2}}\sqrt{\frac{a_2}{a_1}}\ p_F(1-a_3p_F^2)^{1/2} + \cos^{-1}\sqrt{\frac{a_2}{a_1}}\ p_F - \frac{\pi}{2}$$

$$E = \sqrt{a_1}\, p_F(1-a_3p_F^2)^{1/2} + \cos^{-1}\sqrt{\frac{a_2}{a_1}}\ p_F - \frac{\pi}{2} \tag{5.2.22}$$

Where

$$a_3 = \frac{a_2}{a_1} = \frac{g_{00}^{-1}\, c^2}{g_{00}m_0^2c^4} = \frac{g_{00}^{-2}}{m_0^2c^2}$$

$$E = \left[1 - \frac{2MG}{rc^2}\right]^{1/2} p_F\left[1 - \frac{p_F^2\, c^2}{m_0^2c^4\left(1-\frac{2MG}{rc^2}\right)^2}\right]^{1/2}$$

$$+\cos^{-1}\left[\frac{p_F c}{m_0 c^2\left(1-\frac{2MG}{rc^2}\right)}\right]-\frac{\pi}{2} \qquad (5.2.23)$$

It is clear from equation (5.2.23) that stability requires E to be real. This can be satisfied when

$$1-\frac{2MG}{rc^2}>0$$

$$rc^2>2MG$$

$$r>\frac{2MG}{c^2}$$

The critical radius is given by

$$r_c=\frac{2MG}{c^2} \qquad (5.2.24)$$

Thus the radius should be greater than the black hole radius. Also

$$1-\frac{p_F^2\,c^2}{m_0^2c^4\left(1-\frac{2MG}{rc^2}\right)^2}>0$$

$$m_0^2c^4\left(1-\frac{2MG}{rc^2}\right)^2>p_F^2\,c^2$$

Thus

$$m_0c^2\left(1-\frac{2MG}{rc^2}\right)>\pm\,p_Fc$$

$$\left(1-\frac{2MG}{rc^2}\right)>\pm\frac{p_Fc}{m_0c^2}$$

$$rc^2-2MG>\pm\left(\frac{p_Fc}{m_0c^2}\right)rc^2 \qquad (5.2.25)$$

$$\left(1\pm\frac{p_Fc}{m_0c^2}\right)rc^2>2MG$$

$$r>\frac{2MGm_0}{(m_0c^2\pm p_Fc)} \qquad (5.2.26)$$

Thus the critical radius is given by

$$r_c=\frac{2Mm_0G}{(m_0c^2\pm p_Fc)} \qquad (5.2.27)$$

The equilibrium mass also satisfies

$$2MG > rc^2 \pm \left(\frac{p_F c}{m_0 c^2}\right) rc^2$$

$$M < \frac{rc^2}{2G} \pm \frac{p_F rc}{m_0} \tag{5.2.28}$$

Hence the critical maximum mass is given by

$$M_c = \frac{rc^2}{2G} \pm \frac{p_F rc}{m_0} \tag{5.2.29}$$

The equilibrium condition can also be found by minimizing E, where

$$E = mc^2 = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{5.2.30}$$

Assuming the mass to be equal to the rest mass, and the potential to be the Newtonian, one gets

$$m_0 = M \qquad , \qquad \varphi = -\frac{GM}{R} \tag{5.2.31}$$

Therefore

$$E = Mc^2 \left(1 - \frac{2GM}{Rc^2}\right)\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{5.2.32}$$

For small φ and velocity v compared to speed of light c, i.e.

$$\frac{GM}{R} < 1 \qquad , \qquad \frac{v^2}{c^2} < 1$$

One gets

$$E = Mc^2 \left(1 - \frac{2GM}{Rc^2}\right)\left(1 + \frac{GM}{Rc^2} + \frac{1}{2}\frac{v^2}{c^2}\right) = \left(Mc^2 - \frac{2GM^2}{R}\right)\left(1 + \frac{GM}{Rc^2} + \frac{1}{2}\frac{v^2}{c^2}\right)$$

$$E = Mc^2 + \frac{GM^2}{R} + \frac{1}{2}Mv^2 - \frac{2GM^2}{R} - \frac{2G^2M^3}{R^2c^2} - \frac{GM^2v^2}{Rc^2} \tag{5.2.33}$$

The mass which make the energy minimum for constant radius is given by

$$\frac{dE}{dM} = \frac{c^2\left(1 - \frac{2GM}{Rc^2}\right)}{\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{1/2}} + \frac{Mc^2\left(\frac{-2G}{Rc^2}\right)}{\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{1/2}} + \frac{\frac{1}{2}Mc^2\left(1 - \frac{2GM}{Rc^2}\right)\left(\frac{2G}{Rc^2}\right)}{\left(1 - \frac{2GM}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}} \tag{5.2.34}$$

Neglecting the kinetic term yields

$$\frac{dE}{dM}=\frac{\left(c^2-\frac{4MG}{R}\right)\left(1-\frac{2MG}{Rc^2}\right)+\frac{MG}{R}-\frac{2M^2G^2}{R^2c^2}}{\left(1-\frac{GM}{Rc^2}+\frac{1}{2}\frac{v^2}{c^2}\right)^{3/2}}=0 \qquad (5.2.35)$$

This requires

$$c^2-\frac{2MG}{R}-\frac{4MG}{R}+\frac{8M^2G^2}{R^2c^2}+\frac{MG}{R}-\frac{2M^2G^2}{R^2c^2}=0$$

$$c^2-\frac{5MG}{R}+\frac{6M^2G^2}{R^2c^2}=0$$

$$\frac{6G^2}{R^2c^2}M^2-\frac{5G}{R}M+c^2=0$$

$$ax^2+bx+c=0 \quad , \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$M=\frac{\frac{5G}{R}\pm\sqrt{\left(\frac{5G}{R}\right)^2-\frac{24G^2c^2}{R^2c^2}}}{\frac{12G^2}{R^2c^2}}=\frac{R^2c^2}{12G^2}\left(\frac{5G}{R}\pm\sqrt{\frac{G^2}{R^2}}\right)$$

$$M=\frac{R^2c^2}{12G^2}\left(\frac{G}{R}\right)(5\pm1)=\frac{Rc^2}{12G}(5\pm1)$$

$$M=\frac{1}{2}\frac{Rc^2}{G} \quad , \quad \frac{1}{3}\frac{Rc^2}{G} \qquad (5.2.36)$$

For stars one have two forces, pressure force which counter balance the gravity force, thus

$$P=\frac{NKT}{V}=\frac{1}{3}\frac{mv^2}{V} \qquad (5.2.37)$$

Thus the pressure force is given by

$$F_P=PA=\frac{\frac{1}{3}mv^2(4\pi r^2)}{\frac{4\pi}{3}r^3}=\frac{mv^2}{r} \qquad (5.2.38)$$

The gravity force is given by

$$F_g=\frac{GmM}{r^2} \qquad (5.2.39)$$

At equilibrium the two forces counter balances them selves thus

$$F_P=F_g$$

$$\frac{mv^2}{r} = \frac{GmM}{r^2} \quad , \quad mv^2 = \frac{GmM}{r} \tag{5.2.40}$$

If particles are considered as strings with v representing max speed. Thus the average value is given by

$$v_a = \frac{v_m}{\sqrt{2}} \quad , \quad mv_a^2 = \frac{mv_m^2}{2} \tag{5.2.41}$$

Thus

$$mv_a^2 = \frac{mv_m^2}{2} = \frac{1}{2}mv^2 \tag{5.2.42}$$

One thus gets

$$\frac{1}{2}mv^2 = \frac{GmM}{r} = m\varphi \tag{5.2.43}$$

Hence

$$v^2 = 2\varphi \tag{5.2.44}$$

Hence

$$E = \frac{m_0c^2\left(1 + \frac{2\varphi}{c^2}\right)}{\left(1 + \frac{2\varphi - v^2}{c^2}\right)^{1/2}} = m_0c^2\left(1 + \frac{2\varphi}{c^2}\right) \tag{5.2.45}$$

But

$$m_0 = M \quad , \quad \varphi = -\frac{GM}{R} \tag{5.2.46}$$

For attractive force

$$E = M\left(c^2 - \frac{2GM}{R}\right) \tag{5.2.47}$$

$$\frac{dE}{dM} = \left(c^2 - \frac{2GM}{R}\right) + M\left(\frac{-2G}{R}\right) = 0$$

$$-\frac{4GM}{R} + c^2 = 0$$

$$\frac{4GM}{R} = c^2 \tag{5.2.48}$$

$$M = \frac{Rc^2}{4G} \tag{5.2.49}$$

$$M = \frac{R}{2G}c_a^2 = \frac{R}{2G}\left(\frac{c_m}{\sqrt{2}}\right)^2 = \frac{R}{2G}c^2 \tag{5.2.50}$$

For

$$c \to c_a = \frac{c_m}{\sqrt{2}} = \frac{c}{\sqrt{2}}$$

5.3 Conditions of Stars Evolution:

Consider a short range repulsive gravity field derived by some others of the form [130]:

$$\varphi_s = \frac{c_1}{r} e^{-c_0 r} \qquad (5.3.1)$$

$$\varphi_L = -\frac{GM}{r} \qquad (5.3.2)$$

$$\varphi = \varphi_s + \varphi_L = \frac{c_1}{r} e^{-c_0 r} - \frac{GM}{r}$$

$$\varphi = \frac{1}{r}[c_1 e^{-c_0 r} - GM] \qquad (5.3.3)$$

$$E = mc^2 = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \qquad (5.3.4)$$

For small r or strictly speaking small $c_0 r$:

$$e^{-c_0 r} = 1 - c_0 r \qquad (5.3.5)$$

$$\varphi = \frac{1}{r}[c_1 - c_1 c_0 r - GM] \qquad (5.3.6)$$

To make φ finite, one needs

$$c_1 = GM \qquad (5.3.7)$$

Thus

$$\varphi = -\frac{c_1 c_0 r}{r} = -c_0 c_1 = -c_0 GM \qquad (5.3.8)$$

If one assumes that for $r \to 0$ energy is minimum, i.e.

