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Toeplitz and Hankel operators with aHandy formula and Localization Principle

مؤثرات توبليتز وهانكل مع الصيغة المفيدة ومبدأ الموضعية

A Thesis submitted in partial Fulfillment for the Degree of M.Sc. in Mathematics

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Dedication

To my father,

Mother,

Brothers,

Sisters,

And to all family.

Acknowledgments

The Thanks giving metaphors to Allah Almighty who guided me to complete this thesis. I extend my sincere thanks to the teacher of the Almighty Sea mathematics, who has long Taught us, advice rich Prof. Shawgy Hussein Abd Alla. Also I like to thank the princes who stood beside me without laze then my sister and to my older brother, and to all who helped me at the end of this project, to them my sincere thanks and sincere prayers go.

Abstract

We give a decomposition theorem for the Sobolev space of first order on the disc. Using this result, some characterizations for algebraic properties of Toeplitz or small Hankel operators with symbols in $L^{\infty,1}$ are given. We consider Toeplitz and Hankel operators with piecewise continuous generating functions on l^p-spaces and the Banach algbra generated by them. We characterize the pairs of truncated Hankel operators on the model spaces, the asymptotic behavior of the singular values of a compact Hankel operator is determined by the behavior of the symbol in a neighbourhood of its singular support.

الخلاصة

أعطينا مبرهنه التفكيك لأجل فضاء سوبوليف من الرتبة الاولي علي القرص . بإستخدام هذه النتيجة وبعض التشخيصات للخصائص الجبرية لمؤثرات تبوبليتز أو هانكل الصغيره مع الرموز في $L^{\infty.1}$. اعتبرنا مؤثرات تبوليتز و هانكل مع دوال التوليد المستمرة متعدده التعريف علي فضاءات - L^{ρ} و جبر باناخ المولدة بواسطتهما . شخصنا أزواج مؤثرات هانكل الاقتطاعيه علي الفضاءات النموذجية وحددنا السلوك التقاربي للقيم الشاذة لمؤثر هانكل المتراص بواسطة السلوك للرمز في جوار دعامته الشاذة.

The contents

Chapter 1

$L^{\infty,1}$ Symbols on Dirichlet Space

We show that Toeplitz or small Hankel operators with symbols in $\boldsymbol{l}^{\infty,1}$ is ageneralization of the case with the harmonic symbol in $\mathbb{C} \oplus \mathcal{D}$, where $\mathcal{D}$ is the Dirichet on $\mathbb{D}$.

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and dA be the normalized area measure on $\mathbb{D}$. The Sobolev space $L^{2.l} = L^{2.l}$ $(\mathbb{D})$ is the completion of the space of smooth functions u such that

$$\|u\|_{\frac{1}{2}} = \left(\left|\int_{\mathbb{D}} udA\right| + \int_{\mathbb{D}} \left[\frac{\partial u}{\partial z}\right]^2 + \left[\frac{\partial u}{\partial \bar{z}}\right]^2 dA\right)^{\frac{1}{2}} < \infty$$

$L^{2.l}$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{\frac{1}{2}} = \int_{\mathbb{D}} udA \int_{\mathbb{D}} \bar{v}dA + \langle\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}\rangle_{L^2} + \langle\frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial z}\rangle_{L^2}$$

Where the symbol $\langle\cdot,\cdot\rangle_{L^2}$ means the inner product in the Hilbert space $L^2(\mathbb{D}, dA) \in L^{2.l}$ with $f(0) = 0$. Let P be the orthogonal projection from $L^{2.l}$ onto $\mathcal{D}$. Then

$$p(u)(w) = \langle u, k_w\rangle_{\frac{1}{2}}, u \in L^{2.l}$$

Where $k_w(z) = -\log(1 - z\overline{w}) = \sum_{k=1}^{\infty} \frac{z^k \overline{w}^k}{k}$ is a reproducing kernel for $\mathcal{D}$. P is an integral operator represented by

$$P(u)(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z} \frac{\overline{\partial k_w(z)}}{\partial z} dA(z), u \in L^{2.l}. \tag{1}$$

Given a function φ in $L^{2.l}$, the Toeplitz operator $T_{\varphi}: \mathcal{D} \to \mathcal{D}$, the (big) Hankel operator $T_{\varphi}: \mathcal{D} \to \mathcal{D}^{\perp}$ and the small Hankel operator $\Gamma_{\varphi}: \mathcal{D} \to \mathcal{D}$, with symbol φ are densely defined on $\mathcal{D}$ respectively by

$$T_{\varphi} f = (\varphi f), \;\; H_{\varphi} f = (1 - p)(\varphi f), \qquad \Gamma_{\varphi} f = p(J(\varphi f),$$

Where J is the unitary $L^{2.l} \to L^{2.l}$ defined by $Jh(z) = h(\bar{z})$ for $h \in L^{2.l}$, and $\mathcal{D}^{\perp}$ is the orthogonal complement of $\mathcal{D}$ in $L^{2.l}$. Let $L^{\infty}(\mathbb{D})$ denote the algebra of all essentially bounded measurable functions on $\mathbb{D}$ and H^{∞} denote the space of bounded analytic function on $\mathbb{D}$. Define

$$L^{\infty.1} = \left\{ \varphi \in L^{2.l} \colon \varphi, \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}} \in L^{\infty}(\mathbb{D}) \right\}$$

It is well known that Toeplitz operator or (small) Hankel operator with symbol $\varphi \in L^{\infty.1}$is bounded on.

Toeplitz operators and Hankel operators studied on the classical Hardy space H^2 and Bergman space L_a^2. Toeplitz operators on the Dirichiet space have been studied intensively. G.F. Cao considered Fredholm properties of Toeplitz operators with $C^1(\overline{\mathbb{D}})$ symbols in. In case of bounded harmonic symbols in $L^{\infty.1}$, Y.J. Lee in studied the commutativity of two Toeplitz operators while he studied the zero or compactness of finite sum Toeplitz products. Moreover, L.I<. Zhao investigated properties of Hankel operator . We show that Toeplitz operators or small Hankel operators with symbol in $L^{\infty.1}$, is just a generalization of the case with the harmonic symbol in $L^{\infty.1}$. We show that if $f \in L^{\infty.1}$, then there is a harmonic function $f \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$ such that $T_F = T_f \; and \; \Gamma_{\mathrm{f}} = \Gamma_{\mathrm{f}}$ on $\mathcal{D}$. Finally, as a byproduct, based on the above-mentioned work of Lee and Zhao, we obtain algebraic properties of Toeplitz and small Hankel operator with symbols in $L^{\infty.1}$.

The derivative is taken in the distribution sense whereas the derivative is defined in the classical mode, which in turn provides different properties than the properties presented here. We point out that there is a flaw but the result is correct if the Sobolev space $L^{2.1}$ is removed and the Toeplitz operator is directly defined by the integral operator (1).

We will write $f'$for $\frac{\partial f}{\partial z}, \bar{f}'$ for $\frac{\partial \bar{f}}{\partial z},$ when $f \in L^{2.1}$ and $\|\cdot\|_{\mathcal{H}}$ for the norm in a Hulbert space $\mathcal{H}$.

We first need the following result which is probably well known.

Proposition (1.1)[1]. Let E: [0, 1) x [0, 1). Assume that $u \in L^{2.1}$(E). Then the following assertions hold.

(i) For almost all $x \in [0,1), u(x,.)$ is absolutely continuous on $[0,1)$ and $\lim_{y\to 1} u\,(x,y)EL^1[0.1)$.

(ii) For almost all $y \in [0,1), u(.,y)$ is absolutely continuous on $[0,1)$ and $\lim_{x\to 1} u\,(x,y)EL^1[0.1)$

Proof. Due to the lack of an explicit reference, we give a detailed proof. It suffices to prove (i) since (ii) can be treated analogously. Since $u \in L^{2.1}$(E), by Fubini's Theorem for almost all $x \in [0,1)$ the function $\frac{\partial u}{\partial y}(x,.) \in L^2[0,1\,)$. Thus for almost all $x \in [0,1)$, the function

$$\hat{u}(x,y) := \int_0^y \frac{\partial u}{\partial y}(x,t)dt$$

is well defined. Moreover,

$$\int_0^1 \int_0^1 |\hat{u}(x,y)|\, dxdy \leq \int_0^1 \int_0^1 \left|\frac{\partial u}{\partial y}(x,t)\right| dxdt < \infty$$

This implies that $\hat{u} \coloneqq u - \hat{u} \in L^1(E)$. Next we claim that $\frac{\partial \hat{u}}{\partial y} = \frac{\partial u}{\partial y}$ in the sense of distributions. To see this, we let $\{P_k\}_{k\geq 1}$ be a sequence of polynomials on $\mathbb{R}^2$ such that $P_k \to \frac{\partial u}{\partial y} in\ L^2(E) \quad as\ K \to \infty$. Define

$$u_k(x,y) = \int_0^y p_k(x,t)dt$$

Then $\frac{\partial u_k}{\partial y}(x,y) = p_k(x,y)$ in the classieal sense for every $k \in \mathbb{Z}_+$. Since $p_k \to \frac{\partial u}{\partial y}$ in $L^2(E)$, using Fubini's Theorem, after passing to a subsequence we may assume that

$$\lim_{k\to\infty} \int_0^1 \left|p_k(x,t) - \frac{\partial u}{\partial y}(x,t)\right| dt = 0\ for\ almost\ all\ x \in [0,1).$$

It follows that $u_k \to \hat{u}\ in\ L^1(E)$ and the claim follows. There fore $\frac{\partial \hat{u}}{\partial y} = 0$, in the sense of distributions. Thus, we conclude that $\hat{u}$is a function of x. Hence $u(x,.)$ is absolutely continuous on [0,1) for almost all x. Moreover, since for almost all $x \in [0,1), \frac{\partial u}{\partial y}(x,.) \in L^2[0,1)$ we have

$$\lim_{y\to 1} \mathrm{u(x,y)} = \tilde{\mathrm{u}}(x) \int_0^1 \frac{\partial u}{\partial y}(x,t)dt \quad for\ a.e.x$$

Since $\frac{\partial u}{\partial y} \in L^2(E)$ and since $\tilde{u} \in L^1[0,1)$, using Fubini's Theorem again we infer that the limit function $\lim_{y\to 1} u\,(x,y) \in L^1\,[0,1)$. The proof is complete.

Given a function $f \in L^{2,1}$. In the polar coordinates $z = re^{i\theta}$, we have

$$\frac{\partial f}{\partial z} = \frac{1}{2} e^{-i\theta}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} e^{i\theta}\left(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta}\right)$$

It follows that $f(re^{i\theta}) \in L^{2,1}(E)$, where $E\ [0,1)$ x $[0,2\pi)$. By Proposition 1, we see that $f(re^{i\theta})$ is absolutely continuous in r for almost all θ and absolutely continuous in θ for almost all r. In particular, the radial limit $f\,\Big|\partial\mathbb{D} \coloneqq \lim_{r\to 1} f(re^{i\theta})$ exists for almost all θ. Moreover, from the final conclusions of Proposition (1.1) we also have $f|\partial\mathbb{D} \in L^1(\partial\mathbb{D})$. Thus we can define for $k \in \mathbb{Z}$

$$f_k(1) = \frac{1}{2\pi}\int_0^{2\pi} f\Big|\partial\mathbb{D})e^{i\theta})e^{-ik\theta}d\theta$$

The result below roughly says that Toeplitz operators on $\mathcal{D}$ depends only on boundary values of the symbols.

Proposition (1.2)[1]. Let$f \in L^{2,1}$ Then for each $n \in \mathbb{Z}_+$,

$$T_f(z^n) = \sum_{k\in\mathbb{Z}_+} f_{k-n}\,(1)z^k$$

Proof. For $n \in \mathbb{Z}_+$, we have

$$T_f(z^n)(w)P(fz^n)(w) = \langle fz^n, K_w\rangle_{\frac{1}{2}} = \langle\frac{\partial(fz^n)}{\partial z}, \frac{\partial k_w}{\partial z}\rangle_{L^2} = \sum_{k\in\mathbb{Z}_+}\frac{1}{k}\langle\frac{\partial(fz^n)}{\partial z}, \frac{\partial z^k}{\partial z}\rangle_{L^2}\, w^k$$

Hence

$$T_f(z^n)(w) = \sum_{k\in\mathbb{Z}_+} \frac{1}{\pi}\left[\int_0^1\int_0^{2\pi} \frac{\partial f}{\partial z}(re^{i\theta})r^{n+k}e^{i(n-k+1)\theta}dr\,d\theta \right.$$

$$\left. + n\int_0^1\int_0^{2\pi} f\,(re^{i\theta})r^{n+k-1}e^{i(n-k)\theta}dr\,d\theta\right]w^k$$

Notice $\frac{\partial f}{\partial z} = \frac{1}{2}e^{-i\theta}(\frac{\partial f}{\partial r} - \frac{i}{r}\frac{\partial f}{\partial \theta})$ then by Fubini's Theorem we obtain

$$\frac{1}{\pi}\int_0^1\int_0^{2\pi} f\frac{\partial f}{\partial z}(re^{i\theta})r^{n+k}e^{i(n-k+1)\theta}dr\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-k)\theta}\,d\theta\int_0^1 \frac{\partial f}{\partial r}r^{n+k}dr$$

$$- \frac{i}{2\pi}\int_0^1 r^{n+k-1}\,dr\int_0^{2\pi} \frac{\partial f}{\partial \theta}e^{i(n-k)\theta}\,d\theta$$

Using the absolute continuity of f on r and θ, and integration by parts, we get the desired result. The proof is complete.

The following proposition is a characterization for functions in $L^{\infty,1}$. We claim no originality for this result.

Proposition (1.3)[1]. Let f be a meosurable function on $\mathbb{D}$. Then $f \in L^{\infty,1}$ if and only if there exists a continuous function $\tilde{f}$ on $\mathbb{D}$ such that $f = f\ a.e,$ on $\mathbb{D}$ and that

$$\left|\tilde{f}(z) - \tilde{f}(w)\right| \leq M|z - w|\ \forall z, w \in \mathbb{D},$$

Where M > 0 is a constant.

Proof. First assume that $f \in L^{\infty,1}$. Fix a smooth radial nonnegative function p on $\mathbb{C}$ with compact support in $\mathbb{D}$ such that $\int f p(z)dA(z) = 1. For\ \delta > 0$, we define $p_\delta(z) := \delta^{-2}p(|z|/\delta)$. The convolution of a locally integrable function f and p_δ is defined on $\mathbb{D}_\delta := \{z \in \mathbb{D}: d(z,\partial\mathbb{D}) > \delta\}$ as

$$f_\delta(z)(f * p_\delta)(z) := \int_{|w|<\delta} f\ f(z—w)p_\delta(w)dA(w)\ \forall z \in \mathbb{D}_\delta.$$

It is known that f_δ is C^∞ smooth and that f_δ, converges to f in $L^P(\mathbb{D}, dA)\ as\ \delta \to 0$ for $1 \le p < \infty\ if\ f \in L^P(\mathbb{D}, dA)$.

Since $f \in L^{\infty,1}$ and since

$$\frac{\partial f_\delta}{\partial z} = \frac{\partial f}{\partial z} * p_\delta, \qquad \frac{\partial f_\delta}{\partial z} = \frac{\partial f}{\partial \bar{z}} * p_\delta$$

We deduce that for every $\delta > 0$, all the partial derivatives of f_δ are bounded on Ca. Moreover, this bound does not depend on S. It follows that, there is a constant $M > 0$ satisfying

$$|f_\delta(z) - f_\delta(w)| \le M|z - w|\ \forall z, w \in \mathbb{D}$$

We then choose a sequence $\delta_k \to. 0$ such that $f_{\delta_k} \to f$ outside a set $E \subset \mathbb{D}$ of measure 0. Clearly

$$|f(z) - f(w)| \le M|z - w|\ \forall z, w \in \mathbb{D} \backslash E$$

Since E has empty interior, we may extend f to a continuous function $\tilde{f}$ on $\mathbb{D}$ satisfying

$$\left|\tilde{f}(z) - \tilde{f}(w)\right| \le M|z - w|\ \forall z, w \in \mathbb{D}$$

Conversely, suppose that f is a continuous function on $\mathbb{D}$ such that

$$|f(z) - f(w)| \leq M|z - w| \; \forall z, w \in \mathbb{D}$$

Where $M > 0$ is a constant. We have to show $f \in L^{\infty,1}$. For this, we note that by the Rademacher Theorem, $g := \frac{\partial f}{\partial z}$ (in the classical sense) exists almost everywhere on $\mathbb{D}$. Moreover $g \in L^{\infty,1}$.. We define f_δ as in the first part. By the Lebesgue dominated convergence Theorem, we verify that

$$\frac{\partial f_\delta}{\partial z} g * p_\delta.$$

Since the right-hand side converges to $g \; in \; L^1 \; (\mathbb{D}, dA) \; as \; \delta \to 0$, we infer that g is actually the distributional derivative of f. Thus $f \in L^{\infty,1}$.. The proof is complete.

Proposition (1.4)[1]. Let $f \in L^{\infty,1}$. and F be the Poisson extension of $f|\partial\mathbb{D}$. Then $F', \bar{F}' \in H^2 \; and \; F \in \mathbb{C} \oplus \mathcal{D} \oplus \bar{\mathcal{D}}$.

Proof. there exists a constant $M > 0$ such that $f|\partial\mathbb{D}$satisfying

$$f|\partial\mathbb{D}\left(e^{i\theta_1}\right) - f|\partial\mathbb{D}(e^{i\theta_2})| \leq M\left|e^{i\theta_1} - e^{i\theta_2}\right|$$

for every $\theta_1, \theta_2 \in [0, 2\pi.]$ Let $p(r\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$be the Poisson lcernel. Then

$$F(z) \frac{1}{2\pi} \int_0^{2\pi} P\,(r, \theta - t) f|\partial\mathbb{D}\left(e^{it}\right) dt \quad \left(z = re^{i\theta}\right) \qquad (2)$$

is harmonic in $\mathbb{D}$, continuous on the closed unit disk and $F|\partial\mathbb{D} = f|\partial\mathbb{D}$.

Differentiating with respect to θ in both sides of (2), we obtain

$$izF'(z) + \overline{\imath z\bar{F}'(z)} = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{\partial p}{\partial \theta}(r, \theta - t) f|\partial\mathbb{D}\left(e^{it}\right) dt$$

Using integration by parts and the absolute continuity of $f|\partial\mathbb{D}\left(e^{it}\right)$ we get

$$izF'(z) + \overline{\imath z\bar{F}'(z)} = \frac{1}{2\pi}\int_{0}^{2\pi} p\,(r, \theta - t) ie^{it}(f|\partial\mathbb{D})'\left(e^{it}\right) dt$$

Let f'_{+} denote the analytic part of $ie^{it}(f|\partial\mathbb{D})'$and $f'_{-} = ie^{it}(f|\partial\mathbb{D})' - f'_{+}$. Since $(f|\partial\mathbb{D})'$ is bounded on $\partial\mathbb{D}$, we infer that f'_{+} and f'_{-} are in $L^2(\partial\mathbb{D})$. Moreover, since F is harmonic we deduce that

$$F' \frac{\partial F}{\partial z}, \quad F' = \frac{\overline{\partial F}}{\partial \bar{z}}$$

are analytic functions on $\mathbb{D}$. It follows that $izF'(z)$ and $iz\bar{F}'(z)$ are analytic on $\mathbb{D}$. Therefore, by comparing analytic and anti-analytic parts in both sides of the last identity we obtain

$$izF'(z) = \frac{1}{2\pi}\int_{0}^{2\pi} p\,(r, \theta - t) f_{+}\hat{}'\left(e^{it}\right) dt, \qquad \overline{\imath z\bar{F}'(z)}$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} p\,(r, \theta - t) f_{-}\hat{}'\left(e^{it}\right) dt$$

Thus $izF'(z)$and $iz\bar{F}'(z)$in H^2. Since $F', \bar{F}'$ are already analytic functions on $\mathbb{D}$, we conclude that $F'(z)$ and $\bar{F}'(z)$ are in $\bar{F}'(z)$as well. Hence $F \in \mathbb{C}\mathcal{D}\oplus\bar{\mathcal{D}}$. The proof is complete.

Put

$$H_1^\infty = \{\varphi \in \mathcal{D}: \varphi' \in H^\infty\}$$

Clearly $\mathbb{C}\oplus H_1^\infty \oplus \overline{H_1^\infty} \subset L^{\infty.1}$. Moreover, $f \in L^{\infty.1}$ is harmonic if and only if $f \in \mathbb{C}\oplus H_1^\infty \oplus \overline{H_1^\infty}$. Now the following natural que'stion arises: Given $f \in L^{\infty.1}$ let F be the Poisson extension of $f|\partial\mathbb{D}$. Does F belong to $L^{\infty.1}$? The example given at the end of the section provides a negative answer to the above question. Before describing the example, we remark without proof that, by following the same arguments as in Proposition (1.4), we can obtain the following characterization of the boundary function $f|\partial\mathbb{D}$ in order that its harmonic extension F belongs to $L^{\infty.1}$.

Proposition (1.5)[1]. Let $f \in L^{2.1}$ and F be the Poisson extension of $f|\partial\mathbb{D}$ on $\mathbb{D}$. Then $F \in L^{\infty.1}$ if and only $f|\partial\mathbb{D}$ is Lipschitz continuous and both f'_+ and f'_- are in $L^\infty(\partial\mathbb{D})$. Here f'_+ denotes the analytic part of $(f|\partial\mathbb{D})'$ $and f'_- = (f|\partial\mathbb{D})' - f'_+$.

Now we formulate the promised example.

Example (1.6)[1]. Let

$$g(\theta) \sum_{k\in\mathbb{Z}^*} \frac{1}{k^2} e^{ik\theta}$$

Where $\mathbb{Z}^*$ denotes the set of nonzero integers. Then it is easy to see that $g \in C(\partial\mathbb{D})$. Note that the series

$$\sum_{k\in\mathbb{Z}^*} \frac{e^{ik\theta}}{k} = 2i \sum_{k\in\mathbb{Z}_+} \frac{\sin(k\theta)}{k}$$

is the Fourier series of the function $i(\pi-\theta)$. Thus g' is uniformly bounded on $(0,2\pi)$. Hence g is Lipschitz on $[0,2\pi]$, i.e., there exists a constant $M>0$ such that

$$|g(\theta_1)-g(\theta_2)|\leq M|\theta_1-\theta_2|\forall\theta_1,\theta_2\in[0,2\pi]$$

Now we apply a result of Mc Shane to find a function f on $\overline{\mathbb{D}}$ such that $f|\partial\mathbb{D}= g$ and that f is Lipschitz with the same constant M. Using Proposition (1.3), we conclude $f\in L^{\infty.1}$

Next we prove that

$$F(z)\sum_{k\in\mathbb{Z}^*}\frac{z^k}{k^2}+\sum_{k\in\mathbb{Z}_+}\frac{\bar{z}^k}{k^2}$$

the Poisson extension of g, is not in $L^{\infty.1}$. Indeed,

$$zF'(z)\sum_{k\in\mathbb{Z}^*}\frac{z^k}{k^2}=-\log(1-z)$$

Therefore F' is not bounded on $\mathbb{D}$. It follows that $F\notin L^{\infty.1}$ by Proposition (1.1) (this also follows from Proposition (1.3), since neither$g'_+ := i\sum_{k\in\mathbb{Z}^*}\frac{e^{ik\theta}}{k} nor\ g'_- := g'-g'_+$ belongs to $L^{\infty.1}(\partial\mathbb{D})$.

Given a function f in $L^2(\mathbb{D},dA)$, we have the following polar decomposition $f\left(re^{i\theta}\right)=\sum_{k\in\mathbb{Z}^*}e^{ik\theta}f_k(r)$

For almost all $r\in[0,1)$, where $f_k(r)=\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\theta})\,d\theta$, and

$$\|f\|_{L^2}^2=2\sum_{k\in\mathbb{Z}^*}\int_0^1|f_k(r)|^2\,r\,dr<\infty$$

Here $\|\cdot\|_{L^2}$ denotes the $L^2(\mathbb{D}, dA)$-norm. Moreover, if $f \in L^{2,1}$, then by the same argument as , using Proposition (1.1), we can check that $\sum_{|k|\le N} e^{ik\theta} f_k(r)$ converges to f in $L^{2,1}$ as N tends to infinity.

We first give a decomposition of the Sobolev space$L^{2,1}$. Let $\Omega = \Omega_0 + \mathbb{C}$, where

$$\Omega_0 = \left\{\sum_{k\in\mathbb{Z}}\left[f_k(r) - f_k(1)r^{|k|}\right]e^{i\theta} : f\left(re^{i\theta}\right) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r) \in L^{\infty,1}\right\}$$

Notice that the quantities $f_k(1)$ are well defined for all $f \in L^{2,1}$ in view of Proposition (1.1) (see the argument before Proposition (1.2).

Theorem (1.7)[1]. Let Δ_0 denote the closure of Ω_0 in the space $L^{2,1}$and $\Delta = \Delta_0 + \mathbb{C}$. Then $L^{2,1} = \Delta \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$. Moreover,

$$\Delta_0 == \left\{\sum_{k\in\mathbb{Z}}\left[f_k(r) - f_k(1)r^{|k|}\right]e^{i\theta} : f\left(re^{i\theta}\right) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r) \in L^{2,1}\right\}$$

Proof . First we show that $\Omega_0 \perp \mathcal{D}$ and $\Omega_0 \perp \overline{\mathcal{D}}$. For n e Z, we have

$$\langle \sum_{k\in\mathbb{Z}} f_k(1) r^{|k|} e^{ik\theta}, z^n \rangle_{\frac{1}{2}} = n f_n(i)$$

Since$\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right)$ = we get

$$\frac{\partial}{\partial v}\left[f_k(r)e^{ik\theta}\right] = \frac{1}{2} e^{i(k-1)\theta}\left[f_k'(r) + \frac{k}{r} f_k(r)\right]$$

Observe that f_k is absolutely continuous for every k by Proposition (1.3), so we have

$$\langle \sum_{k\in\mathbb{Z}} f_k(r)\, e^{ik\theta}, z^n \rangle_{\frac{1}{2}} = \langle \sum_{k\in\mathbb{Z}} \frac{\partial}{\partial z}\left(f_k e^{ik\theta}\right), nz^{n-1} \rangle_{L^2}$$

$$= \sum_{k\in\mathbb{Z}} \langle \frac{1}{2} e^{i(k-1)\theta} \left[f_k' + \frac{k}{r} f_k \right], nr^{n-1} e^{i(n-1)\theta} \rangle_{L^2}$$

$$= \langle \frac{1}{2} e^{i(n-1)\theta} \left[f_n' + \frac{k}{r} f_n \right], nr^{n-1} e^{i(n-1)\theta} \rangle_{L^2}$$

$$= \int_0^1 \left(f_n' + \frac{k}{r} f_n \right) nr^n dr = nf_n r^n |_0^1 = nf_n(i)$$

It follows that

$$\langle \sum_{k\in\mathbb{Z}} \left[f_k(r) - f_k(1) r^{|k|} \right] e^{ik\theta}, z^n \rangle_{\frac{1}{2}} = 0$$

Since $\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta} \right)$ by an analogous argument we can prove $\Omega_0 \perp \overline{\mathcal{D}}$. The details are omitted.

By combining the last result with Proposition (1.4), we infer $L^{\infty,1} \subset \Omega \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$. Since the set of smooth functions with compact support is dense in $L^{2,1}$, we get the required decomposition for $L^{2,1}$.

Finally, by Proposition (1.2), $T_{f-F}(z^n) = 0$ for every $n \in \mathbb{Z}_+ if\ f \in L^{\infty,1}$ and F is the Poisson extension of $f|\partial\mathbb{D}$. Clearly, if $f \in \Delta_0$, then for every $n \in \mathbb{Z}_+, T_f(z^n) = 0$. It follows that for $f \in L^{2,1}$, there exists a harmonic function $F \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$ such that $T_f(z^n) = T_F(z^n), n \in \mathbb{Z}_+$. Moreor, Proposition (1.2) shows that F is the Poisson extension of $f|\partial\mathbb{D}$ Hence the set in the right side of (2) equals to Δ_0. The proof is complete.

The following Theorem asserts that Toeplitz operator or small Hankel operator with symbol in $f \in L^{\infty,1}$ is just a generalization with the harmonic symbol in $\mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$.

Theorem (1.7)[1]. Let $f \in L^{\infty,1}$ and F be the Poisson extension of $f|\partial\mathbb{D}$. Then

(i) T_F is bounded on $\mathcal{D}$ and $T_F = T_f$;

(ii) Γ_F is bounded on $\mathcal{D}$ and $\Gamma_F = \Gamma_f$.

Proof. (i) By Proposition (1.2), it suffice to show T_F is bounded on $\mathcal{D}$. First we note that $F \in L^{\infty,1}(\mathbb{D})$ and $F' \in H^2$ by Proposition (1.4),

For $g, h \in \mathcal{D}$, we have

$$\langle T_F g, h\rangle_{\frac{1}{2}} = \langle Fg, h\rangle_{\frac{1}{2}} = \langle \frac{\partial(Fg)}{\partial z}, \frac{\partial h}{\partial z}\rangle_{L^2} = \langle F'g, h'\rangle_{L^2} + \langle Fg', h'\rangle_{L^2}$$

This implies

$$\left|\langle T_F g, h\rangle_{\frac{1}{2}}\right| \leq \|F'g\|_{L^2}\|Fg'\|_{L^2}\|h'\|_{L^2} \leq \|F'g\|_{L^2}\|h\|_{\mathcal{D}}\|F\|_{\infty}\|g\|_{\mathcal{D}}\|h\|_{\mathcal{D}}$$

Let $g(z) = \sum_{n\in\mathbb{Z}_+} a_n z^n$ and $F'(z) = \sum_{n\geq 0} b_n z^n$. Then $\|g\|_{\mathcal{D}}^2 = \sum_{n\in\mathbb{Z}_+} n|a_n|^2 < \infty$ and $\|F'\|_{H^2}^2 = \sum_{n\geq 0} n|b_n|^2 < \infty$. Observe that

$$\|F'g\|_{L^2}^2 = \int_{\mathbb{D}} |g(z)F'(z)|^2\, dA(z) = \int_{\mathbb{D}} \left|\sum_{n\in\mathbb{Z}_+} a_n z^n \sum_{n\geq 0} b_n z^n\right|^2 dA(z)$$

$$\int_{\mathbb{D}} \left|\sum_{n\in\mathbb{Z}_+} \left(\sum_{k=0}^{n-1} a_{n-k} b_k\right) z^n\right| dA(z)^2 \sum_{n\in\mathbb{Z}_+} \frac{1}{n+1}\left|\sum_{k=0}^{n-1} a_{n-k} b_k\right|^2$$

Thus using the Cauchy-Schwartz inequality, we obtain

$$\|F'g\|_{L^2}^2 \leq \sum_{n\in\mathbb{Z}_+} \frac{1}{n+1} n \sum_{k=0}^{n-1} |a_{n-k}b_k|^2$$

$$\sum_{n\in\mathbb{Z}_+} \frac{n}{n+1} n \sum_{k=0}^{n-1} |a_{n-k}b_k|^2 \, |b_k|^2 \leq \sum_{n\in\mathbb{Z}_+} \sum_{k=0}^{n-1} |a_{n-k}|^2 \, |b_k|^2$$

$$\sum_{n\in\mathbb{Z}_+}^{n-1} |a_n|^2 \sum_{n\geq 0} |b_n|^2 \leq |b_k|^2 \leq \|g\|_{\mathcal{D}}^2 \|F'\|_{H^2}^2$$

It follows that

$$\left|(T_F g, h)_{\frac{1}{2}}\right| \leq (\|F'\|_{H^2} + \|F\|_\infty \|g\|_{\mathcal{D}} \|h\|_{\mathcal{D}})$$

Therefore T_F is bounded on $\mathcal{D}$.

