Sudan University of Science & Technology

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Geometric Inequalities and Balls in *CR* Sasakian Manifolds with Volume for Sub Riemannian Manifolds

المتباينات الهندسية والكرات في متعددات طيات ساساكيان CRمع

حجم متعددات طيات ريمان الجزئية

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By

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Dedications

To my father, mother, brothers, sisters

and dear friends.

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Abstract

The minimal submanifolds with constant mean curvature and of a sphere with bounded second fundamental form are considered. An intrinsic rigidity theorem from minimal submanifolds with parallel mean curvature in a sphere and the log-Sobolev inequalities for subelliptic operators satisfying a generalized curvature dimension inequality were studied. Stochastic completeness, volume growth, connection, curvature and distance comparison theorem for sub-Riemannian manifolds are shown. The sub-Riemannian curvature dimension inequality, volume doubling property, Poincaré inequality and balls in CR Sasakian manifolds are discussed. We classify the closed minimal submanifolds and geometric inequalities for certain submanifolds in pinched Riemannian manifolds.

الخلاصة

تم اعتبار متعددات الطيات الجزئية الأصغرية مع الانحناء المتوسط الثابت وللكرة مع الصيغة الأساسية الثانية المحددة . درسنا مبرهنة الصلابة الجوهرية من متعددات الطيات الجزئية الأصغرية مع الانحناء المتوسط الموازي في الكرة ومتباينات لوغريثم ــ سوبوليف لأجل المؤثرات الناقصية الجزئية المحققة لمتباينة بعد الانحناء المعمم . أوضحنا التمام التصادفي ونمو الحجم والاتصال والانحناء ومبرهنة مقارنة المسافة لمتعددات ريمانيان ــ الجزئية . تمت مناقشة متباينة بعد انحناء ريمانيان ــ الجزئي وخاصية ازدواجية الحجم ومتباينة بونكارية والكرات في متعددات طيات ساساكيان CR صنفنا متعددات الطيات الجزئية الأصغرية المغلقة والمتباينات الهندسية لاجل متعددات طيات جزئية معينة في متعددات طيات ريمانيان المقروصة.

Introdution

We study submanifolds with constant mean curvature. First, we want to reduce the theory of constant mean curved submanifolds to the theory of minimal submanifolds under fairly general conditions. Second, we study the minimal submanifolds themselves.

Let h be the second fundamental form of an n-dimensional minimal submanifold M of a unit sphere $S^{n+p}(p > 2)$, S be the square of the length of h, and $\sigma(u) = \|h(u,u)\|^2$ for any unit vector $u \in TM$. Simons proved that if $S < n/(2 - 1/p)$ on M, then either $S \equiv 0$, or $S \equiv n/(2 - 1/p)$. Chern, do Carmo, and Kobayashi determined all minimal submanifolds satisfying $S \equiv n/(2 - 1/p)$.

We shall show a rigidity theorem for submanifolds with parallel mean curvature in $S^{n+1}(1)$. Let M be a smooth connected manifold endowed with a smooth measure μ and a smooth locally subelliptic diffusion operator L which is symmetric with respect to μ. We assume that L satisfies a generalized curvature dimension inequality as introduced by Baudoin and Garofalo.

We generalize A. Grigor'yan's volume test for the stochastic completeness of a Riemannian manifold to a sub-Riemannian setting.Let M be a smooth connected manifold endowed with a smooth measure μ and a smooth locally subelliptic diffusion operator L satisfying $L1 = 0$, and which is symmetric with respect to μ.We show that if L satisfies, with a non negative curvature parameter, the generalized curvature inequality introduced, then the following properties hold:
• The volume doubling property;
• The Poincaré inequality;
• The parabolic Harnack inequality.

We first show a generalized Simons integral inequality for closed minimal submanifolds in a Riemannian manifold. Second, we show a pinching theorem for closed minimal submanifolds in a complete simply connected pinched Riemannian manifold, which generalizes the results obtained by S. S. Chern, M. do Carmo, and S. Kobayashi and A. M. Li and J. M. Li respectively. We show global estimates for the sub-Riemannian distance of CR Sasakian manifolds with nonnegative horizontal Webster-Tanaka Ricci curvature.

For a subRiemannian manifold and a given Riemannian extension of the metric, we define a canonical global connection. This connection coincides with both the Levi-Civita connection on Riemannian manifolds and the Tanaka-Webster connection on strictly pseudoconvex CR manifolds. We define a notion of normality generalizing Tanaka's notion for CR manifolds to the subRiemannian case. Under the assumption of normality, we construct local frames that simplify computations in a manner analogous to Riemannian normal coordinates. We study global distance estimates and uniform local volume estimates in a large class of sub-Riemannian manifolds. Our main device is the generalized curvature dimension inequality introduced and its use to obtain sharp inequalities for solutions of the sub-Riemannian heat equation.

The Contents

Chapter 1

Constant Mean Curvature Submanifolds

We study in higher dimensional submanifolds for studing submanifolds with constant mean curvature.

Section (1-1): Submanifolds

We study the submanifolds with constant mean curvature. The object of here is two-fold. We want to reduce the theory of constant mean curved submanifolds to the theory of minimal submanifolds under fairly general conditions. We study the minimal submanifolds themselves.

We shall restrict our attention to surfaces. H. Hopf was the first one who proved that the only constant mean curved surface with genus zero in Euclidean three space is the standard sphere. His idea was then extended and used by Calabi [1] and Chern [2] in the theory of minimal spheres. We shall use this idea again. Klotz and Osserman [9] studied complete surfaces with constant mean curvature in Euclidean three space. The last part of the argument in Theorem (1.1.11) is the same as theirs. This was pointed out to us by Klotz.

Is essentially Chern's presentation of Simon [3], [15] which will be extensively used. Is a generalization of J. Erbacher [8] and Chen and Yano [6]. These are conditions under which a submanifold lies in a totally umbilical submanifold.

The main theorem below is a splitting theorem. It states that for the immersion of a surface with genus zero or a complete surface with non-negative curvature, the splitting of the normal bundle (in the geometric sense) has strong consequences. Suppose the normal bundle is $N_1 \oplus N_2$ and the mean curvature vector lies in N_1. Then we prove if the curvature of the surface is not identically zero, it is either a minimal submanifold of an umbilical submanifold with normal bundle N_2 or a submanifold of a totally geodesic submanifold with normal bundle N_1. If we apply this theorem to full minimal sphere in sphere, it says that the normal bundle cannot split. We also consider the flat case.

We show that every surface with parallel mean curvature in a manifold with constant curvature actually lies in a totally geodesic three space or a minimal surface of an umbilical hypersurface. This essentially reduces the whole theory of surfaces with parallel mean curvature to the theory of minimal surfaces. We note that the theorem was proved by Chen and Ludden [7] under the assumption the surface has constant curvature and the ambient manifold is the Euclidean space.

We consider surfaces with constant mean curvature. The assumption is weaker than the assumption on the parallel mean curvature and we have only partial results. If a sphere or a complete non-negatively curved surface is immersed as a constant mean curved surface

of a four-dimensional constant curved manifold, then either the surface is minimal, a minimal surface of an umbilical hypersurface, or flat. In the last case, the second fundament form is covariant constant. If the ambient manifold is the Euclidean space, this generalizes Hopf's theorem and a theorem of Klotz and Osserman [9]. Using Calabi's theory on holomorphic curve, we show that the hyperbolic plane cannot be minimally immersed in Euclidean space, even locally. Finally we show a similar theorem as above for minimal totally real surfaces in manifolds with constant holomorphic sectional curvature.

Chen proved Theorem (1.1.6). Namely he proved that if N is the Euclidean space, then either M^2 is a minimal surface of an umbilical hypersurface of N or M^2 is a "Hoffman surface". He also proved Theorem (1.1.10) independently. D. Hoffman [17] has also some nice results in this direction. See [18].

We follow closely the exposition in [5]. Let M be an n-dimensional manifold immersed in an $(n + p)-$dimensional Riemannian manifold N. We choose a local field of orthonormal frames e_1, e_2,.. . e_{n+p} in N such that, restricted to M, the vectors e_1, e_2,... ,e_n are tangent to M. We shall make use of the following convention on the ranges of indices:

$$1 \leqq A, B, C, \ldots, \leqq n + p, 1 \leq i, j, k, \ldots \leq n$$
$$n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p$$

and we shall agree that repeated indices are summed over the respective ranges. With respect to the frame field of N chosen above, let ω_1... , ω_{n+p} be the field of dual frames. Then the structure equations of N are given by

$$d\omega_A = -\sum \omega_{AB} \wedge \omega_B,$$
$$\omega_{AB} + \omega_{BA} = 0 \tag{1}$$
$$d\omega_{AB} = -\sum \omega_{AC} \wedge \omega_{CB} + \Phi_{AB}$$
$$\Phi_{AB} = \tfrac{1}{2} \sum K_{ABCD} \omega_C \wedge \omega_D \tag{2}$$
$$K_{ABCD} + K_{ABDC} = 0.$$

We restrict these forms to M. Then

$$\omega_\alpha = 0. \tag{3}$$

Since $0 = d\omega_\alpha = -\sum \omega_{\alpha j} \omega_j$ by Cartan's lemma we may write

$$\omega_{\alpha j} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha \tag{4}$$

From these formulas, we obtain

$$d\omega_i = \sum_i \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \tag{5}$$

$$d\omega_{ij} = -\sum_k \omega_{ij} \wedge \omega_{kl} + \Omega_{ij} \tag{6}$$
$$\Omega_{ij} = \tfrac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l$$
$$R_{ijkl} = K_{ijkl} + \sum_\alpha \left(h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha \right), \tag{7}$$

2

$$d\omega_{\alpha\beta} = -\sum \omega_{\alpha\gamma} \wedge \omega_{\alpha\gamma} + \Omega_{\alpha\beta} \tag{8}$$

$$\Omega_{\alpha\beta} = \tfrac{1}{2} \sum R_{\alpha\beta kl}\, \omega_k \wedge \omega_l$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i \left(h_{ik}^\alpha h_{jl}^\beta - h_{il}^\alpha h_{jk}^\beta \right). \tag{9}$$

The Riemannian connection of M is defined by (ω_{ij}). The form $(\omega_{\alpha\beta})$ defines a connection in the normal bundle of M. We call $\sum h_{ik}^\alpha \omega_i \omega_j e_\alpha$ the second fundamental form of the immersed manifold M. Sometimes we shall denote the second fundamental form by its components h_{ij}^α. We call $\sum_\alpha 1/n (\sum_\alpha h_{ii}^\alpha) e_\alpha$ the mean curvature vector. The length of it is called the mean curvature. An immersion is said to be minimal if the mean curvature vanishes. We take exterior differentiation of (4) and define h_{ijk}^α by

$$\sum h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum h_{il}^\alpha \omega_{li} - \sum h_{lj}^\alpha \omega_{li} + \sum h_{ij}^\beta \omega_{\alpha\beta}. \tag{10}$$

Then

$$\sum (h_{ijk}^\alpha + \tfrac{1}{2} K_{ijk}^\alpha) \omega_j \wedge \omega_k = 0, \tag{11}$$

$$h_{ijk}^\alpha - h_{ikj}^\alpha = K_{\alpha ikj} = -K_{\alpha ijk} \tag{12}$$

We take exterior of (10) and define h_{ijkl}^α by

$$\sum h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha - \sum h_{ljk}^\alpha \omega_{li} - \sum h_{ilk}^\alpha \omega_{lj} - \sum h_{ijl}^\alpha \omega_{lk} + \sum h_{ijk}^\beta \omega_{\alpha\beta}. \tag{13}$$

Then

$$\sum \left(h_{ijkl}^\alpha - \tfrac{1}{2} \sum h_{im}^\alpha R_{mjkl} - \tfrac{1}{2} \sum h_{mj}^\alpha R_{mikl} + \tfrac{1}{2} \sum h_{ij}^\beta R_{\alpha\beta kl} \right) \omega_k \wedge \omega_l = 0, \tag{14}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum h_{im}^\alpha R_{mjkl} + \sum h_{mj}^\alpha R_{mikl} - \sum h_{ij}^\beta R_{\alpha\beta kl} \tag{15}$$

Since (ω_{ij}) defines a connection in the tangent bundle $T = T(M)$ [and, hence a connection in the cotangent bundle $T^* = T^*(M)$ also] and $(\omega_{\alpha\beta})$ defines a connection in the normal bundle $T^\perp = T^\perp(M)$, we have covariant differentiation which maps a section of $T^\perp \otimes T* \otimes \ldots \otimes T* \otimes T*$, (T*: k times), into a section of $T^\perp \otimes T* \otimes \ldots \otimes T* \otimes T*$, $(T^*: k+1$ times). The second fundamental form h_{ij}^α is a section of the vector bundle $T^\perp \otimes T^* \otimes T^*$ and h_{ijk}^α is the covariant derivative of h_{ij}^α. Similarly, h_{ijkl}^α is the covariant derivative of h_{ijk}^α.

Considering $K_{\alpha ijk}$ as a section of $T^\perp \otimes T^* \otimes T^* \otimes T^*$, its covariant derivative $K_{\alpha ijk}$ is defined by

$$\sum K_{\alpha ijkl} \omega_l = dK_{\alpha ijk} - \sum K_{\alpha mjk} \omega_{im} - \sum K_{\alpha imk} \omega_{mj} - \sum K_{\alpha ijm} \omega_{mk} + \sum K_{\beta ijk} \omega_{\alpha\beta}. \tag{16}$$

This covariant derivative of $K_{\alpha ijkl}$ should be distinguished from the covariant derivative of K_{ABCD} as a curvature tensor of N, which will be denoted by $K_{ABCD;E}$ Restricted to M, $K_{\alpha ijk;l}$ is given by

$$K_{\alpha ijk;l} = K_{\alpha ijk;l} - \sum K_{\alpha \beta jk}h_{il}^{\beta} - \sum K_{\alpha i\beta k}h_{jl}^{\beta} - \sum K_{ij\beta}^{\alpha}h_{kl}^{\beta} - \sum K_{mijk}h_{ml}^{\beta} \qquad (17)$$

Now let us assume that N is locally symmetric, i.e., $K_{ABCD;E} = 0$. The Laplacian Δh_{ij}^{α} of the second fundamental form h_{ij}^{α} is defined by

$$\Delta h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha}, \qquad (18)$$

From (12) we obtain

$$\Delta h_{ij}^{\alpha} = \sum_k h_{ijkk}^{\alpha} - \sum_k h_{ijkk}^{\alpha} = \sum_k h_{ijkk}^{\alpha} - \sum_k h_{ijkk}^{\alpha} \qquad (19)$$

From (15) we obtain

$$h_{kijk}^{\alpha} = h_{kikj}^{\alpha} + \sum h_{km}^{\alpha}R_{mijk} + \sum h_{mi}^{\alpha}R_{mkjk} - \sum h_{ki}^{\beta}R_{\alpha \beta jk} \qquad (20)$$

Replace h_{kijk}^{α} in(20) by $h_{kkij}^{\alpha} - K_{\alpha kikj}$ and then substitute the right-hand side of (20) into h_{kijk}^{α} of (19). Then

$$\Delta h_{ij}^{\alpha} = \sum_k (h_{kkij}^{\alpha} - K_{kikj}^{\alpha} - K_{ijkk}^{\alpha})$$

$$+ \sum_k \left(\sum_m h_{km}^{\alpha} R_{mijk} + \sum_m h_{mi}^{\alpha} R_{mkjk} - \sum_{\beta} h_{ki}^{\alpha} R_{\alpha \beta jk} \right) \qquad (21)$$

Now assume e_{α} is the mean curvature vector. Hence

$$\sum_i h_{ij}^{\beta} = 0 \qquad \qquad \text{if} \qquad \beta \neq \alpha. \qquad (22)$$

From (10) we have

$$\sum_{i,k} h_{iik}^{\alpha} \omega_k = ndH$$

$$\sum_{i,k} h_{iik}^{\alpha} \omega_k = ndH\omega_{\beta\alpha} \qquad \qquad \text{if} \qquad \beta \neq \alpha. \qquad (23)$$

Here H is the mean curvature.

From (22) and (13), we have

$$\sum_i h_{iik}^{\alpha} = ndH - \sum_{\beta \neq \alpha} \left(\sum_i h_{iil}^{\beta} \right) \left(\sum_i h_{iil}^{\beta} \right) (nH)^{-1} \quad if \quad H \neq 0$$

$$\sum_i h_{iikl}^{\alpha} = nH_{kl} \quad if\ H = 0. \qquad (24)$$

Substituting (24) into (21), we have

$$\Delta h_{ij}^{\alpha} = nH_{ij} - \varepsilon \sum_{\beta \neq \alpha} \left(\sum_k h_{iil}^{\beta} \right) \left(\sum_k h_{iil}^{\beta} \right) (nH)^{-1} - \sum_k (K_{kikj}^{\alpha} + K_{ijkk}^{\alpha})$$

$$+ \sum_k (\sum_m h_{km}^{\beta} R_{mijk} + \sum_m h_{mi}^{\alpha} R_{mkjk} - \sum_{\beta} h_{ki}^{\alpha} R_{\alpha\beta jk}) \qquad (25)$$

where $\epsilon = 1$ if $H \neq 0$ and $\epsilon = 0$ if $H = 0$.

The vector e_{α} is parallel in the normal bundle of M if the covariant derivative of e_{α} in N is tangent to M. This is equivalent to

$$\omega_{\alpha\beta} = 0. \qquad (26)$$

Then by (23) and (26) we have

$$\Sigma_k h_{iik}^{\beta} = 0 \qquad \text{if } \beta \neq \alpha. \qquad (27)$$

Hence in this case,

$$\Delta h_{ij}^{\alpha} = nH_{ij} - \sum_k (K_{kikj}^{\alpha} + K_{ijkk}^{\alpha})$$

$$+ \sum_k (\sum_m h_{km}^{\beta} R_{mijk} + \sum_m h_{mi}^{\alpha} R_{mkjk} - \sum_{\beta} h_{ki}^{\alpha} R_{\alpha\beta jk}) \qquad (28)$$

$$\Delta h_{ij}^{\alpha} = - \sum_k (K_{kikj}^{\beta} + K_{ijkk}^{\beta})$$

$$+ \sum_k (\sum_m h_{km}^{\beta} R_{mijk} + \sum_m h_{mi}^{\alpha} R_{mkjk} - \sum_{\delta} h_{ki}^{\delta} R_{\alpha\delta jk}) \qquad (29)$$

Let N be the Euclidean space. Then Chen and Yano [6] proved that if there exists a nonzero normal vector field e over M such that M is umbilical with respect to e, then M lies in a sphere with e parallel to the radius vector field. On the other hand, J. Erbacher [8] proved that if N has constant curvature and if the first normal space N_1 of M is invariant under parallel translations with respect to the normal connection, then M is a submanifold of a totally geodesic submanifold of N with dimension $n + l$ where l is the constant dimension of N_1. For the later purpose, we unify and extend these theorems.

Let N_1 be a sub-bundle of the normal bundle. We say that M is umbilical (totally geodesic) with respect to N_1 if M is umbilical (totally geodesic) with respect to any local section of N_1. We say that N_1 is parallel in the normal bundle if it is invariant under the parallel translation in the normal bundle.

Theorem (1.1.1)[33]. Let N be a conformally flat manifold. Let N_1 be a sub- bundle of the normal bundle of M with fiber dimension k. Suppose M is umbilical with respect to N_1 and

N_1 is parallel in the normal bundle. Then M lies in an $n + p - k$ dimensional umbilical submanifold N' of N such that the fiber of N_1 is everywhere perpendicular to N'. If N has constant curvature, the size of N' can be determined. In particular, if M is totally geodesic with respect to N_1, then N' is totally geodesic.

Proof. We first assume N is the euclidean space. Let $\{ e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_k}, e_{\alpha_{k+1}}, \ldots, e_{\alpha_p} \}$ be a local normal frame field of M such that $\{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_k}\}$ span N_1. By assumption, there exists functions $\{f_{\alpha_1}, \ldots, f_{\alpha_k}\}$ such that

$$\omega_{\alpha_1 i} = f_{\alpha_i}\omega_i \tag{30}$$

$for\ j = 1, \ldots, k;\ i = 1, \ldots, n.$ It is easy to see by performing an orthogonal transformation in the normal space, we may assume

$$\omega_{\alpha_1 i} = f\omega_{i'}$$
$$\omega_{\alpha_j i} = 0 \quad for \quad j = 2, \ldots, k \tag{31}$$

The hypothesis that N_1 is parallel in the normal bundle means

$$\omega_{\alpha_i \alpha_j} = 0 \tag{32}$$

for $1 \leq i \leq k, k + 1 \leq j \leq p$. Exterior differentiate the first equation of (31) and use the second equation of (31) and (32), we obtain

$$\mathrm{d}f \wedge \omega_i = 0 \tag{33}$$

This implies f is a constant function.

We assume $f \neq 0$. Now exterior differentiate the second equation of (31), we obtain

$$\omega_{\alpha_j \alpha_1} \wedge \omega_{\alpha_1 i} = 0 \tag{34}$$

for $j = 2, \ldots, k$. Here we use (32) and the second equation of (31). Since $f \neq 0$, the equation (34) implies immediately $\omega_{\alpha_j \alpha_1} = 0$ for all $j = 2, 3, \ldots, k$.

Let X be the position vector of M. Then as f is constant and $\omega_{\alpha_j \alpha_1} = 0$ for all j, one can use the definitions of covariant derivatives and $\omega_{\alpha_j \alpha_1} = 0$ to prove that $X + e_{\alpha_1}/f$ is a constant vector. Hence we have proved that M lies in a sphere e_{α_1} parallel to the radius vector.

By repeating the arguments again, it is easy to see, from the second equation of (31), the multivector $e_{\alpha_2} \wedge e_{\alpha_3} \wedge \ldots \wedge e_{\alpha_k}$ is constant on M. The manifold therefore lies in a linear space perpendicular to the linear spanned by $\{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_k}\}$. Combining these assertions, we see M lies in a sphere with N_1 perpendicular to this sphere.

Let us now turn to the general case. A theorem of Kuiper [10] says that every simply connected conformally flat manifold has a unique conformal immersion onto a domain of Euclidean space. Using this theorem, it suffices to prove Theorem (1.1.1) locally. Let ρ be a smooth function such that the metric $e^{2\rho}\sum_A \omega_A \otimes \omega_A$ on N has zero curvature. Then {

$e^{-\rho}e_A\}$ is an orthonormal frame field with respect to the new metric. The dual frame field is $\{\omega_A^* = e^\rho \omega_A\}$. In this new metric, the structure equations of N are given by

$$d\omega_A^* = -\sum_B \omega_{AB}^* \wedge \omega_B^*$$

(35)

$$\omega_{AB}^* = \omega_{AB} + \rho_A \omega_B.$$

In particular

$$\omega_{AB}^* = \sum_j h_{ij}^\alpha \omega_j - \rho_A \omega_i,$$

(36)

$$\omega_{\alpha\beta}^* = \omega_{\alpha\beta}$$

(37)

on M.

From equation (36), one sees that M is umbilical with respect to e_α in the old metric if M is umbilical with respect to $e^{-\rho}e_\alpha$ in the new metric. From equation (32), e_α is parallel in the normal bundle in the old metric if $e^{-\rho}e_\alpha$ is parallel in the normal bundle in the new metric. The first part of the theorem then follows from the result in euclidean space. The last part follows by examining the sterographic projection.

For the dimension two case, it is more convenient to view M as a complex manifold. We first prove the following theorem.

Theorem (1.1.2)[33]. Let M^2 be a surface in a constant curved manifold. Suppose the normal bundle of M^2 is an orthogonal sum of two subbundles N_1 and N_2 such that both N_1 and N_2 are invariant under the parallel translations in the normal bundle. Suppose the orthogonal projection of the mean curvature vector in N_2 is parallel in the normal bundle. If M^2 has genus zero or if M^2 is a complete non-negatively curved surface with bounded mean curvature, then either

(a) M^2 has curvature identically equal to zero,

(b) M^2 lies in a $k + 2$ dimensional umbilical submanifold of N where k is the fiber dimension of N_1. If the mean curvature vector lies completely in N_1, then the umbilical submanifold can be chosen to be totally geodesic or

(c) M^2 is a submanifold of an $2 + p - k$ dimensional umbilical submanifold with parallel mean curvature vector. If the mean curvature vector lies completely in N_1, then M^2 is actually minimal.

Proof. By considering the two-fold cover of the surface, we may assume M^2 is oriented and the orientation is given by $\omega_1 \wedge \omega_2$. Let

$$\varphi = \omega_1 + i\omega_2$$

(38)

From (1), we have

$$d\varphi = i\omega_{12} \wedge \varphi$$

(39)

The existence of local isothermal coordinates means that we can write, locally,

$$\varphi = \lambda dz$$

(40)

7

Substituting into (39), we obtain

$$(d\lambda - i\lambda\omega_{12}) \wedge dz = 0 \tag{41}$$

For a normal vector ea, we define

$$H_\beta = \frac{h_{11}^\beta - h_{22}^\beta}{2} + ih_{12}^\beta \tag{42}$$

Clearly if φ is changed to $e^{it}\varphi$, H_β is changed to $e^{2it}eH_\beta$. Hence the form $\bar{H}\beta\varphi^2$ is invariant under the change of φ .

Now choose a normal frame field $\{e_{\alpha_1}, e_{\alpha_2}, \ldots, e_{\alpha_k}, e_{\alpha_{k+1}}, e_{\alpha_{k+2}}, \ldots, e_{\alpha_p}\}$ on M such that the first k vectors span the fiber of N_1 and that $e_{\alpha k+1}$ has the same direction as the projected mean curvature vector. Then an application of (10) shows

$$\sum_i h_{iil}^{\alpha j} = 0 , \tag{43}$$

$$dH_{\alpha j} = 2i\omega_{12}H_{\alpha j} - \sum_{l>k} H_{\alpha j}\omega_{\alpha j\alpha_i}$$

$$+ - \sum_i \left(\frac{h_{11l}^\alpha - h_{22l}^\alpha}{2}\right)\omega_l + i\sum_l h_{121\alpha_i}\omega_l \tag{44}$$

when $j > k$.
Combining (43), (44) and (12), we obtain

$$\left(dH_{\alpha j} + \sum_{l>k} H_{\alpha_i}\omega_{\alpha_j\alpha_i} - 2i\omega_{12}H_{\alpha_i}\right) \wedge \bar{\varphi}$$

$$[(h_{121\alpha_j} - h_{121\alpha_j}) + i(h_{221\alpha_j} - h_{212\alpha_j})]\ \omega_1 \wedge \omega_2$$

$$= (K_{\alpha j112} + iK_{\alpha j212})\ \omega_1 \wedge \omega_2 \tag{45}$$

Since N has constant curvature, we have

$$\left(dH_{\alpha j} + \sum_{l>k} H_{\alpha_i}\omega_{\alpha_j\alpha_i} - 2i\omega_{12}H_{\alpha_i}\right) \wedge \bar{\varphi} = 0 \tag{46}$$

Hence

$$d\left(\sum_{j>k}\left(\bar{H}_{\alpha_j}\right)^2\right) + 4i\omega_{12}\left(\sum_{j>k}\left(\bar{H}_{\alpha_j}\right)^2\right) \equiv 0\ mod\varphi \tag{47}$$

From the remark above, it is easy to see $\sum_{j>k}\left(\bar{H}_{\alpha_j}\right)^2\varphi^4$ is a globally defined form on M^2. Using the local isothermal coordinate z introduced in (40), we may write $\sum_{j>k}\left(\bar{H}_{\alpha_j}\right)^2\varphi^4 = f(z)dz^4$. It follows from (41) and (47) that $f(z)$ is holomorphic. This implies

$$\partial\bar\partial \log\left|\sum_{j>k}\left(\bar{H}_{\alpha_j}\right)^2\lambda^4\right| = 0 \tag{48}$$

when $f(z) \neq 0$. On the other hand, it is well known that the Gauss curvature of M is given by

8

$$R = \frac{-\partial \bar{\partial} \log \lambda}{\lambda^4} \tag{49}$$

Combining (48) and (49), we have

$$\Delta \log \left| \sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \right| = 4R \tag{50}$$

where $\Delta = -\partial \bar{\partial} / \lambda^2$ is the Laplacian of M.

Since $f(z)$ is holomorphic, $\sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2$ is either identically zero or has only isolated zeros. An application of Gauss-Bonnet theorem on equation (50) shows that $\sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2$ is identically zero if M^2 has genus zero. Let us now consider the case where M^2 is complete and has non-negative curvature. First of all, it is standard that

$$\sum_{\alpha,i,j} \left(h_{ij}^{\alpha} \right)^2 - 4H^2 = 2(K_{1212} - R). \tag{51}$$

where H is the mean curvature of M^2.

Hence if $R \geqq 0$ and H is bounded, the length of the second fundamental form is also bounded. On the other hand, it is straightforward to see from (50)

$$\Delta \left| \sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \right|^2 \geq 8 \left| \sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \right|^2 R \tag{52}$$

everywhere on M^2. Therefore if $R \geq 0$, $\left| \sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \right|^2$ is a bounded subharmonic function on M^2. A theorem of Blanc, Fiala and Huber [16] states that every complete non-negatively curved surface is parabolic. Hence $\left| \sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \right|^2$ must be a constant. If this constant is nonzero, equation (52) shows $K \equiv 0$.

Hence we may assume $\sum_{j>k} \left(\bar{H}_{\alpha_j} \right)^2 \equiv 0$. We note that formula (46) and a theorem of Chern [2] says that the common zeroes of $\{He_{\alpha_i}\}_{j>k}$ is isolated. Now exterior differentiate the assumption $\omega_{\alpha_i \alpha_j} = 0$ yields

$$-\sum_l \omega_{\alpha_i l} + \Phi_{\alpha_i \alpha_j} = 0 \tag{53}$$

$for\ 1 \leqq i\ \leqq k, k + 1 \leqq j \leqq p.$
By definitions (2) and (4)

$$\sum_l (h_{l1}^{\alpha_i} h_{l2}^{\alpha_j} - h_{l2}^{\alpha_i} h_{l1}^{\alpha_j}) = -K_{\alpha_i \alpha_i 12} \tag{54}$$

If N has constant curvature, the right hand side of the above equation vanishes and the second fundamental form corresponding to e_{α_i} and e_{α_j}, commute. Hence if M^2 is nonumbilical with respect to some e_{α_i}, $1 \leq i \leq k$, the second fundamental corresponding to $\{e_{\alpha_j}\}_{j \geq k+1}$ can be diagonalized simultaneously. On the other hand, $\sum_{j>k} H_{\alpha_j}^2 = 0$ implies

$$\sum_{j>k} \left(\frac{h_{11}^{\alpha_j} - h_{22}^{\alpha_j}}{2} \right) = \sum_{j>k} (h_{12}^{\alpha_j})^2$$

$$\sum_{j>k} \left(\frac{h_{11}^{\alpha_j} - h_{22}^{\alpha_j}}{2} \right) h_{12}^{\alpha_j} = 0. \tag{55}$$

If the second fundamental form corresponding to $\{e_{\alpha_j}\}_{j \geq k+1}$ can be diagonalized simultaneously, one sees from (55) that M^2 is umbilical with respect to N_2. We have therefore proved if M_2 is nonumbilical with respect to N_1, at a point then it is umbilical with respect to N_2 at that point. However, we have remarked that $\{H_{\alpha_j}\}_{j \geq k+1}$ have only isolated common zeroes. These two facts together imply either M^2 is umbilical with respect to N_1 globally or with respect to N_2 globally. Theorem (1.1.2) then follows from Theorem (1.1.1).

Theorem (1.1.3)[33]. Suppose in Theorem (1.1.2), M^2 is flat, N has non-negative curvature and N_2 has trivial normal connection. Then either

(i) M^2 lies in a $2 + k$ umbilical submanifold with normal bundle N_1.

(ii) There are two geodesics g_1 and g_2 in M^2, two umbilical submanifolds U_1 and U_2 in N such that $M^2 = g_1 \oplus g_2 \to U_1 \oplus U_2 \to N$ and the normal bundle of M^2 in $U_1 \oplus U_2$ is N_1. The first immersion preserves the product structure and the second is the standard one.

Proof. From the proof of Theorem (1.1.2), we know that $\left| \sum_{j>i} (\bar{H}_{\alpha_j})^2 \right|^2$ constant. Since the bundle N_2 has trivial normal connection, we can see as in Theorem (1.1.1) that the second fundamental form with respect to local sections in N_2 can be diagonalized simultaneously, i.e., we may assume $h_{12}^{\alpha_j} = 0$ for $j = k + 1, \ldots, p$. Hence we conclude that $\sum_{j>k} (h_{11}^{\alpha_j} - h_{22}^{\alpha_j})^2$ is a constant on M. By taking $e_{\alpha_{k+1}}$ to be the mean curvature vector, we may assume $h_{11}^{\alpha_j} + h_{22}^{\alpha_j} = $ constant for $j > k$. Therefore $\sum_{j>k} \sum_{m,n} (h_{mn}^{\alpha_j})^2$ is constant. On the other hand, a simple computation shows

$$\frac{1}{2}\Delta \left(\sum_{j>k} \sum_{m,n} (h_{mn}^{\alpha_j})^2 \right) = \sum_{j>k} \sum_{m,n,l} (h_{mn}^{\alpha_j})^2 + \sum_{j>k} \sum_{m,n} h_{mn}^{\alpha_j} \Delta h_{mn}^{\alpha_j} \tag{56}$$

Substituting (21) into (56) and noting that $\Sigma_l h^{\alpha_j}_{llmn} = 0$ for $j > k$, we obtain

$$\sum_{j>k}\sum_{m,n,l} \left(h^{\alpha_j}_{mn}\right)^2 = 0 \tag{57}$$

Hence from definition (10), one sees

$$\left(\left(h^{\alpha_j}_{11} - h^{\alpha_j}_{22}\right)\omega_{12} = 0\right.$$

$$dh^{\alpha_j}_{11} + \sum_{i>k} h^{\alpha_i}_{11}\,\omega_{\alpha_j\alpha_i} = 0 \tag{58}$$

$$dh^{\alpha_j}_{22} + \sum_{i>k} h^{\alpha_i}_{22}\,\omega_{\alpha_j\alpha_i} = 0$$

for $j > k$. We already note in Theorem (1.1.2) that either $h^{\alpha_i}_{22} = h^{\alpha_i}_{11}$ on M^2 or the points where $h^{\alpha_i}_{22} = h^{\alpha_i}_{11}$ is isolated. If $h^{\alpha_j}_{22} - h^{\alpha_j}_{11}$ on M^2, then (i) holds by Theorem (1.1.1). So in view of the first equation of (56), we assume $\omega_{12} = 0$. It is also clear than $h^{\alpha_i}_{11}$ and $h^{\alpha_i}_{22}$ are constants for $j > k$ since N_2 has trivial normal connections.

We may assume M^2 to be simply connected. Furthermore, we assume $\Sigma_{j>k}\Sigma_{m,n,l}\left(h^{\alpha_j}_{mn}\right)^2$ is not equal to zero. (If it is zero, then (i) holds.) Hence at every point, we have a well-defined frame $\{e_1, e_2\}$ which diagonalized the second fundamental forms with respect to N_2. Since $\omega_{12} = 0$, the curves defined by $\omega_1=0$, $\omega_2 =0$ respectively define an orthogonal geodesic foliation of M^2. Let g_i be a curve defined by $\omega_i = 0$ for i = 1, 2. Then it is clear that $M = g_1 \oplus g_2$. A lemma in J. Moore [13] shows that there are two geodesic submanifold $\tilde{U}_1$ and $\tilde{U}_2$ of N such that $M = g_1 \oplus g_2 \to \tilde{U}_1 \oplus \tilde{U}_2 = N$ preserving the product structure. Using the fact that $h^{\alpha_j}_{11}$ and $h^{\alpha_j}_{22}$ are constant for $j > k$ and an argument in Theorem (1.1.1), it is easy to see that g. lies in the intersection of $\tilde{U}_i$ with an umbilical submanifold such that N_2 is orthogonal to this umbilical submanifold. The conclusion (ii) then follows easily.

Corollary (1.1.4)[33]. Let M^2 be a complete flat surface in euclidean space. Suppose the normal bundle of M^2 is trivial. Then M^2 is a product immersion described in (ii) of Theorem (1.1.3).

Proof. This follows from Theorem (1.1.3) and the classification of flat surface in Euclidean space of dimension three.

Let M^2 be a surface with parallel mean curvature vector in a constant curved manifold N. In the notation of Theorem (1.1.2), if we take N_1 to be the bundle spanned by the mean curvature vector and N_2 the complement of it in the normal bundle, then it follows from the proof of Theorem (1.1.2) that either M^2 is umbilical with respect to N_1 everywhere or the

second fundamental form can be diagonalized simultaneously. Hence by Theorem (1.1.1), we have

Lemma (1.1.5)[33]. Let M^2 be a surface with parallel mean curvature vector in a constant curved manifold. Then either M^2 is a minimal surface of an umbilical submanifold of N or the second fundamental forms of M^2 can be diagonalized simultaneously.

Theorem (1.1.6)[33]. Let M^2 be a surface with parallel mean curvature vector in a constant curved manifold N. Then either M^2 is a minimal surface of an umbilical hypersurface of N or M^2 lies in a three-dimensional umbilical submanifold of N with constant mean curvature.

Proof. By Lemma (1.1.5), we may assume the second fundamental form of M^2 can be diagonalized simultaneously. Without loss of generality, we may assume the mean curvature vector is nonzero.

Let us now perform an orthogonal transformation in the normal bundle. We note that if $\{e_\alpha\}$ is changed to $\{ a_{\alpha\beta} e_\beta \}$ where $(a_{\alpha\beta})$ is an orthogonal matrix, then $\{h_{ij}^\alpha\}$ is changed to $\{a_{\alpha\beta} h_{ij}^\beta\}$. Since the second fundamental form can be diagonalized simultaneously, we may assume $h_{12}^\alpha = 0$ for all α. The vectors $\sum_\alpha h_{ij}^\alpha e_\alpha$ and $\sum_\alpha h_{ij}^\alpha e_\alpha$ then define two local sections in the normal bundle. If e_3 has the same direction as the mean curvature vector, then $h_{11}^\alpha + h_{22}^\alpha = 0$ for $\alpha > 3$. Hence either $\sum_\alpha h_{11}^\alpha e_\alpha$ or $\sum_\alpha h_{22}^\alpha e_\alpha$ vanishes would imply $\sum_{\alpha>3}(h_{11}^\alpha)^2$ and $\sum_{\alpha>3}(h_{22}^\alpha)^2$ vanishes. But the latter can vanish only at isolated points by the proof of Theorem (1.1.2). (Otherwise Theorem (1.1.6) follows from Theorem (1.1.1) Therefore we can assume neither $\sum_\alpha h_{11}^\alpha e_\alpha$ nor $\sum_\alpha h_{22}^\alpha e_\alpha$ vanishes. Now using the Gram-Schmit orthogonalization process, we may assume the plane spanned by e_3 and e_4 is the plane spanned by $\sum_\alpha h_{11}^\alpha e_\alpha$ and $\sum_\alpha h_{22}^\alpha e_\alpha$, this implies $h_{11}^\alpha = h_{22}^\alpha = 0$ for $\alpha > 4$. Changing the frame e_3 and e_4 again, we may assume e_3 has the same direction as the mean curvature vector.

With all these preparations, we are going to prove

$$\omega_{\alpha_4} = \omega_{\alpha_3} = 0 \tag{59}$$

for all $\alpha > 4$. The last equality follows because e_3 is parallel. It suffices, therefore, to prove $\omega_{\alpha 4} = 0$. First of all, we know from the construction that

$$\omega_{\alpha_i} = 0 \tag{60}$$

$for\ \alpha > 4, i = 1,2.$

Exterior differentiate (60) gives

$$\omega_{\alpha_4} \wedge \omega_{4i} = 0 \tag{61}$$

$for\ \alpha > 4, i = 1,2.$ Hence

$$h_{ii}^4 \omega_{\alpha_4} = 0 \tag{62}$$

$for\ \alpha > 4, i = 1,2.$

We already noted that $h_{11}^4 = -h_{22}^4$ can vanish only at isolated points. Hence (62) implies $\omega_{\alpha_4} = 0$ for $\alpha > 4$. An application of Theorem (1.1.1) shows that M^2 lies in a four-dimensional, totally geodesic submanifold of N. Furthermore, M^2 has constant mean curvature and the normal bundle of M^2 has trivial normal connection.

We are going to complete the proof by changing the normal coordinate again. First of all, e_3 is a global parallel section of the normal bundle and e_4 is therefore also a globally defined parallel section. (By taking a double cover, we can always assume M^2 is oriented.) We first see from the proof of Theorem (1.1.2) that both $(h_{11}^3 - h_{22}^3)^2\lambda^4$ and $h_{11}^4 - h_{22}^4)^2\lambda^4$ are holomorphic functions in the isothermal conditions defined by $\varphi = \lambda dz$. As both $(h_{11}^3 - h_{22}^3)^2$ and $(h_{11}^4 - h_{22}^4)^2$ are real valued, either one of them is identically zero or they differ by a constant factor. The first case implies M^2 is umbilical with respect to e_3 or e_4 and the theorem follows from Theorem (1.1.1). So we assume

$$(h_{11}^3 - h_{22}^3)^2 = c\,(h_{11}^4 - h_{22}^4)^{\,2} \tag{63}$$

for some constant $c \neq 0$. Define

$$\tan\theta = c. \tag{64}$$

Then from (63) h_{11}^3

$$\cos\theta h_{11}^3 + \sin\theta h_{11}^4 = \cos\theta h_{22}^3 + \sin\theta h_{22}^4. \tag{65}$$

On the other hand, we have

$$h_{11}^4 + h_{22}^4 = 0 \tag{66}$$

Therefore

$$\cos\theta h_{11}^3 + \sin\theta h_{11}^4 = -\cos\theta\,\frac{h_{11}^3 + h_{22}^3}{2} \tag{67}$$

which is a constant.

Equations (65) and (67) show that M^2 is umbilical with respect to the normal section $\cos\theta e_3 + \sin\theta e_4$. Furthermore, the eigenvalues of the corresponding second fundamental form is constant. Since e_3 and e_4 are parallel in the normal bundle and θ is constant, it is clear that $\cos\theta e_3 + \sin\theta e_4$ is also parallel in the normal bundle. The theorem then follows from Theorem (1.1.1).

We shall only assume the mean curvature is constant. This fact is weaker than the assumption that the mean curvature vector is parallel. On the other hand, if the mean curvature is a nonzero constant, the mean curvature gives a nonzero global section of the normal bundle which is a topological restriction, we have

Proposition (1.1.7). Let n be a power of two. If M^n is topologically a real projective space and if M^n is embedded in N^{2n} with constant mean curvature, then M^n is actually minimal. Here we assume N is a complete simply connected constant curved manifold.

Now let us consider the case where M^2 has genus zero and M^2 has non-negative curvature. We prove

Theorem (1.1.8). Let M^2 be a topological two sphere or a complete non- negative curved surface immersed in a four-dimensional constant curved manifold N. If M^2 has constant mean curvature, either M^2 is flat, totally umbilic or M^2 is a minimal surface.

Proof. We use the notations above . For each e_α in the normal bundle, we define

$$H_\alpha = \frac{h_{11}^\alpha - h_{22}^\alpha}{2} + i h_{11}^3. \tag{68}$$

As in the proof of Theorem (1.1.2), we know that $\sum_\alpha (\bar{H}_\alpha)^2 \varphi^4$ is globally defined on M^n. Using the local isothermal coordinate z, we can write this form as $f(z) dz^4$. We claim that f is holomorphic.

In fact, let p be an arbitrary point in M^2. Choose a normal frame field such that $\omega_{\delta\delta} = 0$ and e_α has the same direction as the mean curvature vector at the point ρ. As in the proof of Theorem (1.1.2),

$$d\sum_\alpha (\bar{H}_\alpha)^2 - 4i\omega_{12}\left(\sum_\alpha (\bar{H}_\alpha)^2\right) \equiv 0 \ \text{mod}\varphi \tag{69}$$

at p. Hence

$$\frac{\partial f}{\partial \tilde{z}} = 0 \tag{70}$$

at p. Since p is arbitrary, $f(z) dz^4$ is an abelian form of degree 4 on M^2.

If M^2 has genus zero or if M^2 is a complete non-negative curved surface, then as in Theorem (1.1.2), either M^2 is flat or $f \equiv 0$. Suppose $f \equiv 0$. Then

$$\sum_\alpha \left(\frac{h_{11}^\alpha - h_{22}^\alpha}{2}\right)^2 = \sum_\alpha (h_{12}^\alpha)^2$$

$$\tag{71}$$

$$\sum_\alpha \left(\frac{h_{11}^\alpha - h_{22}^\alpha}{2}\right)^2 = \sum_\alpha h_{12}^\alpha$$

Now assume that the codimension is two. Let e_3 be the normal vector which has the same direction as the mean curvature vector. Then we may write

$$\omega_{31} = h_{11}^3 \omega_1, \qquad\qquad \omega_{32} = h_{22}^3 \omega_2,$$
$$\omega_{41} = k_1 \omega_2, \qquad\qquad \omega_{42} = k_1 \omega_1, \tag{72}$$

where

$$K_1^2 = \sum_\alpha (h_{12}^\alpha)^2 \qquad \text{and} \qquad K_1 = \frac{h_{11}^3 - h_{22}^3}{2} \geq 0 \tag{73}$$

Define

$$E_1 = e_1 + ie_2,$$
$$E_2 = e_3 + ie_4. \tag{74}$$

Then

$$DE_1 = i\omega_{12}E_1 + k_1\,\bar{\varphi}E_2 + H\varphi\omega e_3, \tag{75}$$

where H is the mean curvature. By differentiation (75), we have

$$-2ik_1\omega_{12}\varphi E_2 + dk_1\,\bar{\varphi}E_2 - ik_1\,\bar{\varphi}\,\omega_{34}E_2 + H\varphi\omega_{34}e_4 = 0. \tag{76}$$

From (76), if $H \neq 0$,

$$\varphi \wedge \omega_{34} = 0, \tag{77}$$

which implies $\omega_{34} = 0$. We have therefore proved if $H \neq 0$, the mean curvature is parallel in the normal bundle. Applying Theorem (1.1.6), M^2 is a minimal surface in a three-dimensional constant curved manifold. By applying (70) again, one sees M^2 is actually totally umbilic.

Finally let us propose to classify all possible immersions of constant curved surface into a constant curved manifold with constant mean curvature. In case the codimension is 1, these are just standard spheres, planes and cylinders.

Proposition (1.1.9)[33]. Let M be a holornorphic curve in a Kahler manifold with non-negative constant holomorphic sectional curvature c. Suppose M has constant curvature with respect to the induced metric. Then M has strictly positive constant curvature.

Proof. The proof follows from Calabi's theory.

In fact, let $ds^2 = 2F|dz|^2$ be the induced metric on M. Then Calabi [12] proved that there exists a sequence of functions $\{F_k\}_{k=0}^{n+1}$ by setting

$$F_0 = 1,$$
$$F_1 = F,$$

and

$$F_{k+1} = \frac{F_k^2}{F_{k-1}}\left(\frac{d}{dz}\frac{d}{d\bar{z}}\log F_k + \frac{(k+1)}{2}cF\right), \tag{78}$$

for $k = 1, \ldots, n$. For $0 \leq k \leq n$, F_k is non-negative and vanishes only at isolated points. The succeeding function F_{k+1} is defined by (78) away from those points but extends to a real analytic function on all of M. Furthermore the function $F_{n+1} \equiv 0$. On the other hand, it is well known that the Gauss curvature of M is given by

$$K = \frac{1}{F}\frac{d}{dz}\frac{d}{d\bar{z}}\log F. \tag{79}$$

Substituting (79) into (78), one sees that if F is a negative constant, F_{k+1} cannot be zero. This contradiction finishes the proof.

Now for every minimal surface M^2 in E^n Euclidean space, Chern and Osserman [4] defined a Gauss map into CP^{n-1}. This map can be made into a holomorphic mapping such that if $d\tilde{s}^2$ is the metric induced from this map,

$$\frac{d\tilde{s}^2}{ds^2} = -K \tag{80}$$

where K is the Gauss curvature of M^2. Hence if $K = -1$, the Gauss map is actually an isometry. Hence we have a holomorphic curve in CP^{n-1} with constant negative curvature. This is a contradiction by Proposition (1.1.9). We have proved

Theorem (1.1.10)[33]. The hyperbolic space cannot be minimally immersed in Euclidean space, even locally.

In [14], Nomizu and Smyth proved that if M is a compact holomorphic curve in CP^2 whose Gauss curvature satisfies $K \leq \frac{1}{2}$. Then M is the quadric. Let us consider the opposite case, namely the totally real case. A submanifold M of CP^n is called totally real if for every point $x \in M$, $T_x(M)$ is perpendicular to $JT_x(M)$. Here $T_x(M)$ is the tangent space of M at x and J is the complex structure.

If N is a Kahler manifold with constant holomorphic sectional curvature and M is a totally real submanifold, it is straightforward to calculate from (17) and (21) that

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum h_{ij}^\alpha h_{km}^\alpha R_{mijk} + \sum h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} - \sum h_{ij}^\alpha h_{ki}^\beta R_{\alpha\beta jk} \qquad (81)$$

Let us consider minimal totally real surface with codimension 2. Let $e_3 = Je_l$, $e_4 = Je_2$. Then using the fact that N is Kaihler,

$$h_{12}^3 = h_{22}^4,$$
$$h_{22}^3 = h_{12}^4. \qquad (82)$$

Hence by assuming $h_{12}^3 = 0$ and $h_{11}^3 = a$, we can write

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = 8a^2 K + 4a^2 K_{3412} - 8a^4. \qquad (83)$$

Since N has constant holomorphic sectional curvature,

$$K_{3412} = \frac{c}{4}, \qquad (84)$$
$$K_{1212} = \frac{c}{4}. \qquad (85)$$

The Gauss equation then implies

$$2a^2 = \frac{c}{4} - K. \qquad (86)$$

Hence from (83), (84), and (86), we have

$$\sum h_{ij}^\alpha \Delta h_{ij}^\alpha = 8a^2 K + 4a^2 K. \qquad (87)$$

On the other hand, it is straightforward to see

$$\frac{1}{2}\Delta \left(\sum (h_{ij}^\alpha)^2 \right) = \sum (h_{ij}^\alpha)^2 + h_{ij}^\alpha \Delta h_{ij}^\alpha$$
$$= \sum (h_{ij}^\alpha)^2 + 12a^2 K. \qquad (88)$$

Using the fact that

$$\sum_{i,j} (h_{ij}^3)^2 = \sum_{i,j} (h_{ij}^4)^2$$
$$(89)$$
$$\sum_{i,j} h_{ij}^3 h_{ij}^4 = 0$$

16

It is easy to prove

$$\frac{1}{2}\Delta\left(\log\sum(h_{ij}^{\alpha})^2\right) \ = \ \frac{3}{2}K \tag{90}$$

Whenever $\sum(h_{ij}^{\alpha})^2 \neq 0$.

Using the isothermal coordinate and (90), it is not hard to prove that either $\sum\sum(h_{ij}^{\alpha})^2 \equiv 0$ or $\sum(h_{ij}^{\alpha})^2$ vanishes only at isolated points.

If M^2 has genus zero, the Gauss-Bonnet theorem and (90) shows that $\sum(h_{ij}^{\alpha})^2 \equiv 0$, i.e., M^2 is totally geodesic. This is the standard embedding RP^2 into CP^2.

If M^2 is complete and has non-negative curvature, (88) shows that either M^2 is totally geodesic or the curvature $K = 0$ and $\sum(h_{ij}^{\alpha})^2 = 0$. In case M^2 is complete and has nonpositive curvature, we shall use an argument due to Klotz and Osserman [9]. From (90), the curvature of the metric $\left(\sum(h_{ij}^{\alpha})^2\right)^{\frac{1}{3}}ds^2$ is zero. If $\sum(h_{ij}^{\alpha})^2$ is bounded away from zero, this metric is complete and hence M^2 is parabolic. Since equation (90) shows that ,(h is a non-negative superharmonic function, $K \equiv 0$. By equation (86), the quantity $\sum(h_{ij}^{\alpha})^2$ is indeed bounded away from zero if $(c/4) - K \geq a > 0$ for some constant a.

We have proved

Theorem (1.1.11)[33]. Let M^2 be a totally real minimal surface of a Kahler surface with constant holomorphic sectional curvature c. Then

  (i)   If M^2 has genus zero, M^2 is the standard embedding of RP^2 in CP^2.
  (ii)   If M^2 is a complete non-negative curved surface, then M^2 is totally geodesic or flat. In the last case, the second fundamental form is covariant constant.
  (iii) If M^2 is complete non-positive curved with Gauss curvature K and if $(c/4) - K \geq a > 0$ for some constant a, then M^2 is totally geodesic or flat.

Section(1-2): Constant Mean Curvature

We are interested in higher dimensional submanifolds. We show a theorem similar to the Simons' pinching theorem for submanifolds with parallel mean curvature in sphere. Namely, if M^n is a compact submanifold with parallel mean curvature in the sphere S^{n+p} with $p > 1$, and the length of the second fundamental form of M^n is not greater than $n/3 + n^{1/2} - 1/(p - 1)$, then M^n lies in a totally geodesic S^{n+1}. We note that theorems of this form were studied by extra conditions.

We extend a result of Nomizu and Smyth [28]. Nomizu and Smyth classified non-negatively curved hypersurfaces with constant mean curvature. We generalize it to higher codimension. This was done by Yano and Ishihara [32] under a further assumption that the normal bundle is flat. We learned that Smyth has also independently extended their theorem.

17

Then we discuss some simple observations about isometric immersions. For example, a compact hypersurface with non-negative Ricci curvature and constant mean curvature in euclidean space is the standard sphere. This also generalizes the result of Nomizu and Smyth.

We generalize the Hilbert-Liebmann theorem to higher dimensional hypersurfaces. We prove, for example, that a compact convex hypersurface with constant scalar curvature is totally umbilical. Since this theorem is global, it seems to be more natural than the ordinary generalizations. We also discuss the generalization of Efimov's theorem.

We discuss a "quantization phenomenon" of compact minimal submanifolds in sphere. Lawson proved that if M^2 is a non-singular holomorphic curve in CP^n whose curvature K satisfies $1/k \leq K < l/(k-1)$ for some $k, 1 < k \leq n$, then $K \equiv 1/k$. We will show that this phenomenon also occurs for compact minimal submanifolds. in euclidean sphere. The main theorem is a pinching theorem opposite to that of Simons. The bound that we obtain here is sharp. For example, it will be attained by some non-totally geodesic isometric minimal immersion of spheres.

We improve the pinching constant of Simons in the following sense. Simons proved that if the average of the sectional curvatures is greater than $1 - \dfrac{1}{(n-1)\left(2-\frac{1}{p}\right)}$, then the compact minimal submanifold M^n in S^{n+p} must be totally geodesic. We show here that if the sectional curvatures are greater than $p - 1/(2p - 1)$, then the same conclusion holds. The constant $p - l/(2p - 1)$ is always less than Simons' constant than $1 - \dfrac{1}{(n-1)\left(2-\frac{1}{p}\right)}$. It is also less than ½ which is independent of dimension. We also discuss the pinching formulas for complex submanifolds and minimal totally real submanifolds in Kahler manifolds with constant holomorphic sectional curvature. We note that since there is a submersion from sphere to symmetric space of rank one, the corresponding phenomena in sphere also occur in symmetric space of rank one by the techniques of Lawson [25]. In [12] However, the bound so obtained is not good and hence we only discuss complex submanifolds and minimal totally real submanifolds.

We discuss a question asked by Simons [15]. The question of Simons' is the following: Let M^n be a compact minimal submanifold in S^{n+p}. Is it true that *the $(n + 1) - plane$* in R^{n+p+1}, which is spanned by $T(M)_m$ and the radial vector m has non-trivial intersection with every fixed p-plane R^{n+P+1} For p = 1 this was proved by DeGiorgi. Simons [15] and Reilly [31] also obtained partial results for general p. We shall prove it for minimal immersion of S^2 in S^4. Actually a more precise statement will be obtained.

In [15], Simons proved that if M^n is a compact minimal submanifold of the sphere S^{n+p} and if the length of the second fundamental form of M^n is everywhere not larger than

$n/(2 - 1/p)$, then M^n is totally geodesic. The theorem was proved by the following inequality

$$\int_M S\left\{\left(2 - \frac{1}{p}\right)S - n\right\}dV \geq 0 \tag{91}$$

where dV is the volume form of M.

We shall prove a similar pinching theorem for submanifolds with parallel mean curvature by establishing an inequality similar to (91).

Theorem (1.2.1)[36]. Let M^n be an n-dimensional compact submanifold with parallel mean curvature in S^{n+p} with p > 1. If $(3 + n^{\frac{1}{2}} - (p - 1)^{-1})S \leq n$, then M^n lies in a totally geodesic S^{n+p}.

Proof. We shall still use the notations of [33]. Let e_{n+1}, be the normalized mean curvature vector, then

$$\omega_{n+1,\beta} = 0 \tag{92}$$

for all β. Exterior differentiate (92), we obtain

$$\Sigma_i \omega_{n+1,\beta} \wedge \omega_{\beta i} = 0 \tag{93}$$

For each α , let H^α be the matirx (h_{ij}^α). Then (93) implies

$$H^{n+1}H^\alpha = H^\alpha H^{n+1} \tag{94}$$

for all α.

From (9), we see that (94) is equivalent to

$$R_{n+1\alpha kl} = 0 \tag{95}$$

for all α, k, l. Hence, equation (29) gives

$$\Delta h_{ij}^\beta = \sum_{k,m} h_{ij}^\beta R_{mijk} + \sum_{k,m} h_{mi}^\beta R_{mkjk} - \sum_{\substack{k \\ \alpha \neq n+1}} h_{ij}^\alpha R_{\beta\alpha jk} \tag{96}$$

$\beta \neq n + 1$.

The Gauss equation (7) then implies

$$\begin{aligned}
\Delta h_{ij}^\beta &= \sum_{\alpha,k,m} h_{km}^\beta h_{mj}^\alpha h_{ik}^\alpha - \sum_{\alpha,k,m} h_{mi}^\beta h_{mk}^\alpha h_{ij}^\alpha \\
&+ \sum_{\alpha,k,m} h_{mi}^\beta h_{mj}^\alpha h_{kk}^\alpha - \sum_{\alpha,k,m} h_{mi}^\beta h_{mk}^\alpha h_{kj}^\alpha \\
&+ nhh_{mi}^\beta - \sum_{\substack{k \\ \alpha \neq n+1}} h_{ij}^\alpha R_{\beta\alpha jk}
\end{aligned} \tag{97}$$

$for\ \beta \neq n + 1$.

Following Simons' proof of his pinching theorem, one can then prove

19

$$\sum_{\substack{i,j \\ \beta \neq n+1}} h_{ij}^{\beta} \Delta h_{ij}^{\beta} \geq \sum_{\substack{i,j,k,m \\ \beta \neq n+1}} h_{ij}^{\beta} h_{km}^{\beta} h_{mj}^{n+1} h_{ik}^{n+1}$$

s

$$- \sum_{\substack{i,j \\ \beta \neq n+1}} h_{ij}^{\beta} h_{km}^{\beta} h_{nk}^{n+1} h_{ij}^{n+1} + \sum_{\substack{i,j,k,m \\ \beta \neq n+1}} h_{ij}^{\beta} h_{mi}^{\beta} h_{mj}^{n+1} h_{kk}^{n+1}$$

$$- \sum_{\substack{i,j,k,m \\ \beta \neq n+1}} h_{ij}^{\beta} h_{mi}^{\beta} h_{mk}^{n+1} h_{kj}^{n+1}$$

$$+ n \sum_{\substack{i,j,k,m \\ \beta \neq n+1}} (h_{ij}^{\beta})^2 - \left(2 - \frac{1}{p-1}\right)\left[\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^{\beta})^2\right]^2 \tag{98}$$

Now fix a vector e_{β}. Let $\{e_1,..., e_n\}$ be a frame diagonalizing the matrix (h_{ij}^{β}) such that

$$h_{ij}^{\beta} = 0 \tag{99}$$

for $i \neq j$. Then

$$\sum_{i,j,k,m} h_{ij}^{\beta} h_{km}^{\beta} h_{mj}^{n+1} h_{ik}^{n+1} - \sum_{i,j,k,m} h_{ij}^{\beta} h_{km}^{\beta} h_{mk}^{n+1} h_{kj}^{n+1}$$

$$+ \sum_{\substack{i,j \\ \beta \neq n+1}} h_{ij}^{\beta} h_{mi}^{\beta} h_{mj}^{n+1} h_{kk}^{n+1} - \sum_{\substack{i,j,k,m \\ \beta \neq n+1}} h_{ij}^{\beta} h_{mi}^{\beta} h_{mk}^{n+1} h_{kj}^{n+1}$$

$$= \sum_{i,k} h_{ii}^{\beta} h_{ii}^{\beta} h_{ii}^{n+1} h_{kk}^{n+1} - \sum_{i,k} h_{ii}^{\beta} h_{kk}^{\beta} h_{kk}^{n+1} h_{ii}^{n+1}$$

$$+ \sum_{i,k} h_{ii}^{\beta} h_{ii}^{\beta} h_{ii}^{n+1} h_{kk}^{n+1} - \sum_{i,k} h_{ii}^{\beta} h_{ii}^{\beta} h_{ik}^{n+1} h_{ki}^{n+1} \tag{100}$$

On the other hand, from (94)

$$+ \sum_{i,k} h_{ii}^{\beta} h_{kk}^{\beta} h_{ki}^{n+1} h_{ik}^{n+1} = \sum_{i,k} h_{ii}^{\beta} h_{ii}^{\beta} h_{ik}^{n+1} h_{ki}^{n+1} \tag{101}$$

Therefore, (100) is equal to

$$\left(\sum_{i} (h_{ii}^{\beta})^2 h_{ii}^{n+1}\right)\left(\sum_{k} h_{kk}^{n+1}\right) - \left(\sum_{i} h_{ii}^{\beta} h_{ii}^{n+1}\right)^2 \tag{102}$$

The absolute value of this number is not greater than

$$\left(n^{\frac{1}{2}} + 1\right)\left(\sum_{i} (h_{ii}^{\beta})^2\right)\left(\sum_{k} (h_{kk}^{n+1})^2\right) = \left(n^{\frac{1}{2}} + 1\right)\left(\sum_{i,j} (h_{ij}^{\beta})^2\right)\left(\sum_{i,j} (h_{ij}^{n+1})^2\right) \tag{103}$$

by Schwarz inequality.

Hence, from (98), we have

$$\sum_{\substack{i,j \\ \beta \neq n+1}} h_{ij}^\beta \Delta h_{ij}^\beta \geq -\left(n^{\frac{1}{2}} + 1\right)\left(\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2\right)\left(\sum_{i,j} (h_{ij}^{n+1})^2\right)$$

$$+ n\left(\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2\right) - \left(2 - \frac{1}{p-1}\right)\left(\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^{n+1})^2\right)^2$$

$$\geq \left(\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2\right)\left(n - \left(2 - \frac{1}{p-1}\right)S - \left(n^{\frac{1}{2}} + 1\right)S\right) \qquad (104)$$

Now it is straightforward to see

$$\frac{1}{2}\Delta\left(\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2\right) = \sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2 + \sum_{\substack{i,j \\ \beta \neq n+1}} h_{ij}^\beta \Delta h_{ji}^\beta \qquad (105)$$

Therefore, under the assumption $n \geq (3 + n^{\frac{1}{2}}-(p - 1)^{-1})$ S, (104) and (94) shows that $\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2$ is subharmonic on M^n. By the Hopf maximum principle, we see that this function must be a constant and the right hand side of (105) must be zero. In particular

$$\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2 \left(n - \left(3 + n^{\frac{1}{2}} - (p-1)^{-1}\right)S\right) = 0 \qquad (106)$$

If $\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2 = 0$, it is easy to see from a theorem of Erbacher [8] that M lies in a totally geodesic S^{n+1}.

If $n - \left(3 + n^{\frac{1}{2}} - (p-1)^{-1}\right)S = 0$, then all the previous inequalities become equalities and it is not hard to see that these equalities force $\sum_{\substack{i,j \\ \beta \neq n+1}} (h_{ij}^\beta)^2$ to be zero and M lies in a totally geodesic S^{n+1} . This completes the proof of Theorem (1.2.1).

In [28], Nomizu and Smyth proved that if M^n is a non-negatively curved compact hypersurface with constant mean curvature in the euclidean space, or the euclidean sphere, then it is the standard sphere or the product immersion of two spheres. We propose to extend this result to arbitrary codimension. After we have finished our proof, we learned that this was done by Yano and Ishihara [32] by assuming the normal bundle is locally parallelizable.

Theorem (1.2.2)[36]. Let M^n be a compact non-negatively curved manifold immersed in a constantly curved manifold N. Suppose M^n has parallel mean curvature. Then $M^n = M_1 \times M_2 \times .... \times M_k$ such that each M_i is a minimal submanifold of a totally umbilical submanifold N_i (with codimension > 0) and the N_i's are mutually perpendicular along their intersections.