$$\frac{dE}{d\varphi} = 0 \qquad (5.3.9)$$

It follows that

$$\frac{dE}{d\varphi} = \frac{\left(\frac{2}{c^2}\right) m_0 c^2}{\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{1/2}} - \frac{\frac{1}{2} \times \frac{2}{c^2}\left(1 + \frac{2\varphi}{c^2}\right) m_0 c^2}{\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{3/2}} = 0$$

$$\frac{m_0\left(2+\frac{4\varphi}{c^2}-\frac{2v^2}{c^2}-1-\frac{2\varphi}{c^2}\right)}{\left(1+\frac{2\varphi}{c^2}-\frac{v^2}{c^2}\right)^{3/2}}=0$$

$$\frac{2\varphi}{c^2}-\frac{2v^2}{c^2}+1=0$$

$$\varphi=v^2-\frac{c^2}{2} \qquad (5.3.10)$$

For particle at rest

$$v=0$$

$$\varphi=-\frac{c^2}{2} \qquad (5.3.11)$$

For photon

$$v=c$$

$$\varphi=\frac{c^2}{2} \qquad (5.3.12)$$

Since the star is a particle at rest thus (see equation (5.3.8))

$$\varphi=-c_0GM=-\frac{c^2}{2} \qquad (5.3.13)$$

Thus

$$c_0=\frac{c^2}{2GM} \qquad (5.3.14)$$

$$\varphi_s=\frac{GM}{r}e^{-\frac{c^2}{2GM}r} \qquad (5.3.15)$$

It is very interesting to note that for large star having very large mass, such that

$$\frac{c^2}{2GM}r<1 \qquad (5.3.16)$$

$$r<\frac{2GM}{c^2} \qquad (5.3.17)$$

Equation (5.3.15) yields

$$\varphi_s=\frac{GM}{r}\left(1-\frac{c^2}{2GM}r\right)=\frac{GM}{r}-\frac{c^2}{2} \qquad (5.3.18)$$

If

$$\varphi = -\varphi_s = \frac{c^2}{2} - \frac{GM}{r} \tag{5.3.19}$$

Thus the short range gravity potential reduces numerically to long range gravity potential beside zero point gravity potential corresponding to rest mass energy

$$E_0 = m|\varphi| = \frac{1}{2}mc^2 \tag{5.3.20}$$

It is very interesting to note that this is consistent with the zero point energy of harmonic oscillator

$$E_0 = \frac{1}{2}\hbar\omega \tag{5.3.21}$$

Since it means that

$$mc^2 = \hbar\omega \tag{5.3.22}$$

Which conforms with the Einstein and Planck expressions of energy. It also conforms with De Broglie hypothesis that

$$p = mc = \frac{mc^2}{c} = \frac{\hbar\omega}{c} = \frac{h}{\lambda} = \hbar k \tag{5.3.23}$$

This also agrees with quantum hypothesis where

$$\Psi = Ae^{i(kx-\omega t)} = Ae^{\frac{i}{\hbar}(px-Et)} \tag{5.3.24}$$

This means that at early stage of star evolution when

$$r \to 0 \tag{5.3.25}$$

the particles constituting the star be as a string.

It is also very interesting to note that for ordinary classical particle which is at rest equations (5.3.11) and (5.3.4) yields

$$E = \frac{\left(1 + \frac{2\varphi}{c^2}\right)m_0c^2}{\sqrt{1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}}} = \frac{(1-1)m_0c^2}{\sqrt{1 - 1 - \frac{v^2}{c^2}}} = \frac{0}{0}m_0c^2 \tag{5.3.26}$$

While for a photon which obeys quantum laws equations (5.3.12) and (5.3.4) gives

$$E = \frac{2m_0c^2}{\sqrt{2-1}} = 2m_0c^2 \tag{5.3.27}$$

This conforms with the fact that photons can produce particle pairs such that one can consider a star as consisting of photons gas. When minimizing E w.r.t r

$$E = m_0c^2\left(1-\frac{2MG}{rc^2}\right)\left(1-\frac{2MG}{rc^2}-\frac{v^2}{c^2}\right)^{-1/2} \tag{5.3.28}$$

When the star particles speed are small compared to speed of light

$$\frac{v^2}{c^2} \ll 1$$

Thus

$$E = m_0c^2\left(1-\frac{2MG}{rc^2}\right)^{1/2} \tag{5.3.29}$$

$$\frac{dE}{dr} = m_0c^2\left(\frac{2MG}{r^2c^2}\right)\left(\frac{1}{2}\right)\left(1-\frac{2MG}{rc^2}\right)^{-1/2} = \frac{m_0c^2\left(\frac{MG}{r^2c^2}\right)\left(1-\frac{2MG}{rc^2}\right)}{\left(1-\frac{2MG}{rc^2}\right)^{3/2}} = 0$$

Thus the radius which makes E minimum is given by

$$1-\frac{2MG}{rc^2} = 0$$

$$\frac{2MG}{rc^2} = 1$$

$$r = \frac{2MG}{c^2} \tag{5.3.30}$$

(This is the black hole radius)

Thus the potential is given by

$$\varphi = -\frac{MG}{r} = -\frac{MGc^2}{2MG} = -\frac{c^2}{2} \tag{5.3.31}$$

It is very striking to note that this value is typical to the value of φ for

$$r \to 0 \tag{5.3.32}$$

Where in classical limit equation (5.3.30) gives

$$c \to \infty$$

$$r = \frac{2MG}{c^2} \to 0 \tag{5.3.33}$$

Also when energy is minimum

$$E = Mc^2 \to 0 \tag{5.3.34}$$

$$M \to 0$$

$$r = \frac{2MG}{c^2} \to 0 \tag{5.3.35}$$

The stability condition requires also minimization of E with respect to star mass. consider the star as a gas consisting of particles with rest mass m_0. By assuming that each particle is subjected to the effect of attractive gravitational potential

$$\varphi = -\frac{MG}{R} \tag{5.3.36}$$

The energy of m_0 is given by

$$E = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2}$$

$$E = m_0 c^2 \left(1 - \frac{2MG}{Rc^2}\right)\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{5.3.37}$$

minimizing E w.r.t M yields

$$\frac{dE}{dM} = m_0 c^2 \left[\frac{-\frac{2G}{Rc^2}}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{1/2}} + \frac{\left(1 - \frac{2MG}{Rc^2}\right)\left(-\frac{1}{2}\right)\left(\frac{-2G}{Rc^2}\right)}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}}\right] = 0$$

Thus

$$\frac{-\frac{2G}{Rc^2}\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right) + \frac{G}{Rc^2}\left(1 - \frac{2MG}{Rc^2}\right)}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}} = 0$$

If one consider

$$v^2 \ll c^2$$

$$-\frac{2G}{Rc^2}\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right) + \frac{G}{Rc^2}\left(1 - \frac{2MG}{Rc^2}\right) = 0$$

$$-\frac{G}{Rc^2}\left(1 - \frac{2MG}{Rc^2}\right) = 0$$

This requires

$$\frac{2MG}{Rc^2} = 1 \tag{5.3.38}$$

Thus the mass which makes E minimum is

$$M = \frac{Rc^2}{2G} \tag{5.3.39}$$

The gravitational energy takes the form

$$\varphi = -\frac{MG}{R} = -\frac{c^2}{2} \tag{5.3.40}$$

According to general relativity (GR) and the standard big bang model the universe expand or contract when

$$E = + \qquad \text{expansion}$$

$$E = - \qquad \text{contraction}$$

One can use the same argument for the star, on the basis of generalized special relativity expression for energy E, which is given by

$$E^2 = g_{00}^{-1}\, p^2c^2 + g_{00}\, m_0^2c^4 \tag{5.3.41}$$

The conditions of star evolution can be started by adopting classical limit, where

$$\frac{\varphi}{c^2} \ll 1 \qquad , \qquad \frac{v^2}{c^2} \ll 1$$

To get

$$E = m_0c^2\left(1+\frac{2\varphi}{c^2}\right)\left(1+\frac{2\varphi}{c^2}-\frac{v^2}{c^2}\right)^{-1/2}$$

$$E = m_0c^2\left(1+\frac{2\varphi}{c^2}\right)\left(1-\frac{\varphi}{c^2}+\frac{v^2}{2c^2}\right)$$

Neglecting higher order terms, yields

$$E = m_0c^2\left(1-\frac{\varphi}{c^2}+\frac{v^2}{2c^2}+\frac{2\varphi}{c^2}-\frac{2\varphi^2}{c^4}+\frac{\varphi v^2}{c^4}\right)$$

$$E = m_0c^2 + m_0\varphi + \frac{1}{2}m_0v^2$$

$$E = m_0c^2 + V + T_K \tag{5.3.42}$$

Assuming the kinetic energy is due to thermal motion

$$U = \frac{3}{2}KT = \frac{1}{2}m_0v^2 \tag{5.3.43}$$

Assuming also the potential energy of mass m_0 to be

$$V = -\frac{Gm_0M}{R} \tag{5.3.44}$$

Thus the energy E become

$$E = m_0c^2 + \frac{3}{2}KT - \frac{Gm_0M}{R} \tag{5.3.45}$$

The star explode and expand when

$$E = m_0 c^2 + \frac{3}{2} KT - \frac{G m_0 M}{R} > 0 \tag{5.3.46}$$

i.e. when

$$m_0 c^2 + \frac{3}{2} KT > \frac{G m_0 M}{R} \tag{5.3.47}$$

This is quite obvious from the point of view of common since this equation indicates that expansion happen when thermal and rest mass energies exceeds attractive gravity energy. However it collapse and contract when

$$m_0 c^2 + \frac{3}{2} KT < \frac{G m_0 M}{r} \tag{5.3.48}$$

$$E = g_{00}^{1/2} \left(g_{00}^{-2}\, p^2 c^2 + m_0^2 c^4\right)^{1/2} \tag{5.3.49}$$