(ii) Let $g(z) \ = \sum_{n\in\mathbb{Z}_+} a_n z^n \in \mathcal{D}$ and $F(z) = \ \sum_{n\geq 0} b_n z^n + \sum_{n\in\mathbb{Z}_+} b_{-n} \bar{z}^n$. Since

$$P(\bar{z}^n z^m) = \ T_{\bar{z}^n} z^m = z^{m-n}$$

When $m \ > \ n \ > \ 0$ and

$$P(\bar{z}^n z^m) = 0$$

When $O < m \leq n$, thus

$$\Gamma_F g \ P(J(Fg)) \ P(F(\bar{z})g(\bar{z})) \ = \ p\left(\sum_{n\in\mathbb{Z}_+} a_n \bar{z}^n \sum_{n\in\mathbb{Z}_+} b_{-n} z^n\right)$$

$$\sum_{m,n\in\mathbb{Z}_+} a_n b_{-m} p(\,\bar{z}^n z^m) = \sum_{m>n>0} a_n b_{-m} z^{m-n}$$

Again using the Cauchy—Schwartz inequality, we have

$$\|g\|_{\mathcal{D}}^2 = \sum_{k\in\mathbb{Z}_+} \left\| \left(\sum_{n\in\mathbb{Z}_+} a_n b_{-(n+k)} \right) z^k \right\|_{\mathcal{D}}^2 \sum_{k\in\mathbb{Z}_+} k \left| \sum_{n\in\mathbb{Z}_+} a_n b_{-(n+k)} \right|^2$$

$$\sum_{k\in\mathbb{Z}_+} k \sum_{n\in\mathbb{Z}_+} n|a_n|^2 \sum_{n\in\mathbb{Z}_+} \frac{\left|b_{-(n+k)}\right|^2}{n}$$

$$\|g\|_{\mathcal{D}}^2 \sum_{n\in\mathbb{Z}_+} \frac{1}{n} \sum_{k\in\mathbb{Z}_+} (n+k)^2 \left|b_{-(n+k)}\right|^2 \frac{k}{(n+k)^2}$$

$$\|g\|_{\mathcal{D}}^2 \sum_{n\in\mathbb{Z}_+} \frac{1}{n^2} \sum_{k\in\mathbb{Z}_+} (n+k)^2 \left|b_{-(n+k)}\right|^2$$

Because $\bar{F}' \in H^2$ *and* $\|\bar{F}'\|_{H^2}^2 = \sum_{n\geq 0} n^2 \,|b_{-n}|^2 < \infty$, we have

$$\|\Gamma_F g\|_{\mathcal{D}}^2 \leq \|g\|_{\mathcal{D}}^2 \sum_{n\in\mathbb{Z}_+} \frac{1}{n^2} \|\bar{F}'\|_{H^2}^2 < \infty$$

So Γ_F is bounded on $\mathcal{D}$.

It remains to show $\Gamma_f(z^n) = \Gamma_F(z^n)$ for evry $n \in \mathbb{Z}_+$. It is easy to see $J\Omega_0 \subset \Omega_0$, and $z^n\Omega_0 \subset \Omega_0, \bar{z}^n\Omega_0 \subset \Omega_0$ for $n \in \mathbb{Z}_+ \cup \{O\}$. On the other hand, since $f - F \in \Omega_0$ and $P|_{\Omega_0} = 0$, we have

$$\Gamma_f(z^n)\, P\big(J(fz^n)\big) = P\Big(J\big((f - F)Z^n\big)\Big) + \Gamma_F(z^n) = \Gamma_F(z^n) \quad \forall n \in \mathbb{Z}_+.$$

The proof is complete.

Theorem (1.8)[1]. Let$f, g \in L^{\infty,1}$.. Then the following assertions hold.

(a) $T_f T_g = T_g T_f$ if and only if $f, g \in \Omega \oplus \mathcal{D}$ *or* $f, g \in \Omega \oplus \bar{\mathcal{D}}$ or a nontrivial linear combination off, g belongs to Ω.

(b) $T_f T_g = T_{fg} \mathcal{D}\ if$ and only $iff \in \Omega \bigoplus \overline{\mathcal{D}}$or $g \in \Omega \bigoplus \mathcal{D}$.

Theorem (1.9)[1]. Let $f, g \in\ L^{\infty,1}$. Then $\Gamma_f \Gamma_g = \Gamma_g \Gamma_f$ if and only if there exists a constant c such that $f - cg \in \Omega \bigoplus \mathcal{D} \bigoplus \mathbb{C}.\bar{z}$.

Chapter 2

A Handy Formula

The goal of this chapter is to provide a transparent symbol calculus for the Fredholm property and a handy formula for the Fredholm index for operators in this algebra**.**

Section (2.1): Fredholm Index of Toeplitz Plus Hankel Operators

Let $1 < p < \infty$. For a non-empty subset $\mathbb{I}$ of the set $\mathbb{Z}$ of the integers, let $l^p(\mathbb{I})$ denote the complex Banach space of all sequences$x = (x_n)_{n\in\mathbb{I}}$ of complex numbers with norm $\|x\|_p = (\sum_{n\in\mathbb{I}}|x_n|^p)^{\frac{1}{p}} < \infty$. We consider $l^p(\mathbb{I})$ as a closed subspace of $l^p(\mathbb{Z})$ in the natural way and write $P_{\mathbb{I}}$ for the canonical projection from $l^p(\mathbb{Z})$ onto $l^p(\mathbb{I})$. For $\mathbb{I} = \mathbb{Z}^+$, the set of the nonnegative integers, we write l^p and P instead of $l^p(\mathbb{I})$ and $P_{\mathbb{I}}$, respectively. By J we denote the operator on $l^p(\mathbb{Z})$ acting by $(Jx)_n := x_{-n-1}$, and we set $Q := I - P$.

For every Banach space X, let $L(X)$ stand for the Banach algebra of all bounded linear operators on X, and write $K(X)$ for the closed ideal of $L(X)$ of all compact operators. The quotient algebra $L(X)/K(X)$ is known as the Calkin algebra of X. Its importance stems from the fact that the invertibility of a coset $A + K(X)$ of an operator $A \in L(X)$ in this algebra is equivalent to the Fredholm property of A, i.e., to the finite dimensionality of the kernel $\ker A = \{x \in X: Ax = 0\}$and the cokernel $\operatorname{coker} A = X/\operatorname{im} A$ of A, with$\operatorname{im} A = \{Ax: x \in X\}$ referring to the range of A. If A is a Fredholm operator then the difference $\operatorname{ind} A := \dim\ker A - \dim\operatorname{coker} A$ is known as the Fredholm index of A.

For the Fredholm property and a formula for the Fredholm index for operators in the smallest closed subalgebra of $L(l^p)$ which containsall Toeplitz and Hankel operators with piece wise continuous generating function.

Let $\mathbb{T}$ be the complex unit circle. For each function $a \in L^\infty(\mathbb{T})$, let $(a_k)_{k\in\mathbb{Z}}$ denote the sequence of its Fourier coefficients,

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} a\left(e^{i\theta}\right)e^{-ik\theta}\, d\theta.$$

The Laurent operator $L(a)$ associated with $a \in L^\infty(\mathbb{T})$ acts on the space $l^0(\mathbb{Z})$ of all finitely supported sequences on $\mathbb{Z}$ by $(L(a)x)k := \sum_{m\in\mathbb{Z}} a_{k-m}x_m$. (For every$k \in \mathbb{Z}$, there are only finitely many non-vanishing summands in this sum.) We say that a is a multiplier on $l^p(\mathbb{Z})$ if $L(a)_x \in l^p(\mathbb{Z})$ for every $x \in l^0(\mathbb{Z})$ and if

$$\|L(a)\| := \sup\left\{\|L(a)_x\|_p : x \in l^0(\mathbb{Z}), \|x\|_p = 1\right\}$$

is finite. In this case, $L(a)$ extends to a bounded linear operator on $l^p(\mathbb{Z})$ which we denote by $L(a)$ again. The set M^p of all multipliers on$l^p(\mathbb{Z})$ is a Banach algebra under the norm $\|a\|_{M_p} := \|L(a)\|$. We let $M^{\langle p\rangle}$ stand for M^2 if $p = 2$and for the set of all $a \in L^\infty(\mathbb{T})$ which belong to M^r for all r in a certain open neighborhood of p if $p \neq 2$.

It is well known that $M^2 = L^\infty(\mathbb{T})$. Moreover, every function a with boundedtotal variation $\mathrm{Var}(a)$ is in M^p for every p, and the Stechkin inequality

$$\|a\|_{M_p} \leq cp\left(\|a\|_\infty + \mathrm{Var}(a)\right)$$

holds with a constant c_p independent of a. In particular, every trigonometric polynomial and every piecewise constant function on $\mathbb{T}$ are multipliers for every p. We denote the closure inM^p of the algebra $\mathcal{P}$ of all trigonometric polynomials and of the algebra $P\mathbb{C}$ of all piecewise constant functions by C_p and PC_p , respectively.Thus, C_p and PC_p are closed subalgebras of M^p for every p. Note that C_2 is just the algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$, and PC_2 is the algebra $PC(\mathbb{T})$ of all piecewise continuous functions on $\mathbb{T}$. It is well known that $C_p \subseteq C(\mathbb{T})$ and $C_p \subseteq PC_p \subseteq PC(\mathbb{T})$ for every p. In particular, every multiplier $a \in PC_p$ possesses one-sided limits at every point $t \in T$ For definiteness, we agree that $\mathbb{T}$ is oriented counter-clockwise, and we denote the one-sided limit of a at t when approaching t from below (from above) by $a(t^-)$ (by $a(t^+)$).

Let $a \in M^p$. The operators $T(a) := PL(a)P$ and $H(a) := PL(a)QJ$, thought of as acting on $\operatorname{im} P = l^p$ are called the Toeplitz and Hankel operator with generating function a, respectively. It is well known that $\|T(a)\| = \|a\|_{M_p}$ and $\|H(a)\| \leq \|a\|_{M_p}$ for every multiplier $a \in M_p$.

For a sub algebra A of M^p, we let $\mathrm{T}(A)$ and $\mathrm{TH}(A)$ stand for the smallest closed sub algebra of $L(l^p)$ which contains all operators $T(a)$ with $a \in A$ and alloperators $T(a) + H(b)$ with $a, b \in A$, respectively. We will be mainly concerned with the algebras C_p, PC_p, and with their intersections with $M^{\langle p \rangle}$, in place of A. we will state a criterion for the Fredholm property of operators in $\mathrm{TH}(PC_p)$ and derive a formula for the Fredholm index of operators $T(a) + H(b)$ with $a, b \in PC_p$.

The study of the Fredholm property of operators in $\mathrm{TH}(PC_p)$ has a long and involved history. We are going to mention only some of its main stages.

The Fredholm properties of operators in the algebra $\mathrm{T}(PC_p)$ are well understood .

The structure of the algebras $\mathrm{TH}(PC_p)$ is much more involved than that of $\mathrm{T}(PC_p)$. For instance, the Calkin image $\mathrm{T}^\pi(PC) := \mathrm{T}(PC)/K(l^2)$ of $\mathrm{T}(PC)$ is acommutative algebra, whereas that one of $\mathrm{TH}(PC)$ is not. The Calkin image of $\mathrm{TH}(PC)$ was first described by Power. An alternative approach wasdeveloped where it was shown that the algebra $\mathrm{TH}^{\pi}(PC) := \mathrm{TH}(PC)/K(l^2)$ possesses a matrix-valued Fredholm symbol, We take up the approach in order to study the Fredholm properties of operators in $\mathrm{TH}\left(PC_p\right)$for $p \neq 2$.

It should be mentioned that the algebras $\mathrm{TH}(PC_p)$ have close relatives which live on other spaces than l^p, such as the Hardy spaces $H^p(\mathbb{R})$ and the Lebesgue spaces $L^p(\mathbb{R}^+)$. The corresponding algebras were examined (with different methods) Despite these fairlycomplete results for the Fredholm property, a general, transparent and satisfying formula for the Fredholm index of operators in $\mathrm{TH}(PC_p)$ (or on related algebras) was not available until now. Among the particular results which hold under special assumptions we would like to emphasize the following there is derived an index formula for operators of the form $\lambda I + H$ where $\lambda \in \mathbb{C}$ and H is a Hankel operator on $H^p(\mathbb{R})$. Already earlier, some classes of Wiener-Hopf plus Hankel operators were studied in connection with diffraction problems. Note also that the (very hard) invertibility problem for Toeplitz plus Hankel operators is treated.

Finally we would like to mention that algebras like $\mathrm{TH}(PC_p)$ can also be viewed of as sub algebras of algebras generated by convolution-type operators and Carleman shifts changing the orientation. First results in that direction were presented where in particular, a matrix-valued Fredholm symbol was constructed.

We provide a transparent symbol calculus for the Fredholm property as well as a handy formula for the Fredholm index for operators in the algebra $\mathrm{TH}(PC_p)$. The techniques developed and used also allow to handle the corresponding questions for the related algebrason the spaces $H^p(\mathbb{R})$ and $L^p(\mathbb{R}^+)$.

Section (2.2): The Fredholm Property

In what follows, we fix $p \in (1, \infty)$ and consider all operators as acting on l^punless stated otherwise.

We start with recalling the basic results of the Fredholm theory of operators in the algebra $\mathrm{T}(PC_p)$, which are due Gohberg/Krupnik and Duduchava. The functions $f_{\pm 1}(t) := t^{\pm 1}$ are multipliers for every p. It is easy to check that the algebra generated by the Toeplitz operators $T(f_{\pm 1})$contains a dense subalgebra of $K(l^p)$. Thus, the ideal $K(l^p)$ is contained in$\mathrm{T}(C_p)$, hence also in $\mathrm{T}(PC_p)$, and it makes sense to consider the quotient algebra $\mathrm{T}(PC_p)/K(l^p)$. Clearly, if $A \in \mathrm{T}(PC_p)$ and if the coset $A + L(l^p)$ is invertible in $\mathrm{T}(PC_p)/K(l^p)$, then it is also invertible in the Calkin algebra $L(l^p)/K(l^p)$, hence A is a Fredholm operator. The more interesting question is if the converse holds, i.e. if the invertibility of $A + L(l^p)$ in the Calkin algebra implies the invertibilityof $A + K(l^p)$ in $\mathrm{T}(PC_p)/K(l_p)$. If this implication holds for every $A \in T(PC_p)$, one says that $\mathrm{T}(PC_p)/K(l^p)$ is inverse closed in $L(l^p)/K(l^p)$.

Let $\overline{\mathbb{R}}$ denote the two-point compactification of the real line by the points $\pm\infty$(thus $\overline{\mathbb{R}}$ is homeomorphic to a closed interval) and let the function $\mu_p: \mathbb{R} \to \mathbb{C}$ be defined by

$$\mu_p(\lambda) := \big(1 + \coth\big(\pi(\lambda + i/p)\big)\big)/2$$

if $\lambda \in \mathbb{R}$ and by $\mu_p(-\infty) = 0$ and $\mu_p(+\infty) = 1$. Note that when λ runs from$-\infty$ to ∞ then $\mu_p(\lambda)$ runs along a circular arc in $\mathbb{C}$which joins 0 to 1 and passes through the point $(1 - i\cot(\pi/p))/2$. An easy calculation gives $\mu_p(-\lambda) = 1 - \mu_q(\lambda)$, where $1/p + 1/q = 1$. Thus, for fixed $t \in \mathbb{T}$, the values $\Gamma\big(T(a) + K(l^p)\big)(t, \lambda)$

defined in the following Theorem run from $a(t-0)$ to $a(t+0)$ along a circular arc when λ runs from $-\infty$ to ∞.

Theorem (2.2.1)[2] (a) $\mathrm{T}(PC_p)/K(l^p)$ is a commutative unital Banach algebra.

(b) The maximal ideal space of $\mathrm{T}(PC_p)/K(l^p)$ is homeomorphic with the cylinder $\mathbb{T} \times \overline{\mathbb{R}}$, provided with an exotic (non-Euclidean) topology.

(c) The Gelfand transform $\Gamma\colon \mathrm{T}(PC_p)/K(l^p) \to \mathbb{C}(\mathbb{T} \times \mathbb{R})$ of the coset $T(a) + K(l^p)$ with $a \in PC_p$ is

$$(T(a) + K(l^p))(t, \lambda) = a(t-0)\left(1 - \mu_q(\lambda)\right) + a(t+0)\mu_q(\lambda).$$

(d) $\mathrm{T}(PC_p)/K(l^p)$ is inverse closed in $L(l^p)/K(l^p)$.

The topology mentioned in assertion (b) will be explicitly described . Note that this topology is independent of p. Since the cosets $T(a) + K(l^p)$with $a \in PC_p$ generate the algebra $\mathrm{T}(PC_p)/K(l^p)$, the Gelfand transform on$\mathrm{T}(PC_p)/K(l^p)$ is completely described by assertion (c) .Thus, if $A \in \mathrm{T}(PC_p)$,then the coset $A + K(l^p)$ is invertible in $\mathrm{T}(PC_p)/K(l^p)$ if and only if the function $\Gamma\left(A + K(l^p)\right)$ does not vanish on $\mathbb{T} \times \overline{\mathbb{R}}$. Together with assertion (d) this shows that $A \in T(PCp)$is a Fredholm operator if and only if $\Gamma(A + K(l^p))$ does not vanish on $\mathbb{T} \times \overline{\mathbb{R}}$. It is therefore justified to call the function $\mathrm{smb}_p\, A := \Gamma\left(A + K(l^p)\right)$ the Fredholm symbol of A.

The index of a Fredholm operator in $\mathrm{T}(PC_p)$ can be determined my means of its Fredholm symbol. First suppose that $a \in PC_p$ is a piecewise smooth function with only finitely many jumps. Then the range of the function

$$\Gamma(T(a) + K(l^p))(t, \lambda) = a(t^-)(1 - \mu_q(\lambda)) + a(t^+)\mu_q)(\lambda)$$

is a closed curve with a natural orientation, which is obtained from the (essential)range of a by filling in the circular arcs

$$C_q\big(a(t^-), a(t^+)\big) := \left\{a(t^-)\left(1-\mu_q(\lambda)\right) + a(t^+)\mu_q)(\lambda) : \lambda \in \overline{\mathbb{R}}\right\}$$

at every point t $\in T$ where a has a jump. (If the function a is continuous at t,then $C_q\big(a(t^-), a(t^+)\big)$ reduces to the singleton $\{a(t)\}$.) If this curve does not pass through the origin, then we let wind $\Gamma\big(T(a) + K(l^p)\big)$ denote its winding number with respect to the origin, i.e., the integer $1/(2\pi)$ times the growth of the argument of $\Gamma\big(T(a) + K(l^p)\big)$ when t moves along T in positive (= counter-clockwise)direction. If this condition is satisfied then $T(a)$ is a Fredholm operator, and

$$\text{ind}\, T(a) = -\text{wind}\Gamma\big(T(a) + K(l^p)\big)$$

one can extend both the definition of the winding number and the index identity to the case of an arbitrary Fredholm operator in $\mathrm{T}(PC_p)$. One has the following.

Proposition (2.2.2)[2] Let $A \in \mathrm{T}(PC_p)$ be a Fredholm operator. Then

$$\text{ind}\, A = -\text{wind}\Gamma\big(A + K(l^p)\big).$$

We would like to emphasize an important point. The algebra $\mathrm{T}(PC_2)/K(l^2)$is a commutative C^* -algebra, hence the Gelfand transform is an isometric $*$-isomorphism from $\mathrm{T}(PC_2)/K(l^2)$ onto $C(\mathbb{T} \times \overline{\mathbb{R}})$. In particular, the radical of$\mathrm{T}(PC_2)/K(l^2)$ is trivial, and the equality $\text{smb}_2 A = 0$ for some operator $A \in T(PC_2)$ implies that A is compact. For general p it is not known if the radical of $\mathrm{T}(PC_p)/K(l^p)$ is still trivial; it is therefore not known if $\text{smb}_p A = 0$ implies the compactness of A.

In order to state results on the Fredholm property of operators in the Toeplitz-Hankel algebra TH $(PC_p)/\ K(l^p)$. Let $\mathbb{T}_+$ be the set of all points in $\mathbb{T}$ with non-negative imaginary part and set $\mathbb{T}_+^0 := \mathbb{T}_+ \backslash \{-1, 1\}$.

Further let the function $\nu_p : \overline{\mathbb{R}} \to \mathbb{C}$ be defined by

$$\nu_p(\lambda) := (2i \sinh(\pi(\lambda + i/p))) - 1$$

if $\lambda \in \mathbb{R}$ and by $\nu_p(\pm\infty) = 0$. Recall that $1/p + 1/q = 1$.

Theorem (2.2.3)[2] (a) Let a, $b \in PC_p$. Then the operator $T(a) + H(b)$ is Fredholm ifand only if the matrix

$$smb_p \left(T(a) + H(b)\right)(t, \lambda)$$
$$:= \begin{pmatrix} a(t^+)\mu_q(\lambda) + a(t^-)\left(1 - \mu_q(\lambda)\right) & \left(b(t^+) - b(t^-)\right)\nu_q(\lambda) \\ \left(b(\bar{t}^-) - b(\bar{t}^+)\right)\nu_q(\lambda) & a(\bar{t}^-)\left(1 - \mu_q(\lambda)\right) + a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix} \quad (1)$$

is invertible for every $(t, \lambda) \in \mathbb{T}_+^0 \times \overline{\mathbb{R}}$ and if the number

$$\mathrm{smb}_p\left(T(a) + H(b)\right)(t, \lambda :$$
$$= a(t^+)\mu_q(\lambda) + a(t^-)\left(1 - \mu_q(\lambda)\right) + it\left(b(t^+) - b(t^-)\right)\nu_q(\lambda) \quad (2)$$

is not zero for every $(t, \lambda) \in \{\pm 1\} \times \overline{\mathbb{R}}$.

(b) The mapping smb_p defined in assertion (a) extends to a continuous algebra homomorphism from $\mathrm{TH}(PC_p)$ to the algebra $\mathcal{F}$ of all bounded functions on $\mathbb{T}_+ \times \overline{\mathbb{R}}$ with values in $C^{2\times 2}$ on $\mathbb{T}_+^0 \times \overline{\mathbb{R}}$ and with values in $\mathbb{C}$ on $\{\pm 1\} \times \overline{\mathbb{R}}$. Moreover,there is a constant M such that

$$\left\|\mathrm{smb}_p A\right\| := \sup_{(t,\lambda)\in\mathbb{T}_+\times\mathbb{R}} \left\|\mathrm{smb}_p\, A(t, \lambda)\right\|_\infty \le M \inf_{K\in K}(l^p)\, \|A + K\| \quad (3)$$

for every operator $A \in \mathrm{TH}(PC_p)$. Here, $\|B\|_\infty$ refers to the spectral norm of the matrix B.

(c) An operator $A \in \mathrm{TH}(PC_p)$ has the Fredholm property if and only if the function $\mathrm{smb}_p A$ is invertible in $\mathcal{F}$.

(d) The algebra $\mathrm{TH}(PC_p)/K(l^p)$ is inverse closed in $L(l^p)/K(l^p)$.

Before going into the details of the proof, we remark two consequences of Theorem (2.2).

Corollary (2.2.4)[2] Let $a, b \in PC_p$ and $T(a) + H(b)$ a Fredholm operator on l^p. Then(a) the function a is invertible in PC_p, and(b) if b is continuous at ±1, then $T(a) - H(b)$ is a Fredholm operator on l^p.

Proof. If $T(a) + H(b)$ is a Fredholm operator, then the diagonal matrices

$\mathrm{smb_p}\big(T(a) + H(b)\big)(t, \pm\infty) = \mathrm{diag}\Big(a(t^\pm), a(t^\pm)\Big)$are invertible for every $t \in \mathbb{T}_+^0$ and the numbers $smb_p \ \big(T(a) + H(b)\big)(1, \pm\infty) = a(1^\pm)$ and $\mathrm{smb}_p\big(T(a) + H(b)\big)(-1, \pm\infty) = a((-1)^\pm)$ are not zero by assertion(a) of Theorem (2.2.3). Hence, a is invertible as an element of PC. Since the algebra$PC_p$is inverse closed in PC by Proposition (2.2.2) in [2], assertion (a) follows. The proof of assertion (b) is also immediate from the form of the symbol described in Theorem (2.2.3) (a).We devoted to the proof of Theorem (2.2.3). We will need two auxiliary ingredients which we are going to recall first. Let $\mathcal{A}$ be a unital Banach algebra. The center of $\mathcal{A}$ is the set of all elements $a \in \mathcal{A}$ such that $ab = ba$ for all $b \in \mathcal{A}$. A central subalgebra of $\mathcal{A}$ is a closed subalgebra $\mathcal{C}$ of the center ofA which contains the identity element. Thus, $\mathcal{C}$ is a commutative Banach algebra with compact maximal ideal space $M(\mathcal{C})$. For each maximal ideal x of $\mathcal{C}$, consider the smallest closed two-sided ideal I_x of $\mathcal{A}$ which contains x, and letΦ_x refer to the canonical homomorphism from $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\mathcal{I}_x$.

In contrast to the commutative setting, where $C/x \cong \mathbb{C}$ for all $x \in M(C)$, the quotient algebras $\mathcal{A}/\mathcal{I}_x$ will depend on $x \in M(C)$ in general. In particular, it can happen that $I_x = \mathcal{A}$ for certain maximal ideals x. In this case we define that$\Phi_x(a)$ is invertible in $\mathcal{A}/\mathcal{I}_x$ for every $a \in \mathcal{A}$.

Theorem (2.2.5)[2] (Allan's local principle) Let $\mathcal{C}$ be a central subalgebra of the unital Banach algebra $\mathcal{A}$. Then an element $a \in \mathcal{A}$ is invertible if and only if the cosets $\Phi_x(a)$ are invertible in $\mathcal{A}/\mathcal{I}_x$ for each $x \in M(C)$.

Here is the second ingredient. Recall that an idempotent is an element p of an algebra such that $p^2 = p$.

Theorem (2.2.6)[2] (Two idempotents Theorem) Let $\mathcal{A}$ be a Banach algebra with identity element e, let p and q be idempotents in $\mathcal{A}$, and let $\mathcal{B}$ denote the smallest closed sub algebra of $\mathcal{A}$ which contains p, q and e. Suppose that 0 and 1 belong to the spectrum $\sigma_{\mathcal{B}}(pqp)$ of pqp in $\mathcal{B}$ and that 0 and 1 are cluster points of that spectrum. Then

(a) for each point $x \in \sigma_{\mathcal{B}}(pqp)$, there is a continuous algebra homomorphism Φ_x : $\mathcal{B} \rightarrow \mathbb{C}^{2\times 2}$ which acts at the generators of $\mathcal{B}$ by

$$\Phi_x(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi_x(p) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \Phi_x(q) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}$$

Where $\sqrt{x(1 - x)}$ denotes any complex number with $\left(\sqrt{x(1 - x)}\right)^2 = x(1 - x)$.

(b) an element $a \in \mathcal{B}$ is invertible in $\mathcal{B}$ if and only if the matrices $\Phi_x(a)$ are invertible for every $x \in \sigma_{\mathcal{B}}(pqp)$.

(c) if $\sigma_{\mathcal{B}}(pqp) = \sigma_{\mathcal{A}}(pqp)$, then $\mathcal{B}$ is inverse closed in $\mathcal{A}$.

We proceed with the proof of Theorem (2.2.3), which we split into several steps.

Step 1: Localization. For every operator $A \in L(l^p)$, we denote its coset

$A + K(l^p)$ in the Calkin algebra by A^π, and for every multiplier $a \in M^p$, we put$\tilde{a}(t) := a(1/t)$. The identities

$$T(ab) = T(a)T(b) + H(a)H(\tilde{b}) \text{ and } H(ab) = T(a)H(b) + H(a)T(\tilde{b}), \qquad (4)$$

Which hold for arbitrary $a, b \in M^p$, together with the compactness of the Hankel operators $H(c)$ for $c \in C_p$ show that the set C_p of all cosets $T(c)^\pi$ with $c \in C_p$ and $c = \tilde{c}$ forms a central subalgebra of the algebra $\mathrm{TH}(M^p)/K(l^p)$ and, in particular,of the algebra $\mathrm{TH}(PC_p)/K(l^p)$. One can, thus, reify Allan's local principle with$\mathrm{TH}(PC_p)/K(l^p)$ and C_p in place of $\mathcal{A}$ and $\mathcal{C}$, respectively. It is not hard tosee that the maximal ideal space of C_p is homeomorphic to the arc $\mathbb{T}_+$, with $t \in \mathbb{T}_+$ corresponding to the maximal ideal $\{c \in C_p : c(t) = 0\}$ of C_p. We let $\mathcal{J}_t$denote the smallest closed ideal of $TH(PC_p)/K(l^p)$ which contains the maximal ideal t and write $A_t^\pi$for the coset $A^\pi + \mathcal{J}_t$of $A \in \mathrm{TH}(PC_p)$. Instead of $T(a)_t^\pi$and $H(b)_t^\pi$ we often write $T_t^\pi(a)$ and $H_t^\pi(b)$, respectively, and the local quotient algebra $\left(\mathrm{TH}(PC_p)/K(l^p)\right)/\mathcal{J}_t$ is denoted by $\mathrm{TH}_t^\pi(PC_p)$ therefore. By Allan's local principle, we then have

$$\sigma_{\mathrm{TH}} = \cup_{t \in \mathbb{T}_+} \sigma_{\mathrm{TH}_t^\pi}(PC_p)(A_t^\pi) \qquad (5)$$

for every $A \in \mathrm{TH}(PC_p)$.