Proof. Suppose e_{n+1} has the same direction as the mean curvature vector. Assume the second fundamental form of M^n with respect to e_{n+1} has been diagonalized so that the eigenvalues are λ_i. Then it follows from the fact that e_{n+1} is parallel and (28).

$$\tfrac{1}{2}\Delta\left(\sum_i \lambda_i^2\right) = \sum_{i,j,k}\left(h_{ijk}^{n+1}\right)^2 + \frac{1}{2}\sum_{i,j}\left(\lambda_i - \lambda_j\right)^2 R_{ijij} \qquad (107)$$

Since M^n is compact and the right hand side of (107) is non-negative, it follows from Stokes theorem that both of them are identically zero. Hence

$$\sum_{i,j}\left(\lambda_i - \lambda_j\right)^2 R_{ijij} = 0 \qquad (108)$$

$$\sum_{i,j,k}\left(h_{ijk}^{n+1}\right)^2 = 0 \qquad (109)$$

Equation (109) shows λ_i are constants and (108) shows whenever $\lambda_i \neq \lambda_j$, $R_{ijij} = 0$. Without loss of generality, let us assume

$$\lambda_1 = \lambda_2 ... \lambda_{n_1} > \lambda_{n_1+1} = ... = \lambda_{n_2} > ... > \lambda_{n_{k-1}} + 1 = ... = \lambda_{n_k} > .. = \lambda_n \qquad (110)$$

Since e_{n+1} is parallel, $\omega_{n+1,\,i} = \lambda_i\omega_i$ and λ_i's are constants, it follows that

$$(\lambda_i - \lambda_j)\,\omega_{ij} \wedge \omega_j = 0 \qquad (111)$$

for all i, j.

Hence, if $n_{s-1} < i \leq n_s$,

$$d\omega_i = -\sum_{j=n_{s-1}+1}^{n_s} \omega_{ij}\omega_j \qquad (112)$$

It is known that the connection matrix (ω_{ij}) is completely determined by (112). Hence

$$\omega_{ij} = 0 \qquad (113)$$

whenever $\lambda_i \neq \lambda_j$,

It is clear from the equations (112), (113) that $\omega_{n_{k-1}} + 1 = ... = \omega_{n_k} = 0$ defines a totally geodesic foliation of M^n. Since all the λ_i's are constant, the leaves of this foliation are all closed and hence compact.

When k varies, we get different totally geodesic foliation with compact leaves. The leaves of these two foliations are mutually perpendicular to each other and equation (113) shows actually they give a product decomposition of M. (cf. the proof of the decomposition

theorem of [24]). Hence $M = M_1 \times M_2 \times \ldots \times M_k$ such that the second fundamental form of M_i with respect to the mean curvature vector has equal eigenvalues.

Let us now prove that whenever e_i and e_j are tangent to M_i and M_j respectively, then

$$h_{ij}^\alpha = 0. \tag{114}$$

for $\alpha \geq n + 1$ and $i \neq j$. In fact, since e_{n+1} is parallel, the second fundamental form defined by it commutes with all other second fundamental forms, i.e.

$$\lambda_i h_{ij}^\alpha = \lambda_j h_{ji}^\alpha \tag{115}$$

for all $\alpha \geq n + 1$. Hence, if $\lambda_i \neq \lambda_j$, $h_{ij}^\alpha = 0$ and (114) is proved.

If the ambient manifold is the euclidean space, a lemma of J. D. Moore [24] and (114) shows that each M_i lies in a linear subspace N_i and that the N_i's are all mutually perpendicular. Each M_i is umbilical with respect to the mean curvature vector which is parallel in the normal bundle of M_i. ($\omega_{n+1,j} = 0$ on M_i if e_j is perpendicular to M_i). Hence, Theorem (1.1.1) says that M_i is minimal in an umbilical hypersurface of N_i for each i.

If the ambient manifold is the sphere, the theorem also follows by considering the standard embedding of the sphere in the euclidean space. If the ambient manifold is the hyperbolic space, the theorem can be obtained by considering the non-euclidean model of the hyperbolic space. The essential point is that if the tangent space of M_i is spanned by $e_1, \ldots, e_i$, then

$$D(e_1 \wedge \ldots \wedge e_{i_s}) = 0 \tag{116}$$

by (113) and (114).

Let us now discuss the extra-condition imposed by Yano and Ishihara [32].

Theorem (1.2.3)[36]. Let M^n be a subnanifold of the euclidean space N^{n+p}. Suppose M^n has flat normal bundle and parallel mean curvature vector, then there is an open dense subset U of M^n such that each component of U lies in a $2n-$dimensional linear subspace of N^{n+p}.

Proof. We shall prove that for all $p \in M^n$, we can find a dense open set of a neighborhood of p such that each component of this open set lies in a $2n-$dimensional linear subspace of N Take an open neighborhood U of p such that the normal bundle is geometrically trivial on U, i.e., we can find parallel normal vector fields $e_{n+1}, e_{n+2}, \ldots, e_{n+p}$ on U. The matrices (h_{ij}^α) are then mutually commute. Hence, there exists an open dense set $U_1 \subset U$ such that we can find a frame $e_1, e_2, \ldots, e_n$ on U_1 with

$$h_{ij}^\alpha = 0 \tag{117}$$

for $i \neq j$.

By the standard matrix theory, it is easy to see that there exists another open dense set U_2 in U_1 such that on U_2, we can find an orthogonal matrix valued function $(a_{\alpha\beta})$ with the property

$$\sum_{\beta} a_{\alpha\beta} h_{ii}^{\beta} = 0 \tag{118}$$

for $\alpha > 2n$.

Hence, if we replace the normal frame on U_2 by $\{\Sigma_\beta a_{\alpha\beta} e_\beta\}$, we can assume

$$\omega_{\alpha i} = 0 \tag{119}$$

for $\alpha > 2n$ and $i = 1,2,\ldots,n$.

Clearly, we can also assume e_{n+1} is parallel to the mean curvature vector. Then by definition, and (119)

$$\sum_{k} h_{ijk}^{\alpha} \omega_k = \sum_{\beta=n+2}^{2n} h_{ij}^{\beta} \omega_{\alpha\beta} \tag{120}$$

for $\alpha > 2n$.

In particular, $h_{ijk}^{\alpha} = 0$ unless $i = j$. On the other hand, equation (12) says that $h_{ijk}^{\alpha} = h_{ikj}^{\alpha}$ Therefore, $h_{ijk}^{\alpha} = 0$ unless $i = j = k$ and

$$h_{iii}^{\alpha} \omega_i = \sum_{\beta=n+2}^{2n} h_{ii}^{\beta} \omega_{\alpha\beta} \tag{121}$$

for $\alpha > 2n$.

Since e_{n+1} is the mean curvature vector, we have

$$\sum_{i} h_{ii}^{\beta} = 0 \tag{122}$$

for $\beta > n + 1$.

Equations (121) and (122) shows

$$\sum_{i} h_{iii}^{\alpha} \omega_i = 0 \tag{123}$$

which implies

$$h_{iii}^{\alpha} = 0 \tag{124}$$

and

$$\sum_{\beta=n+2}^{2n} h_{ii}^{\beta} \omega_{\alpha\beta} = 0 \tag{125}$$

Change the normal frame on an open dense subset if necessary, we may assume there is a number $k < n$ such that the matrix

$$(h_{iii}^{\alpha})_{\substack{n+z \leq \beta \leq n+k+1 \\ 1 \leq i \leq n}} \tag{126}$$

has rank k and

$$h_{ii}^{\beta} = 0 \tag{127}$$

for $\beta > n + k + 1$. Then

$$\omega_{\alpha i} = 0 \tag{128}$$

for $\alpha > n + k + l$ and $i = 1,2,\ldots,n.$

As in (125), we can derive

$$\sum_{\beta=n+2}^{n+k+1} h_{ii}^{\beta} \omega_{\alpha\beta} = 0 \tag{129}$$

Equations (126), (127) and (129) imply

$$\omega_{\alpha\beta} = 0 \tag{130}$$

for $\alpha > n + k + 1, \beta \leq n + k + 1.$

The theorem of Erbacher [8] (cf. theorem (1)) then says the open set lies in a linear space with dimension $n + k + 1 \leq 2n$.

Corollary (1.2.4)[36]. Let M^n be a submanifold with parallel mean curvature in S^{n+p}. Suppose M^n has flat normal bundle. Then there is an open dense set U in M such that each component of U lies in a totally geodesic $S^{2n+1.}$

Let us discuss the curvature assumption of Nomizu and Smyth.

Proposition (1.2.5)[36]. Let M^n be a subnanifold of another manifold N^{n+p} with constant sectional curvature. Suppose the mean curvature of M is nowhere zero and the Ricci curvature of M is $\geq (n - 1)c$ (or $> (n - 1)c$). Then the second fundamental form of M with respect to the mean curvature is semi-definite (definite).

Proof. Let e_{n+1} be the unit vector in the direction of the mean curvature vector. Diagonalize the second fundamental form so that

$$h_{ij}^{n+1} = \lambda_i \delta_{ij} \tag{131}$$

Then the Gauss equation says

$$\sum_{\alpha} h_{ii}^{\alpha} h_{jj}^{\alpha} - \sum_{\alpha} (h_{ij}^{\alpha})^2 = R_{ijij} - c \tag{132}$$

$$-\sum_{\alpha} (h_{ij}^{\alpha})^2 + n\lambda_i H - \sum_{\alpha} \sum_{j \neq i} (h_{ij}^{\alpha})^2 = Ric(i) - (n - 1)c \tag{133}$$

where $Ric(i)$ is the Ricci curvature of M in direction i.

Suppose $H > 0$, equation (133) and the hypothesis says $\lambda_i \geq 0$ for all i. Furthermore, it is clear that $\lambda_i = 0$ implies $h_{ij}^{\alpha} = 0$ for all j and hence $R_{ijij} = 0$ for all j.

Corollary (1.2.6)[36]. Let M^n be a hypersurface with non-zero mean curvature in a manifold with constant curvature c. If the Ricci curvature of M^n is not less than $(n - 1)c$, then M' is convex.

Corollary (1.2.7)[36]. Let M^n be a compact hypersurface with constant mean curvature in the euclidean space. If M^n has non-negative Ricci curvature, then M^n is an umbilical hypersurface (which is the sphere).

Proof. Since Nomize and Smyth [28] already proved the corollary by assuming the curvature is positive, the assertion follows from this theorem and the proposition.

We generalize some well-known theorems about surfaces to higher dimensional submanifolds. Unlike the ordinary generalization, our theorems are global and nontrivial. First of all, we generalize the Hilbert-Liebmann Theorem which states that the isometric immersion of S^2 in three dimensional euclidean space is rigid.

Theorem (1.2.8)[36]. Let M^n be a compact hypersurface of a manifold with constant curvature c. Suppose M^n has constant scalar curvature and non- negative sectional curvature. Then

i)   if $c \leq 0$ and the Ricci curvature of M^n is larger than $(n-1)c$, M^n is totally umbilical (and isometric to the standard sphere).

ii)   if $c > 0$ and the sectional curvature of M^n is larger than 0, M^n is totally umbilical (and isometric to the standard sphere).

Proof. Let λ_i be the principal curvatures. Since the scalar curvature R is constant, we have, from the Gauss equation

$$-\sum_i \lambda_i^2 + n^2 H^2 = n(n-1)R - (n-1)c = constant \qquad (134)$$

hence

$$\Delta\left(\sum_i \lambda_i^2\right) = n^2 \Delta(H^2) \qquad (135)$$

On the other hand, from (28),

$$\Delta\left(\sum_i \lambda_i^2\right) = \sum_{i,j,k}(h_{ijk})^2 + n\sum_i \lambda_i H_{ii} + \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 R_{ijij} \qquad (136)$$

From (135) and (136) we obtain

$$n\sum_i \left(\sum_{j\neq i} \lambda_j\right) H_{ii} = \sum_{i,j,k}(h_{ijk})^2 + n^2|grad\ H|^2 + \frac{1}{2}\sum_{i,j}(\lambda_i - \lambda_j)^2 R_{ijij} \qquad (137)$$

Since M^2 is compact, there is a point x where H attains its maximal. At this point,

$$H_{ii} \leq 0 \qquad (138)$$
$$(grad\ H)^2 = 0 \qquad (139)$$

On the other hand, we observe that if H is zero at some point, the Ricci curvature at that point will be $(n-1)c$ which is a contradiction to the hypothesis by (133). Hence, we may assume $H > 0$ and by proposition (1.2.5), $\lambda_i > 0$. Therefore, from (137), (138) and (139)

$$\sum_{i,j} (\lambda_i - \lambda_j)^2 R_{ijij} \tag{140}$$

at the point x.

By hypothesis and Proposition (1.2.5), $R_{ijij} > 0$. Equation (140) and this fact implies x is an umbilical point. The value of H^2 at this point is therefore $\lambda_i^2 = R - c$, by the Gauss equation.

Since we know that H^2 attains its maximum at x, H^2 is everywhere not greated than $R - c$. Therefore, it follows from equation (134) that $nH^2 \geq \sum_i \lambda_i^2$ everywhere. On the other hand, the Schwartz inequality says that $nH^2 \leq \sum_i \lambda_i^2$ and $nH^2 = \sum_i \lambda_i^2$ if and only if $\lambda_1 = \lambda_2 = \ldots = \lambda_n$. These last two facts then imply M^n is totally umbilical everywhere.

We discuss the generalizations of the Hilbert-Efimov theorem. Their theorem states that a complete surface with curvature bounded from above by a negative constant cannot be isometrically immersed in three dimensional euclidean space.

Proposition (1.2.9)[36]. Let M^n be a hypersurface of a manifold with constant curvature c. Suppose M^n has sectional curvature $\leq$ c. Then $M^n = M_0 \cup M_1 \cup M_2$ with the following properties: the interior of M_i has codimension i totally geodesic foliation.

The leaves of the foliation are actually totally geodesic in the ambient manifold and the sectional curvature between the normals of the leaf and the tangents of the leaf is equal to c.

Proof. Let λ_i be the principal curvatures of M. Then the Gauss equation shows

$$\lambda_i \lambda_j \leq 0 \tag{141}$$

for $i \neq j$.

A simple calculation then shows at most two principal curvatures of M are non-zero. Let M_i be the set of points of M where the rank of the second fundamental form is i. We shall prove our assertion only for M_2. The rest is trivial.

Assume λ_1, λ_2 are the only non-zero eigenvalues. Then

$$\begin{aligned}
\omega_{n+1,1} &= \lambda_1 \omega_1 \\
\omega_{n+2,2} &= \lambda_2 \omega_2 \\
\omega_{n+1,i} &= 0
\end{aligned} \tag{142}$$

for $i > 2$.

Exterior differentiate the last equation of (142) and simplifying, we obtain

$$\lambda_1 \omega_1 \wedge \omega_{1i} + \lambda_2 \omega_2 \wedge \omega_{2i} = 0 \tag{143}$$

for $i > 2$.

Hence, ω_{1i} and ω_{2k} are linear combinations of forms ω_1 and ω_2 only. Using the defining equations for ω_{ij}, it is then straightforward to see $\omega_1 = 0$, $\omega_2 = 0$ define a foliation on M_2. The leaves of this foliation are totally geodesic in M^n because ω_{1i} and ω_{2i} are zero

on them. They are totally geodesic in the ambient manifold because $\omega_{n+1,i} = 0$ on them also. The last assertion on the curvature follows from the Gauss equation.

Corollary (1.2.10)[36]. Let M^n be a manifold with sectional curvature $\leq c$ and Ricci curvature $< (n-1)c$. If $n > 3$, then M^n cannot be immersed as a hypersurface of a manifold with constant curvature c.

Corollary (1.2.11)[36]. Let M^n be a complete manifold with non-positive sectional curvature. If the scalar curvature of M^n is bounded from above by a negative constant, then M^n cannot be isometrically immersed as a hypersurface in euclidean space.

Proof. It is easy to see from Proposition (1.2.9) and our assumption that $M = M_2$ has codimensional 2 totally geodesic foliation. The leaves of the foliation are actually codimensional 2 linear spaces. Arguments similar to [34] can then be used to prove that $M^n = M^{-2} \times R^{n-2}$ where R^{n-2} is the $n-2$ dimensional euclidean space. Furthermore the immersion of M^n is a product immersion. The corollary then follows from Efimov's theorem.

We believe that Corollary (1.2.4) can be extended to slightly higher codimension. This is true if the normal bundle is locally parallelizable.

Proposition (1.2.12)[36]. Let M^n be a manifold with sectional curvature $\leq c$ and Ricci curvature $< (n-1)c$. If $n > 2^P$, then M^n cannot be immersed in manifold N^{n+p} with constant curvature c and flat normal bundle.

Proof. The assumption implies that the second fundamental forms can be diagonalized simultaneously. For each normal vector e_α, let α_i be the corresponding principal curvatures. The Gauss equation then says

$$\sum_\alpha \alpha_i \alpha_j \leq 0 \tag{144}$$

The n vectors (α_i) lie in the euclidean p-space with $n > 2^P$. By the pigeon box principle, if these vectors satisfy (144), one of them must be zero. This will imply that the Ricci curvature of M^n is zero for some direction which is a contradiction.

We shall prove an inequality opposite to that of Simons and hence derive a quantization theorem similar to that of B. Lawson mentioned in the introduction.

Theorem (1.2.13)[36]. Let M^n be a compact minimal submanifold immersed in a manifold N^{n+p} with constant curvature c. Let S be the length of the second fundamental form of M^n. Let $K(x)$ be the function assigns to each point of M the infinimum of the sectional curvatures of M at that point. Then

$$\int_M S[pn\,(c-2K) - S]dV \geq 0 \tag{145}$$

and *if K* is everywhere non-positive,

$$\int_M S[pn\,(c-K)-S]dV \geq 0 \qquad (146)$$

Proof. It follows from (29) and (9) that

$$\sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha = \sum_{\alpha,i,j,m} h_{ij}^\alpha h_{ij}^\alpha R_{mijk}$$

$$+ \sum_{\alpha,i,j,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}$$

$$- \sum_{\alpha,\beta,i,j,l} h_{ij}^\alpha h_{ki}^\beta (h_{lj}^\alpha h_{lk}^\beta - h_{lk}^\alpha h_{lj}^\beta) \qquad (147)$$

If we denote by H^α the matrix (h_{ij}^α), then by (7) the first two terms together on the right hand side of (147) is equal to the negative of

$$\sum_{\alpha,\beta} tr(H^\alpha H^\beta)^2 - \sum_{\alpha,\beta} tr(H^\alpha)^2 (H^\beta)^2$$

$$+ \sum_{\alpha,\beta} [tr(H^\alpha H^\beta)]^2 - \sum_{\alpha,\beta} (trH^\beta)\ (tr\,(H^\beta)^2\,H^\beta)$$

$$+ (tr\,(H^\beta)^2 - nc\left(\sum_{\alpha,i,j}(h_{ij}^\alpha)^2\right) \qquad (148)$$

Hence for any real number a,

$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha = (1+a)\sum_{\alpha,i,j,k,m} h_{km}^\alpha h_{ij}^\alpha R_{mijk}$$

$$+ (1+a)\sum_{\alpha,i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}$$

$$- (1-a)\sum_{\alpha,\beta} tr(H^\alpha)^2(H^\beta)^2 + (1-a)\sum_{\alpha,\beta} tr\,(H^\alpha H^\beta)^2$$

$$+ a\sum_{\alpha,\beta} [tr(H^\alpha H^\beta)]^2\,a\sum_{\alpha,\beta} tr\,(H^\beta)^2 (tr(H^\beta)^2 H^\beta)$$

$$+ a\sum_{\alpha} (tr\,H^\alpha)^2 - nac\left(\sum_{\alpha,i,j}(h_{ij}^\alpha)^2\right) \qquad (149)$$

It is easy to see

$$\sum_{\alpha}(tr(H^\alpha)^2)^2 \geq \frac{1}{p}\sum_{\alpha}(tr(H^\alpha)^2)^2 \qquad (150)$$

29

$$\sum_{\alpha,\beta} tr\,(H^\alpha)^2(H^\beta)^2 \geq \sum_{\alpha,\beta}[tr(H^\alpha H^\beta)]^2 \tag{151}$$

Since M^n is minimal, it follows from (149), (151) and (152)

$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq (1+a)\sum_{\alpha,i,j,k,m} h_{ij}^\alpha h_{km}^\alpha R_{mijk}$$

$$+(1+a)\sum_{\alpha,i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk}$$

$$+a\frac{S}{p}(S-pnc) \tag{152}$$

for $a \geq 1$.

Now for each α, let α_i be the eignevalues of the matrix (h_{ij}^α). Then

$$2\sum_{i,j,k,m} h_{ij}^\alpha h_{km}^\alpha R_{mijk} + 2\sum_{i,j,k,m} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} = \sum_{i,j}(\alpha_i-\alpha_j)R_{ijij}$$

$$\geq \Sigma_{i,j}(\alpha_i-\alpha_j)^2 K$$

$$= 2nK\sum_{i,j}(h_{ij}^\alpha)^2 \tag{153}$$

Hence, from (152) and (153)

$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq (1+a)nKS + \frac{aS}{p}(S-pnc) = \frac{aS}{p}\left[S-pnc+\frac{np(1+a)}{a}K\right] \tag{154}$$

As in Theorem (1.1.11), we obtain

$$\int_M \left[pnc - \frac{pn(c+a)}{a}K - S\right]dV \geq 0 \tag{155}$$

for $a \geq 1$.

Now (145) follows from (155) by taking $a = 1$. If $K \leq 0$, (146) follows from (155) by letting a approach infinity.

Corollary (1.2.14)[36]. Suppose in Theorem (1.2.3), $S \geq pn(c-2K)(S \geq pn(c-K)$ when $K \leq 0)$. Then the second fundamental form of M^n is covariant constant and M^n is either totally geodesic or $S = pn(c-2K)(S = pn(c-K)$ when $K \leq 0)$.

Corollary (1.2.15)[36]. Let M^n be a compact minimal submanifold of a manifold N^{n+p} with constant curvature c. Let R be the scalar curvature of M^n and K be the function which assigns to each point of M^n the infimum of the sectional curvature at that point. Suppose

$$K \leq R \leq \frac{n-1-p}{n-1}c + \frac{2pK}{n-1}\left(K \leq R \leq \frac{n-1-p}{n-1}c + \frac{pK}{n-1}\,when\,K \leq 0\right)$$

Then either M^n is totally geodesic, or the second fundamental form of M^n is covariant constant and

$$R = \frac{n-1-p}{n-1}c + \frac{2pK}{n-1}\left(R = \frac{n-1-p}{n-1}c + \frac{pK}{n-1} \text{ when } K \leq 0\right)$$

Let us now examine the bound we obtain in Theorem (1.2.3).

In general, let $S^m(r)$ be the m-dimensional sphere with radius r. Then one can construct a standard minimal immersion of $S^{m_1}(m_1/n)^{1/2} \times \ldots \times S^{m_k}(m_k/n)^{1/2})$ into S^{n+k-1} where $n = \sum_{i=1}^{k} m_i$ In fact, if x_i is a point in $S^i(m_i/n)^{1/2}$, i.e., a vector of length $(m_i/n)^{1/2}$ in $\mathbb{R}^{mi+1}$, then $(x_1, x_2, \ldots, x_n)$ defines a minimal immersion of $S^{ml}((m_1/n)^{1/2}) \times \ldots \times S^{m_k}(m_k/n)^{1/2})$ into S^{n+k-1} The length of the second fundamental form is exactly $(k-1)n$, the bound we obtained in Theorem (1.2.13), with $K \equiv 0$. We shall prove that the converse is also true. In fact, if $K \equiv 0$ and $S = npc$, then

$$\sum_{\alpha,\beta} tr\,(H^\alpha)^2(H^\beta)^2 = \sum_{\alpha,\beta}[tr(H^\alpha H^\beta)]^2 \qquad (156)$$

and

$$\sum_{\alpha \neq \beta}[tr(H^\alpha H^\beta)]^2 = 0 \qquad (157)$$

Equality (156) implies $H^\alpha H^\beta = H^\beta H^\alpha$ for all α, β and the normal bundle of M^n is flat. Hence, we can diagonalize $\{H^\alpha\}$ simultaneously. Putting these informations together, we can apply the proof of Theorem (1.2.2) or [13] to see that M^n is an open piece of the product of spheres.

Theorem (1.2.16)[36]. Let M^n be a minimal submanifold in another manifold N^{n+p} with constant curvature c. If M^n has non-negative curvature, $S = pnc$ and $c = 1$, then M^n is an open piece of the product $\pi_i S^{m_i}(m_i/n)^{1/2})$, $n = \sum_{i=1}^{k} m_i$.

Let us consider the case $K = constant > 0$. In [5], Chern, DoCarmo, and Kobayashi gave an isometric minimal immersion of $S^n(2(n+1)/n)^{1/2}$ into S^{n+P} with $p = \frac{1}{2}(n-1)(n+2)$. (Actually, it is an embedding of the real projective space.) The length of the second fundamental form turns out to be $n(n-1)(n+2)/2(n+1)$. This number is exactly $pn(1-2K)$. We suspect the converse may be true, i.e., if $K = constant > 0$ and $S = pn(1-2K)$, then M is the immersion of the standard sphere described above. From the proof of Theorem (1.2.3), we see M^n must satisfy the following strong conditions

(i)

$$\sum_{\alpha,i,j,k}\left(h^\alpha_{ijk}\right)^2 = 0 \qquad (158)$$

(ii) For all normal frame $\{e_\alpha\}$

$$\sum_{i,j}\left(h^\alpha_{ijk}\right)^2 = \frac{S}{p} = n(1-2K) \qquad (159)$$

and

$$\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta} = 0 \tag{160}$$

for $\alpha \neq \beta$.

(iii) For each a, let ai be the eigenvalues of (h_{ij}^{α}) and R_{ijij} the sectional curvature between e_i and e_j. Then

$$(\alpha_i - \alpha_j)(R_{ijij} - K) = 0 \tag{161}$$

(iv)

$$\sum_{\alpha,\beta} tr\,(H^{\alpha})^2 (H^{\beta})^2 = \sum_{\alpha,\beta} tr(H^{\alpha}H^{\beta})^2 = nKS \tag{162}$$

(v) The Gauss equation and (162) imply

$$n - 1 - p \geq (n - 1 - 2p)K \tag{163}$$

Equation (159) implies that if $p = 1, M^n$ is an open piece of the product of two spheres. This was proved in [5]. If $n = 2$, it is easy to see from (161) and (162) that M^n is the Veronese surface. Equation (163) shows that if $\alpha_i \neq \alpha_j$, $R_{ijij} = K$. On the other hand (163) says the matrices $\{(h_{ij}^{\alpha})\}$ are highly non-commutative. These two facts indicate M^n has constant curvature. Then a theorem of DoCarmo and Wallach [20] will prove the assertion.

Now we discuss the case $K < 0$. It seem to be not known whether there is any compact negatively curved minimal submanifold in sphere except for very special case (surface in S^3). The following corollary gives some information:

Corollary (1.2.17)[36]. Let M^n be a compact minimal submanifold of the sphere S^{n+p}. Suppose sectional curvature of M^n is non-positive and bounded from below by $-\frac{n-1-p}{p}$. Then M^n is the standard minimal immersion of the product of circles into S^{n+p}.

Proof. We note that in Theorem (1.2.13), we may take K to be any function which is bounded from above by the sectional curvatures of M at every point. In particular, we may take

$$K = -\frac{n-1-p}{p} \tag{164}$$

in this corollary.

Then the hypothesis of Corollary (1.2.15) is satisfied and therefore $R \equiv 0$. Since the sectional curvature of M^n is everywhere non-positive and the average of them is zero, M^n is a flat manifold. Corollary (1.2.17) then follows from Theorem (1.2.16). Let us remark that if one replaces sectional curvature by Ricci curvature, one can prove inequalities similar to Theorem (1.2.13). In fact, let Ric(x) be the function assigns to each point of M^n the infinimum of the Ricci curvature of M^n at that point, then if M^n is compact,

$$\int_M \left[pnc - \frac{pn(1 + a)}{a}(Ric - (n - 2)c) - S \right] dV \geq 0 \qquad (165)$$

for any $a \geq 1$. This follows because by (7), one can prove that when i $\neq$ j,
$R_{ijij} \geq Ric - (n - 2)c$. Similarly, one can verify that $-\sum_{i,j} \alpha_i \alpha_j R_{ijij} \geq \sum_i \alpha_i^4 - S \sum_i \alpha_i^2$
Therefore we also have $\int_M S[(2p - 2/n + 1)S + 2pRic - pnc] \, dV \geq 0.$

We improve Simons' inequality [15] under sectional curvature restriction (instead of scalar curvature restriction).

Theorem (1.2.18)[36]. Let M^n be a compact minimal submanifold in the sphere S^{n+P}. Suppose the sectional curvature of M^n is everywhere not less than $(p - 1)/(2p - 1)$. Then either M^n is the totally geodesic sphere, the standard immersion of the product of two spheres or the Veronese surface in S^4.

Proof. We first note that it was proved in [5]

$$\sum_{\alpha,\beta} tr\, (H^\alpha)^2 (H^\beta)^2 - \sum_{\alpha,\beta} tr(H^\alpha H^\beta)^2$$

$$\leq \sum_{\alpha,\beta} (tr\, (H^\alpha)^2)(tr(H^\beta)^2) \leq \frac{p - 1}{p} S \qquad (166)$$

and the equality holds if at most two matrices (h_{ij}^α) and (h_{ij}^α) are not zero and these two matrices can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A} = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array} \right)$$

$$\tilde{B} = \left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline & 0 & 0 \end{array} \right) \qquad (167)$$

By taking $0 \leq a \leq 1$ in (149), we obtain

$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq (1 + a)nKS + (1 - a)\frac{p - 1}{p}S^2 + \frac{a}{p}S^2 - naS \qquad (168)$$

If $a = (p - 1)/p$, the right hand side of (168) is $nS/p[(2p - 1)K - (p - 1)]$. If the hypothesis is satisfied, $\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq 0$ and hence (166) and (168) are equalities.

We using the method of B. Lawson, it is possible to generalize Theorems (1.2.13) and (1.2.18) to minimal immersions into symmetric spaces of rank one. However, we shall consider here only immersions into Kahler manifolds with constant holomorphic sectional

curvature. We shall denote our ambient manifold N by $N^{n+p}(c)$ where $n + p$ is the complex dimension and c is the holomorphic sectional curvature.

A straightforward computation then shows that if M^n is a complex manifold or a totally real minimal submanifold,

$$\sum_{\alpha,i,j} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} = \sum_{\alpha,ij,k,m} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk}$$

$$+ \sum_{\alpha,ij,k,m} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk} - \sum_{\substack{\alpha,\beta \\ ij,k}} h_{ij}^{\alpha} h_{ki}^{\beta} R_{\alpha\beta jk} \tag{169}$$

We first discuss the case where M^n is a complex submanifold of $N^{n+p}(c)$. In this case, if e$_\alpha$ is a local section of the normal bundle, then the fundamental form (h_{ij}^{α}) has the following form

$$\begin{pmatrix} A & B \\ \hline B & -A \end{pmatrix}$$

Furthermore, the second fundamental form corresponding to the section Je$_\alpha$ has the form

$$\begin{pmatrix} B & A \\ \hline A & -B \end{pmatrix}$$

Using these two facts, the Gauss equation and (169), it is not hard to see

$$\sum_{\alpha,i,j} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} \geq (1+a) \sum_{\alpha,ij,k,m} h_{ij}^{\alpha} h_{km}^{\alpha} R_{mijk}$$

$$+ (1+a) \sum_{\alpha,ij,k,m} h_{ij}^{\alpha} h_{mi}^{\alpha} R_{mkjk}$$

$$+ a\frac{S^2}{2p} - \frac{n+3}{2} acS - \frac{c}{2}S \tag{170}$$

for $a \geq 1$. Here S is the length of the second fundamental form.

Let K be the function which assigns to each point the infinimum of the section curvature of M^n, then

$$\sum_{\alpha,i,j} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} \geq (1+a)2nKS + \frac{aS^2}{2p} - \frac{(n+3)}{2} acS - \frac{c}{2}S$$

$$= \frac{aS}{2p}\left[S - p(n+3)c - \frac{cp}{a} + \frac{4(1+a)}{a}pnK \right] \tag{171}$$

Hence if M^n is compact,

$$\int_M S\left[-\frac{4(1+a)}{a}pnK + \frac{cp}{a} + p(n+3)c - S\right] \geq 0 \qquad (172)$$

Letting a approach infinity when $K \leq c/(4n)$, we obtain the following.

Theorem (1.2.19)[36]. Let M^n be a compact complex submwnifold of a Kahler manifold $N^{n+p}(c)$ with constant holomorphic sectional curvature c. Let K and S be defined as above. Then

$$\int_M S[p(n+4)c - 8pnk - S] \geq 0 \qquad (173)$$

If furthermore $K \leq c/(4n)$,

$$\int_M S[p(n+3)c - 4pnk - S] \geq 0 \qquad (174)$$

Corollary (1.2.20). Suppose in Theorem (1.2.19), $S \geq p(n+3)c - 8pnK$ [$S \geq p(n+3)c - 4pnK$ if $K < c/(4n)$]. Then the equality actually holds.

Corollary (1.2.21). Suppose in Theorem (1.2.19),
$$R \leq \frac{1}{2n(2n-1)}[n(n+1)c - p(n+3)c + 8pnk]$$

$(R \leq \frac{1}{2n(2n-1)}[n(n+1)c - p(n+3)c + 4pnk]$ when $K \leq c/(4n))$. Then the equality actually holds.

Theorem (1.2.22)[36]. Let M^n be a complex submanifold of a Kahler manifold $N^{n+p}(c)$ with constant holomorphic sectional curvature c. Suppose M^n has non-negative sectional curvature and S is a constant. Then $S < p(n+3)c$ or M^n is totally geodesic.

Proof. Note that from the representation of the second fundamental forms above, it is clear that the second fundamental forms commute to each other if and only if the complex submanifold is totally geodesic. The rest of the proof is similar to that of Theorem (1.2.16).

Theorem (1.2.23)[36]. Let M^n be a compact complex submanifold of the complex projective space Cp^{n+p}. Suppose the curvature of M^n is not less than $[(2p-1)n + 8p - 3]/(16p-4)n$. Then M^n is totally geodesic.

Proof. The proof is similar to that of Theorem (1.2.18). In this case we have
$$\sum_{\alpha,i,j} h_{ij}^\alpha \Delta h_{ij}^\alpha \geq (1+a)2nKS + \frac{aS^2}{2p} - \frac{(n+3)}{2}acS - \frac{c}{2} - (1-a)\left(\frac{2p-1}{2p}\right)S^2 \qquad (175)$$

for $0 \leq a \leq 1$.

Hence if $a = \frac{2p-1}{2p}$ and $K \geq \frac{(2p-1)n+8p-3}{(16p-4)n}$

$$\sum_{\alpha,i,j} h_{ij}^{\alpha} \Delta h_{ij}^{\alpha} \geq 0$$

As in Theorem (1.2.18), this implies $p = 1$, M^n is a complex hypersurface.

Let M^n be a compact minimal submanifold of the sphere S^{n+p}. It was asked by Simons [15] whether the $n + 1 -$plane spanned by the tangent space $T_x(M)$ and the radial vector x has non-trivial intersection with every fixed p-plane in the euclidean space N^{n+p+1}. The assertion was first proved by DeGiorgi for $p = 1$ and partially proved by Simons [15] and Reilly [31]. We observe here that the Calabi-Chern theory can be used to prove the assertion for S^2 in S^4.

Let $G_{2,5}$ be the Grassmann manifold of all oriented two-dimensional planes through origin of the five dimensional euclidean space. Then Chern [20] [21] defined a Gauss map $g: S^2 \to G_{2,5}$ by assigning at each point $x \in S^2$ the oriented plane through the origin which is parallel to the normal plane of S^2 in S^4.

Now $G_{2,5}$ has a natural complex structure defined as follows: Suppose the two-plane be spanned by the orthonormal vectors ξ, η (in that order, as the plane is oriented). Then $\xi + i\eta$ is defined up to a complex constant. Regarding $\xi + i\eta$ as the homogeneous coordinates of a point in the complex projective space C^{P4}, we can consider $G_{2,5}$ as a hyperquadric in C^{P4}.

The Calabi-Chern theory asserts that if S^2 is minimal in S^4, the Gauss map g is holomorphic with respect to the unique conformal structure of S^2. Chern [35] also observed that if $g(S^2)$ lies in a hyperplane of C^{P4}, the minimal sphere must be totally geodesic. The well-known theory of compact holomorphic curve in C^{P4} says that if $g(S^2)$ does not lie in a hyperplane, it has to intersect every hyperplane exactly once.

Let $e_3 = e_3 = (e_3^1, e_3^2, \dots, e_3^5)$ and $e_4 = (e_4^1, e_4^2, \dots, e_4^5)$ be a normal frame at some point x of S^2 such that $e_3 \wedge e_4$ defines the orientation of the normal plane. The condition that $g(x)$ intersects the hyperplane defined by the complex vector $(a_1 + ib_1, a_2 + ib_2, \dots, a_5 + ib_5)$ is equivalent to

$$\sum_j (a_j + ib_j)(e_3^j + ie_4^j) = 0 \tag{176}$$

Hence if $A = (a_1, a_2, \dots, a_5)$ is any vector in the five dimensional euclidean space N^5, there is exactly one point x in S^2 such that A is orthogonal to both e_3 and e_4 at x. (This follows by taking $(a_1, \dots, a_5) = (b_1, b_2, \dots, b_5)$ in (176).)

In conclusion, we have

Theorem (1.2.24)[36]. Let S^2 be any non-totally geodesic minimal sphere in S^4. Suppose S^4 sits in the euclidean space N^5. Then for all four dimensional linear space N^4 in N^5, there is exactly one point $x \in S^2$ such that the normal space at x is parallel to N^4.

The conjecture of Simons is equivalent to say that for all fixed three dimensional linear space passing through the origin, we can find a point x in S^2 such that the normal plane at x has non-trivial projection on this fixed linear space. By taking a four dimensional linear space containing this three dimensional linear space, it is quite easy to see that Simons conjecture follows from Theorem (1.2.24) in this special case.

Chapter 2
Minimal Submanifolds and an Intrinsic Rigidity Theorem

We show that if $\sigma(u) \leq \frac{1}{3}$, then either $\sigma(u) \equiv 0$, or $\sigma(u) \equiv \frac{1}{3}$. All minimal submanifolds satisfying $\sigma(u)$ are determined. A stronger result is obtained if M is odd-dimensional.

Section(2-1): A Sphere with Bounded Second Fundamental Form

Let M be a smooth (i.e. C^∞) compact $n-$dimensional Riemannian manifold minimally immersed in a unit sphere S^{n+p} of dimension $n + p$. Let h be the second fundamental form of the immersion, h is a symmetric bilinear mapping $T_x \times T_x \to T_x^\perp$ for $x \in M$, where T_x is the tangent space of M at x and $T_x^\perp$ is the normal space to M at x. We denote by $S(x)$ the square of the length of h at x. By the equation of Gauss, $S(x) = n(n-1) - \rho(x)$, where $\rho(x)$ is the scalar curvature of M at x. Therefore, $S(x)$ is an intrinsic invariant of M. Let $\Pi: UM \to M$ and UM_x be the unit tangent bundle of M and its fiber over $x \in M$, respectively. We set $\sigma(u) = ||h(u, u)||^2$ for any u in UM. $\sigma(u)$ is not an intrinsic invariant of M. However, like $S(x)$, $\sigma(u)$ is a measure of an immersion from being totally geodesic.

In [42] proved that if $S(x) \leq n/(2-1/p)$ everywhere on M, then either $S(x) \equiv 0$ (i.e. M is totally geodesic), or $S(x) \equiv n(2 - \frac{1}{p})$. In [37], S.-S. Chern, M. do Carmo, and S. Kobayashi determined all minimal submanifolds M of S^{n+p} satisfying $S(x) = n/(2-1/p)$ (for $p = 1$ it was also obtained by B. Lawson [56]). The purpose is to obtain the analogous results for σ(u).

We first describe the following examples of minimal immersions [37, 41].

A. Let $S^m(r)$ be an m-dimensional sphere in R^{m+1} of radius r. We imbed $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ into $S^{m+1} = S^{2m+1}(1)$ as follows. Let $\zeta, \eta \in S^m(\sqrt{\frac{1}{2}})$. Then ζ and η are vectors in R^{m+1} of length $\sqrt{\frac{1}{2}}$. We can consider (ζ, η) as a unit vector in $R^{2m+2} = R^{m+1} \times R^{m+1}$. It is easy to see that $S^m(\sqrt{\frac{1}{2}}) \times S^m(\sqrt{\frac{1}{2}})$ is a minimal submanifold of S^{2m+1}.

B. Let F be the field R of real numbers, the field C of complex numbers, or the field Q of quaternions. Define d by

$$d = \begin{cases} 1, & if \ F = R, \\ 2, & if \ F = C, \\ 4, & if \ F = Q, \end{cases}$$

Let FP^2 denote the projective plane over F. FP^2 is considered as the quotient space of the unit $(3d - 1) - dimensional$ sphere $S^{3d-1}(1) = \{x \epsilon F^3 : \ ^t\bar{x} \cdot x = 1\}$ obtained by identifying x with λx where $\lambda \in F$ such that $|\lambda| = 1$. The canonical metric g_0 in FP^2 is the invariant metric such that the fibering $\pi : S^{3d-1}(1) \to FP^2$ is a Riemannian submersion. The sectional curvature of RP^2 is 1, the holomorphic sectional curvature of CP^2 is 4, and the Q-sectional curvature of QP^2 is 4, with respect to the metric g_0. Let $M(3, F)$ be the vector space of all 3×3 matrices over F and let

$$\mathcal{H}(3, F) = \{A \in M(3, F): A^* = A, \quad trace \ A = 0\}$$

where $A^* = \ ^t\bar{A}$. $\mathcal{H}(3, F)$ is a subspace of $M(3, F)$ of real dimension $3d + 2$. We define the inner product in $\mathcal{H}(3, F) = R^{3d+2}$ by $\langle A, B \rangle = $ ½ trace (AP) for $A, B \in \mathcal{H}(3, F)$. Define a map $\bar{\psi} : S^{3d+1} \to R^{3d+2} = \mathcal{H}(3, F)$ as folows.

$$\bar{\psi}(x) = \begin{bmatrix} |x_1|^2 - \frac{1}{3} & x_1\bar{x}_2 & x_1\bar{x}_3 \\ x_2\bar{x}_1 & |x_2|^2 - \frac{1}{3} & x_2\bar{x}_3 \\ x_3\bar{x}_1 & x_3\bar{x}_2 & |x_3|^2 - \frac{1}{3} \end{bmatrix}$$

for $x = (x_1, x_2, x_3) \in S^{3d-1}$ (l) $\subset F^3$. Then, it is easily verified that $\bar{\psi}$ induces a map ψ: $FP^2 \to R^{3d+2} = \mathcal{H}(3, F)$ such that $\bar{\psi} = \psi$ o π. Direct computation shows that $\psi(FP^2) \subset S^{3d+1}(1/3)$. We blow up the metric g_0 by putting $g = 3g_0$ in FP^2, so that the sectional curvature of RP^2 is ⅓ and the holomorphic sectional curvature (resp. Q-sectional curvature) of CP^2 (resp. QP^2) is $\frac{4}{3}$, with respect to the metric g. Then ψ gives a map $\psi: FP^2 \to S^{3d+1}$ (l). It is proved in [41] that ψ is an isometric minimal imbedding. Thus, we have the following isometric minimal imbeddings:

$$\psi_1: RP^2 \to S^4(l) \quad \text{(the Veronese surface)},$$
$$\psi_2: CP^2 \to S^7(1),$$
$$\psi_3: QP^2 \to {}^{\wedge}S^{13}(l).$$

In a similar manner one may obtain (see [41]) an isometric imbedding of the Cayley projective plane Cay P^2 furnished with the canonical metric (normalized such that the C-sectional curvature equals $\frac{4}{3}$) into $S^{25}(1)$:

$$\psi_4: \text{Cay } P^2 \to S^{25}(1).$$

In addition there is an immersion

$$\psi_1': S^2(\sqrt{3}) \to S^4(1)$$

defined by $\psi_1' = \psi_1$ o π.

For $n, m \geq 0$, let $S^n(1)$ be the great sphere in $S^{n+m}(1)$ given by
$$S^n(1) = \{(x_1, \ldots, x_{n+m+1}) \in S^{n+m}(1) : x_{n+2} = \ldots = x_{n+m+1} = 0\},$$
and $T_{n+m} : S^n(1) \to S^{n+m}(1)$ be the inclusion. For $p = 0, 1, \ldots$, we set
$$\psi_{1,P} = \tau_{4,p} \circ \psi_1 : RP^2 \to S^{4+p},$$
$$\psi_{2,p} = \tau_{7,p} \circ \psi_2 : CP^2 \to S^{7+p},$$
$$\psi_{3,P} = \tau_{13,p} \circ \psi_3 : QP^2 \to S^{13+p},$$
$$\psi_{4,P} = \tau_{25,p} \circ \psi_4 : \text{Cay } P^2 \to S^{25+p,}$$
$$\psi'_{1,P} = \tau_{4,p} \circ \psi'_1 : S^2(\sqrt{3}) \to S^{4+P}.$$

$\phi_{i,p}$ $(i = 1, \ldots, 4; p = 0, 1, \ldots)$, is an isometric minimal imbedding and $\phi'_{1,p}$ $(p = 0, 1, \ldots)$, is an isometric minimal immersion.

Let M be a compact n-dimensional manifold minimally immersed in S^{n+p}. We choose a local field of adapted orthonormal frames in S^{n+p}, that is frames $\{e_1, \ldots, e_{n+p}\}$ such that the vectors $e_1, \ldots, e_n$ are tangent to M. The vectors $e_{n+1}, \ldots, e_{n+p}$ are therefore normal to M. From now on let the indices $a, b, c, \ldots$, run from $1, \ldots, n$, and the indices $\alpha, \beta, \gamma, \ldots$, run from $n + 1, \ldots, n + p$. Let $h = (h_{ab}^\alpha)$ be the second fundamental form of the immersed manifold M, and $\sigma(u) = \|h(u, u)\|^2$ for $u \in UM$. Since the immersion of M into S^{n+p} is minimal, $\sum_a h_{aa}^\alpha = 0$ for all α.

Let $x \in M$. Suppose that $u \in U M_x$ satisfies $\sigma(u) = \max_{v \in UM_x} \sigma(v)$. We shall call u a maximal direction at x. Let $\{e_1, \ldots, e_{n+p}\}$ be an adapted frame at x. Assume that e_1 is a maximal direction at x, $\sigma(e_1) \neq 0$, and $e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$. Because of our choice of e_{n+1},
$$h_{11}^\alpha = 0, \qquad \alpha \neq n + 1. \tag{1}$$
Since e_1 is a maximal direction, we have at the point x for any $t, x^2, \ldots, x^n \in \mathbb{R}$
$$\left\| h\left(e_1 + t\sum_{a=2}^n x^a e_a, e_1 + t\sum_{a=2}^n x^a e_\alpha\right)\right\|^2 \leq \left[1 + t^2\sum_{a=2}^n (x^a)^2\right]^2 (h_{11}^{n+1})^2 \tag{2}$$

Expanding in terms of t, we obtain
$$4t h_{1a}^{n+1} \sum_{a \neq 1} x^a + O(t^2) \leq 0.$$

It follows that
$$h_{1a}^{n+1} = 0, \qquad a = 2, \ldots, n \tag{3}$$
We now choose an adapted frame at $x \in M$ such that in addition to (1) and (3),
$$h_{ab}^{n+1} = 0, \qquad a \neq b. \tag{4}$$
Once more expanding (2) in terms of t, we obtain

$$-2t^2 \left\{ \sum_{a \neq 1} \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{a \neq n+1} (h_{11}^{n+1})^2 \right] (x^a)^2 \right.$$

$$\left. - 4 \sum_{a \neq n+1} \sum_{\substack{a,b \neq 1 \\ a \neq b}} h_{1a} h_{1b} x^a x^b \right\} + O(t^3) \leq 0. \tag{5}$$

It follows that

$$2 \sum_{a \neq n+1} (h_{1a}^{n+1})^2 \leq h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}), \qquad a = 2, \dots, n \tag{6}$$

Let us define a tensor field $H = (H_{abcd})$ on M by the formula

$$H_{abcd} = \sum_a h_{ab}^\alpha h_{cd}^\alpha \tag{7}$$

It is clear that $\sigma(u) = H(u, u, u, u)$.

Lemma (2.1.1)[43]. Let u be a maximal direction at $x \in M$. Assume that $\sigma(u) \neq 0$. Let $e_1, \dots, e_{n+1}$ be an adapted frame at x such that $e_1 = u, e_{n+1} = h(e_1, e_1)/\|h(e_1, e_1)\|$, and $h_{ab}^{n+1} = 0$ for $a \neq b$. At the point x

    (i)   if $p = 1$, then

$$\frac{1}{2}(\Delta H)_{1111} \geq (h_{11}^{n+1})^2 \left[n - \sum_a (h_{aa}^{n+1})^2 \right] \tag{8}$$

  (ii) if $p \geq 2$, then

$$\frac{1}{2}(\Delta H)_{1111} \geq (h_{11}^{n+1})^2 \left[n - n(h_{11}^{n+1})^2 - 2 \sum_a (h_{aa}^{n+1})^2 \right] \tag{9}$$

with equality attained if and only if

$$(h_{11}^{n+1} - h_{aa}^{n+1}) \left[h_{11}^{n+1}(h_{11}^{n+1} - h_{aa}^{n+1}) - 2 \sum_{\alpha \neq n+1} (h_{1a}^\alpha)^2 \right] = 0 \tag{10}$$

and

$$\nabla_a(h_{11}^\alpha)^2 = 0 \tag{11}$$

for all a and all α, where Δ and ∇_a denote the Laplacian and the covariant derivative, respectively.

Proof.

$$\frac{1}{2}(\Delta H)_{1111} = h_{11}^{n+1} (\Delta h)_{11}^{n+1} + \sum_{a,\alpha} (\nabla_a h_{11}^\alpha)^2$$

Using Simons' formula [60] for the Laplacian of the second fundamental form (see also [55]), we obtain

$$\frac{1}{2}(\Delta H)_{1111} = (h_{11}^{n+1})^2\left[n - \sum_a (h_{aa}^{n+1})^2\right] + \sum_{a,\alpha}(\nabla_a h_{11}^\alpha)^2 \quad \text{if } p = 1, \tag{12}$$

and

$$\frac{1}{2}(\Delta H)_{1111} = (h_{11}^{n+1})^2\left[n - n(h_{11}^{n+1})^2 - 2\sum_a (h_{aa}^{n+1})^2\right]$$

$$+ \sum_a h_{11}^{n+1}\left(h_{11}^{n+1} - h_{aa}^{n+1}\right)\left[h_{11}^{n+1}\left(h_{11}^{n+1} - h_{11}^{n+1}\right) - 2\sum_{\alpha \neq n+1}(h_{1a}^\alpha)^2\right] + \sum_{a,\alpha}(\nabla_a h_{11}^\alpha)^2, \quad \text{if } p \geq 2, \tag{13}$$

from which the lemma follows readily by inequality (6).

Lemma (2.1.2)[43]. Let an adapted frame $\{e_1, \ldots, e_{n+p}\}$ at $x \in M$ be as in Lemma (2.1.1).

(i) Assume that $n (= 2m)$ is even. If

$$\sigma(u) \leq \begin{cases} 1, & \text{if } p = 1, \\ \frac{1}{3}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x$$

then $(\Delta H)_{1111} \geq 0$. If equality $(\Delta H)_{1111} = 0$ is attained, then it is possible to renumber $e_1, \ldots, e_{2m}$ such that the following equalities hold

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\,m+1}^{n+1} = \cdots = h_{2m\,2m}^{n+1} = \begin{cases} 1, & \text{if } p = 1, \\ \frac{1}{\sqrt{3}} & \text{if } p \geq 2. \end{cases} \tag{14}$$

(ii) Assume that $n (= 2m + 1)$ is odd. If

$$\sigma(u) \leq \begin{cases} 1 - \dfrac{1}{n}, & \text{if } p = 1, \\ \dfrac{1}{3 - 2/n}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x$$

then $(\Delta H)_{1111} \geq 0$. if equality $(\Delta H)_{1111} = 0$ is attained, then it is possible to renumber $e_1, \ldots, e_{2m+1}$ such that the following equalities hold.

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\,m+1}^{n+1} = \cdots = -h_{2m\,2m}^{n+1}$$

$$\sigma(u) \leq \begin{cases} (1 - \dfrac{1}{n})^{-\frac{1}{2}}, & \text{if } p = 1, \\ \dfrac{1}{(3 - 2/n)^{-\frac{1}{2}}}, & \text{if } p \geq 2, \end{cases} \quad \text{for all } u \in UM_x \tag{15}$$

$$h_{m+1\,m+1}^{n+1} = 0.$$

Proof. Since e_1 is a maximal direction

$$-h_{11}^{n+1} \leq h_{aa}^{n+1} \leq h_{11}^{n+1},_{11}, \qquad a = 2,\ldots, n. \tag{16}$$

Because of minimality of the immersion of M into S^{n+P},

$$\sum_{a=2}^n h_{aa}^{n+1} = -h_{11}^{n+1} \tag{17}$$

It is easily seen that the convex function $f(h_{22}^{n+1},\dots,h_{nn}^{n+1}) = \sum_{a=2}^{n}(h_{aa}^{n+1})^2$ of $(n-1)$ variables $h_{22}^{n+1},\dots,h_{nn}^{n+1}$ subject to the linear constraints (16), (17) attains its maximal value when (after suitable renumbering of $e_1,\dots,e_n$)

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\,m+1}^{n+1} = \cdots = -h_{2m\,2m}^{n+1} \quad,\text{if } n = 2m$$

and

$$h_{11}^{n+1} = \cdots = h_{mm}^{n+1} = -h_{m+1\,m+1}^{n+1} = \cdots = -h_{2m\,2m}^{n+1}$$
$$h_{2m+1\,2m+1}^{n+1} = 0, \ if\ n = 2m+1$$

Therefore, by inequalities (8), (9),

$$\frac{1}{2}(\Delta H)_{1111} \geq \begin{cases} n(h_{11}^{n+1})^2[1-\sigma(e_1)], & if\ p = 1, n = 2m \\ n(h_{11}^{n+1})^2[1-3\sigma(e_1)] & if\ p \geq 2, n = 2m \\ (h_{11}^{n+1})^2[n-(n-1)\sigma(e_1)] & if\ p = 1, n = 2m+1 \\ (h_{11}^{n+1})^2[n-(3n-2)\sigma(e_1)] & if\ p \geq 2, n = 2m+1. \end{cases}$$

Let $L(x)$ be a function on M defined by $L(x) = max_{u \in UMx}\sigma(u)$.

Lemma (2.1.3)[43]. Assume that one of A_1, A_2, A_3, A_4 is satisfied.

(A_1) $p = 1, n$ is even, $\sigma(u) \leq 1$ for all $u \in UM$,

(A_2) $p = 1, n$ is odd, $\sigma(u) \leq 1/(1 - 1/n)$ for all $u \in UM$,

(A_3) $p = 1, n$ is even, $\sigma(u) \leq \frac{1}{3}$ for all $u \in UM$,

(A_4) $p \geq 2$, n is odd, $\sigma(u) \leq 1/(3 - 2/n)$ for all $u \in UM$.

Then $L(x)$ is a constant function on M.

Proof. Following an idea in [39] we prove the lemma using the maximum principle. Clearly $L(x)$ is a continuous function. It suffices to show that $L(x)$ is subharmonic in the generalized sense. Fix $x \in M$ and let e_1 be a maximal direction at x. In an open neighborhood U_x of x within the cut-locus of x we shall denote by $u(y)$ the tangent vector to M obtained by parallel transport of $e_1 = u(x)$ along the unique geodesic joining x to y within the cut-locus of x. Define $g_x(y) = \sigma(u(y))$. Then

$$\Delta g_x(x) = \Delta[H(u(y),u(y),u(y),u(y))]_{y=x}$$
$$= \sum_a(\nabla_a^2 H)(e_1,e_1,e_1,e_1) = (\Delta H)_{1111}(x).$$

If $\|h(e_1,e_1)\| \neq 0$, then by Lemma (2.1.2), $(\Delta H)_{1111}(x) \geq 0$. If $\||h(e_1,e_1)\| = 0$, then $h \equiv 0$ at x. In this case the formula of Simons [60] for Δh shows that $\Delta h = 0$ at x, and therefore

$$(\Delta H)_{1111}(x) = \sum_{a,\alpha}(\nabla_a h_{11}^\alpha)^2 \geq 0.$$

Thus, we obtain that in any case $\Delta g_x(x) = (\Delta H)_{1111}(x) \geq 0$.

For the Laplacian of continuous functions, we have the generalized definition

$$\Delta L = C \lim_{r \to 0}\frac{1}{r^2}\left(\int_{B(x,r)}L \ / \int_{B(x,r)}1 - L(x)\right),$$

where C is a positive constant and $B(x,r)$ denotes the geodesic ball of radius r with the center at x. With this definition L is subharmonic on M if and only if $\Delta L(x) \geq 0$ at each

43

point $x \in M$. Since $g_x(x) = L(x)$ and $g_x \leq L$ on U_x, $\Delta L(x) \geq \Delta g_x(x) \geq 0$. Thus, $L(x)$ is subharmonic and hence constant on M.

Lemma (2.1.4)[43]. Assume that one of B_1, B_2, B_3, B_4 is satisfied.

(B_1) $p = 1$, n is even, $\sigma(u) < 1$ for all $u \in U M$,

(B_2) $p = 1$, n is odd, $\sigma(u) < 1/(1 - 1/n)$ for all $u \in U M$,

(B_3) $p \geq 1$, n is even, $\sigma(u) < \frac{1}{3}$ for all $u \in U M$,

(B_4) $p \geq 2$, n is odd, $\sigma(u) < 1/(3 - 2/n)$ for all $u \in U M$.

Then M is totally geodesic in S^{n+p}.

Proof. Let $x \in M$ and e_1 be a maximal direction at x. Assume that $\sigma(e_1) \neq 0$. Let $g_x(y) = \sigma(u(y))$ be the function defined in the proof of Lemma (2.1.3). By Lemma (2.1.3), g_x (x) is a maximum of g_x. Therefore, $(\Delta H)_{1111}(x) = \Delta g_x(x) \leq 0$. On the other hand, by Lemma (2.1.2), $(\Delta H)_{1111}(x) \geq 0$. Therefore, $(\Delta H)_{1111}(x) = 0$ on M. Hence, by (14) and (15),

$$\sigma(e_1) = \begin{cases} 1, & \text{if } p = 1, n \text{ is even} \\ \dfrac{1}{1-1/n} & \text{if } p = 1, n \text{ is odd}, \\ \dfrac{1}{3}, & \text{if } p \geq 1, n \text{ is even} \\ \dfrac{1}{3-2/n}, & \text{if } p \geq 2, n \text{ is odd}. \end{cases}$$

contradicting the assumptions B_1, B_2, B_3, B_4. Hence, $h(u, u) = 0$ for all $u \in UM$, that is M is totally geodesic in S^{n+p}.

Theorem (2.1.5)[43]. Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+1}. Assume that n ($= 2m$) is even.

(i) If σ (u) < 1 for any $u \in UM$, then M is totally geodesic in S^{n+1}.

(ii) If $Max_{u \in UM} \sigma(u) = 1$, then M is $S^m(\frac{1}{2}) \times S^m(\frac{1}{2})$ minimally imbedded in S^{2m+1} as described above.

Proof. (i) follows from Lemma (2.1.4). We prove (ii). As in the poof of Lemma (2.1.4), we obtain $(\Delta H)_{1111} = 0$. Hence, by (4) and (14),

$$S(x) = \sum_{\alpha, a, b} (h_{ab}^{\alpha})^2 = \sum_a (h_{aa}^{n+1})^2 = n$$

All minimal immersions into S^{n+1} satisfying $S(x) \equiv n$ were found by S.-S. Chern, M. do Carmo, and S. Kobayashi in [37] and B. Lawson in [38]. It is easy to see that among their immersions only $S^m (\sqrt{\frac{1}{2}}) \times S^m (\sqrt{\frac{1}{2}})$ imbedded in S^{2m+1} satisfies the condition $Max_{u \in UM} \sigma(u) = 1$.

Theorem (2.1.6)[43]. Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+1}. Assume that n ($= 2m + 1$) is odd. If $\sigma(u) \leq 1/(1 - 1/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+1}.

Proof. By Lemmas (2.1.3) and (2.1.4), we have to consider only the case $L(x) = Max_{u \in UM}\ \sigma(u) = 1/(1 - \frac{1}{n})$ on M. As in the proof of Lemma (2.1.4), $(\Delta H)_{1111} = 0$. Hence, by (15),

$$S(x) \equiv \sum_{\alpha, a, b} (h^\alpha_{ab})^2 \equiv \sum_{a=1}^{n+1} \frac{1}{1 - 1/n} \equiv n.$$

It is shown in [37] that if M is minimally immersed in S^{n+1} and $S(x) \equiv n$, then h^{n+1}_{aa} may attain at most two different values for $a = 1, \ldots, n$. However, since by (15),

$$h^{n+1}_{11} = \left(\frac{n}{n-1}\right)^{\frac{1}{2}}, \quad h^{n+1}_{m+1\ m+1} = -\left(\frac{n}{n-1}\right)^{\frac{1}{2}}, \quad h^{n+1}_{2m+1\ 2m+1} = 0,$$

we obtain a contradiction, so the equality $Max_{u \in UM}\ \sigma(u) = 1/(1 - \frac{1}{n})$ on UM is impossible.

Theorem (2.1.7)[43]. Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+p}. Assume that $p \geq 2$ and $n(= 2m)$ is even.

(i)   If $\sigma(u) \leq \frac{1}{3}$ for any $u \in UM$, then M is totally geodesic in S^{n+p}.

(ii)   If $Max_{u \in UM}\ \sigma(u) = \frac{1}{3}$, then $\sigma(u) \equiv \frac{1}{3}$ on UM, and the immersion of M into S^{n+p} is one of the imbeddings $\phi_{i,p}$ ($i = 1, \ldots, 4$; $p = 0, 1, \ldots$), or the immersions $\phi'_{1,p}$ ($p = 0, 1, \ldots$), described above.

 Proof. (i) follows from Lemma (2.1.4). We prove (ii). As in the proof of Lemma (2.1.4), we obtain $(\Delta H)_{1111} = 0$. Let the indices $i, j, k, \ldots$, run from $1, \ldots, m$, and let $\bar{\imath}, \bar{\jmath}, \bar{k}, \ldots$, denote $i + m, j + m, k + m, \ldots$, respectively. By (14) we have

$$h^{n+1}_{ii} = -h^{n+1}_{\bar{\imath}\bar{\imath}} = -\frac{1}{\sqrt{3}}, \qquad i = 1, \ldots, m. \tag{18}$$

Since $\|h(e_i, e_i)\|^2 \leq \frac{1}{3}$ and $\| h(e_{\bar{\imath}}, e_{\bar{\imath}})\|^2 \leq \frac{1}{3}$, we obtain

$$h^\alpha_{ii} = -h^\alpha_{\bar{\imath}\bar{\imath}} = 0, \qquad \alpha \neq n + 1, \quad i = 1, \ldots, m. , \tag{19}$$

By (10), $\sum_{\alpha \neq n+1}(h^\alpha_{1\bar{1}})^2 = \frac{1}{3}$. Since each vector e_a, ($a = 1, \ldots, n$), is a maximal direction,

$$\sum_{\alpha \neq n+1} (h^\alpha_{i\bar{\jmath}})^2 == \frac{1}{3} \qquad i, j = 1, \ldots, m \tag{20}$$

Let $u = (e_i + e_j)/\sqrt{2}$. Then.

$$\sigma(u) = \frac{1}{4}\|h(e_i + e_j, e_i + e_j\|^2$$
$$= \frac{1}{4}\|(h^{n+1}_{ii} + h^{n+1}_{jj})e_{n+1} + 2\sum_{\alpha \neq n+1} h^\alpha_{ij}e_\alpha\|^2$$
$$= \frac{1}{3} + \sum_{\alpha \neq n+1} (h^\alpha_{ij})^2 \leq \frac{1}{3}.$$

Therefore,

$$h^\alpha_{ij} = 0, \qquad\qquad \alpha \neq n + 1; \quad i, j = 1, \ldots, m. \tag{21}$$

45

Similarly,

$$h_{ii}^{\alpha} = 0, \qquad\qquad \alpha \neq n + 1; \quad i, j = 1, \ldots, m. \qquad (22)$$

Expansion (5) now takes the form

$$t^2\left(-4 \sum_{\alpha}\sum_{j\neq k} h_{ij}^{\alpha}h_{ik}^{\alpha}x^{\bar{j}}x^{\bar{k}}\right) + O(t^3) \leq 0.$$

It follows that $\sum_{\alpha} h_{ij}^{\alpha}h_{ik}^{\alpha} = 0$ for $j\neq k$. Since each vector e_a is a maximal direction,

$$\sum_{\alpha} h_{i\bar{j}}^{\alpha}h_{i\bar{k}}^{\alpha} = 0 \qquad j \neq k, \qquad\qquad (23)$$

$$\sum_{\alpha} h_{i\bar{k}}^{\alpha}h_{j\bar{k}}^{\alpha} = 0 \qquad i \neq j, \qquad\qquad (24)$$

Once more expanding (2) in terms of t ,

$$2t^3 \sum_{\alpha,j,k,l} (h_{1\bar{k}}^{\alpha}h_{j\bar{l}}^{\alpha} + h_{1\bar{l}}^{\alpha}h_{j\bar{k}}^{\alpha})\, x^{\bar{j}}x^{\bar{k}}x^{\bar{l}} + O(t^4) \leq 0.$$

from which

$$\sum_{\alpha} h_{i\bar{k}}^{\alpha}h_{j\bar{l}}^{\alpha} + h_{i\bar{l}}^{\alpha}h_{j\bar{k}}^{\alpha} = 0 , \qquad i \neq j \text{ or } k \neq l \qquad (25)$$

Using (4) and (19)-(25), we obtain by direct computation that $\sigma(u) = \frac{1}{3}$ for any $u \in UM$. B. O'Neill [40] calls an immersion λ-isotropic if $\|h(u, u)\| = \lambda$ for any $u \in UM$. Therefore, the immersion under consideration is $1/\sqrt{3}$-isotropic.