Consider the case

$$g_{00}^{-2}\; p^2 c^2 \gg m_0^2 c^4 \tag{5.3.50}$$

$$E = g_{00}^{1/2}\, g_{00}^{-1}\, pc \left(1 + g_{00}^2 \frac{m_0^2 c^4}{p^2 c^2}\right)^{1/2}$$

$$E = g_{00}^{-1/2}\, pc \left(1 + \frac{1}{2} g_{00}^2 \frac{m_0^2 c^4}{p^2 c^2}\right)$$

$$E = g_{00}^{-1/2}\, pc \left(\frac{2p^2 c^2 + \; g_{00}^2 m_0^2 c^4}{2p^2 c^2}\right)$$

But

$$\frac{3}{2} KT = \frac{1}{2} m v^2 = \frac{1}{2} \frac{m^2 v^2}{m} = \frac{p^2}{2m}$$

Thus the momentum takes the form

$$p^2 = 3mKT \tag{5.3.51}$$

Therefore

$$E = g_{00}^{-1/2} \left[\frac{6mc^2 KT + \left(1 - \frac{2MG}{Rc^2}\right)^2 m_0^2 c^4}{2pc}\right] \tag{5.3.52}$$

If

$$\varphi = \frac{MG}{Rc^2} < 1 \tag{5.3.53}$$

$$\left(1-\frac{2MG}{Rc^2}\right)^2 = 1-\frac{4MG}{Rc^2}+\frac{4M^2G^2}{R^2c^4} \approx 1-\frac{4MG}{Rc^2} \tag{5.3.54}$$

$$E = g_{00}^{-1/2}\left[\frac{6mc^2KT + {m_0}^2c^4 - \frac{4MGm_0^2c^4}{Rc^2}}{2pc}\right] \tag{5.3.55}$$

Therefore the star explodes when

$$\frac{1}{3}m_0^2c^4 + 2mc^2KT - \frac{4}{3}\frac{MG}{R}\ m_0^2c^2 > 0 \tag{5.3.56}$$

When one assumes

$$m \approx m_0$$

$$\frac{1}{3}m_0c^2 + 2KT > \frac{4}{3}\frac{MGm_0}{R} \tag{5.3.57}$$

But the star collapse when

$$\frac{1}{3}m_0c^2 + 2KT - \frac{4}{3}\frac{MGm_0}{R} < 0 \tag{5.3.58}$$

i.e. when

$$\frac{1}{3}m_0c^2 + 2KT < \frac{4}{3}\frac{MGm_0}{R} \tag{5.3.59}$$

When

$$g_{00}^{-2}\ p^2c^2 \ll m_0^2c^4 \tag{5.3.60}$$

$$E = g_{00}^{1/2}\ m_0c^2\left[1+\frac{g_{00}^{-2}\ p^2c^2}{m_0^2c^4}\right]^{1/2}$$

$$E = g_{00}^{1/2}\ m_0c^2\left[1+\frac{g_{00}^{-2}\ p^2c^2}{2m_0^2c^4}\right]$$

$$E = g_{00}^{1/2}\ m_0c^2\left[\frac{2m_0^2c^4g_{00}^2 + p^2c^2}{2m_0^2c^4}\right]g_{00}^{-2} \tag{5.3.61}$$

Using equation (5.3.51) gives

$$E = g_{00}^{-3/2}\left[\frac{2m_0^2c^4\left(1-\frac{2MG}{Rc^2}\right)^2 + 3mc^2KT}{2m_0c^2}\right] \tag{5.3.62}$$

With the aid of the approximation in equation (5.3.54), one gets

$$E = \frac{g_{00}^{-3/2}}{2m_0c^2}\left[2m_0^2c^4 - \frac{8GM}{Rc^2}\ m_0^2c^4 + 3mc^2KT\right] \tag{5.3.63}$$

$$E = \frac{g_{00}^{-3/2}}{2m_0c^2}\left[2m_0^2c^4 + 3mc^2KT - \frac{8GM}{R}\ m_0^2c^2\right] \tag{5.3.64}$$

Thus explosion is expected when

$$E > 0 \tag{5.3.65}$$

i.e.

$$2m_0^2c^4 + 3mc^2KT - \frac{8GMm_0^2c^2}{R} > 0 \tag{5.3.66}$$

$$\frac{2}{3}m_0c^2 + KT > \frac{8}{3}\ \frac{GMm_0}{R} \tag{5.3.67}$$

While contraction takes place when

$$E < 0 \tag{5.3.68}$$

$$\frac{2}{3}m_0c^2 + KT < \frac{8}{3}\ \frac{GMm_0}{R} \tag{5.3.69}$$

5.4 Evolution of Stars By Kinetic Theory and Quantum Physics:

We employed Gibbs distribution relation and quantum laws to deal with the translational degrees of freedom of the constituent particles, and quantum mechanics to deal with the non-translational degrees of freedom. Let us now discuss ideal gases from a purely quantum mechanical standpoint. It turns out that this approach is necessary to deal with either low temperature or high density gases. At stellar densities which greatly exceed white dwarf densities, the extreme pressures cause electrons to combine with protons to form neutrons. Thus, any star which collapses to such an extent that its radius becomes significantly less than that characteristic of a white dwarf is effectively transformed into a gas of neutrons [87, 88]. Eventually, the mean separation between the neutrons becomes comparable with their wavelength. At this point, it is possible for the degeneracy pressure of the neutrons to halt the collapse of the star [86]. A star which is maintained against gravity in this manner is called a neutron star [125, 128, 129]. it is found that there is a critical mass and critical radius it equivalent the radius of Schwarzschild. above which a neutron star cannot be maintained against gravity. And also cannot be maintained against gravity by degeneracy pressure, and must ultimately collapse to form a black hole. One will discuss in this section the evolution of stars by kinetic theory and quantum physics the basis generalized special relativity.

From the kinetic theory and quantum physics, we can get an equation of star evolution by the pressure force and the force of gravity. for stars one have tow forces, pressure force which counter balance the gravity force, thus

$$P = \frac{1}{3} n m v^2 \quad , \quad m v^2 = 3KT \tag{5.4.1}$$

The number density can be assumed to satisfy Maxwell's distribution

$$n = n_0 e^{-\beta E} \tag{5.4.2}$$

We first consider an ideal gas consisting of a single type of non-relativistic particles. The ideal-gas law for the gas contained in a volume V is commonly written as

$$P = \frac{1}{3} \frac{N}{V} (3KT) = nKT \tag{5.4.3}$$

where $n = \frac{N}{V}$ is the number of particles per unit volume.

Thus the pressure force is given by

$$F_P = PA = nKT(4\pi r^2) = 4\pi nKT r^2 = c_1 r^2 \tag{5.4.4}$$

The gravity force is given by

$$F_g = \int \left(\frac{4\pi}{3} \rho r^3\right) (4\pi r^2 \rho) dr$$

For constant density

$$F_g = \frac{(4\pi)^2}{3} \rho^2 \int_0^r r^5 dr = \frac{1}{6} \frac{(4\pi)^2}{3} \rho^2 r^6$$

Thus

$$F_g = \frac{8}{9} \pi^2 \rho^2 r^6 = c_2 r^6 \tag{5.4.5}$$

Equation of hydrostatic equilibrium requires

$$F_P = F_g \tag{5.4.6}$$

Thus from equations (5.4.4), (5.4.5) and (5.4.6) one gets

$$c_1 r^2 = c_2 r^6 \quad \Rightarrow \quad r = \left(\frac{c_1}{c_2}\right)^{1/4}$$

The critical radius is thus given by

$$r_c = \left(\frac{9nKT}{2\pi\rho^2}\right)^{1/4} \tag{5.4.7}$$

Expansion takes place

$$F_P > F_g \tag{5.4.8}$$

While contraction is observed when

$$F_P < F_g \tag{5.4.9}$$

But according to the laws of quantum mechanics for particle in box the energy is given by

$$E = c_0 V^{-2/3} \tag{5.4.10}$$

At $T = 0$ all quantum states whose energy is less than the Fermi energy E_F are filled. The Fermi energy corresponds to a Fermi momentum $p_F = \hbar k_F$ is thus given by

$$E_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m} \tag{5.4.11}$$

The above expression can be rearranged to give

$$k_F = (3\pi^2 n)^{1/3} = \frac{\Lambda}{\hbar}\left(\frac{N}{V}\right)^{1/3}$$

Where

$$\Lambda = (3\pi^2)^{1/3}\hbar$$

Hence

$$\lambda_F = \frac{2\pi}{k_F} = \frac{2\pi}{(3\pi^2 n)^{1/3}} = \frac{2\pi\hbar}{\Lambda}\left(\frac{V}{N}\right)^{1/3}$$

Which implies that the De-Broglie wavelength λ_F corresponding to the Fermi energy is of order the mean separation between particles $(V/N)^{1/3}$. All quantum states with De-Broglie wavelengths $\lambda > \lambda_F$ are occupied at $T = 0$, whereas all those with $\lambda < \lambda_F$ are empty.