Step 2: Local equivalence of multipliers. Let $a, b \in PC_p$and $t \in \mathbb{T}_+$. We show that if $a(t^{\pm}) = b(t^{\pm})$ and $a(\bar{t}^{\pm}) = b(\bar{t}^{\pm})$, then $T_t^{\pi}(a) = T_t^{\pi}(b)$ and $H_t^{\pi}(a) = H_t^{\pi}(b)$. This fact will be used in what follows in order to replace multipliers by locally equivalent ones. It is clearly sufficient to prove that if $a \in PC_p$ satisfies$a(\bar{t}^{\pm}) = a(\bar{t}^{\pm}) = 0$, then $T^{\pi}(a), H^{\pi}(a) \in \mathcal{J}_t$. We will give this proof for$t \in \mathbb{T}_0^+$;the proof for for $t = \pm 1$ is similar.

Given $\varepsilon > 0$, let $f \in P\mathbb{C}$ such that $\|a - f\|_{M_p} < \varepsilon$. Then there is an open arc$U := (e^{-i\delta}\, t, e^{i\delta} t) \subset \mathbb{T}_+$ such that $|a(s)| < \varepsilon$ almost everywhere on $U \cup \bar{U}$ and suchthat f has at most one discontinuity in each of U and $\bar{U}$. Then $|f(s)| < 2\varepsilon$ for $\in U \cup \bar{U}$. Now choose a real-valued function $\varphi_0 \in C^{\infty}(\mathbb{T})$ such that $\varphi_0(t) = 1$,the support of φ_0 is contained in U, and φ_0 is monotonously increasing on thearc $(e^{-i\delta} t, t)$ and monotonously decreasing on $(t, e^{i\delta} t)$. Set $\varphi := \varphi_0 + \widetilde{\varphi_0}$. Then

$\varphi = \tilde{\varphi}$, and

$$T^{\pi}(f) - T^{\pi}(f\varphi) = T^{\pi}\big(f(1 - \varphi)\big) = T^{\pi}(f)T^{\pi}(1 - \varphi) \in \mathcal{J}_t,$$

$$H^{\pi}(f) - H^{\pi}(f\varphi) = H^{\pi}\big(f(1 - \varphi)\big) = H^{\pi}(f)T^{\pi}(1 - \varphi) \in \mathcal{J}_t.$$

Since $\|f\varphi\|_{\infty} < 2\varepsilon$ and$\mathrm{Var}(f\varphi) < 8\varepsilon$, we conclude that $\|f\varphi\|_{M_p} < 10c_p\varepsilon$ from Stechkin’s inequality. Thus, $\|T^{\pi}(f')\|k < 10c_p\varepsilon$ and $\|H^{\pi}(f\varphi)\| < 10c_p\varepsilon$, with aconstant c_p depending on p only. Thus, $T\hat{}\pi(a)$ differs from the element $T^{\pi}(f) - T^{\pi}(f\varphi) \in \mathcal{J}_t$ by the element $T^{\pi}(a - f) + T^{\pi}(f\varphi)$, which has a norm less than$(1 + 10c_p)\varepsilon$. Since $\varepsilon > 0$ is arbitrary and $\mathcal{J}_t$ is closed, this implies $T^{\pi}(a) \in \mathcal{J}_t$.

Analogously, $H^{\pi}(a) \in \mathcal{J}_t$.

Step 3: The local algebras at $t \in \mathbb{T}_0^+$. We start with describing the local algebras $\mathrm{TH}_t^\pi(PC_p)$ at point $t \in \mathbb{T}_0^+$. Let $\mathcal{X}_t$ denote the characteristic function of the arc in $\mathbb{T}$ which connects t with $\bar{t}$ and runs through the point -1. Clearly , $\mathcal{X}_t \in PC_p$. The crucial observation, which is a simple consequence of the identities(4), is that the operator $T(\mathcal{X}_t) + H(\mathcal{X}_t)$ is an idempotent. Further, let $\varphi_t \in C_p$ beany multiplier such that $0 \leq \varphi_t \leq 1, \varphi_t(t) = 1, \varphi(\bar{t}) = 0$ and $\varphi_t + \tilde{\varphi}_t = 1$. Againby (4), the coset $T_t^\pi(\varphi t)$ is an idempotent.

We claim that the idempotents $p_t := T_t^\pi(\varphi_t)$ and $q_t := T_t^\pi(\mathcal{X}_t) + H_t^\pi(\mathcal{X}_t)$together with the identity element $e := I_t^\pi$ generate the local algebra $\mathrm{TH}_t^\pi(PC_p)$.

Let $a, b \in PC_p$. Then, using step 2,

$$T_t^\pi(a) = a(t^+)T_t^\pi(\mathcal{X}_t\varphi_t) + a(t^-)T_t^\pi\big((1-\mathcal{X}_t)\varphi_t\big) + a(\bar{t}^-)T_t^\pi\big(\mathcal{X}_t(1-\varphi_t)\big) + a(\bar{t}^+)T_t^\pi\big((1-\mathcal{X}_t)(1-\varphi_t)\big). \qquad (6)$$

It is not hard to check that

$$T_t^\pi(\mathcal{X}_t\varphi_t) = p_t q_t p_t,$$

$$T_t^\pi((1-\mathcal{X}_t)\varphi_t) = p_t(e - q_t)p_t,$$

$$T_t^\pi\big(\mathcal{X}_t(1-\varphi_t)\big) = (e-p_t)q_t(e-p_t),$$

$$T_t^\pi\big((1-\mathcal{X}_t)(1-\varphi_t)\big) = (e-p_t)(e-q_t)(e-p_t). \qquad (7)$$

Let us verify the first of these identities, for example. By definition,

$$p_t q_t p_t = T_t^\pi(\varphi_t)T_t^\pi(\mathcal{X}_t)T_t^\pi(\varphi_t) + T_t^\pi(\varphi_t)H_t^\pi(\mathcal{X}_t)T_t^\pi(\varphi_t).$$

Since $T(\varphi_t)$ commutes with $T(\mathcal{X}_t)$ modulo compact operators and $H(\tilde{\varphi}_t)$ is compact, we can use the identities (4) to conclude

$$T_t^\pi(\varphi_t)T_t^\pi(\mathcal{X}_t)T_t^\pi(\varphi_t) = T_t^\pi(\mathcal{X}_t)T_t^\pi(\varphi_t) = T_t^\pi(\mathcal{X}_t\varphi_t).$$

Further, due to the compactness of $H(\varphi_t)$ and $H(\tilde{\varphi}_t)$,

$$T_t^\pi(\varphi_t)H_t^\pi(\mathcal{X}_t)T_t^\pi(\varphi_t) = H_t^\pi(\varphi_t\mathcal{X}_t)T_t^\pi(\varphi_t) = H_t^\pi(\varphi_t\mathcal{X}_t\tilde{\varphi}_t).$$

Since $\varphi_t\mathcal{X}_t\tilde{\varphi}_t$ is a continuous function, $H_t^\pi(\varphi_t\mathcal{X}_t\tilde{\varphi}_t) = 0$. This gives the first of the identities (7). The others follow in a similar way. Thus, (6) and (7) imply that $T_t^\pi(a)$ belong to the algebra generated by $e, p_t$and q_t. Similarly, we write

$$\begin{aligned} H_t^\pi(b) = b(t^+)H_t^\pi(\mathcal{X}_t\varphi_t) + b(t^-)H_t^\pi\big((1-\mathcal{X}_t)\varphi_t\big) + b(\bar{t}^-)H_t^\pi\big(\mathcal{X}_t(1-\varphi_t)\big) \\ + b(\bar{t}^+)H_t^\pi\big((1-\mathcal{X}_t)(1-\varphi_t)\big) \end{aligned} \qquad (8)$$

and use the identities

$$H_t^\pi(\mathcal{X}_t\varphi_t) = p_tq_t(e-p_t),$$

$$H_t^\pi((1-\mathcal{X}_t)\varphi_t) = -p_tq_t(e-p_t),$$

$$H_t^\pi\big(\mathcal{X}_t(1-\varphi_t)\big) = (e-p_t)q_tp_t,$$

$$H_t^\pi\big((1-\mathcal{X}_t)(1-\varphi_t)\big) = -(e-p_t)q_tp_t \qquad (9)$$

to conclude that $H_t^\pi(b)$ also belongs to the algebra generated by e, p_t and q_t.Thus, the algebra $\mathrm{TH}_t^\pi\big(PC_p\big)$ is subject to the two idempotent Theorem.

In order to apply this Theorem we have to determine the spectrum of the coset $p_tq_tp_t = T_t^\pi(\mathcal{X}_t\varphi_t)$ in that algebra. We claim that

$$\sigma_{\mathrm{TH}_t^\pi(PC_p)}\big(T_t^\pi(\mathcal{X}_t\varphi_t)\big) = \{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\} \qquad (10)$$

with $1/p + 1/q = 1$. Let $a_t \in PC_p$ be a multiplier with the following properties:

(a) $a_t$is continuous on $\mathbb{T} \setminus \{t\}$ and has a jump at $t \in \mathbb{T}$.

(b) $a_t(t^+) = \mathcal{X}_t(t^+) = 1$ and $a_t(t^-) = \mathcal{X}_t(t^-) = 0$.

(c) a_t takes values in $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$ only.

(d) a_t is zero on the arc joining $-t$ to t which contains the point 1.

Then, by Theorem (2.2.1), the essential spectrum of the Toeplitz operator $T(a_t)$ in eachof the algebras $L(l^p)/K(l^p)$ and $T(PC_p)/K(l^p)$ is equal to the arc $\{\mu q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$.

Hence, the essential spectrum of $T(a_t)$, now considered as an element of the algebra TH$(PC_p)/K(l^p)$, is also equal to this arc. Hence,

$$\sigma_{\mathrm{TH}_t^\pi}(PC_p)(T_t^\pi(a_t)) \subseteq \{\mu_q(\lambda): \lambda \in \overline{\mathbb{R}}\}$$

by Allan's local principle. Since $T_t^\pi(a_t) = T_t^\pi(\mathcal{X}_t\varphi_t)$, this settles the inclusion $\subseteq$ in (10). For the reverse inclusion, let $b_t \in PC_p$ be a multiplier with the following properties:

(a) b_t is continuous on $\mathbb{T}\backslash\{t\}$ and has a jump at $t \in \mathbb{T}$.

(b) $b_t(t^\pm) = \mathcal{X}_t(t^\pm)$.

(c) b_t takes values not in $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$ on the arc joining $-t$ to t which contains the point -1.

(d) b_t is zero on the arc joining $-t$ to t which contains the point 1.

Then, again by Theorem (2.2.1), the essential spectrum of the Toeplitz operator $T(b_t)$in each of the algebras $L(l^p)/K(l^p)$ and T$(PC_p)/K(l^p)$ is equal to the union of the arc $\{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}$ and the range of b_t. Hence, the essential spectrum of

$T(b_t)$, now considered as an element of the algebra $\mathrm{TH}(PC_p)/K(l^p)$, is also equal to this union. Since b_t is continuous on $\mathbb{T} \setminus \{t\}$ by property (a), we have

$$\sigma_{\mathrm{TH}_s^\pi(PC_p)}\big(T_s^\pi(b_t)\big) = \{b_t(s), b_t(\bar{s})\}$$

for $s \in \mathbb{T}_+^0 \setminus\{t\}$. Since the points $b_t(s)$ and $b_t(\bar{s})$ do not belong to $\{\mu_q(\lambda): \lambda \in \mathbb{R}\}$ by property (c), we conclude that the open arc $\{\mu q(\lambda): \lambda \in \mathbb{R}\}$ is contained inthe local spectrum of $T(b_t)$ at t. Since spectra are closed, this implies

$$\{\mu_q(\lambda): \lambda \in \overline{\mathbb{R}}\} \subseteq \sigma_{\mathrm{TH}_t^\pi(PC_p)}\big(T_t^\pi(b_t)\big).$$

Since $T_t^\pi(b_t) = T_t^\pi(\mathcal{X}_t\varphi_t)$ by property (b), this settles the inclusion $\supseteq$ in (10).

Since $\nu_q(\lambda)^2 = \mu_q(\lambda)\left(1 - \mu_q(\lambda)\right)$, we can choose $\sqrt{\mu_q(\lambda)\left(1 - \mu_q(\lambda)\right)} = \nu_q(\lambda)$.

With this choice and identities (6) – (9) it becomes evident that the two idempotents Theorem associates with the coset $T_t^\pi(a) + H_t^\pi(b)$ the matrix function

$$\lambda \to \begin{pmatrix} a(t^+)\mu_q(\lambda) + a(t^-)(1-\mu_q(\lambda)) & \big(b(t^+) - b(t^-)\big)\nu_q(\lambda) \\ \big(b(\bar{t}^-) - b(\bar{t}^+)\big)\nu_q(\lambda) & a(\bar{t}^-)\left(1-\mu_q(\lambda)\right) + a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix}$$

on $\overline{\mathbb{R}}$.

Step 4: The local algebra at $1 \in \mathbb{T}_+$. Next we are going to consider the local algebra $\mathrm{TH}_1^\pi(PC_p)$ at the fixed point 1 of the mapping $t \to \bar{t}$. Let $f : \mathbb{T} \to \mathbb{C}$denote the function $e^{is} \to 1 - s/\pi$ where $s \in [0, 2\pi)$. This function belongs toPC_p, and it has its only jump at the point $1 \in \mathbb{T}$ where $f(1^\pm) = \pm 1$. Using ideas, it was shown that the Hankel operator$H(f)$belongs to the Toeplitz algebra $\mathrm{T}(PC_p)$ and that its essential spectrum is given by

$$\sigma_{ess}\big(H(f)\big) = \{2i\, \nu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\}. \tag{11}$$

(in fact, this identity was derived with p in place of q, which makes no difference since $\nu_p(-\lambda) = \nu_q(\lambda)$ for every λ.) Let χ_+ denote the characteristic function of the upper half-circle $\mathbb{T}_+$. Since every coset $T_1^{\pi}(a)$ with $a \in PC_p$ is a linear combination of the cosets I_1^{π} and $T_1^{\pi}(\mathcal{X}_+)$ and every coset $H_1^{\pi}(b)$ is amultiple of the coset $H_1^{\pi}(f)$, the local algebra $\mathrm{TH}_1^{\pi}(PC_p)$ is singly generated (as aunital algebra) by the coset $T_1^{\pi}(\mathcal{X}_+)$. In particular, $\mathrm{TH}_1^{\pi}(PC_p)$ is a commutative Banach algebra, and its maximal ideal space is homeomorphic to the spectrum of its generating element. Similar to the proof of (10) one can show that

$$\sigma_{\mathrm{TH}_1^{\pi}(PC_p)}\big(T_t^{\pi}(\mathcal{X}_+)\big) = \{\mu_q(\lambda) : \lambda \in \overline{\mathbb{R}}\} \tag{12}$$

We identify the maximal ideal space of the algebra $\mathrm{TH}_1^{\pi}(PC_p)$ with $\overline{\mathbb{R}}$. The Gelfand transform of $T_t^{\pi}(\mathcal{X}_+)$ is then given by $\lambda \to \mu_q(\lambda)$ due to (12). Let h denote the Gelfand transform of $H_1^{\pi}(f)$. From (4) we obtain

$$H_1^{\pi}(f)^2 = T_1^{\pi}\big(f\,\tilde{f}\big) - T_1^{\pi}(f)T_1^{\pi}(\tilde{f}).$$

The function $f\tilde{f}$is continuous at $1 \in \mathbb{T}$ and has the value -1 there, and the function $f + \tilde{f}$ is continuous at $1 \in \mathbb{T}$ and has the value 0 there. Thus,

$$H_1^{\pi}(f)^2 = -I_1^{\pi} + T_1^{\pi}(f)^2.$$

Since $T_1^{\pi}(f) = T_1^{\pi}(2\mathcal{X}_+ - 1) = 2T_1^{\pi}(\mathcal{X}_+) - I_1^{\pi}$ we conclude that

$$h(\lambda)^2 = \big(2\mu_q(\lambda) - 1\big)^2 - 1 = \Big(\sinh\big(\pi(\lambda + i/q)\big)\Big) - 2$$

if $\lambda \in \mathbb{R}$ and by $h(\pm\infty) = 0$. By (11), this equality necessarily implies that

$$h(\lambda) = \Big(\sinh\big(\pi(\lambda + i/q)\big)\Big) - 1 = 2i\,\nu_q(\lambda)$$

if $\lambda \in \mathbb{R}$ and $h(\pm\infty) = 0$. Combining these results we find that the Gelfand transform of $T_1^\pi(a) + H_1^\pi(b)$ is the function

$$\lambda \to\ a(1^+)\mu_q(\lambda) + a(1^-)\left(1 - \mu_q(\lambda)\right) + i\left(b(1^+) - b(1^-)\right)\nu_q(\lambda).$$

Step 5: The local algebra at $-1 \in \mathbb{T}_+$. It remains to examine the local algebra $\mathrm{TH}_{-1}^\pi(PC_p)$ at the point -1. Let $\Lambda : l^2 \to l^2$ denote the mapping $(x_n)_{n\geq 0} \to 1((-1)^n x_n)_{n\geq 0}$. Clearly, $\Lambda^{-1} = \Lambda$, and one easily checks (perhaps most easily onthe level of the matrix entries, which are Fourier coefficients) that

$$\Lambda^{-1}T(a)\Lambda = T(\hat{a}) \text{ and} \Lambda^{-1}H(a)\Lambda = -H(\hat{a})$$

for $a \in PC_p$, where $\hat{a}(t) := a(-t)$. Thus, the mapping $A \to \Lambda^{-1}A\Lambda$ is an automorphism of the algebra $\mathrm{TH}(PC_p)$, which maps compact operators to compact operators and induces, thus, an automorphism of the algebra $\mathrm{TH}(PC_p)/K(l^p)$. The latter maps the local ideal at 1 to the local ideal at -1 and vice versa and induces,thus, an isomorphism between the local algebras $\mathrm{TH}_1^\pi(PC_p)$ and $\mathrm{TH}_{-1}^\pi(PC_p)$,which sends $T_1^\pi(\mathcal{X}_+)$ to $T_{-1}^\pi(1 - \mathcal{X}_+)$ and $H_1^\pi(\mathcal{X}_+)$ to $-H_{-1}^\pi(1 - \mathcal{X}_+) = H_{-1}^\pi(\mathcal{X}\_+)$,respectively.

Step 6: From local to global invertibility. We have identified the right-handsides of (1) and (2) as the functions which are locally associated with the operator$T(a) + H(b)$ via the two idempotents Theorem and via Gelfand theory for commutative Banach algebras, respectively. It follows from the two idempotents Theorem and from Gelfand theory that the so-defined mappings $\mathrm{smb}_p(t, \lambda)$ extendto a continuous homomorphism from $\mathrm{TH}(PC_p)$ to $\mathbb{C}^{2\times 2}$ or $\mathbb{C}$, respectively,which combine to a continuous homomorphism from $\mathrm{TH}(PC_p)$ to the algebra F. Allan's local principle then implies that the coset $A + K(l^p)$ of an operator$A \in \mathrm{TH}(PCp)$

is invertible in $\mathrm{TH}(PC_p)/K(l^p)$ if and only if its symbol does notvanish. The proof of estimate (3) will base on Mellin homogenization arguments.

Step 7: Inverse closedness. It remains to show that $\mathrm{TH}(PC_p)/K(l^p)$ is an inverse closed subalgebra of the Calkin algebra $L(l^p)/K(l^p)$. We shall prove this fact by using a thin spectra argument as follows: If $\mathcal{A}$ is a unital closed subalgebra of a unital Banach algebra $\mathcal{B}$, and if the spectrum in $\mathcal{A}$ of every element in a dense subset of $\mathcal{A}$ is thin, i.e. if its interior with respect to the topology of $\mathbb{C}$ is empty,then $\mathcal{A}$ is inverse closed in $\mathcal{B}$.

Let $\mathcal{A}_0$ be the set of all operators of the form

$$A := \sum_{i=1}^{l} \prod_{j=1}^{k} \left(T(a_{ij}) + H(b_{ij})\right) \text{with} A_{ij}, b_{ij} \in P\mathbb{C}, \tag{13}$$

and write $\sigma_{ess}^{\mathrm{TH}}(A)$ for the spectrum of A in $\mathrm{TH}(PC_p)/K(l^p)$. Then $\mathcal{A}_0/K(l^p)$ isdense in $\mathrm{TH}(PC_p)/K(l^p)$, and the assertion will follow once we have shown that$\mathrm{TH}(PC_p)/K(l^p)$ is thin for every $A \in \mathcal{A}_0$.

Given A of the form (13), let Ω denote the set of all discontinuities of the

functions a_{ij} and b_{ij}, and put $\widetilde{\Omega} := (\Omega \cup \overline{\Omega}) \cap \mathbb{T}_+$. Clearly, $\tilde{\Omega}$ is a finite set. Bywhat we have shown above,

$$\sigma_{ess}^{TH}(A) = \cup_{(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}} \; \sigma\left(\mathrm{smb}_p(A)(t,\lambda)\right)$$

Where $\sigma(B)$ stands for the spectrum (= set of the eigenvalues) of the matrix B.We write$\sigma_{ess}^{TH}(A)$ as$\sum_1 \cup \sum_2 \cup \sum_3$ where

$$\Sigma_1 := \cup_{(t,\lambda)\in\{-1,1\}\times\overline{\mathbb{R}}} \; \sigma\left(\mathrm{smb}_p(A)(t,\lambda)\right),$$

$$\Sigma_2 := \cup_{(t,\lambda)\in(\mathbb{T}_+^0/\tilde{\Omega})\times\overline{\mathbb{R}}} \ \sigma\left(\mathrm{smb}_p(A)(t,\lambda)\right),$$

$$\Sigma_3 := \cup_{(t,\lambda)\in(\tilde{\Omega}\setminus\{-1,1\})\times\overline{\mathbb{R}}} \ \sigma\left(\mathrm{smb}_p(A)(t,\lambda)\right).$$

It is clear that Σ_1 is a set of measure zero. It is also clear that each set

$$\Sigma_{2,t} := \cup_{\lambda\in\overline{\mathbb{R}}} \ \sigma(\mathrm{smb}_p\ (A)(t,\lambda)) \text{ with } t \ \in \mathbb{T}_+^0 \setminus\tilde{\Omega}$$

has measure zero. Since the functions aij and bij are piecewise constant, the mapping $t \to \Sigma_{2,t}$ is constant on each connected component of $\mathbb{T}_+^0 \setminus\tilde{\Omega}$, and the number of components is finite. Thus, Σ_2 is actually a finite union of sets of measure zero. Since $\tilde{\Omega}$ is finite, it remains to show that each of the sets

$$\Sigma_{3,t} := \cup\_(\lambda \in \overline{\mathbb{R}})\ \sigma(\mathrm{smb}_p\ (A)(t,\lambda))\ with\ t \ \in \tilde{\Omega}\setminus \{-1,1\}$$

has measure zero. For this goal it is clearly sufficient to show that each set

$$\Sigma_{3,t}^0 := \cup_{\lambda\in\mathbb{R}} \ \sigma(\mathrm{smb}_p\ (A)(t,\lambda))\ with\ t \ \in \tilde{\Omega}\setminus \{-1,1\}$$

has measure zero. Let $t \in \tilde{\Omega}\setminus \{-1,1\}$, and write $\mathrm{smb}_p\ (A)(t,\lambda)$ as $\left(c_{ij}(\lambda)\right)_{i,j=1}^2$.The eigenvalues of this matrix are $s_\pm(\lambda) = \left(c_{11}(\lambda) + c_{22}(\lambda)\right)/2 \ \pm \sqrt{r(\lambda)}$where

$$r(\lambda) \ = \ \left(a_{11}(\lambda) + a_{22}(\lambda)\right)^2/4 \ - \ \left(a_{11}(\lambda)a_{22}(\lambda) - \ a_{12}(\lambda)a_{21}(\lambda)\right)$$

and where$\sqrt{r(\lambda)}$ is any complex number the square of which is $r(\lambda)$. Since$r$ is composed by the meromorphic functions coth and $1/$ sinh , the set of zeros of r is discrete. Hence, $\mathbb{R} \setminus \{\lambda \ \in \mathbb{R} : \ r(\lambda) \ = \ 0\}$ is an open set, which as the union of an at most countable family of open intervals. Let I be one of these intervals. Then I can be represented as the union of countably many compact subintervals In such that the intersection $I_n \cap I_m$consists of at most one pointwhenever $n \neq m$ and each set $r(I_n)$ is contained in a domain where a continuousbranch, say f_n, of the function $z \rightarrow \sqrt{z}$ exists. Then $\pm f_n \circ r : I_n \rightarrow \mathbb{C}$ is a continuously differentiable function, which implies that $(\pm f_n \circ r)(I_n)$ is a set of measure zero. Consequently, the associated sets $s_{\pm}(I_n)$ of eigenvalues have measure zero, too. Since the countable union of sets of measure zero has measure zero, we conclude that each set$\Sigma^0_{3,t}$ has measure zero, which finally implies that $\sigma^{TH}_{ess}(A) = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$has measure zero and is, thus, thin. This settles the proof of the inverse closedness and concludes the proof of Theorem (2.2.3).

We would like to mention that there is another proof of the inverse closedness assertion in the previous Theorem which is based on ideas and which works also in other situations.

Section (2.3): An extended Toeplitz algebra

In the proof of the announced index formula for Toeplitz plus Hankel operators,we shall need an extension of the results to certain matrix operators. For $k \in \mathbb{N}$ and X a linear space, we let X_k and $X_{k\times k}$ stand for the linear spaces of all vectors of length k and of all k × k-matrices with entries in X, respectively. If X is an algebra, then $X_{k\times k}$ becomes an algebra under the standard matrix operations. If X is a Banach space, then X_k and $X_{k\times k}$ become Banach spaces with respect to the norms

$$\left\|(x_j)_{j=1}^k\right\| = \sum_{j=1}^k \|x_j\| \text{ and } \left\|(a_{ij})_{i,j=1}^k\right\| = k \sup_{1\le i,j\le k} \|a_{ij}\| . \tag{14}$$

If, moreover, X is a Banach algebra, then $X_{k\times k}$ is a Banach algebra with respect to the introduced norm. Actually, any other norm on X_k and any other compatible matrix norm on $X_{k\times k}$ will do the same job. Note also that if X is a c^*-algebrathere is a unique norm (different from the above mentioned) which makes $X_{k\times k}$ to a c^*-algebra. Since we will not employ a $c^* - ar$guments, the choice (14) will be sufficient for our purposes.

Let $\mathrm{T}^0(PC_p)$ denote the smallest closed subalgebra of $L(l^p(\mathbb{Z}))$ which contains the projection P and all Laurent operators $L(a)$ with $a \in PC_p$. The algebra$\mathrm{T}^0(PC_p)$ contains $\mathrm{T}(PC_p)$ in the sense that the operator $PL(a)P :\ \mathrm{im}\, P \rightarrow \mathrm{im}\, P$can be identified with the Toeplitz operator $T(a)$. For $k \in N$, the matrix algebra $\mathrm{T}^0(PC_p)_{k\times k}$ will be also denoted by $\mathrm{T}^0_{k\times k}(PC_p)$. One can characterize $\mathrm{T}^0_{k\times k}(PC_p)$also as the smallest closed sub algebra of $L(l^p(\mathbb{Z})_k)$ which contains all operators of the form $L(a)\mathrm{diag}\, P + L(b)\mathrm{diag} Q$ with $a, b \in (PC_p)_{k\times k}$, where $Q := I - P,$ $\mathrm{diag} A$ stands for the operator on $L(l^p(\mathbb{Z})_k)$ which has $A \in L(l^p(\mathbb{Z}))$ at each entry of its main diagonal and zeros at all other entries, and where $L(a) = \left(L(a_{ij})\right)_{i,j=1}^{k}$ refers to the matrix Laurent operator with generating function $a = (a_{ij})_{i,j=1}^{k}$. Note that $K(l^p(\mathbb{Z})_k)$ is contained in $T^0_{k\times k}(PC_p)$.

The Fredholm theory for operators in $\mathrm{T}^0_{k\times k}(PC_p)$is well known. We will present it in a form which is convenient for our purposes. Our main tools are againAllan's local principle (Theorem (2.2.5)) and a matrix version of the two idempotents Theorem (Theorem (2.2.6)) . Here is the result.

Theorem (2.3.1)[2] Let $a, b \in (PC_p)_{k\times k}$.

(a) The operator $A := L(a)\mathrm{diag}\, P + L(b)\mathrm{diag} Q$ is Fredholm on $l^p(\mathbb{Z})_k$ if and onlyif the matrix

$(\mathrm{smb}_p\, A)(t,\lambda)$

$$= \begin{pmatrix} a(t-) + \big(a(t^+) - a(t^-)\big)\mathrm{diag}\mu_q(\lambda) & \big(b(t^+) - b(t^-)\big)\mathrm{diag}\nu_q(\lambda) \\ a(t^+) - a(t^-))\ \mathrm{diag}\nu_q(\lambda) & (\,b(t^+) - \big(b(t^+) - b(t^-)\big)\mathrm{diag}\mu_q(\lambda) \end{pmatrix}$$

is invertible for every pair $(t,\lambda) \in \mathbb{T} \times \overline{\mathbb{R}}$.

(b) The mapping smb_p defined in assertion (a) extends to a continuous algebra homomorphism from $\mathrm{T}^0_{k\times k}(PC_p)$ to the algebra $\mathcal{F}$ of all bounded functions on $\mathbb{T} \times \overline{\mathbb{R}}$ with values in $C_{2k\times 2k}$. Moreover, there is a constant M such that

$$\left\|\mathrm{smb}_p A\right\| := \sup_{(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}} \left\|\mathrm{smb}_p\, A(t,\lambda)\right\|_\infty \le M \inf_{K\in K(l^p(\mathbb{Z})_k)} \|A + K\| \qquad (15)$$

for every operator $A \in \mathrm{T}^0_{k\times k}\big(PC_p\big)$.