By Lemma (2.1.1), $\nabla_a h_{11}^{\alpha} = 0$. It follows that $\nabla_a h_{bb}^{\alpha} = 0$. By polarization, $\nabla_a h_{11}^{\alpha} = 0$ or all α, a, b, c. Therefore, the second fundamental form of the immersion is parallel. All λ-isotropic minimal immersions into a unit sphere with parallel second fundamental form were completely classified by K. Sakamoto in [41]. Among his immersions only $\phi_{1,p}$, $\phi_{2,p}$, $\phi_{3,p}$, $\phi_{4,p}$ and $\phi'_{1,p}$ described in above are $1/\sqrt{3}$-isotropic.

Theorem (2.1.8)[43]. Let M be a compact n-dimensional manifold minimally immersed in a unit sphere S^{n+p}. Assume that $p \geq 2$ and $n(= 2m + 1)$ is odd. If $\sigma(u) \leq 1/(3 - 2/n)$ for any $u \in UM$, then M is totally geodesic in S^{n+p}.

Proof. By Lemmas (2.1.3) and (2.1.4), we need only consider the case $L(x) \equiv 1/(3 - 2/n)$ on M. We show that this case cannot occur. Thus, assume that $L(x) \equiv 1/(3 - 2/n)$ on M. As in the proof of Lemma (2.1.4), $(\Delta H)_{1111} = 0$. Let the indices $i, j, k, \ldots,$ run from $1, \ldots, m$, and let $\bar{\imath}, \bar{\jmath}, \bar{k}, \ldots,$ denote $i + m, j + m, k + m, \ldots,$ respectively. By (15),

$$h_{ii}^{n+1} = h_{\bar{\imath}\bar{\imath}}^{n+1} = (3 - \frac{2}{2)^{\frac{1}{2}}}, \quad i = 1, \ldots, m ,$$

$$h_{i\bar{\imath}}^{n+1} = 0 . \qquad\qquad (26)$$

As in the proof of Theorem (2.1.7),

$$h_{ij}^\alpha = h_{\bar{i}\bar{i}}^\alpha = 0, \qquad \alpha \neq n+1; \quad i = 1,\dots, m. \tag{27}$$

Since $h_{nn}^\alpha = -\sum_i h_{ii}^\alpha - \sum_i h_{\bar{i}\bar{i}}^\alpha$,

$$h_{nn}^\alpha = 0. \tag{28}$$

By (10),

$$\sum_\alpha (h_{ij}^\alpha)^2 = \frac{1}{3 - 2/n}, \qquad i = 1,\dots, m \tag{29}$$

$$\sum_\alpha (h_{in}^\alpha)^2 = \frac{1}{2(3 - 2/n)}, \qquad i = 1,\dots, m \tag{30}$$

$$\sum_\alpha (h_{\bar{i}n}^\alpha)^2 = \frac{1}{2(3 - 2/n)}, \qquad i = 1,\dots, m \tag{31}$$

As in the proof of Theorem (2.1.7), we obtain with the help of expansion (2) the following equalities:

$$\sum_\alpha h_{i\bar{j}}^\alpha h_{i\bar{k}}^\alpha = 0, \tag{32}$$

$$\sum_\alpha h_{i\bar{k}}^\alpha h_{j\bar{k}}^\alpha = 0, \tag{33}$$

$$\sum_\alpha h_{i\bar{j}}^\alpha h_{in}^\alpha = 0, \tag{34}$$

$$\sum_\alpha h_{i\bar{j}}^\alpha h_{n\bar{j}}^\alpha = 0, \tag{35}$$

$$\sum_\alpha h_{i\bar{j}}^\alpha h_{i\bar{k}}^\alpha + h_{i\bar{l}}^\alpha h_{j\bar{k}}^\alpha = 0, \qquad i \neq j \text{ or } k \neq 1 \tag{36}$$

$$\sum_\alpha h_{i\bar{k}}^\alpha h_{jn}^\alpha + h_{j\bar{k}}^\alpha h_{in}^\alpha = 0, \qquad i \neq j \tag{37}$$

$$\sum_\alpha h_{i\bar{j}}^\alpha h_{n\bar{k}}^\alpha + h_{i\bar{k}}^\alpha h_{n\bar{j}}^\alpha = 0, \qquad j \neq k \tag{38}$$

$$\sum_\alpha h_{in}^\alpha h_{jn}^\alpha = 0, \qquad i \neq j \tag{39}$$

$$\sum_\alpha h_{\bar{i}n}^\alpha h_{jn}^\alpha = 0, \qquad i \neq j \tag{40}$$

$$\sum_\alpha h_{in}^\alpha h_{\bar{j}n}^\alpha = 0, \tag{41}$$

Let $u = \sum_a u^a e_a \in UM$. Direct computation with the help of (4) and (26)- (41) shows that

$$\sigma(u) = [1- (u^n)^4](3 - 2/n)^{-1}. \tag{42}$$

It follows from (42) that for any $x \in M$, the tangent space T_x of M at x is a direct sum of two mutually orthogonal subspaces $T_x = P_x + Q_x$, where P_x is $2m$−dimensional and is defined by

$$P_x = \{X \in T_x: ||h(X,X)|| = \left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} ||X||^2\}, \tag{43}$$

and Q_x is 1-dimensional and is defined by

$$Q_x = \{X \in T_x: h(X,X) = 0\}. \tag{44}$$

Lemma (2.1.9)[43]. The distributions $P: x \to P_x$ and $Q: x \to Q_x$ are smooth distributions on M.

Proof. It is sufficient to prove that Q is smooth. Let $x_0 \in M$ and $\{e_1, \ldots, e_{n+p}\}$ be a smooth local field of orthonormal adapted frames in a neighborhood U of x_0 such that $e_0(x_0) \in Qx_0$. If U is sufficiently small, there is a unique vector X of the form $X = \sum_{a=1}^{2m} X^a e_a + e_n$ which belongs to Q_x at each point $x \in U$. We prove that $X^a, a = 1, \ldots, 2m$, are smooth functions of x.

By (44), $X^a(x), a = 1, \ldots, 2m$, are a unique solution of the system of equations

$$h^\alpha(X,X) = \sum_{a,b=1}^{2m} h_{ab}^\alpha(x)X^a X^b + 2\sum_{a=1}^{2m} h_{an}^\alpha(x)X^a = 0, \tag{45}$$

$$\alpha = n + 1, \ldots, n + p.$$

At the point x_0 the Jacobian of system (45) is

$$(\partial h^\alpha/\partial X^a) = 2(h_{an}^\alpha), \qquad \alpha = n + 1, \ldots, n + p; \ a = 1, \ldots, 2m.$$

By (30), (31) and (39)-(41), the rows of the matrix (h_{an}^α) are mutually orthogonal nonzero vectors. Hence, rank $(\partial h^\alpha/\partial X^a) = 2m$ at x_0. Therefore, $X^a, a = 1, \ldots, 2m$, are smooth functions of x in a sufficiently small neighborhood of x_0.

We now return to the proof of Theorem (2.1.8). Let $x \in M$. By Lemma (2.1.9), we may choose a smooth family of orthonormal adapted frames $\{e_1, \ldots, e_{n+p}\}$ in some neighborhood U of x such that equations (4), (26)-(41) are satisfied on U. Set

$$N_a = [(3 - 2/n)]^{\frac{1}{2}} \sum_a h_{an}^a, \qquad a = 1, \ldots, 2m$$

By (4), (30), (31), and (39)-(41), the vectors $e_{n+1}, N_1, \ldots, N_{2m}$ are orthonormal. Therefore, with no loss of generality, we may assume that $e_{n+1+a} = Na, a = 1, \ldots, 2m$. Then,

$$h_{in}^{n+1+i} = h_{an}^{n+1+\bar{i}} = [(3 - 2/n)]^{\frac{1}{2}}, \qquad i = 1, \ldots, m, \tag{46}$$

$$h_{in}^\alpha = 0, \qquad \alpha \neq n + 1 + i, i = 1, \ldots, m, \tag{47}$$

$$h_{\bar{i}n}^\alpha = 0, \qquad \alpha \neq n + 1 + i, i = 1, \ldots, m, \tag{48}$$

Let the indices A, B, C run from $1, \ldots, n + p$, and let $\{\omega^A\}$ and $\{\omega_B^A\}$ be the coframe dual to the frame $\{e_A\}$ and the connection forms of the Riemannian con- nection on S^{n+p}, respectively. Then,

$$d\omega^A = \sum_B \omega^B \wedge \omega_B^A , \tag{49}$$

$$d\omega_B^A = \sum_C \omega^C \wedge \omega_C^A + \omega^A \wedge \omega^B , \tag{50}$$

$$\omega^\alpha = 0, \tag{51}$$

$$\omega^\alpha = \sum_b h_{ab}^\alpha \omega^b \tag{52}$$

$$dh_{ab}^\alpha - \sum_c h_{cb}^\alpha \omega_a^c - \sum_c h_{ac}^\alpha \omega_b^c + \sum_\beta h_{ab}^\beta \omega_\beta^\alpha = \sum_c \nabla_c(h_{ab}^\alpha)\omega^c. \tag{53}$$

As in the proof of Theorem (2.1.7), we obtain

$$\nabla_c(h_{ab}^\alpha) = 0, \qquad a, b = 1, \ldots, 2m; \quad c = 1, \ldots, n. \tag{54}$$

Let us take $\alpha = h + 1 + i , a = b = i$ in (53). By (4), (26)-(28), (46),(48), and (45)

$$-2\sum_k h_{ki}^{n+1+i} \omega_i^{\bar k} - \left[2\left(3 - \frac{2}{n}\right)\right]^{-\frac{1}{2}} \omega_i^n + \left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} \omega_{n+1}^{n+1+i} = 0. \tag{55}$$

Analogously, taking $\alpha = n + l + i, a = i, b = j \neq i$ in (53),

$$-2\sum_k h_{kj}^{n+1+i} \omega_j^{\bar k} + \left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} \omega_i^n + \omega_{n+1}^{n+1+i} = 0. \quad i \neq j \tag{56}$$

Summing (56) with respect to $(j \neq i)$ and adding (55), we have

$$-2\sum_{j,k} h_{kj}^{n+1+i} \omega_j^{\bar k} + m\left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} \omega_{n+1}^{n+1+i} - \left[2\left(3 - \frac{2}{n}\right)\right]^{-\frac{1}{2}} \omega_i^n = 0. \tag{57}$$

Let us now take $a = n + 1 + i, a = b = k\grave{}$ in (53). Then,

$$-2\sum_j h_{\bar k j}^{n+1+i} \omega_j^{\bar k} + \left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} \omega_{n+1}^{n+1+i} = 0. \tag{58}$$

Summing (58) with respect to $\bar k$,

$$-2\sum_{j,k} h_{j\bar k}^{n+1+i} \omega_{\bar k}^j + m\left(3 - \frac{2}{n}\right)^{-\frac{1}{2}} \omega_{n+1}^{n+1+i} = 0. \tag{59}$$

Finally, adding (57) to (59), we get

$$\omega_i^n = 0 \tag{60}$$

Analogously, we obtain

$$\omega_{\bar i}^n = 0 \tag{61}$$

Differentiating (60) and using (4), (26)-(28), (46)-(48), and (21), we obtain

$$-\sum_{\alpha,a,b} h_{ia}^\alpha h_{bn}^\alpha \omega^a \wedge \omega^b + \omega^n \wedge \omega^i = 0 \tag{62}$$

Taking the coefficient of $\omega^n \wedge \omega^i$ in (62) we have $-\sum_\alpha (h_{in}^\alpha)^2 + 1 = 0$. By (30), it gives $2(3 - 2/n) = 1$ and therefore $n = 5/4$, yielding a contradiction. Therefore, the equality $max_{u \in UMx}\, \sigma(u) \equiv 1/(3 - 2/n)$ on M is impossible.

Section(2-2). Minimal Submanifolds in a Sphere

Let M be an n-dimensional compact minimal submanifold in a unit sphere S^{n+p} of dimension $n + p$. Denote by $\|\sigma\|^2$ the square of the length of the second fundamental form. S.S. Chern, M. Do Carmo and S. Kobayashi [46] proved that if $\|\sigma\|^2 \le \dfrac{n}{2 - \frac{1}{p}}$ everywhere on M, then either M is totally geodesic or $\|\sigma\|^2 = \dfrac{n}{2 - \frac{1}{p}}$. In the latter case M is either a Clifford hypersurface or a Veronese surface in S^4. In [47] Shen Yibing proved that if $\|\sigma\|^2 \le n/1 + \sqrt{\dfrac{n-1}{2n}}$ everywhere on M, then M is either a totally geodesic submanifold or a Veronese surface in S^4. In [49] Wu Baoqiang and Song Hongzao proved that for a 3-dimensional minimal submanifold M in S^{3+p}, if $\|\sigma\|^2 < 2$ everywhere on M, then M is totally geodesic.

Let M be an n-dimensional compact manifold, $x : M \hookrightarrow S^{n+p}$ a minimal immersion. We choose a local field of orthonormal frames $x, e_1, \ldots, e_n, \ldots, e_{n+p}, \ldots, e_{n+p+1}$ such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M, and $e_{n+p+1} = x$. Let ω^A, $1 \le A \le n + p + 1$, be the field of dual frames. Then the structure equations are given by

$$d\omega^A = \sum \omega^B \wedge \omega_B^A, \qquad \omega_B^A + \omega_A^B = 0,$$

$$d\omega_A^B = \sum \omega_A^C \wedge \omega_C^B, \qquad A, B, C = 1, \ldots, n + p + 1.$$

We restrict these forms to M. Then

$$\omega^\alpha = 0,$$
$$\omega_i^\alpha = \sum h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha,$$

where

$$n + 1 \le \alpha \le n + p, \qquad 1 \le i, j \le n.$$

The invariant

$$\|\sigma\|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2$$

is called the square of the length of the second fundamental form. For each α, let H_α denote the symmetric matrix (h_{ij}^α), and set

$$S_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}$$

Then the $(p \times p)$ −matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1}, \ldots, e_{n+p}$, i.e

$$S_{\alpha\beta} = \begin{cases} S_{\alpha} & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

where we set $S_{\alpha} = S_{\alpha\alpha}$ By definition, $\|\sigma\|^2 = \sum_{\alpha} S_{\alpha}$ .S. S. Chern, M. Do Carmo and S. at Kobayashi [46] obtained the following formula for the Laplacian of $\| \sigma \|^2$:

$$\tfrac{1}{2}\Delta \|\sigma\|^2 = \sum_{i,j,k,\alpha} (h_{ijk}^{\alpha})^2 + n\|\sigma\|^2 - \sum_{\alpha,\beta} N\left(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha}\right) - \sum_{\alpha} S_{\alpha}^2 \qquad (63)$$

The crucial point is to derive an upper bound for $\sum_{\alpha,\beta} N\left(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha}\right) + \sum_{\alpha} S_{\alpha}^2$. in terms of $\|\sigma\|^2$, which will be carried out in the sequent sections.

First of all, let us notice the following two facts, easily to be verified.

A) Let $T = (T_{ij})$ be an, orthogonal, $(n \times n)$ −matrix, and let ${}^{*}A_{\alpha} = TA_{\alpha}{}^{t}T$, $1 \le \alpha \le p$. Then

$$N({}^{*}A_{\alpha}{}^{*}A_{\beta} - {}^{*}A_{\beta}{}^{*}A_{\alpha}) = N(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha})$$
$${}^{*}S_{\alpha\beta} = S_{\alpha\beta},$$
$${}^{*}S = S.$$

B) Let $C = (C_{\alpha\beta})$ be an orthogonal $(p \times p)$ −matrix and let

$$\bar{A}_{\alpha} = \sum_{\beta} C_{\alpha\beta} A_{\beta} \quad 1 \le \alpha, \beta \le p.$$

Then

$$\sum_{\alpha,\beta} N\left(\bar{A}_{\alpha}\bar{A}_{\beta} - \bar{A}_{\beta}\bar{A}_{\alpha}\right) = \sum_{\alpha,\beta} N\left(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha}\right)$$
$$\sum_{\alpha\beta} \bar{S}_{\alpha\beta}^2 = \sum_{\alpha\beta} S_{\alpha\beta}^2$$
$$\bar{S} = S.$$

Let $A_{\alpha} = (a_{ij}^{\alpha})$. According to the fact B, after the transformation by a suitable orthogonal $(p \times p)$ −matrix C, the $(p \times p)$ −matrix $(S_{\alpha\beta})$ may be assumed to be diagonal, i.e.,

$$S_{\alpha\beta} = \sum_{i,j} a_{ij}^{\alpha} a_{ij}^{\beta} = \begin{cases} S_{\alpha} & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases} \qquad (64)$$

Then

$$\sum_{\alpha,\beta} N\left(A_{\alpha}A_{\beta} - A_{\beta}A_{\alpha}\right) + \sum_{\alpha} S_{\alpha}^2 \le \frac{3}{2} S^2 \qquad (65)$$

Proposition (2.2.1)[52]. Let A_1, A_2 be two symmetric $(n \times n)$-matrices. If $S_1 = S_2$, then

$$N\left(A_1 A_2 - A_2 A_1\right) + \sum_{\alpha} S_1^2 \le \frac{3}{2} S_1 S, \qquad (66)$$

and the equality holds if and only if there exists an orthogonal $(n \times n)$-matrix T such that

$$TA_1 \, {}^tT = \sqrt{\tfrac{S_1}{2}}\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline 0 & 0 & \end{array}\right), \qquad TA_2 \, {}^tT = \sqrt{\tfrac{S_1}{2}}\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline 0 & 0 & \end{array}\right).$$

Proof. This proposition follows immediately from Lemma (2.2.2) of Ref. [64] .

Let us now study the case of more than two matrices. Let $A_1,..., A_k$ be symmetric ($n \times n$)-matrices ($k \geq 3$) with

$$S_1 = S_2 = ... = S_k > 0$$

$$S_{\alpha\beta} = \begin{cases} S_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

We may assume that A_1 is diagonal and we denote by $\lambda_1, \lambda_2, ...\lambda_n$ the diagonal entries in A_1. Then

$$S_1 = \lambda_1^2 + ... + \lambda_n^2 ,$$

$$\sum_{\alpha=2}^k N\,(A_1 A_\alpha - A_\alpha A_1) = 2\sum_{i<j}(\lambda_i - \lambda_j)^2 \sum_{\alpha=2}^k (a_{ij}^\alpha)^2$$

Denote $S_1 = ... = S_k = b$. We use Lagrange's method to calculate the maximum of the function

$$F = \frac{2\sum_{i<j}(\lambda_i-\lambda_j)\sum_{\alpha=2}^k(a_{ij}^\alpha)^2 \tfrac{3}{2} S_1^2}{S_1 S}$$

under the constraints

$$\lambda_1^2 + ... + \lambda_n^2 = b$$
$$\sum_{i,j}(a_{ij}^\alpha)^2 = b, \qquad 2 \leq \alpha \leq k.$$

Let

$$\emptyset = F + m_1\left(\sum_i \lambda_i^2 - b\right) + m_2\left(\sum_{ij}(a_{ij}^2)^2 - b\right) + \cdots + m_k\left(\sum_{ij}(a_{i,j}^k)^2 - b\right)$$

Obviously, F attains its maximum $F(\dot{q})$ at some point
and

$$\frac{\partial\emptyset}{\partial a_{ij}^\alpha}(\dot{q}) = 0.$$

Lemma (2.2.2)[52]. For any α, $2 \leq \alpha \leq k$, if there exist two nonzero entries in the set $\{\dot{a}_{ij}^\alpha, i<j\}$, then

$$2\sum_{i<j}(\dot{\lambda}_i - \dot{\lambda}_j)^2(\dot{a}_{ij}^\alpha)^2 \leq \tfrac{3}{2}b^2 \qquad (67)$$

Proof. Without loss of generality, we may assume that $a_{12}^\alpha \neq 0$. Then from $\frac{\partial\emptyset}{\partial a_{ij}^\alpha}(\dot{q}) = 0.$ we

have

$$\frac{(\dot{\lambda}_1 - \dot{\lambda}_2)^2}{kb^2} - \frac{1}{kb}F(\dot{q}) + m_\alpha = 0.$$

For any $\dot{a}_{ij}^\alpha \neq 0$, $i < j$, we have also

$$\frac{(\dot\lambda_i - \dot\lambda_j)^2}{kb^2} - \frac{1}{kb}F(q) + m_\alpha = 0.$$

it follows for that $a_{12}^\alpha \neq 0$, $a_{ij}^\alpha \neq 0$ that

$$(\dot\lambda_i - \dot\lambda_j)^2 = (\dot\lambda_1 - \dot\lambda_2)^2,$$

therefore

$$2\sum_{i<j}(\dot\lambda_i - \dot\lambda_j)^2(\dot a_{ij}^\alpha)^2 = (\dot\lambda_1 - \dot\lambda_2)^2\sum_{i\neq j}(\dot a_{ij}^\alpha)^2 \leq (\dot\lambda_1 - \dot\lambda_2)^2\, b.$$

Assuming $\dot a_{12}^\alpha \neq 0$, we now consider three cases separately.

1) $\dot a_{12}^\alpha \neq 0$ for some $j > 2$. We have

$$(\dot\lambda_1 - \dot\lambda_2)^2 = (\dot\lambda_1 - \dot\lambda_j)^2$$

Again we consider two subcases.

If $\dot\lambda_1 - \dot\lambda_2 = \dot\lambda_1 - \dot\lambda_j$ then

$$b \geq \dot\lambda_1^2 + \dot\lambda_2^2 + \dot\lambda_j^2 = \dot\lambda_1^2 + 2\dot\lambda_2^2,$$

$$(\dot\lambda_1 - \dot\lambda_2)^2 = \dot\lambda_1^2 + \dot\lambda_2^2 - 2\dot\lambda_1\dot\lambda_2 \leq \dot\lambda_1^2 + \dot\lambda_2^2 + \frac{1}{2}\dot\lambda_1^2 + 2\dot\lambda_2^2$$

$$= \frac{3}{2}\left(\dot\lambda_1^2 + \dot\lambda_2^2\right) \leq \frac{3}{2}b.$$

If $\dot\lambda_1 - \dot\lambda_2 = \dot\lambda_1 - \dot\lambda_j$, then $\dot\lambda_j = 2\dot\lambda_1 - \dot\lambda_2$

$$b \geq \dot\lambda_1^2 + \dot\lambda_2^2 + \dot\lambda_j^2 = 5\dot\lambda_1^2 + 2\dot\lambda_2^2 - 4\dot\lambda_1\dot\lambda_2$$

$$(\dot\lambda_1 - \dot\lambda_2)^2 = \dot\lambda_1^2 + \dot\lambda_2^2 - 2\dot\lambda_1\dot\lambda_2 \leq 5\dot\lambda_1^2 + 2\dot\lambda_2^2 - 4\dot\lambda_1\dot\lambda_2$$

$$\leq b < \frac{3}{2}b$$

2) $\dot a_{2j}^\alpha \neq 0$ for some $j > 2$. The same argument as in 1) gives that $(\dot\lambda_1 - \dot\lambda_2) \leq \frac{3}{2}b$.

3) $\dot a_{ij}^\alpha \neq 0$ for some $i > 2, j > i$. We have

$$(\dot\lambda_1 - \dot\lambda_2)^2 = (\dot\lambda_i - \dot\lambda_j)^2 = \frac{1}{2}[(\dot\lambda_1 - \dot\lambda_2)^2 + (\dot\lambda_i - \dot\lambda_j)^2]$$

$$\leq \dot\lambda_1^2 + \dot\lambda_2^2 + \dot\lambda_i^2 + \dot\lambda_j^2 \leq b < \frac{3}{2}b.$$

In summary, we have

$$(\dot\lambda_1 - \dot\lambda_2)^2 \leq \frac{3}{2}b,$$

$$2\sum_{i<j}(\dot\lambda_i - \dot\lambda_j)^2(a_{ij}^\alpha)^2 \leq (\dot\lambda_1 - \dot\lambda_2)^2\, b < \frac{3}{2}b.$$

Lemma (2.2.3)[52]. For any $\alpha, \beta, 2 \leq \alpha < \beta \leq k$, we have

$$2\sum_{i<j}(\dot\lambda_i - \dot\lambda_j)^2\,[(\dot a_{ij}^\alpha)^2 + (\dot a_{ij}^\beta)^2] \leq 3b^2 \tag{68}$$

Proof. If $\sum_{i<j}(\dot a_{ij}^\alpha)^2 = 0$ (or $(\dot a_{ij}^\beta)^2 = 0$), then

$$2\sum_{i<j}(\dot\lambda_i - \dot\lambda_j)^2\,[(\dot a_{ij}^\alpha)^2 + (\dot a_{ij}^\beta)^2]$$

$$= 2\sum_{i<j}(\dot\lambda_i - \dot\lambda_j)^2(\dot a_{ij}^\beta)^2 \leq 2\dot S_1\dot S_\beta = 2b^2 < 3b^2.$$

So, in the following we assume that

$$\sum_{i<j}(\dot{a}_{ij}^{\alpha})^2 \neq 0 \,, \sum_{i<j}(\dot{a}_{ij}^{\alpha})^2 \neq 0$$

We consider two cases separately.

1) There are two nonzero elements in the set $\{a_{ij}^{\alpha}, i<j\}$ and there are also two nonzero elements in the set $\{a_{ij}^{\beta}, i<j\}$. In this case the inequality (68) follows immediately from Lemma (2.2.2).

2) There exists only one nonzero element in the set

$\{\dot{a}_{ij}^{\alpha}, i<j\}$ $\{$or$\{\dot{a}_{ij}^{\beta}, i<j\})$.

Without loss of generality, we may assume that $\dot{a}_{12}^{\alpha} \neq 0$.

If $\lambda_1 - \lambda_2 = 0$, then (68) holds obviously. In the case $\lambda_1 - \lambda_2 \neq 0$ we can prove that

$$\dot{a}_{11}^{\alpha} = \dot{a}_{22}^{\alpha} = \cdots = \dot{a}_{nn}^{\alpha} = 0.$$

In fact, from $\frac{\partial \emptyset}{\partial a_{ij}^{\alpha}}(\dot{q}) = 0.$ we have

$$\frac{(\lambda_1 - \lambda_2)^2}{kb^2} - \frac{1}{kb}F(q) + m_{\alpha} = 0.$$
$$\left(-\frac{1}{kb}F(\dot{q}) + m_{\alpha}\right)\dot{a}_{11}^{\alpha} = 0.$$

It follows that $\dot{a}_{11}^{\alpha} = 0$. Similarly, we have

$$\dot{a}_{22}^{\alpha} = \cdots = \dot{a}_{nn}^{\alpha} = 0.$$

Then the condition $\sum_{i<j}\dot{a}_{ij}^{\alpha}\dot{a}_{ij}^{\beta} = 0$ implies that $\dot{a}_{ij}^{\alpha} = 0$. As $\sum(\dot{a}_{ij}^{\alpha})^2 \neq 0$ suppose that $a_{ij}^{\alpha} \neq 0$ for some i <j, (i,j) $\neq$ (1, 2). We have

$$2\sum_{i<j}(\lambda_i - \lambda_j)^2 [(\dot{a}_{ij}^{\alpha})^2 + (\dot{a}_{ij}^{\beta})^2]$$
$$\leq [(\lambda_1 - \lambda_2)^2 + (\lambda_i - \lambda_j)^2]\, b.$$

We consider three cases separately.

1) $i = 1, j > 2$. We have

$$(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_j)^2 \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_j^2) \leq 3b.$$

2) $i = 2, j > 2$. We have

$$(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_j)^2 \leq 3(\lambda_1^2 + \lambda_2^2 + \lambda_j^2) \leq 3b.$$

3) $2 < i < j$. We have

$$(\lambda_1 - \lambda_2)^2 + (\lambda_i - \lambda_j)^2 \leq 2(\lambda_1^2 + \lambda_2^2 + \lambda_i^2 + \lambda_j^2) \leq 2b.$$

Hence

$$2\sum_{i<j}(\lambda_i - \lambda_j)^2 [(\dot{a}_{ij}^{\alpha})^2 + (\dot{a}_{ij}^{\beta})^2] \leq 3b^2.$$

Lemma (2.2.4)[52]. For any $m-1$ matrices $A_{\alpha_2}, A_{\alpha_3}, \ldots, A_{\alpha_m}$ of the matrices $A_1, A_2, \ldots, A_k, 3 \leq m \leq k$, we have

$$2 \sum_{i<j} (\dot{\lambda}_i - \dot{\lambda}_j)^2 \sum_{h=2}^{m} (\dot{a}_{ij}^{\alpha h})^2 \leqq \frac{3}{2}(m-1)b^2 \tag{69}$$

Proof. We prove Lemma (2.2.4) by mathematical induction on m. By Lemma (2.2.3), the inequality (69) holds for $m = 3$. Assume that (69) is valid for m. Let $A_{\alpha_2}, A_{\alpha_3}, \ldots, A_{\alpha_{m+1}}$ be m matrices from $A_2, \ldots, A_k$ Applying (69) to any $m-1$ matrices of $A_{\alpha_2}, \ldots, A_{\alpha_{m+1}}$1, and adding the obtained m inequalities, we get

$$2(m-1) \sum_{i<j} (\dot{\lambda}_i - \dot{\lambda}_j)^2 \sum_{h=2}^{m+1} (\dot{a}_{ij}^{\alpha h})^2 \leqq \frac{3}{2}(m-1)mb^2$$

which shows that (69) holds for $m + 1$. Lemma (2.2.4) is proved,

By Lemma (2.2.4) we have

$$F(\dot{q}) \leqq \frac{\frac{3}{2}(k-1)b^2 + \frac{3}{2}b^2}{kb^2} = \frac{3}{2}.$$

Since F attains its maximum $F(\dot{q})$ at $\dot{q}$ we obtain the following result.

Proposition (2.2.5)[52]. Let A_1, A_2 ..... A_k be symmetric $(n \times n)$-matrices $(k \leqq 3)$. Suppose that $S_1 = S_2 = ... = S_k > 0$,

$$S_{\alpha\beta} = \begin{cases} S_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

Then

$$\sum_{\alpha=2}^{k} N (A_1 A_\alpha - A_\alpha A_1) + \frac{3}{2} S_1^2 \leqq \frac{3}{2} S_1 S. \tag{70}$$

Using Proposition (2.2.1) and Proposition (2.2.5) we can prove

Proposition (2.2.6)[52]. Let $A_1, A_2, \ldots, A_p$ be symmetric $(n \times n)$−matrices $(k \leqq 2)$. Suppose that $S_1 = max \{S_1, \ldots, S_p\}$

$$S_{\alpha\beta} = \begin{cases} S_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

Then

$$\sum_{\alpha=2}^{k} N (A_1 A_\alpha - A_\alpha A_1) + \frac{3}{2} S_1^2 \leqq \frac{3}{2} S_1 S. \tag{71}$$

and the sign of equality holds if and only if one of the following conditions holds:

    1) $A_1 = A_2 = \ldots = A_p = 0,$

    2) only one of the matrices A_2 ..... A_p, say A_α is different from zero, and $S_\alpha = S_1 > 0$, and there exists an orthogonal $(n \times n)$−matrix T such that

$$TA_1^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & \\ \hline 0 & 0 & \end{array} \right), \qquad TA_\alpha^t T = \sqrt{\frac{S_1}{2}} \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline 0 & 0 & \end{array} \right)$$

Proof. We may assume that $A_1 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \lambda_n \end{pmatrix}$

and

$$S_1 = \lambda_1^2 + \ldots + \lambda_1^2 = b > 0.$$

We wish to maximize the function

$$F = \frac{2 \sum_{i<j} (\lambda_i - \lambda_j) \sum_{\alpha=2}^{k} (a_{ij}^{\alpha})^2 \, S_1^2}{S_1 S}$$

subject to the constraints

$$\lambda_1^2 + \ldots + \lambda_n^2 = b$$
$$0 \leq S_2 \leq b$$
$$\vdots$$
$$0 \leq S_p \leq b.$$

Obviousely, F attains its maximum $F(\dot q)$ at some point

$$\dot q = \left(\lambda_1, \ldots, \lambda_n, \dot a_{ij}^2, \ldots, \dot a_{ij}^p\right).$$

It suffices to consider the following 4 cases separately.

  1) at $\dot q$, $0 < \dot S_2 < b$, $0 < \dot S_3 < b$, ..., $0 < \dot S_p < b$,

For any $\dot a_{ij}^2 \neq 0$ $(i < j, \alpha \geq 2)$, from $\frac{\partial F}{\partial a_{ij}^\alpha}(\dot q) = 0$ we have

$$(\lambda_i - \lambda_j)^2 = \dot S_1 \, F(\dot q).$$

It follows that

$$F(\dot q) = \frac{\dot S_1 F(q)(\dot S_2 + \cdots + \dot S_p) + \dot S_1^2}{\dot S_1 \dot S}$$

Hence

$$F(\dot q) \leq 1 < \frac{3}{2}$$

  2) at $\dot q$, $\dot S_2 = \ldots = \dot S_p = 0$.

We have in this case $F(\dot q) = \frac{\dot S_1^2}{\dot S_1^2} = 1 < \frac{3}{2}$

  3) at $\dot q$, $\dot S_2 = \ldots = \dot S_k = b$, for some $k \geq 2$,
$$\dot S_\alpha = 0 \text{ for } \alpha > k.$$

It follows from Propositions (2.2.1) and (2.2.5) that $F(\dot q) \leq \frac{3}{2}$

  (4) at $\dot q$, $\dot S_2 = \ldots = \dot S_k = b$, for some $k \geq 2$, $k < p$,
$$0 < \dot S_{k+1} < b, \ldots, 0 < \dot S_{k+l} < b, \text{ for } 1 \leq l \leq p - k$$
$$\dot S_\alpha = 0 \text{ for } \alpha > k + l$$

In this case the same argument as in 1) gives

56

$$F(\dot{q}) \leqq \frac{\dot{S}_1 F(\dot{q})(\dot{S}_{k+1}+\cdots+\dot{S}_{pk+1})+\sum_{i<j}(\lambda_i-\lambda_j)^2 \sum_{\alpha=2}^{k}(\dot{a}_{ij}^\alpha)^2+\dot{S}_1^2}{\dot{S}_1\dot{S}}$$

It follows that

$$F(\dot{q}) \leqq \frac{2\sum_{i<j}(\lambda_i-\lambda_j)^2 \sum_{\alpha=2}^{k}(\dot{a}_{ij}^\alpha)^2+\dot{S}_1^2}{\dot{S}_1(\dot{S}_1+\cdots+\dot{S}_k)}.$$

Then we have $F(\dot{q}) \leqq \frac{3}{2}$ by Propositions (2.2.1) and (2.2.5). In summary, we have $F(\dot{q}) \leqq \frac{3}{2}$, therefore the inequality (71) is valid. The conditions for that the sign of equality holds are obvious by Propositions (2.2.1) and (2.2.5).

Lemma (2.2.7)[52]. If $0 < S_p < b$ for some α, $2 \leqq \alpha \leqq p$, then

$$F(\dot{q}) = \frac{\sum\limits_{\substack{\alpha,\beta \\ \beta \neq \alpha}} N(\dot{A}_\alpha\dot{A}_\beta-\dot{A}_\beta\dot{A}_\alpha)+ S_\alpha^2}{\dot{S}_\alpha\dot{S}}. \tag{72}$$

Proof. Without loss of generality, we assume that $\alpha = 2$, and $A_2 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

Then

$$\sum_{\alpha,\beta=1}^{p} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha=1}^{p} S_\alpha^2$$
$$= 4\sum_{i<j}(\lambda_i-\lambda_j)^2 \sum_{\alpha\neq2}^{p}(a_{ij}^\alpha)^2 + S_2^2 + \sum_{\alpha,\beta\neq2} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha=1}^{p} S_\alpha^2$$

We have

$$\frac{\partial F}{\partial \lambda_i}(\dot{q}) = 0. \tag{73}$$

Denote

$$\dot{b}_{ij} = \sum_{\beta\neq2}(\dot{a}_{ij}^\beta)^2.$$

From (73) we obtain

$$2\sum_{k\neq1}(\lambda_1-\lambda_k)\dot{b}_{1k} + \dot{S}_2\lambda_1 = \dot{S}\lambda_1 F(\dot{q})$$
$$2\sum_{k\neq2}(\lambda_2-\lambda_k)\dot{b}_{2k} + \dot{S}_2\lambda_2 = \dot{S}\lambda_2 F(\dot{q})$$
$$\vdots$$
$$2\sum_{k\neq n}(\lambda_n-\lambda_k)\dot{b}_{nk} + \dot{S}_2\lambda_2 = \dot{S}\lambda_n F(\dot{q})$$

Hence

$$2\sum_{k\neq1}(\lambda_1^2-\lambda_1\dot{\lambda}_k)\dot{b}_{1k} + \dot{S}_2\lambda_1^2 = \dot{S}\lambda_1^2 F(\dot{q})$$
$$2\sum_{k\neq2}(\lambda_2^2-\lambda_2\dot{\lambda}_k)\dot{b}_{2k} + \dot{S}_2\lambda_2^2 = \dot{S}\lambda_2^2 F(\dot{q})$$
$$\vdots$$
$$2\sum_{k\neq n}(\lambda_2^2-\lambda_n\dot{\lambda}_k)\dot{b}_{nk} + \dot{S}_2\lambda_n^2 = \dot{S}\lambda_n^2 F(\dot{q}).$$

Adding these inequalities we get

$$F(\dot{q}) = \frac{2\sum_{i<j}(\lambda_i - \lambda_j)^2 \sum_{\beta \neq 2}(\dot{a}_{ij}^\beta)^2 + \dot{S}_2^2}{\dot{S}_2 \dot{S}}.$$

$$= \frac{\sum_{\beta \neq 2} N(\dot{A}_2 \dot{A}_\beta - \dot{A}_\beta \dot{A}_2) + S_2^2}{\dot{S}_2 \dot{S}}.$$

Theorem (2.2.8)[52]. Let A_1, A_2,...,A_p be symmetric $(n \times n)$ −matrices $(p \geqq 2)$. Denote $S_{\alpha\beta} = \text{trace } A_\alpha^t A_\beta$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = S_1 + \dots + S_p$. Then

$$\sum_{\alpha,\beta} N(\dot{A}_\alpha \dot{A}_\beta - \dot{A}_\beta \dot{A}_\alpha) + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2} S^2 \tag{73}$$

and the equality holds if and only if one of the following conditions holds:

1) $A_1 = A_2 = \dots = A_p = 0$,

2) only two of the matrices A_1, A_2, ..., A_p are different from zero. Moreover, assuming $A_1 \neq 0, A_2 \neq 0, A_3 = \dots = A_p = 0$, then $S_1 = S_2$, and there exists an orthogonal $(n \times n)$ − matrix T such that

$$TA_1{}^tT = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline & 0 & 0 \end{array}\right) , TA_2{}^tT = \sqrt{\frac{S_1}{2}}\left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline & 0 & 0 \end{array}\right)$$

Proof. It suffices to consider the following two cases separately.

1) at $\dot{q}$, $\dot{S}_1 = \dot{S}_2 = \dots = \dot{S}_k = b$ for some k,

   $1 \leqq k \leqq p$, and $\dot{S}_\alpha$ S,=0 for $\alpha > k$.

By Propositions (2.2.1), (2.2.5) and (2.2.6) we have

$$F(\dot{q}) = \frac{\sum_{\beta=1}^k (\sum_\alpha N(\dot{A}_\alpha \dot{A}_\beta - \dot{A}_\beta \dot{A}_2) + \dot{S}_\beta^2)}{\dot{S}^2} \leqq \frac{\frac{3}{2} \dot{S} \sum_1^k S_\beta}{\dot{S}^2} = \frac{3}{2}$$

2) at $\dot{q}$, $\dot{S}_1 = \dots = \dot{S}_k = b$ for some k, $1 \leqq k < p$,

   $0 < \dot{S}_{k+1} < b, \dots, 0 < \dot{S}_{k+1} < b \text{ for } 1 \leq l \leq p - k$,

   $S_\alpha = 0 \text{ for } \alpha > k + l$.

By Lemma (2.2.7) we have

$$F(\dot{q}) = \frac{S F(q)(S_{k+1} + \dots + S_{k+1}) + \sum_{\beta=1}^k (\sum_{\alpha=1}^{k+1} N(\dot{A}_\alpha \dot{A}_\beta - \dot{A}_\beta \dot{A}_\alpha) + \dot{S}_\beta^2)}{(\dot{S}_1 + \dots + \dot{S}_{k+1})^2}$$

It follows that

$$F(\dot{q}) = \frac{\sum_{\beta=1}^k (\sum_{\alpha=1}^{k+1} N(\dot{A}_\alpha \dot{A}_\beta - \dot{A}_\beta \dot{A}_\alpha) + \dot{S}_\beta^2)}{(\dot{S}_1 + \dots + \dot{S}_{k+1})^2}$$

By Proposition (2.2.6) we have $F(\dot{q}) \leqq \frac{3}{2}$

In any case we have $F(\dot{q}) \leqq \frac{3}{2}$ Since F attains its maximum at $\dot{q}$ we get

$$\sum_\alpha N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_\alpha \leq \frac{3}{2} S_\alpha^2$$

The conditions for that the sign of equality holds are obvious,

Theorem (2.2.9)[52]. Let M be an n-dimensional compact minimal submanifold in $S^{n+p}, p \geq 2$. Then

$$\int_M \left(\frac{3}{2} ||\sigma||^2 - n \right) ||\sigma||^2 * 1 \geq 0, \tag{74}$$

where *1 denotes the volume element of M.

Proof. Applying the inequality (73), (74) and (63) and integrating over M we get (74).

Theorem (2.2.10)[52]. Let M be an n-dimensional compact minimal submanifold in S^{n+p}, $p \geq 2$. If $||\sigma||^2 \leq \frac{2}{3}n$ everywhere on M, then M is either a totally geodesic submanifold or a Veronese surface in S^4.

Our pinching constant $\frac{2}{3}$ n is better than the pinching constants of [46], [47], [49]. Theorem (2.2.10) can be rewritten in terms of the scalar curvature R of M as follows.

Proof. If $||\sigma||^2 \leqq \frac{2}{3}n$ everywhere, then either $||\sigma||^2 \equiv 0$ or $||\sigma||^2 \equiv \frac{2}{3}n$ by Theorem (2.2.9). When $||\sigma||^2 \equiv \frac{2}{3}n$ we have $h_{ijk}^\alpha = 0$ and we may assume that

$$H_{n+1} = \frac{\sqrt{\frac{2n}{2}}}{2} \left(\begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ \hline 0 & -1 & \\ \hline 0 & & 0 \end{array} \right), \qquad H_{n+2} = \frac{\sqrt{\frac{2n}{3}}}{2} \left(\begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ \hline 1 & 0 & \\ \hline 0 & & 0 \end{array} \right)$$

$H_\alpha = 0$ for $\alpha \geqq n + 3$

by Theorem (2.2.8). The same argument as in [46] shows that $dim\ (M) = 2$, and as M is compact 2 −dimensional minimal surface with $||\sigma||^2 = \frac{4}{3}$, it must be a Veronese surface in S^4.

Chapter 3
Rigidity Theorm and Log-Sobolev Inqualities

We apply the rigidity theorem for submanifolds and we discuss functional inequalities for μ like the Poincaré inequality, the log-Sobolev inequality or the Gaussian logarithmic isoperimetric inequality.

Section(3-1). Submanifolds with Parallel Mean Curvature in a Sphere

We generalize the famous Chern-do Carmo- Kobayashi Rigidity Theorem [53] for minimal submanifolds to general cases. Let M^n be an n −dimensional compact submanifold with parallel mean curvature in a unit sphere $S^{n+p}(1)$, and h its second fundamental form. It follows from the Gauss equations that the square norm of h is given by

$$S = n(n-1) - R + n^2 H^2,$$

where R and H are the scalar curvature and the mean curvature of M respectively. It was proved by Okumura [55, 56] that if the normal bundle of M is fiat, $n \geqq 3$, and $S < 2 + \frac{n^2}{n-1} H^2$, then M is totally umbilical. Yau [58] proved that if $p > 1$, and $S < \frac{n}{3+n^{\frac{1}{2}}-(p-1)^{-1}}$, then M lies in a totally geodesic $S^{n+1}(1)$. [57] improved Yau's result above. We proved that if $p > 1$, and $S < min\left\{\frac{2n}{1+n^{\frac{1}{2}}}, \frac{n}{2-(p-1)^{-1}}\right\}$, then M is a totally umbilical sphere. We shall show a rigidity theorem for submanifolds with parallel mean curvature in $S^{n+1}(1)$ by using a different method, which generalizes the main theorems in [53, 54], and also improves the results in [55,56,57].

Let M^n be an n-dimensional compact manifold immersed in an $(n + p)$ −dimensional unit sphere $S^{n+1}(1)$. We shall make use of the following convention on the range of indices:

$$1 \leqq A, B, C \dots \leqq n + p, 1 \leqq i, j, k, \dots \leqq n, n + 1 \leqq \alpha, \beta, \gamma, \dots \leqq n + p.$$

Choose a local orthonormal frame field $\{e_A\}$ in $S^{n+1}(1)$ such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the dual frame field and the connection l-forms of N respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1}$$

$$h = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_j \otimes \omega_j \otimes e_\alpha, \qquad \xi = \frac{1}{n} \sum_{\alpha,i} h_{ij}^\alpha e_\alpha \tag{2}$$

$$R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{3}$$

$$R_{\alpha\beta kl} = \sum_\alpha (h_{ik}^\alpha h_{il}^\alpha - h_{il}^\alpha h_{ik}^\alpha), \tag{4}$$

where h, ξ, R_{ijkl} and $R_{\alpha\beta kl}$ are the second fundamental form, the mean curvature vector, the curvature tensor and the normal curvature tensor of M respectively. We set

$$S = \|h\|^2, \; H = \|\xi\|, \; H_\alpha = (h_{ij}^\alpha)_{n\times n} \tag{5}$$

Definition (3.1.1)[59]. M is called a submanifold with parallel mean curvature if ξ is parallel in the normal bundle of M. In particular, M is called minimal if H vanishes identically.

We assume that M is a submanifold with paratlel mean curvature $(H\neq 0)$. We choose e_{n+1} such that $e_{n+1}//\xi$, $tr\, H_{n+1} = nH$ and $tr\, H_\beta = 0, n + 2 \leqq n + p$ Set

$$S_H = \sum_{i,j}(h_{ij}^{n+1})^2, \;\; S_1 = \sum_{\substack{i,j \\ \beta\neq n+1}} (h_{ij}^{n+1})^2 \tag{6}$$

We have the following proposition immediately from the definition.

Proposition (3.1.2)[59]. M is a submanifold with parallel mean curvature in $S^{n+p}(1)$ if and only if either $H\equiv 0$, or H is constant and $H_{n+1}\, H_\alpha = H_\alpha H_{n+1}$, for all α.

We denote the covariant derivatives of h_{ij}^α by h_{ijk}^α and h_{ijkl}^α, etc. The Laplacian Δh_{ij}^α of h is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. Following [58], we have

$$\Delta h_{ij}^{n+1} = \sum_{k,m} h_{mk}^{n+1}\, R_{mijk} + \sum_{k,m} h_{im}^{n+1}\, R_{mkjk}, \tag{7}$$

$$\Delta h_{ij}^\beta = \sum_{k,m} h_{mk}^\beta\, R_{mijk} + \sum_{k,m} h_{im}^\beta\, R_{mkjk} + \sum_{\substack{k,m \\ \alpha\neq n+1}} h_{ki}^\alpha\, R_{\alpha\beta jk}, \;\; \beta \neq n + 1 \tag{8}$$

By using Lagrange multiplier method, we have the following

Lemma (3.1.3)[59]. *Let $a_1, \dots, a_n$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = 0$. Then*

$$\left|\sum_i a_i^3\right| \leqq (n-2)[n(n-1)]^{-\frac{1}{2}}a^{\frac{1}{2}}, \tag{9}$$

and the equality holds if and only if at least n - 1 numbers of the a_i's are same with each other.

For a matrix $A = (a_{ij})_{n\times n}$, we denote by $N(A)$ the square norm of A, i,e., $N(A) = tr\,(A^t A) = \sum_{i,j} a_{ij}^2$. Then $N(A) = N(TA^t T)$, for each orthogonal $(n\times n)$-matrix T.

Lemma(3.1.4)[59]. (See [53,54]). Let $A_{n+1}, \dots, A_{n+p}$ be symmetric $(n\times n)$-matrices, Set $S_{\alpha\beta} = tr\,(A_\alpha^t A_\beta), S_\alpha = S_{\alpha\alpha} = N(A_\alpha), S = \sum_\alpha S_\alpha$. Then

$$\sum_{\alpha,\beta} N(A_\alpha A_\alpha - A_\alpha A_\alpha) + \sum_{\alpha,\beta} s_{\alpha\beta}^2 \leq \left(1 + \frac{1}{2}\, sgn\,(p-1)\right)S^2 \tag{10}$$

where sgn $(\cdot)$ is the standard sign function. The equality holds if and only if at most two matrices A_α and A_β are not zero and these two matrices can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}_\alpha$ and $\tilde{A}_\beta$ respectively, where

$$\tilde{A}_\alpha = \begin{pmatrix} \begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array} \end{pmatrix}, \quad \tilde{A}_\beta = \begin{pmatrix} \begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline & 0 & 0 \end{array} \end{pmatrix}.$$

First of all, we define our pinching constants as follows

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)H^4}, \tag{11}$$

$$C(n, p, H) = \begin{cases} \alpha(n, H), & for\ p = 1, or\ p = 2\ and\ H \neq 0, \\ \min\left\{\alpha(n, H), \frac{1}{3}(2n + 5nH^2)\right\}, & for\ p \geqq 3, or\ p = 2\ and\ H = 0. \end{cases} \tag{12}$$

Theorem (3.1.5)[59]. Let M^n be a compact submanifold with parallel mean curvature $(H \neq 0)$ in $S^{n+p}(1)$. If $S \leqq \alpha(n, H)$, then either M is pseudo-umbilical, or $S = S_H = \alpha(n, H)$ and M is the isoparametric hypersurface $S^{n-1}\left(\dfrac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\dfrac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right)$ in a totally geodesic $S^{n+1}(1)$, where $\lambda(n, H) = H + \sqrt{\dfrac{\alpha(n,H) - nH^2}{n(n-1)}}$.

Proof. By (7) and Gauss equations, we have

$$\frac{1}{2}\Delta S_H = \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j} h_{ij}^{n+1}\Delta h_{ij}^{n+1}$$

$$= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + \sum_{i,j,k,m} h_{ij}^{n+1}h_{mk}^{n+1}\left[\delta_{mj}\delta_{ik} - \delta_{mk}\delta_{ij} + \sum_\alpha\left(h_{mj}^\alpha h_{ik}^\alpha - h_{mk}^\alpha h_{ij}^\alpha\right)\right]$$

$$+ \sum_{i,j,k,m} h_{ij}^{n+1}h_{im}^{n+1}\left[\delta_{mj}\delta_{kk} - \delta_{mk}\delta_{jk} + \sum_\alpha\left(h_{mj}^\alpha h_{kk}^\alpha - h_{mk}^\alpha h_{jk}^\alpha\right)\right]$$

$$= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + n\sum_{i,j}(h_{ij}^{n+1})^2 - (\sum_{i,j,k}(h_{ij}^{n+1})^2)^2 - n^2 H^2$$

$$+ nH \sum_{i,j,k} h_{ij}^{n+1}\ h_{jk}^{n+1}h_{kl}^{n+1} - \sum_{i,j,k}(\sum_{i,j}(h_{ijk}^{n+1} - H\delta_{ij})h_{ij}^\beta)^2 \tag{13}$$

Let $\{e_i\}$ be a frame diagonalizing the matrix H_{n+1} such that $h_{ij}^{n+1} = \lambda_i^{n+1}\delta_{ij}$ for all i, j Set

$$f_k = \sum_i (\lambda_i^{n+1})^k, \tag{14}$$

$$\mu_i^{n+1} = H - \lambda_i^{n+1}, i = 1, 2, \dots, n \tag{15}$$

$$B_k = \sum_i (\mu_i^{n+1})^k \tag{16}$$

Then

$$B_1 = 0, B_2 = S_H - nH^2, \tag{17}$$

$$B_3 = 3HS_H - 2nH^3 - f_3. \tag{18}$$

From (12), (16), (17) and Lemma (3.1.3), we get

$$\frac{1}{2}\Delta S_H = \sum_{i,j,k}(h_{ijk}^{n+1})^2 + nH_H - S_H^2 - n^2H^2 + nHf_3 - \sum_{\beta \neq n+1}\left(\sum_i \mu_i^{n+1}h_{ii}^\beta\right)^2$$

$$\geqq \sum_{i,j,k}(h_{ijk}^{n+1})^2 + nH_H - S_H^2 - n^2H^2 + nH\left[3HS_H - 2nH^3 - \frac{n-2}{\sqrt{n(n-1)}}B_2^{\frac{3}{2}}\right] - B_2 S_1$$

$$= \sum_{i,j,k}(h_{ijk}^{n+1})^2 + B_2\left[n + 2nH^2 - S - \frac{n(n-2)}{\sqrt{n(n-1)}}H(S_H - nH^2)^{\frac{1}{2}}\right]$$

$$\geqq \sum_{i,j,k}(h_{ijk}^{n+1})^2 - B_2\left[\sqrt{S - nH^2} + \frac{n(n-2)}{2\sqrt{n(n-1)}}H + \frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2 + 4n(n-1)^2}\right]$$

$$\times \left[\sqrt{S - nH^2} + \frac{n(n-2)}{2\sqrt{n(n-1)}}H - \frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2 + 4n(n-1)^2}\right] \tag{19}$$

On the other hand, the assumption

$$S \leqq \alpha(n, H)$$

is equivalent to

$$\sqrt{S - nH^2} + \frac{n(n-2)}{2\sqrt{n(n-1)}}H - \frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2 + 4n(n-1)^2} \leqq 0, \tag{20}$$

which together with (19) shows that S_H is subharmonic on M. By the Hopf maximum principle, we see that S_H must be a constant. This together with (19) and (20) force that

$$B_2(S_H - nH^2)^{\frac{1}{2}} = B_H(S - nH^2)^{\frac{1}{2}}, \tag{21}$$

$$B_2\left[\sqrt{S - nH^2} + \frac{n(n-2)}{2\sqrt{n(n-1)}}H - \frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2 + 4n(n-1)^2}\right] = 0. \tag{22}$$

If $S_H = nH^2$, then M is a pseudo-umbilical submanifold.

If $S = S_H$ and

$$\sqrt{S - nH^2} + \frac{n(n-2)}{2\sqrt{n(n-1)}}H - \frac{1}{2(n-1)}\sqrt{n^3(n-1)H^2 + 4n(n-1)^2} = 0,$$

then $S = S_H = \alpha(n, H)$, and $S_I = 0$. Consequently M is a hypersurface in a totally geodesic $S^{n+1}(1)$. From (19) we have

$$B_3 = \frac{n-2}{\sqrt{n(n-1)}}B_2^{\frac{3}{2}}. \tag{23}$$

It follows from Lemma (3.1.3) that at least $n-1$ numbers of $\{\mu_i^{n+1}\}$ are same with each other. Without loss of generality, we assume that $\mu_i^{n+1} = \mu$, $k = 1, 2, \ldots, n-1$, and $\mu_n^{n+1} = \bar{\mu}$. Then

$$(n-1)\mu + \mu = 0, \tag{24}$$

$$(n-1)\bar{\mu}^2 + \bar{\mu}^2 = \alpha(n, H) - nH^2 \tag{25}$$

Substituting the solution of equations (24) and (25) with condition $(n-1)\mu^3 + \bar{\mu}^3 > 0$ into (15), we get

$$\lambda_i^{n+1} = H + \sqrt{\frac{\alpha(n,H) - nH^2}{n(n-1)}} , i = 1,2, \dots, n-1$$

$$\lambda_i^{n+1} = H - \sqrt{\frac{(n-1)(\alpha(n,H) - nH^2)}{n}} \tag{26}$$

Hence M is the isoparametric hypersurface

$$S^{n-1}\left(\frac{1}{\sqrt{1 + \lambda^2(n, H)}}\right) \times S^1\left(\frac{\lambda(n, H)}{\sqrt{1 + \lambda^2(n, H)}}\right) \ in \ S^{n+1}(1),$$

where $\lambda(n, H) = H + \sqrt{\frac{\alpha(n,H) - nH^2}{n(n-1)}}$.

Corollary(3.1.6)[59]. Let M^n be a compact hypersurface with constant mean curvature $(H \neq 0)$ in $S^{n+1}(1)$. $if \ S \leqq \alpha(n, H)$, then either M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$

or the isoparametric hypersurface $S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right)$.

If M is a pseudo-umbilical submanifold with nonzero parallel mean curvature and $p \geqq 2$, it is to see from a theorem of [58] that M is a minimal submanifold in

$S^{n+p-1}\left(\frac{1}{\sqrt{1+H^2}}\right)$ with second fundamental form H_α, $\alpha = n + 2, \dots, n + p$. Hence, we have

the following

Theorem (3.1.7)[59]. Let M^n be a compact submanifold with parallel mean curvature $(H \neq 0)$ in $S^{n+p}(1)$. if $S \leqq \alpha(n, H)$, then either M is a totally umbilical sphere, a isoparametric hyperurface in a totally geodesic $S^{n+1}(1)$, or a minimal submanifold in a totally umbilical $S^{n+p-1}\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Theorem (3.1.8)[59]. Let M^n be a compact submanifold with parallel mean curvature in $S^{n+p}(1)$,. If $S \leqq C(n, p, H)$, then either M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$ the

isoparametric hypersurface $S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right)$. in a totally geodesic

$S^{n+1}(1)$,, one of ~ne Clifford minimal hypersutfaces $S^k\left(\sqrt{\frac{k}{n}}\right) \times S^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)$, $k = 1,2\ldots n-1$, in $S^{n+1}(1)$, the Clifford minimal surface

$$S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right) \ in\ S^3 Sn\left(\frac{1}{\sqrt{1+H^2}}\right),$$

or the Veronese surface in $S^4\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Proof. (i) If H = 0, M is minimal. The assertion follows from the main theorems in [53, 54].

(ii) If H ≠ 0 and p = 1, we know from Corollary (2.3.6) that either M is the hypersphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$ or the isoparametric hypersurface $S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right)$.

(iii) If H ≠ 0 and p ≧ 2, it is straightforward to see from (8), Proposition (3.1.2) and Lemma (3.1.4) that

$$\frac{1}{2}\Delta S_1 = \sum_{\substack{i,j,k \\ \beta \neq n+1}} (h_{ijk}^{\beta}) + \sum_{\beta \neq n+1} tr(H_{n+1}H_{\beta})^2 - \sum_{\beta \neq n+1} [tr(H_{n+1}H_{\beta})]^2$$

$$+nH\sum_{\beta \neq n+1} tr(H_{n+1}H_{\beta}^2) - \sum_{\beta \neq n+1} tr(H_{n+1}^2 H_{\beta}^2) + nS_1$$

$$-\sum_{\alpha,\beta \neq n+1} tr(H_{\alpha}H_{\beta} - H_{\beta}H_{\alpha})^2 - \sum_{\beta \neq n+1} [tr(H_{n+1}H_{\beta})]^2 \tag{27}$$

$$\geq \sum_{\substack{i,j,k \\ \beta \neq n+1}} (h_{ijk}^{\beta})^2 + nH\sum_{\beta \neq n+1} tr(H_{n+1}H_{\beta}^2) - \sum_{\beta \neq n+1} [tr(H_{n+1}H_{\beta})]^2$$

$$+ nS_1 - \left(1 + \frac{1}{2} sgn(p-2)\right)S_1^2.$$

We know from Theorem 1that either M is pseudo-umbilical or the isoparametric hypersurface $S^{n-1}\left(\frac{1}{\sqrt{1+\lambda^2(n,H)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{1+\lambda^2(n,H)}}\right)$ in a totally geodesic $S^{n+1}(1)$, If M is pseudo-umbilical, then (27) becomes

$$\frac{1}{2}\Delta S_1 = \sum_{\substack{i,j,k \\ \beta \neq n+1}} (h_{ijk}^{\beta})^2 + (n + nH^2)S_1 - \left(1 + \frac{1}{2} sgn(p-2)\right)S_1^2$$

$$\geq \sum_{\substack{i,j,k \\ \beta \neq n+1}} (h_{ijk}^{\beta})^2 + S_1\left[n + nH^2 - \left(1 + \frac{1}{2} sgn(p-2)\right)(S - nH^2)\right] \geq 0. \tag{28}$$

This shows that S_1 is a constant, and the inequalities above become equalities. It is not hard to see that

$$S_1\left[n + nH^2 - \left(1 + \frac{1}{2} sgn(p-2)\right)(S - nH^2)\right] = 0 \tag{29}$$

65

If $S_1 = 0$, then M lies in a totally geodesic sphere $S^{n+1}(1)$ and M is the totally umbilical sphere $S^n\left(\frac{1}{\sqrt{1+H^2}}\right)$.

If $n + nH^2 - \left(1 + \frac{1}{2} sgn(p-2)\right)(S - nH^2) = 0$, namely

$$S = \left(n + \frac{n}{3} sgn(p-2)\right)(1 - H^2) + nH^2 \quad, \tag{30}$$

then $h_{ijk}^\alpha = 0$ and

$$\sum_{\alpha,\beta\neq n+1} tr(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\beta\neq n+1}\left[tr(H_{n+1}H_\beta)\right]^2 = \left(1 + \frac{1}{2} sgn(p-2)\right)S_1^2.$$

By Lemma (3.1.4) and the same argument as in [71], we conclude that $n = 2$, and the second fundamental form h can be written as follows

(a) $n = 2$ and $p = 2$, $H_3 = H\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$, $H_4 = \sqrt{1 + H^4}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, or

(b) $n = 2$ and $p \geqq 3$, $H_3 = H\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$, $H_4 = \sqrt{\frac{1+H^2}{3}}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, $H_5 = \sqrt{\frac{1+H^2}{3}}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$,

$H_\beta = 0$, $\beta \geqq 6$.

By Theorem (3.1.7), we know that M is a minimal submanifold in $S^{1+p}\left(\frac{1}{\sqrt{1+H^2}}\right)$ with second fundamental form $H_4, \ldots, H_{2+p}$. Therefore, M is the Clifford minimal surface

$S^1\left(\frac{1}{\sqrt{2(1+H^2)}}\right) \times S^1\left(\frac{\lambda(n,H)}{\sqrt{2(1+H^2)}}\right)$ in $S^3\left(\frac{1}{\sqrt{1+H^2}}\right)$ or the Veronese surface in $S^4\left(\frac{1}{\sqrt{1+H^2}}\right)$.

Section(3.2). Sub Elliptic Operators Satisfying a Generalized Curvature Dimension Inequality

Logarithmic Sobolev inequalities, introduced and studied by L. Gross [136], are a major tool for the analysis of finite- or infinite-dimensional spaces, see for instance [118]. The celebrated Bakry–Émerycriterion [122] which is based on the so-called Γ_2 calculus for diffusion operator sprovides a powerful way to establish such inequalities. However this criterion requires some ellipticity property from the diffusion operator and fails to hold even for simple subelliptic diffusion operators like the sub-Laplacian on the Heisenberg group (see [141]). However in the past few years, most of these examples have in common the property that the subelliptic diffusion operator satisfies the generalized curvature dimension inequality that was introduced in [125] in an abstract setting. As we will see, this curvature dimension inequality may also be used to prove the Poincaré inequality, the log-Sobolev inequality or the Gaussian logarithmic isoperimetric inequality for the invariant measure of a subelliptic diffusion operator in some interesting new situations.