According to equation (5.4.11), the Fermi energy at $T = 0$ takes the form

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} = \frac{\Lambda^2}{2m}\left(\frac{N}{V}\right)^{2/3} = c_0 V^{-2/3} \tag{5.4.12}$$

Where

$$\Lambda = (3\pi^2)^{1/3}\hbar$$

$$\frac{dE_F}{dV} = -\frac{2}{3}c_0 V^{-5/3} \qquad (5.4.13)$$

But for spherical body

$$V = \frac{4\pi}{3}r^3$$

Thus

$$\frac{dE_F}{dV} = -\frac{2}{3}c_0\left(\frac{4\pi}{3}r^3\right)^{-5/3} = -\frac{2}{3}\left(\frac{4\pi}{3}\right)^{-5/3} c_0 r^{-5} \qquad (5.4.14)$$

But according to canonical Gibbs's distribution

$$P = n\frac{dE_F}{dV} \qquad (5.4.15)$$

Hence the pressure takes the form

$$P = -\frac{2}{3}\left(\frac{4\pi}{3}\right)^{-5/3} nc_0 r^{-5} = c_1 r^{-5} \qquad (5.4.16)$$

Thus the pressure force is given by

$$F_P = P(4\pi r^2)$$

$$F_P = -\frac{2}{3}\left(\frac{4\pi}{3}\right)^{-5/3} nc_0 r^{-5}(4\pi r^2) = 4\pi c_1 r^{-3}$$

$$F_P = c_2 r^{-3} \qquad (5.4.17)$$

But gravity force is given by

$$F_g = \frac{GmM}{r^2} = \frac{4\pi r^3 \rho GM}{3r^2}$$

Where

$$m = \frac{4\pi}{3}r^3$$

Thus

$$F_g = \frac{4}{3}\pi\rho G m r = c_3 r \qquad (5.4.18)$$

Equation of hydrostatic equilibrium requires

$$F_P = F_g \qquad (5.4.19)$$

i.e.

$$c_2 r^{-3} = c_3 r$$

The critical radius r_c is thus given by

$$r_c^4 = \frac{c_2}{c_3}$$

$$r_c = \left(\frac{c_2}{c_3}\right)^{1/4} = \left(\frac{-(4\pi)^{-5/3}\, nc_0(4\pi)}{2\pi\rho mG\,(3)^{-5/3}}\right)^{1/4} \tag{5.4.20}$$

Expansion takes place

$$F_P > F_g \tag{5.4.21}$$

While contraction happens when

$$F_P < F_g \tag{5.4.22}$$

The conditions of star evolution can be started by adopting classical limit, of generalized special relativity (GSR) energy relation where

$$E = m_0c^2\left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \tag{5.4.23}$$

Considering Newtonian potential and thermal motion

$$\varphi = -\frac{GM}{R} \quad , \quad \frac{1}{2}mv^2 = \frac{3}{2}KT \quad \Rightarrow \quad v^2 = \frac{3KT}{m_0} \tag{5.4.24}$$

$$E = mc^2 = m_0c^2\left(1 - \frac{2GM}{Rc^2}\right)\left(1 - \frac{2GM}{Rc^2} - \frac{3KT}{m_0c^2}\right)^{-1/2} \tag{5.4.25}$$

If the gravitational potential and thermal energy are every where small, so

$$\frac{2GM}{Rc^2} \ll 1 \quad , \quad \frac{3KT}{m_0c^2} \ll 1 \tag{5.4.26}$$

Thus (5.4.25) reduces to

$$E = m_0c^2\left(1 - \frac{2GM}{Rc^2}\right)\left(1 + \frac{GM}{Rc^2} + \frac{3KT}{2m_0c^2}\right) \tag{5.4.27}$$

Neglecting higher order terms, yields

$$E = m_0c^2\left(1 + \frac{GM}{Rc^2} + \frac{3KT}{2m_0c^2} - \frac{2GM}{Rc^2} - \frac{2G^2M^2}{R^2c^4} - \frac{3GMKT}{Rm_0c^4}\right)$$

Thus the energy E become

$$E = m_0c^2 + \frac{3}{2}KT - \frac{GMm_0}{R} \tag{5.4.28}$$

Assuming the kinetic energy is due to thermal motion

$$K.E = \frac{3}{2}KT \tag{5.4.29}$$

Assuming also the potential energy of mass m_0 to be

$$V = -\frac{GMm_0}{R} \tag{5.4.30}$$

Thus equation (5.4.28) gives

$$E = m_0c^2 + K.E + V$$

Thus the expression of energy includes the total kinetic energy of the degenerate electrons (the kinetic energy of the ion is negligible), the rest energy m_0c^2 and the gravitational potential energy V. Let us assume, for the sake of simplicity, that the density of the star is its uniform. The total energy of a star is its gravitational potential energy, its internal energy and its kinetic energy (due to bulk motions of gas inside the star, not the thermal motions of the gas particles).

Using the hypothesis of universe expansion, the star explode and expand when the energy E is positive

$$E = m_0c^2 + \frac{3}{2}KT - \frac{Gm_0M}{R} > 0 \tag{5.4.31}$$

i.e.

$$m_0c^2 + \frac{3}{2}KT > \frac{Gm_0M}{R} \tag{5.4.32}$$

This is quite obvious from the point of view of common sense because this equation indicates that expansion happen when thermal and rest mass energies exceeds attractive gravity energy. However it collapse and contract when the energy E is negative, this requires

$$m_0c^2 + \frac{3}{2}KT < \frac{Gm_0M}{R} \tag{5.4.33}$$

Thus collapse take place when gravity energy exceeds thermal one.

Can be obtained the critical radius, using the following energy for generalized special relativity

$$E = m_0c^2\left(1 - \frac{2GM}{rc^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{3KT}{m_0c^2}\right)^{-1/2}$$

$$c_1 = -GM$$

$$E = m_0 c^2 \left(1 + \frac{2c_1}{rc^2}\right)\left(1 + \frac{2c_1}{rc^2} - \frac{3KT}{m_0 c^2}\right)^{-1/2}$$

$$E = m_0 c^2 (1 + c_2 r^{-1})\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right)^{-1/2} \qquad (5.4.34)$$

Where

$$c_2 = \frac{2c_1}{c^2}$$

The critical radius of the star requires minimizing the total energy E and can be found by using the conditions for minimum value, i.e.

$$\frac{dE_r}{dr} = \frac{-m_0 c^2 c_2\, r^{-2}}{\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right)^{1/2}} + \frac{\frac{1}{2} m_0 (1 + c_2 r^{-1})(c^2 c_2 r^{-2})}{\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right)^{3/2}} = 0$$

$$\frac{-m_0 c^2 c_2\, r^{-2}\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right) + \frac{1}{2} m_0 (1 + c_2 r^{-1})(c^2 c_2 r^{-2})}{\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right)^{3/2}} = 0$$

$$-m_0 c^2 c_2\, r^{-2}\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right) + \frac{1}{2} m_0 (1 + c_2 r^{-1})(c^2 c_2 r^{-2}) = 0$$

$$m_0 (1 + c_2 r^{-1})(c^2 c_2 r^{-2}) = 2 m_0 c^2 c_2\, r^{-2}\left(1 + c_2 r^{-1} - \frac{3KT}{m_0 c^2}\right)$$

$$1 + c_2 r^{-1} = 2 + 2c_2 r^{-1} - \frac{6KT}{m_0 c^2}$$

$$2c_2 r^{-1} - c_2 r^{-1} = \frac{6KT}{m_0 c^2} - 1$$

$$c_2 r^{-1} = \frac{6KT - m_0 c^2}{m_0 c^2}$$

When temperature is neglected, i.e. when

$$T = 0$$

One gets

$$c_2 r^{-1} = -1$$

$$r = -c_2 = \frac{-2c_1}{c^2} = \frac{2MG}{c^2}$$

The critical radius is thus given by

$$r_c = \frac{2GM}{c^2} \tag{5.4.35}$$

(This is the black hole radius)

Using the generalized special relativity energy relation

$$E = m_0c^2\left(1 - \frac{2GM}{rc^2}\right)\left(1 - \frac{2GM}{rc^2} - \frac{3KT}{m_0c^2}\right)^{-1/2} \tag{5.4.36}$$

For star having spherical shape:

$$-GM = -G\left(\frac{4\pi}{3}\rho r^3\right) = -\frac{4\pi}{3}G\rho r^3 = c_3r^3 \tag{5.4.37}$$

$$E = m_0c^2\left(1 + \frac{2c_3r^2}{c^2}\right)\left(1 + \frac{2c_3r^2}{c^2} - \frac{3KT}{m_0c^2}\right)^{-1/2}$$

$$E = m_0c^2(1 + c_4r^2)\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right)^{-1/2} \tag{5.4.38}$$

Where

$$c_4 = \frac{2c_3}{c^2}$$

The radius of the star r that dimension which reduces the total energy E and his can be found by using the minimum energy condition that has to be less energy as soon as possible, i.e.

$$\frac{dE_r}{dr} = 0 \tag{5.4.39}$$

$$\frac{dE_r}{dr} = \frac{m_0c^2(2c_4r)}{\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right)^{1/2}} - \frac{m_0c^2(1 + c_4r^2)\left(\frac{1}{2}\right)(2c_4r)}{\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right)^{3/2}} = 0$$

$$\frac{2m_0c^2c_4r\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right) - m_0c^2c_4r(1 + c_4r^2)}{\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right)^{3/2}} = 0$$

$$2m_0c^2c_4r\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right) - m_0c^2c_4r(1 + c_4r^2) = 0$$

$$2m_0c^2c_4r\left(1 + c_4r^2 - \frac{3KT}{m_0c^2}\right) = m_0c^2c_4r(1 + c_4r^2)$$

$$2\left(1 + c_4 r^2 - \frac{3KT}{m_0 c^2}\right) = 1 + c_4 r^2$$

$$2 + 2c_4 r^2 - \frac{6KT}{m_0 c^2} = 1 + c_4 r^2$$

$$c_4 r^2 = \frac{6KT}{m_0 c^2} - 1$$

$$r^2 = \frac{6KT - m_0 c^2}{m_0 c^2 c_4} = \frac{6KT - m_0 c^2}{2m_0 c_3}$$

The minimum radius

$$r = \left(\frac{6KT - m_0 c^2}{2m_0 c_3}\right)^{1/2}$$

For r to be real

$$6KT > m_0 c^2 \qquad (5.4.40)$$

$$m_0 < \frac{6KT}{c^2}$$

Thus the critical mass is given by

$$m_{0_c} = \frac{6KT}{c^2}$$

Hence for equilibrium

$$m_0 < m_{0_c}$$

Using equation (5.4.37)

$$c_3 = \frac{4\pi}{3} G\rho$$

The critical radius is thus given by

$$r_c = \left(\frac{6KT - m_0 c^2}{\frac{8}{3} m_0 \pi G\rho}\right)^{1/2} \qquad (5.4.41)$$

$$\frac{d^2E}{dr^2}=\left\{\frac{2m_0c^2c_4}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{1/2}}-\frac{2m_0c^2c_4^2r^2}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}-\frac{m_0c^2c_4(1+3c_4r^2)}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}\right.$$

$$\left.+\frac{3m_0c^2c_4^2r^2(1+c_4r^2)}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{5/2}}\right\}$$

$$\frac{d^2E}{dr^2}=\frac{2m_0c^2c_4\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}-\frac{[2m_0c^2c_4^2r^2+m_0c^2c_4(1+3c_4r^2)]}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}$$

$$+\frac{3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}$$

$$=\frac{m_0c^2c_4-6c_4KT-3m_0c^2c_4^2r^2+3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}$$