(c) An operator $A \in \mathrm{T}^0_{k\times k}\big(PC_p\big)$ has the Fredholm property on $l^p(\mathbb{Z})_k$ if and only if the function $\mathrm{smb}_p\, A$ is invertible in $\mathcal{F}$.

(d) The algebra $\mathrm{T}^0_{k\times k}\big(PC_p\big)/K(l^p(\mathbb{Z})_k)$ is inverse closed in the Calkin algebra $L(l^p(\mathbb{Z})_k)/K(l^p(\mathbb{Z})_k)$.

(e) If $A \in \mathrm{T}^0_{k\times k}\big(PC_p\big)$ is a Fredholm operator, then

$$\mathrm{ind}\, A = -\mathrm{wind}\ (\det \mathrm{smb}_p\, A(t,\lambda)/(\det a_{22}(t,\infty)\ \det a_{22}(t,-\infty)))$$

Where $\mathrm{smb}_p\, A = \big(a_{ij}\big)^2_{i,j=1}$ with $k \times k$-matrix-valued functions a_{ij} .

It is a non-trivial fact that the function

$$W : \mathbb{T} \times \overline{\mathbb{R}}, (t,\lambda) \longmapsto \det \mathrm{smb}_p\, A(t,\lambda)/(\det a_{22}(t,\infty)\ det\ a_{22}(t,-\infty))$$

forms a closed curve in the complex plane. Thus, the winding number of W is well defined if A is a Fredholm operator.

We devoted to the proof of Theorem (2.3.1).

Step 1: Spline spaces. We start with recalling some facts about spline spaces and operators there on . Let $\chi_{[0,1]}$ denote the characteristic function of the interval $[0, 1] \subset \mathbb{R}$ and, for $\in \mathbb{N}$, let S_n denote the smallest closed subspace of $L^p(\mathbb{R})$ which contains all functions

$$\varphi_{k,n}(t) := \chi_{[0,1]}(nt - k), \qquad t \in \mathbb{R},$$

where $k \in \mathbb{Z}$. The space $l^p(\mathbb{Z})$ can be identified with each of the spaces S_n in the sense that a sequence (x_k) is in $l^p(\mathbb{Z})$ if and only if the series

$\sum_{k\in\mathbb{Z}} x_k \varphi_{k,n}$ converges in $L^p(\mathbb{R})$ and that

$$\left\| \sum x_k \varphi_{k,n} \right\|_{L^p(\mathbb{R})} = n^{-1/p} \|(x_k)\|_{l^p(\mathbb{Z})}$$

in this case. Thus, the linear operator

$$E_n : l^p(\mathbb{Z}) \to S_n \subset L^p(\mathbb{R}), (x_k) \longmapsto n^{1/p} \sum x_k \varphi_{k,n},$$

and its inverse $E_{-n} : L^p(\mathbb{R}) \supset S_n \to l^p(\mathbb{Z})$ are isometries for every n. Further we define operators

$$L_n : L^p(\mathbb{R}) \to S_n, \qquad u \longmapsto n \sum_{k\in\mathbb{Z}} \langle u, \varphi_{k,n} \rangle \varphi_{k,n}$$

with respect to the sesqui-linear form $\langle u, v \rangle := \int_{\mathbb{R}} u\bar{v} dx$, where $u \in L^p(\mathbb{R})$ and $v \in L^q(\mathbb{R})$ with $1/p + 1/q = 1$. It is easy to see that every L_n is a projection operator with norm 1 and that the L_n converge strongly to the identity operator on $L^p(\mathbb{R})$ as $n \to \infty$. Finally we set

$$Y_t : l^p(\mathbb{Z}) \to l^p(\mathbb{Z}), \qquad (x_k) \mapsto (t^{-k}x_k) \text{for } t \in \mathbb{T}.$$

Clearly, Y_t is an isometry, and $Y_t^{-1} = Y_{t^{-1}}$. One easily checks that $Y_t^{-1}L(a)Y_t = L(a_t)$ with $a_t(s) = a(ts)$ for every multiplier a, which implies in particular that$Y_t^{-1}\mathrm{T}^0(PC_p)Y_t = \mathrm{T}^0(PC_p)$.

Step 2: Some homomorphisms. It is shown that, for every $A \in \mathrm{T}^0(PC_p)$ and every $t \in \mathbb{T}$, the strong limit

$$\mathrm{smb}_t A := s - \lim_{n\to\infty} E_n Y_t^{-1} A Y_t E_{-n} L_n$$

exists and that the mapping smbt is a bounded unital algebra homomorphism.

This homomorphism can be extended in a natural way to the matrix algebra $\mathrm{T}^0_{k\times k}(PC_p)$. We denote this extension by $\mathrm{smb}_t A$ again.

In order to characterize the range of the homomorphism smb_t, we have to introduce some operators on $L^p(\mathbb{R})$. Let χ_+ stand for the characteristic function of the interval $\mathbb{R}^+ = [0,\infty)$ and $\chi_+ I$ for the operator of multiplication by χ_+.

Further, $S_{\mathbb{R}}$ refers to the singular integral operator

$$(S_{\mathbb{R}}f)(t) := \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{f(s)}{s - t} ds,$$

with the integral understood as a Cauchy principal value. Both $\chi_+ I$ and $S_{\mathbb{R}}$ are bounded on $L^p(\mathbb{R})$, and $S_{\mathbb{R}}^2 = I$. Thus, the operators $P_{\mathbb{R}} := (I + S_{\mathbb{R}})/2$ and $Q_{\mathbb{R}} := I - P_{\mathbb{R}}$ are bounded projections on $L^p(\mathbb{R})$. We let $\sum_k^p(\mathbb{R})$ stand for the smallest closed subalgebra of $L(L^p(\mathbb{R})_k)$ which contains the operators diag $\chi_+ I$, diag$S_{\mathbb{R}}$, and all operators of multiplication by constant $k \times k$-matrix valued functions.

Theorem (2.3.2)[2]. Let $t \in \mathbb{T}$. Then

(a) $\text{smb}_t \text{diag}\, P = \text{diag}\chi_+ I$.

(b) $\text{smb}_t\, L(a) = a(t^+)\text{diag}Q_{\mathbb{R}} + a(t^-)\text{diag}P_{\mathbb{R}}$ for $a \in (PC_p)_{k\times k}$.

(c) $\text{smb}_t K = 0$ for every compact operator K.

(d) smb_t maps the algebra $\mathrm{T}^0_{k\times k}(PC_p)$ onto $\sum_k^p(\mathbb{R})$.

(e) The algebra $\sum_k^p(\mathbb{R})$ is inverse closed in $L(L^p(\mathbb{R})_k)$.

Assertion (c) of the previous Theorem implies that every mapping smb_t induces anatural quotient homomorphism from $\mathrm{T}^0(PC_p)/K(l^p(\mathbb{Z}))$ to $\sum_1^p(\mathbb{R})$. We denote this quotient homomorphism by smb_t again. It now easily seen that the estimate (15) holds for every $A \in \mathrm{T}^0_{k\times k}(PC_p)$ (with the constant $M = 1$ for $k = 1$).

Step 3: The Fredholm property. Since the commutator $L(a)P - PL(a)$ is compact for every $a \in C_p$, the algebra $\mathcal{C}_p := \{\text{diag}\, L(a): a \in C_p\}/K(l^p(\mathbb{Z})_k)$ lies in the center of the algebra $\mathcal{A} := \mathrm{T}^0_{k\times k}(PC_p)/K(l^p(\mathbb{Z})_k)$. It is not hard to see that$\mathcal{C}_p$ is isomorphic to C_p;hence the maximal ideal space of $\mathcal{C}_p$ is homeomorphic to the unit circle $\mathbb{T}$. In accordance with Allan's local principle, we introduce the local ideals $\mathcal{J}_t$ and the local algebras $\mathcal{A}_t := \mathcal{A}/\mathcal{J}_t\ at\ t \in \mathbb{T}$.

By Theorem (2.3.2) (b), the local ideal $\mathcal{J}_t$ lies in the kernel of smbt. We denote the related quotient homomorphism by smb_t again. Thus, smb_t is an algebra homomorphism from $\mathcal{A}_t$ onto $\sum_k^p(\mathbb{R})$, which sends the local cosets containing the operators $\operatorname{diag} P$ and $L(a)$ with $a \in \left(PC_p\right)_{k\times k}$ to $\operatorname{diag} \chi_+ I$ and $a(t^+)\operatorname{diag}Q_{\mathbb{R}} + a(t^-)\operatorname{diag}P_{\mathbb{R}}$, respectively. this homomorphism is injective, i.e., it is an isomorphism between $\mathcal{A}_t$ and $\sum_k^p(\mathbb{R})$.

Since $P_{\mathbb{R}}$ and $\operatorname{diag}\chi_+ I$ are projections, the algebra $\sum_k^p(\mathbb{R})$ is subject to the two projections Theorem with coefficients, as derived in [5]. Alternatively, this algebra can be described by means of the Mellin symbol calculus. In each case, the result is that an operator of the form

$$(a^+\operatorname{diag}\chi_+ I \ + \ a^-\operatorname{diag}\chi_- I)\operatorname{diag}P_{\mathbb{R}} \\ + \ (b + \operatorname{diag}\chi_+ I \ + \ b - \operatorname{diag}\chi_- I)\operatorname{diag}Q_{\mathbb{R}} \qquad (16)$$

Where $\chi_- := 1 - \chi_+$ and $a^{\pm}, b^{\pm} \in \mathbb{C}_{k\times k}$ is invertible if and only if the $(2k) \times (2k)$-matrix-valued function

$$\lambda \mapsto \begin{pmatrix} a + \operatorname{diag}(1 - \mu_p(\lambda)) + a^-\operatorname{diag}\mu_p(\lambda) & (b^+ - b^-)\, diag\ v_p(\lambda) \\ (a^+ - a^-)\operatorname{diag}v_p(\lambda) & b + \operatorname{diag}\mu_p(\lambda) + b - \operatorname{diag}(1 - \mu_p(\lambda)) \end{pmatrix}$$

is invertible at each point $\lambda \in \overline{\mathbb{R}}$. Note that the function

$$\lambda \mapsto a^+\operatorname{diag}(1 - \mu_p(\lambda)) + a^-\operatorname{diag}\mu_p(\lambda)$$

is continuous on $\overline{\mathbb{R}}$ and that this function connects a^+ with a^- if λ runs from$-\infty$ to $+\infty$. For the sake of index computation, one would prefer to work with afunction which connects a^- with a^+ if λ increases. Since $\mu_p(-\lambda) = 1 - \mu_q(\lambda)$ and $v_p(-\lambda) = v_q(\lambda)$ with q satisfying $1/p + 1/q = 1$, we obtain that the operatorA in (16) is invertible if and only if the matrix function

$$\lambda \mapsto \begin{pmatrix} a^+\operatorname{diag}\mu_q(\lambda) + a^-\operatorname{diag}(1 - \mu^q(\lambda)) & (b^+ - b^-)\operatorname{diag}v_q(\lambda) \\ (a^+ - a^-)\operatorname{diag}v_q(\lambda) & b^+\operatorname{diag}(1 - \mu^q(\lambda)) + b^-\operatorname{diag}\mu^q(\lambda) \end{pmatrix}$$

is invertible on $\overline{\mathbb{R}}$. This observation, together with the local principle, implies that the coset $L(a)\mathrm{diag}\, P + L(b)\mathrm{diag}\, Q + K(l^p(\mathbb{Z})_k)$ is invertible in the quotient algebra $\mathrm{T}^0_{k\times k}(PC_p)/K(l^p(\mathbb{Z})_k)$ if and only if the matrix function in assertion (a) of Theorem (2.3.1) is invertible. In particular, this gives the "if"-part of assertion (a).

The "only if"-part of this assertion follows from the inverse closedness assertion (d), which can be proved using ideas, where inverse closedness issues of two projections algebras with coefficients are studied. The proof of assertions (b)and (c) of Theorem (2.3.1) is then standard.

Step 4: The index formula. It remains to prove the index formula (e). First we have to equip the cylinder $\mathbb{T} \times \overline{\mathbb{R}}$ with a suitable topology, which will be different from the usual product topology. We provide $\mathbb{T}$ with the counter-clockwise orientation and $\overline{\mathbb{R}}$ with the natural orientation given by the order $<$. Then the desired topology is determined by the system of neighborhoods $U(t_0, \lambda_0)$ of the point $(t_0, \lambda_0) \in \mathbb{T} \times \overline{\mathbb{R}}$, defined by

$$U(t_0, -\infty) = \{(t,\lambda) \in \mathbb{T} \times \overline{\mathbb{R}}: |t - t_0| < \delta, t \prec t_0\} \cup \{(t_0, \lambda) \in \mathbb{T} \times \overline{\mathbb{R}} : \lambda < \varepsilon\},$$

$$U(t_0, +\infty) = \{(t,\lambda) \in \mathbb{T} \times \overline{\mathbb{R}}: |t - t_0| < \delta, t_0 \prec t\} \cup \{(t_0, \lambda) \in \mathbb{T} \times \overline{\mathbb{R}} : \varepsilon < \lambda\}$$

if $\lambda_0 = \pm\infty$ and by

$$U(t_0, \lambda_0) = \{(t_0, \lambda) \in \mathbb{T} \times \overline{\mathbb{R}} : \lambda_0 - \delta_1 < \lambda < \lambda_0 + \delta_2\}$$

if $\lambda_0 \in \mathbb{R}$, where $\varepsilon \in \mathbb{R}$ and $\delta, \delta_1, \delta_2$ are sufficiently small positive numbers, and where $t \prec s$ means that t precedes s with respect to the chosen orientation of$\mathbb{T}$. Note that the cylinder $\mathbb{T} \times \overline{\mathbb{R}}$, provided with the described topology, is just a homeomorphic image of the cylinder $\mathbb{T} \times [0,1]$, provided with the Gohberg-Krupnik topology. The latter has been shown by Gohberg and Krupnik to be homeomorphic to the maximal ideal space of the commutative Banach algebra$\mathrm{T}(PC_p)/K(l^p)$. If one identifies $\mathbb{T} \times [0,1]$ with$\mathbb{T} \times \overline{\mathbb{R}}$, then the Gelfand transform of a coset $A + K(l^p)$ of $A \in \mathrm{T}(PC_p)$ is just the function $\Gamma(A)$ defined in Theorem (2.2.1).

It is an important point to mention that while the function $\mathrm{smb}_p\, A$ for $A \in\in \mathrm{T}^0_{k\times k}(PC_p)$ is not continuous on $\mathbb{T} \times \overline{\mathbb{R}}$ (just consider the south-east entry of$\mathrm{smb}_p\, (L(a)P + L(b)Q)$), the function

$$(t,\lambda) \longmapsto \det \mathrm{smb}_p\, A(t,\lambda)/(det\, a_{22}(t,\infty)\, \det a_{22}(t,-\infty)$$

is continuous on $\mathbb{T} \times \overline{\mathbb{R}}$. This non-trivial fact was observed by Gohberg and Krupnik in a similar situation when studying the Fredholm theory for singular integral operators with piecewise continuous coefficients. We will establish the index formula by employing a method which also goes back to Gohberg and Krupnik and is known as linear extension. This method has found its first applications in the Fredholm theory of one-dimensional singular integral equations. We will use this method in the slightly different context of Toeplitz plus Hankel operators.

Let $\mathcal{B}$ be a unital ring with identity element e. With every $h \times r$-matrix $\beta := \left(b_{jl}\right)_{j,l=1}^{h,r}$ with entries in $\mathcal{B}$, we associate the element

$$el(\beta) = \sum_{j=1}^{h} b_{j1} \dots b_{jr} \in B \tag{17}$$

generated by β and call the b_{jl} the generators of $el(\beta)$. For each element of this form, there is a canonical matrix $\mathrm{ext}(\beta) \in \mathcal{B}_{s\times s}$ with $s = h(r + +1) + 1$ with entries in the set $\{0, e, b_{jk} : 1 \le j \le h, 1 \le k \le r\}$ and with the property that

$el(\beta)$ is invertible in $\mathcal{B}$ if and only if $\text{ext}(\beta)$ is invertible in $\mathcal{B}_{s\times s}$. Actually, a matrix with this property can be constructed as follows. Let

$$\text{ext}(\beta) := \begin{pmatrix} Z & X \\ Y & 0 \end{pmatrix} = \begin{pmatrix} e_{h(r+1)} & 0 \\ W & e \end{pmatrix} \begin{pmatrix} e_{h(r+1)} & 0 \\ 0 & el(\beta) \end{pmatrix} \begin{pmatrix} Z & X \\ 0 & e \end{pmatrix} \quad (18)$$

where e_l denotes the unit element of $\mathcal{B}_{l\times l}$,

$$Z := e_{h(r+1)} + \begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & B_r \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $B_j := \text{diag}\,(b_{1j}, b_{2j}, \dots, b_{hj}), X$ is the column $-(0, \dots, 0, e, \dots, e)^T$ with hr zeros followed by h identity elements, Y is the row $(e, \dots, e, 0, \dots, 0)$ with h identity elements followed by hr zeros, and $W := (M_0, M_1, \dots, M_r)$ with $M_0 :=$ $(e, \dots, e)$ consisting of h identity elements and

$M_j := (b_{11}b_{12} \dots b_{1j}, b_{21}b_{22} \dots b_{2j}, \dots, b_{h1}b_{h2} \dots b_{hj})$ for $j = 1, \dots, r$. The matrix $\text{ext}(\beta)$ in (18) is called the linear extension of $el(\beta)$.

Since the outer factors on the right-hand side of (18) are invertible, it follows indeed that $el(\beta)$ is invertible in $\mathcal{B}$ if and only if its linear extension $\text{ext}(\beta)$ isinvertible in $\mathcal{B}_{s\times s}$. As a special case we obtain that if the b_{jl} are bounded linear operators on some Banach space B, then $el(\beta)$ is a Fredholm operator on Bif and only if $\text{ext}(\beta)$ is a Fredholm operator on $L(B)_{s\times s} = L(B_s)$. Moreover,ind $el(\beta) = \text{ind}\,\text{ext}(\beta)$ is this case.

We shall apply this observation for $B = l^p(\mathbb{Z})_k$ and for the generating operators

$$b_{jl} := L(c_{jl})\text{diag}\, P + L(d_{jl})\text{diag}\, Q \; with \; c_{jl}, d_{jl} \in (PC_p)_{k\times k}. \quad (19)$$

Put $\beta := \left(b_{jl}\right)_{j,l=1}^{h,r}, \gamma \coloneqq \left(L\left(c_{jl}\right)\right)_{j,l=1}^{h,r}$ $and\ \delta \coloneqq \left(L\left(d_{jl}\right)\right)_{j,l=1}^{h,r}$. The linear extensions of γ and δ are Laurent operators again; thus ext$(\gamma) = L(c)$ and ext$(\delta) = L(d)$with piecewise continuous multipliers c and d. Moreover,

$$ext(\beta) = L(c)\text{diag}\, P + L(d)\text{diag}\, Q. \tag{20}$$

If $el(\beta)$ is a Fredholm operator then, by Theorem (2.3.1) (a), the matrices $c(t^{\pm})$ and $d(t^{\pm})$ are invertible for every $t \in \mathbb{T}$. Hence, c and d are invertible in $\left(PC_p\right)_{ks\times ks}$.

This fact together with the above observation implies that the operator $el(\beta)$is Fredholm on $l^p(\mathbb{Z})_k$ if and only if its linear extension ext(β) is Fredholm on$l^p(\mathbb{Z})_{ks}$, which on its hand holds if and only if the Toeplitz operator $T(d^{-1}c)$ is Fredholm on l^p_{ks}, and that the Fredholm indices of the operators $el(\beta), ext(\beta)$and $T(d^{-1}c)$ coincide in this case. The symbol of the Toeplitz operator $T(d^{-1}c)$is the function

$$\text{smb}_p\ (T(d^{-1}c))(t,\lambda) = (d^{-1}c)(t^{+})\text{diag}\mu_q(\lambda) + (d^{-1}c)(t^{-})\text{diag}\ (1-\mu_q(\lambda))$$

(which stems from the matrix-version of Theorem (1.7)), and smb_p (ext(β)) $=: \left(a_{ij}\right)_{i,j=1}^{2}$ is related with $\text{smb}_p\ (T(d^{-1}c))$ via

$$\det \text{smb}_p\ (T(d^{-1}c))(t,\lambda)$$
$$= \det(\text{smb}_p\text{ext}(\beta))(t,\lambda)/(\det a_{22}(t,\infty)\ \det a_{22}(t,-\infty))$$

as can be checked directly. This fact can finally be used to derive the index formula for Fredholm operators of the form $el(\beta)$ with the entries of β given by (19).

Since the operators $el(\beta)$ lie dense in $\mathrm{T}^0_{k\times k}(PC_p)$, the index formula for aFredholm operator in this algebra follows by a standard approximation argument.

To carry out this argument one has to use the estimate

$$\left\|\mathrm{smb}_p\ el(\beta)\right\| \leq M \inf_{K\in K(l^p(\mathbb{Z})_k)} \|el(\beta)\ +\ K\|$$

With M independent of β, which is an immediate consequence of (15).

Section (2.4): The index formula for $T\ +\ H$-operators

We provide an index formula for Fredholm operators of the form $T(a)\ +\ H(b)$ on l^p where a, b are multipliers in PC_p with a finite set ofdiscontinuities. We start with a couple of Lemma.

Lemma (2.4.1)[2] If a $\in C(\mathbb{T})\ \cap M^{\langle p\rangle}$, then $H(a)$ is compact on l^p.

Proof. It is shown that $C(\mathbb{T})\ \cap\ M^{\langle p\rangle} \subseteq C_p$ (in fact It is shown there that the closure of $C(\mathbb{T})\ \cap M^{\langle p\rangle}$ in the multiplier norm equals C_p) and that $H(a)$ is compact on l^p if $a\ \in C_p$.

For a subset Ω of $\mathbb{T}$, let $PC(\Omega)$ stand for the set of all piecewise continuousfunctions which are continuous on $T\setminus\Omega$, and put $PC_{\langle p\rangle}(\Omega) := PC(\Omega) \cap M^{\langle p\rangle}$.

Thus, $C_{\langle p\rangle} := PC_{\langle p\rangle}(\emptyset) = C(\mathbb{T})\ \cap\ M^{\langle p\rangle}$. We concludes that $PC_{\langle p\rangle}(\Omega)\ \subseteq\ PC_p$ if Ω is finite.

In what follows, we specify $\Omega_0 :=\ \{\tau_1, \ldots, \tau_m\}$ to be a finite subset of $\mathbb{T}\setminus\{\pm 1\}$ and put $\Omega := \Omega_0 \cup\ \{\pm 1\}$. Let $\varphi_0 \in C_{\langle p\rangle}$ be a multiplier which satisfies $\varphi = \tilde{\varphi}$,takes its values in $[0, 1]$, and is identically 1 on a certain neighborhood of $\{-1, 1\}$and identically 0 on a certain neighborhood of $\Omega_0 \cup \overline{\Omega_0}$. Moreover, we suppose that $\varphi_0^2 + \varphi_1^2 = 1$ where $\varphi_1 \coloneqq 1 - \varphi_0$.

Lemma (2.4.2)[2] Let $c \in PC_{\langle p \rangle}(\{-1,1\})$ and $d \in PC_{\langle p \rangle}(\Omega_0)$. Then the operators $H(c)T(d) - H(cd\varphi_0)$ and $T(c)H(d) - H(cd\varphi_1)$ are compact on l^p.

Proof. We write $H(c)T(d) = H(c)T(d)T(\varphi_0) + H(c)T(d)T(\varphi_1)$ with

$$H(c)T(d)T(\varphi_0) = H(c)\left(T(d\varphi_0) - H(d)H(\widetilde{\varphi_0})\right)$$

$$= H(cd\varphi_0) - T(c)H(\widetilde{d\varphi_0}) - H(c)H(d)H(\varphi_0),$$

$$H(c)T(d)T(\varphi_1) = H(c)T(\varphi_1)T(d) + H(c)\left(T(d)T(\varphi_1) - T(\varphi_1)T(d)\right)$$

$$= \left(H(c\varphi_1) - T(c)H(\widetilde{\varphi_1})\right)T(d)$$

$$+ H(c)H(d)H(\widetilde{\varphi_1}) - H(\varphi_1)H(\tilde{d}).$$

The operators $H(\widetilde{d\varphi_0}), H(\widetilde{\varphi_0}), H(c\varphi_1), H(\varphi_1)\ and\ H(\widetilde{\varphi_1})$ are compact by Lemma (2.4.1), which gives the first assertion. The proof of the second assertion proceeds similarly.

Lemma (2.4.3)[2] Let $a_0, b_0 \in PC_{\langle p \rangle}(\{-1,1\})$ and $a_1, b_1 \in PC_{\langle p \rangle}(0)$. Then the operator

$$\left(T(a_0) + H(b_0)\right)\left(T(a_1) + H(b_1)\right) - \left(T(a_0 a_1) + H(a_1 b_0 \varphi_0) + H(a_0 b_1 \varphi_1)\right)$$

is compact on l^p.

Proof. We write $\left(T(a_0) + H(b_0)\right)\left(T(a_1) + H(b_1)\right)$ as

$$T(a_0)T(a_1) + T(a_0)H(b_1) + H(b_0)T(a_1) + H(b_0)H(b_1)$$

$$= T(a_0 a_1) + K_1 + H(a_0 b\_1 \varphi_1) + K_2 + H(b_0 a_1 \varphi_0) + K_3 + K_4$$

where $K_1 := T(a_0)T(a_1) - T(a_0 a_1)$ and $K_4 := H(b_0)H(b_1) = T(b_0)T(\widetilde{b_1}) - T(b_0\widetilde{b_1})$ are compact on l^p , and $K_2 := T(a_0)H(b_1) - H(a_0 b_1 \varphi_1)$ and $K_3 := H(b_0)T(a_1) - H(b_0 a_1 \varphi_0)$ are compact by Lemma (2.4.2).

The following proposition provides us with a key observation; it will allow us to separate the discontinuities in Ω_0 and $\{-1, 1\}$.

Proposition (2.4.4)[2] Let $a, b \in PC_{\langle p\rangle}(\Omega)$. If the operator $T(a) + H(b)$ is Fredholm on l^p, then there are functions $a_0, b_0 \in PC_{\langle p\rangle}(\{-1,1\})$ and $a_1, b_1 \in PC_{\langle p\rangle}(\Omega_0)$such that $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$ are Fredholm operators on l^p and thedifference

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - (T(a) + H(b))$$

is compact.

Proof. If $T(a) + H(b)$ is Fredholm on l^p , then a is invertible in PC_p by Corollary(2.2.4) (a). Since the maximal ideal space of PC_p is independent on p and $a \in PC_{\langle p\rangle}$,one even has $a - 1 \in PC_{\langle p\rangle}$.

Let U and V be open neighborhoods of $\{-1, 1\}$ and $\Omega_0 \cup \overline{\Omega_0}$, respectively,

such that clos $U \cap$ clos $V = \emptyset$. We will assume moreover that $U = U_{-1} \cup U_1$ is the union of two open arcs such that $\pm 1 \in U_{\pm 1}$, and that $V = V_+ \cup V_-$ is the union oftwo open arcs such that $V_+ \subseteq \mathbb{T}_+^0$ and $V_- \subseteq \mathbb{T} \setminus \mathbb{T}_+^0$. Note that these conditions imply that clos$U_{-1} \cap$ clos$U_1 = \emptyset$.

Now we choose a continuous piecewise (with respect to a finite partition of $(\mathbb{T})$ linear function c on $\mathbb{T}$ which is identically 1 on clos , coincides with a on ∂U,and does not vanish on $\mathbb{T} \setminus U$. This function is of bounded total variation; thus$c \in$

$C(\mathbb{T}) \cap M^{\langle p \rangle}$, whence $c \in C_p$ as mentioned in the proof of Lemma (2.4.1). Put $a_0 := a\chi_U + c\chi_{T\setminus U}$. Then $a_0 \in PC_{\langle p \rangle}$ and $a_0^{-1} \in PC_{\langle p \rangle}$. Further, set $a_1 := a_0^{-1}a$.

The function a_1 is identically 1 on U and coincides with a on ∂U . Since $PC_{\langle p \rangle}$ isan algebra, a_1 belongs to $PC_{\langle p \rangle}$. Finally, set $b_0 := b\varphi_0$ and $b1 := b\varphi_1$, with $\varphi_0$and φ_1 as in front of Lemma (2.4.2).

The above construction guarantees that $a_0, b_0 \in PC_{\langle p \rangle}(\{-1,1\})$ and $a_1, b_1 \in PC_{\langle p \rangle}(\Omega_0)$, and the operator

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - \big(T(a_0a_1) + H(a_1b_0\varphi_0) + H(a_0b_1\varphi_1)\big)$$

is compact on l^p by Lemma (2.4.3). The functions $(a_1 - 1)b_0\varphi_0$ and $(a_0 - 1)b_1\varphi_1$vanish identically on a certain neighborhood of Ω by their construction. Hence,the Hankel operators $H((a_1 - 1)b_0\varphi_0)$ and $H((a_0 - 1)b_1\varphi_1)$ are compact by Lemma(2.4.1), which implies that the operator

$$(T(a_0) + H(b_0))(T(a_1) + H(b_1)) - \big(T(a_0a_1) + H(b_0\varphi_0) + H(b_1\varphi_1)\big)$$

is compact. Since $a_0a_1 = a$ and $b_0\varphi_0 + b_1\varphi_1 = b(\varphi_0^2 + \varphi_1^2) = b$, and since $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$ are Fredholm operators on l^p by Theorem (2.2.3), the assertion follows.

By the previous proposition,

$$\text{ind}\,(T(a) + H(b)) = \text{ind}\,(T(a_0) + H(b_0)) + \text{ind}\big(T(a_1) + H(b_1)\big).$$

Since $H(b_0) \in \mathrm{T}(PC_p)$ as already mentioned, and since an index formula forFredholm operators in $\mathrm{T}(PCp)$ is known, the determination of $\text{ind}\big(T(a_0) +$

$H(b_0))$ is no serious problem. The following Theorem provides uswith a basic step on the way to compute the index of $T(a_1) + H(b_1)$.

Theorem (2.4.5)[2] Let $a, b \in PC_{\langle p \rangle}(\Omega_0)$. If one of the operators $T(a) \pm H(b)$ isFredholm on l^p, then the other one is Fredholm on l^p, too, and the Fredholm indices of these operators coincide.