M will be a C^∞ connected finite-dimensional manifold endowed with a smooth measure μ and a second-order diffusion operator L on M, locally subelliptic in the sense of [132] (see also [140]), satisfying $L1 = 0$,

$$\int_M fLg \, d\mu = \int_M gLf d\mu, \qquad \int_M fLf \, d\mu \leq 0,$$

for every $f, g \in C_0^\infty(M)$. We indicate with $\Gamma(f) := \Gamma(f, f)$ the carré du champ, that is the quadratic differential form defined by

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \qquad f, g \in C_0^\infty(M). \tag{31}$$

An absolutely continuous curve $\gamma : [0, T] \to M$ is said to be subunit for the operator L iffor every smooth function $f : M \to R$ we have $\left|\frac{d}{dt} f(\gamma(t))\right| \sqrt{(\Gamma f)(\gamma(t))}$. We then define thesubunit length of γ as $\ell_s(\gamma) = T$. Given $x, y \in M$, we indicate with

$$S(x, y) = \{\gamma : [0, T] \to M \mid \gamma \text{ is subunit for } L, \quad \gamma(0) = x, \gamma(T) = y\}.$$

we assume that

$$S(x, y) \neq \emptyset, \qquad \text{for every } x, y \in M.$$

Under such assumption it is easy to verify that

$$d(x, y) = \inf \{\ell_s(\gamma) \mid \gamma \in S(x, y), \tag{32}$$

defines a true distance on M. Furthermore, it is known that

$$d(x, y) = \sup\{|f(x) - f(y)| \mid f \in C^\infty(M), \ \|\Gamma(f)\|_\infty \leq 1\}, \quad x, y \in M. \tag{33}$$

We assume that the metric space (M, d) is complete.

In addition to the differential form (84), we assume that M is endowed with another smooth symmetric bilinear differential form, indicated with Γ^Z, satisfying for $f, g \in C^\infty(M)$

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),$$

and $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$.

We make the following assumptions:

(H.1) There exists an increasing sequence $h_k \in C_0^\infty(M)$ such that $h_k \nearrow 1$ on M, and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0, \quad \text{as } k \to \infty$$

(H.2) For any $f \in C^\infty(M)$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

As it has been proved in [121], the assumption (H.1) which is of technical nature, implies in particular that L is essentially self-adjoint on $C_0^\infty(M)$. The assumption (H.2) is more subtle and is crucial for the validity of most the subsequent results: It is discussed in details in [121] in several geometric examples. Let us consider

$$\Gamma_2(f, g) = \frac{1}{2}[L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)], \tag{34}$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)] \qquad (35)$$

As for Γ and Γ^Z, we will freely use the notations $\Gamma_2(f) = \Gamma_2(f,f), \Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

Definition (3.2.1)[161]. We say that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ if there exist constants $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0, and\ 0 < d \leq \infty$ such that the inequality

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f)$$

holds for every $f \in C^\infty(M)$ and every $\nu > 0$, where Γ_2 and Γ_2^Z are defined by (34) and (35).

Theorem (3.2.2)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ with $\rho_1 > 0, \rho_2 > 0$ and $\kappa \geq 0$.

(i) The measure μ is finite and the following Poincaré inequality holds

$$\int_M f^2 d\mu - \left(\int_M f\ d\mu\right)^2 \leq \frac{\kappa + \rho_2}{\rho_1\rho_2} \int_M \Gamma(f)d\mu \qquad f \in D(L).$$

(ii) *If μ is a probability measure, that is $\mu(M) = 1$, then for $f \in C_0(M)$,*

$$\int_M f^2 \ln f^2 d\mu - \int_M f^2 d\mu \ln \int_M f^2 d\mu$$

$$\leq \frac{2(\kappa + \rho_2)}{\rho_1\rho_2}\left(\int_M \Gamma(f)d\mu + \frac{\kappa + \rho_2}{\rho_1}\int_M \Gamma^Z(f)d\mu\right).$$

Theorem (3.2.3)[161]. Assume that the measure μ is a probability measure and that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$. Assume moreover that

$$\int_M e^{\lambda d^2(x_0,x)}\ d\mu(x) < +\infty,$$

for some $x_0 \in M$ and $\lambda > \frac{\overline{\rho_1}}{2}$, then there is a constant $\rho_0 > 0$ such that for every function $f \in C_0^\infty(M)$,

$$\int_M f^2 \ln f^2 d\mu - \int_M f^2 d\mu\ \ln \int_M f^2 d\mu \leq \frac{2}{\rho_0}\int_M \Gamma(f)d\mu.$$

Adapting some methods of Bobkov, Gentil and Ledoux [127], we prove an analogue of an Otto–Villani theorem [148]. We recall that L^2-Wasserstein distance of two measures $\nu_1\ and\ \nu_2$ on M is defined by

$$W_2(v_1, v_2)^2 = \inf_{\Pi} \int_M d^2(x, y) d\Pi(x, y)$$

where the infimum is taken over all coupling of v_1 and v_2 that is on all probability measures Π on $M \times M$ whose marginals are respectively v_1 and v_2.

Theorem (3.2.4)[161].. Assume that the measure μ is a probability measure and that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0, \kappa \leq 0$. If the quadratic transportation cost inequality

$$W_2(\mu, v)^2 \leq cEnt_\mu\left(\frac{dv}{d\mu}\right) \tag{36}$$

is satisfied for every absolutely continuous probability measure v with a constant $c < \frac{2}{\rho_1}$, then the following modified log-Sobolev inequality

$$Ent_\mu(f) \leq C_1 \int_M \frac{\Gamma(f)}{f} d\mu + C_2 \int_M \frac{\Gamma^Z(f)}{f} d\mu$$

holds for some constants C_1 and C_2 depending only on c, ρ_1, κ, ρ_2.

Theorem (3.2.5)[161]. Assume that the measure μ is a probability measure, that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$ and that μ satisfies the log-Sobolev inequality:

$$\int_M f^2 \ln f^2 d\mu - \int_M f^2 d\mu \ln \int_M f^2 d\mu \leq \frac{2}{\rho_0} \int_M \Gamma(f) d\mu, \qquad f \in C_0^\infty(M) \tag{37}$$

for all smooth functions $f \in C_0^\infty(M)$. Let A be a set of the manifold M which has a finite perimeter $P(A)$ and such that $0 \leq \mu(A) \leq \frac{1}{2}$, then

$$P(A) \geq \frac{\ln 2}{4\left(3 + \frac{2\kappa}{\rho_2}\right)} \min\left(\sqrt{\rho_0}, \frac{\rho_0}{\sqrt{\bar{\rho}_1}}\right) \mu(A) \left(\ln\frac{1}{\mu(A)}\right)^{\frac{1}{2}}.$$

Let us now turn to the fundamental question of examples to which our above results apply. See [125].

A first observation is that if M is an n-dimensional complete Riemannian manifold and L is the Laplace–Beltrami operator, the assumptions (H.1) and (H.2) hold trivially with Γ^Z =0. Indeed, the assumption (H.1) is satisfied as a consequence of the completeness of M and the assumption (H.2) is trivially satisfied. In this example, the generalized curvature dimension inequality $CD(\rho_1, 1, 0, n)$ is implied by (and it is in fact equivalent to) the assumption that the Ricci curvature of M satisfies the lower bound $Ric \geq \rho_1$.

Besides Laplace–Beltrami operators on complete Riemannian manifolds with Ricci curvature bounded from below, awide class of examples is given by sub-Laplacians on Sasakian manifolds. Let M be a complete strictly pseudo convex CR Sasakian manifold with real dimension $2n + 1$. Let θ be a pseudo-hermitian form on M with respect to which the Levi form is positive definite. The kernel of θ determines a horizontal bundleH. Denote now by T the Reeb vector field on M, i.e., the characteristic direction of θ. We denote by ∇ the Tanaka–Webster connection of M. We recall that the CR manifold (M, θ) is called Sasakian if the pseudo-hermitian torsion of ∇ vanishes, in the sense that $T(T, X) = 0$, for every $X \in H$. For instance the standard CR structures on the Heisenberg group H_{2n+1} and the sphere S^{2n+1} are Sasakian. On CR manifolds, there is a canonical subelliptic diffusion operator which is called the CR sub-Laplacian. It plays the same role in CR geometry as the Laplace–Beltrami operator does in Riemannian geometry. We have the following result that shows the relevance of the generalized curvature dimension inequality.

Proposition (3.2.6)[161]. (See [125].) Let (M, θ) be a CR manifold with real dimension $2n + 1$ and vanishing Tanaka–Webster torsion, i.e., a Sasakian manifold. If for every $x \in M$ the Tanaka–Webster Ricci tensor satisfies the bound

$$Ric_x(v, v) \geq \rho_1 |v|^2,$$

for every horizontal vector $v \in H_x$, then, for the CR sub-Laplacian of M, the curvature dimension inequality $CD\left(\rho_1, \frac{d}{4}, 1, d\right)$ holds with $d = 2n$ and $\Gamma^Z(f) = (Tf)^2$.

In addition to sub-Laplacians on Heisenberg groups, more generally, the sub-Laplacian on any Carnot group of step 2 has been shown to satisfy the generalized curvature dimension inequality $CD(0, \rho_2, \kappa, d)$, for some values of the parameters ρ_2 and κ. Let us mention that recently [126], study sub-Laplacians in infinite-dimensional Heisenberg type groups and show that a generalized curvature dimension inequality is satisfied with $d = +\infty$. In that case the assumption (H.1) is of course not satisfied but is somehow replaced by the existence of nice and uniform finite-dimensional approximations, so that with suitable modifications the results of this topic may be used. For infinite-dimensional situations, see [139].

Another interesting example, which has been highlighted by several works is given by the Grushin operator on R^{2n}. It is defined by

$$L = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\|x\|^2}{2} \frac{\partial^2}{\partial y_i^2} \right)$$

Where $\|x\|^2 = x_1^2 + \cdots + x_n^2$ for $x = (x_1, \ldots, x_n) \in R^n$. This operator admits the Lebesgue measure λ as invariant and symmetric measure. If we set $X_i = \frac{\partial}{\partial x_i}$, $Y_{i,j} = x_j \frac{\partial}{\partial y_i}$ and $Z_i = \frac{\partial}{\partial y_i}$, we can write this operator as

$$L = \sum_{i=1}^{n} X_i^2 + \sum_{i,j=1}^{n} Y_{i,j}^2 = -\sum_{i=1}^{n} X_i^* X_i - \sum_{i=1}^{n} Y_{i,j}^* Y_{i,j}.$$

The only non-zero Lie bracket relations are

$$[X_i, Y_{i,j}] = Z_j \qquad for\ 1 \leq i,j \leq n.$$

This algebra structure is then exactly the one of a Carnot group of step 2 and the criterion $CD(0, \rho_2, \kappa, n + n^2)$ therefore holds with $\Gamma^Z(f, f) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y_i}\right)^2$ and some constant ρ_2.

Alsoit is easy to see that assumptions (H.1) and (H.2) are satisfied in that case. Let us however observe that more general Grushin operators are considered in [160], and that they cannot be handled at the moment with our methods, since their Lie algebra correspond to a Carnot group of step higher than 2. We mention that some close results are obtained in [137] for Fokker–Planck type operators. In those examples, that typically does not satisfy the generalized curvature dimension inequality studied, the hypoellipticity of the operator stems from its first order part; a situation radically different from the examples discussed above.

We assume that the operator L satisfies the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 > 0, \rho_1 > 0, \kappa \geq 0$.

The main tool to prove the fore mentioned theorems, is the heat semigroup $P_t = e^{tL}$, which is defined using the spectral theorem. Since L satisfies the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$, this semigroup is stochastically complete (see [125]), i.e. $P_t 1 = 1$. Moreover, thanks to the hypoellipticity of L, for $f \in L^p(M), 1 \leq p \leq \infty$, the function $(t, x) \to P_t f(x)$ is smooth on $M \times (0, \infty)$ and

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y)$$

where $p(x, y, t) = p(y, x, t) > 0$ is the so-called heat kernel associated to P_t.

Henceforth, we denote

$$C_b^\infty(M) = C^\infty(M) \cap L^\infty(M).$$

For $\varepsilon > 0$ we denote by A_ε the set of functions $f \in C_b^\infty(M)$ such that

$$f = g + \varepsilon,$$

for some $\varepsilon > 0$ and some $g \in C_b^\infty(M)$, $g \geq 0$, such that $g, \sqrt{\Gamma(g)}, \sqrt{\Gamma^Z(g)} \in L^2(M)$. As show in [125], this set is stable under the action of P_t, i.e., if $f \in A_\varepsilon$, then $P_t f \in A_\varepsilon$.

Our goal is to prove Theorem (3.2.2). In that direction, we first establish a useful gradient bound for P_t.

Proposition (3.2.7)[161]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$. For $x \in M, t \geq 0$ one has

$$(P_t f)\Gamma(\ln P_t f) + \frac{\kappa + \rho_2}{\rho_1}(P_t f)\Gamma^Z(\ln P_t f)$$

$$\leq e^{-2\frac{\rho_1\rho_2 t}{\kappa+\rho_2}}\left(P_t(f\Gamma(\ln f)) + \frac{\kappa+\rho_2}{\rho_1}P_t(f\Gamma^Z(\ln f))\right)$$

Proof. Let us fix $T > 0$ once time for all in the following proof. Given a function $f \in A_\varepsilon$, for $0 \leq t \leq T$ we introduce the entropy functionals

$$\emptyset_1(x,t) = (P_{T-t}f)(x)\Gamma (\ln P_{T-t}f)(x),$$
$$\emptyset_2(x,t) = (P_{T-t}f)(x)\Gamma^Z(\ln P_{T-t}f)(x),$$

which are defined on $M \times [0,T]$. As it has been proved in [125], a direct computation shows that

$$L\emptyset_1 + \frac{\partial\emptyset_1}{\partial t} = 2(P_{T-t}f)\Gamma 2(\ln P_{T-t}f),$$

and

$$L\emptyset_2 + \frac{\partial\emptyset_2}{\partial t} = 2(P_{T-t}f)\Gamma_2^Z(\ln P_{T-t}f).$$

Let us observe that for the second equality the hypothesis (H.2) is used in a crucial way.

Consider now the function

$$\emptyset(x,t) = a(t)\emptyset_1(x,t) + b(t)\emptyset_2(x,t)$$
$$= a(t)(P_{T-t}f)(x)\Gamma(\ln P_{T-t}f)(x) + b(t)(P_{T-t}f)(x)\Gamma^Z(\ln P_{T-t}f)(x),$$

where a and b are two non-negative functions that will be chosen later. Applying the generalized curvature dimension inequality $CD(\rho_1,\rho_2,\kappa,\infty)$, we obtain

$$L\emptyset + \frac{\partial\emptyset}{\partial t} = a'(P_{T-t}f)\Gamma(\ln P_{T-t}f) + b'(P_{T-t}f)\Gamma^Z(\ln P_{T-t}f)$$
$$+ 2a(P_{T-t}f)\Gamma_2(\ln P_{T-t}f) + 2b(P_{T-t}f)\Gamma_2^Z(\ln P_{T-t}f)$$
$$\geq \left(a' + 2\rho_1 a - 2\kappa\frac{a^2}{b}\right)(P_{T-t}f)\Gamma(\ln P_{T-t}f)$$
$$+ (b' + 2\rho_2 a)(P_{T-t}f)\Gamma^Z(\ln P_{T-t}f).$$

Let us now chose

$$b(t) = e^{-2\frac{\rho_1\rho_2 t}{\kappa+\rho_2}}$$

and

$$a(t) = -\frac{b'(t)}{2\rho_2},$$

so that

$$b' + 2\rho_2 a = 0$$

and

$$a' + 2\rho_1 a - 2\kappa\frac{a^2}{b} = 0.$$

With this choice, we get

$$LØ + \frac{\partial Ø}{\partial t} \geq 0,$$

and therefore from a comparison theorem for parabolic partial differential equations we have

$$P_T(Ø(\cdot, T))(x) \geq Ø(x, 0).$$

Since,

$$Ø(x, 0) = a(0)(P_T f)(x)\Gamma (lnP_T f)(x) + b(0)(P_T f)(x)\Gamma^Z(lnP_T f)(x)$$

And

$$P_T(Ø(\cdot, T))(x) = a(T)P_T(f\Gamma(lnf))(x) + b(T)P_T(f\Gamma^Z(lnf))(x),$$

A similar proof as above also provides the following:

Proposition(3.2.8)[161]. Let $f \in L^2(M)$ such that $f \in C^\infty(M)$ and $\Gamma(f), \Gamma^Z(f) \in L^1(M)$. $For\ x \in M, t \geq 0$ one has

$$\Gamma(P_t f) + \frac{\kappa + \rho_2}{\rho_1}\Gamma^Z(Ptf) \leq e^{\frac{-2\rho_1\rho_2 t}{\kappa+\rho_2}}\left(P_t(\Gamma(f)) + \frac{\kappa + \rho_2}{\rho_1}P_t(\Gamma^Z(f))\right).$$

Proof. We introduce

$$Ø_1(x, t) = \Gamma(P_{T-t}f)(x),$$
$$Ø_2(x, t) = \Gamma^Z(P_{T-t}f)(x),$$

and observe that

$$LØ_1 + \frac{\partial Ø_1}{\partial t} = 2\Gamma_2(P_{T-t}f),$$

and

$$LØ_2 + \frac{\partial Ø_2}{\partial t} = 2\Gamma_2^Z(P_{T-t}f),$$

The conclusion is then reached by following the lines of the proof of Proposition (3.2.7).

A first interesting consequence of the above functional inequalities is the fact that $\rho_1 > 0$ implies that the invariant measure is finite.

Corollary(3.2.9)[161]. The measure μ is finite, i.e. $\mu(M) < +\infty$ and for every $x \in M, f \in L^2(M)$,

$$P_t f(x) \to_{t \to +\infty} \frac{1}{\mu(M)} \int_M f\ d\mu.$$

Proof. Let $f, g \in C_0^\infty(M)$, we have

$$\int_M (P_t f - f)g d\mu = \int_0^t \int_M \left(\frac{\partial}{\partial s} P_s f\right) g d\mu ds = \int_0^t \int_M (LP_s f)g\ d\mu ds$$

$$= -\int_0^t \int_M \Gamma(P_s f, g)d\mu ds.$$

By means of Proposition (3.2.8), and Cauchy–Schwarz inequality, we find

$$\left| \int_M (P_t f - f)g\, d\mu \right|$$

$$\leq \left(\int_0^t e^{\frac{-2\rho_1\rho_2 s}{\kappa+\rho_2}}\, ds \right) \sqrt{ \|\Gamma(f)\|_\infty + \frac{\kappa+\rho_2}{\rho_1} \|\Gamma^Z(f)\|_\infty } \int_M \Gamma(g)^{\frac{1}{2}}\, d\mu. \qquad (38)$$

Now it is seen from spectral theorem that in $L^2(M)$ we have a convergence $P_t f \to P_\infty f$, where $P_\infty f$ belongs to the domain of L. Moreover $LP_\infty f = 0$. By hypoellipticity of L we deduce that $P_\infty f$ is a smooth function. Since $LP_\infty f = 0$, we have $\Gamma(P_\infty f) = 0$ and therefore $P_\infty f$ is constant.

Let us now assume that $\mu(M) = +\infty$. This implies in particular that $P_\infty f = 0$ because no constant besides 0 is in $L^2(M)$. Using then (3.2.7) and letting $t \to +\infty$, we infer

$$\left| \int_M fg\, d\mu \right| \leq \left(\int_0^{+\infty} e^{\frac{-2\rho_1\rho_2 s}{\kappa+\rho_2}}\, ds \right) \sqrt{ \|\Gamma(f)\|_\infty + \frac{\kappa+\rho_2}{\rho_1} \|\Gamma^Z(f)\|_\infty } \int_M \Gamma(g)^{\frac{1}{2}}\, d\mu.$$

Let us assume $g \geq 0, g \neq 0$ and take for f the sequence hn from assumption (H.1). Letting $n \to \infty$, we deduce

$$\int_M g\, d\mu \leq 0,$$

which is clearly absurd. As a consequence $\mu(M) < +\infty$.

The invariance of μ implies then

$$\int_M P_\infty f\, d\mu = \int_M f\, d\mu,$$

and thus

$$P_\infty f = \frac{1}{\mu(M)} \int_M f\, d\mu.$$

Finally, using the Cauchy–Schwarz inequality, we find that for $x \in M, f \in L^2(M), s, t, \tau \geq 0,$

$$|P_{t+\tau}f(x) - P_{s+\tau}f(x)| = |P_\tau(P_t f - P_s f)(x)|$$

$$= \left| \int_M p(\tau, x, y)(P_t f - P_s f)(y)\mu(dy) \right|$$

$$\leq \int_M p(\tau, x, y)^2\, \mu(dy)\, \|P_t f - P_s f\|_2^2$$

$$\leq p(2\tau, x, x)\|P_t f - P_s f\|_2^2.$$

Thus, we also have

$$P_t f(x) \to_{t \to +\infty} \frac{1}{\mu(M)} Mf\, d\mu.$$

74

We also deduce a spectral gap inequality:

Corollary (3.2.10)[161]. For every f in the domain of L,

$$\int_M f^2 d\mu - \left(\int_M f d\mu\right)^2 \leq \frac{\kappa + \rho_2}{\rho_1 \rho_2} \int_M \Gamma(f) d\mu.$$

Proof. We use an argument close to one found in [151]. Let $f \in C^\infty(M)$ with a compact support. By Proposition (3.2.8)m we have for $t \geq 0$

$$\int_M \Gamma(P_t f, P_t f) d\mu \leq C(f) e^{\frac{-2\rho_1\rho_2 t}{\kappa+\rho_2}},$$

with

$$C(f) = \int_M \Gamma(f, f) + \frac{\kappa + \rho_2}{\rho_1} \Gamma^Z(f, f) d\mu.$$

By the spectral theorem, one has

$$\int_M \Gamma(P_t f, P_t f) d\mu = \int_0^\infty \lambda e^{-2\lambda t} dE_\lambda(f)$$

and

$$\int_M \Gamma(f, f) d\mu = \int_0^\infty \lambda dE_\lambda(f)$$

where dE_λ is the spectral measure associated to $-L$. Thus, by Holder inequality, for $0 \leq s \leq t$

$$\int_M \Gamma(P_t f, P_t f) d\mu = \int_0^\infty \lambda e^{-2\lambda s} dE_\lambda(f) \leq \left(\int_0^\infty \lambda e^{-2\lambda s} dE_\lambda(f)\right)^{\frac{s}{t}} \left(\int_0^\infty \lambda \, dE_\lambda(f)\right)^{\frac{t-s}{t}}$$

$$\leq C(f)^{\frac{s}{t}} e^{\frac{-2\rho_1\rho_2 s}{\kappa+\rho_2}} \left(\int_M \Gamma(f, f) d\mu\right)^{\frac{t-s}{t}}$$

Letting $t \to \infty$ gives

$$\int_M \Gamma(P_t f, P_t f) d\mu \leq e^{\frac{-2\rho_1\rho_2 s}{\kappa+\rho_2}} \int_M \Gamma(f, f) d\mu$$

for all C^∞ function with a compact support. Since this space is dense in the domain of the Dirichlet form, it implies the desired Poincaré inequality.

We also deduce a modified log-Sobolev inequality that involves a vertical term:

Corollary (3.2.11)[161]. Let us assume $\mu(M) = 1$. For $f \in C_0(M)$,

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \leq \frac{2(\kappa + \rho_2)}{\rho_1\rho_2}\left(\int_M \Gamma(f) d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_M \Gamma^Z(f) d\mu\right).$$

Proof. Let $g \in A_\varepsilon$. We have

$$\int_M g \ln g \, d\mu - \int_M g d\mu \ln \int_M g d\mu = -\int_0^{+\infty} \frac{\partial}{\partial t} \int_M P_t g \ln P_t g \, d\mu dt$$

$$= -\int_0^{+\infty} \int_M L P_t g \ln P_t g \, d\mu dt$$

$$= \int_0^{+\infty} \int_M \frac{\Gamma(P_t g)}{P_t g} \, d\mu dt$$

$$= \int_0^{+\infty} \int_M P_t g \, \Gamma(\ln P_t g) d\mu dt$$

$$\leq \int_0^{+\infty} e^{\frac{-2\rho_1 \rho_2 t}{\kappa + \rho_2}} dt \int_M \left(g\Gamma(\ln g) + \frac{\kappa + \rho_2}{\rho_1} g\Gamma^Z(\ln g) \right) d\mu$$

$$\leq \frac{2(\kappa + \rho_2)}{\rho_1 \rho_2} \int_M \left(\frac{\Gamma(g)}{g} + \frac{\kappa + \rho_2}{\rho_1} \frac{\Gamma^Z(g)}{g} \right) d\mu.$$

Let now $f \in C_0(M)$ and consider $g = \varepsilon + f^2 \in A_\varepsilon$. Using the previous inequality and letting $\varepsilon \to 0$, yields

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 d\mu \ln \int_M f^2 d\mu$$

$$\leq \frac{2(\kappa + \rho_2)}{\rho_1 \rho_2} \int_M \Gamma(f) d\mu + \frac{\kappa + \rho_2}{\rho_1} \int_M \Gamma^Z(f) d\mu.$$

We assume that the operator L satisfies the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$. We shall denote $\bar{\rho}_1 = max(-\rho_1, 0)$.

We show Theorem (3.2.3).

Proposition (3.2.12)[161]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$. For $x \in M$, $t > 0$ one has

$$tP_t f(x)\Gamma(\ln P_t f)(x) + \rho_2 t^2 P_t f(x)\Gamma^Z(\ln P_t f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t\right)[P_t(f \ln f)(x) - P_t f(x) \ln P_{T-t} f(x)].$$

Proof. We may assume $\rho_1 \geq 0$. We proceed similarly to the proof of Proposition (3.2.7). Let $f \in A_\varepsilon, 0 \leq t \leq T$ and

$$\emptyset_1(x, t) = (P_{T-t}f)(x)\Gamma(\ln P_{T-t} f)(x),$$
$$\emptyset_2(x, t) = (P_{T-t}f)(x)\Gamma^Z(\ln P_{T-t} f)(x),$$

As before, we consider the function

$\emptyset(x, t) = a(t)\emptyset_1(x, t) + b(t)\emptyset_2(x, t)$
$\qquad = a(t)(P_{T-t}f)(x)\Gamma(\ln P_{T-t}f)(x) + b(t)(P_{T-t}f)(x)\Gamma^Z(\ln P_{T-t}f)(x),$

where a and b are to be later chosen. As already seen, applying the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$, we obtain

$$L\emptyset + \frac{\partial \emptyset}{\partial t} \geq \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b}\right)(P_{T-t}f)\Gamma(\ln P_{T-t} f)$$

$$+ (b' + 2\rho_2 a)\,(P_{T-t}f)\Gamma^Z(\ln P_{T-t}f).$$

The idea is now to chose the functions a and b in such a way that

$$b' + 2\rho_2 a = 0$$

and

$$a' + 2\rho_1 a - 2\kappa\frac{a^2}{b} \geq C$$

where C is a constant independent from t. This leads to the candidates

$$a(t) = \frac{1}{\rho_2}(T - t)$$

and

$$b(t) = (T - t)^2,$$

for which we obtain

$$C = -\frac{1}{\rho_2} - \frac{2\kappa}{\rho_2^2} + \frac{2\rho_1}{\rho_2}T.$$

For this choice of a and b, we obtain

$$L\emptyset + \frac{\partial\emptyset}{\partial t} \geq C(P_{T-t}f)\Gamma\,(\ln P_{T-t}f).$$

The comparison principle for parabolic partial differential equations leads then to

$$P_T(\varphi(\cdot,T))(x) \geq \emptyset(0,x) + C\int_0^T P_t((P_{T-t}f)\Gamma\,(\ln P_{T-t}f))\,(x)dt.$$

It is now seen that

$$\int_0^T P_t((P_{T-t}f)\Gamma\,(\ln P_{T-t}f)\,(x)dt\ =\ P_T(f\ln f)(x) - P_T f(x)\ln P_T f\,(x),$$

which yields

$$T\,P_T f(x)\Gamma(\ln P_T f)(x) + \rho_2 T^2 P_T f(x)\Gamma^Z(\ln P_T f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2} + 2\bar\rho_1\,T\right)[\ P_T\,(f\ln f)(x) - P_T f(x)\ln P_T f\,(x)]$$

Using a similar argument, we may prove the following:

Proposition (3.2.13)[161]. Let $f \in C_0^\infty(M)$, then for $x \in M, t > 0$ one has

$$t\Gamma(P_t f)(x) + \rho_2 t^2\Gamma^Z(P_t f)(x) \leq \frac{1}{2}\left(1 + \frac{2\kappa}{\rho_2} + 2\bar\rho_2 t\right)[P_t(f^2)\,(x) - P_t f(x)^2]$$

As a consequence, we get the following useful regularization bound that will be later used:

Corollary (3.2.14)[161]. Let $f \in C_0^\infty(M)$, then for all $t > 0$,

$$\left\|\sqrt{\Gamma(P_t f)}\right\|_\infty \leq \left(\frac{\dfrac{1}{2} + \dfrac{\kappa}{\rho_2} + \bar\rho_1 t}{t}\right)^{\frac{1}{2}}\|f\|_\infty.$$

An important by-product of the reverse log-Sobolev inequality that was proved in the previous section (Proposition 3.2.12) is the following inequality that was first observed by F.Y. Wang [154] in a Riemannian framework.

Proposition (3.2.15)[161]. Let $\alpha > 1$. For $f \in L^\infty(M), f \geq 0, t > 0, x, y \in M,$

$$(P_t f)^\alpha(x) \leq P_t(f)^\alpha(y) \exp\left(\frac{\alpha}{\alpha - 1}\left(\frac{1 + \frac{\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t}\right) d^2(x, y)\right)$$

Proof. We first assume $f \in A_\varepsilon$.

Consider a subunit curve $\gamma: [0, T] \to M$ such that $\gamma(0) = x, \gamma(T) = y$. Let $\alpha > 1$ and $\beta(s) = 1 + (\alpha - 1)\frac{s}{T}, 0 \leq s \leq T$. Let

$$\emptyset(s) = \frac{\alpha}{\beta(s)} \ln P_t f^{\beta(s)}(\gamma(s)), \quad 0 \leq s \leq T,$$

where $t > 0$ is fixed. Differentiating with respect to s and using then Proposition (3.2.12) yields

$$\varphi'(s) \geq \frac{\alpha(\alpha - 1)}{T\beta(s)^2} \frac{P_t\left(f^{\beta(s)} \ln f^{\beta(s)}\right) - \left(P_t f^{\beta(s)}\right) \ln P_t f^{\beta(s)}}{P_t f^{\beta(s)}} - \frac{\alpha}{\beta(s)}\sqrt{\Gamma\left(\ln P_t f^{\beta(s)}\right)}$$

$$\geq \frac{\alpha(\alpha - 1)t}{T\beta(s)^2(1 + \frac{\kappa}{\rho_2} + 2\bar{\rho}_1 t)} \Gamma(\ln \ln P_t f^{\beta(s)} - \frac{\alpha}{\beta(s)}\sqrt{\Gamma\left(\ln P_t f^{\beta(s)}\right)}.$$

Now, for every $\lambda > 0$,

$$-\sqrt{\Gamma\left(\ln P_t f^{\beta(s)}\right)} \geq -\frac{1}{2\lambda^2}\Gamma\left(\ln P_t f^{\beta(s)}\right) - \frac{\lambda^2}{2}.$$

If we chose

$$\lambda^2 = \frac{(1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t)}{2(\alpha - 1)t} T\beta(s)$$

we infer

$$\varphi'(s) \geq -\frac{\alpha(1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t)}{4(\alpha - 1)t} T.$$

Integrating from 0 to L yields

$$\ln P_t(f^\alpha)(y) - \ln(P_t f)^\alpha(x) \geq -\frac{\alpha(1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t)}{4(\alpha - 1)t} T^2.$$

Minimizing then T^2 over the set of subunit curves such that $\gamma(0) = x$ and $\gamma(T) = y$ gives the claimed result.

If $f \in L^\infty(M), f \geq 0$, then for $\varepsilon > 0$, n0, and $\tau > 0$, the function $\varepsilon + h_n P_t f \in A_\varepsilon$, where $h_n \in C_0^\infty(M)$ is an increasing, non-negative, sequence that converges to 1. Letting then $\varepsilon \to 0, n \to \infty$ and $\tau \to 0$ proves that the inequality still holds for $f \in L^\infty(M)$.

An easy consequence of the Wang inequality of Proposition (3.3.15) is the following log-Harnack inequality.

Proposition (3.2.16)[161]. For $f \in L^\infty(M)$, $\inf f > 0, t > 0, x, y \in M$,

$$P_t(\ln f)(x) \leq \ln P_t(f)(y) + \left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} \right) d^2(x, y).$$

The proof of this result appears in [158] where a general study of these Harnack inequalities is done. For the sake of completeness, we reproduce the argument here.

Proof. Applying Proposition (3.2.15) to the function $f^{\frac{1}{2^n}}$ for $\alpha=2^n$, we ge

$$P_t\left(f^{2^{-n}}\right)(x) \leq \left(P_t(f)(y)\right)^{2^{-n}} \exp\left(\frac{1}{2^n - 1} \left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} \right) d^2(x, y) \right).$$

Now, since $2^{-n} \to 0$ as $n \to \infty$, by the dominated convergence theorem,

$$P_t(\ln f)(x) = \lim_{n\to\infty} P_t\left(\frac{f^{2^{-n}} - 1}{2^{-n}} \right)(x)$$

$$\leq \lim_{n\to\infty} \left[\frac{(P_t f(y))^{2^{-n}} \exp\left(\frac{1}{2^n-1} \left(\frac{1+\frac{2\kappa}{\rho_2}+2\bar{\rho}_1 t}{4t} \right) d^2(x,y) \right) - 1}{2^{-n}} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(P_t f(y))^{2^{-n}} - 1}{2^{-n}} + (P_t f(y))^{2^{-n}} \frac{\exp\left(\frac{1}{2^n-1} \left(\frac{1+\frac{2\kappa}{\rho_2}+2\bar{\rho}_1 t}{4t} \right) d^2(x,y) \right) - 1}{2^{-n}} \right]$$

$$= \ln(P_t f(y) + \left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} \right) d^2(x, y).$$

When μ is a probability measure, the above log-Harnack inequalities implies the following lower bound for the heat kernel.

Corollary (3.2.17)[161]. Assume that μ is a probability measure, then for $t > 0, x, y \in M$,

$$p_{2t}(x, y) \geq \exp\left(- \frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} d^2(x, y) \right).$$

Proof. Again, we reproduce an argument of Wang [159]. By applying Proposition (3.2.16) to the function $f(\cdot) = p_t(x,\cdot)$ and integrating over the manifold, one gets

$$\int_M p_t(x, z) \ln p_t(x, z) d\mu(z) \leq \ln \int_M p_t(y, z) p_t(x, z) d\mu(z) + \frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} d^2(x, y).$$

Now, by Jensen inequality, $\int_M p_t(y, z) p_t(x, z) d\mu(z) \geq 0$ thus

$$\ln p_{2t}(x,y) \geq -\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} d^2(x,y).$$

With Wang's inequality in hands, we can prove a log-Sobolev inequality provided the square integrability of the distance function.

Theorem(3.2.18)[161]. Assume that the measure μ is a probability measure and that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$. Assume moreover that

$$\int_M e^{\lambda d^2(x_0,x)} \, d\mu(x) < +\infty,$$

for some $x_0 \in M$ and $\lambda > \frac{\bar{\rho}_1}{2}$, then there is a constant $C > 0$ such that for every function $f \in C_0^\infty(M)$,

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \leq C \int_M \Gamma(f) d\mu$$

Proof. Let $\alpha > 1$ and $f \in L^\infty(M), f \geq 0$. From Proposition (3.2.15), by integrating with respect to y, we have

$$\int_M f^\alpha(y) d\mu(y) \geq (P_t f)^\alpha(x) \int_M \exp\left(-\frac{\alpha}{\alpha-1}\left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t}\right) d^2(x,y)\right) d\mu(y)$$

$$\geq (P_t f)^\alpha(x) \int_{B(x_0,1)} \exp\left(-\frac{\alpha}{\alpha-1}\left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t}\right) d^2(x,y)\right) d\mu(y)$$

$$\geq \mu(B(x_0,1)) (P_t f)^\alpha(x) \exp\left(-\frac{\alpha}{\alpha-1}\left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t}\right) d^2(x_0,x) + 1\right).$$

As a consequence, we get

$$(P_t f)(x) \leq \frac{1}{\mu(B(x_0,1))^{\frac{1}{\alpha}}} \exp\left(\frac{\alpha}{\alpha-1}\left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t}\right) d^2(x_0,x) + 1\right) \|f\|_{L^\alpha}.$$

Therefore if

$$\int_M e^{\lambda d^2(x_0,x)} \, d\mu(x) < +\infty,$$

for some $x_0 \in M$ and $\lambda > \frac{\bar{\rho}_1}{2}$,, then we can find $1 < \alpha < \beta$ and $t > 0$ such that

$$\|P_t f\|_{L^\beta} \leq C_{\alpha,\beta} \|f\|_{L^\alpha}.$$

for some constant $C_{\alpha,\beta}$. This implies the supercontractivity of the semigroup $(P_t)_t \geq 0$ and therefore from Gross' theorem (see [119]), a defective logarithmic Sobolev inequality is satisfied, that is there exist two constants $A, B > 0$ such that

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 d\mu \, \ln \int_M f^2 d\mu \le A \int_M \Gamma(f) d\mu \quad B \int_M f^2 d\mu, \; f \in C_0^\infty(M).$$

Now, since moreover the heat kernel is positive and the invariant measure a probability, we deduce from the uniform positivity improving property that L admits a spectral gap. That is, a Poincaré inequality is satisfied. It is then classical (see [118]), that the conjunction of a spectral gap and a defective logarithmic Sobolev inequality implies the log- Sobolev inequality (i.e. we may actually take $B = 0$ in the above inequality).

If we take the dimension in the generalized curvature dimension inequality, we may obtain an upper bound for the log-Sobolev constant under the assumption that the curvature parameter ρ_1 is positive.

Theorem (3.2.19)[161]. Assume that the measure μ is a probability measure and that L satisfies the generalized curvature dimension inequality $CD(\rho_1,\rho_2,\kappa,d)$ for some $\rho_1 > 0, \rho_2 > 0, \kappa \ge 0$ and $d \ge 1$. For every function $f \in C_0^\infty(M)$,

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 d\mu \, \ln \int_M f^2 d\mu \le C \int_M \Gamma(f) d\mu$$

with

$$C = \frac{3(\rho_2 + \kappa)}{\rho_1 \rho_2} \left(1 + \Phi\left(\frac{d}{2}\left(1 + \frac{3\kappa}{2\rho_2} \right) \right) \right),$$

where

$$\Phi(x) = (1 + x)\ln(1 + x) - x \ln x.$$

Proof. It is proved in [125] that the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 > 0, \rho_2 > 0, \kappa \ge 0$ and $d > 0$ implies the following upper bound for the heat kernel: For $x, y \in M$ and $t > 0$,

$$p(x, y, t) \le \frac{1}{\left(1 - e^{-\frac{2\rho_1 \rho_2 t}{3(\rho_2 + \kappa)}} \right)^{\frac{d}{2}(1 + \frac{3\kappa}{2\rho_2})}}.$$

Therefore, from Davies' theorem (in [129]), for $f \in C_0^\infty(M)$, we obtain the following defective log-Sobolev inequality which is valid for every $t > 0$,

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 d\mu \, \ln \int_M f^2 d\mu$$

$$\le 2t \int_M \Gamma(f) d\mu - d\left(1 + \frac{3\kappa}{2\rho_2} \right) \ln\left(1 - e^{-\frac{2\rho_1 \rho_2 t}{3(\rho_2 + \kappa)}} \right) \int_M f^2 d\mu.$$

The previous heat kernel upper bound also implies that $-L$ has a spectral gap of size at least $\frac{2\rho_1 \rho_2}{3(\rho_2 + \kappa)}$. Therefore, the following Poincaré inequality holds

$$\int_M f^2 \, d\mu - \left(\int_M f \, d\mu \right)^2 \leq \frac{3(\rho_2 + \kappa)}{2\rho_1 \rho_2} \int_M \Gamma(f) \, d\mu \, .$$

If we combine the two previous inequalities using Rothaus' inequality and then chose the optimal t, we get the result.

Theorem (3.2.20)[161]. Assume that the measure μ is a probability measure and that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ for some $\rho_1 > 0, \rho_2 > 0, \kappa \geq 0$ and $d > 0$.

(i) The metric space (M, d) is compact if and only if a log-Sobolev inequality

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \leq C \int_M \Gamma(f) \, d\mu, \quad f \in C_0^\infty(M) \qquad (39)$$

is satisfied for some $C > 0$.

(ii) Moreover, if (M, d) is compact with diameter D then, there is a constant $CD(\rho_1, \rho_2, \kappa, d)$ such that

$$D \leq \frac{C(\rho_1, \rho_2, \kappa, d)}{min(1, \rho_0)}$$

where $\frac{2}{\rho_0}$ is the smallest constant C such that (125) is satisfied.

Proof. If M is compact, then

$$\int_M e^{\lambda d^2(x_0, x)} \, d\mu(x) < +\infty,$$

for every $x_0 \in M$ and $\lambda > \frac{\bar{\rho}_1}{2}$. Therefore, from Theorem (3.2.18), a log-Sobolev inequality is satisfied.

Let us now assume that

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \leq \frac{2}{\rho_0} \int_M \Gamma(f) \, d\mu, \quad f \in C_0^\infty(M)$$

is satisfied.

Here we only sketch the proof, since we may actually follow quite closely an argument from Ledoux [143]. The key is to note that the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ for some $\rho_1 \in R, \rho_1 > 0, \rho_2 > 0, \kappa \geq 0 \text{ and } d > 0$ implies a Li–Yau type inequality. In particular for $0 < t \leq 1$ and a positive function f

$$0 \leq A \frac{LP_t f}{P_t f} + \frac{B}{t}$$

where A and B are some explicit positive constants depending only on $\rho_1, \rho_2, \kappa, d$. Since $\frac{LP_t f}{P_t f} = \partial_t \ln P_t f$, integrating between t and 1 yields, with $\gamma = \frac{B}{A}$,

$$P_t f \leq \frac{1}{t^\gamma} P_1 f \quad \text{for all} \quad 0 < t \leq 1.$$

Using now the equivalence between the log-Sobolev inequality and the hypercontractivity of the heat semigroup due to Gross, we find that for $1 < p < q < \infty$

$$\|P_t f\|_q \leq \|f\|_p$$

as soon as $e^{2\rho_0 t} \geq \frac{q-1}{p-1}$. Therefore, for $t = 1, p = 2$ and $q = 1 + e^{2\rho_0}$,

$$\|P_t f\|_q \leq \frac{1}{t^\gamma}\|P_1 f\|_q \leq \frac{1}{t^\gamma}\|f\|_2 \qquad \text{for } 0 < t \leq 1.$$

Such a semigroup estimate implies a Sobolev inequality

$$\|f\|_r^2 \, 8(\|f\|_2^2 + \left\|\sqrt{\Gamma(f,f)}\right\|_2^2)$$

for some $r > 2$. Finally, the conjunction of the logarithmic Sobolev inequality and of the above Sobolev inequality implies an entropy-energy inequality that may be used to prove that the diameter is bounded. Carefully tracking the constants leads to the desired bound for the diameter.

We shall examine the links between the log-Sobolev inequality and some transportations cost inequalities. First, it is well known that the log-Sobolev inequality implies some transportation inequalities in a general "metric" setting. Conversely, on a weighted Riemannian manifold, under the hypothesis that the Bakry–Émery curvature is bounded from below, the converse implication holds true.

We shall study how some transportation inequalities can, if the generalized curvature dimension inequality is satisfied, imply a log-Sobolev inequality. Unfortunately, we were only able to establish a partial converse in the sense that the log-Sobolev inequality we obtain involves a term with Γ^Z.

we assume $\mu(M) = 1$.

Let us begin with some notations. For a positive function f on M, we write

$$Ent_\mu(f) = \int_M f \ln f \, d\mu - \int_M f \, d\mu \ln \int_M f \, d\mu.$$

We recall that the L^2-Wasserstein distance of two measures v_1 and v_2 on M is given by

$$W_2(v_1, v_2)^2 = \inf_\Pi \int_M d^2(x,y) d\Pi(x,y)$$

where the infimum is taken over all coupling of v_1 and v_2 that is on all probability measures Π on $M \times M$ whose marginals are respectively v_1 and v_2.

Proposition (3.2.21)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \ , \rho_2 > 0, \kappa \geq$. Let f be a non-negative function on M such that $\int_M f \, d\mu = 1$ and set $dv = f \, d\mu$. Then, for any $t > 0$,

$$Ent_\mu(P_t f) \leq \left(\frac{1 + \dfrac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} \right) W_2(\mu, \nu)^2.$$

Proof. Let $t > 0$ and f be a positive function on M such that $\int_M f \, d\mu = 1$. The log-Harnack inequality of Proposition(3.2.16) applied to the function $P_t f$ gives then

$$P_t (\ln P_t f)(x) \leq \ln P_{2t}(f)(y) + \frac{1}{s} d^2(x, y),$$

with

$$s = \frac{4t}{1 + \dfrac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}$$

For x fixed, by taking the infimum with respect to y on the right-hand side of the last inequality, we obtain

$$P_t (\ln P_t f)(x) \leq Q_s (\ln P_{2t} f)(x)$$

where Q_s is the infimum-convolution semigroup:

$$Q_s(\varphi)(x) = \inf_{y \in M} \left\{ \varphi(y) + \frac{1}{2s} d(x, y)^2 \right\}.$$

Setting $\varphi = \ln P_{2t} f$, by Jensen inequality

$$\int_M \varphi d\mu = \int_M \ln P_{2t} f \, d\mu \leq \ln \left(\int_M P_{2t} f \, d\mu \right) = 0,$$

thus

$$P_t (\ln P_t f)(x) \leq Q_s(\varphi)(x) - \int_M \varphi d\mu .$$

Since by symmetry:

$$Ent_\mu(P_t f) = \int_M f \, P_t (\ln P_t f) \, d\mu$$

one finally gets

$$Ent_\mu(f_t) \leq \sup_\psi \left\{ \int_M Q_s(\psi)(x) d\nu - \int_M \psi d\mu \right\}.$$

where the supremum is taken over all bounded measurable functions ψ and where the measure ν is defined by $\frac{d\nu}{d\mu} = f$. By Monge–Kantorovich duality,

$$\sup_\psi \left\{ Q_s(\psi)(x) - \int_M \psi d\mu \right\} = \inf_\Pi \int_M T(x, y) d\Pi(x, y)$$

where the infimum is taken over all coupling of μ and ν and where the cost T is just

$$T(x,y) = \frac{1}{s}d^2(x,y).$$

Therefore the latter infimum is equal to $\frac{1}{s}W_2(\mu,v)^2$.

The following lemma may be proved in the very same way as Proposition (3.2.7)

Lemma(3.2.22)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0,\ \kappa \geq 0.$ Let $\varepsilon > 0$ and $f \in A_\varepsilon$. For $x \in M, t \geq 0$ one has

$$P_t f\Gamma(\ln P_t f) + P_t f\Gamma^Z(\ln P_t f) \leq e^{2\alpha t}(P_t(f\Gamma(\ln f)) + P_t(f\Gamma^Z(\ln f))), \quad t \geq 0,$$

where $\alpha = -min(\rho_2, \rho_1 - \kappa, 0)$.

Theorem(3.2.23)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ for some $\rho_1 \in R, \rho_2 > 0,\ \kappa \geq 0$. If the quadratic transportation cost inequality

$$W_2(\mu, v)^2 \leq c\ Ent_\mu\left(\frac{dv}{d\mu}\right) \tag{40}$$

is satisfied for every absolutely continuous probability measure v with a constant $c < \frac{\bar{\rho}_1}{2}$, then the following modified log-Sobolev inequality

$$Ent_\mu(f)C_1\int_M \frac{\Gamma(f)}{f}d\mu + C_2\int_M \frac{\Gamma^Z(f)}{f}d\mu, \quad f \in A_\epsilon, \epsilon > 0,$$

holds for some constants C_1 and C_2 depending only on $c, \rho_1, \kappa, \rho_2$.

Proof. Let $f \in A_\varepsilon$ such that $\int_M f\ d\mu = 1$, by the diffusion property, we have

$$\frac{d}{dt}Ent_\mu(P_t f) = -I(P_t f)$$

with

$$I(P_t f) = \int_M \frac{\Gamma(P_t f)}{P_t f}d\mu.$$

From Lemma (3.2.22), we have

$$\frac{\Gamma(P_t f)}{P_t f} \leq e^{2\alpha t}(P_t(f\Gamma(\ln f) + P_t(f\Gamma^Z(\ln f))),$$

which implies, by integration over the manifold M,

$$I(P_t f) \leq e^{2\alpha t}\left(\int_M \frac{\Gamma(f)}{f}d\mu + \int_M \frac{\Gamma^Z(f)}{f}d\mu\right).$$

As a consequence,

$$Ent_\mu(f) \leq \int_0^T I(P_t f)dt + Ent_\mu(P_T f)$$

$$\leq \left(\int_0^T e^{2\alpha t}\, dt \right) \left(\int_M \frac{\Gamma(f)}{f}\, d\mu + \int_M \frac{\Gamma^Z(f)}{f}\, d\mu \right) + Ent_\mu(P_T f).$$

We now use Proposition (3.3.21) and infer

$$Ent_\mu(f) \leq \left(\int_0^T e^{2\alpha t}\, dt \right) \left(\int_M \frac{\Gamma(f)}{f}\, d\mu + \int_M \frac{\Gamma^Z(f)}{f}\, d\mu \right)$$
$$+ \left(\frac{1 + \frac{2\kappa}{\rho_2} + 2\bar{\rho}_1 t}{4t} \right) W_2(\mu, v)^2,$$

where $dv = f\, d\mu$. Using the assumption $W_2(\mu, v)^2 \leq c\, Ent_\mu(f)$ and choosing T big enough finishes the proof.

We assume that the measure μ is a probability measure, that is $\mu(M) = 1$, and we show how the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ together with a log-Sobolev inequality implies alogarithmic isoperimetric inequality of Gaussian type. The method used here is very close from the one in Ledoux [142].

We first need to precise what we mean by the perimeter of a set in our subelliptic setting: This is essentially done in [134].

We observe that, given any point $x \in M$ there exists an open set $x \in U \subset M$ in which the operator L can be written as

$$L = -\sum_{i=1}^m X_i^* X_i, \tag{41}$$

where the vector fields X_i have Lipschitz continuous coefficients in U, and X_i^* indicates the formal adjoint of X_i in $L^2(M, d\mu)$.

We indicate with $F(M)$ the set of C^1 vector fields which are subunit for L. Given a function $f \in L^1_{loc}(M)$, which is supported in U we define the horizontal total variation of f as

$$Var(f) = \sup_{\varphi \in F(M)} \int_U f \left(\sum_{i=1}^m X_i^* \varphi_i \right) d\mu,$$

where on U, $\varphi = \sum_{i=1}^m \varphi_i X_i$. For functions not supported in U, $Var(f)$ may be defined by using a partition of unity. The space

$$BV(M) = \{ f \in L^1(M) \mid Var(f) < \infty \},$$

endowed with the norm

$$\|f\|_{BV(M)} = \|f\|_{L^1(M)} + Var(f),$$

is a Banach space. It is well known that $W^{1,1}(M) = \{f \in L^1(M) | \sqrt{\Gamma f} \in L^1(M)\}$ is a strict subspace of $BV(M)$ and when $f \in W^{1,1}(M)$ one has in fact

$$Var(f) = \|\sqrt{\Gamma f}\|_{L^1(M)}.$$

Given a measurable set $E \subset M$ we say that it has finite perimeter if $1_E \in BV(M)$. In such casethe perimeter of E is by definition

$$P(E) = Var(1_E).$$

We will need the following approximation result,

Lemma (3.2.24)[161]. Let $f \in BV(M)$, then there exists a sequence $\{f_n\}_{n \in N}$ of functions in $C_0^\infty(M)$ such that:

(i) $\|f_n - f\|_{L^1(M)} \to 0$;

(ii) $\int_M \sqrt{\Gamma(f_n)}\, d\mu \to Var(f)$.

After this digression, we now state the follwing theorem.

Lemma (3.2.25)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$, let $f \in C_0^\infty(M)$, then for all $t > 0$

$$\|f - P_t f\|_1 \left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t\right) \sqrt{t} \ \|\Gamma(f)\|_1. \tag{42}$$

Proof. First, since the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ holds true, by Corollary(3.2.14), for all $g \in C_0^\infty(M)$ and for all $0 < t \le t_0$,

$$\|\Gamma(P_t g)\|_\infty \le \left(\frac{\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t_0}{t}\right) \|g\|_\infty$$

Therefore, by duality, for every positive and smooth function f, every smooth function g such that $\|g\|_\infty \le 1$ and all $0 < t \le t_0$,

$$\int_M g\,(f - P_t f)\,d\mu = -\int_0^t \int_M g L P_s f \, d\mu \, ds$$

$$= \int_0^t \int_M \Gamma(P_s g, f)\,d\mu ds$$

$$\le \|\Gamma(f)\|_1 \int_0^t \|\Gamma(P_s g)\|_\infty \, ds$$

$$\le \left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t_0\right) \sqrt{t} \ \|\Gamma(f)\|_1$$

Theorem(3.2.26)[161]. Assume that L satisfies the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, \infty)$ and that μ satisfies the log-Sobolev inequality:

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \le \frac{2}{\rho_0} \int_M \Gamma(f) d\mu, \tag{43}$$

87

for all smooth functions $f \in C_0^\infty(M)$. Let A be a set of the manifold M which has a finite perimeter $P(A)$ and such that $0 \leq \mu(A) \leq \frac{1}{2}$, then

$$P(A) \geq \frac{\ln 2}{4\left(3 + \frac{2\kappa}{\rho_2}\right)} \; min\left(\sqrt{\rho_0}, \frac{\rho_0}{\sqrt{\bar{\rho}_1}}\right) \mu(A) \left(\ln\frac{1}{\mu(A)}\right)^{\frac{1}{2}}.$$

Proof. Let A be a set with finite perimeter. Applying Lemma (3.2.26) to smooth functions approximating the characteristic function 1_A as in Lemma (3.2.24) gives

$$\|1_A - P_t 1_A\|_1 \leq \left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t\right) \sqrt{t} \; P(A).$$

By symmetry and stochastic completeness of the semigroup,

$$\|1_A - P_t 1_A\|_1 = \int_A (1 - P_t 1_A)d\mu + \int_{A^c} P_t(1_A)d\mu$$

$$= \int_A (1 - P_t 1_A)d\mu + \int_A (P_t 1_{A^c})d\mu$$

$$= 2\left(\mu(A) - \int_A P_t(1_A)d\mu\right)$$

$$= 2\left(\mu(A) - \left\|P_{\frac{t}{2}}(1A)\right\|_2^2\right).$$

Now we can use the hypercontractivity constant to bound $\left\|P_{\frac{t}{2}}(1A)\right\|_2^2$. Indeed, from Gross' theorem it is well known that the logarithmic Sobolev inequality

$$\int_M f^2 \ln f^2 \, d\mu - \int_M f^2 \, d\mu \ln \int_M f^2 \, d\mu \leq \frac{2}{\rho_0}\int_M \Gamma(f)d\mu, \quad f \in C_0^\infty(M)$$

is equivalent to hypercontractivity property

$$\|P_t f\|_q \leq \|f\|_p$$

for all f in $L^p(M)$ whenever $1 < p < q < \infty$ and $e^{\rho_0 t} \geq \sqrt{\frac{q-1}{p-1}}$

Therefore, with $p(t) = 1 + e^{-\rho_0 t} < 2$, we get,

$$\left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t\right) \sqrt{t} \; P(A) \geq 2(\mu(A) - \mu(A)^{\frac{2}{p(t)}})$$

$$\geq 2\mu(A)(1 - \mu(A)^{\frac{1-e^{-\rho_0 t}}{1+e^{-\rho_0 t}}}).$$

Since for $x > 0$

$$1 - e^{-x} \min\left(\frac{x}{2}, \frac{1}{2}\right) \text{ and } \frac{1 - e^{-x}}{1 + e^{-x}} \geq \min\left(\frac{x}{4}, \frac{1}{2}\right),$$

$$\mu(A)^{\frac{1-e^{-\rho_0 t}}{1+e^{-\rho_0 t}}} \leq \exp\left(-\min\left(\frac{\rho_0 t}{4}, \frac{1}{4}\right)\ln\frac{1}{\mu(A)}\right),$$

88

$$1 - \mu(A)^{\frac{1-e^{-\rho_0 t}}{1+e^{-\rho_0 t}}} \leq \min\left(\min\left(\frac{\rho_0 t}{8}, \frac{1}{4}\right) \ln\frac{1}{\mu(A)}, \frac{1}{2}\right)$$

Therefore for all $t > 0$,

$$P(A) \geq \frac{2}{\left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t\right)\sqrt{t}} \mu(A) \min\left(\min\left(\frac{\rho_0 t}{8}, \frac{1}{4}\right) \ln\frac{1}{\mu(A)}, \frac{1}{2}\right). \qquad (44)$$

With $t_0 = min(\frac{1}{\rho_0}, \frac{1}{\bar{\rho}_1})$, for $0 < t \leq t_0$, we have

$$P(A) \geq \frac{2}{\left(\frac{1}{2} + \frac{\kappa}{\rho_2} + \bar{\rho}_1 t\right)\sqrt{t}} \mu(A) \min\left(\frac{\rho_0 t}{8} \ln\frac{1}{\mu(A)}, \frac{1}{2}\right)$$

Now, if $\mu(A)$ is small enough, i.e. $\mu(A) \leq e^{-4}$, we can chose $t = \frac{4t_0}{\ln\frac{1}{\mu(A)}} \leq t_0$ so that $\min\left(\frac{\rho_0 t}{8} \ln\frac{1}{\mu(A)}, \frac{1}{2}\right) = \frac{\rho_0 t_0}{2}$ and then get

$$P(A) \geq \frac{\rho_0 \sqrt{t_0} \mu(A)(\ln\frac{1}{\mu(A)})^{\frac{1}{2}}}{3 + \frac{2\kappa}{\rho_2}}.$$

When $0 \leq \mu(A) \leq \frac{1}{2}$, we can apply (3.2.26) with $t = t_0$ and since $\ln\frac{1}{\mu(A)} \geq \ln 2$,

$$\min\left(\frac{\rho_0 t}{8} \ln\frac{1}{\mu(A)}, \frac{1}{2}\right) \leq \frac{\rho_0 t_0 \ln 2}{2}$$

and thus

$$P(A) \geq \frac{\ln 2 \, \rho_0 \sqrt{t_0} \mu(A)}{2\left(3 + \frac{2\kappa}{\rho_2}\right)}$$

Noticing $\ln\frac{1}{\mu(A)} \leq 4$ if $\mu(A) \geq e^{-4}$, we obtain that for every A with $0 \leq \mu(A) \leq \frac{1}{2}$,

$$P(A) \geq \frac{\rho_0 \sqrt{t_0} \mu(A)(\ln\frac{1}{\mu(A)})^{\frac{1}{2}} \ln 2}{4\left(3 + \frac{2\kappa}{\rho_2}\right)}$$

Keeping in mind that $t_0 = min(\frac{1}{\rho_0}, \frac{1}{\bar{\rho}_1})$.

Chapter 4
Stochastic Completenss and Sub-Riemannian Curvature Dimension Inequality

We give a different proof of (and extend) a theorem in Baudoin and Garofalo stating that when a smooth, complete and connected manifold satisfies the generalized curvature-dimension inequality introduced, then the manifold turns out to be stochastically complete. The key ingredient is the study of dimension dependent reverse log-Sobolev inequalities for the heat semigroup and corresponding non-linear reverse Harnack type inequalties. The results apply in particular to all Sasakian manifolds whose horizontal Webster–Tanaka–Ricci curvature is nonnegative, all Carnot groups of step two, and to wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is nonnegative.

Section (4-1). Volume Growth in Sub-Riemannian Manifolds

By Baudoin and Garofalo [164] a generalizsation of the curvature- dimension inequality was introduced of sub-Riemannian manifolds. It proved, among other results, that if a smooth manifold M satisfies such generalized curvature-dimension inequality with a finite bound from below on the curvature parameter, then the stochastic completeness of the heat semigroup follows. Such result extended to a sub-Riemannian setting a classical 1975 result by Yau, see [174].

We generalize this result in [164]. Namely, we extend a result by Grigor'yan (see Theorem 11.8 in [68]) that gives a condition on the growth of the volume of balls that guarantees stochastic completeness. We establish a point wise estimate of the volume of the metric balls when the manifold satisfies the curvature-dimension inequality. Once that these results are established, the stochastic completeness proved in [164] will follow as a special case.

It is worth mentioning that the strength of Grigor'yan's theorem is that it only requires the volume condition to hold at one particular point. This is of special importance in the sub-Riemannian setting, since obtaining point wise estimates of the volume of the metric balls is an easier task than establishing the uniform control provided by the Bishop-Gromov comparison theorem. We should however mention by Baudoin et al. [163], in which a global doubling property has been proved when the generalized curvature dimension inequality holds below with a nonnegative curvature parameter. A detailed exposition on the subject of curvature-dimension inequalities and Ricci-lower bounds for sub-Riemannian manifolds can be found in [164].

We establish with the help of a Harnack inequality some new volume estimates when the manifold satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$, with a negative curvature parameter ρ_1. We generalize Grigor'yan's theorem to a metric setting

and show that the estimates obtained below imply the stochastic completeness of any manifold satisfying the generalized curvature-dimension inequality.

One of the most important identities in Riemannian geometry is the one due to Bochner. The latter states that if M is an n-dimensional Riemannian manifold with Laplacian Δ, for any $f \in C^\infty(M)$ one has

$$\Delta(|\nabla f|^2) = 2\|\nabla^2 f\|^2 + 2\langle \nabla f, \nabla \Delta f \rangle + 2Ric(\nabla f, \nabla f), \qquad (1)$$

where Ric indicates the Ricci tensor of M. Consider the following differential forms on functions $f, g \in C^\infty(M)$,

$$\Gamma(f, g) = \frac{1}{2}(\Delta(fg) - f\Delta g - g\Delta f) = (\nabla f, \nabla g),$$

and

$$\Gamma_2(f, g) = \frac{1}{2}[\Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f)].$$

When $f = g$, we simply write $\Gamma(f) = \Gamma(f, f), \quad \Gamma_2(f) = \Gamma_2(f, f)$. In terms of these functionals, Bochner's identity can be reformulated as

$$\Gamma_2(f) = 2\|\nabla^2 f\|^2 + Ric(\nabla f, \nabla f).$$

Since the Cauchy-Schwarz inequality gives $\|\nabla^2 f\|^2 \geq \frac{1}{n}(\nabla f)^2$, it is clear that if the Riemannian Ricci tensor on M is bounded from below by $\rho_1 \in R$, then we obtain the so-called curvature-dimension inequality $CD(\rho_1, n)$:

$$\Gamma_2(f) \geq \frac{1}{n}(\nabla f)^2 + \rho_1\Gamma(f), \qquad (2)$$

where $f \in C^\infty(M)$. One should notice that by combining Theorem 1.3 in [173] with Proposition 3.3 in [162] the following result is obtained: on a complete n-dimensional Riemannian manifold M the inequality $CD(\rho_1, n)$ is actually equivalent to $Ric \geq \rho_1$.

Baudoin and Garofalo in [164] introduced a generalization of the curvature-dimension inequality (2) which has proved successful in extending to some sub-Riemannian settings several results from Riemannian geometry.

Here is a brief description of their framework which is the same we are going to work with. See [164].

Consider a smooth connected manifold M endowed with a smooth measure μ and a smooth second-order diffusion operator L, which is assumed to be locally sub-elliptic, with real coefficients and satisfying:

(i) $L1 = 0$;

(ii) $\int_M f\, Lg\, d\mu = \int_M g Lf\, d\mu$;

(iii) $\int_M fLf\, d\mu \leq 0$,

for every $f, g \in C_0^\infty(M)$, where $C_0^\infty(M)$ denotes the set of smooth and compactly supported functions $f : M \rightarrow R$.

There is a notion of "length of a gradient" canonically associated to L. Consider in fact the bilinear differential form:

$$\Gamma(f, g) = \frac{1}{2} (L(fg) - fLg - gLf),$$

where $f, g \in C^\infty(M)$, and set

$$\Gamma(f) = \Gamma(f, f).$$

There is also a canonical distance d associated with L which is continuous and defines the topology of M. It is given by

$$d(x, y) = \sup\{|f(x) - f(y)| \,|\, f \in C^\infty(M), \|\Gamma(f)\|_\infty \leq 1\}, \quad (3)$$

where for a function g on M we have let $\|g\|_\infty = \text{ess sup}_M |g|$. It is assumed that the metric space (M, d) be complete.

For the purposes it will be necessary to work with yet another distance on M. Such distance is based on the notion of subunit curve introduced Fefferman and Phong in [166]. Here is a brief description of such metric. A result of [171] shows that, given any point $x \in M$, there exists an open neighborhood of x, $U_x \subset M$, in which the operator L can be written as

$$L = - \sum_{i=1}^{m} X_i^* \, X_i,$$

where the vector fields X_i have Lipschitz continuous coefficients in U, and X_i^* indicates the formal adjoint in $L^2(M, d\mu)$. Such representation of L is not unique, and the number of vector fields X_i varies with the representation. However, m is bounded from above by the dimension of M. A tangent vector $v \in T_x M$ is called subunit for L at x if $v = \sum_{i=1}^{m} a_i X_i$, with $\sum_{i=1}^{m} a_i^2 \leq 1$. The notion of subunitvector does not depend on the local representation of L. Furthermore, a Lipschitz $path \, \gamma : [0, T] \rightarrow M$ is called subunit for L if $\gamma'(t)$ is subunit for L at $\gamma(t)$ for a.e. $t \in [0, T]$. The subunit length of γ is defined as $s(\gamma) = T$. The set of subunit paths joining x to y in M is denoted by $S(x, y)$. We assume that $S(x, y) = \emptyset$ for every $x, y \in M$, so that

$$d_s(x, y) = \inf\{l_s(\gamma) \,|\, \gamma \in S(x, y)\}, \quad (4)$$

defines a true distance on M. We can work indifferently with either one of the distances d and d_s since

$$d(x, y) = d_s(x, y).$$

In addition to Γ, we assume that there exists another first-order bilinear form Γ^Z satisfying for $f, g, h \in C^\infty(M)$:

(i) $\Gamma^Z(fg, h) = f \Gamma^Z(g, h) + g\Gamma^Z(f, h)$;

(ii) $\Gamma^Z(f) = \Gamma^Z(f,f) \geq 0$.