For maximum values

$$\frac{d^2E}{dr^2}<0 \tag{5.4.42}$$

$$\frac{m_0c^2c_4-6c_4KT-3m_0c^2c_4^2r^2+3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}}{\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{3/2}}<0$$

$$m_0c^2c_4-6c_4KT-3m_0c^2c_4^2r^2+3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}<0$$

$$\left[3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}\right]<[6c_4KT+3m_0c^2c_4^2r^2-m_0c^2c_4]$$

$$\left[3m_0c^2c_4^2r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}\right]<\left[m_0c^2c_4\left(\frac{6KT}{m_0c^2}+3c_4r^2-1\right)\right]$$

$$\left[c_4r^2(1+c_4r^2)\left(1+c_4r^2-\frac{3KT}{m_0c^2}\right)^{-1}\right]<\left[\frac{2KT}{m_0c^2}+c_4r^2-\frac{1}{3}\right]$$

$$c_4 r^2(1 + c_4 r^2) < \left(\frac{2KT}{m_0 c^2} + c_4 r^2 - \frac{1}{3}\right)\left(1 + c_4 r^2 - \frac{3KT}{m_0 c^2}\right)$$

When temperature is neglected, i.e. when

$$T = 0$$

$$c_4 r^2(1 + c_4 r^2) < \left(c_4 r^2 - \frac{1}{3}\right)(1 + c_4 r^2)$$

$$c_4 r^2 + c_4^2 r^4 < c_4 r^2 + c_4^2 r^4 - \frac{1}{3} - \frac{1}{3} c_4 r^2$$

$$\frac{1}{3} c_4 r^2 + \frac{1}{3} < 0$$

$$c_4 r^2 + 1 < 0$$

$$r^2 < -\frac{1}{c_4}$$

$$r < \left(\frac{1}{c_4}\right)^{1/2}$$

Where

$$c_4 = \frac{2c_3}{c^2} = \qquad , \qquad c_3 = \frac{4\pi G\rho}{3}$$

$$c_4 = \frac{8\pi G\rho}{3c^2} \tag{5.4.43}$$

$$r < c\left(\frac{3}{8\pi G\rho}\right)^{1/2}$$

While contraction takes place when

$$r < \sqrt{3}\, c(8\pi G\rho)^{-1/2} \tag{5.4.44}$$

For minimum values

$$\frac{d^2E}{dr^2} > 0 \tag{5.4.45}$$

Thus explosion is expected when

$$r > \sqrt{3}\, c(8\pi G\rho)^{-1/2} \tag{5.4.46}$$

Thus the critical radius is given by

$$r_c = \sqrt{3}\, c(8\pi G\rho)^{-1/2} \tag{5.4.47}$$

5.5 Generation of Elementary Particles Inside Black Holes:

Our vision for the beginning of universe is on the basis of the fundamental forces unify at the beginning of time. Unification forces leads to the answer to the most important questions in the cosmology: How did the creation and how space and time begin. A proper unification of all interactions should include the fundamental constants, representing the four basic Interactions, gravitational, electromagnetic beside nuclear interactions. An appropriate combinations of the physical constant that determine interaction nature provide a proper description of the anticipated unified interaction. We have found that these fundamental constants describe completely our universe, at all stages. One will discuss in this section generation of elementary particles inside black holes at Planck's time.

Generalized special relativistic energy (GSR) expression, beside ordinary Newtonian gravity potential are given by

$$E = m_0 c^2 \left(1 + \frac{2\varphi}{c^2}\right)\left(1 + \frac{2\varphi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2} \quad (5.5.1)$$

Where the Newtonian potential takes the form

$$\varphi = -\frac{MG}{R} \quad (5.5.2)$$

$$E = m_0 c^2 \left(1 - \frac{2MG}{Rc^2}\right)\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{-1/2} \quad (5.5.3)$$

Minimizing E w.r.t M yields

$$\frac{dE}{dM} = m_0 c^2 \left[\frac{-\frac{2G}{Rc^2}}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{1/2}} + \frac{\left(1 - \frac{2MG}{Rc^2}\right)\left(-\frac{1}{2}\right)\left(\frac{-2G}{Rc^2}\right)}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}}\right] = 0$$

Thus

$$\frac{-\frac{2G}{Rc^2}\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right) + \frac{G}{Rc^2}\left(1 - \frac{2MG}{Rc^2}\right)}{\left(1 - \frac{2MG}{Rc^2} - \frac{v^2}{c^2}\right)^{3/2}} = 0$$

If one consider

$$v^2 \ll c^2$$

$$-\frac{2G}{Rc^2}\left(1-\frac{2MG}{Rc^2}-\frac{v^2}{c^2}\right)+\frac{G}{Rc^2}\left(1-\frac{2MG}{Rc^2}\right)=0$$

$$-\frac{G}{Rc^2}\left(1-\frac{2MG}{Rc^2}\right)=0$$

This requires

$$\frac{2MG}{Rc^2}=1$$

$$2MG=Rc^2 \tag{5.5.4}$$

Thus the mass which makes E minimum is

$$M=\frac{Rc^2}{2G} \tag{5.5.5}$$

Consider also the generalized special relativity energy E equilibrium condition by minimizing E w.r.t r. From equation (5.5.3), when the star particles speed are small compared to speed of light

$$\frac{v^2}{c^2}\ll 1$$

Thus

$$E=m_0c^2\left(1-\frac{2MG}{rc^2}\right)^{1/2} \tag{5.5.6}$$

$$\frac{dE_r}{dr}=m_0c^2\left(\frac{2MG}{r^2c^2}\right)\left(\frac{1}{2}\right)\left(1-\frac{2MG}{rc^2}\right)^{-1/2}$$

$$\frac{dE_r}{dr}=\frac{m_0c^2\left(\frac{MG}{r^2c^2}\right)\left(1-\frac{2MG}{rc^2}\right)}{\left(1-\frac{2MG}{rc^2}\right)^{3/2}}=0$$

Thus the radius which makes E minimum is given by

$$1-\frac{2MG}{rc^2}=0$$

The critical radius is thus given by

$$r_c=\frac{2MG}{c^2} \tag{5.5.7}$$

(This is the black hole radius)

But the critical mass is given by equation (5.5.7), i.e.

$$M = m_c = \frac{c^2 r_c}{2G} \tag{5.5.8}$$

Hence from (5.5.8)

$$2m_c G = r_c c^2 \tag{5.5.9}$$

The condition governing the equilibrium of the universe, from (5.5.9) and (5.5.4) we get

$$\frac{m_c R}{M r_c} = 1 \tag{5.5.10}$$

Where M and R are the mass and radius of the universe respectively. The mass of the universe ($M = 2.2 \times 10^{56}\ g$) and the radius ($R = 1.6 \times 10^{28}\ cm$).

According to generalized general relativity (GGR) there is a short range repulsive gravitational force beside long range attractive gravity force given by [130]:

$$\varphi_s = \frac{c_1}{r} e^{-\frac{r}{r_c}} \tag{5.5.11}$$

$$\varphi_L = -\frac{GM}{r} \tag{5.5.12}$$

$$\varphi = \varphi_s + \varphi_L = \frac{c_1}{r} e^{-\frac{r}{r_c}} - \frac{GM}{r}$$

$$= \frac{1}{r}\left[c_1 e^{-\frac{r}{r_c}} - GM\right] \tag{5.5.13}$$

For small radius r or strictly speaking small $\frac{r}{r_c}$:

$$e^{-\frac{r}{r_c}} = 1 - \frac{r}{r_c} \tag{5.5.14}$$

Hence

$$\varphi = \frac{1}{r}\left[c_1 - c_1 \frac{r}{r_c} - GM\right] \tag{5.5.15}$$

To secure finite self energy φ at small r, one requires

$$c_1 = GM \tag{5.5.16}$$

Thus the star self energy is given by

$$\varphi = -\frac{c_1}{r_c} = -\frac{GM}{r_c} \tag{5.5.17}$$

Since the star is a particle at rest thus the minimization of E requires (see equations (5.5.2), (5.5.4) and (5.5.17))

$$\varphi = -\frac{c_1}{r_c} = -\frac{c^2}{2} \tag{5.5.18}$$

For photon ($v = c$) thus one gets

$$\varphi = \frac{c^2}{2} \tag{5.5.19}$$

From equations (5.5.17) and (5.5.18)

$$\varphi = -\frac{GM}{r_c} = -\frac{c^2}{2} \tag{5.5.20}$$

Thus the critical radius is given by

$$r_c = \frac{2GM}{c^2} \tag{5.5.21}$$

This is the black hole radius.

Since r_c should be small as shown by equation (5.5.14). Thus requires

$$r_c < 1$$

$$\frac{2GM}{c^2} < 1$$

$$M < \frac{c^2}{2G} \tag{5.5.22}$$

Thus there is a critical mass

$$M_c = \frac{c^2}{2G} \tag{5.5.23}$$

Above it the particle rest mass energy cannot be formed form potential.

We see from equation (5.5.4) that the present radius of the universe should be

$$R_0 = \frac{2GM_0}{c^2} \sim 10^{28}\ cm \tag{5.5.24}$$

Which conforms with observations.

Consider a star as consisting of photons gas, such that the critical radius is related to the wave number according to the relation

$$p = m_0 c = \hbar k = \frac{\hbar}{r_c} \qquad , \qquad k = \frac{1}{r_c} \tag{5.5.25}$$

For oscillating string the energy takes the form

$$E_{r_c} = m_0 c^2 = \frac{\hbar c}{r_c} \tag{5.5.26}$$

Hence

$$r_c = \frac{\hbar}{m_0 c} \tag{5.5.27}$$

The photon which obeys quantum laws equations (5.5.19) and (5.5.1) gives

$$E = \frac{2m_0 c^2}{\sqrt{2-1}} = 2m_0 c^2 \tag{5.5.28}$$

This conforms with the fact that photons can produce particle pairs.