Proof. By Corollary (2.2.4) (b), the operators $T(a) + H(b)$ and $T(a) - H(b)$ are Fredholm operators on l^p only simultaneously. It remains to prove that their indices coincide. Recall from the introduction that $T(a) = PL(a)P \; and \; H(a) = PL(a)QJ$. Thus, the index equality will follow once we have constructed a Fredholm operator D such that the difference

$$D(PL(a)P + PL(b)QJ + Q) - (PL(a)P - PL(b)QJ + Q)D \qquad (21)$$

is compact. The following construction of D is a modification .

(Note that the compactness of the operator (21) also provides an alternate proof of the simultaneous Fredholm property of the operators $T(a) \pm H(b)$.)

A function $c \in M_p$ is called even (resp. odd) if $c = \tilde{c}$ (resp. $c = -\tilde{c}$) or, equivalently, if $JL(c)J = L(c)$ (resp. $JL(c)J = -L(c)$). Every function $c \in C_p$ can be written as a sum of an even and an odd function in a unique way: $c = (c + \tilde{c})/2 + (c - \tilde{c})/2$. Let θ_o and θ_e be an odd and an even function in $C(\mathbb{T}) \cap M^{\langle p \rangle}$ i, respectively, and assume that θ_e vanishes at all points of Ω_0 (and,hence, at all points of $\overline{\Omega_0}$). Put

$$D := PL(\theta_o + \theta_e)P + QL(\theta_o - \theta_e)Q. \qquad (22)$$

We will later specify the functions θ_o and θ_e such that D becomes a Fredholm operator. First note that

$$JPL(\theta_o+\theta_e)PJ=-QL(\theta_o-\theta_e)Q, JQL(\theta_o-\theta_e)QJ=-PL(\theta_o+\theta_e)P,$$

whence $JDJ=-D$ and $JD+DJ=0$. Next we show that D commutes with the operator $PL(a)P+PL(b)Q+Q$ up to a compact operator. Since the Toeplitz operators $PL(\theta_o+\theta_e)P$ and $PL(a)P$ commute modulo a compact operator, it remains to show that D commutes with $PL(b)Q$ up to a compact operator. The latter fact follows easily from the identity

$$\begin{aligned}DPL(b)Q-PL(b)QD&=PL(\theta_o+\theta_e)PL(b)Q-PL(b)QL(\theta_o-\theta_e)Q\\&=PL(\theta_o+\theta_e)L(b)Q-PL(\theta_o+\theta_e)QL(b)Q-PL(b)L(\theta_o-\theta_e)Q\\&+PL(b)PL(\theta_o-\theta_e)Q\\&=2PL(\theta_e b)Q-PL(\theta_o+\theta_e)QL(b)Q+PL(b)PL(\theta_o-\theta_e)Q\end{aligned}$$

and from the compactness of the operators $PL(\theta_e b)Q$ and $PL(\theta_o\pm\theta_e)Q$ by Lemma (2.4.1) (note that $\theta_e b\in C(\mathbb{T})\cap M^{\langle p\rangle}$). The compactness of the operator (21) is then aconsequence of the identity

$$\begin{aligned}D(PL(a)P+PL(b)QJ+Q)-(PL(a)P-PL(b)QJ+Q)D\\=DPL(a)P-PL(a)PD+DPL(b)QJ+PL(b)QJD\\=DPL(a)P-PL(a)PD+(DPL(b)Q-PL(b)QD)J\end{aligned}$$

and of the compactness of the commutators $[D,PL(a)P]$ and$[D,PL(b)Q]$.

Finally we show that the functions θ_e and θ_o can be specified such that the operator D in (22) is a Fredholm operator on l^p. Set $\theta_o(t):=|t^2-1|^2$ for $t\in\mathbb{T}$.

Then θ_o is an even function in $C^\infty(\mathbb{T})$ and $\theta_o:=\chi_{\mathbb{T}_+}\theta_o-\chi_{-}\mathbb{T}_{-}\theta_o$ is an odd function in $C(\mathbb{T})\cap M^{\langle p\rangle}$. Further,

$$\theta_e(t) := i\prod_{j=1}^{m}\left|t-\tau_j\right|^2\left|t-\overline{\tau_j}\right|^2, \qquad t \in \mathbb{T}$$

Defines an even function $\theta_e \in C(\mathbb{T}) \cap M^{\langle p\rangle}$ which vanishes at the points of Ω_0.

Since θ_o and $i\theta_e$ are real-valued functions, we conclude that $\theta_o \pm \theta_e$ are invertible in $C(\mathbb{T}) \cap M^{\langle p\rangle}$, which implies that D is a Fredholm operator as desired.

Now we are in a position to derive an index formula for a Fredholm operator of the form $T(a) + H(b)$ with $a, b \in PC_{\langle p\rangle}(\Omega_0)$. We make use of the well-known identity

$$\begin{pmatrix} PL(a)P + PL(b)QJ + Q & 0 \\ 0 & PL(a)P - PL(b)QJ + Q \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} I & J \\ I & -J \end{pmatrix}\begin{pmatrix} PL(a)P + Q & PL(b)Q \\ JPL(b)QJ & J(PL(a)P + Q)J \end{pmatrix}\begin{pmatrix} I & I \\ J & -J \end{pmatrix}, \quad (23)$$

where the outer factors in (23) are the inverses of each other. Thus, if one of the operators $T(a) \pm H(b) = PL(a)P \pm PL(b)QJ$ is a Fredholm operator, then so is the other, and the Fredholm indices of these operators coincide. Hence the middle factor

$$\begin{pmatrix} PL(a)P + Q & PL(b)Q \\ JPL(b)QJ & J(PL(a)P + Q)J \end{pmatrix} = \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix}$$

in (23) is a Fredholm operator, and

$$\operatorname{ind}\big(T(a) + H(b)\big) = \frac{1}{2}\operatorname{ind}\begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix}$$

$$= \frac{1}{2}\operatorname{ind}\begin{pmatrix} PL(a)P & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q \end{pmatrix}.$$

For the latter identity note that the operator

$$A := \begin{pmatrix} PL(a)P + Q & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q + P \end{pmatrix} \in L(l^p(\mathbb{Z})_2)$$

has the complementary subspaces $L_1 := \{(Qx_1, Px_2) : (x_1, x_2) \in l^p(\mathbb{Z})_2\}$ and $L_2 := \{(Px_1, Qx_2) : (x_1, x_2) \in l^p(\mathbb{Z})_2\}$ of $l^p(\mathbb{Z})_2$ as invariant subspaces and that A acts on L_1 as the identity operator and on L_2 as the operator

$$A_0 := \begin{pmatrix} PL(a)P & PL(b)Q \\ QL(\tilde{b})P & QL(\tilde{a})Q \end{pmatrix}$$

Let the function $W : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{C}$ be defined by

$$W(t, \lambda) = \det \operatorname{smb}_p A_0(t, \lambda)/(\tilde{a}(t, \infty)\tilde{a}(t, -\infty)) .$$

Since $T(a) + H(b)$ is Fredholm, W does not pass through the origin, and Theorem (2.3.1) entails that $\operatorname{ind} A_0 = -\operatorname{wind} W$. Thus,

$$\operatorname{ind}\big(T(a) + H(b)\big) = -\frac{1}{2} \operatorname{wind} W.$$

We are going to show that actually

$$\operatorname{ind}\,(T(a) + H(b)) = -\operatorname{wind} T + W, \tag{24}$$

where the right-hand side is defined as follows. The compression of W onto $\mathbb{T}_+ \times \overline{\mathbb{R}}$ is a continuous function the values of which form a closed oriented curve in $\mathcal{C}$ which starts and ends at $1 \in \mathbb{C}$ and does not contain the origin. The winding number of this curve is denoted by $\operatorname{wind} \mathbb{T}_+ W$. Analogously, we define $\operatorname{wind} T_- W$.

For the proof of (24) we suppose for simplicity that a and b have jumps only at the points t_1 and $\overline{t_1}$ where $t_1 \in \mathbb{T}_+^0$. If t moves along $\mathbb{T}_+$ from 1 to t_1 (resp.on $\mathbb{T}_-$ from 1 to $\overline{t_1}$), then the values of $W(t, \lambda) = a(t)/\tilde{a}(t) = a(t)/a(t)$ move continuously from 1 to $a(t_1^-)/a(\overline{t_1}^+)$ (resp. from 1 to $a(\overline{t_1}^+)/a(t_1^-)$) . Using that $W(t,\lambda) = W(\bar{t},\lambda)^{-1}$ for $t \in \mathbb{T} \setminus \{-1,1\}$, one easily concludes that

$$[\arg W]_{1\to t_1 \subset \mathbb{T}_+} = [\arg W]_{\overline{t_1}\to 1 \subset T_-}$$

where the numbers on the left- and right-hand side stand for the increase of the argument of W if t moves in positive direction along the arc from 1 to t_1 in $\mathbb{T}_+$and along the arc from $\overline{t_1}$ to 1 in $\mathbb{T}_-$, respectively. Analogously,

$$[\arg W]_{-1\to \overline{t_1} \subset \mathbb{T}_-} = [\arg W]_{t_1\to -1 \subset \mathbb{T}_+}.$$

Consider

$$\begin{aligned} W(t_1,\lambda)/(a(\overline{t_1}^+)a(\overline{t_1}^-)) \\ &= [a(t_1^+)\mu_q(\lambda) + a(t_1^-)(1 - \mu_q(\lambda))]\,[a(\overline{t_1}^+)\mu_q(\lambda) + a(\overline{t_1}^-)(1 \\ &\quad - \mu_q(\lambda))] - (b(t_1^+) - b(t_1^-))(b(\overline{t_1}^+) - b(\overline{t_1}^-))\,\mu_q(\lambda)(1 - \mu_q(\lambda)) \end{aligned}$$

and the related expression for $W(\overline{t_1},\lambda)/(a(t_1^+)a(t_1^-))$. Then

$$[\arg W]_{\mathcal{C}_q\left(a(t_1^-),a(t_1^+)\right)} = [\arg W]_{\mathcal{C}_q\left(a(\overline{t_1}^-),a(\overline{t_1}^+)\right)}$$

Because $W(t_1,\lambda)/(a(\overline{t_1}^+)a(\overline{t_1}^-)) = W(t_1,\lambda)/(a(t_1^+)a(t_1^-))$. So we arrive at the equality $\text{wind}_{\mathbb{T}_+}W = \text{wind}_{\mathbb{T}_-}W$, whence (24) follows.

Now suppose that $a,b \in PC_{\langle p\rangle}$ are continuous on $\mathbb{T}\setminus\{-1,1\}$. Then we definea function $W : \mathbb{T}_+ \times \overline{\mathbb{R}}$ by

$$\begin{aligned} W(t,\lambda) &= \Big(a(t^+)\mu_q(\lambda) + a(t^-)\left(1 - \mu_q(\lambda)\right) \\ &\qquad + it\big(b(t^+) - b(t^-)\big)v_q(\lambda)\Big)\,a^{-1}\left(\pm 1^{\mp}\right) \end{aligned}$$

if $t = \pm 1$ and by $W(t,\lambda) = a(t)/a(t)$ if $t \in \mathbb{T}_+^0$. The function W is continuous and determines a closed curve which starts and ends at $1 \in \mathbb{C}$. If $T(a) + H(b)$ is a Fredholm operator, then this curve does not pass through the origin and possesses, thus, a well defined winding number.

Since $T(a) + H(b)$ is in $\mathrm{T}(PC_p)$ and the symbol $V : \mathbb{T} \times \overline{\mathbb{R}} \to \mathbb{C}$ of this operator relative to the algebra $\mathrm{T}(PC_p)$ is known (it is just given by

$$V(t,\lambda) = a(t^+)\mu_q(\lambda) + a(t^-)(1 - \mu_q(\lambda)) + it\big(b(t^+) - b(t^-)\big)v_q(\lambda)$$

if $t = \pm 1$ and by $V(t,\lambda) = a(t)$ if $t \in \mathbb{T}\backslash\{-1,1\}$) and since $\operatorname{ind} T(a) = -\mathrm{wind}_{\mathbb{T}} V$,one can again prove that $\mathrm{wind}_{\mathbb{T}} V = \mathrm{wind}_{\mathbb{T}_+} W$ by comparing the increments of the arguments as above.

Now we look at the factorization given by Proposition (2.4.4) and denote by W_0 and W_1 the above defined function $W : \mathbb{T}_+ \times \overline{\mathbb{R}}$ for the operators $T(a_0) + H(b_0)$ and $T(a_1) + H(b_1)$, respectively. It is easy to see that $W_0 W_1$ coincides with the function W for the operator $T(a) + H(b)$. Summarizing, we get

Theorem (2.4.6)[2] Let $a, b \in PC_{\langle p \rangle}$ and $T(a) + H(b)$ a Fredholm operator on l^p. Then

$$\operatorname{ind}(T(a) + H(b)) = -\mathrm{wind}_{\mathbb{T}_+} W_0 - \mathrm{wind}_{\mathbb{T}_+} W_1 = -\mathrm{wind}_{\mathbb{T}_+} W$$

with W, W_0 and W_1 defined as above.

We want to sketch an approach to derive an index formula for an arbitrary Fredholm operator $A \in \mathrm{TH}(PC_p)$. With A, we associate the function $W(A) : \mathbb{T}_+ \times \overline{\mathbb{R}} \to \mathbb{C}$ defined by

$$W(A)(t,\lambda) = \begin{cases} \text{smb}_p\, A(t,\lambda)/\text{smb}_p\, A(t,\mp\infty) & \text{if } t = \pm 1 \\ \det \text{smb}_p\, A(t,\lambda)/(a_{22}(t,\infty)a_{22}(t,-\infty)) & \text{if } t \neq \pm 1 \end{cases}$$

where we wrote $smb_p \quad A(t,\lambda) = \left(a_{ij}(t,\lambda)\right)_{i,j=1}^{2}$ for $t \in \mathbb{T}_+^0$. For $A = T(a) + H(b)$,this definition coincides with that one from the previous section.

Theorem (2.4.7)[2] If $A \in \text{TH}(PC_p)$ is a Fredholm operator, then

$$\text{ind}\, A = -\text{wind}_{\mathbb{T}_+} W(A). \tag{25}$$

We devoted to the proof of this Theorem. It will become evident from this proof that $W(A)$ traces out a closed oriented curve which does not pass through the origin; so the winding number of $W(A)$ is well defined.

We start with the observation that Theorem (2.2.3) remains true for matrix-valued multipliers, $b \in \left(PC_p\right)_{k\times k}$: just replace $\mu_q, 1-\mu_q$ and v_q by the corresponding $k \times k$ -diagonal matrices $\text{diag}\mu_q, \text{diag}\left(1-\mu_q\right)$ and $\text{diag}v_q$, respectively. Also Proposition (2.2.2) holds in the matrix setting: If

$$T(a) + H(b) := (\text{diag}\, P)L(a)(\text{diag}\, P) + (\text{diag}\, P)L(b)(\text{diag}\, QJ)$$

is a Fredholm operator, then the identity

$$\text{ind}\big(T(a) + H(b)\big) = -\text{wind}\, W(T(a) + H(b))$$

still holds if one replaces in the above definition of W all scalars by the determinants of the corresponding matrices. These facts follow in a similar way as their scalar counterparts.

Now let $a_{jl}, b_{jl} \in PC_p$, consider the $h \times r$-matrix $\beta \coloneqq \left(T\left(a_{jl}\right) + H\left(b_{jl}\right)\right)_{j,l=1}^{h,r}$,and associate with β the operator

$$A \coloneqq el(\beta) = \sum_{j=1}^{h} (T(a_{j1}) + H(b_{j1})) \cdots (T(a_{jr}) + H(b_{jr})) \in \mathrm{TH}(PC_p)$$

as in (17). Further set $\gamma \coloneqq \left(L\left(a_{jl}\right)\right)_{j,l=1}^{h,r}$ and $\delta \coloneqq \left(L\left(b_{jl}\right)\right)_{j,l=1}^{h,r}$. The linear extensions of γ and δ are Laurent operators again; thus $\mathrm{ext}(\gamma) = L(a)$ and$\mathrm{ext}(\delta) = L(b)$ with certain multipliers $a, b \in \left(PC_p\right)_{s\times s}$ with $s = h(r + 1) + 1$.

Moreover, these extensions are related with the extension of β by

$$\mathrm{ext}(\beta) = T(\mathrm{ext}(\gamma)) + H(\mathrm{ext}(\delta)) = T(a) + H(b) \in L(l_s^p)$$

(note that $H(1) = 0$). We noticed that if $el(\beta)$ is Fredholm, then(and only then) $\mathrm{ext}(\beta)$ is Fredholm and $\mathrm{ind}\, el(\beta) = \mathrm{ind}\,\mathrm{ext}(\beta)$. Further, if $el(\beta)$is a Fredholm operator, then the matrices $a(t^{\pm})$ are invertible for every $t \in \mathbb{T}$.

Hence, a is invertible in $\left(PC_p\right)_{s\times s}$. Now consider

$$\mathrm{smb}_p el(\beta) = \sum_{j=1}^{h} \mathrm{smb}_p(T(a_{j1}) + H(b_{j1})) \cdots \mathrm{smb}_p(T(a_{jr}) + H(b_{jr})).$$

Let $t \neq \pm 1$. Then $\mathrm{smb}_p(T(a) + H(b))(t, \lambda)$ is a matrix of size $2s \times 2s$. We put the rows and columns of this matrix in a new matrix according to the following rules: If $j \leq h(r + 1) + 1$, then the j th row of the old matrix becomes the $2j - 1$ th row of the new one, whereas if $j > h(r + 1) + 1$, the j th row of the old matrix becomes the $2(j-h(r+1)-1)$ th row of the new matrix. The columns of $\mathrm{smb}_p(T(a)\ +\ H(b))(t,\lambda)$ are rearranged in the same way. The matrix obtained in this way is just $mb_p\ el(\beta)(t,\lambda)$. By these manipulations,

$$\mathrm{smb}_p el(\beta)(t,\lambda)=\mathcal{P}\mathrm{smb}_p\big(T(a)+H(b)\big)(t,\lambda)\mathcal{P}^T$$

with a certain permutation matrix $\mathcal{P}$ and its transpose $\mathcal{P}^T$. Hence,

$$\det \mathrm{smb}_p(T(a)+H(b))(t,\lambda)=\det \mathrm{smb}_p(el(\beta))(t,\lambda)$$

for $t\neq\pm1$. For $t\ =\ \pm1$ we do not change the matrix $\mathrm{smb}_p(T(a)+H(b))(t,\lambda)$.

For $t\neq\pm1$, we write $\mathrm{smb}_p(T(a)+H(b)(t,\lambda)=\big(a_{mn}(t,\lambda)\big)_{m,n=1}^{2}$ and

$$\mathrm{smb}_p(T(a_{jl})+H(b_{jl}))(t,\lambda))=(a_{mn}^{jl}(t,\lambda))_{m,n=1}^{2}.$$

Then

$$\mathrm{smb}_p el(\beta)(t,\pm\infty)=\sum_{j=1}^{h}\prod_{l=1}^{r}\begin{pmatrix} a_{11}^{jl}(t,\pm\infty) & 0\\ 0 & a_{22}^{jl}(t,\pm\infty)\end{pmatrix}$$

and it follows that

$$\det a_{22}(t,\pm\infty)=\det \mathrm{ext}(\rho(t,\pm\infty))$$

Where $(t,\pm\infty):=\Big(a_{22}^{jl}(t,\pm\infty)\Big)_{j,l=1}^{hr}$. It is now easy to see that

$$W(el(\beta))(t,\lambda)\ =\ W(T(a)+H(b))(t,\lambda)=W(\mathrm{ext}(\beta))(t,\lambda)$$

for all $(t,\lambda)\in\mathbb{T}_+\times\overline{\mathbb{R}}$, which implies that ind $el(\beta)=-\mathrm{wind}_{\mathbb{T}_+}W(el(\beta))$ and, thus, settles the proof of the index formula (25) for a dense subset of Fredholm operators in $\mathrm{TH}\big(PC_p\big)$.

Finally, we are going to prove estimate (3), i.e., we will show that there is aconstant M such that

$$\left\|\mathrm{smb}_p A\right\|_\infty \leq M \inf\{\|A+K\|: K \text{ compact}\} \qquad (26)$$

for every operator $A \in \mathrm{TH}(PC_p)$. Once this estimate is shown, the validity of the index formula (25) for an arbitrary Fredholm operator in $\mathrm{TH}(PC_p)$ will follow by standard approximation arguments .

To prove (26), we consider $\mathrm{TH}(PC_p)$ as a subalgebra of the smallest closed sub algebra $\mathrm{T}_J^0(PC_p)$ of $L(l^p(\mathbb{Z}))$ which contains all Laurent operators $L(a)$ with$a \in PC_p$, the projection P, and the flip J. The homomorphism smbt defined cannot be extended to the algebra $\mathrm{T}_J^0(PC_p)$ unless $t = \pm 1$. Instead, we are going to use ideas and introduce a related family of homomorphisms smbt,t with $t \in \mathbb{T}_+^0$ from $\mathrm{T}_J^0(PCp)$ onto $\left(\sum_1^p(\mathbb{R})\right)_{2\times 2}$. A crucial observation is that the strong limit

$$\mathrm{smb}_{t,\bar{t}}\, A := s - \lim_{n\to\infty} \begin{pmatrix} A_{t,n.0,0} & A_{t,n,0,1} \\ A_{t,n,1,0} & A_{t,n,1,1} \end{pmatrix} \qquad (27)$$

with $A_{t,n,i,j} := E_n Y_t^{-1} L(\chi_{\mathbb{T}^+}) J^i A J^j L(\chi_{T^+}) Y_t E_{-n} L_n$ exists for every operator $A \in \mathrm{T}_J^0(PC_p)$ and every $t \in \mathbb{T}_+^0$.

Theorem (2.4.8)[2] Let $t \in \mathbb{T}_+^0$. Then the mapping smbt,t is a bounded homomorphism from $\mathrm{T}_J^0(PC_p)$ onto $\left(\sum_1^p(\mathbb{R})\right)_{2\times 2}$. In particular,

(a) $\mathrm{smb}_{t,\bar{t}}\, P = \mathrm{diag}\,(\chi_+ I, \chi_- I)$ with $\chi_- = 1 - \chi_+$,

(b) $\mathrm{smb}_{t,\bar{t}}\, L(a) = \mathrm{diag}(a(t^+)Q_{\mathbb{R}} + a(t^-)P_{\mathbb{R}}, a(\bar{t}^-)Q_{\mathbb{R}} + a(\bar{t}^+)P_{\mathbb{R}})$for $a \in PC_p$,

(c) $\mathrm{smb}_{t,\bar{t}}\, K = 0$ for every compact operator K,

(d) $\mathrm{smb}_{t,\bar{t}} J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.

Sketch of the proof. The existence of the strong limits of the operators in

(a) - (d) and their actual values follow by straightforward computation. Let us check assertion (a), for instance. For $A = P$, the strong limits of the diagonal elements of the matrix (27) exist and are equal to $\chi_+ I$ and $\chi_- I$ by Theorem (2.3.2) (a) and since $JPJ = Q$. Now consider the 01-entry of that matrix. It is$L(\chi_{\mathbb{T}^+})PJ = JL(\chi_{\mathbb{T}^-})Q$ and thus

$$E_n Y_t^{-1} L(\chi_{\mathbb{T}^+}) PJL(\chi_{\mathbb{T}^+}) Y_t E_{-n} L_n$$
$$= (E_n Y_t^{-1} J Y_t E_{-n})(E_n Y_t^{-1} L(\chi_{\mathbb{T}^-}) QL(\chi_{\mathbb{T}^+}) Y_t E_{-n} L_n). \qquad (28)$$

The first factor on the right-hand side is uniformly bounded with respect to n,whereas the second one tends strongly to 0 by Theorem (2.3.2) (note that $\chi_{\mathbb{T}^-}(t) = 0$ for $t \in \mathbb{T}_+^0$). Thus, the sequence of the operators (28) tends strongly to zero. The strong convergence of the 10-entry to zero follows analogously.

Another straightforward calculation shows that the mappings $\mathrm{smb}_{t,\bar{t}}$ are algebra homomorphisms and that these mappings are uniformly bounded with respect to $t \in \mathbb{T}_+^0$. Thus, the mappings $\mathrm{smb}_{t,\bar{t}}$ are well-defined on a dense subalgebra of $\mathrm{T}_J^0(PC_p)$, and they extend to (uniformly bounded with respect to t) homomorphisms on all of $\mathrm{T}_J^0(PC_p)$ by continuity.

By assertion (c) of the previous Theorem, every mapping $\mathrm{smb}_{t,\bar{t}}$ induces a quotient homomorphism on $\mathrm{T}_J^0(PC_p)/K(l^p(\mathbb{Z}))$ in a natural way. We denote this homomorphism by $\mathrm{smb}_{t,\bar{t}}$ again.

Now we are ready for the last step. Let $t \in \mathbb{T}_+^0$ and $a, b \in PC_p$. From Theorem (2.4.8) we conclude that then the operator $\mathrm{smb}_{t,\bar{t}}(T(a) + H(b))$ is given bythe matrix

$$\begin{pmatrix} \chi_+(a(t^+)Q_{\mathbb{R}} + a(t^-)P_{\mathbb{R}})\chi_+ I & \chi_+(b(t^+)Q_{\mathbb{R}} + b(t^-)P_{\mathbb{R}})\chi_- I \\ \chi_-(b(\bar{t}^-)Q_{\mathbb{R}} + b(\bar{t}^+)P_{\mathbb{R}})\chi_+ I & \chi_-(a(\bar{t}^-)Q_{\mathbb{R}} + a(\bar{t}^+)P_{\mathbb{R}})\chi_- I \end{pmatrix}$$

acting on $L^p(\mathbb{R})_2$. This matrix operator has the complementary subspaces

$$L_1 \coloneqq \{(\chi_- f_1, \chi_+ f_2) : f_1, f_2 \in L^p(\mathbb{R})\}, L_2 \coloneqq \{(\chi_+ f_1, \chi_- f_2) : f_1, f_2 \in L^p(\mathbb{R})\}$$

of $L^p(\mathbb{R})_2$ as invariant subspaces, and it acts as the zero operator on L_1. So wecan identify $\mathrm{smb}_{t,\bar{t}}(T(a) + H(b))$ with its restriction to L_2, which we denote byA_0 for brevity.

The space L_2 can be identified with $L^p(\mathbb{R})$ in a natural way. Under this identification, the operator A_0 can be identified with the operator

$$A_1 \coloneqq \chi_+(a(t^+)Q_{\mathbb{R}} + a(t^-)P_{\mathbb{R}})\chi_+ I + \chi_+(b(t^+)Q_{\mathbb{R}} + b(t^-)P_{\mathbb{R}})\chi_- I$$

$$+\chi_-(b(\bar{t}^-)Q_{\mathbb{R}} + b(\bar{t}^+)P_{\mathbb{R}})\chi_+ I + \chi_-(a(\bar{t}^-)Q_{\mathbb{R}} + a(\bar{t}^+)P_{\mathbb{R}})\chi_- I$$

which belongs to $\sum^p(\mathbb{R})$. It is well known and not hard to check that the algebra $\sum^p(\mathbb{R})$ is isomorphic to $\sum^p_{2\times 2}(\mathbb{R}_+)$, where the isomorphism$\eta$ acts on the generating operators of $\sum^p(\mathbb{R})$ by

$$\eta(S_{\mathbb{R}}) = \begin{pmatrix} S_{\mathbb{R}_+} & H_\pi \\ -H_\pi & -S_{\mathbb{R}_+} \end{pmatrix} \quad and \quad \eta(\chi_+ I) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

with H_π referring to the Hankel operator

$$(H_\pi \varphi)(s) := \frac{1}{\pi i} \int_{\mathbb{R}_+} \frac{\varphi(t)}{t + s} dt$$

on $L^p(\mathbb{R}_+)$. The entries of the matrix $\eta(A_1)$ are Mellin operators, and the value of the Mellin symbol of $\eta(A_1)$ at $(t,\lambda) \in \mathbb{T}_+^0 \times \overline{\mathbb{R}}$ is the matrix

$$\begin{pmatrix} a(t^+)\mu_q(\lambda) + a(t^-)\left(1-\mu_q(\lambda)\right) & \left(b(t^+)-b(t^-)\right)v_q(\lambda) \\ \left(b(\bar{t}^-)-b(\bar{t}^+)\right)v_q(\lambda) & a(\bar{t}^-)(1-\mu_q(\lambda)) + a(\bar{t}^+)\mu_q(\lambda) \end{pmatrix},$$

which evidently coincides with $\mathrm{smb}_p(T(a)+H(b))(t,\lambda)$ given in (1). Summarizing the above arguments we conclude that the homomorphisms

$$A + K(l^p) \longmapsto (\mathrm{smb}_p\, A)(t,\lambda)$$

are uniformly bounded with respect to $(t,\lambda) \in \mathbb{T}_+^0 \times \overline{\mathbb{R}}$, which finally implies the estimate (26).

Chapter 3

Hankel Operators

The mode space whose products result in truncated Toeplitz operators when the inner function u a certain symmetric property.

In 2007, D. Sarason defined truncated Toeplitz operators (TTO) as the compression of Toeplitz operators to invariant subspaces for the backward shift on the Hardy space H^2. Toeplitz matrices can be interpreted as truncated Toeplitz operators on finite dimensional model spaces. Recently, C. Cu defined truncated Hankel operators (THO) as the compression of Hankel operators to invariant subspaces for the backward shift and proved a number of algebraic properties of them. Some of the properties reveal the relation between the THO's and TTO's. We will consider when the product of' two THO's becomes a TTO.

$Let\ L^2 = L^2(\mathbb{T})$be the set of all square-integrable functions on the unit circle 1' in the complex plane $\mathbb{C}$ and $H^2 = H^2(\mathbb{T})$ be the corresponding Hardy space, i.e., the closed linear span of the analytic polynomials inL^2. The space H^∞ is defined by $H^\infty \coloneqq H^\infty(\mathbb{T}) \cap L^\infty(\mathbb{T})$. A function $\theta \in H^\infty$ is called inner $if\ |\theta(z)| =$ almost everywhere on the unit circle T.