Similarly to the Riemannian case, we introduce the following second-order differential forms:

$$\Gamma_2(f,g) = \frac{1}{2}[L\,\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)],$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)],$$

and we let $\Gamma_2(f) = \Gamma_2(f,f), \Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

The following definition was introduced in [164] and it occupies a central role in the developments in that work. It is a generalization of the above mentioned curvature-dimension inequality (2).

Definition (4.1.1)[175]. We shall say that M satisfies the generalized curvature-dimension inequality $CD(\rho_1,\rho_2,\kappa,d)$ with respect to L if there exist constants $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$, and $d \geq 2$ such that the inequality

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f) \tag{5}$$

holds for every $f \in C^\infty(M)$ and every $\nu > 0$.

From now on we set

$$D = d\left(1 + \frac{3\kappa}{2\rho_2}\right) \tag{6}$$

where ρ_2, κ, d are the parameters in (5).

We emphasize that the parameter ρ_1 plays the role of a lower bound on a sub-Riemannian version of the Ricci tensor, see [164]. We now introduce the general assumptions we will be working with.

Hypothesis (4.1.2)[175]. There exist an increasing sequence $h_k \in C_0^\infty(M)$ such that $h_k \nearrow 1$ on M, and

$$\|\Gamma(h_k)\|_\infty + \|\Gamma^Z(h_k)\|_\infty \to 0, \qquad as \qquad k \to \infty.$$

Hypothesis (4.1.3)[175]. For any $f \in C^\infty(M)$ one has

$$\Gamma(f,\Gamma^Z(f)) = \Gamma^Z(f,\Gamma(f))$$

Hypothesis (4.1.4)[175]. There exist $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$, and $d \geq 2$, such that M satisfies the generalized curvature-dimension inequality $CD(\rho_1,\rho_2,\kappa,d)$ with respect to L.

Under these hypothesis it was proved in [164] that L is an essentially self-adjoint operator on $C_0^\infty(M)$ whose Friedrichs extension (that we continue to denote by L) is the generator of a strongly continuous semigroup of contractions on $L^2(M)$, which we denote $P_t = e^{tL}$. Since the semigroup $(P_t)_t \geq 0$ issub-Markovian we have

$$P_t 1 \leq 1.$$

By Hörmander's theorem [169], $(t,x) \to P_t f(x)$ is smooth on $M \times (0,\infty)$ and

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y),$$

where $p(x, y, t) > 0$ is the so called heat kernel associated to P_t. Such function is smooth and symmetric, i.e.,

$$p(x, y, t) = p(y, x, t).$$

By the semigroup property for every $x, y \in M$ and $0 < s, t$ we have

$$p(x, y, t + s) = \int_M p(x, z, t) p(z, y, s) d\mu(z)$$

$$= \int_M p(x, z, t) p(y, z, s) d\mu(z) \tag{7}$$

$$= P_s(p(x, \cdot, t))(y).$$

In [164] it was proved that under the Hypotheses (4.1.2),(4.1.3) and (4.1.4) the following result holds.

Theorem (4.1.5)[175]. The manifold M is stochastically complete with respect to the semigroup $\{P_t\}_{t>0}$, i.e.,

$$P_t 1 \leq 1.$$

The objective of this note is to provide a different proof, and a generalization of Theorem (4.1.5), by proving that Grigor'yan's test for stochastic completeness can be extended to the present setting.

One should point out here that there is alarge class of sub-Riemannian manifolds that satisfy the above inequality. Such class includes all CR Sasakian manifolds, all Carnot groups with step two, and a wide sub-class of principal bundles, see [164].

The main goal is proving an estimate of the volume of the metric balls when the curvature-dimension inequality (5) holds, with a curvature parameter $\rho_1 < 0$. Before proving such estimate, we need to establish a Harnack inequality for non-negative solutions of the heat equation $H = L - \dfrac{\partial}{\partial t}$ on M which are of the form $u(x, t) = P_t f(x)$, for some $f \in C^\infty(M) \cap L^\infty(M)$. This inequality is a consequence of the following generalization of the celebrated Li-Yau inequality [170], whose proof can be found in [164],

Proposition (4.1.6) [175]. Assume that the manifold M satisfies (5). Let $f \in C_0^\infty(M)$, with $f \geq 0$, then the following inequality holds for $t > 0$:

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^Z(\ln P_t f) \leq \left(\frac{D}{d} + \frac{2|\rho_1|}{3} t\right) \frac{L P_t f}{P_t f} + \frac{d\rho_1^2}{6} t + \frac{|\rho_1|D}{2} + \frac{D^2}{2dt}.$$

Hereafter, we will denote $C_b^\infty(M) = C^\infty(M) \cap L^\infty(M)$.

Theorem (4.1.7) [175]. (Harnack inequality) Assume that (5) holds with $\rho_1 < 0$. Let $f \in C_b^\infty(M)$ be such that $f \geq 0$, and consider $u(x,t) = P_t f(x)$. For every $(x,s),(y,t) \in M \times (0,\infty)$ with $s < t$ one has with D as in (6)

$$u(x,s) \leq u(y,t) \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{d(x,y)^2}{4(t-s)}\left(\frac{D}{d} + \frac{2|\rho_1|}{3}t\right)\frac{3d|\rho_1|(t-s)}{4}\right). \tag{8}$$

Proof. Let f be as in the statement of the theorem, and consider $f_n = h_n f$, where $h_n \in C_0^\infty(M)$ is an increasing sequence with $0 \leq h_n \leq 1$, and $h_n \nearrow 1$ on M. By the monotone convergence theorem we have $u_n(x,t) = P_t f_n(x) \nearrow u(x,t) = P_t f(x)$ for every $(x,t) \in M \times (0,\infty)$. Since $u_n = \frac{\partial u_n}{\partial t}$, Proposition (4.1.6) gives us

$$\Gamma(\ln u_n) \leq \left(\frac{D}{d} + \frac{2|\rho_1|}{3}t\right)\frac{\partial \ln u_n}{\partial t} + \frac{d\rho_1^2}{6}t + \frac{D|\rho_1|}{2} + \frac{D^2}{2dt}.$$

This implies that

$$-\left(\frac{D}{d} + \frac{2|\rho_1|}{3}t\right)\frac{\partial \ln u_n}{\partial t} \leq -\Gamma(\ln u_n) + \frac{d\rho_1^2}{6}t + \frac{D|\rho_1|}{2} + \frac{D^2}{2dt} \tag{9}$$

Fix two points $(x,s),(y,t) \in M \times (0,\infty)$ with $s < t$. Let $\gamma(\tau), 0 \leq \tau \leq T$, be a subunitary path such that $\gamma(0) = y, \gamma(T) = x$. Let $\alpha(\tau), 0 \leq \tau \leq T$, be the path in $M \times (0,\infty)$ defined by

$$\alpha(\tau) = \left(\gamma(\tau), t + \frac{s-t}{T}\tau\right)$$

so that $\alpha(0) = (y,t), \alpha(T) = (x,s)$. We have

$$\ln\frac{u_n(x,s)}{u_n(y,s)} = \int_0^T \frac{d}{d\tau}\ln u_n(\alpha(\tau))d\tau$$

$$\leq \int_0^T \left[\Gamma(\ln u_n(\alpha(\tau)))^{\frac{1}{2}} - \frac{t-s}{T}\frac{\partial \ln u_n}{\partial t}(\alpha(\tau))\right]d\tau$$

Then for any $\epsilon > 0$

$$\ln\frac{u_n(x,s)}{u_n(y,s)} \leq T^{\frac{1}{2}}\left(\int_0^T \frac{d}{d\tau}\ln u_n(\alpha(\tau))d\tau\right)^{\frac{1}{2}} - \frac{t-s}{T}\int_0^T \frac{\partial \ln u_n}{\partial t}(\alpha(\tau))d\tau$$

$$\leq \frac{1}{2\epsilon}T + \frac{\epsilon}{2}\int_0^T \Gamma(\ln u_n(\alpha(\tau)))d\tau - \frac{t-s}{T}\int_0^T \frac{\partial \ln u_n}{\partial t}\alpha((\tau))d\tau \tag{10}$$

Set $\beta(\tau) = \left(\frac{D}{d} + \frac{2|\rho_1|}{3}\left(t + \frac{s-t}{T}\tau\right)\right)$ for $0 \leq \tau \leq T$. From (41) we get

$$-\frac{t-s}{T}\int_0^T \frac{\partial \ln u_n}{\partial t}\alpha\left(\left(\tau\right)\right)d\tau \le -\frac{t-s}{T}\int_0^T \frac{\Gamma(\ln u_n\left(\alpha\left(\tau\right)\right))}{\beta\left(\tau\right)}d\tau + \frac{d\rho_1^2(t-s)}{6T}\int_0^T \frac{t+\frac{s-t}{T}\tau}{\beta\left(\tau\right)}d\tau$$

$$+\frac{(t-s)D^2}{2dT}\int_0^T \frac{d\tau}{\beta\left(\tau\right)t+\frac{s-t}{T}\tau}$$

$$+\frac{D|\rho_1|(t-s)}{2T}\int_0^T \frac{d\tau}{\beta\left(\tau\right)}$$

Choose $\epsilon > 0$ such that

$$\frac{\epsilon}{2} = \frac{(t-s)}{\beta(0)T},$$

hence from (4) we obtain

$$\ln\frac{u_n(x,s)}{u_n(y,s)} \le \frac{T^2\beta(0)}{4(t-s)} + \frac{3|d\rho_1|(t-s)}{4} + \frac{D}{2}\ln\left(\frac{t}{s}\right)$$

If we now minimize over all sub-unitary paths joining y to x, and we exponentiate, we obtain

$$u_n(x,s) \le u_n(y,t)\left(\frac{t}{s}\right)^{\frac{D}{2}}\exp\left(\frac{d(x,y)^2\beta\left(0\right)}{4(t-s)} + \frac{3|d\rho_1|(t-s)}{4}\right)$$

Letting $n \to \infty$ in this inequality we finally obtain (8)

We can now extend this inequality to the heat kernel.

Corollary (4.1.8) [175]. Let $p(x,y,t)$ be the heat kernel on M. For every $x,y,z \in M$ and every $0 \le s \le t < \infty$ one has

$$p(x,y,s) \le p(x,z,t)\left(\frac{t}{s}\right)^{\frac{D}{2}}\exp\left(\frac{d(x,y)^2}{4(t-s)}\left(\frac{D}{d}+\frac{2|d\rho_1|}{3}t\right) + \frac{3d|\rho_1|(t-s)}{4}\right).$$

Proof. The idea of the proof is to write the heat kernel in terms of the heat semi- group in order to apply the above Harnack inequality. Due to the hypoellipticity of L we have that $p(x,\cdot, +\tau) \in C^\infty(M \times (-\tau, \infty))$ for $\tau > 0$ and $x \in M$ fixed. Because of (7)

$$p(x,y,s+\tau) = P_s\left(p(x,\cdot,\tau)\right)(y)$$

and

$$p(x,z,t+\tau) = P_s(p(x,\cdot,\tau))(z).$$

Consider as in the proof of the previous theorem $u_n(y,t) = P_t(h_n\,p(x,\cdot,\tau))(y)$, where $h_n \in C_0^\infty(M), 0 \le h_n \le 1, and\ hn \nearrow 1$. Applying the Harnack inequality (8) we obtain

$$P_t(h_n\,p(x,\cdot,\tau))(y) \le P_t(h_n\,p(x,\cdot,\tau))$$

$$(z)\left(\frac{t}{s}\right)^{\frac{D}{2}}\exp\left(\frac{d(x,y)^2}{4(t-s)}\left(\frac{D}{d}+\frac{2|\rho_1|}{3}t\right) + \frac{3d|\rho_1|(t-s)}{4}\right)$$

By the monotone convergence theorem, we obtain by letting $n \to \infty$

$$p(x, y, s + \tau) \leq p(x, z, t + \tau) \left(\frac{t}{s}\right)^{\frac{D}{2}}$$

The corollary follows by letting $\tau \to 0$.

To introduce our next result we recall that in [163] it was proved the following pointwise estimate of the volume of the balls in the special case of positive curvature, namely $\rho_1 \geq 0$.

Proposition (4.1.9) [175]. Assume that (6) holds with $\rho_1 \geq 0$ on (M, d). Then, for every $x \in M$ and every $R_0 > 0$ there is a constant $C(d, \kappa, \rho_2) > 0$ such that, with D as in (6),

$$\mu(B(x, R)) \leq \frac{C(d, \kappa, \rho_2)}{R_0^D p(x, x, R_0^2)} R^D \quad, R \geq R_0.$$

The next result generalizes Proposition (4.1.9) to the negative curvature case.

Proposition (4.1.10) [175]. Assume that (5) holds with $\rho_1 \geq 0$. There exists a constant $C(d, \kappa, \rho_2) > 0$ such that, given R_0, for every $x \in M$ and every $R \geq R_0$ one has

$$\mu(B(x, R)) \leq C(d, \kappa, \rho_2) \exp \frac{2d|\rho_1| R_0^2}{R_0^2 p(x, x, R_0^2)} R^D \exp(2d|\rho_1|R^2).$$

Theorem (4.1.11) [175]. If (M, g) is a complete Riemannian manifold with $Rc \geq (n - 1) K$, where $K \in R$. Then for any $x \in M, r > 0, \frac{\mu(B(x,r))}{\mu_K(B_r)}$ is non-increasing in r. Hence,

$$\mu(B(x, r)) \leq \mu_K(B_r),$$

where $\mu_K(B_r)$ denotes the volume of the ball B_r in the space form of constant curvature K. At this point one should remember that the volume of a ball in the space form of constant curvature $K < 0$ is given by

$$\mu_K(B_R) = \omega_n \int_0^R \frac{\sinh^{n-1}\sqrt{-K}r}{\sqrt{-K}} \, dr.$$

This implies, in particular, that when r is large enough we obtain the following bound

$$\mu(B(x, r)) \leq C_1 \exp(C2r), \qquad C_1, C_2 > 0.$$

The proof of the Bishop comparison theorem uses the theory of Jacobi fields. Since the exponential map in a sub-Riemannian space (M, d) is in general not a local diffeomorphism in a neighborhood of the point at which it is based (see[172]), the use of Jacobi fields in this more general setting presently encounters some serious obstructions. Instead, function analytical tools, like those developed in [164], are emerging to tackle these type of geometric problems.

As an interesting application of Proposition (4.1.10) we obtain a growth estimate of the volume for CR Sasakian manifolds. Let M be a non degenerate CR manifold of real hypersurface type and dimension $2n + 1$, where $n \geq 1$. Let θ be a contact form on M with respect to which the Levi form L_θ is positive definite. The kernel of θ determines the

horizontal bundle H. Denote by Z the Reeb vector field on M and by ∇ the Tanaka-Webster connection on M. The pseudo-hermitian torsion with respect to ∇ is
$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$
Since the CR manifold is Sasakian, we must have
$$T(Z,X) = 0,$$
for every $X \in H$. The following result was proved in [164].

Theorem (4.1.12) [175]. Assume that the Tanaka-Webster Ricci tensor is bounded from below by $\rho_1 \in R$ on smooth functions, that is
$$Ric(\nabla_H f, \nabla_H f) \geq \rho_1 \|\nabla_H f\|^2. \tag{11}$$
Then, M satisfies the generalized curvature-dimension inequality $CD\left(\rho_1, \frac{n}{2}, 1, 2n\right)$, i.e.,
$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{2n}(Lf)^2 + \left(\rho_1 - \frac{1}{\nu}(f)\right) + \frac{n}{2}\Gamma^Z(f),$$
for every $f \in C^\infty(M)$ and any $\nu > 0$.

As a consequence of Theorem (4.1.12) and of Proposition (4.1.10) we obtain the following result. Let us note that in the present case we obtain from (6)
$$D = d\left(1 + \frac{3\kappa}{2\rho_2}\right) = 2n + 6.$$

Proposition (4.1.13) [175]. Let M be a complete Sasakian manifold with Tanaka-Webster Ricci tensor satisfying (12) with $\rho_1 < 0$. There exists a constant $C(n) > 0$ such that, given R_0, for every $x \in M$ and every $R \geq R_0$ one has
$$\mu(B(x,R)) \leq C(n)\,exp\,\frac{\left(4n|\rho_1|R_0^2\right)}{R_0^D p(x,x,R_0^2)}R^D\,exp\left(4n|\rho_1|R^2\right)$$

We recall that a manifold (M,d) is stochastically complete when the heat semigroup satisfies $P_t 1 = e^{tL}1 = 1$ for every $t > 0$. This is equivalent to the condition
$$\int_M p(x,y,t)d\mu(y) = 1,$$
for all $x \in M$ and $t > 0$. It is well-known, see [248] for instance, that the stochastic completeness is equivalent to the fact that, for a given $T > 0$, the only bounded solution of the Cauchy problem:
$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu & in\ M \times (0,T), \\ u|_{t=0} = 0. \end{cases} \tag{12}$$
is the trivial one. Here, we are looking for a solution $u \in C^\infty(M \times (0,T))$, and the initial condition means that $u(x,t) \to 0$ locally uniformly in $x \in M$ as $t \to 0$. The stochastic completeness will follow from the following beautiful result due to Grigor'yan in the Riemannian case.

Lemma (4.1.14) [175]. Let $u \in C(M \times [a,b])$ be a solution of $\Delta u - ut = 0$ in $M \times (a,b)$ satisfying for some $x_0 \in M$ and for all $r > 0$

$$\int_a^b \int_{B(x_0,r)} u(x,t)^2 \, d\mu(x) dt \leq e^{f(r)},$$
(13)

where f is a positive function on $(0, \infty)$. Then, for any $r > 0$ which satisfies the condition

$$b - a \leq \frac{r^2}{8f(4r)},$$
(14)

one has

$$\int_{B(x_0,r)} u(x,t)^2 d\mu(x) \leq \int_{B(x_0,r)} u(x,t)^2 d\mu(x) + \frac{4}{r^2}.$$
(15)

Proof. The proof is completely analogous to the original one of Grigor'yan in the Riemannian case, and thus we confine ourselves to mention some necessary facts, and then refer to [168] for details. Consider the function $\rho(x) = (d(x,x_0) - r)_+$ defined on M. Set $s := 2b - a$ and consider the function

$$\eta(x,t) := \frac{\rho^2(x)}{4(t-s)}.$$

Notice that $\eta(x,t)$ is defined on $M \times [a,b]$, due to the fact that $s \notin [a,b]$. Sincethe function $y \to d(y,x_0)$ is Lipschitz continuous (with respect to the Carnot-Carathéodory distance), by the Rademacher theorem in [167], one can conclude that ρ belongs to the Sobolev space

$$W_H^{1,\infty}(M) = \{f \in L^\infty(M) \mid (f) \in L^\infty(M)\},$$

and furthermore $\Gamma(\rho)^{1/2} \leq 1$. This implies that for $t \in [a,b]$,

$$\Gamma(\eta) \leq \rho^2 \frac{4}{(t-s)^2},$$

and we thus have

$$\eta_t + \Gamma(\eta) \leq 0.$$
(16)

For $r > 0$, define the function $\varphi(x)$ by

$$\varphi(x) = \min\left(\left(3 - \frac{d(x,x_0)}{r}\right)_+, 1\right).$$

We notice here $\varphi \in Lip(M)$ and it has the following properties:
(i) $\varphi \equiv 1$ on $B(x_0, 2r)$ and $\varphi \equiv 0$ outside $B(x_0, 3r)$.
(ii) $\Gamma(\varphi)^{1/2} \leq 1/r$

Consider the function $u\varphi^2 e^\eta$ as a function of x for fixed $t \in [a,b]$. Notice that since by [167] we know that $u\varphi^2 e^\eta$ belongs to $W_H^{1,\infty}(M)$, and supp φ is compact, then such function belongs to $\dot{W}_H^{1,2}(M)$. We can thus multiply the heat equation

$$u_t = Lu$$

by $u\varphi e^\eta$ and then integrate it over $[a, b] \times M$ to obtain

$$\frac{1}{2} \int_a^b \int_M (u^2)_t \, \varphi^2 e^\eta d\mu dt = \int_a^b \int_M (Lu) u\varphi^2 e^\eta \, d\mu dt \qquad (17)$$

The time integral in the left hand side can be computed as follows:

$$\frac{1}{2} \int_a^b (u^2)_t \, \varphi^2 e^\eta dt = \frac{1}{2} [u^2 \varphi^2 e^\eta]_a^b - \frac{1}{2} \int_a^b \eta_t u^2 \, \varphi^2 e^\eta dt \qquad (18)$$

We can write the spatial integral on the right hand side of (17) as

$$\int_M (Lu) u\varphi^2 e^\eta \, d\mu = - \int_M \Gamma(u, u\varphi^2 e^\eta) \, d\mu$$

Observe

$$-\Gamma(u, u\varphi^2 e^\eta) \le \left(-\frac{1}{2}\Gamma(u) + \Gamma(u)^{1/2}\Gamma(\eta)^{1/2}|u| \right) \varphi^2 e^\eta + 2\Gamma(\varphi)u^2 e^\eta.$$

If we replace (17) into (18), and we use (16), we now obtain

$$\left[\int_M u^2 \, \varphi^2 e^\eta d\mu \right]_a^b \le - \int_a^b \int_M \left(\Gamma(u)^{1/2}|u| - \Gamma(u)^{1/2} \right)^2 \varphi^2 e^\eta$$

$$+ 4 \int_a^b \int_M \Gamma(\varphi) u^2 \varphi^2 e^\eta d\mu dt$$

and hence,

$$\left[\int_M u^2 \, \varphi^2 e^\eta d\mu \right]_a^b \le 4 \int_a^b \int_M \Gamma(\varphi) u^2 \varphi^2 e^\eta d\mu dt \qquad (19)$$

The properties of φ imply that from inequality (19) we obtain

$$\int_{B_r} u^2(x, b) e^{\eta(x,b)} \, d\mu \le \int_{B_{4r}} u^2(x, a) e^{\eta(x,b)} \, d\mu + \frac{4}{r^2} \int_a^b \int_{B_{4r} \setminus B_{2r}} u^2 e^\eta \, d\mu dt. \quad (20)$$

Theorem (4.1.15) [175]. Let u be a solution to the Cauchy problem (12). Assume that there exist an increasing function $f: (0, \infty) \to (0, \infty)$, such that

$$\int^\infty \frac{rdr}{f(r)} = \infty . \qquad (21)$$

If for some $x_0 \in M$ and for all $r > 0$ one has

100

$$\int_0^T \int_{B(x_0,r)} u^2(x,t)d\mu(x)dt \leq \exp(f(r)) \qquad (22)$$

Then, $u \equiv 0$ in $(0,T) \times M$.

Proof. Since we have nothing to add to the proof of Theorem 11.9 in [168].

Using Theorem (4.1.15) we can now establish the following generalization of Grigor'yan's criterion for stochastic completeness. Define the volume function $V(x,r)$ on the manifold (M,d) by

$$V(x,r) = \mu(B(x,r)),$$

where $B(x,r)$ is a metric ball. Since (M,d) is a complete metric space we have $V(x,r) < \infty$ for all $x \in M$ and $r > 0$.

Theorem (4.1.16) [175]. If for some point $x_0 \in M$ there exists $R_0 > 0$ such that

$$\int_{R_0}^{\infty} \frac{rdr}{\ln V(x_0,r)} = \infty, \qquad (23)$$

then M is stochastically complete.

Proof. If we can show that the only bounded solution to the Cauchy problem (12) is $u \equiv 0$ then the stochastic completeness will follow. This is because if $P_t 1 \neq 1$, then the function $u_1 = 1 - P_t 1$ is a non-trivial bounded solution to (12). Now, if u is a bounded solution of (12), then if we set $M := \sup|u|$ we obtain for $T < \infty$

$$\int_0^T \int_{B(x_0,r)} u^2(x,t)d\mu(x) \leq M^2 TV(x_0,r) = \exp(f(r)),$$

where

$$f(r) := \ln(M^2 TV(x_0,r)).$$

From (25) the function f satisfies (21). Therefore, by Theorem (4.1.15), we conclude $u \equiv 0$. By combining Theorem (4.1.15) with Propositions (4.1.10) and (4.1.9) we now recapture (with a different approach) the mentioned stochastic completeness result in [164].

Proposition (4.1.17) [175]. Suppose that the curvature dimension (5) hold with $\rho_1 \in R$. Then, M is stochastically complete.

Proof. It clearly suffices to consider the case $\rho_1 < 0$. In such case, for every $x_0 \in M$ and every $r \geq R_0$ we obtain by Proposition (4.1.10)

$$V(x_0,r) \leq C(d,\kappa,\rho_2) \exp \frac{(2d|\rho_1|R_0^2)}{R_0^2 p(x_0,x_0,R_0^2)} r^D \exp(2d|\rho_1|r^2)$$

$$= C_{x_0} r^D \exp(2d|\rho_1|r^2).$$

This gives for every $r \geq R_0$,

$$\ln V(x_0, r) \leq A_{x_0} + D \ln r + Br^2.$$

This clearly implies that

$$\int_{R_0}^{\infty} \frac{r\, dr}{\ln V(x_0, r)} = \infty \ .$$

The desired conclusion follows by Proposition (4.1.10).

Section(4-2). Volume Doubling Property and the Poincaré Inequality

A fundamental property of a measure metric space (X, d, μ) is the so-called doubling condition stating that for every $x \in X$ and every $r > 0$ one has

$$\mu(B(x, 2r)) \leq C_d\, \mu(B(x, r)), \qquad (24)$$

for some constant $C_d > 0$, where $B(x, r) = \{ y \in X \mid d(y, x) < r \}$. As it is well-known, such property is central for the validity of covering theorems of Vitali–Wiener type, maximal function estimates, and it represents one of the central ingredients in the development of analysis and geometry on metric measure spaces, see for instance [176,189,193,194,200–202]. Another fundamental property is the Poincaré inequality which claims the existence of constants $C_p > 0$ and $a \geq 1$ such that for every Lipschitz function f on $B(x, ar)$ one has

$$\int_{B(x,r)} |f - f_B|^2 d\mu \ \leq C_p r^2 \int_{B(x,ar)} g^2 d\mu \ , \qquad (25)$$

where we have let $f_B = \mu(B)^{-1} \int_B f d\mu$, with $B = B(x, r)$. In the right-hand side of (25) the function g denotes an upper gradient for f.

One basic instance of a measure metric space supporting (24) and (25) is a complete n-dimensional Riemannian manifold M with nonnegative Ricci tensor. In such case (24) follows with $C_d = 2^n$ from the Bishop-Gromov comparison theorem, whereas (25) was proved by Buser [185], with $a = 1$ and $g = |\nabla f|$.

Beyond the classical Riemannian case two situations of considerable analytic and geometric interest are CR and sub-Riemannian manifolds. For these classes global inequalities such as (24) and (25) are mostly terra incognita. The purpose is taking a first step in filling this gap in the class of sub-Riemannian manifolds that satisfy the generalized curvature dimension inequality introduced in [183]. Our main result, Theorem (4.2.8) below, constitutes a sub-Riemannian counterpart of the case in which Ricci ≥ 0 (for this aspect, see e.g. Theorem (4.2.10) below).

To introduce the results, we recall that a n-dimensional Riemannian manifold M with Laplacian Δ is said to satisfy the curvature-dimension inequality $CD(\rho_1, n)$ if there exists $\rho_1 \in R$ such that for every $f \in C^\infty(M)$ one has

$$\Gamma_2(f) \geq \frac{1}{n} (\Delta f)^2 + \rho_1 |\nabla f|^2 , \tag{26}$$

where

$$\Gamma_2(f) = \frac{1}{2}(\Delta|\nabla f|^2 - 2\langle \nabla f, \nabla(\Delta f)\rangle).$$

This notion was introduced by Bakry and Emery [261], and it was further developed in [178–180,210,216–221]. What is remarkable about the curvature-dimension inequality (27) is that it holds on a Riemannian manifold M if and only if $Ric \geq \rho_1$. It follows that such notion could be taken as an alternative characterization of Ricci lower bounds.

This point of view was recently taken up by [183], where a new sub-Riemannian curvature-dimension inequality was introduced. Such new inequality was shown to constitute a very robust tool for developing a Li–Yau type program in some large classes of sub-Riemannian manifolds. We develop our program even further, and in a different direction, by proving that the generalized curvature- dimension inequality introduced in [183] can be successfully used to establish global inequalities such as (24) and (25) above.

We now introduce the relevant framework. We consider measure metric spaces (M, d, μ), where M is C^∞ connected manifold endowed with a C^∞ measure μ, and d is a metric canonically associated with a C^∞ second-order diffusion operator L on M with real coefficients. We assume that L is locally subelliptic on M in the sense of [195], and that moreover:

(i) $L1 = 0$;

(ii) $\int_M f\, Lg\, d\mu = \int_M g L f d\mu$;

(iii) $\int_M f L f d\mu \leq 0$,

for every $f, g \in C_0^\infty(M)$. The following distance is canonically associated with the operator L:

$$d(x,y) = \sup\{|f(x) - f(y)|\,|\,f \in C^\infty(M), \|\Gamma(f)\|_\infty \leq 1\}, \quad x, y \in M, \tag{27}$$

where for a function g on M we have let $\|g\|_\infty$ ess $\sup_M |g|$.

Given the manifold M and the diffusion operator L, similarly to [181] we consider the quadratic functional $\Gamma(f) = \Gamma(f, f)$, where

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf), \quad f, g \in C^\infty(M), \tag{28}$$

is known as le carré du champ. One should in fact think of $\Gamma(f)$ as the square of the length of the gradient of f along the so-called horizontal directions. We remark that Γ depends only

on the diffusion operator L, and in this sense it is canonical. Notice that $(f) \geq 0$ and that $\Gamma(1) = 0$.

Unfortunately, in sub-Riemannian geometry the canonical bilinear form does not suffice to develop the Li–Yau program. To circumvent this obstruction, we further suppose that M is equipped with a symmetric, first-order differential bilinear form $\Gamma^Z : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, satisfying

$$\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h).$$

We make the assumption that $\Gamma^Z(f) = \Gamma^Z(f, f) \geq 0$ (one should notice that $\Gamma^Z(1) = 0$). Roughly speaking, in a sub-Riemannian manifold $\Gamma^Z(f)$ represents the square of the length of the gradient of f in the directions of the commutators. We emphasize that, in the above general formulation, the bilinear form Γ^Z is not canonical since, unlike the form Γ, a priori it has no direct correlation to the diffusion operator L. If should however find reassuring that, in all the concrete geometric examples encompassed, the choice of the form Γ^Z can be shown to be, in fact, canonical.

To clarify this important point we pause for a moment to discuss a basic class of three-dimensional models which have been analyzed. Given a $\rho_1 \in R$ we consider a Lie group $G(\rho_1)$ whose Lie algebra g admits a basis of generators X, Y, Z satisfying the commutation relations

$$[X, Y] = Z, \quad [X, Z] = -\rho_1 Y, \quad [Y, Z] = \rho_1 X. \quad (29)$$

The group $G(\rho_1)$ can be endowed with a natural CR structure θ withrespect to which the Reeb vector field is given by $-Z$. A sub-Laplacian on $G(\rho_1)$ with respect to such structure is thus given by $L = X^2 + Y^2$. The pseudo-hermitian Tanaka–Webster torsion of $G(\rho_1)$ vanishes, and thus $(G(\rho_1), \theta)$ is a Sasakian manifold. In the smooth manifold $M = G(\rho_1)$ with sub-Laplacian L we introduce the differential forms Γ and Γ^Z defined by

$$\Gamma(f, g) = XfXg + YfYg, \Gamma^Z(f, g) = ZfZg .$$

It is worth observing that, since as we have said $-Z$ is the Reeb vector field of the CR structure θ, then the above choice of Γ^Z is canonical. It is also worth remarking at this point that for the CR manifold $(G(\rho_1), \theta)$ the Tanaka–Webster horizontal sectional curvature is constant and equals ρ_1. For instance, when G is the 3-dimensional Heisenberg group H^1, with real coordinates (x, y, t), and generators of the Lie algebra $X = \partial_x - \frac{y}{2}\partial_t$, $Y = \partial_y + \frac{x}{2}\partial_t$, $Z = \partial_t$, then (30) holds with $\rho_1 = 0$. In [183] two other special instances of the model CR manifold $G(\rho_1)$ were discussed in detail, namely $SU(2)$, and $SL(2, R)$, corresponding, respectively, to the cases $\rho_1 = 1$ and $\rho_1 = -1$. Given the first-order bilinear forms Γ and Γ^Z on M, we now introduce the following second-order differential forms:

$$\Gamma_2(f, g) = \frac{1}{2}[L\Gamma (f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)], \quad (30)$$

$$\Gamma_2^Z(f,g) = \frac{1}{2}\left[L\Gamma^Z(f,g) - \Gamma^Z(f,Lg) - \Gamma^Z(g,Lf)\right]. \qquad (31)$$

Observe that if $\Gamma^Z \equiv 0$, then $\Gamma_2^Z \equiv 0$ as well. As for Γ and Γ^Z, we will use the notations $\Gamma_2(f) = \Gamma_2(f,f), \Gamma_2^Z(f) = \Gamma_2^Z(f,f)$.

The next definition, which we are taking from [183], is the central character .

Definition (4.2.1)[225]. (Generalized curvature-dimension inequality) Let $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$, and $m > 0$. We say that M satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ if the inequality

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \geq \frac{1}{m}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f) \qquad (32)$$

holds for every $f \in C^\infty(M)$ and every $\nu > 0$.

Proposition (4.2.2). The sub-Laplacian L on the Lie group $G(\rho_1)$ satisfies the generalized curvature-dimension inequality $CD(\rho_1, \frac{1}{2}, 1, 2)$.

The essential new aspect of the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ with respect to the Riemannian inequality $CD(\rho_1, n)$ in (26) is the presence of the a priori non-intrinsic bilinear forms Γ^Z and Γ_2^Z .As in [183], to be able to handle these non-intrinsic forms we will assume throughout the following hypothesis (Hypothesis (4.2.3)), (Hypothesis (4.2.4)) and (Hypothesis (4.2.5)). Even if they will not be mentioned explicitly in every individual result.

Hypothesis (4.2.3) [225]. There exists an increasing sequence $h_k \in C_0^\infty(M)$ such that $h_k \nearrow 1$ on M, and

$$||\Gamma(h_k)||_\infty + ||\Gamma^Z(h_k)||_\infty \to 0, \qquad as\ k \to \infty.$$

Hypothesis (4.2.4) [225]. For any f $\in C^\infty(M)$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

Hypothesis (4.2.5) [225]. The heat semigroup generated by L, which will denoted P_t throughout the section , is stochastically complete that is, for t ≥ 0, $P_t 1 = 1$ and for every $f \in C_0^\infty(M)$ and $T \geq 0$, one has

$$\sup_{t \in [0,T]} ||\Gamma(P_t f)||_\infty\ ||\Gamma^Z(P_t f)||_\infty < +\infty.$$

In addition to (Hypothesis (4.2.3)–(Hypothesis (4.2.5), throughout we also assume that:

Hypothesis (4.2.6) [225]. Given any two points $x, y \in M$, there exist a subunit curve (in the sense of [195]), joining them.

Hypothesis (4.2.7) [225]. The metric space (M, d) is complete.

We note that in the geometric examples encompassed by the framework (for a detailed discussion of these examples see [183]), (Hypothesis (4.2.3)) is equivalent to assuming that (M, d) be a complete metric space, i.e., (Hypothesis (4.2.7)). The assumption (Hypothesis (4.2.6)) is for instance fulfilled when the operator L satisfies the finite rank condition of the

Chow–Rashevsky theorem. When (Hypothesis (4.2.6)) holds, definition above provides a true distance, and the metric space (M, d) is a length-space in the sense of Gromov. The hypothesis (Hypothesis (4.2.4)) is of a geometric nature. For instance, all CR manifolds which are Sasakian satisfy it. It is important to mention that the hypothesis (Hypothesis (4.2.5)) has been shown in [183] to be a consequence of the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ in the large class of sub-Riemanniann manifolds with transverse symmetries of Yang-Mills type. Such class encompasses Riemannian structures, CR Sasakian structures, and Carnot groups of step two. Therefore, the assumption (Hypothesis (4.2.5)) should not be seen as restrictive if we assume that the curvature dimension inequality is satisfied. We can also observe that the stochastic completeness of P_t is intimately related to the volume growth of large metric balls and has been extensively studied (see [198,216]). The following is the central result.

Theorem (4.2.8)[225]. Suppose that the generalized curvature-dimension inequality hold for some $\rho_1 \geq 0$. Then, there exist constants $C_d, C_p > 0$, depending only on $\rho_1, \rho_2, \kappa, m$, for which one has for every $x \in M$ and every $r > 0$:

$$\mu(B(x, 2r)) \leq C_d \, \mu(B(x, r)); \tag{33}$$

$$\int_{B(x,r)} |f - f_B|^2 d\mu \leq C_d r^2 \int_{B(x,r)} \Gamma(f) d\mu, \tag{34}$$

for every $f \in C^1(\bar{B}(x, r))$.

We note explicitly that the possibility of having the same ball in both sides of (34) is due to the above mentioned fact that (M, d) is a length-space. This follows from the assumption (41) below (which guarantees that (M, d) is a Carnot–Carathéodory space), and from Proposition (4.2.11) in [191] (which states that every Carnot–Carathéodory space is a length-space). Once we know that (M, d) is a length-space, we can follow the arguments in Jerison [206] on the local Poincaréin equality to replace the integral on a larger ball in the right-hand side of (34) with an integral on the same ball $B(x, r)$ as in the left-hand side, see [196]. To put Theorem (4.2.8) in the proper perspective we note that, be sides the already cited case of a complete Riemannian manifold having $Ric \geq 0$, the only genuinely sub-Riemannian manifolds in which (33) and (34) are presently known to simultaneously hold are stratified nilpotent Lie groups, aka Carnot groups, and, more in general, groups with polynomial growth. In Carnot groups the doubling condition (33) follows from a simple rescaling argument based on the non-isotropic group dilations, from the group left-translations and form the fact that the push-forward to the group of the Lebesgue measure on the Lie algebra is a bi-invariant Haarmeasure. For more general Lie groups with polynomial growth Varopoulos gave an elementary proof of the Poincaré inequality (34) in [221]. In which they establish two-sided global Gaussian bounds in a Lie group with

polynomial growth. As it is well known, such bounds are equivalent to the doubling condition and the Poincaré inequality.

It is worth mentioning at this point that, when L is a sum of square of vector fields like in Hörmander's work on hypoellipticity [205], then a local (both in $x \in X$ and $r > 0$) doubling condition was proved in [211]. In this same framework, a local version of the Poincaré inequality was proved by Jerison [206]. But no geometry is of course involved in these fundamental local results. The novelty of our work is in the global character of the estimates (33) and (34).

In order to elucidate some of the new geometric settings covered, we recall that one of the main motivations for [211] was understanding boundary value problems coming from several complex variables and CR geometry. In connection with CR manifolds we mention that in [183] the first and third named proved the following result.

Theorem (4.2.9)[225]. Let (M, θ) be a complete CR manifold with real dimension $2n + 1$ and vanishing Tanaka–Webster torsion, i.e., a Sasakian manifold. If for every $x \in M$ the Tanaka–Webster Ricci tensor satisfies the bound

$$Ric_x (v, v) \geq \rho_1 |v|^2,$$

for every horizontal vector $v \in H_x$, then the curvature-dimension inequality $CD\left(\rho_1, \frac{n}{2}, 1, 2n\right)$ holds.

By combining Theorem (4.2.8) with Theorem (4.2.9) we obtain the following result which provides a large class of new geometric examples which are encompassed by our results, and which could not be previously covered by the existing works.

Theorem (4.2.10)[225]. Let M be a Sasakian manifold of real dimension $2n + 1$. If for every $x \in M$ the Tanaka–Webster Ricci tensor satisfies the bound $Ric_x \geq 0$, when restricted to the horizontal sub bundle H_x, then there exist constants $C_d, C_p > 0$, depending only on n, for which one has for every x ∈ M and every r > 0:

$$\mu(B(x, 2r)) \leq C_d \, \mu(B(x, r)); \tag{35}$$

$$\int_{B(x,r)} |f - f_B|^2 d\mu \leq C_p r^2 \int_{B(x,r)} |\nabla_H f|^2 d\mu. \tag{36}$$

In (36) we have denoted with $\nabla_H f$ the horizontal gradient of a function $f \in C^1(\bar{B}(x, r))$. Concerning Theorem (4.2.10) we mention that in [176] Agrachev and Lee, with a completely different approach from us, have obtained (35) and (36) for three-dimensional Sasakian manifolds.

Once Theorem (4.2.8) is available, then from the work of Grigor'yan [197] and Saloff–Coste [214] it is well-known that, in a very general Markov setting, the conjunction of (33) and (36) is equivalent to Gaussian lower bounds and uniform Harnack inequalities for the

heat equation $L - \partial_t$. For the relevant statements we refer the reader to Theorems (4.2.24) and (4.2.25) below.

Another basic result which follows from Theorem (4.2.8) is a generalized Liouville type theorem, see Theorem (4.2.29) below, stating that, for any given $N \in \mathbf{N}$,

$$\dim H_N (M, L) < \infty, \tag{37}$$

where we have indicated with $H_N(M, L)$ the linear space of L-harmonic functions on M with polynomial growth of order $\leq N$ with respect to the distance d.

In closing we mention that the framework is analogous to that [183], where two of us have used the generalized curvature-dimension inequality in Definition (4.2.1) to establish various global properties such as:

(i)   An a priori Li–Yau gradient estimate for solutions of the heat equation $L - \partial_t$ of the form $u(x, t) = P_t f(x)$, where $P_t = e^{tL}$ is the heat semigroup associated with L;

(ii) A scale invariant Harnack inequality for solutions of the heat equation of the form $u = P_t f$, with $f \geq 0$;

(iii) A Liouville type theorem for solutions of $Lf = 0$ on M;

(iv) Off-diagonal upper bounds for the fundamental solution of $L - \partial_t$;

(v)   A Bonnet–Myers compactness theorem for the metric space (M, d).

As for the ideas involved in the proof of Theorem (4.2.8) we mention that our approach is purely analytical and it is exclusively based on some new entropy functional inequalities for the heat semigroup. Our central result in the proof of Theorem (4.2.8) is a uniform Hölder estimate of the caloric measure associated with the diffusion operator L. Such estimate is contained in Theorem (4.2.22) below, and it states the existence of an absolute constant $A > 0$, depending only the parameters in the inequality $CD(\rho_1, \rho_2, \kappa, d)$, such that for every $x \in M$, and $r > 0$,

$$P_{Ar^2}\left(1_{B(x,r)}\right)(x) \geq \frac{1}{2}. \tag{38}$$

Here, for a set $E \subset M$, we have denoted by 1_E its indicator function. Once the crucial estimate (38) is obtained, with the help of the Harnack inequality

$$P_s f(x) \leq P_t f(y) \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{D}{m} \frac{d(x,y)^2}{4(t-s)}\right), \quad s < t, \tag{39}$$

that was proved in [183] (for an explanation of the parameter D see (45) below), the proofs of (33), (34) become fairly standard, and they rely on a powerful circle of ideas that may be found.

The proof of (38) which represents the main novel contribution of the present work is rather technical. We mention that the main building block is a dimension dependent reverse logarithmic Sobolev inequality in Proposition (4.2.12) below. We stress here that, even in

the Riemannian case, which is of course encompassed, such estimates are new and lead to some delicate reverse Harnack inequalities which constitute the key ingredients in the proof of (38). Still in connection with the Riemannian case, it is perhaps worth noting that, although as we have mentioned, in this setting the inequalities (24), (25) are of course well-known, nonetheless our approach provides a new perspective based on a systematic use of the heat semigroup. The more PDE oriented might in fact find somewhat surprising that one can develop the whole local regularity starting from a global object such the heat semigroup. This in a sense reverses the way one normally proceeds, starting from local solutions.

We mention that in [184] two of us have obtained a purely analytical proof of (38) for complete Riemannian manifolds with Ric ≥ 0. The approach, which is based on a functional inequality much simpler than the one found, is completely different from that of Theorem (4.2.22) below and cannot be adapted to the non-Riemannian setting .

Hereafter, M will be a C^∞ connected manifold endowed with a smooth measure μ and a second-order diffusion operator L on M with real coefficients, locally subelliptic, satisfying $L1 = 0$ and

$$\int_M fLg\, d\mu = \int_M gLf d\mu, \qquad \int_M fLf d\mu \leq 0,$$

for every $f, g \in C_0^\infty(M)$. We indicate with $\Gamma(f)$ the quadratic differential form defined by (28) and denote by $d(x, y)$ the canonical distance associated with L as in (28) .

There is another useful distance on M which in fact coincides with $d(x, y)$. Such distance is based on the notion of subunit curve introduced by Fefferman and Phong [195], see also [207]. By a result in [213], given any point $x \in M$ there exists an open set $x \in U \subset M$ in which the operator L can be written as

$$L = -\sum_{i=1}^m X_i^* X_i, \tag{40}$$

where the vector fields X_i have Lipschitz continuous coefficients in U, and X_i^* indicates the formal adjoint of X_i in $L^2(M, d\mu)$. We remark that such local representation of L is not unique. A tangent vector $v \in T_x M$ is called subunit for L at x if $v = \sum_{i=1}^m a_i X_i(x)$, with $\sum_{i=1}^m a_i^2 \leq 1$. It turns out that the notion of subunit vector for L at x does not depend on the local representation (40) of L. A Lipschitz path $\gamma : [0, T] \to M$ is called subunit for L if $\gamma'(t)$ is subunit for L at $\gamma(t)$ for a.e. $t \in [0, T]$. We then definethe subunit length of γ as $s(\gamma) = T$. Given $x, y \in M$, we indicate with
$S(x, y) = \{\gamma : [0, T] \to M \mid \gamma \text{ is subunit for } L, \gamma(0) = x, \gamma(T) = y\}$.
We remark explicitly that the assumption (H.(4.3.6)) can be reformulated by saying that
$$S(x, y) = \emptyset, \quad \text{for every} \quad x, y \in M. \tag{41}$$

Now, it is easy to verify that (72) implies that for any $x, y \in M$ one has
$$d_s(x, y) = \inf\{l_s(\gamma) \mid \gamma \in S(x, y)\} < \infty, \qquad (42)$$
and therefore (42) defines a true distance on M (once we have the finiteness of ds the other properties defining a distance are easily verified). Furthermore, in Lemma 5.43 in [186] it is proved that
$$d(x, y) = d_s(x, y), \qquad x, y \in M. \qquad (43)$$
Therefore, also d is a true distance on M and, in view of (43), we can work indifferently with either one of the distances d or d_s.

In closing, we mention if L is in the form $L = \sum_{i=1}^{m} X_i^2 + X_0$, with vector fieldswhich are C^∞ and satisfying the so-called Hörmander's finite rank condition on the Lie algebra, then the Theorem of Chow–Rashevsky guarantees the validity of (Hypothesis (4.2.6)). If moreover L has real-analytic coefficients, we know that L is hypoelliptic if and only if it satisfies Hörmander's finite rank condition. Therefore, in this situation, the hypoellipticity of L would guarantee the validity of (Hypothesis (4.2.6)). For generalizations of the cited result in [192] to more general hypoelliptic operators with real-analytic coefficients, see [212].

We collect some results from [183] which will be needed. In the framework below, L is essentially self-adjoint on $C_0^\infty(M)$. Due to the hypoellipticity of L, the function $(t, x) \to P_t f(x)$ is smooth on $(0, \infty) \times M$ and

$$P_t f(x) = \int_M p(x, y, t) f(y)\, d\mu(y), \quad f \in C_0^\infty(M)$$

where $p(x, y, t) = p(y, x, t) > 0$ is the so-called heat kernel associated to P_t. We denote
$$C_b^\infty(M) = C^\infty(M) \cap L^\infty(M).$$
For $\varepsilon > 0$ we also denote by A_ε the set of functions $f \in C_b^\infty(M)$ such that
$$f = g + \varepsilon,$$
for some $\varepsilon > 0$ and some $g \in C_b^\infty(M), g \geq 0$, such that $g, \sqrt{\Gamma(g)}, \sqrt{\Gamma^Z(g)} \in L^2(M)$. As shown in [183], this set is stable under the action of P_t, i.e., if $f \in A_\varepsilon$, then $P_t f \in A_\varepsilon$. Letusfix $x \in M$ and $T > 0$.Given a function $f \in A_\varepsilon$, for $0 \leq t \leq T$ we introduce the entropy functionals
$$\Phi_1(t) = P_t((P_{T-t}f)(\ln P_{T-t}f))(x),$$
$$\Phi_2(t) = P_t(P_{T-t}f) Z(\ln P_{T-t}f)(x).$$
For later use, we observe here that
$$\frac{d}{dt} P_t((P_{T-t}f) \ln P_{T-t}f)(x) = P_t((P_{T-t}f)\Gamma(\ln P_{T-t}f)(x) = \Phi_1(t),$$
and thus, with the above notations,

$$\int_0^T \Phi_1(t)dt = P_T(f \ln f)(x) - P_T f(x) \ln P_T f(x). \tag{44}$$

For the sake of brevity, we will often omit reference to the point $x \in M$, and write for instance $P_T f$ instead of $P_T f(x)$. This should cause no confusion.

The main source of the functional inequalities that will be studied in the present work is the following result that was proved in [183]:

Theorem (4.2.11)[225]. Let $a, b : [0, T] \to [0, \infty)$ and $\gamma : [0, T] \to R$ be C^1 functions. For $\varepsilon > 0$ and $f \in A_\varepsilon$, we have

$$a(T)P_T(f\Gamma(\ln f)) + b(T)P_T(f\Gamma^z(\ln f))$$
$$-a(0)(P_T f)\Gamma(\ln P_T f) - b(0)(P_T f)\Gamma^z(\ln P_T f)$$
$$\geq \int_0^T \left(a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - 4\frac{a\gamma}{m} \right) \Phi_1 ds + \int_0^T (b' + 2\rho_2 a) \Phi_2 ds$$
$$+ \frac{4}{m} \int_0^T a\gamma ds \, LP_T f - \frac{2}{m} \int_0^T a\gamma^2 \, ds \, P_T f.$$

Henceforth, we let

$$D = \left(1 + 3\frac{\kappa}{2\rho_2} \right) m. \tag{45}$$

The following scale invariant Harnack inequality for the heat kernel was also proved in [183].

Proposition (4.2.12)[225]. Let $p(x, y, t)$ be the heat kernel on M. For every $x, y, z \in M$ and every $0 < s < t < \infty$ one has

$$p(x, y, s) \leq p(x, z, t) \left(\frac{t}{s} \right)^{\frac{D}{2}} \exp \left(\frac{D}{m} \frac{d(y, z)^2}{4(t - s)} \right)$$

A basic consequence of this Harnack inequality is the control of the volume growth of balls centered at a given point.

Proposition (4.2.13)[225]. For every $x \in M$ and every $R_0 > 0$ there is a constant $C(m, \kappa, \rho_2) > 0$ such that,

$$\mu(B(x, R)) \leq \frac{C(m, \kappa, \rho_2)}{R_0^D p(x, x, R_0^2)} R^D, \qquad R \geq R_0.$$

Proof. Fix $x \in M$ and $t > 0$. Applying Proposition (4.2.12) to $p(x, y, t)$ for every $y \in B(x, \sqrt{t})$ we find

$$p(x, x, t) \leq 2^{\frac{D}{2}} e^{\frac{D}{4m}} p(x, y, 2t) = C(m, \kappa, \rho_2) p(x, y, 2t).$$

Integration over $B(x, \sqrt{t})$ gives

$$p(x,x,t)\mu(B(x,\sqrt{t})) \le C(m,\kappa,\rho_2) \int_{B(x,\sqrt{t})} p(x,y,2t)\,d\mu(y) \le C(m,\kappa,\rho_2),$$

where we have used $P_t 1 \le 1$. This gives the on-diagonal upper bound

$$p(x,x,t) \le \frac{C(m,\kappa,\rho_2)}{\mu\left(B(x,\sqrt{t})\right)}. \tag{46}$$

Let now $t > \tau > 0$. Again, from the Harnack inequality of Proposition (4.2.12), we have

$$p(x,x,t) \ge p(x,x,\tau)\left(\frac{\tau}{t}\right)^{\frac{D}{2}}.$$

The inequality (46) finally implies the desired conclusion by taking $t = R^2$ and $\tau = R_0^2$.

We derive some functional inequalities which will play a fundamental role in the proof of Theorem (4.2.17) below.

Proposition (4.2.14)[225]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$. For $x \in M, t, \tau > 0$, and $C \in R$, one has

$$\frac{\tau}{\rho_2} P_t(f\Gamma(\ln f))(x) + \tau^2 P_t(f\Gamma^Z(\ln f))(x)$$

$$+ \frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)[P_t(f\ln f)(x) - P_t f(x)\ln P_t f(x)]$$

$$\ge \frac{t+\tau}{\rho_2} P_t f(x)\Gamma(\ln P_t f)(x) + (t+\tau)^2 P_t f(x)\Gamma^Z(\ln P_t f)(x)$$

$$+ \frac{4Ct}{\rho_2 m} L P_t f(x) - \frac{2C^2}{m\rho_2}\ln\left(1 + \frac{t}{\tau}\right) P_t f(x).$$

Proof. Let $T, \tau > 0$ be arbitrarily fixed. We apply Theorem (4.2.11) with $\rho_1 = 0$, in which we choose

$$b(t) = (T + \tau - t)^2, \quad a(t) = \frac{1}{\rho_2}(T + \tau - t), \quad \gamma(t) = \frac{C}{T + \tau - t}, \quad 0 \le t \le T.$$

With such choices we obtain

$$\begin{cases} a' - 2\kappa\frac{a^2}{b} - 4\frac{a\gamma}{m} \equiv \frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right), \\ b' + 2\rho_2 a \equiv 0, \\ \int_0^T \frac{4a\gamma}{m} = \frac{4CT}{\rho_2 m'}, \\ and \\ -\int_0^T \frac{2a\gamma^2}{m} = -\frac{2C^2}{m\rho_2}\ln\left(1 + \frac{t}{\tau}\right). \end{cases} \tag{47}$$

Keeping (75) in mind, we obtain the sought for conclusion with T in place of t. The arbitrariness of $T > 0$ finishes the proof.

Corollary (4.2.15)[225]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$. For $x \in M, t > 0$ one has

$$t P_t f(x)\Gamma(\ln P_t f)(x) + \rho_2 t^2 P_t f(x)\Gamma^Z(\ln P_t f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2}\right) [P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)].$$

Proof. We first apply Proposition (4.2.14) with $C = 0$, and then we let $\tau \to 0^+$ in the resulting inequality.

We may actually improve Corollary (4.2.15) and obtain the following crucial dimension dependent reverse log-Sobolev inequality.

Theorem (4.2.16)[225]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$, then for every $C \geq 0$ and $\delta > 0$, one has for $x \in M, t > 0$,

$$t P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right) [P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)]$$

$$- \frac{4C}{\rho_2 m} \frac{t}{1+\delta} L P_t f(x) + \frac{2C^2}{\rho_2 m} \ln\left(1 + \frac{1}{\delta}\right) P_t f(x). \qquad (48)$$

Proof. For $x \in M, t, \tau > 0$, we apply Proposition (4.2.14) to the function $P_t f$ instead of f. Recalling that $P_t(P_t f) = P_{t+\tau} f$, we obtain, for all $C \in R$,

$$\frac{\tau}{\rho_2} P_t(P_\tau f \Gamma(\ln P_\tau f)(x) + \tau^2 P_t(P_\tau f \Gamma^Z(\ln P_\tau f)(x)$$

$$+ \frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right) [P_t(P_\tau f \ln P_\tau f)(x) - P_{t+\tau} f(x) \ln P_{t+\tau} f(x)]$$

$$\geq \frac{t+\tau}{\rho_2} P_{t+\tau}(f(x) \Gamma(\ln P_{t+\tau} f)(x) + (t+\tau)^2 P_{t+\tau} f(x) \Gamma^Z(\ln P_{t+\tau} f)(x)$$

$$+ \frac{4C}{\rho_2 m} t \, L P_{t+\tau} f(x) - \frac{2C^2}{\rho_2 m} \ln\left(1 + \frac{1}{\tau}\right) P_{t+\tau} f(x) \qquad (49)$$

Invoking Proposition (4.2.15) we now find for every $x \in M, \tau > 0$,

$$\tau P_\tau f(x) \Gamma(\ln P_\tau f)(x) + \rho_2 \tau^2 P_\tau f(x) \Gamma^Z(\ln P_\tau f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2}\right) [P_\tau(f \ln f)(x) - P_\tau f(x) \ln P_\tau f(x)].$$

If we now apply P_t to this inequality, we obtain

$$\tau P_t f(x) \Gamma(\ln P_\tau f)(x) + \rho_2 \tau^2 P_t(P_\tau f \Gamma^Z(\ln P_\tau f)(x)$$

$$\leq \left(1 + \frac{2\kappa}{\rho_2}\right) [P_{t+\tau}(f \ln f)(x) - P_t(P_\tau f \ln P_\tau f)(x)].$$

We use this inequality to bound from above the first two terms in the left-hand side of (49), obtaining

$$\frac{1 + \frac{2\kappa}{\rho_2}}{\rho_2} P_{t+\tau}(f \ln f)(x) + \frac{4C}{\rho_2 m} P_t(P_\tau f \ln P_\tau f)(x)$$

113

$$-\frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)P_{t+\tau}f(x)\ln P_{t+\tau}f(x)$$

$$\geq \frac{t+\tau}{\rho_2}P_{t+\tau}f(x)\Gamma(\ln P_{t+\tau}f)(x) + (t+\tau)^2 P_{t+\tau}f(x)\Gamma^Z(\ln P_{t+\tau}f)(x)$$

$$+\frac{4C}{\rho_2 m}t\, LP_{t+\tau}f(x) - \frac{2C^2}{\rho_2 m}\ln\left(1 + \frac{1}{\tau}\right)P_{t+\tau}f(x)$$

Consider the convex *function* $\Phi(s) = s\ln s$, $s > 0$. Thanks to Jensen's inequality, we have for any $\tau > 0$ and $x \in M$

$$\Phi(P_\tau f(x)) \leq P_\tau(\Phi(f))(x),$$

which we can rewrite

$$P_\tau f(x)\ln P_\tau f(x) \leq P_\tau(f\ln f)(x).$$

For $C \geq 0$, applying P_t to this inequality we find

$$\frac{4C}{\rho_2 m}P_t(P_\tau f\ln P_\tau f)(x) \leq \frac{4C}{\rho_2 m}P_{t+\tau}(f\ln f)(x).$$

We therefore conclude, for $C \geq 0$,

$$\frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)[P_{t+\tau}(f\ln f)(x) - P_{t+\tau}f(x)\ln P_{t+\tau}f(x)]$$

$$\geq \frac{t+\tau}{\rho_2}P_{t+\tau}f(x)\Gamma(\ln P_{t+\tau}f)(x) + (t+\tau)^2 P_{t+\tau}f(x)\Gamma^Z(\ln P_{t+\tau}f)(x)$$

$$+\frac{4C}{\rho_2 m}t\, LP_{t+\tau}f(x) - \frac{2C^2}{\rho_2 m}\ln\left(1 + \frac{1}{\tau}\right)P_{t+\tau}f(x)$$

If in the latter inequality we now choose $\tau = \delta t$, we find:

$$\frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)[P_{t+\delta t}(f\ln f)(x) - P_{t+\delta t}f(x)\ln P_{t+\delta t}f(x)]$$

$$\geq \frac{t+\delta t}{\rho_2}P_{t+\delta t}f(x)\Gamma(\ln P_{t+\delta t}f)(x) + (t+\delta t)^2 P_{t+\delta t}f(x)\Gamma^Z(\ln P_{t+\delta t}f)(x)$$

$$+\frac{4C}{\rho_2 m}t\, LP_{t+\delta t}f(x) - \frac{2C^2}{\rho_2 m}\ln\left(1 + \frac{1}{\tau}\right)P_{t+\delta t}f(x)$$

Changing $(1 + \delta)t$ into t in the latter inequality, we finally conclude:

$$\frac{t}{\rho_2}P_t f(x)\Gamma(\ln P_t f)(x) + t^2 P_t f(x)\Gamma^Z(\ln P_t f)(x)$$

$$\leq \frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)[P_t(f\ln f)(x) - P_t f(x)\ln P_t f(x)]$$

$$-\frac{4C}{\rho_2 m}\frac{t}{1+\delta}LP_t f(x) - \frac{2C^2}{\rho_2 m}\ln\left(1 + \frac{1}{\delta}\right)P_t f(x)$$

This gives the desired conclusion (48).

Our principal objective is proving the following result.

As a first step, we prove a small time asymptotics result interesting in itself. In what follows for a given set $A \subset M$ we will denote by 1_A its indicator function.

Proposition (4.2.17)[225]. Given $x \in M$ and $r > 0$, let $f = 1_{B(x,r)^c}$. One has,

$$\liminf_{s \to 0^+} (-s \ln P_s f(x)) \geq \frac{r^2}{4}.$$

Proof. To prove the proposition it will suffice to show that

$$\liminf_{t \to 0^+} (t \ln P_t f(x)) \geq -\frac{r^2}{4}$$

Let $0 < \varepsilon < r$. By the Harnack inequality of Proposition (4.2.12) and the symmetry of the heat kernel, we have for $y \in M$ and $z \in B(x, \varepsilon)$,

$$p(x, y, t) \leq p(z, y, (1 + \varepsilon)t) 2^{D/m} e^{\frac{D\varepsilon}{4mt}}.$$

Therefore, multiplying the above inequality by $f(y) = 1_{B(x,r)^c}(y)$ and then integrating with respect to y, we obtain

$$P_t f(x) \leq \left(P_{(1+\varepsilon)t} f\right)(z) 2^{D/m} e^{\frac{D\varepsilon}{4mt}}.$$

By integrating now with respect to $z \in B(x, \varepsilon)$, we get

$$P_t f(x) \leq \frac{2^{D/m} e^{\frac{D\varepsilon}{4mt}}}{\mu(B(x, \varepsilon))} \int_M 1_{B(x,\varepsilon)}(z)(P_{(1+\varepsilon)t} f)(z) d\mu(z).$$

Now, from Theorem 1.1 in [204] (for which normalization differs from us by a factor 1/2 because he considers the semigroup $e^{tL/2}$), we obtain:

$$\lim_{t \to 0} t \ln \int_M 1_{B(x,\varepsilon)}(z)(P_{(1+\varepsilon)t} f)(z) d\mu(z) = -\frac{(r - \varepsilon)^2}{4(1 + \varepsilon)}.$$

This yields therefore

$$\liminf_{t \to 0^+} (t \ln P_t f(x)) \leq -\frac{(r - \varepsilon)^2}{4(1 + \varepsilon)} + \frac{D\varepsilon}{4m}.$$

We conclude by letting $\varepsilon \to 0$.

As a second step toward the proof of Theorem (4.2.22) we investigate some of the consequences of the reverse log-Sobolev inequality in theorem (4.2.16) for functions f such that $0 \leq f \leq 1$ (later, we will apply this to indicator functions).