Newton's law of potential gives

$$E_{r_c} = U(r) = -G\frac{m_1 m_2}{r} \tag{5.5.29}$$

Gravity force is also given by

$$F = -G\frac{m_1 m_2}{r^2}.\frac{\underline{r}}{r} \tag{5.5.30}$$

If

$$m_1 = m_2 = m_c$$

Thus (5.5.26) and (5.5.29) given

$$E_{r_c} = \frac{G\,m_c^2}{r_c} = \frac{\hbar c}{r_c} \tag{5.5.31}$$

Therefore

$$\hbar c = G m_c^2 \tag{5.5.32}$$

Hence

$$m_c = \left(\frac{\hbar c}{G}\right)^{1/2} \tag{5.5.33}$$

Where

$\hbar = 1.05 \times 10^{-27}$ *erg. s* , $c = 3 \times 10^{10}$ *cm.s*$^{-1}$, $G = 6.67 \times 10^{-8}$ *erg. cm. g*$^{-1}$

The critical mass m_c is equal

$$m_c = \left(\frac{\hbar c}{G}\right)^{1/2} \sim 2.2 \times 10^{-5}\text{ g} \tag{5.5.34}$$

(Equivalent Planck's mass)

Which matches the proposed value. The same equation applies to Planck's length, namely

$$R_P = \frac{G_P M_P}{c^2} \sim 10^{-33}\ cm \tag{5.5.35}$$

(Planck's length)

At distances smaller than this scale the gravitational interaction should be stronger than the quantum effects [131].

Also the critical distance r_c is equal

$$r_c = \frac{\hbar}{m_c c} = \left(\frac{G\hbar}{c^3}\right)^{1/2} \sim 1.6 \times 10^{-33}\ cm \tag{5.5.36}$$

(Equivalent the length of Planck)

One can calculate the critical density σ_c of the material when the particles are considered as a hollow sphere surrounded by thin layer or membrane. In this case the surface density is given by

$$\sigma = \frac{m_c}{A} \quad , \quad m_c = \frac{\hbar}{r_c c} \quad , \quad A = 4\pi r_c{}^2 \tag{5.5.37}$$

$$\sigma = \left(\frac{\hbar}{r_c c}\right)\left(\frac{1}{4\pi r_c{}^2}\right) = \frac{\hbar}{4\pi r_c^3 c} \tag{5.5.38}$$

Where

$$m_c = \frac{\hbar}{r_c c} \tag{5.5.39}$$

$$\sigma = \frac{m_c}{4\pi r_c^2} \sim 6.7 \times 10^{59}\ \text{g}.cm^{-2} \tag{5.5.40}$$

Thus the critical density satisfies

$$\sigma_c = \frac{m_c}{r_c^2} = \left(\frac{c^7}{G^3\hbar}\right)^{1/2} \sim 8.4 \times 10^{60}\ \text{g}.cm^{-2} \tag{5.5.41}$$

Where

$$\sigma_c = 4\pi\sigma$$

According to this model the universe began at a time and specific place, at the critical point (r_c , t_c), where all fundamental forces are unified into a single force. The Planck time is thus given by

$$t_c = \frac{r_c}{c} = \left(\frac{G\hbar}{c^3}\right)^{1/2}\left(\frac{1}{c}\right) = \left(\frac{G\hbar}{c^5}\right)^{1/2} \sim 5.4 \times 10^{-44}\ s \tag{5.5.42}$$

(Equivalent Planck's time)

The value speed of light c at the critical point (r_c , t_c).

$$c = \frac{r_c}{t_c} \sim 3 \times 10^{10}\ cm.s^{-1} \tag{5.5.43}$$

Began creation of the universe at the critical point (r_c , t_c), and show the fundamental constants such as ($\hbar$, c, G) known values, since that time and keep as it is without any change, the structure of the our universe is sensitive to precise degree to less change in these fundamental constants. The status of the universe at different stages is shown to be described in terms of the constants ($\hbar$, c, G) only. This masterly organization of the universe is the result for precise tuning arbitrator. The acceleration was great, which is equal to

$$a_c = R_c = \frac{c}{t_c} \tag{5.5.44}$$

Where R_c critical curvature (the maximal acceleration occurred at Planck's time).

From a purely dimensional argument one can constant a quantum acceleration from the set of fundamental constants ($\hbar$, c, G) to be valid at Planck's time, and according to our hypothesis, an analogous acceleration of the form [132]:

$$a_c = \frac{c}{t_c} = \frac{r_c}{t_c^2} = \left(\frac{c^7}{G\hbar}\right)^{1/2} \sim 5.7 \times 10^{53}\ cm.s^{-2} \tag{5.5.45}$$

Getting limited value to a larger curvature or maximal acceleration in the relation (5.5.43) resolved the problem singular behavior. and the matching bending dimensions to pry acceleration are consistent with the principles of general relativity. Conform to the critical value of the acceleration a_c in this relation with the researches results [133, 134].This acceleration on unwavering c constants, and associated critical point (r_c , t_c). The existence of this greatest acceleration confirms the occurrence of stretch accelerator of the universe at the beginning of time [135]. The acceleration declining at critical value a_c generates the force to attract at the beginning of time, when the universe takes its way to expansion, and this explains why the presence of the cosmic force of the overall attraction.

The critical force F_c as follows

$$F_c = m_c a_c = \frac{c^4}{G} \sim 1.25 \times 10^{49}\ dyne \tag{5.5.46}$$

We can find critical energy E_c that unites all fields be the rank of ($\sim 10^{19}\ GeV$):

$$E_c = m_c c^2 = \left(\frac{\hbar c^5}{G}\right)^{1/2} \sim 10^{19}\ GeV \qquad (5.5.47)$$

5.6 Discussion:

In this work generalized special relativity energy relation (5.2.1) is used by assuming the average momentum p to be related to the maximum momentum p_F, beside the ordinary expression for Newton gravity potential (see equations (5.2.2), (2.5.3) and (5.2.4)) to get the expression for E. The radius which make E minimum in (5.2.8) requires maximum mass given by (5.2.13). The condition for maximum mass resembles Chandrasekhar limit. However the expression does not depend on the gravitational constant G.

Using the same steps used in the conventional general relativity theory, E is integrated over the momentum p. A useful expression for E was found in (5.2.23). The equilibrium condition requires, here, E to be real. This makes the critical radius to be dependent on G and h as shown by equations (5.2.27) and (5.2.9). equation (5.2.26) shows that this is the minimum radius which secures equilibrium.

But according to equations (5.2.28) and (5.2.29) the maximum critical mass depends two on G and h. The pence of these tow parameters reflects the quantum gravitational nature of the Steller mass.

The equilibrium condition is also studied by considering the effect of pressure force in relation to centrifugal force. Surprisingly equations (5.2.38) shows that the pressure force act as a centrifugal force which counter balance the gravity force. By treating particles as strings it was shown by equation (5.2.37) that equilibrium takes place when kinetic and potential energy balances each other. The mass which makes E minimum also tackled in equations (5.2.34), (2.5.35) and (5.2.36). The mass at which E is minimum is given by equation conforms with that of black hole (see equation (5.2.50)).

According to generalized general relativity theory there is a short range repulsive gravitational force given by equation (5.3.1). In this work one assumes the ordinary. Attractive gravity force beside the repulsive force (see equation (5.3.3)). To find mass self energy, by assuming it resulting from gravity potential energy only, one considers the potential behavior when r approaches zero. The finiteness of φ requires the constant parameter c_1 to be given by (5.3.7). This makes φ finite and constant as shown by equation (5.3.8). Since the matter self energy is the minimum energy, thus one needs minimization of E. Minimizing E, by using generalized special relativity expression in (5.3.9) required φ to be related to the speed of light in vacuum. If one believes in Einstein energy-mass relation and Plank energy expressions for photon, equation (5.3.20). Shows that mass self energy expression resembles the zero point energy of harmonic oscillator (see equation (5.3.21)). This indicates that the matter building blocks are strings. It is also very interesting to note that the physical restrictions imposed on c_0 and c_1, indicates, according to equation (5.3.15), that the field becomes string when the mass increases and the distance decreases. This conforms with common sense and physical intuition.

Surprisingly the minimization of energy w.r.t radius and mass lead to the same radius of black hole and string nature of building blocks. This means that black holes are the state of matter where energy is minimum w.r.t to potential, mass and radius (see equations (5.3.29) – (5.3.40)).

The star evolution to be came black hole or supernova is also tackled for slow speed and weak potential. Using different approximations. For weak field and slow speed (see (5.3.42)), high speed and large momentum compared to the rest mass (see (5.3.50)) and small momentum compared to the rest mass (see (5.3.60)), the star evolution gives the same scenario. For all approximation supernova is observed when rest mass energy and thermal energy exceeds attractive gravity energy as shown by equations (5.3.47), (5.3.57) and (5.3.67). In contrary the star became a black hole when thermal and mass energy exceeds attractive gravity force. This result conforms with common sense and general relativity approach. The incorporation of rest mass beside thermal energy may be related to the fact that in the vicinity of Centre of mass repulsive gravity act as a repulsive force against attractive force.

The equilibrium radius can be found by using ordinary expression for thermal pressure (see equation (5.4.3)) and the ordinary Newtonian force relation for star having constant density. Assuming pressure and attractive gravity force to balance each other, one can find temperature and density dependent critical radius r_c, where r_c increases as temp. increase and decreases as density increase. This conforms with the fact that thermal pressure force causes contraction. The same result can be found by using quantum mechanical relation for pressure (see equations (5.4.11) - (5.4.17)), but here the increase of density increases r_c as shown by equation (5.4.20).

Using generalized special relativity energy relation the condition of expansion requires the thermal energy to exceed gravity energy, while contraction requires gravity energy to be more than thermal energy which agrees with previous models. By assuming gravity Newtonian potential relation and thermal energy for generalized special relativity energy relation (see equation (5.4.34)). The minimum energy requires the existence of critical temperature depend Ent mass similar to Chandrasekhar mass. where the star mass should be less than this mass to attain minimum energy.

Generalized special relativity energy relation used to find the mass and radius at which the energy is minimum. The two conditions leads to relate critical mass and radius to the mass and universe radius. This relation is typical to that obtained by Ibrahim and others [136], (see equation (5.5.10)).