For $\emptyset \in L^\infty$, the Toeplitz operator $T_\emptyset$ onH^2 is defined by

$$T_\emptyset f = p(\emptyset f),$$

Where P is the orthogonal projection of L^2onto H^2. The Hankel operator $H_\emptyset : H^2 \to H^2$ with symbol $\emptyset \in L^\infty$ is defined by

$$H_{\emptyset} f = J(I - P)(\emptyset f),$$

where J denotes the unitary map on L^2 defined by $(Jf)(\mathcal{Z}) = \bar{\mathcal{Z}} f(\bar{\mathcal{Z}})$

For a non constant inner function u, define the model space K_u^2 by

$$K_u^2 := H^2 \ominus uH^2.$$

It is known that the dimension of K_u^2 is finite if and only if u is a finite Blaschke product and in that case, dim K_u^2 equals the number of zeros of u counting multiplicity. The dimension of K_u^2 is also called the degree of the inner function u and is denoted by deg it. If it is not a finite Blaschke product, we say that the degree of it is infinite. The following set equality is easily verified

$$K_u^2 = u\overline{\mathcal{Z} K_u^2} \qquad (1)$$

For a function $\emptyset \in L^2(\mathbb{T})$, the truncated Toeplitz operator $A_{\emptyset}$ on K_u^2 is defined by

$$A_{\emptyset} f = P_u(\emptyset f), \text{for } f \in K_u^2,$$

Where P_u denotes the orthogonal projection of L^2 onto K_u^2. For a function $\emptyset \in L^2(\mathbb{T})$, a truncated Hankel operator $B_{\emptyset}$ on K_u^2 is defined by

$$B_{\emptyset} f = P_u J(I - P)\emptyset f, \text{for each } f \in K_u^2.$$

It is easy to see that $B_{\emptyset}$, does not depend on the analytic part of the symbol function$\emptyset$. So, we often assume $\emptyset \in \emptyset\overline{H^2}$when is the symbol function of a then $B_{\emptyset}$ or $A_{\emptyset}$ can be an unbounded operator. Since we are mainly concerned with bounded operators, we denote the set of all bounded truncated Toeplitz operators truncated Hankel operator. If $\emptyset \in L^2(\mathbb{T})$ is not an essentially bounded function, by $\mathfrak{H}(K_u^2)$ and the set of all bounded truncated Hankel operators by $\mathfrak{T}(K_u^2)$.

If the inner function u is z^n, then $\{l, z, z^2, \ldots, z^{n-1}\}$ forms an orthonormal basis for K_u^2. With respect to this basis, $A_\emptyset$and $B_\emptyset$ can be represented as a Toeplitz matrix and a Hankel matrix, respectively:

$$A_\emptyset = \begin{pmatrix} a_0 & a-1 & a-2 & \cdots & a_{-n+1} \\ a_1 & a_0 & a-1 & \ddots & \vdots \\ a_2 & a_1 & a_0 & \ddots & a-2 \\ \vdots & \ddots & \ddots & \ddots & a-1 \\ a_{n-1} & \cdots & a_2 & a_1 & a_0 \end{pmatrix}$$

and

$$B_\emptyset = \begin{pmatrix} a-1 & a-2 & a-3 & \cdots & a_{-n} \\ a-2 & a-3 & a-4 & \ddots & \vdots \\ a-3 & a-4 & a-5 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-2n+2} \\ a_{-n} & \cdots & \ldots & a_{-2n+2} & a_{-2n+1} \end{pmatrix}$$

Where a1's are the i-th Fourier coefficients of the symbol function $\emptyset$.

The followings facts are easily verified:

$$A_\emptyset = P_u T_\emptyset | K_u^2 \ and \ A_\emptyset - =$$

$$B_\emptyset = P_u H_\emptyset | K_u^2 \ and \ B_\emptyset^* = B_{\widehat{\emptyset}}, \tag{2}$$

where, the$\widehat{\emptyset}\mathcal{Z} := \overline{\emptyset(\mathcal{Z})}$.

The reproducing kernel for H^2 is defined to be $k_\lambda = \frac{1}{1-\lambda_z}$and k_λ, satisfies $(f, k_\lambda) = f(\lambda)$ for every function $f \in H^2$. The reproducing kernel for K_u^2 is defined to be $K_\lambda^u = P_u k_\lambda = \frac{1-\overline{u(\lambda)}u(z)}{1-\lambda_z}$ Note that $K_\lambda^u \in K_u^2$ and we have $\langle g(z), K_\lambda^u(z) \rangle = g(\lambda)$ for every$g \in K_u^2$. A conjugation Con $L^2(\mathbb{T})$ is defined by

$$(Cf)(z) = u(z)\overline{zf(z)} \ for \ f \in L^2$$

and it is easy to verify that C is a conjugate linear map and an isometry on L^2 satisfying $C^2 = \text{I}$. It is easy to verify that C maps uH^2 onto $\bar{z}H^2, \bar{z}H^2$ onto uH^2 and K_u^2 onto itself. For convenience, Cf will be denoted ly$\tilde{f}$. One can verify

$$\widetilde{k_\lambda^u}(z) = CK_\lambda^u(z)\frac{u(z) - u(\lambda)}{z - \lambda}$$

The compressed shift S_u on K_u^2 is defined by $S_u \coloneqq A_z = P_u S|K_u^2$. For functions $f, g \in L^2, f \otimes g$ denotes a rank one operator defined by $(f \otimes g)h = \langle h, g\rangle f$, for $h \in L^2$. The following properties can be verified:

$$(i)(f \otimes g)^* = g \otimes f$$

$$(ii)\ A(f \otimes g)B = (Af) \otimes (B^* g), for\ bounded\ linear\ operators\ A\ and\ B\ on\ L^2. (3)$$

We have two Lemmas

Lemma (3.1)[3].

(a) for, λ in $\mathbb{D}$,

$$S_u^* k_\lambda^u = \bar{\lambda} K k_\lambda^u - \overline{u(\lambda)}\widetilde{k_0^u}, S_u \widetilde{k_\lambda^u} = \lambda \widetilde{k_\lambda^u} - u(\lambda) k_0^u.$$

(b) For $\lambda \in \mathbb{D}\backslash\{O\}$,

$$S_u k_\lambda^u = \frac{1}{\lambda} k_\lambda^u - \frac{1}{\lambda}, S_u^* \widetilde{k_0^u}, = \frac{1}{\lambda}\widetilde{k_\lambda^u} - \frac{1}{\lambda} k_0^u$$

Lemma (3.2)[3].

$$I - S_u S_u^* = K_0^u \otimes K_0^u\ and\ I - S_u S_u^* = \widetilde{k_0^u} \otimes \widetilde{k_0^u}$$

C. Gu gave a nice characterizatjon of truncated Hankel operators as D. Sarason did for truncated Toeplitz operators.

Theorem (3.3)[3]. A bounded operator B on K_u^2 is a truncated Hankel operator if and only if there eristfunctions $f, g \in K_u^2$, such that

$$B - S_u^* B S_u^* = g \otimes K_0^u + \widetilde{k_0^u} \otimes f$$

For a complex number c, we define the modified compressed shift by $S_{u,c} by\ S_{u,c} = S_u + c(K_0^u \otimes \widetilde{k_0^u})$.

Lemma (3.4)[3]. If a bounded operator A on K_u^2 commutes with $S_{u,c}$ for some complex number c, then A is a truncated Toeplitz operator.

The analog of Lemma (3.4) for TTO' is the following:

Lemma (3.5)[3]. For complex numbers $\alpha$and β, if $S_{u,\alpha}^* B = B S_{u,\beta},\ or\ S_{u,\beta}, B = B S_{u,\beta}^*$, then B is a truncated Hankel operator.

Proof. We will show that $S_{u,\alpha}^* B = B S_{u,\beta}$ implies B is a truncated Hankel operator. The proof of other case is similar. In view of the above Theorem, we will show that there exist $f, g \in K_u^2$ such that

$$B - S_{u,\alpha}^* B = B S_{u,\beta} g \otimes K_0^u + \widetilde{k_0^u} \otimes f$$

By multiplying S, on the right side of equation $S_{u,\alpha}^* B = B S_{u,\beta}$ we have

$$S_u^* B S_u^* + \left(\widetilde{k_0^u} \otimes \alpha S_u B^* K_0^*\right) = B S_u S_u^* + \left(\beta B K_0^u \otimes S_u \widetilde{k_0^u}\right) \qquad (4)$$

Since $S_u S_u^* = I - K_0^u \otimes K_0^u\ and\ S_u \widetilde{k_0^u} = -u(0) k_0^u$by Lemma (3.1) reduces to

$$B - S_u^* B S_u^* = \left(\left(1 + \beta \overline{u(0)}\right) B K_0^u \otimes K_0^u\right) + (\widetilde{k_0^u} \otimes \alpha S_u B^* K_0^u).$$

The proof is complete.

N. Sedlock gave a nice necessary and sufficient condition that the product of two truncated Toplitz operators is again a truncated Toeplitz operator.

Theorem (3.6)[3]. Suppose $A_1, A_2 \in \mathfrak{T}(K_u^2) If A_1 A_2 \in \mathfrak{T}(K_u^2)$, then one of two cases holds:

(i) Either $A_1\ or\ A_2$ is equal to λI for some $\lambda \in \mathbb{C}$.

(ii) Both $A_1\ and\ A_2$ commute with$S_{u,c}$ for some $c \in\ \mathbb{C}, where S_{u,c}$ is defined by $S_{u,c} = S_u + c(K_0^u \otimes \widetilde{k_0^u})$.

The main purpose of this paper is to characterize the pairs of truncated Hankel operators whose products are truncated Toeplitz operators. Since it turns out that many of the algebraic properties of THO's are more complicated than the case of TTO's, we are going to give a partial answer to the above problem using some interesting results. Let a bounded operator D on K_u^2 be defined by $D := B_{\overline{u}}$.

Lemma (3.7)[3]. $For\ \varphi \in \overline{zH^2}$,

$$B_\varphi S_u - S_u^* B_\varphi = \widetilde{k_0^u} \otimes P_u H_\varphi^* u - P_u H_\varphi u \otimes \widetilde{k_0^u}$$

An inner function u is called real symmetric if $u = \hat{u}$. The following Lemma shows interesting relation between TTO's and THO's when n is real symmetric.

Lemma (3.8)[3]. If $u = \hat{u}$and $D = B_{\overline{u}} \in \mathfrak{H}\,(K_u^2)$, then

(a) $D^* = D\ and\ D^2 = I.$

$(b) \mathfrak{H}\,(K_u^2) = \mathfrak{H}\,(K_u^2) = \mathfrak{T}\,(K_u^2). For\ \psi \in K_u^2 + \overline{K}_u^2 D A_\psi = B_{\overline{u}\psi} and\ A_\psi D = B_{\overline{u}\psi(\bar{z})}$

(c) $D k_\lambda^u = \widetilde{K_{\frac{u}{\lambda}}}\ and\ D\widetilde{k_\lambda^u} = k_{\frac{u}{\lambda}}$.

Theorem (3.9)[3]. If u — ii, the product of two truncated Hankel operators $B_1, B_2 \in \mathfrak{H}(K_u^2)$ becomes a truncated Toeplitz operator if and only if one of the following cases holds:

(i) Either B_1 *or* B_2 is equal to λD for some $\lambda \in \mathbb{C}$.

(ii) There eristsc $c \in \mathbb{C}$such that

$$S_{u,c}, B_1 = B_1 S_{u,\bar{c}}^* \, and \, S_{u,\bar{c}}^* B_2 = B_2 S_{u,c}$$

Proof. If either B_1 *or* B_2 is a constant multiple of D, then (b) of Lemma (3.8) shows $B_1 B_2$ is a TTO. Now observe

$$S_{u,}, B_1 B_2 = B_1 S_{u,\bar{c}}^* B_2 = B_1 B_2 S_{u,c}$$

Thus, by Lemma (3.3), we conclude that $B_1 B_2$ is a TTO. The proof of sufficiency is complete.

For the proof of necessity assume that neither B_1 nor B_2 is a constant multiple of the identity and $B_1 B_2$ = A for some TTO $A \in \mathfrak{T}(K_u^2). Put \, A_1 = B_1 D$ and $A_2 = DB_2, then \, A_1, A_2 \in \mathfrak{T}(K_u^2)$ by Lemma (3.8). Note that

$$A_1 A_2 = B_1 DDB_2 B_1 B_2,$$

by property (b) of Lemma (3.8), and $A_1, A_2 = A \in \mathfrak{T}(K_u^2)$. By Theorem (3.2), if none of A_1 and A_2 is the multiple of identity, then $A_1, A_2 \, e \, \{S_{u,c}\}$' for some complex number c. First, we will show the second equation of (2). Then by Theorem (3.5), A_2 commutes with some modified shift $s_{u,c}$ .Thus we have

$$S_{u,c}, DB_2 = DB_2 S_{u,c} \, .$$

Claim. $S_{u,\bar{c}}^* = DS_{u,c}$

By a straightforward calculation, we have

$$S_{u,\bar{c}}^{*} — DS_{u,c}^{*} = S_{u}^{*}D — DS_{u} + c(\widetilde{k_{0}^{u}} \otimes K_{0}^{u}) - c(DK_{0}^{u} \otimes \widetilde{k_{0}^{u}}).$$

Since $P_{u}H_{\bar{u}}^{*}u = P_{u}H_{\bar{u}}u = 0$, by Lemma (2.2), $S_{u}^{*}D — DS_{u} = 0$. Also, by Lemma (3.8) (c), $Dk_{0}^{u} = \widetilde{k_{0}^{u}}. Hence\ S_{u,\bar{c}}^{*}D = DS_{u,c}$

By claim we have

$$DS_{u,c}D = S_{u,\bar{c}}^{*}, \tag{5}$$

so that

$$S_{u,\bar{c}}^{*}B_{2} = DS_{u,c}DB_{2} = B_{2}S_{u,c}$$

By the same argument, we also have $S_{u,c}B_{1} = B_{1}S_{u,\bar{c}}^{*}$

In fact, the shape of the symbol functions of the THO's B_{1}, B_{2} and the resulting TTO $B_{1}B_{2}$ can be concretely determined.

Lemma (3.10)[3]. A bounded operator A on K_{u}^{2} commutes with if and only if $A = A_{\phi}$ with $= f + \frac{c}{1+\overline{cu(0)}}\overline{S_{u}\tilde{f}}$' for some $f \in K_{u}^{2}$. Moreover, $f = \left(1 + \overline{cu(0)}\right)Ak_{0}^{u}$.

Theorem (3.11)[3]. The product $B_{1}B_{2}$ of two THO's is a TTO, with none of B_{i} a constant multiple of $D = B_{\bar{u}}$, then there are complex number c and analytic functions $f_{1}, f_{2} \in K_{u}^{2}$ such that $B_{1} = B_{\bar{u}\phi_{1}(\bar{z})}$, where $\phi_{1} = f_{1} + \frac{c}{1+\overline{cu(0)}}\overline{S_{u}\tilde{f}_{1}}$ and $B_{2} = B_{u\phi_{2}}$where $\phi_{2} = f_{2} + + \frac{c}{1+\overline{cu(0)}}\overline{S_{u}\tilde{f}_{2}}$ Moreover, the symbol of the resulting TTO is $A_{\phi_{1}}f_{2}$, withϕ_{1} and f_{2} as above, that is, $B_{\bar{u}\phi_{1}(\bar{z})}B_{\bar{u}\phi_{2}} = A_{A\phi_{1}f_{2}}$.

Proof. If $B_{1}B_{2}$is a TTO, then by Theorem (3.9),we know

$$S_{u,c}B_{1} = B_{1}S_{u,\bar{c}}^{*} \ and\ S_{u,\bar{c}}^{*}B_{2} = B_{2}S_{u,c} \tag{6}$$

fbi some $c \in \mathbb{C}$. Multiplying D to the right side of the first equation of (6), we have $S_{u,c}B_1D = B_1DDS^*_{u\bar{c}}D = B_1DS_{u,c}$, where the second equality comes from (5). Therefore, Lemma 2.8 implies $B_1D = A_{\phi_1} = f_1 + \frac{c}{1+\overline{cu(0)}}\overline{S_u\tilde{f}_1}$ for some $F_1 \in K_u^2$. Using (b) of Lemma (3.8) S, $B_1 = A_{\phi_1}D = B_{\bar{u}\phi_{1(\bar{z})}}$

To get the symbol of the THO B_2, note that the second equation of (6) implies $S_{u,c}DB_2 = DS^*_{u,\bar{c}}DDB_2 = DS^*_{u,\bar{c}}B_2 = DB_2S^*_{u,\bar{c}}$ where we used (6) for the first equality. Thus DB_2 commutes with$S_{u,c}$ Again, by the use of (b) of Lemma (3.8) , we have $DB_2 = A_{\phi_2}, i.e., B_2 = DA_{\phi_2} = B_{\bar{u}\phi_2}$ where $\phi_2 = f_2 + + \frac{c}{1+\overline{cu(0)}}\overline{S_u\tilde{f}_2}$ for sonic $f_2 \in K_u^2$.

Now let $B_{\bar{u}\phi_{1(\bar{z})}}B_{\bar{u}\phi_2} = A_\psi$. commutes with $S_{u,c}$ By Lemma (3.10), $\psi = g + \frac{c}{1+\overline{cu(0)}}\overline{S_u\tilde{g}}$ where $g = \left(1 + \overline{cu(O)}\right)A_\psi, k_0^u$. Recall that $A_\psi = B_{\bar{u}\Phi_{1(\bar{z})}}B_{\bar{u}\Phi_2} = B_{\bar{u}\Phi_{1(\bar{z})}}DDB_{\bar{u}\Phi_2} = A_{\Phi_1}, A_{\Phi_1}$. Thus we have $A_\psi k_0^u = A_{\Phi_1}A_{\Phi_2}k_0^u$. Note that both A_{Φ_1} and A_{Φ_2} commute with $S_{u,c}$ and by Lemma (3.10), $A_{\Phi_2}k_0^u = \frac{1}{1+\overline{cu(0)}}f_2$

So we have $g = 1 + \overline{cu(0)}A_{\Phi_1}A_{\Phi_2}k_0^u = 1 + \overline{cu(0)}A_{\Phi_1}\frac{1}{1+\overline{cu(0)}}f_2 = A_{\Phi_1}f_2$ as desired.

Using the method we used before, we can also get some analogous results on the product of a TTO and a THO. The proof is skipped.

Theorem (3.12)[3]. Let u be a real symmetric inner function, $A \in \mathfrak{T}(K_u^2)$ *and* $B \in \mathfrak{H}(K_u^2)$. Then AB is a truncated Hankel operator on K_u^2 if and only if one of the following conditions holds:

(a) Either A is a constant multiple of the identity or B is a constant multiple of $D = B_{\overline{u}}$.

(b) A commutes with and $S_{u,c}B = BS^*_{u,\bar{c}}$ for some complex number c.

Note that under the second condition, the resulting THO AB satisfies $S_{u,c}B = BS^*_{u,\bar{c}}$.

Using Theorem (3.9), we can get an interesting condition that the product of two Hankel matrices be a Toeplitz matrix. To apply the Theorem to the matrix case, let $u = z^N$. Obviously, u is a real symmetric function $(u = \hat{u}\,). Let\, B_1 = B_\varphi\ and\, B_2 = B_\psi$, where $\varphi = a_1\bar{z} + a_2\bar{z}^2 + \cdots, \psi = b_1\bar{z} + b_2\bar{z}^2 + \cdots$. Their with respect to the standard ordered basis of $K^2_{z^N}$,

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & aN \\ a_2 & a_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2N-2 \\ aN & \cdots & a_2N-2 & a_2N-2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & bN \\ b_2 & b_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_2N-2 \\ bN & \cdots & b_2N-2 & b_2N-2 \end{pmatrix}$$

and

$$S_{u,c} = \begin{pmatrix} 0 & \cdots & 0 & c \\ 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

From the equation $S_{u,c}B_1 = B_1S^*_{u,\bar{c}}$ a direct comparison of the matrix multiplications gives

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & \begin{matrix} aN \\ \frac{1}{c}a_1 \end{matrix} \\ a_2 & a_3 & \ddots & \\ \vdots & \ddots & \ddots & \vdots \\ aN & \cdots & \cdots 1 & \\ & & \overline{c} & aN-1 \end{pmatrix}$$

If$c \neq 0$, or ,we have $a_1 = \cdots = a_{N-1} = 0 \; if \; c = 0$. Therefore we have either $\varphi = a_1\bar{z} + \cdots + a_N\bar{z}^N + \frac{1}{c}(a_1\bar{z}^{N+1} + \cdots + a_{N-1}\bar{z}^{2N-1}$depending on whether c = 0 or not. Similarly, $S^*_{u,\bar{c}}B_2 = S_{u,c}B_2$, gives

$$\begin{pmatrix} b_1 & b_2 & \cdots & bN \\ b_2 & b_3 & \ddots & cb_1 \\ \vdots & \ddots & \ddots & \vdots \\ bN & \cdots & \cdots cb_{N-1} & \end{pmatrix}$$

So we have $\psi = b_1\bar{z} + b_2\bar{z}^2 + \cdots + b_N\bar{z}^N + c(b_1\bar{z}^{N+1} + b_2\bar{z}^{N+2} + \cdots + b_{N-1}\bar{z}^{2N-1})$

Summarizing the above, we have the following Theorem that characterizes the condition that the product of two Hankel matrices become a Toeplitz matrix. Indeed, this result can also be worked out by an appropriate manipulation of Theorem (3.2).

Theorem (3.13)[3]. Suppose $B_1, B_2 \; are \; N$ Hankel matrices. If $B_1B_2$is a Toeplitz matri:x, then one of two cases holds:

(i) Either $B_1 \; or \; B_2$ is of the form

$$\begin{pmatrix} 0 & \cdots & 0 & \\ \vdots & & & 0 \\ \vdots & & & \vdots \\ & \cdots & \cdots & 0 \end{pmatrix}$$

(ii) There exists $c \in \mathbb{C}$ such that

$$B_1 = \begin{pmatrix} a_1 & a_2 & \cdots & \frac{a_N}{1} \\ a_2 & a_3 & \ddots & \frac{1}{c}a_1 \\ \vdots & \ddots & \ddots & \vdots \\ a_N & \cdots & \cdots \frac{1}{c}a_{N-1} & \end{pmatrix} \; and \; B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & b_3 & \ddots & cb_1 \\ \vdots & \ddots & \ddots & \vdots \\ b_N & cb_1 & \cdots cb_{N-1} & \end{pmatrix}$$

or

$$B_1 = \begin{pmatrix} 0 & \cdots & 0 & a_N \\ \vdots & 0 & \ddots & a_{N+1} \\ 0 & \ddots & \ddots & \vdots \\ a_N & a_{N+1} & \cdots & a_{2N-1} \end{pmatrix} \quad and \quad B_2 = \begin{pmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ b_N & 0 & \cdots & 0 \end{pmatrix}$$

We conclude with anaturally arising questions.

Chapter 4

Localization principle

We discuss the localization principle which says that the contributions of disjoint , parts of the singular support to this asymptotic behavior are independent of each other .We apply this principle to hankel integral operators and to infinite Hankel matrices. In both cases, we describe a wide class of Hankel operators with power-like asymptotics of singular values .The leading term of this asymptotics is found explicitly.

Section (4.1): Compact Hankel Operators

Hankel operators admit variousunitarily equivalent descriptions. We start by recalling the definition of Hanke loperators on the Hardy class $H^2(\mathbb{T})$. Here $\mathbb{T}$ is the unit circle in the complex plane,equipped with the normalized Lebesgue measure $dm(\mu) = (2\pi i\mu)^{-1}d\mu, \mu \in \mathbb{T}$; the Hardy class $H^2(\mathbb{T}) \subset L^2(\mathbb{T})$ is de_ned in the standard way as the subspace of $L^2(\mathbb{T})$ spanned by the functions $1\mu, \mu^2, \dots \dots$. Let $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ be the orthogonal projection onto $H^2(\mathbb{T})$, and let W be the involution in $L^2(\mathbb{T})$ defined by $(Wf)(\mu) = f(\overline{\mu})$. For a function $\omega \in L^\infty(\mathbb{T})$, which is called a symbol, the Hankel operator $H(\omega)$ is defined by the relation

$$H(w)f = P_+(\omega W f). \tag{1}$$

Recall that the singular values of a compact operator H are defined by the relation $s_n(H) = \lambda_n(|H|)$, where $\{\lambda_n(|H|)\}_{n=1}^{\infty}$ is the non-increasing sequence of eigenvalues of the compact positive operator $|H| = \sqrt{H^*H}$ (enumerated with multiplicities taken into account). The study of singular values of compact Hankel operators has a long history and is linked to rational approximation, control theory and other subjects, Singular values $sn(H(\omega))$ of a Hankel operator with a symbol $\omega \in C^{\infty}(\mathbb{T})$ decay faster than any power of n^{-1} as $n \to \infty$.On the other hand, singularities of ωgenerate a slower decay of singular values. Here we will be interested in the case when the singular values behave as some power of n^{-1}. Optimal upper estimates on singular values of Hankel operators are due to V. Peller . He found necessary and sufficient conditions on !for the estimate

$$s_n\big(H(\omega)\big) \leq Cn^{-\alpha}$$

for some $\alpha > 0$. These conditions are stated in terms of the Besov-Lorentz classes.

It is natural to expect that the asymptotic behavior of singular values is determined by the behaviour of the symbol ω in a neighbourhood of its singular support. We justify this thesis and show that the contributions of the disjoint components of the singular support of ω to the asymptotics of the singular values of $H(\omega)$ are independent of each other. We use the term "localization principle" for this fact. This principle is well understood of the study of the essential spectrum and of the absolutely continuous spectrum of non-compact Hankel operators. Our aim here is to bring this principle to the fore in the question of the asymptotics of singular values of compact Hankel operators.

We combine the localization principle to determine the asymptoticsof singular values of Hankel operators of various natural classes. In particular, for Hankel matrices with oscillating matrix elements we show that the contributions of different oscillating terms to the asymptotics of singular values are independent of each other. We also establish similar results for Hankel integral operators whose integral kernels have a singularity at some finite point $t_0 \geq 0$ and several oscillatingterms at infinity.

Recall that the singular support $\operatorname{sing\,supp} \omega$ of a function $\omega \in L^\infty(\mathbb{T})$ is defined the smallest closed set $X \subset T$ such that $\omega \in C^\infty\,(T \setminus X)$. Localization principle is stated as follows.

Theorem (4.1.1)[4]. Let $\omega_1, \omega_2, \dots, \omega_L$ be bounded functions on T such that

$$\operatorname{sing\,supp} \omega_\ell \cap \operatorname{sing\,supp} \omega_j = \emptyset, \ell = j. \tag{2}$$

Set $\omega = \omega_1 + \cdots + \omega_L$.Then for all $p > 0$ we have the relations

$$\limsup_{n\to\infty} n s_n\big(H(\omega)\big)^p = \sum_{\ell=1}^{L} \limsup_{n\to\infty} n s_n\big(H(\omega_\ell)\big)^p, \tag{3}$$

$$\liminf_{n\to\infty} n s_n\big(H(\omega)\big)^p = \sum_{\ell=1}^{L} \liminf_{n\to\infty} n s_n\big(H(\omega_\ell)\big)^p, \tag{4}$$

The upper and lower limits in this Theorem usually coincide. However, we prefer to work with these limits separately both because it is more general and because it is technically more convenient. The limits in the right-hand sides of (3), (4) may be infinite; in such cases the left-hand sides of (3) or(4) are also infinite. It is not excluded in Theorem (4.1.1) that the singular support of each ω_ℓ consists of one point only. In fact, this is exactly this case that we will see in our applications.

Theorem (4.1.1) can be equivalently stated in terms of the counting functions. For a compact operator H, the singular value counting function isdefined by

$$n(\varepsilon, H) = \#\{n : s_n(H) > \varepsilon\}, \qquad \varepsilon > 0. \tag{5}$$

We have

$$\lim_{n\to\infty} \sup n s_n\,(H)^p \; = \lim_{\varepsilon\to 0} \sup \varepsilon^p n\,(\varepsilon; H)$$

and similarly for the lower limits. It follows that (3), (4) may be restated as

$$\lim_{\varepsilon\to 0} \sup \varepsilon^p (\varepsilon; H(\omega)) = \lim_{\varepsilon\to 0} \sup \varepsilon^p n(\varepsilon; H)\,,$$

$$\lim_{\varepsilon\to 0} \inf \varepsilon^p n(\varepsilon; H(\omega)) \; = \sum_{\ell=1}^{L} \lim_{\varepsilon\to 0} \inf \varepsilon^p n(\varepsilon; H(\omega))$$

In particular, in the case when all upper limits in the right-hand sides coincide with the lower limits and are finite, we have

$$n\big(\varepsilon; H(\omega)\big) = \sum_{\ell=1}^{L} n\,\big(\varepsilon; H(\omega)\big) + \; o(\varepsilon^{-p});\; \varepsilon \to \; 0.$$

Our proof of Theorem (4.1.1) consists of two steps. The first one is to check that under the assumption (2) the operators $H(\omega_\ell)$ are asymptotically orthogonal in the sense that for all $j \; \neq \ell$ and all $\alpha \; > \; 0$ we have

$$s_n\left(H(\omega_\ell)^* H\big(\omega_j\big)\right) = \; O(n^{-\alpha}),$$

$$s_n\big(H(\omega_\ell) H\big(\omega_j\big)^*\big) \; = \; O(n^{-\alpha});\; n \to \infty. \qquad (6)$$

This result follows from the reduction of the products of Hankel operators in (6) to integral operators in $L^2(\mathbb{T})$ with smooth kernels.

The second step is to show that (6) implies relations (3) and (4). This fact is not specific for Hankel operators. In order to get some intuition into its proof, let us suppose for a moment that the operators $H(\omega_\ell)$ are pairwise orthogonal in the sense that

$$H\big(\omega_j\big)^* H(\omega_\ell) = \; 0 \; and \; H\big(\omega_j\big) H(\omega_\ell)^* \; = \; 0, \qquad \forall j \; \neq \ell,. \qquad (7)$$

Then

$$\mathrm{Ran}H(\omega_j) \perp \mathrm{Ran}H(\omega_\ell) and\ \mathrm{Ran}H(\omega_j)^* \perp \mathrm{Ran}H(\omega_\ell)^*;\ \forall j \neq \ell.$$

Thus, representing the sum $H(\omega) = H(\omega_1) + \cdots + H(\omega_L)$ as a "block-diagonal" operator acting from $\overline{\bigoplus_{\ell=1}^{L} \mathrm{Ran}H(\omega_\ell)^*}\ to\ \overline{\bigoplus_{\ell=1}^{L} \mathrm{Ran}H(\omega_\ell)}$, we conclude that

$$n(\varepsilon; H(\omega)) = \sum_{\ell=1}^{L} n(\varepsilon; H(\omega_\ell)),\ \forall \varepsilon > 0.$$

The orthogonality condition (7) is too strong. In fact, an operator theoretic result, Theorem (4.1.2), shows that the asymptotic orthogonality (6) ensures the relations (3), (4) for $p = 1/\alpha$.