Proposition (4.2.18)[225]. Let $\varepsilon > 0$, $f \in A_\varepsilon, \varepsilon \leq f \leq 1$, and consider the function $u(x, t) = \sqrt{-\ln P_t f(x)}$. Then, with the convention that $\frac{1}{0} = +\infty$, we have

$$2t u_t + u + \left(1 + \sqrt{\frac{D^*}{2}}\right) u^{1/3} + \sqrt{\frac{D^*}{2}} u^{-1/3} \geq 0,$$

where
$$D^* = m\left(1 + \frac{2\kappa}{\rho_2}\right).$$

Proof. Noting that we have
$$\frac{t}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x) \geq 0$$
$$\leq \frac{1}{\rho_2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)[P_t(f \ln f)(x) - P_t f(x) \ln P_t f(x)]$$
$$-\frac{4C}{\rho_2 m}\frac{t}{1+\delta} LP_t f(x) - \frac{2C^2}{\rho_2 m}\ln\left(1 + \frac{1}{\delta}\right)P_t f(x)$$

applying the inequality (48) in Theorem (4.2.16), we obtain that for all $C \geq 0$,
$$\frac{m}{2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)P_t(f \ln f)(x) - \frac{m}{2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)(P_t f)\ln P_t f$$
$$-\frac{2Ct}{1+\delta} LP_t f(x) - \frac{C^2}{\delta} P_t f \geq 0,$$

where we used the fact that
$$\ln\left(1 + \frac{1}{\delta}\right) \leq \frac{1}{\delta}.$$

On the other hand, the hypothesis $0 \leq f \leq 1$ implies $f \ln f \leq 0$. After dividing both sides of the above inequality by $P_t f$, we thus find
$$-\frac{m}{2}\left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{m}\right)\ln P_t f - \frac{2Ct}{1+\delta}\frac{LP_t f}{P_t f} + \frac{C^2}{\delta} \geq 0.$$

Dividing both sides by $C > 0$, this may be re-written
$$-\frac{D^*}{2C}\ln P_t f - 2\ln P_t f - \frac{2t}{1+\delta}\frac{LP_t f}{P_t f} + \frac{C}{\delta} \geq 0 \qquad (50)$$

We now minimize the left-hand side of (50) with respect to C. The minimum value is attained in
$$C = \sqrt{-\frac{\delta D^*}{2}\ln P_t f}.$$

Substituting this value in (50), we obtain
$$\sqrt{-\frac{\delta D^*}{2}}\sqrt{-\ln P_t f} - 2P_t f - \frac{2t}{1+\delta}\frac{LP_t f}{P_t f} + \frac{C}{\delta} \geq 0.$$

With $u(x,t) = \sqrt{-\ln P_t f(x)}$, and noting that $u_t = -\frac{1}{2u}\frac{LP_t f}{P_t f}$, we can re-write this inequality as follows,

116

$$\sqrt{\frac{\delta D^*}{2\delta}} + u + \frac{2t}{1+\delta} u_t \geq 0,$$

or equivalently,

$$2tu_t + u + \delta u + (1+\delta)\sqrt{\frac{D^*}{2\delta}} \geq 0.$$

If we choose

$$\delta = \frac{1}{u^{2/3}},$$

we obtain the desired conclusion.

We now introduce the function $g : (0,\infty) \rightarrow (0,\infty)$ defined by

$$g(v) = \frac{1}{v + \left(1 + \sqrt{\frac{D^*}{2}}\right) v^{1/3} + \sqrt{\frac{D^*}{2}} v^{-1/3}}. \tag{51}$$

One easily verifies that

$$\lim_{v\to 0^+} \sqrt{\frac{D^*}{2}}\, v^{-\frac{1}{3}} g(v) = 1, \quad \lim_{v\to\infty} vg(v) = 1.$$

These limit relations show that $g \in L^1(0,A)$ for every $A > 0$, but $g \notin L^1(0,\infty)$. Moreover, if we set

$$G(u) = \int_0^u g(v)\, dv,$$

then $G'(u) = g(u) > 0$, and thus $G: (0,\infty) \rightarrow (0,\infty)$ is invertible. Furthermore, as is seen from (42), as $u \to \infty$ we have

$$G(u) = \ln u + C_0 + R(u), \tag{52}$$

where C_0 is a constant and $\lim_{u\to\infty} R(u) = 0$. At this point we notice that, in terms of the function $g(u)$, we can re-express the conclusion of Proposition (4.2.18) in the form

$$2tu_t + \frac{1}{g(u)} \geq 0.$$

Keeping in mind that $g(u) = G'(u)$, we thus conclude

$$\frac{dG(u)}{dt} = G'(u)u_t \geq -\frac{1}{2t}. \tag{53}$$

From this identity we now obtain the following basic result.

Corollary (4.2.19)[225]. Let $f \in L^\infty(M), 0 \leq f \leq 1$, then for any $x \in M$ and $0 < s < t$,

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq G\left(\sqrt{-\ln P_s f(x)}\right) - \frac{1}{2}\ln\left(\frac{t}{s}\right).$$

Proof. If $f \in A_\epsilon$ for some ϵ, the inequality is a straightforward consequence of the above results. In fact, keeping in mind that $u(x,t) = \sqrt{-\ln P_t f(x)}$, in order to reach the desired conclusion all we need to do is to integrate (53) between s and t. Consider now $f \in L^\infty(M)$, $0 \le f \le 1$. Let $h_n \in C_0^\infty(M)$, with $0 \le h_n \le 1$, and $h_n \nearrow 1$. For $n \ge 0, \tau > 0$ and $\epsilon > 0$, the function

$$(1 - \varepsilon)P_\tau(h_n f) + \varepsilon \in A_\varepsilon.$$

Therefore,

$$G\left(\sqrt{-\ln P_t f\,(1-\varepsilon)P_\tau(h_n f) + \varepsilon(x)}\right)$$
$$\ge G\left(\sqrt{-\ln P_s f\,(1-\varepsilon)P_\tau(h_n f) + \varepsilon(x)}\right) - \frac{1}{2}\ln\left(\frac{t}{s}\right).$$

Letting $\epsilon \to 0, \tau \to 0$ and finally $n \to \infty$, we obtain the desired conclusion for f. This completes the proof.

Combining Corollary (4.2.19) with Proposition (4.2.17) we obtain the following key estimate.

Proposition (4.2.20)[225]. Let $x \in M$ and $r > 0$ be arbitrarily fixed. There exists $C_0^* \in R$, independent of x and r, such that for any $t > 0$,

$$\left(\sqrt{-\ln P_t 1_{B(x,r)^c}(x)}\right) \ge \ln\frac{r}{\sqrt{t}} + C_0^*.$$

Proof. Re-write the inequality claimed in Corollary (4.2.19) as follows

$$G\left(\sqrt{-\ln P_t f(x)}\right) \ge G\left(\sqrt{-\ln P_s f(x)}\right) + \ln\sqrt{s} - \ln\sqrt{t},$$

where we have presently let $f(y) = 1_{B(x,r)^c}(y)$. Since for this function we have, from Proposition (4.2.17),
$\lim_{s\to 0^+}(-\ln P_s f(x)) = \infty$, using (52) we see that, for $s \to 0+$, the latter inequality is equivalent to

$$G\left(\sqrt{-\ln P_t f(x)}\right) \ge \ln\sqrt{-\ln P_s f(x)} - \ln\sqrt{t} + C_0 + R\left(\sqrt{-\ln P_s f(x)}\right).$$

We now take the lim inf as $s \to 0^+$ of both sides of this inequality. Applying Proposition (4.2.17) we deduce

$$G\left(\sqrt{-\ln P_t f(x)}\right) \ge \ln\frac{r}{2} - \ln\sqrt{t} + C_0 = \ln\frac{r}{\sqrt{t}} + C_0^*,$$

where we have let $C_0^* = C_0 - \ln 2$. This establishes the desired conclusion.
We are now in a position to prove the central result.

Theorem (4.2.21)[225]. There exists a constant $A > 0$ such that for every $x \in M$, and $r > 0$,

$$P_{Ar^2}\left(1_{B(x,r)}\right)(x) \ge \frac{1}{2}.$$

Proof. By the stochastic completeness of M we know that $P_t 1 = 1$. Therefore,

118

$$P_{Ar^2}\left(1_{B(x,r)}\right)(x) = 1 - P_{Ar^2}\left(1_{B(x,r)^c}\right)(x).$$

We conclude that the desired estimate is equivalent to proving that there exists an absolute constant $A > 0$ such that

$$\left(\sqrt{\ln 2}\right) \leq \left(\sqrt{-\ln P_{Ar^2}\left(1_{B(x,r)^c}\right)(x)}\right),$$

or, equivalently,

$$G\left(\sqrt{\ln 2}\right) \leq G\left(\sqrt{-\ln P_{Ar^2}\left(1_{B(x,r)^c}\right)(x)}\right) \tag{54}$$

At this point we invoke Proposition (4.2.20), which gives

$$G\left(\sqrt{-\ln P_{Ar^2}\left(1_{B(x,r)^c}\right)(x)}\right) \geq \ln\left(\frac{1}{\sqrt{A}} + C_0^*\right)$$

It is thus clear that, letting $A \to 0^+$, we can certainly achieve (54), thus completing the proof.

Theorem (4.2.22)[225]. (Global doubling property) The metric measure space (M, d, μ) satisfies the global volume doubling property. More precisely, there exists a constant $C_1 = C_1(\rho_1, \rho_2, \kappa, d) > 0$ such that for every $x \in M$ and every $r > 0$,

$$\mu(B(x, 2r)) \leq C_1 \mu(B(x, r)).$$

Proof. The argument which shows how to obtain Theorem (4.2.22) from Theorem (4.2.21) was developed independently by Grigor'yan [197] and by Saloff-Coste [214], and it is by now well-known. However, since it is short for the sake of completeness in what follows we provide the relevant details. From the semigroup property and the symmetry of the heat kernel we have for any $y \in M$ and $t > 0$

$$p(y, y, 2t) = \int_M p(y, z, t)^2 d\mu(z).$$

Consider now a function $h \in C_0^\infty(M)$ such that $0 \leq h \leq 1, h \equiv 1$ on $B(x, \sqrt{t}/2)$ and $h \equiv 0$ outside $B(x, \sqrt{t})$. We thus have

$$P_t h(y) = \int_M p(y, z, t)h(z)d\mu(z) \leq \left(\int_M p(y, z, t)^2 d\mu(z)\right)^{\frac{1}{2}} \left(\int_M h(z)^2 d\mu(z)\right)^{\frac{1}{2}}$$

$$\leq p(y, y, 2t)^{\frac{1}{2}} \mu(B(x, \sqrt{t}))^{\frac{1}{2}}.$$

If we take $y = x$, and $t = r^2$, we obtain

$$P_{r^2}\left(1_{B(x,r)}\right)(x)^2 \leq P_{r^2} h(x)^2 \leq p(x, x, 2r^2) \mu(B(x, r)). \tag{55}$$

At this point we use Theorem (4.2.21), which gives for some $0 < A < 1$, (the fact that we can choose $A < 1$ is clear from the proof of Theorem (4.2.21)

$$P_{Ar^2}\big(1_{B(x,r/2)}\big)(x) \geq \frac{1}{2}, \qquad x \in M, \quad r > 0.$$

Combining this estimate with the Harnack inequality in Proposition (4.2.12) and with (56), we obtain the following on-diagonal lower bound

$$p(x,x,2r^2) \geq \frac{C^*}{\mu\big(B(x,r)\big)}, \qquad x \in M, \qquad r > 0. \tag{56}$$

Applying Proposition (4.2.12) we find for every $y \in B(x,\sqrt{t})$,

$$p(x,x,t) \leq Cp(x,y,2t).$$

Integration over $B(x,\sqrt{t})$ gives

$$p(x,x,t)\mu(B(x,\sqrt{t})) \leq C \int_{B(x,\sqrt{t})} p(x,y,2t)d\mu(y) \leq C,$$

where we have used $P_t 1 \leq 1$. Letting $t = r^2$, we obtain from this the on-diagonal upper bound

$$p(x,x,r^2) \leq \frac{C}{\mu\big(B(x,r)\big)}. \tag{57}$$

Combining (56) with (57) we finally obtain

$$\mu\big(B(x,2r)\big) \leq \frac{C}{p(x,x,4r^2)} \leq \frac{CC'}{p(x,x,2r^2)} \leq C^{**}\mu\big(B(x,r)\big),$$

where we have used once more Proposition (4.2.12) (with $y = z = x$), which gives

$$\frac{p(x,x,2r^2)}{p(x,x,4r^2)} \leq C'$$

and we have let $C^{**} = CC'(C^*)^{-1}$. This completes the proof.

It is well-known that Theorem (4.2.22) provides the following uniformity control at all scales.

Theorem (4.2.23)[225]. With C_1 being the constant in Theorem (4.2.22), let $Q = \log_2 C_1$. For any $x \in M$ and $r > 0$ one has

$$\mu(B(x,tr)) \geq C_1^{-1} t^Q \mu(B(x,r)), \qquad 0 \leq t \leq 1.$$

The purpose is to establish some optimal two-sided bounds for the heat kernel $p(x,y,t)$ associated with the subelliptic operator L. Such estimates are reminiscent of those obtained by Li and Yau for complete Riemannian manifolds having $Ric \geq 0$. As a consequence of the two-sided Gaussian bound for the heat kernel, we will derive the Poincaré inequality and the local parabolic Harnack inequality thanks to well-known results in the works [193,197,198,214,216–218].

We assume, once again, that the assumptions (Hypothesis (4.2.3))–(Hypothesis (4.2.7)) be satisfied, and that the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ hold, with $\rho_1 \geq 0$. Here is the main result.

Theorem (4.2.24)[225]. For any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(m, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \to 0^+$, such that for every $x, y \in M$ and $t > 0$ one has

$$\frac{C(\varepsilon)^{-1}}{\mu\left(B(x, \sqrt{t})\right)} exp\left(-\frac{Dd(x, y)^2}{m(4 - \varepsilon)t}\right) \leq p(x, y, t) \leq \frac{C(\varepsilon)}{\mu\left(B(x, \sqrt{t})\right)} exp\left(-\frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

Proof. We begin by establishing the lower bound. First, from Proposition (4.2.12) we obtain for all $y \in M, t > 0$, and every $0 < \varepsilon < 1$,

$$p(x, y, t) \geq p(x, x, \varepsilon t)\varepsilon^{\frac{D}{2}} exp\left(-\frac{D}{m}\frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

We thus need to estimate $p(x, x, \varepsilon t)$ from below. But this has already been done in (4.2.23). Choosing $r > 0$ such that $2r^2 = \varepsilon t$, we obtain from that estimate

$$p(x, x, \varepsilon t) \geq \frac{C^*}{\mu(B(x, \sqrt{\varepsilon/2}\,\sqrt{t}))} \quad , x \in M, t > 0.$$

On the other hand, since $\sqrt{\varepsilon/2} < 1$, by the trivial inequality $\mu(B(x, \sqrt{\varepsilon/2}\sqrt{t})) \leq \mu(B(x, \sqrt{t}))$, we conclude

$$p(x, y, t) \geq \frac{C^*}{\mu(B(x, \sqrt{t}))}\varepsilon^{\frac{D}{2}} exp\left(-\frac{D}{m}\frac{d(x, y)^2}{(4 - \varepsilon)t}\right)$$

This proves the Gaussian lower bound.

For the Gaussian upper bound, we first observe that the following upper bound is proved in [183]:

$$p(x, y, t) \geq \frac{C(m, \kappa, \rho_2, \varepsilon')}{\mu(B(x, \sqrt{t}))^{\frac{1}{2}}\mu(B(x, \sqrt{t}))^{\frac{1}{2}}} exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon')t}\right)$$

At this point, by the triangle inequality and Theorem (4.2.23) we find.

$$\mu(B(x, \sqrt{t})) \leq \mu(B(y, d(x, y) + \sqrt{t}))$$

$$\leq C_1\mu(B(y, \sqrt{t}))\left(\frac{d(x, y) + \sqrt{t}}{\sqrt{t}}\right)^Q.$$

This gives

$$\frac{1}{\mu(B(y, \sqrt{t}))} \leq \frac{C_1}{\mu(B(y, \sqrt{t}))}\left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^Q.$$

Combining this with the above estimate we obtain

$$p(x, y, t) \geq \frac{C_1^{1/2}(m, \kappa, \rho_2, \varepsilon')}{\mu(B(y, \sqrt{t}))}\left(\frac{d(x, y)}{\sqrt{t}} + 1\right)^{Q/2} exp\left(-\frac{d(x, y)^2}{(4 + \varepsilon')t}\right)$$

121

If now $0 < \epsilon < 1$, it is clear that we can choose $0 < \varepsilon' < \varepsilon$ such that

$$\frac{C_1^{1/2}(m,\kappa,\rho_2,\varepsilon')}{\mu(B(y,\sqrt{t}))}\left(\frac{d(x,y)}{\sqrt{t}}+1\right)^{Q/2} exp\left(-\frac{d(x,y)^2}{(4+\varepsilon')t}\right)$$
$$\leq \frac{C^*(m,\kappa,\rho_2,\varepsilon)}{\mu(B(y,\sqrt{t}))} exp\left(-\frac{d(x,y)^2}{(4+\varepsilon')t}\right)$$

where $C^*(m,\kappa,\rho_2,\epsilon)$ is a constant which tends to ∞ as $\epsilon \to 0^+$. The desired conclusion follows by suitably adjusting the values of both ε' and of the constant in the right-hand side of the estimate.

With Theorems (4.2.22) and (4.2.24) in hands, we can now appeal to the results in [193,197,208,214,216-218]. From the developments it is by now well-known that strictly regular local Dirichlet spaces we have the equivalence between:

(i) A two sided Gaussian bounds for the heat kernel (like in Theorem 4.2.24);

(ii) The conjunction of the volume doubling property and the Poincaré inequality (see Theorem 4.2.25);

(iii) The parabolic Harnack inequality (see Theorem 4.2.27).

For uniformly parabolic equations in divergence form the equivalence between (i) and (iii) was first proved in [193]. The fact that (i) implies the volume doubling property is almost straight forward, the argument may be found in [215]. The fact that (i) also implies the Poincaré inequality relies on a beautiful and general argument by Kusuoka and Stroock [208]. The equivalence between (ii) and (iii) originates from [197,214] and has been worked out of strictly local regular Dirichlet spaces in [218]. Finally, the fact that (ii) implies (i) is also proven in [218].

We obtain the following weaker form of Poincaré inequality. We already know the volume doubling property since we proved it to obtain the Gaussian estimates.

Theorem (4.2.25)[225]. There exists a constant $C = C(m,\kappa,\rho_2) > 0$ such that for every $x \in M, r > 0$, and $f \in C^\infty(M)$ one has

$$\int_{B(x,r)} |f(y) - f_r|^2\, d\mu(y) \leq Cr^2 \int_{B(x,2r)} \Gamma(f)(y)d\mu(y),$$

where we have let $f_r = \frac{1}{\mu(B(x,r))} \int_{B(x,2r)} B(x,r)\, fd\mu.$

Since thanks to Theorem (4.2.22) the space (M,μ,d), where $d = d(x,y)$ indicates the sub-Riemannian distance, is a space of homogeneous type, and it is also a length-space in the sense of Gromov, arguing as in [206] we now conclude with the following result.

Corollary (4.2.26)[225]. There exists a constant $C^* = C^*(m,\kappa,\rho_2) > 0$ such that for every $x \in M, r > 0$, and $f \in C^\infty(M)$ one has

$$\int_{B(x,r)} |f(y) - f_r|^2 \, d\mu(y) \leq C^*r^2 \int_{B(x,r)} \Gamma(f)(y) \, d\mu(y),$$

Furthermore, the following scale invariant Harnack inequality for local solutions holds.

Theorem (4.2.27)[225]. If u is a positive solution of the heat equation in a cylinder of the form $Q = (s, s + \alpha r^2) \times B(x, r)$ then

$$\sup_{Q-} u \leq C \inf_{Q+} u, \tag{58}$$

where for some fixed $0 < \beta < \gamma < \delta < \alpha < \infty$ and $\eta \in (0,1)$,

$$Q- = (s + \beta r^2, s + \gamma r^2) \times B(x, \eta r), \quad Q+ = (s + \delta r^2, s + \alpha r^2) \times B(x, \eta r).$$

Here, the constant C is independent of x, r and u, but depends on the parameters m, κ, ρ_2, as well as on $\alpha, \beta, \gamma, \delta$ and η.

In [183] were able to establish a Yau type Liouville theorem stating that when M is complete, and the generalized curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, m)$ holds for $\rho_1 \geq 0$, then there exist no bounded solutions of $Lf = 0$ on M besides the constants. Note that this result is weaker thanYau'soriginal Riemannian result in [223] since this author only assumes a one-side bound. However, as a consequence of Theorems (4.2.22) and (4.2.25) we can now remove such limitation and obtain the following complete sub-Riemannian analogue of Yau's Liouville theorem.

Theorem (4.2.28)[225]. There exist no positive solutions of $Lf = 0$ on M besides the constants.

We can now prove much more. In [190] Colding and Minicozzi obtained a complete resolution of Yau's famous conjecture that the space of harmonic functions with a fixed polynomial growth at infinity on an open manifold with $Ric \geq 0$ is finite dimensional. A fundamental discovery is the fact that such property can be solely derived from the volume doubling condition and the Neumann–Poincaré inequality. However, at the time [190] was written the only application of such theorem that could be given was to Lie groups with polynomial volume growth.

If we combine Theorem (4.2.22) and Corollary (4.2.26) above, we can considerably broaden the scope of Colding and Minicozzi's result and generalize it to the geometric framework covered. We obtain in fact the following generalization of Yau's conjecture. Given a fixed base point $x_0 \in M$, and a number $N \in \mathbb{N}$, we will indicate with $HN(M, L)$ the linear space of all solutions of $Lf = 0$ on M such that there exist a constant $C < \infty$ for which

$$|f(x)| \leq C(1 + d(x, x_0)^N), \qquad x \in M.$$

Theorem (4.2.29)[225]. For every $N \in \mathbb{N}$ one has: $\dim \mathcal{H}_N(M, L) < \infty$.

Corollary(4.2.30)[291]. For every $x \in M$ and every $\epsilon \geq 0$ there is a constant $C(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon) > 0$ such that,

$$\mu\big(B(x, (R_0 + \epsilon))\big) \leq \frac{C(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon)}{R_0^D p(x, x, R_0^2)} (R_0 + \epsilon)^D, \qquad \epsilon \geq 0.$$

Proof. Fix $x \in M$ and $\epsilon \geq 0$. Applying to $p(x, x + \epsilon, 1 + \epsilon)$ for every $(x + \epsilon) \in B(x, \sqrt{1 + \epsilon})$ we find

$$p(x, x, 1 + \epsilon) \leq 2^{\frac{D}{2}} e^{\frac{D}{4(1+\epsilon)}} p(x, x + \epsilon, 2(1 + \epsilon))$$
$$= C(1 + \epsilon, 1 + \epsilon, 2(1 + \epsilon)) p(x, x + \epsilon, 2(1 + \epsilon)).$$

Integration over $B(x, \sqrt{1 + \epsilon})$ gives

$$p(x, x, x + \epsilon)\mu(B(x, \sqrt{1 + \epsilon}) \leq C(1 + \epsilon, 1 + \epsilon, 1 + \epsilon) \int_{B(x, \sqrt{x+\epsilon})} p(x, x + \epsilon, 2(1 + \epsilon)) \, d\mu(x + \epsilon)$$

$$\leq C(1 + \epsilon, 1 + \epsilon, 2(1 + \epsilon)),$$

where we have used $P_{1+\epsilon}1 \leq 1$. This gives the on-diagonal upper bound

$$p(x, x, s + \epsilon) \leq \frac{C(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon)}{\mu\big(B(x, \sqrt{s + \epsilon})\big)}. \tag{59}$$

Let now $\tau + \epsilon > 0$. Again, from the Harnack inequality, we have

$$p(x, x, \tau + \epsilon) \geq p(x, x, \tau)\left(\frac{\tau}{\tau + \epsilon}\right)^{\frac{D}{2}}.$$

Corollary(4.2.31)[291]. Let $\varepsilon > 0$ and the sequence $f_m \in A_\varepsilon$. For $x \in M, \tau + \epsilon > 0$, and $C \in R$, one has

$$\frac{\tau}{(1 + 2\epsilon)} P_{1+\epsilon}(f_m \Gamma(\ln f_m))(x) + \tau^2 P_{1+\epsilon}(f_m \Gamma^{X+2\epsilon}(\ln f_m))(x)$$

$$+ \frac{1}{1 + \epsilon}\left(3 + \frac{4C}{1 + \epsilon}\right) [P_{1+\epsilon}(f_m \ln f_m)(x) - P_{1+\epsilon}f_m(x) \ln P_{1+\epsilon} f_m(x)]$$

$$\geq \frac{2\tau + \epsilon}{(1 + 2\epsilon)} P_{1+\epsilon} f_m(x)\Gamma(\ln P_{1+\epsilon}f_m)(x) + (2\tau + \epsilon)^2 P_{\tau+\epsilon} f_m(x)\Gamma^{X+2\epsilon}(\ln P_{\tau+\epsilon} f_m)(x)$$

$$+ \frac{4C}{(1 + 2\epsilon)} L P_{\tau+\epsilon} f_m(x) - \frac{2C^2}{(1 + \epsilon)(1 + 2\epsilon)} \ln\left(1 + \frac{\tau + \epsilon}{\tau}\right) P_{\tau+\epsilon}f_m(x).$$

Proof. Let $T, \tau > 0$ be arbitrarily fixed. With $\epsilon > 0$, in which we choose

$$(a + \epsilon)(1 + \epsilon) = (T + \epsilon)^2, \quad a(1 + \epsilon) = \frac{1}{1 + 2\epsilon}(T + \epsilon), \quad \gamma(\tau + \epsilon) = \frac{C}{T + \epsilon}, 0$$

$$\leq \tau + \epsilon \leq T.$$

With such choices we obtain

$$\begin{cases} a' - 2(1+\epsilon)\dfrac{a^2}{a+\epsilon} - 4\dfrac{a\gamma}{(1+\epsilon)} \equiv \dfrac{1}{(1+2\epsilon)}\left(1 + \dfrac{2(1+\epsilon)}{(1+2\epsilon)} + \dfrac{4C}{(1+\epsilon)}\right), \\ (1+\epsilon)' + 2(1+2\epsilon)a \equiv 0, \\ \displaystyle\int_0^T \dfrac{4a\gamma}{(1+\epsilon)} = \dfrac{4CT}{(1+2\epsilon)(1+\epsilon)}, \\ \qquad and \\ -\displaystyle\int_0^T \dfrac{2a\gamma^2}{(1+\epsilon)} = -\dfrac{2C^2}{(1+\epsilon)(1+2\epsilon)}\ln\left(1 + \dfrac{1+2\epsilon}{1+\epsilon}\right). \end{cases} \qquad (60)$$

We obtain the sought for conclusion with T in place of $\tau + \epsilon$. The arbitrariness of $T > 0$ finishes the proof.

Corollary(4.2.32)[291]. Let $\epsilon > 0$ and the sequence $f_m \in A_\varepsilon$. For $x \in M,\ \epsilon \geq -1$ one has

$$(1+\epsilon)P_{1+\epsilon}f_m(x)\Gamma(\ln P_{1+\epsilon}\, f_m)(x) + (1+2\epsilon)(1+\epsilon)^2 P_{1+\epsilon}f_m\,(x)\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}f_m)(x)$$

$$\leq \left(1 + \frac{2(1+\epsilon)}{(1+2\epsilon)}\right)[P_{1+\epsilon}(\,f_m \ln f_m\,)(x) - P_{1+\epsilon}f_m(x)\ln P_{1+\epsilon}\,f_m(x)].$$

Proof. We first apply with $C = 0$, and then we let $1 + \epsilon \to 0^+$ in the resulting inequality.

We may actually improve Corollary (4.2.32) and obtain the following crucial dimension dependent reverse log-Sobolev inequality.

Corollary(4.2.33)[291]. Let $\epsilon > 0$ and the sequence $f_m \in A_\epsilon$, then for every $C \geq 0$ and $\delta > 0$, one has for $x \in M, \epsilon \geq -1$,

$$(1+\epsilon)P_{1+\epsilon}f_m(x)\Gamma(\ln P_{1+\epsilon}\, f_m)(x) + (1+\epsilon)^2 P_{1+\epsilon}f_m\,(x)\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}f_m)(x)$$

$$\leq \left(1 + \frac{2(1+\epsilon)}{(1+2\epsilon)} + \frac{4C}{(1+\epsilon)}\right)[P_{1+\epsilon}(f_m \ln f_m\,)(x) - P_{1+\epsilon}f_m(x)\ln P_{1+\epsilon}\,f_m(x)]$$

$$-\frac{4C}{(1+2\epsilon)}\frac{1}{1+\delta}LP_{1+\epsilon}f_m(x) + \frac{2C^2}{(1+2\epsilon)(1+\epsilon)}\ln\left(1 + \frac{1}{\delta}\right)P_{1+\epsilon}f_m(x). \qquad (61)$$

Proof. For $x \in M, \epsilon \geq -1$, we apply to the function $P_{1+\epsilon}f_m$ instead of f_m. Recalling that $P_{1+\epsilon}(P_{1+\epsilon}f_m) = P_{2(1+\epsilon)}f_m$, we obtain, for all $C \in R$,

$$\frac{1+\epsilon}{(1+2\epsilon)}P_{1+\epsilon}(P_{1+\epsilon}f_m\Gamma(\ln P_{1+\epsilon}\, f_m\,)(x) + (1+\epsilon)^2 P_{1+\epsilon}(P_{1+\epsilon}f_m\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}f_m)(x)$$

$$+\frac{1}{(1+2\epsilon)}\left(1 + \frac{2(1+\epsilon)}{\rho_2} + \frac{4C}{(1+\epsilon)}\right)[P_{1+\epsilon}(\,P_{1+\epsilon}f_m \ln P_{1+\epsilon}f_m\,)(x) - P_{2(1+\epsilon)}f_m(x)\ln P_{2(1+\epsilon)}\,f_m(x)]$$

$$\geq \frac{P_{2(1+\epsilon)}}{(1+2\epsilon)}P_{2(1+\epsilon)}(f_m(x)\Gamma(\ln P_{2(1+\epsilon)}\, f_m)(x) + \left(2(1+\epsilon)\right)^2 P_{2(1+\epsilon)}f_m(x)\Gamma^{X+2\epsilon}(\ln P_{2(1+\epsilon)}f_m)(x)$$

$$+\frac{4C}{(1+2\epsilon)(1+\epsilon)}(1+\epsilon)LP_{2(1+\epsilon)}f_m(x) - \frac{2C^2}{(1+2\epsilon)(1+\epsilon)}\ln\left(\frac{2+\epsilon}{1+\epsilon}\right)P_{2(1+\epsilon)}f_m(x) \qquad (62)$$

We now find for every $x \in M, \epsilon \geq -1$,

$$(1+\epsilon)P_{1+\epsilon}f_m(x)\Gamma(\ln P_{1+\epsilon}\, f_m)(x) + (1+2\epsilon)(1+\epsilon)^2 P_{1+\epsilon}f_m(x)\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}f_m)(x)$$

$$\leq \left(1 + \frac{2(1+\epsilon)}{1+2\epsilon}\right)[P_{1+\epsilon}(\,f_m \ln f_m\,)(x) - P_{1+\epsilon}f_m(x)\ln P_{1+\epsilon}f_m(x)].$$

If we now apply $P_{1+\epsilon}$ to this inequality, we obtain

$$(1 + \epsilon)P_{1+\epsilon}f_m(x)\Gamma(\ln P_{1+\epsilon}\, f_m)(x) + (1$$
$$+ 2\epsilon)(1 + \epsilon)^2 P_{1+\epsilon}(P_{1+\epsilon}f_m\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}\, f_m)(x))$$
$$\leq \left(1 + \frac{2(1 + \epsilon)}{(1 + 2\epsilon)}\right)[P_{2(1+\epsilon)}(f_m \ln f_m)(x) - P_{1+\epsilon}(P_{1+\epsilon}f_m \ln P_{1+\epsilon}f_m)(x)].$$

We use this inequality to bound from above the first two terms, obtaining

$$\frac{1 + \dfrac{2(1 + \epsilon)}{(1 + 2\epsilon)}}{(1 + 2\epsilon)}P_{2(1+\epsilon)}(f_m \ln f_m)(x) + \frac{4C}{(1 + 2\epsilon)(1 + \epsilon)}\, P_{1+\epsilon}(P_{1+\epsilon}f_m \ln P_{1+\epsilon}f_m)(x)$$

$$- \frac{1}{(1 + 2\epsilon)}\left(1 + \frac{2(1 + \epsilon)}{(1 + 2\epsilon)} + \frac{4C}{(1 + \epsilon)}\right)P_{2(1+\epsilon)}f_m(x)\ln P_{2(1+\epsilon)}f_m(x)$$

$$\geq \frac{2(1 + \epsilon)}{(1 + 2\epsilon)}\, P_{2(1+\epsilon)}f_m(x)\Gamma(\ln P_{2(1+\epsilon)}f_m)(x)$$

$$+ \left(2(1 + \epsilon)\right)^2 P_{2(1+\epsilon)}f_m(x)\Gamma^{X+2\epsilon}(\ln P_{2(1+\epsilon)}f_m)(x)$$

$$+ \frac{4C}{(1 + 2\epsilon)}\, LP_{2(1+\epsilon)}f_m(x) - \frac{2C^2}{(1 + 2\epsilon)(1 + \epsilon)}\ln\left(\frac{2 + \epsilon}{1 + \epsilon}\right)P_{2(1+\epsilon)}f_m(x)$$

Consider the convex function $\Phi(1 + \epsilon) = (1 + \epsilon)\ln s,\ \epsilon \geq -1$. Thanks to Jensen's inequality, we have for any $\epsilon \geq -1$ and $x \in M$

$$\Phi(P_{1+\epsilon}f_m(x)) \leq P_{1+\epsilon}(\Phi(f_m))(x),$$

which we can rewrite

$$P_{1+\epsilon}f_m(x)\ln P_{1+\epsilon}\, f_m(x) \leq P_{1+\epsilon}(f_m \ln f_m)(x).$$

For $C \geq 0$, applying $P_{1+\epsilon}$ to this inequality we find

$$\frac{4C}{(1 + 2\epsilon)(1 + \epsilon)}\, P_{1+\epsilon}(P_{1+\epsilon}f_m \ln P_{1+\epsilon}\, f_m)(x) \leq \frac{4C}{(1 + 2\epsilon)(1 + \epsilon)}P_{2(1+\epsilon)}(f_m \ln f_m)(x).$$

We therefore conclude, for $C \geq 0$,

$$\frac{1}{1 + 2\epsilon}\left(1 + \frac{2(1 + \epsilon)}{1 + 2\epsilon} + \frac{4C}{1 + \epsilon}\right)[P_{2(1+\epsilon)}(f_m \ln f_m)(x) - P_{2(1+\epsilon)}f_m(x)\ln P_{2(1+\epsilon)}\, f_m(x)]$$

$$\geq \frac{2(1 + \epsilon)}{(1 + 2\epsilon)}P_{2(1+\epsilon)}f_m(x)\Gamma(\ln P_{2(1+\epsilon)}\, f_m)(x) + \left(2(1 + \epsilon)\right)^2 P_{2(1+\epsilon)}f_m(x)\Gamma^{X+2\epsilon}(\ln P_{2(1+\epsilon)}f_m)(x)$$

$$+ \frac{4C}{(1 + 2\epsilon)}LP_{2(1+\epsilon)}f_m(x) - \frac{2C^2}{(1 + 2\epsilon)(1 + \epsilon)}\ln\left(\frac{2 + \epsilon}{1 + \epsilon}\right)P_{2(1+\epsilon)}f_m(x)$$

If in the latter inequality we now choose $(1 + \epsilon) = \delta(1 + \epsilon)$, we find:

$$\frac{1}{(1 + 2\epsilon)}\left(2 + \epsilon + \frac{4C}{(1 + \epsilon)}\right)[P_{(1+\epsilon)+\delta(1+\epsilon)}(f_m \ln f_m)(x)$$

$$- P_{(1+\epsilon)+\delta(1+\epsilon)}f_m(x)\ln P_{(1+\epsilon)+\delta(1+\epsilon)}\, f_m(x)]$$

$$\geq \frac{(1 + \epsilon) + \delta(1 + \epsilon)}{(1 + 2\epsilon)} P_{(1+\epsilon)+\delta(1+\epsilon)} f_m(x) \Gamma(\ln P_{(1+\epsilon)+\delta(1+\epsilon)} f_m)(x)$$

$$+ \left((1 + \epsilon) + \delta(1 + \epsilon)\right)^2 P_{(1+\epsilon)+\delta(1+\epsilon)} f_m(x) \Gamma^{X+2\epsilon}(\ln P_{(1+\epsilon)+\delta(1+\epsilon)} f_m)(x)$$

$$+ \frac{4C}{(1 + 2\epsilon)} L P_{(1+\epsilon)+\delta(1+\epsilon)} f_m(x) - \frac{2C^2}{(1 + 2\epsilon)(1 + \epsilon)} \ln\left(\frac{2 + \epsilon}{1 + \epsilon}\right) P_{(1+\epsilon)+\delta(1+\epsilon)} f_m(x)$$

Changing $(1 + \delta)(1 + \epsilon)$ into $(1 + \epsilon)$ in the latter inequality, we finally conclude:

$$\frac{1 + \epsilon}{(1 + 2\epsilon)} P_{1+\epsilon} f_m(x) \Gamma(\ln P_{1+\epsilon} f_m)(x) + (1 + \epsilon)^2 P_{1+\epsilon} f_m(x) \Gamma^{X+2\epsilon}(\ln P_{1+\epsilon} f_m)(x)$$

$$\leq \frac{1}{(1 + 2\epsilon)}\left(1 + \frac{2(1 + \epsilon)}{(1 + 2\epsilon)} + \frac{4C}{(1 + \epsilon)}\right)[P_{1+\epsilon}(f_m \ln f_m)(x)$$

$$- P_{1+\epsilon} f_m(x) \ln P_{1+\epsilon} f_m(x)]$$

$$- \frac{4C}{(1+2\epsilon)}\frac{1}{1+\delta} L P_{1+\epsilon} f_m(x) - \frac{2C^2}{(1+2\epsilon)(1+\epsilon)} \ln\left(1 + \frac{1}{\delta}\right) P_{1+\epsilon} f_m(x) .$$

Corollary(4.2.34)[291]. Given $x \in M$ and $\epsilon \geq -1$, let the sequence $f_m = 1_{B(x,1+\epsilon)^c}$. One has,

$$\lim_{s \to 0^+} \inf (-s \ln P_s f_m(x)) \geq \frac{(1 + \epsilon)^2}{4} .$$

Proof. To prove the proposition it will suffice to show that

$$\lim_{1+\epsilon \to 0^+} \inf\left((1 + \epsilon) \ln P_{1+\epsilon} f_m(x)\right) \geq -\frac{(1 + \epsilon)^2}{4}$$

Let $0 < \epsilon < 1$. By the Harnack inequality of Proposition 2.2 and the symmetry of the heat kernel, we have for $(x + \epsilon) \in M$ and $(x + 2\epsilon) \in B(x, \varepsilon)$,

$$p(x, x + \epsilon, 1 + \epsilon) \leq p(x + 2\epsilon, x + \epsilon, (1 + \epsilon)^2) 2^{D/1+\epsilon} e^{\frac{D\varepsilon}{4(1+\epsilon)^2}} .$$

Therefore, multiplying the above inequality by $f_m(x + \epsilon) = 1_{B(x,1+\epsilon)^c}(x + \epsilon)$ and then integrating with respect to $x + \epsilon$, we obtain

$$P_{1+\epsilon} f_m(x) \leq \left(P_{(1+\epsilon)^2} f_m\right)(x + 2\epsilon) 2^{D/1+\epsilon} e^{\frac{D\varepsilon}{4(1+\epsilon)^2}} .$$

By integrating now with respect to $(x + 2\epsilon) \in B(x, \varepsilon)$, we get

$$P_{1+\epsilon} f_m(x) \leq \frac{2^{D/1+\epsilon} e^{\frac{D\varepsilon}{4(1+\epsilon)^2}}}{\mu(B(x, \varepsilon))} \int_M 1_{B(x,\varepsilon)}(x + 2\epsilon)(P_{(1+\epsilon)^2} f_m)(x + 2\epsilon) d\mu(x + 2\epsilon).$$

Now, (for which normalization differs from us by a factor 1/2 because he considers the semigroup $e^{(1+\epsilon)L/2}$), we obtain:

$$\lim_{(1+\epsilon) \to 0} (1 + \epsilon) \ln \int_M 1_{B(x,\varepsilon)}(x + 2\epsilon)(P_{(1+\epsilon)^2} f_m)(x + 2\epsilon) d\mu(x + 2\epsilon) = -\frac{1}{4(1 + \varepsilon)} .$$

This yields therefore

$$\lim_{(1+\varepsilon)\to 0^+} \inf\left((1+\varepsilon)\ln P_{1+\varepsilon}f_m(x)\right) \le \frac{D\varepsilon - 1}{4(1+\epsilon)}.$$

We conclude by letting $\varepsilon \to 0$.

Corollary(4.2.35)[291]. Let $\varepsilon > 0$, the sequence $f_m \in A_\varepsilon$, $\varepsilon \le f_m \le 1$, and consider the function $u(x, 1 + \varepsilon) = \sqrt{-\ln P_{1+\varepsilon}f_m(x)}$. Then, with the convention that $\frac{1}{0} = +\infty$, we have

$$2(1+\epsilon)u_{1+\epsilon} + u + \left(1 + \sqrt{\frac{D^*}{2}}\right)u^{1/3} + \sqrt{\frac{D^*}{2}}u^{-1/3} \ge 0,$$

where

$$D^* = (1+\epsilon)\left(1 + \frac{2(1+\epsilon)}{1+2\epsilon}\right).$$

Proof. Noting that we have

$$\frac{1+\epsilon}{1+2\epsilon}P_{1+\epsilon}f_m(x)\Gamma(\ln P_{1+\epsilon}f_m)(x) + (1+\epsilon)^2 P_{1+\epsilon}f_m(x)\Gamma^{X+2\epsilon}(\ln P_{1+\epsilon}f_m)(x) \ge 0$$

$$\le \frac{1}{1+2\epsilon}\left(1 + \frac{2(1+\epsilon)}{1+2\epsilon} + \frac{4C}{1+\epsilon}\right)[P_{1+\epsilon}(f_m \ln f_m)(x)$$
$$- P_{1+\epsilon}f_m(x)\ln P_{1+\epsilon}f_m(x)]$$
$$- \frac{4C}{1+2\epsilon}\frac{1}{1+\delta}LP_{1+\epsilon}f_m(x) - \frac{2C^2}{(1+2\epsilon)(1+\epsilon)}\ln\left(1 + \frac{1}{\delta}\right)P_{1+\epsilon}f_m(x)$$

we obtain that for all $C \ge 0$,

$$\frac{1+\epsilon}{2}\left(1 + \frac{2(1+\epsilon)}{1+2\epsilon} + \frac{4C}{1+\epsilon}\right)P_{1+\epsilon}(f_m \ln f_m)(x)$$
$$- \frac{1+\epsilon}{2}\left(1 + \frac{2(1+\epsilon)}{1+2\epsilon} + \frac{4C}{1+\epsilon}\right)(P_{1+\epsilon}f_m)\ln P_{1+\epsilon}f_m$$
$$- \frac{2C(1+\epsilon)}{1+\delta}LP_{1+\epsilon}f_m(x) - \frac{C^2}{\delta}P_{1+\epsilon}f_m \ge 0,$$

where we used the fact that

$$\ln\left(1 + \frac{1}{\delta}\right) \le \frac{1}{\delta}.$$

On the other hand, the hypothesis $0 \le f_m \le 1$ implies $f_m \ln f_m \le 0$. After dividing both sides of the above inequality by $P_{1+\epsilon}f_m$, we thus find

$$-\frac{1+\epsilon}{2}\left(1 + \frac{2(1+\epsilon)}{1+2\epsilon} + \frac{4C}{1+\epsilon}\right)\ln P_{1+\epsilon}f_m - \frac{2C(1+\epsilon)}{1+\delta}\frac{LP_{1+\epsilon}f_m}{P_{1+\epsilon}f_m} + \frac{C^2}{\delta} \ge 0.$$

Dividing both sides by $C > 0$, this may be re-written

128

$$-\frac{D^*}{2C}\ln P_{1+\epsilon}f_m - 2\ln P_{1+\epsilon}f_m - \frac{2(1+\epsilon)}{1+\delta}\frac{LP_{1+\epsilon}f_m}{P_{1+\epsilon}f_m} + \frac{C}{\delta} \geq 0 \qquad (63)$$

We now minimize the left-hand side of (63) with respect to C. The minimum value is attained in

$$C = \sqrt{-\frac{\delta D^*}{2}\ln P_{1+\epsilon}f_m} \ .$$

Substituting this value in (63), we obtain

$$\sqrt{-\frac{\delta D^*}{2}}\sqrt{-\ln P_{1+\epsilon}f_m} - 2P_{1+\epsilon}f_m - \frac{2(1+\epsilon)}{1+\delta}\frac{LP_{1+\epsilon}f_m}{P_{1+\epsilon}f_m} + \frac{C}{\delta} \geq 0.$$

With $u(x, 1+\epsilon) = \sqrt{-\ln P_{1+\epsilon}f_m\,(x)}$, and noting that $u_{1+\epsilon} = -\frac{1}{2u}\frac{LP_{1+\epsilon}f_m}{P_{1+\epsilon}f_m}$, we can re-write this inequality as follows,

$$\sqrt{\frac{\delta D^*}{2\delta}} + u + \frac{2(1+\epsilon)}{1+\delta}u_{1+\epsilon} \geq 0,$$

or equivalently,

$$2(1+\epsilon)u_{1+\epsilon} + u + \delta u + (1+\delta)\sqrt{\frac{D^*}{2\delta}} \geq 0.$$

Finally, if we choose

$$\delta = \frac{1}{u^{2/3}},$$

we obtain the desired conclusion.

We now introduce the functions $(f_m + \epsilon) : (0, \infty) \rightarrow (0, \infty)$ defined by

$$(f_m + \epsilon)(v) = \frac{1}{v + \left(1 + \sqrt{\frac{D^*}{2}}\right)v^{1/3} + \sqrt{\frac{D^*}{2}}\,v^{-1/3}}. \qquad (64)$$

One easily verifies that

$$\lim_{v\to 0^+}\sqrt{\frac{D^*}{2}}\,v^{-\frac{1}{3}}(f_m + \epsilon)(v) = 1, \quad \lim_{v\to\infty}v(f_m + \epsilon)(v) = 1.$$

These limit relations show that the sequence $(f_m + \epsilon) \in L^1(0, 1+\epsilon)$ for every $\epsilon \geq -1$, but $(f_m + \epsilon) \notin L^1(0, \infty)$. Moreover, if we set

$$G(u) = \int_0^u (f_m + \epsilon)(v)\,dv,$$

then $G'(u) = (f_m + \epsilon)(u) > 0$, and thus $G: (0, \infty) \to (0, \infty)$ is invertible. Furthermore, as is seen from (64), as $u \to \infty$ we have

$$G(u) = \ln u + C_0 + R(u), \tag{65}$$

where C_0 is a constant and $\lim_{u\to\infty} R(u) = 0$. At this point we notice that, in terms of the functions $(f_m + \epsilon)(u)$ we can re-express the conclusion of Corollary(4.2.35) in the form

$$2(1 + \epsilon)u_{1+\epsilon} + \frac{1}{(f_m + \epsilon)(u)} \geq 0.$$

Keeping in mind that $(f_m + \epsilon)(u) = G'(u)$, we thus conclude

$$\frac{dG(u)}{dt} = G'(u)u_{1+\epsilon} \geq -\frac{1}{2(1 + \epsilon)}. \tag{66}$$

From this identity we now obtain the following basic result .

Corollary(4.2.36)[291]. Let the sequence $f_m \in L^\infty(M), 0 \leq f_m \leq 1$, then for any $x \in M$ and $0 < s < s + \epsilon$,

$$G\left(\sqrt{-\ln P_{s+\epsilon}f_m(x)}\right) \geq G\left(\sqrt{-\ln P_s f_m(x)}\right) - \frac{1}{2}\ln\left(\frac{s + \epsilon}{s}\right).$$

Proof. If $f_m \in A_\varepsilon$ for some ε, the inequality is a straightforward consequence of the above results. In fact, keeping in mind that $u(x, s + \epsilon) = \sqrt{-\ln P_{s+\epsilon}f_m(x)}$, in order to reach the desired conclusion all we need to do is to integrate (66) between s and $s + \epsilon$. Consider now $f_m \in L^\infty(M), 0 \leq f_m \leq 1$. Let $(f_m + 2\epsilon)_{1+\epsilon'} \in C_0^\infty(M)$, with $0 \leq (f_m + 2\epsilon)_{1+\epsilon'} \leq 1$, and $(f_m + 2\epsilon)_{1+\epsilon'} \nearrow 1$. For $\epsilon' \geq 0$ and $\varepsilon > 0$, the function

$$(1 - \varepsilon)P_{1+\epsilon}((f_m + 2\epsilon)_{1+\epsilon'}f_m) + \varepsilon \in A_\varepsilon.$$

Therefore,

$$G\left(\sqrt{-\ln P_{1+\epsilon}f_m(1 - \varepsilon)P_{1+\epsilon}((f_m + 2\epsilon)_{1+\epsilon'}f_m) + \varepsilon(x)}\right)$$

$$\geq G\left(\sqrt{-\ln P_s f_m(1 - \varepsilon)P_{1+\epsilon}((f_m + 2\epsilon)_{1+\epsilon'}f_m) + \varepsilon(x)}\right) - \frac{1}{2}\ln\left(\frac{s + \epsilon}{s}\right).$$

Letting $\varepsilon \to 0$, and finally $\epsilon' \to \infty$, we obtain the desired conclusion for f_m.

Corollary(4.2.37)[291]. Let $x \in M$ and $\epsilon \geq -1$ be arbitrarily fixed. There exists $C_0^* \in R$, independent of x and $1 + \epsilon$, such that for any $\epsilon \geq -1$,

$$\left(\sqrt{-\ln P_{1+\epsilon}1_{B(x,1+\epsilon)^c}(x)}\right) \geq \ln\sqrt{1 + \epsilon} + C_0^*.$$

Proof. Re-write the inequality claimed in Corollary (4.2.36) as follows

$$G\left(\sqrt{-\ln P_{s+\epsilon}f_m(x)}\right) \geq G\left(\sqrt{-\ln P_s f_m(x)}\right) + \ln\sqrt{s} - \ln\sqrt{s + \epsilon},$$

where we have presently let $f_m(x + \epsilon) = 1_{B(x,1+\epsilon)^c}(x + \epsilon)$. Since for this function we have, from Corollary(4.2.34),

$\lim_{s\to0^+} (-\ln P_s f_m(x)) = \infty$, using (65) we see that, for $s\to0+$, the latter inequality is equivalent to

$$G\left(\sqrt{-\ln P_{s+\epsilon} f_m\ (x)}\right) \geq \ln \sqrt{-\ln P_s f_m\ (x)} - \ln \sqrt{s + \epsilon}\ + C_0\ +\ R\left(\sqrt{-\ln P_s f_m\ (x)}\right).$$

We now take the lim inf as $s \to 0^+$ of both sides of this inequality. Applying Corollary(4.2.34), we deduce

$$G\left(\sqrt{-\ln P_{1+\epsilon} f_m\ (x)}\right) \geq \ln \frac{1 + \epsilon}{2} - \ln \sqrt{s + \epsilon}\ + C_0 = \ln \sqrt{1 + \epsilon} + C_0^*,$$

where we have let $C_0^* = C_0 - \ln 2$. This establishes the desired conclusion.

We are now in a position to show the central result.

Corollary(4.2.38)[291].. There exists a constant $\epsilon \geq -1$ such that for every $x \in M$,

$$P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)}\right)(x) \geq \frac{1}{2}\ .$$

Proof. By the stochastic completeness of M we know that $P_{1+\epsilon} 1 = 1$. Therefore,

$$P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)}\right)(x)\ =\ 1 -\ P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)^c}\right)(x).$$

We conclude that the desired estimate is equivalent to proving that there exists an absolute constant $\epsilon > 0$ such that

$$\left(\sqrt{\ln 2}\right) \leq \left(\sqrt{-\ln P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)^c}\right)(x)}\right),$$

or, equivalently,

$$G\left(\sqrt{\ln 2}\right) \leq G\left(\sqrt{-\ln P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)^c}\right)(x)}\right) \tag{67}$$

At this point we which gives

$$G\left(\sqrt{-\ln P_{(1+\epsilon)^3}\left(1_{B(x,1+\epsilon)^c}\right)(x)}\right) \geq \ln\left(\frac{1}{\sqrt{1 + \epsilon}} + C_0^*\right)$$

It is thus clear that, letting $1 + \epsilon \to 0^+$, we can certainly achieve (67), thus completing the proof.

We have the following ,

Corollary(4.2.39)[291]. (Global doubling property) The metric measure space (M, d, μ) satisfies the global volume doubling property. More precisely, there exists a constant $C_1 = C_1(1 + \epsilon, 1 + 2\epsilon, 1 + \epsilon, 1 + \epsilon) > 0$ such that for every $x \in M$ and every $\epsilon \geq -1$,

$$\mu(B(x, 2(1 + \epsilon))) \leq C_1 \mu(B(x, 1 + \epsilon)).$$

Proof. The argument which shows was developed independently by Grigor'yan [197] and by Saloff-Coste [214], and it is by now well-known. However, since it is short for the sake of completeness in what follows we provide the relevant details. From the semigroup property and the symmetry of the heat kernel we have for any $(x + \epsilon) \in M$ and $\epsilon \geq -1$

$$p(x + \epsilon, x + \epsilon, 2(x + \epsilon))\ =\ \int_M p\ (x + \epsilon, x + 2\epsilon, 1 + \epsilon)^2 d\mu(x + 2\epsilon).$$

Consider now the functions $(f_m + 2\epsilon) \in C_0^\infty(M)$ such that $0 \le f_m + 2\epsilon \le 1$, $(f_m + 2\epsilon) \equiv 1$ on $B(x, \sqrt{1+\epsilon}/2)$ and $(f_m + 2\epsilon) \equiv 0$ outside $B(x, \sqrt{1+\epsilon})$. We thus have

$$P_{1+\epsilon}(f_m + 2\epsilon)(x + \epsilon) = \int_M p(x+\epsilon, x+2\epsilon, 1+\epsilon)(f_m + 2\epsilon)(x+2\epsilon)d\mu(x+2\epsilon)$$

$$\le \left(\int_M p(x+\epsilon, x+2\epsilon, 1+\epsilon)^2 d\mu(x+2\epsilon)\right)^{\frac{1}{2}} \left(\int_M (f_m + 2\epsilon)(x+2\epsilon)^2 d\mu(x+2\epsilon)\right)^{\frac{1}{2}}$$

$$\le p(x+\epsilon, x+\epsilon, 2(1+\epsilon))^{\frac{1}{2}} \mu(B(x, \sqrt{1+\epsilon}))^{\frac{1}{2}} .$$

If we take $\epsilon = 0$, and $\epsilon = -1$, we obtain

$$P_{(1+\epsilon)^2}(1_{B(x,r)})(x)^2 \le P_{(1+\epsilon)^2}(f_m + 2\epsilon)(x)^2 \le p(x, x, 2(1+\epsilon)^2) \mu(B(x, 1+\epsilon)). \quad (68)$$

At this point we use Corollary(4.2.38) which gives for some $0 < \epsilon < 1$, (the fact that we can choose $\epsilon < 1$ is clear from the proof of Corollary(4.2.38).

$$P_{(1+\epsilon)^3}(1_{B(x,1+\epsilon/2)})(x) \ge \frac{1}{2}, \qquad x \in M, \quad \epsilon \ge -1.$$

Combining this estimate with the Harnack, we obtain the following on-diagonal lower bound

$$p(x, x, 2(1+\epsilon)^2) \ge \frac{C^*}{\mu(B(x, 1+\epsilon))}, \qquad x \in M, \qquad \epsilon \ge -1. \quad (69)$$

we find for every $x + \epsilon \in B(x, \sqrt{1+\epsilon})$,

$$p(x, x, 1+\epsilon) \le Cp(x, x+\epsilon, 2(1+\epsilon)).$$

Integration over $B(x, \sqrt{1+\epsilon})$ gives

$$p(x, x, 1+\epsilon)\mu(B(x, \sqrt{1+\epsilon})) \le C \int_{B(x,\sqrt{1+\epsilon})} p(x, x+\epsilon, 2(1+\epsilon))d\mu(x+\epsilon) \le C,$$

where we have used $P_{1+\epsilon}1 \le 1$. Letting $\epsilon = -1$, we obtain from this the on-diagonal upper bound

$$p(x, x, (1+\epsilon)^2) \le \frac{C}{\mu(B(x, 1+\epsilon))}. \quad (70)$$

Combining (69) with (70) we finally obtain

$$\mu\left(B(x, 2(1+\epsilon))\right) \le \frac{C}{p(x, x, 4(1+\epsilon)^2)} \le \frac{CC'}{p(x, x, 2(1+\epsilon)^2)} \le C^{**}\mu\left(B(x, 1+\epsilon)\right),$$

(with $x + \epsilon = x + 2\epsilon = x$), which gives

$$\frac{p(x, x, 2(1+\epsilon)^2)}{p(x, x, 4(1+\epsilon)^2)} \le C'$$

and we have let $C^{**} = CC'(C^*)^{-1}$. This completes the proof.

It is well-known that Corollary(4.2.39) provides the following uniformity control at all scales.

Corollary(4.2.40)[291].. For any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \to 0^+$, such that for every $x, (x + \epsilon) \in M$ and $\epsilon \geq 0$ one has

$$\frac{C(\varepsilon)^{-1}}{\mu\left(B(x, \sqrt{1 + \epsilon})\right)} \exp\left(-\frac{Dd(x, x + \epsilon)^2}{(1 + \epsilon)^2(4 - \varepsilon)}\right) \leq p(x, x + \epsilon, 1 + \epsilon)$$

$$\leq \frac{C(\varepsilon)}{\mu\left(B(x, \sqrt{1 + \epsilon})\right)} \exp\left(-\frac{d(x, x + \epsilon)^2}{(4 - \varepsilon)(1 + \epsilon)}\right).$$

Proof. We begin by establishing the lower bound. First, we obtain for all $(x + \epsilon) \in M, \epsilon \geq 0$, and every $0 < \varepsilon < 1$,

$$p(x, x + \epsilon, 1 + \epsilon) \geq p(x, x, \epsilon(1 + \epsilon))\varepsilon^{\frac{D}{2}}\exp\left(-\frac{D}{1 + \epsilon}\frac{d(x, x + \epsilon)^2}{(4 - \varepsilon)(1 + \epsilon)}\right).$$

We thus need to estimate $p(x, x, \epsilon(1 + \epsilon))$ from below. But this has already been done in (69). Choosing $\epsilon \leq -1$ such that $2(1 + \epsilon)^2 = \epsilon(1 + \epsilon)$, we obtain from that estimate

$$p(x, x, \epsilon(1 + \epsilon)) \geq \frac{C^*}{\mu(B(x, \sqrt{\varepsilon/2}\,\sqrt{1 + \epsilon}))} \quad, x \in M, \quad \epsilon > -1$$

On the other hand, since $\sqrt{\varepsilon/2} < 1$, by the trivial inequality $\mu(B(x, \sqrt{\varepsilon/2}\sqrt{1 + \epsilon})) \leq \mu(B(x, \sqrt{1 + \epsilon}))$, we conclude

$$p(x, x + \epsilon, 1 + \epsilon) \geq \frac{C^*}{\mu(B(x, \sqrt{1 + \epsilon}))}\varepsilon^{\frac{D}{2}}\exp\left(-\frac{D}{1 + \epsilon}\frac{d(x, x + \epsilon)^2}{(4 - \varepsilon)(1 + \epsilon)}\right)$$

This proves the Gaussian lower bound.

For the Gaussian upper bound, we first observe that the following upper bound is proved in [183]:

$$p(x, x + \epsilon, 1 + \epsilon) \geq \frac{C(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon')}{\mu(B(x, \sqrt{1 + \epsilon}))^{\frac{1}{2}}\mu(B(x, \sqrt{1 + \epsilon}))^{\frac{1}{2}}} \exp\left(-\frac{d(x, x + \epsilon)^2}{(4 + \varepsilon')(1 + \epsilon)}\right)$$

At this point, by the triangle inequality we find.

$$\mu(B(x, \sqrt{1 + \epsilon})) \leq \mu(B(x + \epsilon, d(x, x + \epsilon) + \sqrt{1 + \epsilon}))$$

$$\leq C_1\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))\left(\frac{d(x, x + \epsilon) + \sqrt{1 + \epsilon}}{\sqrt{1 + \epsilon}}\right)^Q.$$

This gives

$$\frac{1}{\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))} \leq \frac{C_1}{\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))}\left(\frac{d(x, x + \epsilon)}{\sqrt{1 + \epsilon}} + 1\right)^Q.$$

Combining this with the above estimate we obtain

$p(x, x + \epsilon, 1 + \epsilon)$

$$\geq \frac{C_1^{1/2}(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon')}{\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))} \left(\frac{d(x, x + \epsilon)}{\sqrt{1 + \epsilon}} + 1\right)^{Q/2} exp\left(-\frac{d(x, x + \epsilon)^2}{(4 + \varepsilon')(1 + \epsilon)}\right)$$

If now $0 < \varepsilon < 1$, it is clear that we can choose $0 < \varepsilon' < \varepsilon$ such that

$$\frac{C_1^{1/2}(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon')}{\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))} \left(\frac{d(x, x + \epsilon)}{\sqrt{1 + \epsilon}} + 1\right)^{Q/2} exp\left(-\frac{d(x, x + \epsilon)^2}{(4 + \varepsilon')(1 + \epsilon)}\right)$$

$$\leq \frac{C^*(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon)}{\mu(B(x + \epsilon, \sqrt{1 + \epsilon}))} exp\left(-\frac{d(x, x + \epsilon)^2}{(4 + \varepsilon')(1 + \epsilon)}\right)$$

where $C^*(1 + \epsilon, 1 + \epsilon, 1 + 2\epsilon, \varepsilon s)$ is a constant which tends to ∞ as $\varepsilon \to 0^+$. The desired con- clusion follows by suitably adjusting the values of both ε' and of the constant in the right-hand side of the estimate.

Chapter 5

Pinched Riemannian Manifolds and Geometric Inqualities with Sub Riemannian Balls

We obtain a distribution theorem for the square norm of the second fundamental form of M under the assumption that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold. We give some geometric inequalities for a submanifold with parallel second fundamental form in a pinched Riemannian manifold and the distribution for the square norm of its second fundamental form. In particular, large sub-Riemannian balls are comparable to Riemannian balls.

Section (5-1): Closed Minimal Submanifolds

Let M^n be an $n-$dimensional oriented closed minimal submanifold in an $(n + p) -$dimensional manifold N^{n+P}. We denote the square norm of the second fundamental form of M by S. In the case that the ambient manifold N is the Euclidean sphere $S^{n+P}(1)$, it is well known [227] that if $S \leq n/(2 - 1/p)$ on M, then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Further discussions in this regard have been carried out by many other ([228,230,233,234,237], etc.). A. M. Li and J. M. Li [231] have improved the pinching constant above to $\frac{2}{3}n$ for the case ≥ 3. But all these results were obtained under the assumption that the ambient manifolds possess very nice symmetry.

We establish a generalized Simons integral inequality for minimal submanifolds in a Riemannian manifold, and prove a pinching theorem for minimal submanifolds in a complete simply connected pinched Riemannian manifold, which does not possess symmetry in general. The proof uses some equations and inequalities naturally associated to the second fundamental form of M, the curvature tensor of N, and their covariant derivatives. Since we do not assume that N^{n+p} is a sphere, the maximum principle and the estimate for ΔS in [227, 231] cannot be applied here, and the trick of constructing a differentiable 1-form and using integral estimates seems essential. Finally, a distribution theorem for S is obtained under the assumption that M is a minimal submanifold with parallel second fundamental form in a Riemannian manifold.

Let M^n be an n-dimensional Riemannian manifold immersed in an $(n + p) -$ dimensional Riemannian manifold N^{n+p}. We shall make use of the following convention on the range of indices:

$$1 < A, B, C, \ldots, \leq n + p, \quad 1 < i, j, k, \ldots, \leq n,$$
$$n + 1 < \alpha, \beta, \gamma, \ldots < n + p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in N such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of N respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha, \tag{1}$$

$$h = \sum_{\alpha,i,j} h_{ij}^\alpha \omega_j \otimes \omega_j \otimes e_\alpha, \qquad \xi = \frac{1}{n} \sum_{\alpha,i} h_{ii}^\alpha e_\alpha, \tag{2}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{3}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta), \tag{4}$$

where $h, \xi, R_{\alpha\beta kl}, R_{ijkl}$, and K_{ABCD} are the second fundamental form, the mean curvature vector, the normal curvature tensor, the curvature tensor of M, and the curvature tensor of N respectively. We define

$$S = \|h\|^2, \qquad H = \|\xi\|, \qquad H_\alpha = (h_{ij}^\alpha)_{n\times n}.$$

M is called minimal if H vanishes identically. Therefore, if M is minimal, its scalar curvature is given by

$$R = \sum_{\alpha,i} K_{ijij} - S.$$

Now we define the covariant derivatives of h_{ij}^α, denoted by h_{ijk}^α and h_{ijkl}^α respectively, as

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_s h_{sj}^\alpha \omega_{is} + \sum_s h_{is}^\alpha \omega_{js} + \sum_\beta h_{ij}^\beta \omega_{\alpha\beta},$$

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_s h_{sjk}^\alpha \omega_{is} + \sum_s h_{isk}^\alpha \omega_{js} + \sum_s h_{ijs}^\alpha \omega_{ks} + \sum_\beta h_{ijk}^\beta \omega_{\alpha\beta},$$

Then we have

$$h_{ijk}^\alpha - h_{ikj}^\alpha = K_{\alpha ikj}, \tag{5}$$

and the Ricci formula

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_s h_{sj}^\alpha R_{sikl} + \sum_s h_{is}^\alpha R_{sjkl} + \sum_s h_{ij}^\beta R_{\beta\alpha kl} \tag{6}$$

Considering $K_{\alpha ijk}$ as a section of $T^\perp(M) \otimes T^*(M) \otimes T^*(M) \otimes T^*(M)$, we also define its covariant derivative $K_{\alpha ijkl}$ as

$$\sum_l K_{\alpha ijkl} \omega_l = dK_{\alpha ijk} + \sum_s K_{\alpha sjk} \omega_{is} + \sum_s K_{\alpha isk} \omega_{js}$$

$$+ \sum_{s} K_{\alpha ijs}\omega_{ks} + \sum_{\beta} K_{\beta ijk}\omega_{\alpha\beta}$$

M is called a submanifold with parallel second fundamental form if $h^{\alpha}_{ijk} \equiv 0$ for all i, j, k, α. The Laplacian Δh^{α}_{ij} of the second fundamental form h is defined by $\Delta h^{\alpha}_{ij} = \sum_{k} h^{\alpha}_{ijkk}$. In the next section, we sometimes also use $\nabla_k h^{\alpha}_{ij}$ to denote h^{α}_{ijk}, etc.