It is also very interesting to note that according to equations (5.5.11) - (5.5.21) that the stars having short and long range gravity force have finite self energy that is formed when the radius is very small, provided that the mass should be less than a critical value. This means that only elementary particles having very small radius and very small mass can have self energy due to the transformation of potential field energy to rest mass energy, where equations (5.5.17), (5.5.19) and (5.5.20) gives:

$$V = m\varphi = \frac{mc^2}{2} = \frac{GMm}{r_c}$$

It is very interesting to note that the radius for self energy is that of black holes, which can be considered as a vent producing elementary particle. It is also very interesting note that, using quantum oscillator and relativistic energy expressions (5.5.25) and (5.5.26) beside Newtonian potential relation a useful expressions for Planck's mass, length and time are obtained in equations (5.5.34), (5.5.36) and (5.5.42). the numerical values of these parameters agrees with standard values.

5.7 Conclusion:

The generalized special relativity can successfully describe stars equilibrium condition. It shows that the equilibrium conditions are related to certain critical mass and radius values, beside effects of pressure, centrifugal force and attractive gravity force, similar to that obtained by general relativity.

generalized special relativity model is successful in describing the star evolution. It shows that the mass building blocks are strings. It also shows that black holes have minimum no zero radius. It also shows that stars became black holes when attractive force dominates, while it became supernova when thermal energy dominates. It also shows that the elementary particles have finite self energy when the radius becomes vanishingly small.

Gibbs distribution relations, quantum laws beside generalized special relativity energy relations are used to study star equilibrium conditions. It was shown that equilibrium conditions requires certain critical value for the radius typical to that of the black hole. The critical mass is shown to depend on temperature. The equilibrium also requires rest and thermal energy to be equal to potential energy. The fact that the rest mass energy is joined with thermal energy comes from the fact that rest mass energy can be converted to thermal energy.

Using generalized special relativity, quantum mechanics and Newton's laws of gravitation it is shown that elementary particles are created inside black holes at Planck time.

References:

[1] A. Einstein, “The Foundation of the General Relativity”, The Principle of Relativity, Dover Publications, Inc. Dover, 1952, pp.111-164.

[2] R. M. Wald,(1984), “General Relativity”, University of Chicago Press, Chicago, doi:10.7208
/Chicago/9780226870373.001.0001

[3] C. W. Misner, K. S. Thorne and J. A. Wheeler, “Gravitation”, W. H. Freeman & Co, San Francisco, 1973.

[4] S. Hawking, “A Brief History of Time”, Time Magazine, 1999.

[5] M. Carmeli, “Classical Fields: General Relativity and Gurage Theory”, John Wily and Sons, New York, 1992.

[6] E. Alvares, “Quantum Gravity: An Introduction to Some Recent Results”, Reviews of Modern Physics,Vol.61,No. 3, 1989, pp.561-604. doi: 10.1103/RevModphys.61.561

[7] J. A. Wheeler, “Battelle Rencontres: 1967 Lectures in Mathematics and Physics”, Benjamin, New York, 1968.

[8] J. M. Maldacena, “The Large N Limit of Super Conformal Field Theories and Super Gravity”, International Journal of Theoretical Physics, Vol. 38, No. 4, 1999, pp 111 3-1133. doi:10.1023/A:
1026654312961

[9] B. S. DeWitt, “Quantum Theory of Gravity”, Physical Review, Vol. 160, No. 5, 1967, pp. 1113-1148. doi:10.1103/PhysRev.160.1113.

[10] S. W. Hawking and D. N. Page, “Operator Ordering and the Flatness of the Universe”, Nuclear Physics B, Vol.264 , 1986, pp. 185-196. doi:10.1016/0550-3213(86)90478-5.

[11] M. Dirar, Ali EL-Tahir and M. H. Shaddad, “A Generalized Field Cosmology”, Modern Physics Letters A,Volo.37,1998,pp. 3025-3031.
doi:10.1142/S0217732398003211

[12] Q. Birkhoff, (1923), “Relativity And Modern Physics”, Harvard University Press, Cambridge, Mass.

[13] C. Lanczos, "Electricity as a Natural Property of Riemannian Geometer", Physical Review, Vol. 39, No. 4, 1932, pp. 716-736. doi:10.1103/PhysRev.39.716

[14] Ali El-Tahir, (1982) "A Generalized Variational Principle of Gravitation", Ph.D. Theses, City University, London.

[15] Ali El-Tahir, "A Generalized Metric of Gravitation", International Journal of Modern Physics, Vol. 7, No. 13,1992,p.3133.doi:10.1142/s0217751x92200140x.

[16] M.B. King, Tapping the zero-point energy, Paraclete Pub., Utah (1989).

[17] J. Davidson, "The Secret of the Creative Vacuum", Ed. C.W. Daniel, Saffron Walden, England (1989).

[18] H.E. Puthoff, "Source of Vacuum Electromagnetic zero-point Energy", Phys. Rev. A40 (1989) 4857-4862.

[19] D. Savicks,(2002), "Relation Between Newtonian Mechanics, General Relativity and Quantum Mechanics", Am. J. Phy 70 .

[20] R. Adler, M. Bazir, Schiffer, (1960), "Introduction to General Relativity", Mc Graw Hill, New York.

[21] P. Tiwari, (1984), "Physics of Free Power Generation", Crest Pub. House, New Delhi.

[22] S. Inomata, Paradigm of New Science, Ed. G. Shuppan Pub., Tokyo (1987).

[23] H.E. Puthoff, "The Energetic Vacuum", implications for energy research, Speculate. Sci. Tech. 13, No.3 (1990) 247-257.

[24] J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939)

[25] E. W. Kolb and M. S. Turner, "The Early Universe", Addison-Wesley, New York Publishing Company, 1990.

[26] R.M. Wald. "Black Holes and Thermodynamics. In Black Holes and Relativistic Stars", ed. R.M. Wald. University of Chicago Press, Chicago, 1998.

[27] Bowler M.G, (1976), "Gravitation and Relativity", Pergamum Press, Oxford.

[28] Marianne Frei Berger, "What is Time, Plus magazine", plus.maths.org, July, 2011.

[29] A. A. Logunov,(1983), "Gravitation and Elementary Particle Physics", Mir Publishers, Moscow.

[30] Weinberg. S, “Gravitation and Cosmology”, John Wiley and Sons, New York 1972.

[31] M. Dirar , Ali-Eltahir, “A New Derivation of the Generalized Field Equation with a Source Term” , ICTP Preprint. IC/96/158.

[32] S. L. Shapiro and S. A. Teukolsky, “Black Holes, White Dwarfs, and Neutron Stars”, Wiley, New York, 1983.

[33] A. Ashtekar, J. Baez and K. Krasnov, “Quantum Geometry of Isolated Horizons and Black Hole Entropy”, To appear.

[34] F. K. Lamb, “In Frontiers of Stellar Evolution”, ed. D. L. Lambert, (Astron. Soc. Pac. Conf. Ser., San Francisco, 1991), Vol. 20, p. 299.

[35] H. Heiselberg and V. Padharipande, Annu. Rev. Nucl. Part. Sci. 50, 481 (2000).

[36] P. M. Woods and C. Thompson, to appear in Compact Stellar X-ray Sources, eds. W. H. G. Lewin and M. van der Klis (Astro -ph/0406133) (2004).

[37] J.D. Bekenstein, “Black Holes and Entropy”, Phys. Rev. D7 (1973) 2333-2346.

[38] J.M. Bardeen, B. Carter and S.W. Hawking, “The Four Laws of Black Hole Mechanics. Common”, Math. Phys. 31 (1973) 161-170.

[39] R.M. Wald, (1994), “Quantum Field Theory in Curved Space-time and Black Hole Thermodynamics”, University of Chicago Press, Chicago.

[40] A. Ashtekar, J. Baez, A. Cori chi and K. Krasnov, “Quantum Geometry and Black Hole Entropy”, Phys. Rev. Lett. 80 (1998) 904-907.A. Ashtekar, A. Corichi and K. Krasnov. Isolated Black Holes: The Classical Phase Space. To appear.

[41] A. Ashtekar, C. Beetle and S. Fair Hurst, The Mechanics of Isolated Horizons”, To appear.

[42] A. Ashtekar, (1986), “New Variables for Classical and Quantum Gravity”, Phys. Rev. Lett. 57 2244-2247.

[43] M. Dirar , Ali - Eltahir, “A New Derivation of the Generalized Field Equation with a Source Term”, ICTP Preprint. IC/96/158

[44] E. W. Kolb and M. S. Turner, “The Early Universe”, Addison-Wesley, New York Publishing Company, 1990.

[45] H. Stephani, (1990), “General Relativity”, Cambridge University Press, Cambridge.

[46] M. Kenmoku, Yuko Okamoto and Kaszuyasu Shigemoto, Phys. Rev.D.48, 578 (1993).

[47] M. Dirar, M. H. Eisa1, (2013), “Gravitational Self Energy Mass and Gravitational Radiation Quantization within the Framework of the Generalized General Relativity”, International Journal of Astronomy and Astrophysics,, 3, 131-136.

[48] Dong Lai, “Tidal Stabilization of Neutron Stars and White Dwarfs Theoretical Astrophysics”, 130-33, California Institute of Technology Pasadena, CA 91125 (Accepted for publication in Physical Review Letters, 1996)

[49] G. Dillon, (2013), “Gravitational Self-Energy and Black Holes in Newtonian Physics”, University di Genova INFN, Sezione di Genova arXiv:1303.2577Vl.

[50] Paolo Christillin, (2011), “Spooky black holes : gravitational self energy and space curvature”, University di Pisa, arXiv:1112.1793V1.

[51] Abhay Ashtekar, Isolated Horizons, (1998), “A Generalization of Black Hole Mechanics”, University Park, PA 16802, ar Xiv: gr/9812065V1.December 17.

[52] R. Adler, M. Basin, and M. Schliffer, (1975), “Introduction to General Relativity”, (McGraw –Hill Book Company, Tokyo.

[53] R. Adler, M. Schiffer, “Introduction to General Relativity”, see the box experiment pp.146, (McGraw -Hill, 1975).