Representing Hankel operators in the basis $\{\mu^j\}_{j=0}^{\infty}\ in\ H^2(\mathbb{T})$, one obtains the class of infinite Hankel matrices of the form $\{h(j + k)\}_j^{\infty}$; $k = 0$ in the space $\ell^2(\mathbb{Z}_+)$.

We give an application of the localization principle to such Hankel matrices Although the localization principle in the form stated above (Theorem (4.1.1) is quite natural, this application looks far less obvious.

Theorem (4.1.3) can be equivalently stated in terms of Hankel operators $H(\omega)$ acting in the Hardy space $H_+^2(\mathbb{R})$ of functions analytic in the upperhalf-plane. In this case the symbol $\omega(x)$ is a function of $x \in R$. This leads to new results for Hankel operators defined as integral operators in the space $L^2(\mathbb{R}_+)$.

We will refer to the Hankel operators in $H^2(\mathbb{T})$ and in $\ell^2(\mathbb{Z}_+)$ as to the discrete case, and to the Hankel operators in $H_+^2(\mathbb{R})$ and in $L^2(\mathbb{R}_+)$ as to the continuous case. We will use boldface font for objects associated with the continuous case. We have tried to make exposition in the discrete and continuous cases parallel as much as possible.

Recall that for a bounded operator H, the non-zero parts of the operators

$$(-|H|) \oplus |H| \text{and} \quad \begin{pmatrix} 0 & H \\ H^* & 0 \end{pmatrix}$$

are unitarily equivalent. Therefore various spectral results for $|H(\omega)|$ are equivalent to those for the self-adjoint Hankel operator with the matrix valued symbol

$$\begin{pmatrix} 0 & \omega(\mu) \\ \overline{\omega(\overline{\mu})} & 0 \end{pmatrix}$$

Some forms of localization principle are known in the study of the continuous spectrum of $|H(\omega)|$. The idea of separation of singularities of the symbol goes back to the S. R. Power on the essential spectrum specess of Hankel operators with piecewise continuous symbols ω. Let $a_j \in \mathbb{T}$ be the points where ω has the jumps

$$\kappa(a_j) = \lim_{\varepsilon \to +0} \omega(a_j e^{i\varepsilon}) \lim_{\varepsilon \to +0} \omega(a_j e^{-i\varepsilon}) \neq 0$$

Although Power was interested in the essential spectrum of $H(\omega)$, it follows from the matrix version of his results that

$$\text{spec}_{ess}(|H(\omega)|) = [0, M], \qquad M = \frac{1}{2} \sup_{a_j \in \mathbb{T}} |\kappa(a_j)|, \tag{8}$$

Where the supremum is taken over all points a_j where ω has a jump.

A description of the absolutely continuous spectrum of $|H(\omega)|$ with piecewise continuous symbol ω follows from the matrix version of the results of Howland where the trace class method of scattering theory was used. In both cases, under some mild additional assumptions, including the condition that ω has finitely many jumps, it can be shown that

$$\mathrm{spec}_{ac}(|H(\omega)|) \bigcup_{a_j \in \mathbb{T}} \left[0\,, \frac{1}{2}\left|\kappa\left(a_j\right)\right|\right] \tag{9}$$

Every term in the right-hand side of (9) gives its own band of the absolutely continuous spectrum of multiplicity one. Thus, formula (9) can be regarded asthe continuous spectrum analogue of the localisation principle discussed the contributions of different jumps of ω to $\mathrm{spec}_{ac}(|H(\omega)|)$ are independent of each other. Formulas (8) and (9) are consistent with each other.

We prove the localization principle in the discrete case and also state and prove its analogue in the continuous case. We describe the applications of localization principle to the Hankel operators acting in $\ell^2(\mathbb{Z}_+)$; We give applications to integral Hankel operators in $L^2(\mathbb{R}_+)$; We consider integral Hanke loperators with kernels with local singularities in R+.

For $\omega \in L^2(\mathbb{T})$, the Fourier coefficients of ω are denoted as usual by

$$\widehat{\omega}(j) = \int_{\mathbb{T}} \omega(\mu)\mu^{-j} dm(\mu), \qquad j \in \mathbb{Z}$$

We will consistently make use of the following constant, which appears in our asymptotic formulas:

$$v(\omega) = 2^{-\alpha}\pi^{1-2\alpha}\left(B\left(\frac{1}{2\alpha}, \frac{1}{2}\right)\right)^{\alpha}, \qquad \alpha > 0, \tag{10}$$

Here $B(\cdot,\cdot)$ is the Beta function. We make a standing assumption that the exponents $p > 0$ and $\alpha > 0$ are related by $\alpha = 1/p$.

We prove Theorem (4.1.1). We also prove a similar statement, Theorem (4.1.2), for Hankel operators in the Hardy space $H_+^2(\mathbb{R})$ of functions analytic in the upper half-plane.

Let B be the algebra of bounded operators in a Hilbert space H, and let S_∞ be the ideal of compact operators in B. For $p > 0$, the weak Schatten class $S_{p,\infty}$ consists of all compact operators A such that

$$\sup_n n s_n(A)^p < \infty.$$

The subclass $S_{p,\infty}^0 \subset S_{p,\infty}$ is defined by the condition

$$\lim_{n\to\infty} n s_n(A)^p = 0$$

It is well known that both $S_{p,\infty}$ and $S_{p,\infty}^0$ are ideals of B; in particular, they are linear spaces. $A \in S_{p,\infty}\left(\text{or } A \in S_{p,\infty}^0\right)$ if and only if the same is true for its adjoint A^*.We set $S_0 = \cap_{p>0} S_{p,\infty}$, that is,

$$A \in S_0 \Leftrightarrow s_n(A) = O(n^{-\alpha}), \qquad n \to \infty, \forall \alpha > 0 \tag{11}$$

First we recall a classical result in perturbation theory on the spectral stability of singular values.

Lemma (4.1.2)[4]. Let $A \in S_\infty$ and $B \in S_{p,\infty}^0$ for some $p > 0$. Then

$$\lim_{n\to\infty} \sup n s_n(A+B)^p = \lim_{n\to\infty} \sup n s_n(A)^p \tag{12}$$

$$\lim_{n\to\infty} \inf n s_n(A+B)^p = \lim_{n\to\infty} \inf n s_n(A)^P \tag{13}$$

Lemma (4.1.2) is stated in a slightly more general form than usual because we do not require that $A \in S_{p,\infty}$ and hence the limits in(12) and (13) may be infinite; in this case Lemma(4.1.2) means that both sides in (12) and (13) are infinite simultaneously. Note that if $A \notin S_{p,\infty}$, then the expression (12) is infinite, but the expression (13) may be finite. Lemma (4.1.2) can also be equivalently stated in terms of the singular value counting functions $n(\varepsilon, A)$defined by (5).

$$A \in S_{p,\infty}, \qquad B \in S_{p,\infty} \quad \Rightarrow A^*B \in S_{\frac{p}{2},\infty}, \qquad AB^*S_{\frac{p}{2},\infty} \tag{14}$$

We say that the operators A and B in $S_{p,\infty}$ are asymptotically orthogonal if the class $S^0_{p/2,\infty}$ in the right side of (14) can be replaced by its subclass $S^0_{p/2,\infty}$. The following Theorem allows us to study singular values of sums of asymptotically orthogonal operators. This result is the key operator theoretic ingredient of our construction.

Theorem (4.1.3)[4]. Let $p > 0$. Assume that $A_1, \dots, A_L \in S_\infty$ and

$$A_\ell^* A_j \in S^{\,0}_{p/2,\infty}, \qquad A_\ell A_j^* S_{p/2,\infty} \tag{15}$$

Then for $A = A_1 + \cdots + A_L$, we have

$$\lim_{n\to\infty} \sup n s_n(A)^p \sum_{\ell=1}^{L} \lim_{n\to\infty} \sup n s_n(A_\ell)^p \tag{16}$$

$$\lim_{n\to\infty} \inf n s_n(A)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} \inf n s_n(A_\ell)^p \tag{17}$$

Proof. Let us prove the first relation (16); the second one is proven in the same way. We argue in terms of counting functions (5). For an operator $A \in S_\infty$, letus denote

$$\Delta_{\mathrm{p}}(A) = \lim_{\varepsilon\to 0}\sup \varepsilon^{1/p}(\varepsilon; A)$$

(this limit may be infinite). Then our aim is to prove that

$$\Delta_{\mathrm{p}}(A) = \sum_{\ell=1}^{L} \Delta_{\mathrm{p}}\,(A_\ell), \tag{18}$$

which is (16) in different notation. Put

$$H^L = \underbrace{H \oplus \dots \oplus H}_{L\ terms}$$

and let $A_0 \;=\;$ diag$\{A_1, \dots, A_L\}\; in\; H^L, i.e.$,

$$A_0(f_1, \dots, f_L) = (A_1 f_1, \dots, A_L f_L)$$

Since

$$A_0^* A_0 = \mathrm{diag}\{A_1^* A_1, \dots, A_L^* A_L\} \tag{19}$$

we see that

$$n(\varepsilon; A_0) = \sum_{\ell=1}^{L} n(\varepsilon, A_\ell)$$

and therefore

$$\Delta_{p/2}(A_0^* A_0) = \sum_{\ell=1}^{L} \Delta_{p/2}\,(A_\ell^* A_\ell) = \sum_{\ell=1}^{L} \Delta_p\,(A_\ell) \tag{20}$$

Next, let $J :\; H^L \to\; H$ be the operator given by

$$J(f_1, \dots, f_L) = f_1 + \dots + f_L \text{so that} J^* f \;= (f, \dots, f).$$

Then

$$JA_0(f_1, \ldots, f_L) = A_1 f_1 + \cdots + A_L f_L$$

and

$$(JA_0)^* f = (A_1^* f, \ldots, A_L^* f).$$

It follows that

$$(JA_0)(JA_0)^* f = (A_1 A_1^* + \cdots + A_L A_L^*) f \quad (22)$$

and the operator $(JA_0)^*(JA_0)$ is a "matrix" in H^L given by

$$(JA_0)^*(JA_0) = \begin{pmatrix} A_1^* A_1 & A_1^* A_2 & \ldots A_1^* A_L \\ A_2^* A_1 & A_2^* A_2 & \ldots A_2^* A_L \\ \vdots & \vdots & \ddots \quad \vdots \\ A_L^* A_1 & A_L^* A_2 & \ldots A_L^* A_L \end{pmatrix} \quad (23)$$

According to (20) and (23) we have

$$(JA_0)^*(JA_0) - A_0^* A_0 \in S_{p/2,\infty}^0 \quad (24)$$

Indeed, the "matrix" of the operator in (23) has zeros on the diagonal, and its off-diagonal elements are given by $A_\ell^* A_j$, $\ell \neq j$. Thus (24) follows from the first assumption (5). Therefore Lemma (4.1.5) implies that

$$\Delta_p = ((JA_0)^*(JA_0)) = \Delta_p = (A_0^* A_0)$$

or

$$\Delta_p = \left((JA_0)(JA_0)^*\right) = \Delta_p = (A_0^* A_0) \quad (25)$$

Because for any compact operator T the non-zero singular values of T^*T and TT^*coincide.

Further, since $AA^* = \sum_{\ell,j=1}^{L} A_\ell A_j^*$; it follows from (11) and the second assumption (5) that

$$AA^* - (JA_0)(JA_0)^* = \sum_{j\neq\ell} A_\ell A_j^* \in S_{p/2,\infty}^0$$

Using again Lemma (4.1.2) from here we obtain

$$\Delta_p(A) = \Delta_{\frac{p}{2}}(AA^*) = \Delta_{\frac{p}{2}}\big((JA_0)(JA_0)^*\big).$$

Combining the last equality with (24), we see that $\Delta_p(A) = \Delta_{p/2}(A_0^* A_0)$. Thus (9) yields the relation (12).

Corollary (4.1.4)[4]. Under the assumption (15) we have

$$\lim_{n\to\infty} ns_n(A)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} ns_n(A_\ell)^p$$

provided the limits in the right-hand side exist.

Under slightly more restrictive assumptions Theorem (4.1.2) appeared first

Our proof is quite different from that of (2).

First we state two well-known facts that will be needed below. We recall that the Hankeloperators $H(\omega)$ are defined by the class S_0 is defined by (10)

Lemma (4.1.5)[4].(i) Let K be an integral operator in $L^2(\mathbb{T})$ with an integral kernel of the class $C^\infty(\mathbb{T} \times \mathbb{T})$.Then $K \in S_0$.

(ii) Let $\omega \in C^\infty(\mathbb{T})$; then $H(\omega) \in S_0$.

Proof. Part (i) is a classical fact; it can be obtained, for example, by approximating the integral kernel of K by trigonometric polynomials. This yields a fast approximation of K by finite rank operators.

Part (ii) is also well-known; let us show that it follows from part (i). It will be convenient to consider the projection P_+ here as an operator acting from $L^2(\mathbb{T})$to $L^2(\mathbb{T})$ (rather than from $L^2(\mathbb{T})$to $H^2(\mathbb{T})$). Recall that P_+ acts according to the formula

$$(P_+f)(\mu) = \lim_{\epsilon \to +0} \int_{\mathbb{T}} \frac{f(\mu)}{\mu' - (1-\epsilon)\mu} \mu' dm(\mu') \tag{26}$$

and that W is the involution $(Wf)(\mu) = f(\overline{\mu})$. We have to prove that the operator$P_{+\omega}WP_+ \; in \; L^2(\mathbb{T})$ belongs to the class S_0. Since P_+WP_+ is a rank one operator (projection onto constants), it suffices to check that

$$P_+\omega WP_+ - \omega P_+WP_+ = [P_+, \omega]WP_+ \in S_0. \tag{27}$$

It follows from (26) that the commutator $[P_+, \omega]$ is an integral operator in $L^2(\mathbb{T})$

with the kernel

$$\frac{\omega(\mu')\omega(\mu)}{\mu' - \mu}\mu', \qquad \mu, \mu' \in T.$$

This is a C^∞ function, and so $[P_+, \omega] \in S_0$which implies (27).

The following assertion allows us to separate the contributions of different singularities of the symbol. Essentially, this is a verywell known argument.

Lemma (4.1.6)[4]. Let $\omega_1, \omega_2 \in L^\infty(\mathbb{T})$ be such that sing supp ω_1 sing supp $\omega_2 = \emptyset$.

Then

$$H(\omega_1)^* H(\omega_2) \in S_0, \qquad H(\omega_1) H(\omega_2)^* \in S_0.$$

Proof. Let κ_1, κ_2 be functions in $C^\infty(\mathbb{T})$ with disjoint supports such that

$$(1-\kappa_k)\omega_k \in C^\infty(\mathbb{T}), \qquad k = 1, 2.$$

By Lemma (4.1.5)(ii), we have

$$H\big((1-\kappa_k)\omega_k\big) \in S_0,$$

and hence it suffices to show that

$$H(\kappa_1\omega_1)^* H(\kappa_2\omega_2) \in S_0;\ H(\kappa_1\omega_1) H(\kappa_2\omega_2)^* \in S_0. \tag{28}$$

It follows that

$$H(\kappa_1\omega_1)^* H(\kappa_2\omega_2) f = P_+ W \overline{\omega_1} (\overline{\kappa_1} P_+ \kappa_2) \omega_2 W f, f \in H^2(\mathbb{T}):$$

Since the supports of κ_1 and κ_2 are disjoint, the operator $\overline{\kappa_1} P_+ \kappa_2$ has a C^∞-smooth integral kernel

$$\frac{\overline{\kappa_1(\mu)}\kappa_2(\mu')}{\mu' - \mu}\mu', \qquad \mu, \mu' \in T;$$

and so by Lemma (4.1.5) (i) it belongs to the class S_0. This ensures the first inclusionin (28). In view of the obvious identity

$$H(\omega)^* = H(\omega_*)\ where\ \omega_*(\mu) = \overline{\omega(\overline{\mu})};$$

the second inclusion (28) follows from the first one of Theorem(4.1.1). Proof. Let us apply the abstract Theorem (4.2.1) to the Hankel operators $A_\ell = H(\omega_\ell), \ell = 1, \dots, L$. Lemma (4.1.6) implies that the asymptotic orthog-onality condition (26) is satisfied. Therefore the asymptotic relations (3) and (4) follow directly from (27) and (28).

Hankel operators can also be defined in the Hardy space $H^2_+(\mathbb{R})$ of functions analytic in the upper half-plane. We denote by the unitary Fourier transform on $L^2(\mathbb{R})$,

$$\hat{u}(t) = (\Phi u)(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-ixt}dx.$$

Let $H^2_+(\mathbb{R}) \subset L^2(\mathbb{R})$ be the Hardy class,

$$H^2_+(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : \hat{u}(t) = 0 \; for \; t < 0\};$$

and let $P_+ : L^2(\mathbb{R}) \to H^2_+(\mathbb{R})$ be the corresponding orthogonal projection. Let

W be the involution in$L^2(\mathbb{R})$, $(\mathrm{Wf})(\mathrm{x}) = \mathrm{f}(-\mathrm{x})$. For $\omega \in L^\infty(\mathbb{R})$, the operator

$H(\omega) \; in \; H^2_+(\mathbb{R})$is defined by the formula

$$H(\omega)f = P_+(\omega W f), \qquad f \in H^2_+(\mathbb{R}). \tag{29}$$

There is a unitary equivalence between the Hankel operators $H(\omega)$ defined in

$H^2(\mathbb{T})$by formula (1) and the Hankel operators $H(\omega)$ defined in $H^2_+(\mathbb{R})$ by formula (29). Indeed, let

$$w = \frac{z - i/2}{z + i/2} \;, z = \frac{i}{2}\frac{1 + w}{1 - w} \tag{30}$$

be the standard conformal map sending the real line onto the unit circle, and let

$U : H^2(\mathbb{T}) \to H^2_+(\mathbb{R})$be the corresponding unitary operator defined by

$$(Uf)(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{x+i/2}\ f\left(\frac{x-i/2}{x+i/2}\right)\ ;\ (U^*f\ )(\mu) =\ i\sqrt{2\pi}\frac{1}{1-\mu}f\ \left(\frac{i}{2}\frac{1+\mu}{1-\mu}\right).$$

Then

$$UH(\omega)U^* \ =\ H(\omega), if\ \omega(x) =\ -\frac{x-\frac{i}{2}}{x+\frac{i}{2}}\omega\left(\frac{x-\frac{i}{2}}{x+\frac{i}{2}}\right). \tag{31}$$

So the localization principle stated for $H(\omega)$ can be automatically mapped to operators $H(\omega)$. This is discussed below.

Symbols $\omega(x)$ of Hankel operators $H(\omega)$have the exceptional points $x\ =\ +\infty$ and $x\ =\ -\infty$; it will be convenient to identify these two points. The real line with such identification will be denoted $\mathbb{R}_*$.We write $\omega \in\ C(\mathbb{R}_*)$ if $\omega \in\ C(\mathbb{R})$ and

$$\lim_{x\to\infty} w(x) = \lim_{x\to\infty} w(x)$$

where both limits are supposed to exist. Similarly, we write $\omega \in\ C^\infty(\mathbb{R}_*)$ if $\omega \in C^\infty(\mathbb{R})$ and, for all $m\ =\ 0, 1, \ldots$,

$$\lim_{x\to\infty} \omega^{(m)}(x) = \lim_{x\to\infty} \omega^{(m)}(x) \tag{32}$$

In particular, the point $x\ = \infty$ belongs to the singular support of ω if for at least one $m \geq 0$ the relation (32) fails (i.e. if one of the limits does not exist or if the limits are not equal).

Let us state the localization principle for Hankel operators in $H^2_+(\mathbb{R})$.

Theorem (4.1.7)[4]. Let $\omega_\ell \in\ L^\infty(\mathbb{R}), \ell\ =\ 1, \ldots, L\ <\ 1$, be such that

$$\text{sing supp}\ \omega_\ell\ \cap\ \text{sing supp}\omega_j\ =\ \emptyset; \ell \neq\ j.$$

Set $\omega = \omega_1 + \cdots + \omega_L$.Then for all $p > 0$ we have the relations

$$\lim_{n\to\infty} \sup ns_n\big(H(\omega)\big)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} \sup ns_n\big(H(\omega_\ell)\big)^p,$$

$$\lim_{n\to\infty} \inf ns_n\big(H(\omega)\big)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} \inf ns_n\big(H(\omega_\ell)\big)^p.$$

Observe that formulas (30) establish a one-to-one correspondence between the unit circle $\mathbb{T}$ and the real axis $\mathbb{R}_*$ with the points $x = +\infty$ and $x = -\infty$ identified.

They yield also the one-to-one correspondence between the singular supports of the symbols $\omega(\mu\ )$ and $\omega(x)$ linked by equality (31). Thus, Theorem (4.1.7) is a direct consequence of Theorem (4.1.1).

Section (4.2): Applications of Localization Principle and Local Singularities of the Kernel

For a sequence $\{h(j)\}_{j=0}^{\infty}$ of complex numbers, the Hankel operator $\Gamma(h)$ in the space $\ell^2(\mathbb{Z}_+)$ is formally defined by the "infinite matrix"$\{h(j + k)\}_{j,k=0}^{\infty}$:

$$(\Gamma(h)u)(j) = \sum_{k=0}^{\infty} h(j+k)u(k), \ \ u = \{u(k)\}_{k=0}^{\infty} \qquad (33)$$

The Hankel operators $\Gamma(h)$ in $\ell^2(\mathbb{Z}_+)$ and $H(\omega)$ in $H^2(\mathbb{T})$ are related as follows. Let

$$F : f \mapsto \big\{\hat{f}(j)\big\}_{j=0}^{\infty}, F : H^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z}_+);$$

be the discrete Fourier transform. Then the matrix elements of $H(\omega)$ in the orthonormal basis $\{\mu^j\}_{j=0}^{\infty}$are

$$\left(H(\omega)\mu^j\,,\mu^k\right)_{L^2(\mathbb{T})} = \widehat{\omega}(j + k), \qquad j,k \geq 0;$$

so that

$$(h) = FH(\omega)F^* \; if \; \widehat{\omega}(j) = h(j), \qquad j \geq 0. \tag{34}$$

Since (34) involves only the coefficients with $j \geq 0$, for a given sequence h the symbol ω is not uniquely defined.

We considered compact self-adjoint Hankel operators, corresponding to sequences of real numbers of the type

$$q(j) = j^{-1}(\log j)^{-\alpha} + \text{error term}, \qquad j \to \infty\,, \tag{35}$$

Where $\alpha > 0$. Under the appropriate assumptions on the error term, we proved that the positive eigenvalues of the Hankel operator $\Gamma(q)$ have the asymptotics

$$\lambda_n^+\big(\Gamma(q)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty\,,$$

where the coefficient $v(\alpha)$ is defined in (10). For negative eigenvalues, we have

$$\lambda_n^-\big(\Gamma(q)\big) = o(n^{-\alpha})\text{as } n \to \infty.$$

our analysis was based on the asymptotic form (35) and did not involve symbols directly. We check that if $q(j) = j^{-1}(\log j)^{-\alpha}$, then a symbol σ of $\Gamma(q)$ can be chosen such that singsupp $\sigma = \{1\}$.

Theorem (4.1.1) allows us to find the asymptotics of singular values for more general "oscillating" sequences of the type

$$h(j) = \sum_{\ell=1}^{L} b_\ell j^{-1}(\log j)^{-\alpha} \xi_\ell^{-j} \ + \text{errorterm}, \ \ j \to \infty \tag{36}$$

Where $\xi_1, \dots, \xi_L \in \mathbb{T}$ are distinct points and $b_1, \dots, b_L \in C$ are arbitrary coefficients. It is easy to see that the symbol corresponding to the ℓ'th term in (36)equals $b_\ell \sigma$ ($\mu = \xi_\ell$). Hence its singular support consists of one point $\{\xi_\ell\}$, and so we are in the situation described by the localization principle for $p = 1/\alpha$.The error term in (36) is treated by using the estimates on singular values of Hankel operators.

Notice that the operators $\Gamma(h)$ corresponding to sequences h of the class (36)are in general not self-adjoint. We have information about the asymptotics of their singular values, but not of their eigenvalues.

In order to state our requirements on the error term in (36). Let

$$M(\alpha) = \begin{cases} [\alpha] + 1, & if \ \alpha \geq 1/2, \\ 0, & if \ \alpha \geq 1/2, \end{cases} \tag{37}$$

Where $[\alpha]$ is the integer part of α . For a sequence $h = \{h(j)\}_{j=0}^{\infty}$,we define iteratively the sequences $h^{(m)} = \{h^{(m)}(j)\}_{j=0}^{\infty}, m = 0, 1, 2, \dots$, by setting $h^{(0)}(j) = h(j)$ and

$$h^{(m+1)}(j) = h^{(m)}(j + 1) - h^{(m)}(j), \quad j \geq 0. \tag{38}$$

Note that if $h(j) = j^{-1}(log\, j)^{-\alpha}$ for sufficiently large j, then for all$m \geq 1$ the sequences $h^{(m)}$ satisfy

$$h^{(m)}(j) = O(j^{-1-m}(log\, j)^{-\alpha}), \quad j \to \infty. \tag{39}$$

Now we are in a position to state precisely our result on Hankel operators with matrix elements (36).

Theorem (4.2.1)[4]. Suppose that a sequence g satisfies

$$g^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \qquad j \to \infty, \tag{40}$$

for some $\alpha > 0$ and for all $m = 0, 1, \dots, M(\alpha)$. Then

$$s_n\big(\Gamma(g)\big) = o(n^{-\alpha}), \qquad n \to \infty \tag{41}$$

We also have a result with O instead of oin both (40) and (41), but we do not use it in this paper. Observe that for $\alpha < {}^1\!/_2$ we need only the estimate on g, whereas for $\alpha \geq {}^1\!/_2$ we also need estimates on the iterated differences $g^{(m)}$.

Theorem (4.2.2)[4]. Let $\alpha > 0$, and let the "model sequence" *q* defined by

$$q(j) = j^{-1}(\log j)^{-\alpha} \tag{42}$$

for all sufficiently large j (the values q(j) for any finite number of j are unimportant). Then

$$s_n\big(\Gamma(q)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty,$$

Where $v(\alpha)$ is given by (10).

This result corresponds to a particular case of Theorem (4.2.6) with $L = 1$, $\zeta_1 = 1$, $b_1 = 1$.

In order to combine the contributions of different terms in (46), we use the localization principle To that end, we have to identify the singular support of the symbol corresponding to the model sequence q in Theorem (4.2.1). To be definite, we put $q(0) = q(1) = 0$ and define $q(j)$ by formula (42) for all $j \geq 2$.We need to find a function σ such that its Fourier coefficients $\hat{\sigma}(j) = q(j)$ for $j \geq 0$. The choice of σ is not unique. We will choose σ corresponding to the odd extension of the sequence $q(j)$ to the negative j.

Lemma (4.2.3)[4]. Let $\alpha \geq 0$, and let q be given by (41); set

$$\sigma(\mu) = \sum_{j=2}^{\infty} q(j)\left(\mu^j - \mu^{-j}\right), \qquad \mu \in \mathbb{T}. \tag{43}$$

Then $\sigma \in L^\infty(\mathbb{T})$ and $\sigma \in C^\infty(\mathbb{T} \setminus \{1\})$.

Proof. Note that for all $\mu \in \mathbb{T}$, the series (43) converges absolutely if $\alpha > 1$ and conditionally if $\alpha \leq 1$.

First, we check that $\sigma \in L^\infty(\mathbb{T})$.We write $\mu = e^{i\theta}$, $\theta \in (-\pi, \pi]$. For $\theta \neq 0$, we set $N = [(2|\theta|)^{-1}]$ and write $\sigma = \sigma_1 + \sigma_2$, where

$$\sigma_1(\mu) = \sum_{j=2}^{N} q(j)\left(\mu^j - \mu^{-j}\right), \qquad \sigma_2(\mu) = \sum_{j=N+1}^{\infty} q(j)\left(\mu^j - \mu^{-j}\right).$$

Using the bounds $q(j) \leq (\log 2)^{-1} j^{-1}$ and

$$\left|\mu^j - \mu^{-j}\right| = 2|\sin(j\theta)| \leq 2j|\theta|$$

For σ_1, we obtain the estimate

$$|\sigma_1(\mu)| \leq 2|\theta| \sum_{j=2}^{N} jq(j) \leq 2(\log 2)^{-1}|\theta|N \leq (\log 2)^{-1}.$$

In order to estimate σ_2, let us use summation by parts:

$$(\mu-1)\sum_{j=N+1}^{\infty} q(j)\mu^j$$

$$= \sum_{j=N+1}^{\infty} q(j)(\mu^{j+1}-\mu^j) = -\sum_{j=N+1}^{\infty} q^{(1)}(j)\mu^{j+1} - q(N+1)\mu^{N+1} \quad (44)$$

Where $q^{(1)}(j)$ is defined by (38). By (39), we have$q^{(1)}(j) = O(j^{-2}),\ j \to \infty$, and hence

$$\left|(\mu-1)\sum_{j=N+1}^{\infty} q(j)\mu^j\right| \le C_1\left(\sum_{j=N+1}^{\infty} j^{-2} + N^{-1}\right) \le C_2 N^{-1}.$$

It follows that

$$|\sigma_2(\mu)| \le 2\left|\sum_{j=N+1}^{\infty} q(j)\mu^j\right| \le \frac{2C_2}{N|\mu-1|} = \frac{2C_2}{[(2|\theta|)^{-1}]|e^{i\theta}-1|} \le C.$$

Thus $\sigma_2 \in L^\infty(\mathbb{T})$.