For a matrix $A = (a_{ij})_{n \times n}$ we denote by $N(A)$ the square norm of A, i.e., $N(A) = tr(A'A) = \sum_{i,j} a_{ij}^2$. Then $N(A) = N(TA^tT)$, for each orthogonal $(n \times n)$ −matrix T.

Proposition (5.1.1)[238]. (see [307, 311]). Let $A_{n+1}, A_{n+1}, \ldots, A_{n+p}$ be symmetric $(n \times n)$-matrices. Denote $S_{\alpha\beta} = tr(A^t_\alpha A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = \sum_\alpha S_\alpha$. Then

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S^2_{\alpha\beta} \le \left(1 + \frac{1}{2} sgn(p-1)\right) S^2, \tag{7}$$

where $sgn(\cdot)$ is the standard sign function, and the equality holds if and only if at most two matrices A_α and A_β are not zero and these two matrices can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}_\alpha$ and $\tilde{A}_\beta$ respectively, where

$$\tilde{A}_\alpha = \begin{pmatrix} \begin{array}{cc|c} 1 & 0 & \\ & & 0 \\ 0 & -1 & \\ \hline 0 & 0 & 0 \end{array} \end{pmatrix}, \qquad \tilde{A}_\beta = \begin{pmatrix} \begin{array}{cc|c} 0 & 1 & \\ & & 0 \\ 1 & 0 & \\ \hline 0 & 0 & 0 \end{array} \end{pmatrix}$$

Proposition (5.1.2)[238]. (see [309]). Let N be an $(n + p)$ −dimensional Riemannian manifold. If $a \le K_N \le b$ at a point $x \in N$, then, at this point,

(i) $|K_{ACBC}| \le \frac{1}{2}(b-a)$, $for\ A \ne B$.

(ii) $|K_{ABCD}| \le \frac{2}{3}(b-a)$, $for\ A, B, C, D$ distinct with each other.

From now on, we assume that M^n is a minimal submanifold in N^{n+P}. By (5), (6), and the minimality of M, we have

$$\Delta h^{\alpha}_{ij} = -\sum_{k}\left(K_{\alpha kikj} + K_{\alpha ijkk}\right) + \sum_{k,m} h^{\alpha}_{mi} R_{mkjk}$$
$$+ \sum_{k,m} h^{\alpha}_{km} R_{mijk} + \sum_{k,\beta} h^{\beta}_{ki} R_{\alpha\beta kj.} \tag{8}$$

Substituting (3) and (4) into the above, (8) becomes

$$\Delta h^{\alpha}_{ij} = -\sum_{k}\left(K_{\alpha kikj} + K_{\alpha ijkk}\right) + \sum_{k,m}\left(h^{\alpha}_{mi} R_{mkjk} + h^{\alpha}_{mk} R_{mijk}\right) + \sum_{k,\beta} h^{\beta}_{ki} R_{\alpha\beta kj.}$$
$$+ \sum_{m,k,\beta}\left(h^{\alpha}_{mi} h^{\beta}_{mj} h^{\beta}_{kk} + 2 h^{\alpha}_{km} h^{\beta}_{ki} h^{\beta}_{mj} - h^{\alpha}_{km} h^{\beta}_{km} h^{\beta}_{ij} - h^{\alpha}_{mi} h^{\beta}_{mk} h^{\beta}_{kj} - h^{\alpha}_{mj} h^{\beta}_{ki} h^{\beta}_{mk}\right).$$

Therefore,

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha$$

$$= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha kikj} + h_{ij}^\alpha K_{\alpha ijkk})$$

$$+ \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{mkik} + h_{mk}^\alpha h_{ij}^\alpha K_{mijk}) \tag{9}$$

$$+ \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha\beta kj} - \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta$$

$$- \sum_{i,j,k,l,\alpha,\beta} (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta).$$

Put

$$S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$$

Then the $(p \times p)-$matrix $(S_{\alpha\beta})$ is symmetric and can be assumed to be diagonal for a suitable choice of $\{e_\alpha\}$, i.e.,

$$S_{\alpha\beta} = S_\alpha \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta.$$

By the definition, $S = \sum_\alpha S_\alpha$ From (9) we have

Lemma (5.1.3)[238]. Denote

$$A = -\sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_\alpha S_{\alpha'}^2$$

$$B = \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{mkik} + h_{mk}^\alpha h_{ij}^\alpha K_{mijk}) + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha\beta kj'}$$

$$C = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha kikj} + h_{ij}^\alpha K_{\alpha ijkk})$$

Then

$$\tfrac{1}{2}\Delta S = A + B + C. \tag{10}$$

Let *a(x)* and *b(x)* denote the infimum and the supremum of the sectional curvature of *N* at a point *x* respectively. Now we derive a lower bound for B in terms of *a, b*, and *S*.

Lemma (5.1.4)[238]. $B \geq nbS - [n + \frac{2}{3}(p - 1)(n - 1)^{1/2}](b - a)S.$

Proof. Fix a vector e_α. Let $\{e_i\}$ be a frame diagonalizing the matrix (h_{ij}^α) such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij'} \quad 1 \leq i, j \leq n.$$

Then

$$\sum_{i,j,k,m} h_{mj}^{\alpha} h_{ij}^{\alpha} K_{mkik} + \sum_{i,j,k,m} h_{mk}^{\alpha} h_{ij}^{\alpha} K_{mijk} + \sum_{i,j,k,\beta} h_{ij}^{\alpha} h_{ki}^{\beta} K_{\alpha\beta kj}$$

$$= \sum_{i,j,k,\alpha} (\lambda_i^{\alpha})^2 K_{ikik} + \sum_{i,j,k,\alpha} \lambda_k^{\alpha} \lambda_i^{\alpha} K_{kiik} + \sum_{i,k,\beta} h_{ki}^{\beta} \lambda_i^{\alpha} K_{\alpha\beta ki}. \tag{11}$$

By Proposition (5.1.2), we have

$$|K_{\alpha\beta ki}| < \tfrac{2}{3}(b-a) \quad for \quad \alpha \neq \beta, \quad i \neq k.$$

Hence, for fixed α, one sees

$$\sum_{i,k,\beta} h_{ki}^{\alpha} \lambda_i^{\alpha} K_{\alpha\beta ki} \geq - \sum_{\alpha \neq \beta, i \neq k.} \tfrac{2}{3}(b-a)|h_{ki}^{\beta} \lambda_i^{\alpha}|$$

$$\geq - \sum_{\alpha \neq \beta, i \neq k.} \tfrac{1}{3}(b-a)[(n-1)^{1/2}(h_{ki}^{\beta})^2 + (n-1)^{-1/2}(\lambda_i^{\alpha})^2]$$

$$\geq -\frac{1}{3}(n-1)^{1/2}(b-a)\sum_{\alpha \neq \beta} tr\, H_{\beta}^2 - \frac{1}{3}(n-1)^{\frac{1}{2}}(p-1)(b-a)tr\, H_{\alpha}^2 \tag{12}$$

On the other hand, we have

$$\sum_{i,k}(\lambda_i^{\alpha})^2 K_{ikik} + \sum_{i,j,k,\alpha} \lambda_k^{\alpha} \lambda_i^{\alpha} K_{kiik}$$

$$= \frac{1}{2}\sum_{i,k}(\lambda_i - \lambda_k)^2 K_{ikik} \geq \frac{1}{2}a\sum_{i,k}(\lambda_i - \lambda_k)^2 = na\, trH_{\alpha}^2. \tag{13}$$

Substituting (12) and (13) into (14), we obtain

$$B \geq \sum_{\alpha}[na\, trH_{\alpha}^2 - \frac{1}{3}(n-1)^{1/2}(b-a)\sum_{\alpha \neq \beta} tr\, H_{\beta}^2$$

$$-\tfrac{1}{3}(n-1)^{1/2}(p-1)(b-a)trH_{\alpha}^2]$$

$$= nbS - [n + \tfrac{2}{3}(p-1)(n-1)^{\frac{1}{2}}(b-a)S. \tag{14}$$

We shall next estimate the integral of C.

Lemma (5.1.5). $\int_M C \geq -\frac{1}{72}pn(n-1)(26n-25)\int_M (b-a)^2.$

Proof. Note that

$$- \sum_{i,j,k,\alpha}(h_{ik}^{\alpha} K_{\alpha jijk} + h_{ij}^{\alpha} K_{\alpha ijkk})$$

$$= - \sum_{i,j,k,\alpha} \nabla_k(h_{ik}^{\alpha} K_{\alpha jij} + h_{ij}^{\alpha} K_{\alpha ijk}) + \sum_{i,j,k,\alpha}(h_{ikk}^{\alpha} K_{\alpha jij} + h_{ijk}^{\alpha} K_{\alpha ijk}).$$

We define a differentiable 1- form as

$$\omega = \sum_{i,j,k,\alpha}(h_{ik}^{\alpha} K_{\alpha jij} + h_{ijk}^{\alpha} K_{\alpha ijk})\omega_k. \tag{15}$$

It follows that
$$\text{div } \omega = \sum_{i,j,k,\alpha} \nabla_k (h_{ik}^\alpha K_{\alpha jij} + h_{ij}^\alpha K_{\alpha ijk}).$$

Thus
$$C = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ikk}^\alpha K_{\alpha jij} + h_{ijk}^\alpha K_{\alpha ijk}) - \text{div } \omega.$$

Since M is minimal, we have
$$\sum_i h_{iij}^\alpha = 0 \quad \text{for all } j, \alpha. \tag{16}$$

From (5), (16), and Proposition (5.1.2), we have
$$\sum_{i,j,k,\alpha} h_{ikk}^\alpha K_{\alpha jij} = \sum_{i,j,k,\alpha} (h_{kki}^\alpha - K_{\alpha kik}) K_{\alpha jij} = -\sum_{i,\alpha} \left(\sum_j K_{\alpha jij} \right)^2$$
$$\geq -\frac{1}{4} pn(n-1)^2 (b-a)^2. \tag{17}$$

On the other hand, by Proposition (5.1.2), we have
$$\sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} h_{ijk}^\alpha K_{\alpha ijk}$$
$$\geq -\frac{1}{4} \sum_{i,j,k,\alpha} (K_{\alpha ijk})^2$$
$$\geq -\frac{1}{4} \sum_{\alpha} \sum_{i,j,k \text{ distinct}} (K_{\alpha ijk})^2 - \frac{1}{2} \sum_{\alpha} \sum_{i \neq j} (K_{\alpha iji})^2 \tag{18}$$
$$\geq -\frac{1}{4} pn(n-1)(n-2)(b-a)^2 - \frac{1}{8} pn(n-1)(b-a)^2.$$

So
$$C \geq -\frac{1}{72} pn(n-1)(26n-25)(b-a)^2 - \text{div } \omega \tag{19}$$
and by using Green's divergence theorem, we get
$$\int_M C \geq -\frac{1}{72} pn(n-1)(26n-25) \int_M (b-a)^2. \tag{20}$$

Lemma (5.1.5) follows.

Now we define
$$D(n,p) = n + \frac{2}{3}(p-1)(n-1)^{1/2},$$
$$E(n,p) = \frac{1}{72} Pn(n-1)(26n-25).$$

Theorem (5.1.6)[238]. (Generalized Simons inequality). Let M^n bean n-dimensional oriented closed minimal submanifold in an $(n + p)$ −dimensional Riemannian manifold N^{n+P}. Denote the infimum and the supremum of the sectional curvature of N at a point x by $a(x)$ and $b(x)$ respectively. Then

$$\int_M \left[nbS - (1 + \tfrac{1}{2} sgn(p - 1))S^2 - D(n, p)(b - a)S - E(n, p)(b - a)^2 \right] \le 0.$$

Proof. Combining Proposition (5.1.1), Lemma (5.1.3) and (5.1.4), we obtain

$$\tfrac{1}{2}\Delta S \ge nbS - \left(1 + \tfrac{1}{2} sgn(p - 1)\right) S^2 - \left[n + \tfrac{2}{3}(p - 1)(n - 1)^{1/2}\right](b - a)S + C. \quad (21)$$

Integrating both sides of (21) and applying Lemma (5.1.5), we have

$$\int_M \left[nbS - (1 + \tfrac{1}{2} sgn(p - 1))S^2 - D(n, p)(b - a)S - E(n, p)(b - a)^2 \right] \le 0. \quad (22)$$

This completes the proof of Theorem (5.1.6).

Denote

$$\alpha(n, p) = \tfrac{1}{12}[pn(n - 1)(52n - 50)]^{1/2},$$

$$\beta(n, p) = n + \tfrac{2}{3}(p - 1)(n - 1) + \tfrac{1}{12}[pn(n - 1)(52n - 50)]^{1/2}.$$

We are now in a position to prove

Theorem (5.1.7)[238]. There is a number $\delta(n, p)$ with $0 < \delta(n, p) < 1$ such that if there exists an oriented closed minimal submanifold M^n in a complete simply connected manifold N^{n+P} with $\delta(n, p) < K_N < 1$ and

$$\alpha(n, p)(1 - c) \le S \le n - \tfrac{1}{3}n \, sgn(p - 1) - \beta(n, p)(1 - c),$$

where c is the infimum of the sectional curvature of N, then either M is the unit sphere $S^n(1)$, one of the Clifford minimal hypersurfaces $S^k\left(\sqrt{\tfrac{k}{n}}\right) \times S^{n-k}(\sqrt{n - k)/n})$, $k = 1, 2, \ldots, n - 1$, in $S^{n+1}(1)$, or the Veronese surface in $S^4(1)$. Moreover, $N = S^{n+p}(1)$.

Proof. Since

$$c \le a(x) \le b(x) < 1,$$

(22) gives

$$\int_M \left[nS - (1 + \tfrac{1}{2} sgn(p - 1))S^2 - D(n, p)(1 - c)S - E(n, p)(1 - c)^2 \right] \le 0. \quad (23)$$

Take

$$\delta(n, p) = 1 - n(3 - sgn(p - 1))(3D(n, p) + 6E^{1/2}(n, p))^{-1}.$$

Then

$$\alpha(n, p)(1 - c) \le n - \frac{1}{3}n \, sgn(p - 1) - \beta(n, p)(1 - c).$$

From the assumption

$$\alpha(n,p)(l - c) \le S \le n - \frac{1}{3}n\,\text{sgn}(p - 1) - \beta(n,p)(1 - c), \tag{24}$$

we see that

$$nS - \left(1 + \frac{1}{2}sgn(p - 1)\right)S^2 - D(n,p)(1 - c)S - E(n,p)(1 - c)^2 \ge 0 \tag{25}$$

Therefore, all inequalities in (17), (18), (22), and (25) are actually equalities. This implies $1 - c = b - a = 0$ and N is a complete simply connected Riemannian manifold with constant curvature 1. Hence $N = S^{n+P}(1)$. This together with (23) and (25) gives

$$S = 0 \quad or \quad S = n - \frac{1}{3}n\,\text{sgn}(p - 1).$$

Furthermore, the previous inequalities become equalities, and it is not hard to see from Proposition (5.1.1) that either M is the unit sphere $S^n(1)$, one of the Clifford hypersurfaces $S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{n-k)/n})$, $k = 1, 2, \ldots, n - 1$, or the Veronese surface. This proves Theorem (5.1.7).

Theorem (5.1.8)[238]. Let M^n be an oriented closed minimal submanifold with parallel second fundamental form in a Riemannian manifold N^{n+P}. Then

(i) $S \le pnd + F(n,p)(d - c)$, where $F(n,p) = \frac{2}{3}p(p - 1)(n - 1)^{1/2}$ and d is the supremum of the sectional curvature of N,

(ii) if $\delta'(n,p) \le K_N \le 1$, here

$$\delta'(n,p) = 1 - n\big(3 - sgn(p - 1)\big)\Big[3n + 2(p - 1)(n - 1)^{\frac{1}{2}}\Big]^{-1},$$

then either M is totally geodesic or $n - \frac{1}{3}n\,sgn(p - 1) - D(n,p)(1 - c) \le S \le pn + F(n,p)(1 - c)$.

Proof. From the proof of Lemma (5.1.5) we have

$$C = -\sum_{i,j,k,\alpha} \nabla_k \left(h_{ik}^\alpha K_{\alpha jij} + h_{ik}^\alpha K_{\alpha ijk}\right).$$

It is easy to see from (5) that $K_{\alpha ijk} = 0$, for all i, j, k, α. So

$$C = 0. \tag{26}$$

Since $\frac{1}{2}\Delta S = \sum(h_{ijk}^\alpha)^2 + \sum h_{ij}^\alpha \Delta h_{ij}^\alpha = 0$, S is a constant. This together with (10) and (16) implies

$$A + B = 0. \tag{27}$$

Obviously,

$$\sum_{\alpha,\beta} N\big(H_\alpha H_\beta - H_\beta H_\alpha\big) + \sum_\alpha S_\alpha^2 \ge S^2/p \tag{28}$$

For fixed α, similar to the estimate of lower bound for B , we have

$$LHS \ of \ (11) \ = \ \sum_{i,k} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i,k} \lambda_k^\alpha \lambda_i^\alpha K_{kiik} + \sum_{i,k,\beta} h_{ki}^\beta \lambda_i^\alpha K_{\alpha\beta ki}$$

$$\leq nd \ tr \ H_\alpha^2 + \frac{1}{3}(p-1)(n-1)^{1/2}(d-c) tr \ H_\alpha^2$$

$$+\frac{1}{3}(n-1)^{1/2}(d-c) \sum_{\beta\neq\alpha} tr \ H_\beta^2.$$

This gives

$$B \leq n \ dS + \frac{2}{3}p(p-1)(n-1)^{1/2}(d-c)S. \tag{29}$$

It follows from (27), (28), and (29) that

$$ndS + \frac{2}{3}(p-1)(n-1)^{1/2}(d-c)S \geq S^2/p.$$

This yields

$$S \leq p \ nd + \frac{2}{3}p(p-1)(n-1)^{1/2}(d-c). \tag{30}$$

If $\delta'(n,p) < K_N \leq 1$, it is not hard to see from the definition of $\delta'(n,p)$ that

$$n - \frac{1}{3}n \ sgn \ (p-1)D(n,p)(1-c) > 0. \tag{31}$$

By (27), Proposition (5.1.1), and Lemma (5.1.4), we get

$$nS - (1 + \frac{1}{S} sgn(p-1))S^2 - D(n,p)(1-c)S \leq 0, \tag{32}$$

which together with (30) implies that either S = 0 or $n - \frac{1}{3}n \ sgn(p-1) - D(n,p)(1-$
$c) \leq S \leq pn + F(n,p)(1-c)$. This completes the proof of Theorem (5.1.8).

Section(5-2). Certain Submanifolds in a Pinched Riemannian Manifold

It seems interesting to generalize the famous Simons' pinching theorem to general cases. It is well known [244] that if M^n is a compact minimal submanifold of the sphere $S^{n+p}(1)$ and if the square norm of the second fundamental form of M^n, denoted by S, is everywhere less than $\frac{n}{2-\frac{1}{p}}$, then M^n is totally geodesic. There are many further discussions in this regard in the literature [239, 240, 242, 243]. Yau [248,249] proved that if $p > 1$ and $< \frac{n}{3+n^{1/2}-(p-1)^{-1}}$, then M^n lies in a totally geodesic $S^{n+1}(1)$. In [245], Xu proved that if $p > 1$ and $S < \min\{\frac{2n}{1+n^{1/2}}, \frac{n}{2-(p-1)^{-1}}\}$, then M^n is a totally umbilical sphere which improves Yau's result above. Thereafter, Xu also gave a sharp pinching constant $C(n,p,H)$ in [246] which is larger than the ones in [245] and [248, 249]. Precisely, if M^n is a compact submanifold with parallel mean curvature in $S^{n+p}(1)$ and if $S < C(n,p,H)$, then M^n is the totally umbilical sphere $S^n(\frac{1}{(1+H^2)^{1/2}})$. Here the pinching constants are defined by

$$\alpha(n, H) = n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} (n^2 H^4 + 4(n-1)H^2)^{\frac{1}{2}},$$

$$C(n, p, H) = \begin{cases} \alpha(n, H), \text{ for } p = 1, \text{ or } p = 2 \text{ and } H \neq 0, \\ \min\{\alpha(n, H), \frac{1}{3}(2n + 5nH^2)\}, \text{ for } p \geq 3, \text{ or } p = 2 \text{ and } H = 0. \end{cases}$$

All these results were obtained under the condition that the ambient spaces possess very nice symmetry.

However, the existence of parallel second fundamental form imposes nice properties to submanifolds whatever the ambient spaces are. The aim is to obtain the distribution for the square norm of the second fundamental form of a submanifold with parallel second fundamental form in a pinched Riemannian manifold. We get the estimate of upper bound for the square norm of the second fundamental form under the above assumption. Moreover, we establish a generalized Simons-type inequality which derives a quantization phenomenon.

We give a quick recall of some preliminaries of the geometry of submanifolds. Let M^n be an n-dimensional connected Riemannian manifold immersed in an $(n + p)-$ dimensional Riemannian manifold N^{n+p}. We shall make use of convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n + p, \quad 1 \leq i, j, k, \cdots \leq n,$$
$$n + 1 \leq \alpha, \beta, \gamma, \cdots \leq n + p.$$

Choose a local field of orthonormal frames $\{e_A\}$ in N such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ and $\{\omega_{AB}\}$ be the field of dual frames and the connection 1-forms of N, respectively. Restricting these forms to M, we have

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha, \tag{33}$$

$$h = \sum_j h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha, \quad \xi = \frac{1}{n} \sum_j h_{ij}^\alpha e_\alpha, \tag{34}$$

$$R_{ijkl} = K_{ijkl} + \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{35}$$

$$R_{\alpha\beta kl} = K_{\alpha\beta kl} + \sum_i (h_{ik}^\alpha h_{il}^\beta - h_{il}^\alpha h_{ik}^\beta), \tag{36}$$

where $h, \xi, R_{\alpha\beta kl}$, and K_{ABCD} are the second fundamental form, the mean curvature vector, the normal curvature tensor, the curvature tensor of M, and the curvature tensor of N, respectively. We define

$$S = \|h\|^2, \quad H = \|\xi\|, \quad H_\alpha = (h_{ij}^\alpha)_{n \times n}.$$

Now we define the covariant derivatives of h_{ij}^α, denoted by h_{ijk}^α and h_{ijkl}^α, as

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_s h^\alpha_{sj}\omega_{is} + \sum_s h^\alpha_{is}\omega_{js} + \sum_\beta h^\beta_{ij}\omega_{\alpha\beta},$$

$$\sum_l h^\alpha_{ijkl}\omega_l = dh^\alpha_{ijk} + \sum_s h^\alpha_{sjk}\omega_{is} + \sum_s h^\alpha_{isk}\omega_{js} + \sum_s h^\alpha_{ijs}\omega_{ks} + \sum_\beta h^\beta_{ijk}\omega_{\alpha\beta},$$

respectively. Then we have

$$h^\alpha_{ijk} - h^\alpha_{ikj} = K_{\alpha ikj}, \tag{37}$$

and the Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_s h^\alpha_{sj}R_{sikl} + \sum_s h^\alpha_{is}R_{sjkl} + \sum_s h^\beta_{ij}R_{\beta\alpha kl} \tag{38}$$

Considering $K_{\alpha ijk}$ as a section of $T^\perp(M) \otimes T^*(M) \otimes T^*(M) \otimes T^*(M)$, we also define its covariant derivative $K_{\alpha ijkl}$ as

$$\sum_l K_{\alpha ijkl}\omega_l = dK_{\alpha ijk} + \sum_s K_{\alpha sjk}\omega_{is} + \sum_s K_{\alpha isk}\omega_{js} + \sum_s K_{\alpha ijs}\omega_{ks} + \sum_\beta K_{\beta ijk}\omega_{\alpha\beta}$$

We say that M is a submanifold with parallel second fundamental form if $h^\alpha_{ijk} \equiv 0$ for all i, j, k, α. The Laplacian Δh^α_{ij} of the second fundamental form h is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$.

For a matrix $A = (a_{ij})_{n\times n}$ we denote by $N(A)$ the square norm of A, that is, $N(A) = tr(A'A) = \sum_{i,j} a^2_{ij}$.

Moreover, we quote the following two propositions from [239,242] and [241], respectively.

Proposition (5.2.1)[250]. Let $A_{n+1}, A_{n+2}, \ldots, A_{n+p}$ be symmetric $(n \times n)$- matrices. Denote $S_{\alpha\beta} = tr(A^t_\alpha A_\beta)$, $S_\alpha = S_{\alpha\alpha} = N(A_\alpha)$, $S = \sum_\alpha S_\alpha$. Then

$$\sum_{\alpha,\beta} N(A_\alpha A_\beta - A_\beta A_\alpha) + \sum_{\alpha,\beta} S^2_{\alpha\beta} \leq \left(1 + \frac{1}{2} sgn(p-1)\right)S^2, \tag{39}$$

where $sgn(\cdot)$ is the standard sign function,

Proposition (5.2.2) [250]. Let N be an $(n + p)$ −dimensional Riemannian manifold. If $a \leq K_N \leq b$ at point $x \in N$, then, at this point,

(i) $|K_{ACBC}| \leq \frac{1}{2}(b - a)$, $for\ A \neq B$.

(ii) $|K_{ABCD}| \leq \frac{2}{3}(b - a)$, $for\ A, B, C, D$ distinct with each other.

From now on, we assume that M^n is a connected submanifold with parallel second fundamental form. We choose e_{n+1} such that $e_{n+1} \parallel \xi$, $tr\ H_{n+1} = nH$, and $trH_\beta = 0$, $n + 2 \leq \beta \leq n + p$. Hence, M^n has constant mean curvature since $\sum h^{n+1}_{iik}\omega_k = ndH$ (d here represents the exterior differential which is not the one in the pinching constants). By direct computation, we have

$$\Delta h_{ij}^\alpha = -\sum_k \left(K_{\alpha kikj} + K_{\alpha ijkk}\right) + \sum_{k,m} h_{mi}^\alpha R_{mkjk} + h_{mk}^\alpha R_{mijk}) + \sum_{k,\beta} h_{ki}^\beta R_{\alpha\beta kj}. \quad (40)$$

Plugging (35) and (36) into (40), we get

$$\Delta h_{ij}^\alpha = -\sum_k \left(K_{\alpha kikj} + K_{\alpha ijkk}\right) + \sum_{k,m} h_{mi}^\alpha R_{mkjk} + h_{mk}^\alpha R_{mijk}) + \sum_{k,\beta} h_{ki}^\beta R_{\alpha\beta kj}.$$

$$+ \sum_{m,k,\beta} \left(h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta + 2\, h_{km}^\alpha h_{ki}^\beta h_{mj}^\beta - h_{km}^\alpha h_{km}^\beta h_{ij}^\beta - h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta - h_{mj}^\alpha h_{ki}^\beta h_{mk}^\beta\right).$$

Thus,

$$\frac{1}{2}\Delta S = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 + \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha$$

$$= \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha kikj} + h_{ij}^\alpha K_{\alpha ijkk})$$

$$+ \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{mkik} + h_{mk}^\alpha h_{ij}^\alpha K_{mijk}) + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha\beta kj}$$

$$- \sum_{i,j,k,l,\alpha,\beta} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta - \sum_{i,j,k,l,\alpha,\beta} (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta)(h_{il}^\alpha h_{jl}^\beta - h_{jl}^\alpha h_{il}^\beta).$$

$$+ \sum_{i,j,m,k,\alpha,\beta} h_{ij}^\alpha h_{kl}^\alpha h_{ij}^\beta h_{kl}^\beta$$

Set

$$S_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta$$

Then the $(p \times p)$ −matrix $(S_{\alpha\beta})$ is symmetric. Therefore, we have

Lemma (5.2.3) [250]. Denote

$$A = -\sum_{\alpha,\beta} N\left(H_\alpha H_\beta - H_\beta H_\alpha\right) + \sum_{\alpha,\beta} S_{\alpha\beta}^2,$$

$$B = \sum_{i,j,k,m,\alpha} (h_{mj}^\alpha h_{ij}^\alpha K_{mkik} + h_{mk}^\alpha h_{ij}^\alpha K_{mijk}) + \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha\beta kj},$$

$$C = \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2 - \sum_{i,j,k,\alpha} (h_{ij}^\alpha K_{\alpha kikj} + h_{ij}^\alpha K_{\alpha ijkk})$$

$$D = \sum_{i,j,m,k,\alpha,\beta} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^\beta h_{kk}^\beta = nH \sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1}.$$

Then

$$\frac{1}{2}\Delta S = A + B + C + D. \quad (41)$$

It is easy to see from (37) that $K_{\alpha ijk} = 0$, for all i, j, k, α. Hence, we have

146

Lemma (5.2.4) [250]. $C = 0$.

Let $a(x)$ and $b(x)$ be the infimum and the supremum of the sectional curvature of N at a point x, respectively. We shall estimate B.

Lemma (5.2.5) [250]. $B \geq nbS - \frac{2}{3}(p - 1)(n - 1)^{\frac{1}{2}}](b - a)S - n^2 aH^2$.

Proof. Fix a vector e_α. Let $\{e_i\}$ be a frame diagonalizing the matrix (h_{ij}^α) such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}, \qquad 1 \leq i,j \leq n.$$

Then

$$\sum_{i,j,k,m} h_{mj}^\alpha h_{ij}^\alpha K_{mkik} + \sum_{i,j,k,m} h_{mk}^\alpha h_{ij}^\alpha K_{mijk} + \sum_{i,j,k,\beta} h_{ij}^\alpha h_{ki}^\beta K_{\alpha\beta kj}$$

$$= \sum_{i,j,k,\alpha} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i,j,k,\alpha} \lambda_k^\alpha \lambda_i^\alpha K_{kiik} + \sum_{i,k,\beta} h_{ki}^\beta \lambda_i^\alpha K_{\alpha\beta ki}. \qquad (42)$$

By Proposition (5.2.2),

$$|K_{\alpha\beta ki}| < \frac{2}{3}(b - a) \qquad \text{for all} \quad \alpha \neq \beta, i \neq k.$$

Hence, for fixed α,

$$\sum_{i,k,\beta} h_{ki}^\alpha \lambda_i^\alpha K_{\alpha\beta ki} \geq - \sum_{\alpha \neq \beta, i \neq k.} \frac{2}{3}(b - a)|h_{ki}^\beta \lambda_i^\alpha|$$

$$\geq - \sum_{\alpha \neq \beta, i \neq k.} \frac{1}{3}(b - a)[(n - 1)^{1/2}(h_{ki}^\beta)^2 + (n - 1)^{-1/2}(\lambda_i^\alpha)^2]$$

$$\geq -\frac{1}{3}(n - 1)^{1/2}(b - a)\sum_{\alpha \neq \beta} tr\, H_\beta^2 - \frac{1}{3}(n - 1)^{1/2}(p - 1)(b - a)tr\, H_\alpha^2$$

On the other hand, one sees

$$\sum_{i,k} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i,j,k,\alpha} \lambda_k^\alpha \lambda_i^\alpha K_{kiik}$$

$$= \frac{1}{2}\sum_{i,k} (\lambda_i^\alpha - \lambda_k^\alpha)^2 K_{ikik} \geq \frac{1}{2}a \sum_{i,k}(\lambda_i^\alpha - \lambda_k^\alpha)^2 = na\, trH_\alpha^2 - a(tr\, H_\alpha)^2$$

Plugging the above two inequalities into (42), we obtain

$$B \geq \sum_\alpha [na\, trH_\alpha^2 - \frac{1}{3}(n - 1)^{1/2}(b - a)\sum_{\alpha \neq \beta} tr\, H_\beta^2 - \frac{1}{3}(n - 1)^{\frac{1}{2}}(p - 1)(b - a)trH_\alpha^2] - n^2 aH^2$$

$$= nbS - \frac{2}{3}(p - 1)(n - 1)^{\frac{1}{2}}(b - a)S - n^2 aH^2. \qquad (43)$$

For fixed α, similar to the estimate of lower bound for B, we have

$$LHS\, of\, (42) = \sum_{i,k} (\lambda_i^\alpha)^2 K_{ikik} + \sum_{i,k} \lambda_k^\alpha \lambda_i^\alpha K_{kiik} + \sum_{i,k,\beta} h_{ki}^\beta \lambda_i^\alpha K_{\alpha\beta ki}$$

$$\leq nd\, tr\, H_\alpha^2 + \frac{1}{3}(p - 1)(n - 1)^{1/2}(b - a)tr\, H_\alpha^2$$

$$+\frac{1}{3}(n-1)^{1/2}(b-a)\sum_{\beta\neq\alpha} tr\ H_\beta^2 - n^2 aH^2$$

This gives

Lemma (5.2.6) [250] . $B \leq n\ dS + \frac{2}{3}p(p-1)(n-1)^{1/2}(b-a)S - n^2 aH^2$.

We next shall estimate D.

Lemma (5.2.7) [250]. $|D| \leq n|H|S^{\frac{3}{2}}$.

Proof. Following the proof of Lemma (5.2.5), for fixed α, let $\{e_i\}$ be a frame diagonalizing the matrix (h_{ij}^α) such that $h_{ij}^\alpha = 0$ if $i \neq j$. Then we have

$$\sum_{i,j,m} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1} = \sum_i (h_{ij}^\alpha)^2 h_{ii}^{n+1} \tag{44}$$

The absolute value of this number is not greater than

$$\sqrt{\sum_i \left((h_{ij}^\alpha)^2\right)^2 \cdot \sum_i (h_{ii}^{n+1})^2}$$

by Schwarz inequality. In fact, for fixed α, this is less than

$$\sqrt{(\sum_{i,j}(h_{ij}^\alpha)^2)^2 \cdot (\sum_i(h_{ii}^{n+1})^2)} \leq (\sum_{i,j}(h_{ij}^\alpha)^2)\sqrt{S} = S_\alpha\sqrt{S}.$$

Hence,

$$|D| \leq n|H|.|\sum_{i,j,m,\alpha} h_{ij}^\alpha h_{mi}^\alpha h_{mj}^{n+1}| \leq n|H|.\sum_\alpha (S_\alpha\sqrt{S}) = n|H|S^{\frac{3}{2}}.$$

Moreover, it is obvious that

$$|D| \leq nS(\frac{tH^2}{2} + \frac{S}{2t}), \tag{45}$$

for any $t > 0$.

We define the pinching constants as follows. Set

$$E(p) = 1 + \frac{1}{2}sgn(p-1),$$

$$F(n,p,c,d) = nc - \frac{2}{3}(p-1)(n-1)^{1/2}(d-c),$$

$$G(n,p,c,d) = nd + \frac{2}{3}(p-1)(n-1)^{1/2}(d-c),$$

$$J(n,d,H) = n^2 H^2 d,$$

$$K(n,c,H) = n^2 H^2 c,$$

$$Q_1(n,p,c,d,H) = \frac{F - \frac{n^2|H|}{4E} - \sqrt{\left(F - \frac{n^2|H|}{4E}\right)^2 - 4(1+|H|)EJ}}{2(1+|H|)E}$$

$$Q_2(n,p,c,d,H) = \frac{F - \frac{n^2|H|}{4E} + \sqrt{\left(F - \frac{n^2|H|}{4E}\right)^2 - 4(1 + |H|)EJ}}{2(1 + |H|)E}$$

$$R(n,p,c,d,H) = \frac{G + \frac{n^2p(1 + |H|)|H|}{4} + \sqrt{\left(G - \frac{n^2p(1 + |H|)|H|}{4}\right)^2 - \frac{4K}{(1 + |H|)p}}}{\frac{2}{(1 + |H|)p}}.$$

It is obvious that $Q_1(n,p,c,d,H) \le Q_2(n,p,c,d,H)$.

We give an estimate of the upper bound of the square norm of the second fundamental form of the above submanifold.

Theorem (5.2.8) [250]. Let M^n be an n-dimensional submanifold with parallel second fundamental form in an $(n + p)$ −dimensional Riemannian manifold N^{n+p}. Denote by c and d the infimun and the supremum of the sectional curvature of N^{n+p}, respectively. If $R(n,p,c,d,H) \ge 0$, then S is bounded, that is, $S \le R(n,p,c,d,H)$.

Proof. It is obvious that

$$-A = \sum_{\alpha,\beta} N(H_\alpha H_\beta - H_\beta H_\alpha) + \sum_{\alpha,\beta} S^2_{\alpha\beta} \ge \frac{S^2}{p}. \tag{46}$$

Under the assumption of $\Delta S = 0$, Lemma (5.2.4) and Lemma (5.2.6), together with Lemma (5.2.7) give

$$GS - K + n|H|S^{\frac{3}{2}} \ge \frac{S^2}{p}. \tag{47}$$

In the case M^n is minimal, this shows that $GS \ge \frac{S^2}{p}$ which means $S \le Gp$. For $H \ne 0$, using (45), it follows that

$$GS - K + nS\left(\frac{tH^2}{2} + \frac{S}{2t}\right) \ge \frac{S^2}{p},$$

that is,

$$\left(\frac{1}{p} - \frac{n}{2t}\right)S^2 - \left(G + \frac{ntH^2}{2}\right)S + K \le 0. \tag{48}$$

Let $t = \frac{np(1+|H|)}{2|H|}$. Then (48) becomes

$$\frac{1}{(1 + |H|)p}S^2 - \left(G - \frac{n^2p(1 + |H|)|H|}{4}\right)S + K \le 0 \tag{49}$$

Therefore, we obtain that

$$S \le R(n,p,c,d,H).$$

Xu's previous result in [327] is recovered when M^n is minimal.

Corollary (5.2.9) [250]. Under the assumption of Theorem (5.2.8), if M^n is minimal, then

$$S \leq pnd + \frac{2}{3}p(p-1)(n-1)^{1/2}(d-c).$$

In addition, we also establish a generalized Simons-type inequality.

Theorem (5.2.10) [250]. (Generalized Pinching Theorem) Let M^n be an n-dimensional submanifold with parallel second fundamental form in an $(n+p)$−dimensional Riemannian manifold N^{n+p}. Denote by c *and* d the infimun and the supremum of the sectional curvature of N^{n+p} respectively. If $Q_1(n,p,c,d,H) \geq 0$ and $S < Q_2(n,p,c,d,H)$, then we have $S \leq Q_1(n,p,c,d,H)$.

Proof. Since M^n is a submanifold with parallel second fundamental form, $\frac{1}{2}\Delta S = 0$. From Proposition (5.2.1), we have

$$A \geq -ES^2. \tag{50}$$

Lemma (5.2.5) yields

$$B \geq naS - \frac{2}{3}(p-1)(n-1)^{1/2}(b-a)S - n^2 aH^2$$
$$\geq ncS - -\frac{2}{3}(p-1)(n-1)^{1/2}(d-c)S - n^2 dH^2$$
$$= FS - J. \tag{51}$$

Therefore, we obtain

$$0 = A + B + C + D \geq -ES^2 + FS - J - n|H|S^{\frac{3}{2}}. \tag{52}$$

If M^n is minimal, that is, $H = 0$, then $0 = \frac{1}{2}\Delta S \geq -ES^2 + FS$.

We claim that Theorem (5.2.10) holds in this case by direct check.

For $H \neq 0$, using (45), we have

$$0 = \frac{1}{2}\Delta S \geq -ES^2 + FS - nS\left(\frac{tH^2}{2} + \frac{S}{2t}\right) - J.$$

Equivalently,

$$\left(E + \frac{n}{2t}\right)S^2 - \left(F - \frac{ntH^2}{2}\right)S + J \geq 0. \tag{53}$$

Let $t = \frac{n}{2|H|E}$. Then (53) becomes

$$(|H| + 1)ES^2 - (F - \frac{n^2|H|}{4E})S + J \geq 0. \tag{54}$$

The inequality (45) shows that $S \leq Q_1(n,p,c,d,H)$ or $S \geq Q_2(n,p,c,d,H)$. We thus obtain that $0 \leq S < Q_2(n,p,c,d,H)$ implies $0 \leq S \leq Q_1(n,p,c,d,H)$.

From Theorem (5.2.10), we have

Corollary (5.2.11) [250]. Under the assumption of Theorem (5.2.10), if M^n is minimal, then M^n is totally geodesic.

Section(5-3). CR Sasakian Manifolds

Let M be a complete strictly pseudo-convex CR Sasakian manifold with real dimension $2n + 1$. Let θ be a pseudo-hermitian form on M with respect to which the Levi form L_θ is positive definite. The kernel of θ determines a horizontal bundle H. Now denote by T the Reeb vector field on M, i.e., the characteristic direction of θ. We denote by ∇ the Tanaka-Webster connection of M.

We recall that the CR manifold (M, θ) is called Sasakian if the pseudo-hermitian torsion of ∇ vanishes, in the sense that $T(T, X) = 0$, for every $X \in H$. For instance the standard CR structures on the Heisenberg group H_{2n+1} and the sphere S^{2n+1} are Sasakian. In every Sasakian manifold the Reeb vector field T is a sub-Riemannian Killing vector field
.

We consider the family of scaled Riemannian metrics g_τ, $\tau > 0$, such that for $X, Y \in H$,

$$g_\tau(X, Y) = d\theta(X, JY), \qquad g_\tau(X, T) = 0, \qquad g_\tau(T, T) = \frac{1}{\tau^2}, \tag{55}$$

where J is the complex structure on M. We denote by d_τ the distance corresponding to the Riemannian structure g_τ and by d the sub-Riemannian distance on M. It is well known that $d_\tau(x, y) \to d(x, y)$ when $\tau \to 0$. Our goal is to prove the following theorem:

To put things in perspective, estimates between the sub-Riemannian distance and Riemannian ones have been extensively studied (see [254], [255], [256], [257], [258]). But in these cited works, such estimates are local in nature. Theorem (5.3.1) is the first result that gives global and uniform estimates for a large class of sub-Riemannian metrics. It is consistent with the well known Nagel-Stein-Wainger estimate [257], which implies that, at small scales, $(x, y) \leq C d_\tau(x, y)^{1/2}$, and shows that, at large scales, due to curvature effects we have $d(x, y) \simeq d_\tau(x, y)$.

We first recall some results that will be needed in the sequel and that can be found in [251] and [252]. We denote by Δ the sub-Laplacian on M and by ∇^H the horizontal gradient. For smooth functions $f : M \to R$, set

$$\Gamma_2(f) = \frac{1}{2}[\Delta\|\nabla^H f\|^2 - 2\langle\nabla^H f, \nabla^H \Delta f\rangle] \tag{56}$$

and

$$\Gamma_2^T(f) = \frac{1}{2}[\Delta(Tf)^2 - 2(Tf)(T\Delta f)]. \tag{57}$$

The following result was obtained in [252] by means of a Bochner type formula. **Theorem (5.3.1) [259].** Assume that for every $\in H$,

$$R(X, X) \geq 0.$$

Then for every $f \in C^\infty(M)$ and any $\nu > 0$,

$$\Gamma_2(f) + \nu\Gamma_2^T(f) \geq \frac{1}{2n}(\Delta f)^2 - \frac{1}{\nu}\|\nabla^H f\|^2 + \frac{n}{2}(Tf)^2.$$

We denote by $p(t, x, y)$ the heat kernel of M, that is, the fundamental solution of the heat equation $\frac{\partial f}{\partial t} = \Delta f$. The following global lower and upper bounds were proved in [251].

Theorem (5.3.2) [259]. Assume that for every $\in H$,
$$R(X, X) \geq 0.$$
For any $0 < \epsilon \leq 1$ there exists a constant $C(\varepsilon) = C(n, \varepsilon) > 0$, which tends to ∞ as $\epsilon \to 0^+$, such that for every $x, y \in M$ and $t > 0$ one has
$$\frac{C(\varepsilon)^{-1}}{\mu(B(x, \sqrt{t}))} \exp\left(-\left(1 + \frac{3}{n}\right)\frac{d(x, y)^2}{(4 - \varepsilon)t}\right)$$
$$\leq p(t, x, y) \leq \frac{C(\varepsilon)}{\mu(B(x, \sqrt{t}))} \exp\left(-\frac{d(x, y)^2}{(4 - \varepsilon)t}\right).$$

In the above theorem, d is the sub-Riemannian distance, $B(x, \sqrt{t})$ is the sub- Riemannian ball with center x and radius $\sqrt{t}$, and μ is the volume corresponding to the volume form $\theta \wedge (d\theta)^n$.

From now on and in the sequel we assume that for every $X \in H$, $R(X, X) \geq 0$. We first have the following Li-Yau type estimatefor the heat kernel.

Proposition (5.3.3) [259]. For $t > 0$,

$$\|\nabla^H \ln P_t\|^2 + \frac{n}{3} t(T \ln P_t)^2 \left(1 + \frac{3}{n}\right)\frac{\Delta P_t}{P_t} + \frac{n\left(1 + \frac{3}{n}\right)^2}{t}.$$

Proof. The result is essentially proved in [252], but due to the simplicity of the ar- gument, we reproduce, without the details, the proof by sake of completeness. Fix $T > 0$ and consider the functional
$$\Phi(t) = \frac{3}{n}(T - t)^2 P_t \left(\frac{\|\nabla^H P_{T-t}\|^2}{P_{T-t}}\right) + (T - t)^3 P_t \left(\frac{(TP_{T-t})^2}{P_{T-t}}\right),$$
where P_t is the heat semigroup associated with Δ. Since T is a Killing vector field, for any smooth function f we have
$$\langle \nabla^H f, \nabla^H (Tf)^2\rangle = (Tf)(T\|\nabla^H f\|^2).$$
Differentiating Φ and using the above yield
$$\Phi'(t) = \frac{6}{n}(T - t)^2 P_t(P_{T-t}(\Gamma_2(\ln P_{T-t})) + 2(T - t)^3 P_t(P_{T-t}\Gamma_2^T(\ln P_{T-t}))$$
$$- \frac{6}{n}(T - t)P_t\left(\frac{\|\nabla^H P_{T-t}\|^2}{P_{T-t}}\right) - 3(T - t)^2 P_t\left(\frac{(TP_{T-t})^2}{P_{T-t}}\right).$$

From Theorem (5.3.1), we have
$$\frac{6}{n}(T - t)^2 P_{T-t}\Gamma_2(\ln P_{T-t}) + 2(T - t)^3 P_{T-t}\Gamma_2^T(\ln P_{T-t})$$

152

$$\geq \frac{3}{n^2}(T-t)^2 P_{T-t}(\Delta \ln P_{T-t})^2 - \frac{18}{n^2}(T-t)P_{T-t}\|\nabla^H \ln P_{T-t}\|^2$$
$$+ 3(T-t)^2 P_{T-t}(T \ln P_{T-t})^2.$$

Therefore we obtain

$$\Phi'(t) \geq \frac{3}{n^2}(T-t)^2 P_t(P_{T-t}(\Delta \ln P_{T-t})^2)$$
$$- \left(\frac{18}{n^2} + \frac{6}{n}\right)(T-t)P_t(P_{T-t}\|\nabla^H \ln P_{T-t}\|^2).$$

Now, for every $\gamma(t)$, we have

$$((\Delta \ln P_{T-t}))^2 \geq 2\gamma(t)\Delta \ln P_{T-t} - \gamma(t)^2$$
$$\geq 2\gamma(t)\left(\frac{\Delta P_{T-t}}{P_{T-t}} - \|\nabla^H \ln P_{T-t}\|^2\right) - \gamma(t)^2.$$

Therefore we get

$$\frac{3}{n^2}(T-t)^2 P_t(P_{T-t}(\Delta \ln P_{T-t})^2) \geq \frac{6}{n}(T-t)^2\gamma(t)(\Delta P_T - P_t(P_{T-t}\|\nabla^H \ln P_{T-t}\|^2))$$
$$- \frac{3}{n^2}(T-t)^2\gamma(t)^2 p_T$$

This implies

$$\Phi'(t) \geq \frac{3}{n^2}(T-t)^2\gamma(t)\Delta P_T - \frac{3}{n^2}(T-t)^2\gamma(t)^2 p_T$$
$$- \left(\frac{18}{n^2} + \frac{6}{n} + \frac{6}{n^2}(T-t)\gamma(t)\right)(T-t)P_t(P_{T-t}\|\nabla^H \ln P_{T-t}\|^2)$$

Choosing $\gamma(t) = -\frac{n+3}{T-t}$ then leads to

$$\Phi'(t) \geq -\frac{6(n+3)}{n^2}(T-t)\Delta p_T - \frac{3}{n^2}(n+3)^2.$$

By integrating the last inequality from 0 to T, we obtain

$$-\Phi(0) \geq -\frac{3(n+3)}{n^2}T^2\Delta p_T - \frac{3}{n^2}(n+3)^2 T,$$

which is the required inequality.

We can deduce from the previous Li-Yau type inequality the following Harnack inequality.

Theorem (5.3.4) [259]. For $x, y, z \in M, s < t$,

$$p(s, x, y) \leq p(t, x, z)\left(\frac{t}{s}\right)^{n+3} exp\left(\left(1 + \frac{3}{n}\right)\left(\frac{1}{4(t-s)} + \frac{\frac{3}{n}\tau^2 \ln\frac{t}{s}}{4(t-s)^2}\right)d_\tau(x, y)^2\right), \quad s < t,$$

where d_τ denotes the Riemannian metric introduced in (55).

Proof. From Proposition (5.3.3)

$$\|\nabla^H \ln P_t\|^2 \leq \left(1 + \frac{3}{n}\right)\frac{\Delta P_t}{P_t} + \frac{\left(1 + \frac{3}{n}\right)^2}{t}$$

and

$$\frac{3}{n}t(T \ln P_t)^2 \leq \left(1 + \frac{3}{n}\right)\frac{\Delta P_t}{P_t} + \frac{\left(1 + \frac{3}{n}\right)^2}{t}.$$

Therefore we have that for every $\tau > 0$,

$$\|\nabla^H \ln P_t\|^2 + \tau^2(T \ln P_t)^2 \leq \left(1 + \frac{3\tau^2}{nt}\right)\left(1 + \frac{3}{n}\right)\frac{\Delta P_t}{P_t} + \frac{\left(1 + \frac{3}{n}\right)^2}{t}\left(1 + \frac{3\tau^2}{nt}\right). \quad (58)$$

Now let $x, y, z \in M$ and let $\gamma: [s, t] \to M$, $s < t$, be an absolutely continuous path such that $\gamma(s) = y, \gamma(t) = z$. We first write (58) in the form

$$g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) \leq a(u)\frac{\Delta P_t}{P_t} + b(u), \quad (59)$$

where ∇^τ denotes the Riemannian gradient of the metric g^τ; that is,

$$g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) = \|\nabla^H \ln P_t\|^2 + \tau^2(T \ln P_t)^2$$

and

$$a(u) = \left(1 + \frac{3\tau^2}{nu}\right)\left(1 + \frac{3}{n}\right),$$

$$b(u) = \frac{n\left(1 + \frac{3}{n}\right)^2}{u}\left(1 + \frac{3\tau^2}{nu}\right).$$

Let us now consider

$$\varphi(u) = \ln p_u(x, \gamma(t)).$$

We compute

$$\varphi'(u) = (\partial_u \ln p_u(x, \gamma(u)) + g_\tau(\nabla^\tau \ln p_u(x, \gamma(u)), \gamma'(u)).$$

Now, for every $\lambda > 0$, we have

$$g_\tau\left(\nabla^\tau \ln p_u(x, \gamma(u)), \gamma'(u)\right) \geq -\frac{1}{2\lambda^2}g_\tau(\nabla^\tau \ln p_u, \nabla^\tau \ln p_u) - \frac{\lambda^2}{2}g_\tau(\gamma(u), \gamma'(u)).$$

Choosing $\lambda = \sqrt{\frac{a(u)}{2}}$ and then using (59) yield

$$\varphi'(u) \geq -\frac{b(u)}{a(u)} - \frac{1}{4}a(u)g_\tau(\gamma'(u), \gamma'(u)).$$

By integrating this inequality from s to t we get as a result:

$$\ln p(t, x, y) - \ln p(s, x, z) \geq -\int_s^t \frac{b(u)}{a(u)}\,du - \frac{1}{4}\int_s^t a(u)\,g_\tau(\gamma'(u), \gamma'(u))\,du.$$

We now minimize the quantity $\int_s^t a(u)\, g_\tau(\gamma'(u), \gamma'(u))du$ over the set of absolutely continuous paths such that $\gamma(s) = y, \gamma(t) = z$. By using reparametrization of paths, it is seen that

$$\int_s^t a(u)\, g_\tau(\gamma'(u), \gamma'(u))du \geq \frac{d_\tau^2(x, y)}{\int_s^t \frac{dv}{a(v)}},$$

with equality achieved for $\gamma(u) = \sigma\left(\frac{\int_s^u \frac{dv}{a(v)}}{\int_s^t \frac{dv}{a(v)}}\right)$, where $\sigma: [0,1] \to M$ is a unit geodesic for the distance d_τ that joins y *and* z. As a conclusion we get

$$p(s, x, y) \leq \exp\left(\int_s^t \frac{b(u)}{a(u)}\, du + \frac{d_\tau^2(y, z)}{4\int_s^t \frac{dv}{a(v)}}\right) p(t, x, z).$$

Finally, from Cauchy-Schwarz inequality, we have

$$\int_s^t \frac{dv}{a(v)} \geq \frac{(t-s)^2}{\int_s^t a(v)dv},$$

and thus

$$p(s, x, y) \leq \exp\left(\int_s^t \frac{b(u)}{a(u)}\, du + \frac{d_\tau^2(y, z) \int_s^t a(v)dv}{4(t-s)^2}\right) p(t, x, z).$$

Theorem (5.3.5) [259]. Let R be the Ricci curvature of the Webster-Tanaka connection ∇. If for every $\in H$,

$$R(X, X) \geq 0,$$

then for every $x, y \in M$,

$$d_\tau(x, y) \leq d(x, y) \leq A_n d_\tau(x, y) + B_n \sqrt{\tau} d_\tau(x, y)^{1/2},$$

where A_n and B_n are two positive universal constants depending only on n.

Proof. The inequality $d_\tau(x, y) \leq d(x, y)$ is straightforward. We now prove the second inequality. From Theorem (5.3.4) and Theorem (5.3.2),

$$p(s, x, y) \geq \frac{1}{2^{n+3}}\, p(t/2, x, x)exp\left(-\left(1 + \frac{3}{n}\right)\left(\frac{1}{2t} + \frac{3\ln 2\tau^2}{nt^2}\right) d_\tau(x, y)^2\right),$$

$$\geq \frac{1}{2^{n+3}} \frac{C_0(n)}{\mu(B(x, \sqrt{t}))}\, exp\left(-\left(1 + \frac{3}{n}\right)\left(\frac{1}{2t} + \frac{3\ln 2\tau^2}{nt^2}\right) d_\tau(x, y)^2\right).$$

From the Gaussian upper bound of Theorem (5.3.2) and the previous lower bound, we deduce that for all $t > 0$,

$$\ln \frac{2^{n+3}C(\varepsilon)}{C_0(n)} + \left(\frac{1}{2}\left(1 + \frac{3}{n}\right) d_\tau(x,y)^2 - \frac{d(x,y)^2}{4+\varepsilon}\right)\frac{1}{t}$$

$$+ \left(1 + \frac{3}{n}\right)\left(\frac{3\ln 2}{n}\right) \tau^2 d_\tau^2(x,y)\frac{1}{t^2} \geq 0.$$

We now choose $t = \tau d_\tau(x,y)$ and obtain

$$d(x,y)^2 \leq (4+\varepsilon)\left(\ln\frac{2^{n+3}C(\varepsilon)}{C_0(n)} + + \left(1 + \frac{3}{n}\right)\left(\frac{3\ln 2}{n}\right)\right)\tau d_\tau(x,y)$$

Chapter 6
Connection and Curvature with Volume and Distance

We establish Bianchi Identities and symmetries for the associated curvatures. Next we study subRiemannian notions of the Ricci curvature and horizontal Laplacian, establishing general Bochner type identities. Finally we explore sub Riemannian generalizations of the Bonnet-Myers theorem, providing some new results and some new proofs and interpretations of existing results. As a consequence, we obtain a Gromov type precompactness theorem for the class of sub-Riemannian manifolds whose generalized Ricci curvature is bounded from below .

Section(6-1). Subriemannian Geometry

A fundamental tool in Riemannian geometry is the Levi-Civita connection. As the device which permits us to glue local differential equations into global ones, it is the key ingredient in most modern descriptions of curvature and geodesics and underlies many computational methods in differential geometry. The Tanaka-Webster connection, ([265], [266]) plays a similar role in the study of strictly pseudo convex CR manifolds.

There has been much recent effort to define such geometrically useful connections in sub-Riemannian geometry. All work, including this one, has operated under the assumption that the subRiemannian metric on the horizontal bundle has been extended to a Riemannian metric on the whole space. This allows us to define a vertical bundle. Previous work has been inherently local, depending on a choice of frame. Usually some additional geometric and topological restrictions have been required. In [261], [262] a subRiemannian connection was defined under the assumption of a global frame for the vertical bundle. In [260], a connection is defined under a strong tensorial condition, referred to as strict normality the assumption of the existence of a frame of vertical Killing fields. All of these examples required a *a priori* choice of frame for the vertical bundle and so do not define global connections in general.

This lack of a global covariant derivative scheme means that the study of the relationships between subelliptic PDE and subRiemannian manifolds has been by necessity local in nature. Recently there has been some effort addressing this need.

In [260], several global curvature results such as Myer's theorem have been extended to certain step 2 subRiemannian manifolds.

We propose a new globally defined connection to facilitate this process. We shall work assumption that a global complement to the horizontal bundle has been chosen. For any Riemannian metric extending the sub Riemannian metric and preserving this decomposition, we shall define a canonical, global metric compatible connection such that the horizontal and vertical bundles are parallel. In the special cases of Riemannian and strictly pseudoconvex pseudohermitian manifolds, this connection will coincides with the

Levi-Civita and the Tanaka-Webster connections respectively. Furthermore any covariant derivative of any horizontal vector field will be seen to be independent of the choice of Riemannian extension. Thus for a subRiemannian manifold with vertical complement, there is a canonical method for taking covariant derivatives of horizontal vector fields.

We define the connection and explore its basic properties and how they relate to bracket structures of the underlying horizontal and vertical bundles. We introduce a tool similar to Riemannian normal coordinates, to aid computation. We consider the associated curvature tensors and their symmetries. SubRiemannian equivalents of the Bianchi identities are introduced and proved. We establish some Bochner-type formulas for general subRiemannian manifolds and show how the analytic framework developed by Baudoin and Garofalo generalizes to the category strictly normal subRiemannian manifolds. We compare the sub Riemannian connection to the Levi-Civita connection for metric extensions. We then use this to provide a new interpretation and proof of an existing subRiemannian Bonnet-Myers theorem as well as providing new results.

We shall use the following definition:

Definition (6.1.1) [269]. A sub Riemannian manifold is a smooth manifold M, a smooth constant rank distribution $HM \subset TM$ and a smooth inner product $\langle \cdot, \cdot \rangle$ on HM. The bundle HM is known as the horizontal bundle.

We should remark here that we are not assuming any conditions on the horizontal bundle other than constant rank. Unless otherwise stated, we are not even assuming that it bracket generates.

Definition (6.1.2) [269]. A subRiemannian manifold with complement, henceforth sRC-manifold, is a subRiemannian manifold together with a smooth bundle VM such that $HM \oplus VM = TM$. The bundle VM is known as the vertical bundle.

Two sRC-manifolds M, N are sRC-isometric if there exists a diffeomorphism $\pi: M \to N$ such that $\pi_* HM = HN, \pi_* VM = VN$ and $\langle \pi_* X, \pi_* Y \rangle_N = \langle X, Y \rangle_M$ for all horizontal vectors X, Y.

Definition (6.1.3) [269]. A sRC-manifold $(M, HM, VM, \langle \cdot, \cdot \rangle)$ is r-graded if there are smooth constant rank bundles $V^{(j)}, 0 < j \leq r$, such that

$$VM = V^{(1)} \oplus \cdots \oplus V^{(r)}$$

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] \subseteq HM \oplus V^{(j)} \oplus V^{(j+1)} \qquad (1)$$

for all $0 \leq j \leq r$. Here we have adopted the convention that $V^{(0)} = HM$ and $V^{(k)} = 0$ for $k > r$.

The grading is j-regular if

$$HM \oplus V^{(j)} \oplus [HM, V^{(j)}] = HM \oplus V^{(j)} \oplus V^{(j+1)} \qquad (2)$$

and equiregular if is j-regular for all $0 \leq j \leq r$.

A metric extension for an r-graded vertical complement is a Riemannian metric g of $\langle \cdot, \cdot \rangle$ that makes the split

$$TM \;=\; HM \underset{1 \leq j \leq r}{\oplus} V^{(j)}$$

orthogonal.

We shall denote a section of $V^{(k)}$ by $X^{(k)}$ and set

$$\hat{V}^{(j)} \;=\; \underset{k \neq j}{\oplus} V^{(k)}.$$

If a metric extension has been chosen then b $\hat{V}^{(j)} = (V^{(j)})^{\perp}$. the orthogonal com-plement of $V^{(j)}$. For convenience, we shall often also extend the notation $\langle \cdot, \cdot \rangle$ to whole tangent space using it interchangeably with g.

Definition (6.1.4) [269]. The unique 1-grading on each sRC-manifold,

$$V^{(1)} \;=\; V M$$

is known as the basic grading.

Example (6.1.5) [269]. A Carnot group (of step r) is a Lie group, whose Lie algebra g is stratified in the sense that

$$g \;=\; g_0 \oplus \cdots g_{r-1}, \qquad [g_0, g_j] \;=\; g_{j+1} \qquad j \;=\; 1 \ldots r, \qquad g_r \;=\; 0$$

together with a left-invariant metric $\langle \cdot, \cdot \rangle$ on HM, the left-translates of g_0.

The vertical bundle $V M$ consists of the left-translates of $g_1 \oplus \cdots g_{r-1}$. In addition to the basic grading, there is then a natural equiregular $r - 1 -$grading defined by setting $V^{(j)}$ to be the left-translates of g_j.

Definition (6.1.6) [269]. If a metric extension g has been chosen, we define

$$B(X, Y, Z) \;=\; (L g z)(X, Y) \;=\; Z g(X, Y) \;+\; g([X, Z], Y) \;+\; g([Y, Z], X)$$

for vector fields X, Y, Z

Unfortunately B is not tensorial in general and so cannot be viewed as a map on vectors rather than vector fields. However, we can define a symmetric tensor $B^{(j)}$ by setting

$$B^{(j)}(X, Y, T) \;=\; B(X, Y, T)$$

for $X, Y \in V^{(j)}, T \in b \hat{V}^{(j)}$ and declaring $B^{(j)}$ to be zero on the orthogonal complement of $V^{(j)} \times V^{(j)} \times \hat{V}^{(j)}$. We can then contract these to tensors $C^{(j)} : TM \times TM \to V^{(j)}$ defined by

$$g\big(C^{(j)}(X, Y), Z^{(j)}\big) \;=\; B^{(j)}\big(X, Z^{(j)}, Y\big) \tag{3}$$

Additionally, we can define j-traces, by

$$tr_{(j)} B^{(j)}(Z) \;=\; \sum B^{(j)}\big(E_i^{(j)}, E_i^{(j)}, Z\big)$$

where $\{E_i^{(j)}\}$ are (local) orthonormal frames for $V^{(j)}$.

Definition (6.1.7) [269]. Suppose that M is an r-graded sRC-manifold with metric extension g.

 (i) The metric extension is j-normal with respect to the grading if $B^{(j)} \equiv 0$.

 (ii) The metric extension is strictly normal with respect to the grading if it is j-normal for all $0 \leq j \leq r$.

Example (6.1.8) [269]. Let M be the 4 dimensional Carnot group with Lie algebra induced by the global left invariant vector fields X, Y, T, S with bracket structures

$$[X, Y] = T, [X, T] = S$$

and all others being zero. Then $B(T, S, X) = -1$ with all others vanishing. Now M admits an equiregular 2-grading defined by

$$V^{(1)} = \langle T \rangle, \quad V^{(2)} = \langle S \rangle.$$

Let g be the metric making the global frame orthonormal. Then g is strictly normal with respect to this 2-grading.

It should be remarked that this metric is not 1-regular with respect to the basic grading. For then we get $\tilde{B}^{(0)} \equiv 0$ *but* $\tilde{B}^{(1)}(T, S, X) = -1$. Thus the metric is 0-normal but not strictly normal with respect to the basic grading.

Example (6.1.9) [269]. Any step r Carnot group with a bi-invariant metric extension is strictly normal with respect to the equiregular $r - 1 -$grading, but is only 0-normal with respect to the basic grading.