[54] A. S. Edding, (1924), “The Mathematical Theory of Relativity”, Cambridge University, Press.

[55] M. Gasperini, (2010), “Relativity General” e Teoria Della Gravitation, Springer.

[56] Ya. B. Zel'dovich and I. D. Novikov, (1971), “Relativistic Astrophysics”, University of Chicago Press, Chicago, Vol. 1.

[57] A. Ashtekar, (1987), “New Hamiltonian Formulation of General Relativity”,
Phys. Rev. D36. 1587-1602.

[58] L. Janossy, (1971), “Theory of Relativity Based on Physical Reality”, Akad'emiai Kaido', Budapest.

[59] Bowler M.G, (1976), “Gravitation and Relativity”, Pergamon Press, Oxford.

[60] Janos Polonyi, (2016), “Introduction to General Relativity”, University of Strasbourg , Strasbourg, France.

[61] A . Al-Mosallami, “Generalization of Schwarzschild Metric”, Malaya University , Kliala Lumpur, Malaysia.

[62] R. Penrose, phs. Rev. Letters, 14,P.57(1967). S.W. Hawking, Proc. Roy. Soc, 294A,P.511, (1966).

[63] S. Chandrasekhar, Mon. Not. Roy. Astron. Soc, 95,P.207,(1935)

[64] David. Halliday, “Robert Resnick Physics”, Third edition, (John Wiley and sons, New York, 1978).

[65] Ya. B. Zeldovich and I.D. Novikov, (1983), “The Structure and Evolution of the Universe”, Chicago University Press, Chicago.

[66] Ya. B. Zeldovich and I. D. Novikov, (1971), “Stars and Relativity, Dover Publications”, Inc. Mineola, New York.

[67] A. Ritter, Wiedemann , Ann, 11,332 (1880), E. Betti, Nuoro Cimento, 7, 26 (1880).

[68] T.R. Oppenhermer. G. M . Volkoff , Phys. Rev, 55,374 (1939).

[69] C.W. Misner and H.S. Zapolsky, phys. Rev. Letters, 12,635 (1964).

[70] F. Hoyle and W.A. Fowler, Mon. Not. Roy. Astron. Soc. 125, 169 (1963).

[71] K. Schwarzschild, Sitzung – sberichte Preuss. Akad. Wiss, 424,1916.

[72] H. Bondi, Proc. Roy, soc (London), A281, 39 (1964).

[73] S.A. Bludman and M. Ruderman, Phys. Rev, 170, 1176. (1968).

[74] G. S. Bisnovatyi-Kogan , Astrofizka, 5,223 (1969).

[75] F. Hoyle , W. Afowler , Nature, 213, 373 (1967).

[76] J.D. Bekenstein, “Black Holes and Entropy”, Phys. Rev. D7 (1973) 2333-2346.

[77] R. C. Tolman , Proc. Nat. Acad. Sei, U.S.A 20,3(1934).

[78] J. R. Oppenheimer and H. Snyder, phys. Rev. 56.455 (1939).

[79] B. Cartcr, Phys. Rev. letters, 26, 331 (1971).

[80] F. Hoyle and W. Fowler, Nature, (1963).

[81] F. J. Dyson, Comments Astrophysics. And space Phys, 1.75 (1969).

[82] C. Leibowitz and W. Israel , Phys. Rev, 1,326 (1970).

[83] R. Penrose , Phys. Rev. letters, 14.57 (1965).
[84]W. H, (1971), "Gravitational Radiation From Vibrating Black Holes has been Considered", Press, AP, J. 170,L105.
[85] A. G. W. Cameron, Nature 229, (1971).
[86] O.R. Pols. (2011), "Stellar Structure and Evolution", Astronomical Institute Utrecht.
[87] R. Fitzpatrick, (2001), "Thermodynamics and Statistical Mechanics", University of Texas at Austin.
[88] F. Reif, (1965), "Fundamentals of Statistical and Thermal Physics", (McGraw-Hill, New York NY).
[89] D. Park, (1992), "Introduction to Quantum Theory", 3rd Edition, (McGraw-Hill, New York NY,).
[90] Penrose, R. (1965), "Gravitational Collapse and Space-time Singularities", Phys. Rev. Lett, 14,57–59.
[91] Olmo, G.J. "Introduction to Palatini Theories of Gravity and Nonsingular Cosmologies", In Open Questions in Cosmology, Gonzalo, J.O., Ed, Intech Publishing: Rijeka, Croatia, 2012.
[92] Chandrasekhar, S, (1992) "The Mathematical Theory of Black Holes", Oxford University Press: New York, NY, USA.
[93] Olmo, G. J, "Quantum Gravity Phenomenology", J. Cosmol. Astropart. Phys. 2011, 2011, 018.
[94] Braginsky V.L., et al. (1957), "Laboratory Experiments to Test Relativistic Gravity Physical Review",D, 15, 2047- 2068. S. Chu. doi:10.1103/PhysRevD.15.2047.
[95] T. Damour, G. Veneziano, (2000), "Self-gravitating Fundamental Strings and Black Holes", Nucl. Phys B568, 93–119. http://www.arXiv.org/abs/hep-th/9907030hep th/9907030.
[96] J. R. Oppenheimer and G. Volkoff, Phys. Rev. 55,374 (1939).
[97] G. t'Hoft, (2009), "Introduction to the Theory of Black Holes", Lectures at the Utrecht University ITP-UU-09/11.

[98] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Math.-Phys. Tech. 424 (1916).

[99] R.M. Wald, (1998), "Black Holes and Thermodynamics In Black Holes and Relativistic Stars", ed. R.M. Wald. University of Chicago Press, Chicago.

[100] J. R. Oppenheimer and G. Volkoff, Phys. Rev. 55,374 (1939).

[101] R. C. Tolman, Relativity, (1934), "Thermodynamics and Cosmology", 1st Edition, Oxford University Press, Oxford.

[102] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Math.-Phys. Tech. 424 (1916).

[103] K. S. Thorne, (1998), "In Black Holes and Relativistic Stars", edited by R.M. Wald University of Chicago Press, Chicago.

[104] J. R. Oppenheimer and G. Volkoff, Phys. Rev. 55,374 (1939).

[105] S. Chandrasekhar,(1992), "The Mathematical Theory of Black Holes", Oxford University Press, Oxford.

[106] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Math.-Phys. Tech. 424 (1916).

[107] Lobo, F.S.N, Olmo, G.J.; Rubiera-Garcia, D. Crystal clear lessons on the microstructure of space time and modified gravity. Phys. Rev. D 2015, 91, 124001.

[108] Gravitational Waves from Neutron Stars with Large Toroidal B-fields, September 2002.

[109] Bowler M.G, (1976), "Gravitation and Relativity", (Pergamon Press, Oxford).

[110] R. Adler, M. Schiffer, (1975) "Introduction to General Relativity", see the box experiment pp.146, (Mc Grow-Hill).

[111] M. Dirar,(1998),"Applications of the Generalized Field Equations to Energy and Cosmological Problems", Ph.D. Thesis, Khartoum University, Khartoum.

[112] M. Dirar, Ali EL-Tahir and M. H. Shaddad, "Short Range Gravitational Field and the Red Shift of Quasars," Inter- national Journal of Theoretical Physics, Vol. 36, No. 6, 1997, pp. 1395-1400. doi:10.1007/BF02435932

[113] M. Y. Shargawi, (2012), "Derivation of Quantum Cosmological Model", Ph.D. Thesis, Sudan University of Science and Technology, Khartoum.

[114] E. W. Kolb and M. S. Turner, (1990), “The Early Universe”, Addison-Wesley, New York Publishing Company.
[115] D.R. Brill. Gowdy, “Quantization of General Relativity”, Rep. Phys.33, 413, (1970).
[116] A. A. Logunov, (1983), “Gravitation and Elementary Particle Physics”, Mir Publishers, Moscow.
[117] M. Kenmoku, Yuko Okamoto and Kaszuyasu Shigemoto, Phys. Rev.D.48, 578 1993).
[118] Chongming Xu, G. F. R. Ellis, Xuejun Wu and Ali El-Tahir, S. Afr. J. Phys. 15, 5 (1992).
[119] W. M. Suen and P. R. Anderson, Phs. Rev. D.35, 2940 (1987).
[120] M. B. Miji'c, M. S. Morris and W. M. Suen, Phys. Rev.D.34, 2934(1986).
[121] C. Lanczos, Phys. Rev. 31, 716 (1932).
[122] Ali Eltahir, (1982), Ph. D. Thesis ,City University, London.
[123] Derk, F. Lowden, (1982), “An Introduction to Tensor Calculus and Relativity”, (John Wiley and Sons, New York)
[124] Marianne Frei Berger, (2011), “What is Time, Plus magazine”, (plus.maths.org).
[125] M. Born, Einstein’s theory of relativity, Dover, 1962.
[126] A. Osborne, (1979), “Gravitation and Dynamical System”, PhD Thesis, The City University, London.
[127] Misner, K. S. Thone, J. A .Wheeler, (1974), “Gravitation”, Freeman & Co. San Francisco.
[128] S. Chandrasekhar, Mon . Not. Roy. Astron. Soc, 95,P.207, 1935.
[129] Schwarzschild, (1987), “Probing the Early Universe with QuasarLight”, Physics Today 40; Nov. 17-20.
[130] I. H. Hassan, (1998), M.Sc. Thesis, Sudan University of Science and Technology, Khartoum.
[131] Janos Polonyi, (2013), “Basic Equation of General Relativity”, University of Strasbourg, France.

[132] Arbab. I. Arbab, (2004), “Quantization of Gravitation System and its Cosmological Consequences”, ar Xiv. gr- qc/0309040V2
[133] R. G. Torrome, (2007), “On the Maximal Universal Acceleration”, Germany.
[134] C. Castro, (2005), “The Extended Relativity Theory in Clifford”, PHASE spaces with a lower and upper scale Foundations of Physics”, 35, no. 6971.
[135] D. Pavon, (2006), “Holographic Dark Energy and Late Cosmic Acceleration”, Spain.
[136] I. H. Hassan,(2007), Ph.D. Thesis, Sudan University of Science and Technology, Khartoum.