It remains to prove that $\sigma \in C^M(\mathbb{T}\backslash\{1\})$ for any $M \in \mathbb{N}$. Choose $\mu \in \mathbb{T}$ and put $a(j) = \mu^j$; then, by (39), $a^{(M+1)}(j) = (\mu - 1)^{M+1}\mu^j$. Similarly to (44), by a repeated summation by parts procedure, we obtain the identity

$$(\mu-1)^{m+1}\sum_{j=2}^{\infty} q(j)\mu^j = \sum_{j=2}^{\infty} q(j)a^{(M+1)}(j)$$

$$= (-1)^{M+1}\sum_{j=2}^{\infty} q^{(M+1)}(j)a(j) + p_M(\mu) \quad (45)$$

With some polynomial p_M. Since, by (39), $q^{(M+1)}(j) = O(j^{-2-M})$ as $j \to \infty$ and $a(j) = \mu^j$, the function of μ in the right-hand side of (44) is in $C^M(\mathbb{T})$. It follows that $\sigma \in C^M(\mathbb{T} \setminus \{1\})$ and hence $\sigma \in C^\infty(\mathbb{T} \setminus \{1\})$.

Theorem (4.2.4)[4]. Let the function $\sigma(\mu)$ be defined by formula (43) where $q(j)$ are given by (42) and $\alpha > 0$. Then the asymptotic relation holds

$$s_n\big(H(\sigma)\big) = v(\sigma)n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty,$$

where $v(\alpha)$ is given by (10).

For a parameter $\zeta \in \mathbb{T}$, let R_ζ be the "rotation by ζ " operator:

$$(R_\zeta f)(\mu) = f(\mu/\zeta).$$

Obviously, R_ζ is a unitary operator in $L^2(\mathbb{T})$ and in $H^2(\mathbb{T})$.Similarly, let V_ζ be the multiplication by ζ^{-j} :

$$\big(V_\zeta u\big)(j) = \zeta^{-j}\, u(j)\,.$$

Obviously, V_ζ is a unitary operator in $\ell^2(Z_+)$.

Lemma (4.2.5)[4]. (i) For arbitrary $\zeta \in \mathbb{T}$ and $\omega \in L^\infty(\mathbb{T})$, we have

$$H\big(R_\zeta \omega\big) = R_\zeta H(\omega) R_\zeta\,.$$

In particular, if $H(\omega)$ is compact, then

$$s_n\Big(H\big(R_\zeta\omega\big)\Big) = s_n\big(H(\omega)\big), \qquad \forall n \geq 1.$$

(ii) For any sequence h such that $\Gamma(h)$ is bounded, we have

$$\Gamma\big(V_\zeta h\big) = V_\zeta \Gamma(h) V_\zeta\,.$$

In particular, if $\Gamma(h)$ is compact, then

$$s_n\left(\Gamma(V_\zeta h)\right) = s_n(\Gamma(h)), \qquad \forall n \geq 1.$$

Proof. Since

$$P + R_\zeta = R_\zeta P_+ \qquad \text{and} \qquad R_\zeta W R_\zeta = W,$$

the first assertion is a direct consequence of the definition (1) of the Hankel operator $H(\omega)$ in $H^2(\mathbb{T})$. The second assertion immediately.

Theorem (4.2.6)[4]. Let $\alpha > 0$, *let* $\zeta_1, \dots, \zeta_L \epsilon \mathbb{T}$ be distinct numbers, and let

$b_1, \dots, b_L \epsilon \mathbb{C}$. Let h be a sequence of complex numbers such that

$$h(j) = \sum_{\ell=1}^{L} \left(b_\ell j^{-1}(\log j)^{-\alpha} + g_\ell(j)\right)\zeta_\ell^{-j}, \qquad j \geq 2, \tag{46}$$

where the error terms $g_\ell, \ell = 1, \dots, L$,satisfy the estimates

$$g_\ell^{(m)}(j) = o(j^{-1-m}(\log j)^{-\alpha}), \qquad j \to \infty, \tag{47}$$

for all $m = 0, 1, \dots, M(\alpha)$ ($M(\alpha)$ is given by (37). Then the singular values of the Hankel operator $\Gamma(h)$ in $\ell^2(\mathbb{Z}_+)$ defined by formula (33) satisfy the asymptotic relation

$$s_n(\Gamma(h)) = c\, n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty, \tag{48}$$

where

$$c = v(\alpha)\left(\sum_{\ell=1}^{L} |b_\ell|^{1/\alpha}\right)^{\alpha} \tag{49}$$

and the coefficient $v(\alpha)$ is given by formula (10).

This result means that asymptotically the singular value counting function of the operator $\Gamma(h)$ is the sum of such functions for every term in the right-hand side of (46).

Proof. Let the symbol $\sigma(\mu)$ be defined by relation (43) and let

$$\omega_{\ell}(\mu) = \sum_{\ell=1}^{L} \omega_{\ell}(\mu) \quad \text{where} \quad \omega_{\ell}(\mu) = b_{\ell}\sigma(\mu/\zeta_{\ell}). \qquad (50)$$

According to Theorem (4.2.4) and Lemma (4.2.5)(i) we have

$$s_n\big(H(\omega_{\ell})\big) = |b_{\ell}| v(\alpha) n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty.$$

It follows from Lemma (4.2.2) that $\omega_{\ell} \in L^{\infty}(\mathbb{T})$ and $\omega_{\ell} \in C^{\infty}(\mathbb{T} \setminus \zeta_{\ell})$. Since $\zeta_1, \dots, \zeta_L$ are distinct points, the localisation principle is applicable to the sum (50). This yields

$$\lim_{n\to\infty} n s_n\big(H(\omega_{\ell})\big)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} n s_n\big(H(\omega_{\ell})\big)^p = v(\alpha)^p \sum_{\ell=1}^{L} |b_{\ell}|^p, \quad p = 1/\alpha. \qquad (51)$$

Note that, by the definition (50),

$$\widehat{\omega_{\ell}}(j) = b_{\ell}\hat{\sigma}(j)\zeta^{-j}$$

and hence according to formula (43)

$$\widehat{\omega_{\ell}}(j) = \sum_{\ell=1}^{L} b_{\ell} j^{-1} (\log j)^{-\alpha} \zeta_{\ell}^{-j} =: h_{\ell}(j), \qquad j \geq 2$$

Set $h_\ell(0) = h_\ell(1) = 0$. Since the operators $H(\omega_\ell)$ and $\Gamma(h_\ell)$ are unitarily equivalent, it follows from (51) that

$$\lim_{n\to\infty} n s_n\big(\Gamma(h_\ell)\big)^p = v(\alpha)^p \sum_{\ell=1}^{L} |b_\ell|^p. \qquad (52)$$

Next, we consider the error term

$$g(j) = \sum_{\ell=1}^{L} g_\ell(j)\zeta_\ell^{-j}$$

in (47). According to condition (48) it follows from Theorem (4.2.3) that $s_n\big(\Gamma(g_\ell)\big) = o(n^{-\alpha})$ as $n \to \infty$. By Lemma (4.2.5)(ii), we also have $s_n(\Gamma(V_{\zeta_\ell} g_\ell)) = o(n^{-\alpha})$ and hence

$$s_n(\Gamma(g)) = o(n^{-\alpha})\ as \quad n \to \infty \qquad (53)$$

Since

$$\Gamma(h) = \Gamma(h_\ell) + \Gamma(g),$$

we can use Lemma (4.3.1) with $A = \Gamma(h_\ell)$ *and* $B = \Gamma(g)$. The required relations (49), (40) follow from (52) and (53).

Integral Hankel operators $\Gamma(h)$ in the space $L^2(\mathbb{R}_+)$ are defined by the relation

$$(\Gamma(h)u)(t) = \int_0^\infty h(t + s)u(s)ds, u \in C_0^\infty(\mathbb{R}_+), \qquad (54)$$

where at least $h \in L^1_{loc}(\mathbb{R}_+)$; this function is called the kernel of the Hankel operator $\Gamma(h)$. Under the assumptions on h below the operators $\Gamma(h)$ are compact.

Similarly to the discrete case, Hankel operators in the Hardy space $H_+^2(\mathbb{R})$ are unitarily equivalent to integral operators $\Gamma(h)$ in the space $L_2(\mathbb{R}_+)$:

$$\Phi H(\omega)\,\Phi^* = \Gamma(h) \quad if \quad h(t) = \frac{1}{\sqrt{2\pi}}\widehat{\omega}(t) \quad for\, t > 0. \qquad (55)$$

The Fourier transform $\widehat{\omega}$ of $\omega \in L^\infty(\mathbb{R})$ should in general be understood in the sense of distributions (for example, on the Schwartz class $S'(\mathbb{R})$) and the precise meaning of (55) is given by the equation

$$(H(\omega)\Phi^* u, \Phi^* u) = (\Gamma(h)u, u), \qquad u \in C_0^\infty(\mathbb{R}_+).$$

A function $\omega(x)$ satisfying (55) is known as a symbol of the Hankel operator $\Gamma(h)$.

In the discrete case, the spectral asymptotics of $\Gamma(h)$ is determined by the asymptotic behaviour of the sequence $h(j)$ as $j \to \infty$.In the continuous case, the behaviour of the kernel $h(t)$ for $t \to \infty$ and for $t \to 0$ as well as the local singularities of $h(t)$at positive points t contribute to the spectral asymptotics of $\Gamma(h)$. In the following result we exclude local singularities. We denote $\langle x \rangle = \sqrt{1 + |x|^2}$.

The proof of Theorem (4.1.4) follows the scheme of the proof of Theorem (4.1.3). The only new point is that now we have to additionally establish the correspondence between symbols singular at infinity and kernels singular at $t = 0$.

Let us state the analogues of Theorems (4.1.4) and (4.2.3).

Theorem (4.2.7)[4]. Let $\alpha > 0$,and let the number $M = M(\alpha)$ be given by (38). Suppose thatg $\in L_{loc}^\infty(\mathbb{R}_+)$if $\alpha < 1/2$ and g $\in C^M(\mathbb{R}_+)$ if $\alpha = 1/2$.

Assume that

$$g^{(m)}(t) = o(t^{-1-m}\langle logt\rangle^{-\alpha}) \; as \; t \to 0 \; \; and \; as \; t \; \to \infty \qquad (56)$$

for all $m = 0, 1, \dots, M$.Then

$$s_n\big(\Gamma(g)\big) = o(n^{-\alpha}), \qquad n \to \infty \qquad (57)$$

We also have a result with O instead of oin (56) and (57), although we will not need it here. Observe that for $\alpha < 1/2$ we need only the estimate on g, whereas for $\alpha \geq 1/2$ we also need estimates on the derivatives g(m). Next, we define model kernels q_0, q_∞. Choose some non-negative functions $x_0, x_\infty \in C^\infty(\mathbb{R})$such that

$$x_0(x) = \begin{cases} 1 \text{ for} |x| \leq c_1, \\ 0 \text{ for} |x| \geq C_2, \end{cases} x_\infty(x) = \begin{cases} 0 \text{ for} |x| \leq C_1, \\ 1 \text{ for } |x| \geq C_2, \end{cases}$$

for some $0 < c_1 < c_2 < 1$ *and* $1 < C_1 < C_2$.

Theorem (4.2.8)[4]. For $\alpha > 0$,set

$$q_0(t) = x_0(t)t^{-1}\big(\log(1t)\big)^{-\alpha}; \; q_\infty(t) = x_\infty(t)t^{-1}(\log t)^{-\alpha}, t > 0. (58)$$

Then

$$s_n\big(\Gamma(q_0)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), s_n\big(\Gamma(q_\infty)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), n \to \infty,$$

Where $v(\alpha)$ given by (10).

Of course, this result corresponds to particular cases of Theorem (4.2.8) $L = 1, b_0 = 1, b_1 = 0$ and $b_0 = 0, b_1 = 1$.

In order to put together the contributions of different terms in (63) and (64), we use the localization principle in the form of Theorem (4.1.7). To that end, we need to determine the singular supports of the symbols corresponding to the model kernels q_0, q_∞. Again, we will choose functions σ_0, σ_∞ whose Fourier transform coincides with the odd extension of q_0, q_∞ to the real line. The proof below is very similar to that of Lemma (4.2.2).

Lemma (4.2.9)[4]. Let σ_0, σ_∞ be defined by

$$\sigma_0(x) = 2i\int_0^\infty q_0(t)\sin(xt)dt, \qquad \sigma_\infty(x) = 2i\int_0^\infty q_\infty(t)\sin(xt)dt, x \in R, (59)$$

Where $q_0(t)$ and $q_\infty(t)$ are given by (58) with $\alpha \geq 0$. Then $\sigma_0; \sigma_\infty \in L^\infty(\mathbb{R})$ and $\sigma_0 \in C^\infty(\mathbb{R}), \sigma_\infty \in C^\infty(\mathbb{R}_* \setminus \{0\})$.

Proof. Note that for all $x \in \mathbb{R}$, the first integral in (59) converges absolutely while the second one converges absolutely for $\alpha > 1$ and conditionally for $\alpha \geq 1$.

Since the integral in (59) of σ_0 is taken over a finite interval, we can differentiate this integral with respect to x arbitrary many times. Hence

$\sigma_0 \in C^\infty(\mathbb{R})$. To prove that $\sigma_\infty \in C^\infty(\mathbb{R}_* \setminus \{0\})$, we integrate by parts $2M + 2$ times in the definition (59):

$$\sigma_\infty(x) = 2i(-1)^{M+1}x^{-2M-2}\int_0^\infty q_\infty^{(2M+2)}(t)\sin(xt)dt.$$

Since $q_\infty^{(2M+2)}(t) = O(|t|^{-2M-3}) as\ |t| \to \infty$, we see that $\sigma_\infty \in C^m(\mathbb{R} \setminus \{0\})$ and $\sigma_\infty^{(m)}(x) \to 0\ for\ m = 0, 1, \ldots, 2M$ as $|x| \to \infty$. Finally, we use that M is arbitrary.

It remains to prove that the functions σ_0 andσ_1 are bounded. Below $\kappa = 0$ or $\kappa = \infty$. We may suppose that $x > 0$. Write $\sigma_\kappa = \sigma_\kappa^{(1)} + \sigma_\kappa^{(2)}$, where

$$\sigma_\kappa^{(1)}(x) = 2i\int_0^{1/x} q_\kappa(t)\sin(xt)dt, \sigma_\kappa^{(2)}(x) = 2i\int_{1/x}^\infty q_\kappa(t)\sin(xt)dt.$$

Since $|\sin(xt)| \leq xt$, for σ_1 we have the estimate

$$\left|\sigma_\kappa^{(1)}(x)\right| \le 2x \int_0^{1/x} q_\kappa(t) t dt \le C$$

Because $q_\kappa(t)t$ are bounded functions. For $\sigma_\kappa^{(2)}$, integrating by parts once, we get

$$\sigma_\kappa^{(2)}(x) = -\frac{2i}{x} \int_{1/x}^{\infty} q_\kappa(t) \big(\cos(xt)\big)' dt$$

$$= \frac{2i}{x} q_\kappa(1/x) \cos 1 + \frac{2i}{x} \int_{1/x}^{\infty} q_\kappa'(t) \cos(xt) dt.$$

The first term in the right-hand side is bounded because $q_\kappa(t)t$ are bounded functions. The second term is also bounded because the functions $q_\kappa'(t)t^2$ are bounded.

It easily follows from Lemma (4.2.9) that

$$\Phi H(\sigma_0)\Phi^* = \Gamma(q_0) and\, \Phi H(\sigma_\infty)\Phi^* = \Gamma(q_\infty). \quad (60)$$

Indeed, in view of relation (55), we only have to check that

$$\frac{1}{\sqrt{2\pi}} \tilde{\sigma}_0(t) = q_0(t); \frac{1}{\sqrt{2\pi}} \tilde{\sigma}_\infty(t) = q_\infty(t);\ t > 0, \quad (61)$$

where the Fourier transform is understood, for example, in the class of distributions $S(\mathbb{R}')$. According to the first formula in (59), the function $(2\pi)^{-1/2}\sigma_0$ is the Fourier transform of the "extended" distribution $q_0^{(ext)}$ defined by the equation

$$\langle q_0^{(ext)}, \varphi \rangle = \int_0^{\infty} q_0(t) \big(\overline{\varphi}(t) - \overline{\varphi}(-t)\big) dt.$$

Thus $(2\pi)^{-1/2}\hat{\sigma}_0(t) = q_0^{(ext)}(t)$ which coincides with $q_0(t)$ for $t > 0$. The second equality (61) is obvious because $q_\infty \in S(\mathbb{R})'$.

The next assertion is a direct consequence of relations (60) and Theorem (4.2.1). Theorem (4.2.4). Let the functions σ_0 and σ_∞ be defined by formulas (59) where $q_0(t)$ and $q_\infty(t)$ are given by (58) and $\alpha > 0$. Then the asymptotic relations hold

$$s_n\big(H(\sigma_0)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), s_n\big(H(\sigma_\infty)\big) = v(\alpha)n^{-\alpha} + o(n^{-\alpha}), n \to \infty,$$

Where $v(\alpha)$ is given by (10)

For a parameter $a \in \mathbb{R}$, let R_a be the shift

$$(R_a f)(x) = f(x - a).$$

Obviously, R_a is a unitary operator in $L^2(\mathbb{R})$ and $H_+^2(\mathbb{R})$. Of course, now R_a is not a rotation, but we keep the letter R in order to maintain the analogy between the discrete and continuous cases.

Similarly, let V_a be the multiplication operator

$$(V_a u)(t) = e^{-iat}u(t), t > 0.$$

Obviously, V_a is a unitary operator in $L^2(\mathbb{R}_+)$.

Recall that the Hankel operators $H(\omega)$ in $H^2(\mathbb{R})$ were defined by formula (21).

Lemma (4.2.10)[4]. For arbitrary $a \in \mathbb{R}$, we have the following statements:

(i) For any $\omega \in L^\infty(\mathbb{R})$, we have

$$H(R_a\omega) = R_a H(\omega) R_a.$$

In particular, if $H(\omega)$ is compact, then

$$s_n\big(H(R_a\omega)\big) = s_n\big(H(\omega)\big), \forall n \geq 1:$$

(ii) Suppose that $\Gamma(h)$ is bounded; then

$$\Gamma(V_a h) = V_a \Gamma(h) V_a.$$

In particular, if $\Gamma(h)$ is compact, then

$$s_n\big(\Gamma(V_a h)\big) = s_n\big(\Gamma(h)\big), \forall n \geq 1:$$

Proof. Since

$$P_+ R_a = R_a P_+ \text{and } R_a W R_a = W,$$

the first assertion is a direct consequence of the Hankel operator $H(\omega)$ in $H^2(\mathbb{R})$. The second assertion immediately.

Theorem (4.2.11)[4]. Let $\alpha > 0$, let $a_1, \dots, a_L \in \mathbb{R}$ be distinct numbers and let $b_0, b_1, \dots, b_L \in \mathbb{C}$. Let the number $M = M(\alpha)$ be given by (38). Suppose that $h \in L^\infty_{loc}(\mathbb{R})$ if $\alpha < 1/2$ and $h \in C^M(\mathbb{R}_+)$ if 1/ 2. Assume that

$$\mathrm{h}(t) = \sum_{\ell=1}^{L} (\mathrm{b}_\ell t^{-1} (\log t)^{-\alpha} + \mathrm{g}_\ell(t) e^{-i a_\ell t}, \quad t \geq 2, \tag{62}$$

$$\mathrm{h}(t) = (\mathrm{b}_0 t^{-1} \left(\log {}^1/_t\right)^{-\alpha} + \mathrm{g}_0(t), \qquad t \leq {}^1/_2, \tag{63}$$

where the error terms g_ℓ satisfy the estimates

$$\mathrm{g}_\ell^{(m)}(t) = o(t^{-1-m} \langle logt \rangle^{-\alpha}), \qquad m = 0, \dots, M(\alpha), \tag{64}$$

As $t \to \infty$ for $\ell = 1, \dots, L$ and as $t \to 0$ for $\ell = 0$. Then the singular values of the integral Hankel operator $\Gamma(h)$ in $L^2(\mathbb{R}_+)$ satisfy the asymptotic relation

$$s_n\big(\Gamma(h)\big) = c\, n^{-\alpha} + o(n^{-\alpha}), \qquad n \to \infty, \tag{65}$$

Where

$$c = v(\alpha) \left(\sum_{\ell=0}^{L} |\mathrm{b}_\ell|^{1/\alpha} \right)^{\alpha} \tag{66}$$

and the coefficient $v(\alpha)$ is given by formula (10).

The proof in the continuous case follows the same general outline as in the discrete case with the only difference that the singularity of the kernel h(t) at $t = 0$ has to be treated separately. It corresponds to the singularity of the symbol $\omega(x)$ at infinity.

We consider kernels h(t) that have a singularity at some positive point and admit representation (62) for large t. It turns out that, similarly to Theorem (4.1.5) the contributions of the singularities of these two types to the asymptotics of singular values are independent of each other.

Proof. Let the symbols $\sigma_0(x)$ and $\sigma_\infty(x)$ be defined by relations (59) and let

$$\omega_\ell(x) = \omega_0(x) + \sum_{\ell=1}^{L} \omega_0\,(x) where\ \omega_0(x) = b_0\sigma_0(x), \omega_\ell(x)$$
$$= b_\ell\sigma_\infty\left(x - a^\ell\right). \tag{67}$$

According to Theorem (4.2.1) and Lemma (4.2.5) (i) we have $s_n\big(H(\omega_\ell)\big) = |b_\ell| v(\alpha) n^{-\alpha} + o(n^{-\alpha}),\ n \to \infty$, for all $\ell = 0, 1, \dots, L$. It follows from Lemma (4.2.2) that $\omega_\ell \in L^\infty(\mathbb{R})$ for all $\ell =$ $0, 1, \dots, L, \omega_0 \in C^\infty(\mathbb{R})\, Z$and $\omega_\ell \in C^\infty(\mathbb{R}_* \setminus a_\ell)$ for $\ell = 1, \dots, L$. Since $a_1, \dots, a_L$ are distinct points, the localisation principle (Theorem (4.2.11)) is applicable to the sum (67). This yields

$$\lim_{n\to\infty} ns_n\big(H(\omega_\ell)\big)^p = \sum_{\ell=1}^{L} \lim_{n\to\infty} ns_n\big(H(\omega_\ell)\big)^p = v(\alpha)^p \sum_{\ell=1}^{L} |b_\ell|^p, p = \frac{1}{\alpha}. \tag{68}$$

Note that, by its definition (67), $\widehat{\omega}_0(t) = b_0\hat{\sigma}_0(t)$,

$$\widehat{\omega}_\ell(t) = b_\ell\sigma_\infty(t) e^{-ia_\ell t}; \ell = 1, \dots, L,$$

and hence according to formula (61)

$$\omega_\ell(t) = b_0 x_0(t) t^{-1} |\log t|^{-\alpha} + \sum_{\ell=1}^{L} b_\ell x_\infty(t) t^{-1} |\log t|^{-\alpha} e^{ia_\ell t} =: h_\ell(t), t$$
$$> 0.$$

In view of relation (55) it now follows from (68) that

$$\lim_{n\to\infty} n s_n\big(\Gamma(h_\ell)\big)^p \;=\; v(\alpha)^p \sum_{\ell=0}^{L} |b_\ell|^p.$$

Next, we consider the error term

$$g(t) = h(t) - h_\ell\,(t) \;=\; \mathrm{g}_0(t) \;+ \sum_{\ell=1}^{L} \mathrm{g}_\ell(t) e^{-i a_\ell t}$$

where all functions $\mathrm{g}_\ell(t), \ell \;=\; 0, 1, \dots, L,$ satisfy the condition (64) both for $t \to 0$ and $t \to \infty$. It follows from Theorem (4.2.6) and Lemma (4.2.5) (ii) that $sn(H(\mathrm{g}_\ell)) \;= o(n^{-\alpha})$ and hence

$$s_n\big(H(\mathrm{g})\big) = \; o(n^{-\alpha}) as\; n \to \infty. \tag{69}$$

Since

$$H(h) = \; H(h_\ell) + \; H(g),$$

we can use Lemma (4.1.1) with $A \;=\; H(h\_\ell)$ and $B \;=\; H(\mathrm{g})$. The required relations (65), (46) follow from (68) and (69).

The localization principle shows that the results on the asymptotics of singular values of different Hankel operators can be combined provided that the singular supports of their symbols are disjoint. This idea has already been illustrated by Theorems (4.1.1) and (4.2.6). Here we apply the same arguments to kernels $h(t)$ satisfying condition (62) as $t \;\to \infty$ and singular at some point $t_0 > \; 0.$ below $1_+ \;\; (t)$ is the characteristic function of $\mathbb{R}_+$.

The effect of local singularities of $h(t)$ on the asymptotics of singular values of the corresponding Hankel operator $\Gamma(h)$ was studied. We need the following result

Lemma (4.2.12)[4]. Let $t_0 > \; 0, m \in \mathbb{Z}_+$ and

$$\mathrm{a}_m(t) = \; (t_0 - \; t)^m 1 + (t_0 - \; t). \tag{70}$$

Then $\mathrm{Ker}\Gamma(\mathrm{a}_m) \;=\; L^2(t_0, \infty)$ and

$$\Gamma(\mathrm{a}_m)|_{L^2(0,t_0)} \;\;=\; m!\, A_m^{-1}$$

Where the self-adjoint operator A_m in $L^2(0, t_0)$ is defined by the differential expression

$$(A_m u)(t) = (-1)^{m+1} u^{(m+1)}(t_0 - t)$$

and the boundary conditions

$$u(t_0) = \cdots = u(m)(t_0) = 0. \tag{71}$$

Note that the operator A_m^2 is given by the differential expression

$$(A_m^2 u)(t) = (-1)^{m+1} u^{(2m+2)}(t)$$

and the boundary conditions (70) and

$$u^{(m+1)}(0) = \cdots = u^{(2m+1)}(0) = 0.$$

Thus A_m^2 is a regular differential operator and the asymptotics of its eigenvalues is given by the Weyl formula. Therefore the following result is an immediate consequence of Lemma (4.2.12).

Corollary (4.2.13)[4]. Let the function $\mathrm{a}_m(t)$ be given by formula (70). Then

$$s_n\big(\Gamma(\mathrm{a}_m)\big) = m!\, t_0^{m+1}(\pi n)^{-m-1}\big(1 + O(n^{-1})\big), n \to \infty. \tag{72}$$

Notice that formula (72) was obtained much earlier by a completely different method.

We also note the explicit formula for the symbol $\tau_m(x)$ of the operator $\Gamma(\mathrm{a}_m)$:

$$\tau_m(x) = m!\,(ix)^{-m-1}\left(e^{it_0x} - \sum_{k=0}^{m} \frac{1}{k!}(it_0x)^k\right), x \in \mathbb{R}. \tag{73}$$

Obviously,$\tau_m \in C^\infty(R)\ and\ \tau_m(x)$ is an oscillating function as$|x| \to \infty$.

We are now in a position to consider the general case.

Theorem (4.2.14)[4]. Let $t_0 > 0, m \in \mathbb{Z}_+$and $\beta \in C$. Set

$$h_m(t) = \beta(t_0 - t)^m 1_+(t_0 - t) + h(t)$$

Where $h(t)$ satisfies the assumptions of Theorem (4.2.10) with $b_0 = 0$ and $\alpha = m + 1$. Then the singular values of the operator $\Gamma(h_m)$ satisfy the asymptotic relation

$$s_n\big(\Gamma(h_m)\big) = c_m n^{-m-1} + o(n^{-m-1})$$

with

$$c_m = \left(\pi^{-1} t_0 (m!\,|\beta|)^{1\alpha} + v(\alpha)^{1/\alpha} \sum_{\ell=1}^{L} |b_\ell|^{1/\alpha}\right)^{\alpha}, \qquad \alpha = m + 1,$$

and $v(\alpha)$ defined by (10).

Proof . It is almost the same as that of Theorem (4.2.6). Let us use notation (70). The asymptotics of the singular values of the operator $\Gamma(\mathrm{a}_m)$ is given by formula (72). The operator $\Gamma(h)$ satisfies the assumptions of Theorem (4.2.6) so that the asymptotics of its eigenvalues is given by formula (65). The symbol (73) of the operator $\Gamma(\mathrm{a}_m)$ is singular only at infinity. Neglecting the terms satisfying the assumptions of Theorem (4.2.1) and using Lemma (4.2.2), we see that the singular support of the symbol of the operator $\Gamma(h)$ consists of the points $a_1, \dots, a_L \in \mathbb{R}$. Therefore applying Theorem (4.2.11), we conclude the proof.

Observe that we have excluded the term (63) singular at $t = 0$ in Theorem (4.2.12) cause the corresponding symbol is singular at the same point $x = 1$ as the function (73). In this case one might expect that the contributions of singularities of h(t) at $t = 0$ and $t = t_0 > 0$ are not independent of each other. In any case, our technique does not allow us to treat this situation.

Let us discuss the operator $\Gamma(\mathrm{a}_m)$ in the representation $\ell^2(\mathbb{Z}_+)$, that is, the operator

$$FU^* H(\tau_m) UF^* = \Gamma(\mathrm{g}_m).$$

Here $\mathrm{g}_m(j)$ are the Fourier coefficients of the function $\tau_m(\mu)$ linked to $\tau_m(x)$ by formula (30). Making the change of variables (23) in (73), we see that $\tau_m(\mu)$ is an oscillating function as $\mu \to 1$. Therefore the asymptotics of its Fourier coefficients $\mathrm{g}_m(j)$ is determined by the stationary phase method which yields:

$$\mathrm{g}_m(j) \sim m!\,\pi^{-12} 2^{-(2m+1)/4} j^{-(2m+5)/4} \cos\!\big(2\sqrt{2j} - \pi(2m + 1)/4\big).$$

Note that these sequences decay faster as $j \to \infty$ than the coefficients (39) where $\alpha = m + 1$. Never the less due to the oscillating factor their contribution to the asymptotics of singular values is of the same order.

List of Symbols

Symbol		page
$L^{\infty,1}$	Lebesgue space	1
$L^{2.l}$	Sobdev space	1
L^2	Hilberl space	1
$\oplus$	Direct sum	1
H^2	Hardy space	2
L_a^2	Bergman space	2
$a.e$	almost every where	6
L^p	Lebesgue space	7
H^∞	Hardy space	10
ℓ^p	Helbert space of sequences	18
ker	kernel	18
im	imaginary	18
dim	dimension	18
ind	index	18
sup	Supremum	19
H^p	Hardy space	21
inf	infimum	25
diag	diagonal	26
ess	essential	33
det	determinant	40
ext	extension	46
arg	argument	57
TTO	Truncated Toeplitz operators	66
THO	Truncated Hankel operators	66
$\ominus$	Direct difference	67
$\otimes$	Tensor product	69
sn	Singular values	79
supp	support	80
spec	Spectrum	83
loc	local	104

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