Example (6.1.10) [269]. Let (M, J, η) be a strictly pseudoconvex pseudohermitian manifold, (see [265]) with characteristic vector field T such that $\eta(T) = 1, Td\eta = 0$. The horizontal bundle HM is defined to be the kernel of the 1-form η. An immediate consequence of the defining properties of T is that $[T, HM] \subset HM$. When J is extended to TM by defining $JT = 0$, the Levi metric

$$g(A, B) = d\eta(A, JB) + \eta(A)\eta(B)$$

can be viewed as an extension of the subRiemannian metric $\langle X, Y \rangle = d\theta(X, JY)$ with $VM = \langle T \rangle$. As VM is one dimensional, the basic grading is the only grading admitted and since $[T, HM] \subset HM$ we see $B^{(1)} = 0$ trivially. Thus the Levi metric is always 1-normal and so strict normality is equivalent to 0-normality. However, the Jacobi Identity coupled with $[T, HM] \subset HM$ implies

$$\langle [T, X], Y \rangle = -\langle [[T, X], JY], T \rangle = \langle [[X, JY], T], T \rangle + \langle [[JY, T], X], T \rangle$$
$$= T\langle X, Y \rangle + \langle [JY, T], JX \rangle$$

This implies that 0-normality is equivalent to $\langle [Y, T], X \rangle = -\langle [T, JY], JX \rangle$. But this equivalent to [T,JY] = J[T,Y] which is Tanaka's definition of normal for a strictly pseudoconvex pseudohermitian manifold, [265].

The tensors $C^{(j)}$ provide the essential ingredient for the definition of our connections. The idea boils down to using the Levi-Civita connection on each component $V^{(j)}$ and using projections of the Lie derivative for mixed components. In general, this will not produce a metric compatible connection, but we can use the tensors $C^{(j)}$ to adjust appropriately.

Lemma (6.1.11) [269]. If g is an extension of an r-graded sRC-manifold, then there exists a unique connection $\nabla^{(r)}$ such that

(i) g is metric compatible

(ii) $V^{(j)}$ is parallel for all j

(iii) $Tor^{(r)}(V^{(j)}, V^{(j)}) \subseteq \hat{V}^{(j)}$ for all j

(iv) $\langle Tor^{(r)}(X^{(j)}, Y^{(k)}), Z^{(j)}\rangle, = \langle Tor^{(r)}(Z^{(j)}, Y^{(k)}), X^{(j)}\rangle$ for all j, k

Furthermore, if X, Y are horizontal vector fields, then $\nabla^{(r)}X$ and $Tor^{(r)}(X, Y)$ are independent of the choice of grading and extension g. (They do however depend on choice of VM)

Proof. For a vector field Z, we denote the orthogonal projections of Z to $V^{(j)}$ by Z_j. Define a new connection $\nabla^{(r)}$ as follows: for X, Y, Z sections of $V^{(j)}$ and T a section of $\hat{V}^{(j)}$ set

$$\langle \nabla_X Y, \hat{V}^{(j)}\rangle = 0$$

$$\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle)$$
$$- \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle$$
$$\nabla_T Y = [T, Y]j + \frac{1}{2} C^{(j)}(Y, T)$$

for X, Y, Z horizontal vector fields and T, U, W vertical vector fields. It's easy to check that this defines a connection with the desired properties. Futhermore if $\overline{\nabla}$ is the Levi-Civita connection for g, then for sections X, Y of $V^{(j)}$,

$$\nabla_X Y = (\overline{\nabla}_X Y)_j$$

For uniqueness, suppose that connections ∇ and ∇' satisfy the required properties and set $A(W, Z) = \nabla_W Z - \nabla'_W Z$. Then for sections X, Y, Z of $V^{(j)}$, since the torsion terms are inb $\hat{V}^{(j)}$ we see

$$\langle A(X, Y), Z\rangle = -\langle Y, A(X, Z)\rangle = -\langle Y, A(Z, X)\rangle$$
$$= \langle A(Z, Y), X\rangle = \langle A(Y, Z), X\rangle$$
$$= -\langle Z, A(X, Y)\rangle$$

Similarly if T is a section of $\hat{V}^{(j)}$,

$$\langle A(T, X), Y\rangle = -\langle X, A(T, Y)\rangle = -\langle X, Tor(T, Y) - Tor'(T, Y)\rangle$$
$$= -\langle Tor(T, X) - Tor'(T, X), Y\rangle$$
$$= -\langle A(T, X), Y\rangle$$

Thus A = 0. Thus this connection ∇ is the unique connection with the desired properties. The required independence from g follows easily from (4).

Corollary (6.1.12) [269]. If M admits an r-grading, then

(i) $Tor^{(r)}(V^{(j)}, V^{(j)}) = 0$ if and only if $V^{(j)}$ is integrable.

(ii) $Tor^{(r)}(HM, V^{(j)}) \subset HM \oplus V^{(j)} \oplus V^{(j+1)}$ for all j

If the r-grading is j-normal then

$$Tor^{(r)}(TM, V^{(j)}) \subset \hat{V}^{(j)}$$

If the r-grading is 0-normal and j-normal then

$$Tor^{(r)}(HM, V^{(j)}) \subseteq V^{(j+1)}$$

with equality holding if and only if the grading is j-regular.

Example (6.1.13) [269]. Suppose that HM has global orthonormal frame $\{X_i\}$ and VM has global orthonormal frame $\{T_\beta\}$ with the following bracket identities:

$$[X_i, X_j] = c_{ij}^k X_k + c_{ij}^\alpha T_\alpha$$
$$[X_i, X_\beta] = c_{i\beta}^k X_k + c_{i\beta}^\alpha T_\alpha$$
$$[X_\gamma, X_j] = c_{\gamma\beta}^k X_k + c_{\gamma\beta}^\alpha T_\alpha$$

Then using the basic grading and connection we have

i. VM is normal if and only if $c_{i\beta}^k = - c_{k\beta}^i$

ii. g is strictly normal if and only if $c_{i\beta}^\alpha = -c_{i\alpha}^\beta$ and $c_{i\beta}^k = - c_{k\beta}^i$

iii. g is vertically rigid if and only if $\sum c_{i\beta}^\beta = 0$

and

iv. $\nabla_{X_i} X_j = \frac{1}{2}\left(c_{ij}^k + c_{ij}^k + c_{ij}^k\right)X_k$, $Tor(X_i, X_j) = -c_{ij}^\alpha T_\alpha$

v. $\nabla_{T_\beta} X_j = \frac{1}{2}\left(c_{k\beta}^j + c_{j\beta}^k\right) X_k$

vi. $\nabla_{X_j} T_\beta = \frac{1}{2}\left(c_{i\beta}^\alpha + c_{i\alpha}^\beta\right) T_\alpha$

vii. $\nabla_{T_\gamma} T_\beta = \frac{1}{2}\left(c_{\gamma\beta}^\alpha + c_{\alpha\gamma}^\beta + c_{\alpha\beta}^\gamma\right) T_\gamma$, $Tor(T_\gamma, T_\beta) = -c_{\gamma\beta}^k X_k$

viii. $Tor(X_k, T\beta) = -\frac{1}{2}\left(c_{k\beta}^j + c_{j\beta}^k\right) X_k - \frac{1}{2}\left(c_{j\beta}^\alpha + c_{j\alpha}^\beta\right) T_\alpha.$

To illustrate some important behavior, we shall highlight a group particular cases of the previous example

Example (6.1.14) [269]. Let M be the 4 dimensional Carnot group of Example (6.1.8). Using the basic grading, we can easily compute that

$$\nabla_X T = S - \frac{1}{2}S = \frac{1}{2}S, \nabla_X S = 0 - \frac{1}{2}T = -\frac{1}{2}T,$$

$$Tor(X, Y) = -T, \quad Tor(X, T) = -\frac{1}{2}S, \quad Tor(X, S) = -\frac{1}{2}T$$

All other covariant derviatives of frame elements vanish. That the basic covariant derivatives of the natural vertical frame do not vanish is typical of non-step 2 Carnot groups.

However if we use the more refined 2-grading, then all covariant derivatives of the frame elements vanish and the only non-trivial behavior occurs in the torsion

$$\text{Tor}^{(2)}(X,Y) = -T, \quad \text{Tor}^{(2)}(X,T) = -S, \quad \text{Tor}^{(2)}(X,S) = 0$$

Example (6.1.15) [269]. Let $M = R^4$ with the following global orthonormal frames for HM and VM

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} + \sin x \frac{\partial}{\partial t} - \cos x \frac{\partial}{\partial s}$$

$$T = \cos x \frac{\partial}{\partial s} + \sin x \frac{\partial}{\partial t}, \quad S = -\sin x \frac{\partial}{\partial t} + \cos x \frac{\partial}{\partial s}$$

Then $[X,Y] = T = -[X,S], [X,T] = S$ with all other commutators vanishing. It's then easy to check that this is a strictly normal extension for the basic grading and that the only non-trivial covariant derivatives are then $\nabla_X T = S$ and $\nabla_X S = T$. This is an example of a flat, equiregular, strictly normal sRC-manifold with step size > 2.

Example (6.1.16) [269]. Let (M, J, η) be a strictly pseudoconvex pseudohermitian manifold. The Tanaka-Webster connection is the unique connection such that $\eta, d\eta$ and J are parallel and the torsion satisfies

$$\text{Tor}(X,Y) = d\eta(X,Y)T, \quad \text{Tor}(T, JX) = -J\text{Tor}(T, X)$$

The only defining property of the basic connection not clearly satisfied by the Tanaka-Webster connection is torsion symmetry. But if we pick X, Y as any horizontal vector fields then the Jacobi identity implies

$$
\begin{aligned}
0 &= \eta\left([T, [X, JY]] + [JY, [T, X]] + [X, [JY, T]]\right) \\
&= -T\langle X, Y\rangle + \langle [T, X], Y\rangle + \langle [JY, T], JX\rangle \\
&= -\langle X, \nabla_T Y\rangle + \langle \text{Tor}(T, X), Y\rangle - \langle \nabla_T JY, JX\rangle + \langle \text{Tor}(T, JY), JX\rangle \\
&= \langle \text{Tor}(T, X), Y\rangle + \langle \text{Tor}(T, JY), JX\rangle \\
&= \langle Tor(T, X), Y\rangle - \langle \text{Tor}(T, Y), X\rangle.
\end{aligned}
$$

Thus the Tanaka-Webster connection satisfies the requirements of the basic connection.

One of the key computational tools when using the Levi-Civita connection is the existence of Riemannian normal coordinates in a neighborhood of any given point. As HM is non-integrable in every interesting example, we cannot expect to find a similarly useful coordinate system in the subRiemannian case. However, when the extension is normal, we can guarantee the existence of a local orthonormal horizontal frame with computationally nice properties at any particular point p.

Definition (6.1.17) [269]. If M is r-graded, then an orthonormal frame $\{E_i^{(j)}\}$ for $V^{(j)}, 0 \leq j \leq r$, defined in a neighborhood of p is $\nabla^{(r)}$-normal at p if

$$\left(\nabla^{(r)} E_i^{(j)}\right)_{|p} = 0$$

Lemma (6.1.18) [269]. Suppose g is a j-normal r-grading. Then there exists a $\nabla^{(r)}$-normal frame for $V^{(j)}$ at every $p \in M$.

Proof. Let $v_1^{(k)}, \ldots, v_{nk}^{(k)}$ be orthonormal vectors spanning $v_p^{(k)}$ and let $\{x_{(k)}^i\}$ be the coordinates near p induced by the exponential map of $\nabla^{(r)}$ at p using this frame. Then certainly $\nabla^{(r)}_{c_i \partial_{x^i}}(c_i \partial_{x^i}) = 0$ at p whenever the coefficients c_i are constant. Considering $\nabla^{(r)}_{\partial_i + \partial_j}(\partial_i + \partial_j)$ in particular, this implies that for all i, j at p

$$0 = \nabla^{(r)}_{\partial_i} \partial_j + \nabla^{(r)}_{\partial_j} \partial_i = 2\nabla_{\partial_i} \partial_j + \mathrm{Tor}^{(r)}(\partial_i, \partial_j).$$

Now $\left(\partial_{x_{(j)}^i}\right)_{|p} \in v_p^{(j)}$. Since torsion is tensorial and $\mathrm{Tor}^{(r)}(TM, V^{(j)}) \subset \hat{V}^{(j)}$ by Corollary (6.1.12), this implies that

$$\left(\nabla^{(r)}_{\partial_i} \partial_{x_{(j)}^i}\right)_{|p} \in \hat{V}_p^{(j)} \tag{5}$$

for all i.

Now in a small neighbourhood of p define $Z_k^{(j)} = (\partial_{x^{(j)}}^k)_0$, i.e. the orthogonal projection of $\partial_{x^{(j)}}^k$ onto $V^{(j)}$. Set $T_k^{(j)} = \partial_{x_{(j)}} - Z_k^{(j)}$. We clearly have linear independence near p and so $Z_1^{(j)}, \ldots, Z_{nj}^{(j)}$ is a local frame for $V^{(j)}$.

Now for any vector field Y,

$$\nabla^{(r)}_Y Z_i^{(j)} = \nabla^{(r)}_Y(\partial_{x^{(j)}}^i - T_i^{(j)}) = \nabla^{(r)}_Y \partial_{x^{(j)}}^i - \nabla^{(r)}_Y T_i^{(j)}$$

The first term on the right is in $\hat{V}^{(j)}$ by (5). The last term is in $\hat{V}^{(j)}$ everywhere as $T_i^{(j)}$ is a section of $\hat{V}^{(j)}$ which is parallel. But $\nabla_Y Z_i^{(j)}$ is in $V^{(j)}$ as $V^{(j)}$ is parallel. This implies that $\nabla^{(r)} Z_i^{(j)} = 0$ at p.

Now from metric compatibility, we see that $Y \langle Z_i^{(j)}, Z_k^{(j)}\rangle_{|p} = 0$ for each i, k, so an easy induction argument shows that if we apply the Gram-Schmidt algorithm to $Z_1^{(j)}, \ldots, Z_{nj}^{(j)}$ we obtain an orthonormal frame with the same property at p.

Corollary (6.1.19) [269]. If the grading is strictly normal, then near any point $p \in M$, there is a graded orthonormal frame $X_i^{(j)}$ for TM such that $\left(\nabla X_i^{(j)}\right)_{|p} = 0$.

Definition (6.1.20) [269]. The sub Riemannian curvature tensors for a sRC-manifold with extension g are defined by

$$R(A, B)C = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C$$

and

$$Rm^s(A,B,C,D) = \langle R(A,B)C,D\rangle$$

We note that for any vectors $A, B \in TM$, the restriction of the (1,1)-tensor $R(A,B)$ to HM is independent of the choice of extension g.

This definition immediately yields notions of flatness in subRiemannian geometry.

Definition (6.1.21) [269]. We say that an M is horizontal flat if $Rm^s(\cdot,\cdot,HM,\cdot) = 0$ for any extension g. A particular extension is vertically flat if $Rm^s(\cdot,\cdot,VM,\cdot) = 0$ or flat if $Rm^s = 0$.

Lemma (6.1.22) [269]. A sRC-manifold is horizontally flat if and only if in a neighborhood of every point $p \in M$ there is a local orthonormal frame $\{E_i\}$ for HM such that $\nabla E_i = 0$. If HM is integrable, this local frame can be chosen to be a coordinate frame.

A similar result holds for a vertically flat extension g and VM.

Example (6.1.23) [269]. Every step r Carnot group is horizontally flat for the basic grading and flat for the r -1-grading. The sRC-manifolds considered in Example (6.1.8) and Example (6.1.15) are both flat.

It is useful to define the following

Definition (6.1.24) [269]. If S is any set and $F: S^k \to L$ is any map into a vector space L, we define ℓF to be the sum of all cyclic permutations of F. For example if k = 3, then

$$\ell F(X,Y,Z) = F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y)$$

An example of the cyclic construction in action is a compressed form of the Jacobi Identity for vector fields, namely

$$\ell[X,[Y,Z]] = 0$$

We shall use it primarily to efficiently describe symmetries of the curvature tensor.

We also introduce

Definition (6.1.25) [269]. The second-order torsion of ∇ is the (3,1)-tensor

$$\mathsf{TOR}_2(A,B,C) = Tor(A,Tor(B,C))$$

We are now in a position to discuss the fundamental questions of curvature symmetries. Many of the properties of the Riemannian curvature tensor go through unchanged, with exactly the same proof. In particular,

Lemma (6.1.26) [269]. The subRiemannian curvature tensor always has the following symmetries

ix. $Rm^s(A,B,C,D) = -Rm^s(A,B,D,C)$

x. $Rm^s(A,B,C,D) = -Rm^s(B,A,C,D)$

xi. $Rm^s(TM,TM,HM,VM) = 0$

However, many symmetry properties of the Riemannian curvature tensor require additional assumptions in the subRiemannian case. Most of these symmetries are naturally related to the Bianchi Identities.

Lemma (6.1.27) [269]. (Algebraic Bianchi Identites). For any vector fields X, Y, Z,

$$\ell R(X,Y)Z \;=\; -\ell\mathrm{TOR}_2(X,Y,Z) \;+\; \ell\,(\nabla\mathrm{Tor})(X,Y,Z).$$

Furthermore

(a) if $X,Y,Z \in V^{(j)}$ then
$$\ell(\nabla\mathrm{Tor})(X,Y,Z) \in \hat{V}^{(j)}$$

(b) if $X,Y,Z \in V^{(j)}$ and the grading is j-normal, then
$$-\ell\,\mathrm{TOR}_2(X,Y,Z) \in \hat{V}^{(j)}$$

(c) if $X,Y \in V^{(j)}$, the grading is j-normal and $\hat{V}^{(j)}$ is integrable then
$$-\ell\,\mathrm{TOR}_2(X,Y,Z) \in \hat{V}^{(j)}$$

Proof. The first part of the lemma is a standard result from differential geometry, but for completeness we shall present a short proof

$$
\begin{aligned}
\ell R(X,Y)Z &= \ell(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\\
&= \ell(\nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z\\
&= \ell(\nabla_Z([X,Y] + \mathrm{Tor}(X,Y)) - \nabla_{[X,Y]}Z)\\
&= \ell\,([Z,[X,Y]] + \mathrm{Tor}(Z,[X,Y]) + (\nabla_Z\mathrm{Tor}(X,Y))\\
&= \ell\,(\mathrm{Tor}(Z,[X,Y]) + \mathrm{Tor}((\nabla_Z X,Y) + \mathrm{Tor}(X,(\nabla_Z Y)) + \ell(\nabla\mathrm{Tor})(X,Y,Z)\\
&= \ell\,(\mathrm{Tor}(Z,[X,Y] - (\nabla_X Y - (\nabla_Y X)) + \ell\,(\nabla\mathrm{Tor})(X,Y,Z)\\
&= -\ell\,\mathrm{TOR}_2(X,Y,Z) + \ell(\nabla\mathrm{Tor})(X,Y,Z)
\end{aligned}
$$

The remaining parts consist of analyzing the terms $\ell\,\mathrm{TOR}_2$ and $\ell(\nabla\mathrm{Tor})$. Since these are tensorial, we can compute using normal and seminormal frames. First let X,Y,Z be elements of a seminormal frame for $V^{(j)}$ at p, then
$$\ell(\nabla\mathrm{Tor})(X,Y,Z) \;=\; \ell(\nabla_X\mathrm{Tor}(Y,Z))$$

But each torsion piece must be in $\hat{V}^{(j)}$. As this is bundle parallel, we have established (a). Now, if we assume the frame is j-normal, then we can instead use a normal frame at p. If X is an element of this frame then $\mathrm{Tor}(X,TM) \subset \hat{V}^{(j)}$ and it is easy to check that (b) holds.

Assume a j-normal grading and that X,Y are elements of a j-normal frame at p, but Z is an arbitrary vector field. Then
$$-\ell\,\mathrm{TOR}_2(X,Y,Z) \;=\; \ell\mathrm{Tor}(X,\mathrm{Tor}(Y,Z)) \;=\; \mathrm{Tor}(Z,\mathrm{Tor}(X,Y))$$

But
$$(\mathrm{Tor}(Z,\mathrm{Tor}(X,Y)))_j \;=\; -[Z_{\hat{j}},\mathrm{Tor}(X,Y)]_j$$

which vanishes if $\hat{V}^{(j)}$ is integrable. Thus (c) holds.

Corollary (6.1.28) [269]. (Horizontal Algebraic Bianchi Identity). If X,Y,Z,W are horizontal vector fields and $V\,M$ is normal, then
$$\langle \ell R(X,Y)Z,W\rangle \;=\; 0$$

If $V\,M$ is also integrable, then this can be relaxed to any three of X,Y,Z,W horizontal.

Corollary (6.1.29) [269]. If VM is normal and X,Y,Z,W are horizontal vector fields then
$$Rm^s(X,Y,Z,W) \;=\; Rm^s(Z,W,X,Y)$$

If V M is also integrable, then this can be relaxed to any three of X,Y,Z,W horizontal.

Proof. A straightforward computation shows that

$$2Rm^s(C,A,B,D) - 2Rm^s(B,D,C,A) = \ell\langle\ell R(A,B)C,D\rangle$$

The result then follows from the horizontal algebraic Bianchi Identity,

Lemma (6.1.30) [269]. (Differential Horizontal Bianchi Identites). For any vector fields X,Y,Z,V

$$\ell(\nabla_W R(X,Y))Z = \ell(R(Tor(X,V),Y))Z$$

Furthermore, if VM is normal and integrable and $X,Y,Z,W,V \in HM$ then

$$\nabla Rm^s(X,Y,Z,W,V) + \nabla Rm^s(X,Y,W,V,Z) + \nabla Rm^s(X,Y,V,Z,W) = 0$$

Proof. Again, the first part is a standard result that can be derived as follows

$$\begin{aligned}
(\nabla_W R)(X,Y)Z &= \nabla_W R(X,Y)Z - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z \\
&\quad - R(X,Y)\nabla_W Z \\
&= [\nabla_V, R(X,Y)]Z - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z
\end{aligned}$$

Thus, recalling the Jacobi identity applies to operators, we see

$$\begin{aligned}
\ell\left((\nabla_W)R(X,Y)\right)Z &= \ell\left([\nabla_W, R(X,Y)]\right)Z \\
&\quad - \ell\left(R(\nabla_W X,Y)\right)Z - \ell\left(R(X,\nabla_W Y)\right)Z \\
&= \ell\left([\nabla_V,[\nabla_X,\nabla_Y]] - [\nabla_V,\nabla[X,Y]]\right)Z \\
&\quad - \ell(R(\nabla_W X,Y))Z + \ell(R(\nabla_X V,Y))Z \\
&= -\ell\left([\nabla_V,\nabla_{[X,Y]}]\right)Z + \ell(R([X,V] + Tor(X,V),Y))Z \\
&= -\ell\left([\nabla_V,\nabla_{[X,Y]}]\right)Z + \ell(R(Tor(X,V),Y))Z \\
&\quad + \ell\left([\nabla_{[X,Y]},\nabla_Y] - \nabla_{[X,V],Y]}\right)Z \\
&= -\ell\left([\nabla_V,\nabla_{[X,Y]}]\right)Z + \ell(R(Tor(X,V),Y))Z \\
&\quad + \ell\left([\nabla_{[X,Y]},\nabla_V]Z\right) \\
&= \ell(R(Tor(X,V),Y))Z
\end{aligned}$$

To see the second part, we note that as VM is normal Corollary (6.1.29) implies that the required identity is equivalent to

$$\nabla Rm^s(Z,W,X,Y,V) + \nabla Rm^s(W,V,X,Y,W,Z) + \nabla Rm^s(V,Z,X,Y\,W) = 0$$

Choose X,Y,Z,W,V to be elements of a normal from for $HM = V^{(0)}$ at p, then

$$\begin{aligned}
\nabla Rm^s(Z,W,X,Y,V) &+ \nabla Rm^s(W,V,X,Y,W,Z) + \nabla Rm^s(V,Z,X,Y\,W) \\
&= \langle\ell\left((\nabla_V)R(Z,W)\right)X,Y\rangle \\
&= \langle\ell\left(R(Tor(Z,V),W)\right)X,Y\rangle
\end{aligned}$$

But by Corollary (6.1.29)

$$Rm^s(Tor(Z,V),W,X,Y) = Rm^s(X,Y,Tor(Z,V),W) = 0$$

as $Tor(Z,W)$ is vertical.

Definition (6.1.31) [269]. We define the subRiemannian Ricci curvature of ∇ by

$$Rc^s(A,B) = \sum_k Rm^s(A,X_k,X_k,B)$$

where $\{X_k\}$ is any horizontal orthonormal frame. The horizontal scalar curvature is defined by

$$S_0 = tr_{(0)}Rc^s = Rc^s(X_k,X_k)$$

It should be noted that the scalar curvature is independent of the choice of extension g as is the Ricci curvature restricted to horizontal vector fields.

It should be remarked here, that in general the Ricci curvature for the canonical connection is not symmetric. However, using Corollary (6.1.29) and elementary properties of the connection, we can immediately deduce

Lemma (6.1.32) [269]. If VM is normal and $X,Y \in HM$ then

$$Rc^s(X,Y) = Rc^s(Y,X)$$

If VM is normal and integrable then

$$Rc^s(VM,HM) = 0$$

Proof. The first follows from the corollary to the horizontal Bianchi Identity. For the second, we apply the corollary to the horizontal Bianchi Identity, to see that

$$Rc^s(U,X) = Rm^s(E_k,U,X,E_k) = Rm^s(X,E_k,E_k,U) = 0.$$

Lemma (6.1.33) [269]. (Contracted Bianchi Identity). Suppose VM is normal and integrable, then for any horizontal X

$$\nabla_X S_0 = 2\sum (\nabla Rc^s)(E_j,X,E_j)$$

where E_i is an orthonormal frame for HM. Equivalently

$$\nabla_0 S_0 = 2tr_{(0)}(\nabla Rc^s)$$

Proof. Let X be any element of a normal frame at p. Apply the differential Bianchi Identity to E_i, E_j, E_j, E_i, X and sum over i and j.

As a quick and easy consequence of this identity, we get a subRiemannian version of a result of Schur, that whenever the Ricci tensor is conformally equivalent to the metric then the manifold is Einstein.

Corollary (6.1.34) [269]. Suppose that M is a connected sRC-manifold such that HM bracket generates, $dim(HM) = d > 2$ and that VM is normal and integrable. If

$$Rc^s(X,Y) = \lambda\langle X,Y\rangle$$

for horizontal all vectors X,Y then λ must be constant.

Proof. Let E_i be a normal frame at $p \in M$. Then at p,

$$S_0 = Rc^s(E_i,E_i) = \lambda d$$

But

$$2tr_{(0)}(\nabla Rc^s)(E_j) = 2\nabla_i Rc^s(E_j,E_i) = 2E_j\lambda$$

Since $E_j S_0 = 2tr_{(0)}(\nabla Rc^s)(E_j)$, we must have $d = 2$ or $E_j\lambda = 0$. Thus all horizontal vector fields annihilate λ. As HM bracket generates, this implies that λ is constant

   One of our purposes is to use Bochner type results to study the relationships between curvature, geometry and topology on subRiemannian manifolds. To use this theory, we shall need a geometrically defined subelliptic Laplacian.

Definition (6.1.35) [269]. For a tensor τ, the horizontal gradient of τ is defined by

$$\nabla_0\tau = \nabla_{E_i}\tau \otimes E_i,$$

where E_i^* is the dual to E_i.

   The horizontal Hessian of τ is defined by

$$\nabla_0^2\tau(X,Y) = (\nabla_X\nabla_Y - \nabla_{\nabla_X Y})\tau$$

for $X,Y \in HM$ and zero otherwise.

The symmetric horizontal Hessian of τ is defined by

$$\nabla_0^{2,\text{sym}}\tau(X,Y) = \frac{1}{2}\left(\nabla_0^2\tau(X,Y) + \nabla_0^2\tau(X,Y)\right)$$

Finally, the horizontal Laplacian of τ is defined by

$$\Delta_{(0)}\tau = tr_{(0)}(\nabla_0^2\tau) = (\nabla_{E_i}\nabla_{E_i} - \nabla_{\nabla_{E_i}E_i})\tau$$

   The Laplacian on a Riemannian manifold has a rich and interesting L^2-theory. To replicate this for sRC-manifolds, it is necessary to choose a metric extension. This metric extension then yields a volume form and we have meaningful L^2- adjoints. Unfortunately, the horizontal Laplacian defined here, does not always behave as nicely as the Riemannian operator. However, if we make a mild assumption on the metric extension, much of the theory can be generalized.

Definition (6.1.36). For a metric extension of an r-grading we define a 1-form R_g by

$$R_g(v) = \sum_{j>0}\sum_i B^{(j)}(E_i^{(j)}, E_i^{(j)}, v_0)$$

where $E_i^{(j)}$ is an orthonormal frame for $V^{(j)}$.

   We say that a complement VM is vertically rigid if there exists a metric extension g such that

$$R_g \equiv 0.$$

Lemma (6.1.37) [269]. For an orientable sRC-manifold, the following are equivalent
 (i) VM is vertically rigid
 (ii) There exists a volume form dV on M such that for any horizontal vector field

$$div\, X = tr_0\nabla X = \langle\nabla_{e_i}X, e_i\rangle$$

where e_i is an orthonormal frame for HM.

(iii) Every metric extension g is vertically conformal to a metric $\tilde{g}$ with $R_{\tilde{g}} \equiv 0$. Furthermore, if HM bracket generates, then the volume form in (b) is unique up to constant multiplication.

Proof. To show that (a) implies (b), we first note that for the particular metric extension g with $R_g \equiv 0$, we have

$$\sum_{j>0} \sum_i \langle Tor(E_i^{(j)}, X, E_i^{(j)}) \rangle = 0.$$

Now we recall the standard result (see [263]) that since ∇ is metric compatible and HM is parallel, the divergence operator for the metric volume form g satisfies

$$\text{div}_g X = tr(\nabla + Tor)(X)$$
$$= tr_0 \nabla X + \sum_{j>0} \sum_i \langle Tor(E_i^{(j)}, X, E_i^{(j)}) \rangle$$
$$= tr_0 \nabla X - R_g(X)$$
$$= tr_0 \nabla X$$

Thus we can set $dV = dV_g$.

To show (ii) implies (iii), we consider metrics vertically conformal to an arbitrary extension g,

$$g\lambda = \begin{cases} g, & on \ HM \\ e^\lambda g, & on \ V \ M \end{cases}.$$

Now if $V_g = e^\mu dV$, then set $\lambda = -\dfrac{\mu}{dim(V \ M)}$ so $dV_{g\lambda} = dV$. Then for horizontal X

$$tr_0 \nabla X - R_{g\lambda}(X) = \text{div}_{g\lambda} X = \text{div} \ X = tr_0 \nabla X$$

so $R_{g\lambda} \equiv 0$.

Since (c) trivially implies (a), the equivalence portion of the proof is complete.

For the uniqueness portion, we note that if $\Omega = e^\lambda dV$ then for any horizontal X, we have

$$\text{div}_\Omega X = \text{div} \ X - X(\lambda).$$

If the two divergences agree on horizontal vector fields and HM bracket generates, this immediately implies that λ is a constant.

For an orientable, vertically rigid sRC-manifold, there is then a 1-dimensional family of volume forms for which $\text{div} \ X = tr_0 \nabla X$. We shall often refer to such a volume form as a rigid volume form. Vertical rigidity therefore gives us a canonical notion of integration on a sRC-manifold that does not depend on the choice of metric extension.

As an immediate consequence, we have

Lemma (6.1.38) [269]. Suppose that M is orientable and VM is vertically rigid. Then on functions,

$$\Delta_{(0)} = E_i^2 + div \ E_i = -\nabla_0^* \nabla_0$$

where the divergence and L^2 adjoint are taken with respect to a rigid volume form.

Thus on a vertically rigid sRC-manifold, the horizontal Laplacian behaves qualitatively in a similar fashion to the Riemannian Laplacian.

Theorem (6.1.39) [269]. If F is a closed vector field and F_k is the projection of F to $V^{(k)}$ then

$$\frac{1}{2} \Delta_{(0)} |F_j|^2 = Rc^s(F_j, F_0) + |\Delta_{(0)} F_j|^2$$
$$+ \sum_i \left(\langle E_i, \nabla^2 F_j(F_j, E_i) \rangle \right.$$
$$- 2\langle \nabla_{E_i} F, Tor(E_i, F_j) \rangle + \langle F, (\nabla Tor)(F_j, E_i, E_i) \rangle$$
$$\left. -\langle F, TOR_2(E_i, E_i, F_j) \rangle \right)$$

where $\{E_i\}$ is any orthonormal horizontal frame.

Before we prove this result, we introduce some terms and notation. Firstly, we define $J : TM \times TM \to TM$ by

$$\langle J(A, Z), B \rangle = \langle Tor(A, B), Z \rangle \tag{7}$$

Next we recall that a vector field F is closed if

$$A \to \langle F, A \rangle$$

is a closed 1-form. It is then easy to check that F is closed if and only if for all vector fields A, B

$$\langle \nabla_B F, A \rangle = \langle \nabla_A F, B \rangle - \langle J(B, F), A \rangle = \langle \nabla_A F, B \rangle + \langle J(A, F), B \rangle$$

Proof. Set $u = \frac{1}{2}|F_j|^2$, then

$$\langle \Delta_{(0)} u, Y \rangle = \langle \nabla_Y F_j, F_j \rangle = \langle \nabla_Y F, F_j \rangle$$
$$= \langle \nabla_{F_j} F_0, Y \rangle + \langle J(F_j, F), Y \rangle \tag{8}$$

So

$$\Delta_{(0)} u = \nabla_{F_j} F_0 + J(F_j, F)_0.$$

Next we need some preliminaries. Firstly, for horizontal X, Y

$$\nabla^2 u(X, Y) = X\langle Y, \nabla_u u \rangle - \langle \nabla_X Y, \nabla \rangle = \langle Y, \nabla_X \nabla_0 u \rangle$$
$$= \langle Y, \nabla_X \nabla_{(0)} u \rangle \tag{9}$$

Secondly,

$$\langle \nabla_X J(F_j, F), X \rangle = X\langle J(F_j, F), X \rangle - \langle J(F_j, F), \nabla_X X \rangle$$
$$= X\langle F, Tor(F_j, X) \rangle - \langle J(F_j, F), \nabla_X X \rangle$$
$$= \langle \nabla_X F, Tor(F_j, X) \rangle + \langle F, \nabla_X Tor(F_j, X) \rangle$$
$$- \langle F, Tor(F_j, \nabla_X X) \rangle$$

Now we can begin the main computation. For horizontal X

$$\nabla^2 u(X, Y) = \langle \nabla_X \nabla_0 u, X \rangle = \langle \nabla_X \nabla_{F_j} F_0, X \rangle + \langle \nabla_X J(F_j, F), X \rangle$$
$$= R(X, F_j, F_0, X) + \langle \nabla_{F_j} \nabla_X F_0, X \rangle + \langle \nabla_{[X, F_j]} F_0, X \rangle$$
$$+ \langle \nabla_X J(F_j, F), X \rangle$$

$$
\begin{aligned}
&= R(X, F_j, F_0, X) \ + \ \langle \nabla_{F_j} \nabla_X F_0, X \rangle \\
&\quad + \langle \nabla_{\nabla_X F_j - \nabla_{F_j} X - Tor(X,F_j)} F, X \rangle \ + \ \langle \nabla_X J(F_j, F), X \rangle \\
&= R(X, F_j, F_0, X) \ + \ \langle \nabla_{F_j} \nabla_X F_0 - \nabla_{\nabla_{F_j} X} F_0, X \rangle \\
&\quad + \langle \nabla_X F_j, \nabla_X F_j \rangle \ + \langle F, Tor(X, \nabla_X F_j) \rangle \\
&\quad - \langle \nabla_X F, Tor(X, F_j) \rangle \ - \ \langle F, Tor(X, Tor(X, F_j)) \rangle \\
&\quad + \langle \nabla_X J(F_j, F), X \rangle \\
&= R(X, F_j, F_0, X) \ + \ \langle X, \nabla^2 F_0(F_j, X) \rangle + \ |\nabla_X F_j|^2 \\
&\quad - 2\langle \nabla_X F, Tor(X, F_j) \rangle \ + \ \langle F, (\nabla Tor)(F_j, X, X) \rangle \\
&\quad - \langle F, TOR_2(X, X, F_j) \rangle
\end{aligned}
$$

Now we let X range over the frame E_i and take a sum.

To apply this theorem, we make the following observations

Lemma (6.1.40) [269]. If $F = \nabla f$ then
$$
\langle E_i, \nabla^2 F_0(F_j, E_i) \rangle = \nabla_{(j)} f, \nabla_{(j)} \Delta_{(0)} f
$$

Proof. By (9),
$$
\begin{aligned}
(\nabla_Z \nabla^2 f)(X, X) &= Z\langle X, \nabla_X F_0 \rangle \ - \ \langle \nabla_Z X, \nabla_X F_0 \rangle \ - \ \langle X, \nabla_{\nabla_Z X} F_0 \rangle \\
&= \langle X, \nabla_Z \nabla_X F_0 \rangle \ - \ \langle X, \nabla_{\nabla_Z X} F_0 \rangle \\
&= \langle X, \nabla^2 F_0(Z, X) \rangle
\end{aligned}
$$

so
$$
\begin{aligned}
\sum_i \left(E_i, \nabla^2 F_0(F_j, E_i) \right) &= \sum_i (\nabla_{F_j} \nabla^2 f)\,(E_i, E_i) \\
&= \sum_k \langle \nabla f, U_k^{(j)} \rangle \,(\nabla_{U_k^{(j)}} \nabla_{(0)} f \\
&\quad - \sum_{i,m} \langle \nabla_{U_k^{(j)}} E_i, E_m \rangle \,(\nabla^2 f(E_m, E_i) \ + \ \nabla^2 f(E_i, E_m))) \\
&= \nabla_{(j)} f, \nabla_{(j)} \Delta_{(0)} f
\end{aligned}
$$

as the latter term is skew-symetric in i and m.

Definition (6.1.41) [269]. The Baudoin-Garofalo tensor for an sRC-manifold with metric extension is the unique symmetric 2-tensor such that
$$
R(A, A) = Rc^s(A_0, A_0) \ + \ \langle A, tr_{(0)}(\nabla Tor)(A_0) \rangle
$$
$$
+ \frac{1}{4} \sum_{i,j} |\langle Tor(E_i, E_j), A \rangle|^2 \tag{10}
$$

Note that from standard polarization arguments this defines
$$
R(A, B) = \frac{1}{4} (R(A + B, A + B) - R(A - B, A - B))
$$

Corollary (6.2.42) [269]. If g is strictly normal with respect to the basic grading and VM is integrable then

$$\frac{1}{2}\Delta_{(0)}\left|\nabla_{(0)}f\right|^2 - \langle\nabla_{(0)}f,\nabla_{(0)}\Delta_{(0)}f\rangle =$$

$$R(\nabla f,\nabla f) + \left\|\nabla_{(0)}^{2,sym}f\right\|^2 - 2\langle\nabla_{E_i}\nabla_{(1)}f,Tor(E_i,\nabla_{(0)}f)\rangle$$

$$\frac{1}{2}\Delta_{(0)}\left|\nabla_{(1)}f\right|^2 - \langle\nabla_{(1)}f,\nabla_{(1)}\Delta_{(0)}f\rangle = \left\|\nabla_{(0)}\nabla_{(1)}f\right\|^2$$

Proof. Most of this result follows immediately from noticing that the strictly normal condition eliminates many of the torsion terms from Theorem (6.1.39) and then applying Lemma (6.1.40). The rest consists of analyzing the $\left\|\nabla_{(0)}^2 f\right\|^2$. First note

$$\nabla^2 f(E_i,E_j) = E_iE_jf - (\nabla_{E_i}E_j)f$$

$$= \frac{1}{2}\left(E_iE_jf - (\nabla_{E_i}E_j)f\right) + \frac{1}{2}\left(E_jE_if - (\nabla_{E_j}E_i)f\right)$$

$$+ \frac{1}{2}\left([E_i,E_j]f + \left(\nabla_{E_j}E_i\right)f - (\nabla_{E_i}E_j)f\right)$$

$$= \frac{1}{2}\nabla^2 f\left(E_i,E_j\right) + \nabla^2 f\left(E_j,E_i\right) - \frac{1}{2}Tor(E_i,E_j)f$$

From this we immediately obtain,

$$\left\|\nabla_{(0)}^2 f\right\|^2 = \left\|\nabla_{(0)}^{2,sym}f\right\|^2 + +\frac{1}{4}\sum_{i,j}|\langle Tor(E_i,E_j),\nabla f\rangle|^2$$

Definition (6.1.43) [269]. The torsion bounds of M are the defined by

$$\kappa_{ij}^m = \sup\left\{|Tor(X^{(i)},X^{(j)})_m|^2 : |X^{(i)}|,|X^{(j)}| \le 1\right\}$$

Noting that $0 \le \kappa_{ij}^m \le +\infty$.

To obtain topological and geometric information from this result, we follow the technique developed by Baudoin and Garofalo in [340]. We define symmetric bilinear forms by

$$\Gamma_{(j)}(f,g) = \langle\nabla_{(j)}f,\nabla_{(j)}g\rangle$$

$$\Gamma_{(j)}^2(f,g) = \Delta_{(0)}\Gamma_{(j)}(f,g) - \Gamma_{(j)}(\Delta_{(0)}f,g) - \Gamma_{(j)}(f,\Delta_{(0)}g)$$

If g is strictly normal then it is easy to check that

$$\Gamma_{(0)}\left(f,\Gamma_{(1)}(f,f)\right) = \Gamma_{(1)}\left(f,\Gamma_{(0)}(f,f)\right) \tag{11}$$

and we obtain the following result

Theorem (6.1.44) [269]. Suppose g is strictly normal for the basic grading and VM is integrable. If

$$\kappa_{00}^1 < \infty$$

and there exist constants $\rho_1 \in R$ and $\rho_2 > 0$ such that

$$R(A,A) \ge \rho_1\|A_0\|^2 + \rho_2\|A_1\|^2$$

then for $\kappa = dim(HM)\kappa_{00}^1$, the generalized curvature-dimension inequality

$$\Gamma_{(0)}^2 + \nu\Gamma_{(1)}^2 \geq \frac{1}{\dim HM}(\Delta_{(0)}f)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma_{(0)}(f,f) + \rho_2\Gamma_{(1)}(f,f)$$

holds for every $f \in C^\infty(M)$ and $\nu > 0$.

Proof. As

$$\left\|\nabla_{(0)}^{2,sym} f\right\|^2 \geq \sum_i (\nabla^2 f(E_i, E_i))^2 \geq \frac{1}{\dim HM}(\Delta_{(0)}f)^2$$

this follows immediately from Corollary (6.1.42) and the elementary identity that

$$2\sum_i \left|\nabla_{E_i}\nabla_{(1)}f , \mathrm{Tor}(E_i, \nabla_{(0)}f)\right| \leq \nu\left\|\nabla_{(0)}\nabla_{(1)}\right\|^2 f + \frac{\kappa}{\nu}\left\|\nabla_{(0)}f\right\|^2$$

It was shown in [340], that under the additional mild hypothesis that there exists a sequence $h_k \in C_c^\infty(M)$ of increasing functions that converge pointwise to 1 everywhere and satisfy

$$\left\|\Gamma_{(0)}(h_k, h_k)\right\|_\infty + \left\|\Gamma_{(1)}(h_k, h_k)\right\|_\infty \to 0,$$

the generalized curvature inequality has a wide variety of topological, geometric and analytical consequences. In our case, this hypothesis is automatically satisfied as $\Gamma_{(0)}(f,f) + \Gamma_{(1)}(f,f) = |\nabla f|^2$. We shall focus on their subRiemannian generalization of the Bonnet-Myers theorem.

Theorem (6.1.45) [269]. (Baudoin-Garofalo). If the generalized curvature inequality is satisfied with $\rho_1 > 0$ and the above hypothesis holds together with (11) then M is compact. Combining this with Theorem (6.1.44), provides the following generalization of the examples considered in [260]

Theorem (6.1.46) [269]. Under the same conditions as Theorem (6.1.44), if $\rho_1 > 0$ then M is compact.

A common theme in the early development of sub Riemannian geometry was the use of Riemannian approximations. More precisely, a Riemannian extension $g = g_0 \oplus g_1$ was chosen and then re-scaled as $g^\lambda = \lambda \oplus \lambda^2 g_1$. The behavior of these Riemannian metrics was then studied as $\lambda \to \infty$. The idea is that blowing up the vertical directions makes movement in these direction prohibitively expensive so the Riemannian geodesics should converge to the subRiemannian geodesics. Unfortunately, this is problematic for the study of the effects of curvature as this re-scaling makes the vertical curvatures much larger than the horizontal ones. However, useful information can be derived from this approach if instead we let $\lambda \to 0$.

We compute the Ricci and sectional curvatures of these scaled Riemannian metrics in terms of the basic connection. For simplicity, we shall specialize to the case where dim $VM = 1$ and so the only basic grading applies. We shall be able to provide alternative

proofs to some of the results above and see the nature of the obstructions when the conditions are weakened.

To proceed, we fix a sRC-manifold M and choose a Riemannian extension $g = g_0 \oplus g_1$. The basic connection will always be in terms of this metric. Throughout, $E_1, \ldots, E_d$ will represent an orthonormal frame for HM with respect to g and U will represent a unit length vector in VM, again with respect to g.

We refine the J operator introduced earlier by defining

$$\langle J^1(A, B), C \rangle = \langle Tor(A, C), B_1 \rangle,$$
$$\langle J^0(A, B), C \rangle = \langle Tor(A, C), B_0 \rangle \tag{12}$$

Lemma (6.1.47) [269]. For any sRC-manifold (with no restriction on dimVM) the Levi-Civita connection associated to g can be computed from the basic connection for g as follows

$$\bar{\nabla}_X Y = \nabla_X Y - \frac{1}{2} Tor(X, Y) + J^1(X, Y)$$
$$\bar{\nabla}_T T = \nabla_T T - \frac{1}{2} J^0(T, T)$$
$$\bar{\nabla}_T X = \nabla_T X + \frac{1}{2} J^0(X, T) - Tor(T, X)_1 \tag{13}$$
$$\bar{\nabla}_X T = \nabla_X T + \frac{1}{2} J^0(X, T) - Tor(X, T)_0$$

From these it is a straightforward, if brutal, computation to show that

Corollary (6.1.48) [269]. If X, Y are horizontal vector fields and T is a vertical vector field then

$$\overline{Rm}(X, Y, Y, X) = Rm^s(X, Y, Y, X) - \frac{3}{4} |Tor(X, Y)|^2$$
$$- \langle J^1(Y, Y), J^1(X, X) \rangle + |J^1(X, Y)|^2 \tag{14}$$
$$\overline{Rm}(T, X, X, T) = Rm^s(T, X, X, T) + \frac{1}{4} |J^0(X, T)|^2$$
$$+ \langle \nabla Tor(T, X, X) - Tor(X, Tor(X, T)), T \rangle$$
$$+ \langle \nabla Tor(X, T, T), X \rangle - |Tor(X, T)_0|^2 \tag{15}$$
$$\overline{Rm}(X, Y, T, X) = Rm^s(X, Y, T, X) + \frac{1}{2} \langle \nabla Tor(Y, X, X), T \rangle$$
$$+ \langle \nabla Tor(X, T, Y) - \nabla Tor(Y, T, X), X \rangle \tag{16}$$

While this is far from a complete list of curvature terms, if we use properties of both Riemannian and subRiemannian curvatures and polarization identities, it is suffcient to compute all sectional and Ricci curvatures for the case dim $VM = 1$.

Provided that we only use constants for our re-scaling, it is easy to verify that the covariant derivatives for the basic connection associated to the re-scaled metric are unchanged from the base metric. Thus, paying careful attention to how each term scales, we can compute the Riemannian Ricci curvatures for the metrics $g^\lambda = \lambda \oplus \lambda^2 g_1$.

For $Y \in HM$ and $T \in VM$, with inner products and norms computed in the unscaled metric

$$\overline{Rc}^\lambda (Y,Y) = \lambda^0 [Rc(Y,Y) + \langle \nabla Tor(U,Y,Y), U \rangle - \langle TOR_2(Y,Y,U), U \rangle]$$

$$+ \lambda^2 \left[-\frac{1}{2} \sum_i |Tor(E_i, Y)|^2 \right]$$

$$+ \lambda^{-2} [\langle \nabla Tor(Y,U,U), Y \rangle - |Tor(Y,U)_0|^2 \tag{17}$$

$$+ \sum_i (|J^1(E_i,Y)|^2 - \langle J^1(E_i, E_i), J1(Y,Y) \rangle)]$$

$$\overline{Rc}^\lambda (Y,T) = \lambda^0 \left[\sum_i \langle \nabla Tor(E_i, T, Y) - \nabla Tor(Y,T,E_i), E_i \rangle \right]$$

$$+ \lambda^2 \left[\frac{1}{2} \langle tr_0 \nabla Tor(Y), T \rangle \right] \tag{18}$$

$$\overline{Rc}^\lambda(T,T) = \lambda^0 \left[\sum_i \langle \nabla Tor(E_i, T, T), E_i \rangle - |Tor(E_i, T)_0|^2 \right]$$

$$+ \lambda^2 \left[\sum_i \langle \nabla Tor(T, E_i, E_i) - TOR_2(E_i, E_i, T), T \rangle \right]$$

$$+ \lambda^4 \frac{1}{4} |J^0(E_i, T)|^2 \tag{19}$$

For the case of a strictly normal sRC-manifold, these formulae greatly simplify to

$$\overline{Rc}^\lambda(Y,Y) = \lambda^0 Rc(Y,Y) - \frac{\lambda^2}{2} \sum_i |Tor(E_i, Y)|^2$$

$$\overline{Rc}^\lambda(Y,T) = \frac{\lambda^2}{2} \langle tr_0 \nabla Tor(Y), T \rangle \tag{20}$$

$$\overline{Rc}^\lambda(T,T) = \frac{\lambda^4}{4} \sum_i |J^0(E_i, T)|^2 = \frac{\lambda^4}{4} \sum_{i,j} |Tor(E_i, E_j)|^2$$

and so

$$R(T + Y, T + Y) = \lim_{\lambda \to 0} \overline{Rc}^\lambda (Y + \lambda^{-2} T, Y + \lambda^{-2} T) \tag{21}$$

Next we note that if T is unit length with respect to the base metric then for any smooth function

$$\nabla^\lambda f = \nabla_0 f + \lambda^{-2}(T f) T$$

which means that the Baudoin-Garofalo tensor applied to ∇f can expressed as a limit of Riemannian Ricci curvatures as follows:

$$R(\nabla f, \nabla f) = \lim_{\lambda \to 0} \overline{Rc}^\lambda (\nabla^\lambda f, \nabla^\lambda f)$$

Theorem (6.1.49) [269]. Under the same conditions as Theorem (6.1.4) and the added assumption that $\dim VM = 1$, there are constants $\lambda, c > 0$ such that

$$\overline{Rc}^\lambda(A, A) \geq cg^\lambda(A, A)$$

for all vectors A.

Proof. Split $A = A_0 + A_1$ and then note that

$$\overline{Rc}^\lambda(A, A) = R(A_0 + \lambda^2 A_1, A_0 + \lambda^2 A_1) - \frac{\lambda^2}{2}\sum_i |Tor(E_i, A_0)|^2$$

$$\geq \left(\rho_1 - \frac{\lambda^2 \kappa}{2}\right) g^\lambda(A_0, A_0) + \rho_2 \lambda^2 g^\lambda(A_1, A_1)$$

Thus for very small $\lambda > 0$, we can take $c = \min\left\{\lambda^2 \rho_2, \rho_1 - \frac{\lambda^2 \kappa}{2}\right\} > 0$.

Combining this with the classical Myers theorem yields

Corollary (6.1.50) [269]. Under the same conditions as Theorem (6.1.44) and the added assumption that $\dim VM = 1$, M is compact and has finite fundamental group.

If we do not restrict to the strictly normal case, then this Riemannian approach immediately has problems. If we send $\lambda \to \infty$, we see that $\overline{Rc}^\lambda(Y, Y) \to -\infty$, so any Riemannian results for positive Ricci curvature will immediately be lost. Since there are very few topological consequences of negative Ricci curvature, this approach is unlikely to bear fruit. If however we let $\lambda \to 0$, then we run into the issue that the subRiemannian Ricci curvature for the horizontal terms isn't the dominant term. Instead we must deal with the symmetric 2-tensors

$$B(X, Y) = \langle \nabla Tor(X, U, U), Y \rangle - Tor(X, U)_0, Tor(Y, U)_0$$

$$K(X, Y) = \sum_i (\langle J^1(E_i, X), J^1(E_i, Y)\rangle - \langle J^1(E_i, E_i), J^1(X, Y)\rangle) \quad (23)$$

where again U is a unit length vertical vector. The tensor B is a genuine sRC- invariant when $\dim VM = 1$, but has no good invariant generalization when $\dim VM = 1$. However K is only a vertically conformal sRC-invariant. With these caveats in mind, we do however obtain the following theorem

Theorem (6.1.51) [269]. Let M be an sRC-manifold with $\dim VM = 1$ and bounded curvature and torsion. If there are constants $a, b > 0$ such that for all horizontal vectors Y,

$$tr_0 B \geq a$$

$$B(Y, Y) + K(Y, Y) \geq b|Y|^2 \quad (24)$$

then M is compact and has finite fundamental group.

Proof. The condition of bounded curvature implies that for small λ there will be some, possibly large, constant M such that

$$2\overline{Rc}^{\lambda}(T,Y) \leq 2M \, |T||Y| \leq \frac{a}{4}|T|^2 + \frac{4M^2}{a}|Y|^2 \, .$$

Since $tr_0 B \geq a$ globally, for suffciently small λ, we will have

$$\overline{Rc}^{\lambda}(T,T) \geq \frac{a}{2}|T|^2$$

and since $K(Y,Y) + B(Y,Y) \geq b|Y|^2$, again for small λ, we have

$$\overline{Rc}^{\lambda}(Y,Y) \geq \frac{b}{2\lambda^2}|Y|^2$$

But then for small enough λ

$$\overline{Rc}^{\lambda}(T + Y, T + Y) \geq \left(\frac{b}{2\lambda^2} - \frac{4M^2}{a}\right)|Y|^2 + \frac{a}{4}|T|^2$$

For very small λ, both coeffcients will be postive, so

$$\overline{Rc}^{\lambda}(T + Y, T + Y) \geq c|T + Y|^2$$

for some positive constant c. The result then follows from the classical Myers theorem.

This is a purely sub Riemmanian result as the conditions are trivially false when restricted to Riemannian manifolds. However, it is somewhat unsatisfactory in nature. It would seem reasonable to conjecture that for sRC-manifolds (or at least those that are in some sense nearly strictly normal) that there would be some sort of analogue of Theorem (6.1.46) where the dominant terms are genuine subRiemannian Ricci tensors. However, it appears that to prove it will be necessary to create new subRiemannian techniques such as the heat kernel methods of [340] rather than fall back on existing Riemannian methods. we expect the basic connection developed to provide a solid computational foundation for such techniques.

Section(6-2): Comparison Theorems for Sub-Riemannian Manifolds

We study volume and distance comparison estimates on sub-Riemannian manifolds that satisfy the generalized curvature dimension inequality introduced in [270]. We in particular prove a global doubling property in the possibly negative curvature case which complements the volume estimates obtained in [269], where the curvature was always supposed to be non negative. The distance estimates we obtain, and the methods to prove them are new, but in the non negatively curved Sasakian case that was treated in [268]. As a consequence of the global doubling property, we obtain a Gromov type precompactness theorem for the class of sub-Riemannian manifolds that satisfy the generalized curvature dimension inequality and, as a consequence of the distance comparison theorem, we obtain Fefferman-Phong type subelliptic estimates.

To put the results we obtain in perspective, let us point out that distance and volume estimates in sub-Riemannian geometry have been extensively studied. But most of the obtained results are of local nature. Let (M,g) be a smooth and connected Riemannian

manifold. Let us assume that there exists on M a family of vector fields $\{X_1, \cdots X_d\}$ that satisfy the bracket generating condition. We are interested in the sub-Riemannian structure on M which is given by the vector fields $\{X_1, \cdots X_d\}$. In sub-Riemannian geometry the Riemannian distance d_R of M is most of the times confined to the background. There is another distance on M, that was introduced by Caratheodory in his seminal paper [275], which plays a central role. A piecewise C^1 curve $\gamma : [0, T] \rightarrow M$ is called subunitary at x if for every $\xi \in T_x^* M$ one has

$$g(\gamma'(t), \xi)^2 \leq \sum_{i=1}^{d} g(X_i(\gamma(t)), \xi)^2.$$

We define the subunit length of γ as $\ell_s(\gamma) = T$. If we indicate with $S(x, y)$ the family of subunit curves such that $\gamma(0) = x$ and $\gamma(T) = y$, the fundamental accessibility theorem of Chow-Rashevsky the connectedness of M implies that $S(x, y) \neq \emptyset$ for every $x, y \in M$, see [276], [285]. This allows to define the sub-Riemannian distance on M as follows
$$d(x, y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x, y)\}.$$
We refer to the cited contribution of Gromov to [272], by Bellaiche in the same volume. Another elementary consequence of the Chow-Rashevsky theorem is that the identity map $i : (M, d) \rightarrow (M, d_R)$ is continuous and thus, the topologies of d_R and d coincide. Several fundamental properties of the metric d have been discussed by Nagel, Stein and Wainger [283]. The following local distance comparison theorem was proved in [283].

Theorem (6.2.1)[290]. (Nagel-Stein-Wainger, [283]). For any connected set $\Omega \subset M$ which is bounded in the distance d_R there exist $K = K(\Omega) > 0$, and $\epsilon = \epsilon(\Omega) > 0$, such that
$$d(x, y) \leq C d_R(x, y)^\epsilon, \quad x, y \in \Omega.$$
The following result also proved in [283] provides a uniform local control of the growth of the metric balls in (M, d).

Theorem (6.2.2) [290]. (Nagel-Stein-Wainger, [283]). For any $x \in M$ there exist constants $C(x), R(x) > 0$ such that with $Q(x) = \log_2 C(x)$ one has
$$\mu(B(x, tr)) \geq C(x)^{-1} t^{Q(x)} \mu(B(x, r)), \quad 0 \leq t \leq 1, 0 < r \leq R(x).$$
Given any compact set $K \subset M$ one has
$$\inf_{x \in K} C(x) > 0, \quad \inf_{x \in K} R(x) > 0.$$
These theorems and the methods used to prove them are local in nature. The goal is to obtain global analogues for a large class of sub-Riemannian manifolds.

To fix the ideas, let us present the main results of Sasakian manifolds but we stress that the class of sub-Riemannian structures to which our results apply is much larger than the class of Sasakian manifolds.

Let M be a complete strictly pseudo convex CR manifold with real dimension $2n + 1$. Let θ be a pseudo-Hermitian form on M with respect to which the Levi form L_θ is positive

definite and thus defines a Riemannian metric g on M (the Webster metric). The kernel of θ defines a horizontal bundle $\mathcal{H}$. The triple $(M, \mathcal{H}, g)$ is a sub-Riemannian manifold. The CR structure on M is said to be Sasakian if the Reeb vector field of θ is a sub-Riemannian Killing vector field. Let us denote by Ric_∇ the Ricci curvature tensor of the Tanaka-Webster connection on M. We prove the following global version of Theorems (6.2.1) and (6.2.2)

Theorem (6.2.3) [290]. Let M be a complete Sasakian manifold. Let us assume that there exists $K \in R$, such that for every $V \in H$,

$$Ric_\nabla(V, V) \geq -K\|V\|^2,$$

then:

(i) Distance comparison theorem) There exists a constant $C = C(n, K) > 0$, such that for every $x, y \in M$,

$$d(x, y) \leq C \, max\{d_R(x, y), \sqrt{d_R(x, y)}\}.$$

(ii) Uniform local volume doubling property) For every $R > 0$, there exists a constant $C = C(R, n, K) > 0$ such that for any $x \in M$, with $Q = \log_2 C$ one has

$$\mu(B(x, tr)) \geq C^{-1} t^Q \mu(B(x, r)), \quad 0 \leq t \leq 1, 0 < r \leq R.$$

The dependency of the constant C on R in the volume estimate is described more precisely in Theorem (6.2.23).

The method we use to approach these types of results are heat equation techniques and sharp Gaussian bounds for the heat kernel relying on the methods developed in [269] and [270]. We find it convenient to work in the context of a local Dirichlet space associated to a subelliptic diffusion operator. This abstract presentation has the advantage to encompass in the same framework many relevant examples of different nature.

We introduce the framework of [270] and recall the generalized curvature dimension inequality that is going to be our main device. We study sharp Harnack inequalities for solutions of the sub-Riemannian heat equations. The main novelty here with respect to [270] and [269] is that these Harnack inequalities involve a family of distances that interpolate between the sub-Riemannian distance and the Riemannian one. We devoted to the proof of the uniform volume doubling property. We skip most of the details in some of the proofs since the methods are close to the methods of [269]. However, due to the more general setting, several computations are more involved. We establish through sharp upper Gaussian bounds for the heat kernel the distance comparison theorem. And shows how the distance comparison theorem is used to prove subelliptic estimates. The fact that the sub-Riemannian distance behaves as $\sqrt{d_R(x, y)}$ for close x, y implies that the operator L is subelliptic of order 1/2. Finally we establishes a sub-Riemannian Gromov type precompactness theorem which is obtained as a consequence of our volume estimates.

We consider a measure metric space (M, d, μ), where M is C^∞ connected manifold endowed with a C^∞ measure μ, and d is a metric canonically associated with a C^∞ second-order diffusion operator L on M with real coefficients. We assume that L is locally subelliptic on M in the sense of [277], and that moreover:

(i) $L1 = 0$;

(ii) $\int_M f\, Lg\, d\mu = \int_M g\, Lf\, d\mu$;

(iii) $\int_M f\, Lf\, d\mu \le 0$,

for every $f, g \in C_0^\infty(M)$. The quadratic functional $\Gamma(f) = \Gamma(f, f)$, where

$$\Gamma(f, g) = \frac{1}{2}\,(L(fg) - fLg - gLf),\, f, g \in C^\infty(M), \tag{25}$$

is known as le carr´e du champ. Notice that $\Gamma(f) \ge 0$ and that $\Gamma(1) = 0$.

An absolutely continuous curve $\gamma : [0, T] \to M$ is said to be subunit for the operator Γ if for every smooth function $f : M \to R$ we have $|\frac{d}{dt} f(\gamma(t))| \le \sqrt{(\Gamma f)(\gamma(t))}$. We then define the subunit length of γ as $\ell_s(\gamma) = T$. Given $x, y \in M$, we indicate with

$S(x, y) = \{\gamma : [0, T] \to M \,|\, \gamma$ is subunit for $\Gamma, \gamma(0) = x, \quad \gamma(T) = y\}$.

We assume that

$$S(x, y) \ne \emptyset, \qquad \text{for every} \quad x, y \in M.$$

Under such assumption it is easy to verify that

$$d(x, y) = \inf\{\ell_s(\gamma) \,|\, \gamma \in S(x, y)\}, \tag{26}$$

defines a true distance on M. Furthermore, it is known that

$$d(x, y) = \sup\{|f(x) - f(y)| \,|\, f \in C^\infty(M), \|\Gamma(f)\|_\infty \le 1\}, x, y \in M. \tag{27}$$

Throughout we assume that the metric space (M, d) is complete.

In addition to Γ, we assume that there exists another first-order bilinear form Γ^Z satisfying for $f, g, h \in C^\infty((M)$:

(i) $\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h)$;

(ii) $\Gamma^Z(f) = \Gamma^Z(f, f) \ge 0$.

We introduce the following second-order differential forms:

$$\Gamma_2(f, g) = \frac{1}{2}[L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)],$$

$$\Gamma_2^Z(f, g) = \frac{1}{2}L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)$$

and we let $\Gamma_2(f) = \Gamma_2(f, f)$, $\Gamma_2^Z(f) = \Gamma_2^Z(f, f)$.

We also introduce a family of control distances d_τ for $\tau \ge 0$. Given $x, y \in M$, let us consider

$S_\tau(x, y) = \{\gamma : [0, T] \to M \,|\, \gamma$ is subunit for $\Gamma + \tau^2\Gamma^Z, \gamma(0) = x, \gamma(T) = y\}$.

A curve which is subunit for Γ is obviously subunit for $\Gamma + \tau^2\Gamma^Z$, therefore $S_\tau(x, y) \ne \emptyset$. We can then define

$$d_\tau(x, y) = \inf\{\ell_s(\gamma) \mid \gamma \in S_\tau(x, y)\}. \tag{28}$$

Note that $d(x, y) = d_0(x, y)$ and that, clearly: $d_\tau(x, y) \le d(x, y)$.

The following definition was introduced in [270].

Definition (6.2.4) [290]. We shall say that M satisfies the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ if there exist constants $\rho_1 \in R, \rho_2 > 0, \kappa \ge 0$, and $d > 0$ such that the inequality

$$\Gamma_2(f) + \nu\Gamma_2^Z(f) \ge \frac{1}{d}(Lf)^2 + \left(\rho_1 - \frac{\kappa}{\nu}\right)\Gamma(f) + \rho_2\Gamma^Z(f) \tag{29}$$

holds for every $f \in C^\infty(M)$ and every $\nu > 0$.

Let us observe right-away that if $\rho'_1 \ge \rho_1$, then $CD(\rho'_1, \rho_2, \kappa, d) \Rightarrow CD(\rho_1, \rho_2, \kappa, d)$. To provide with some perspective on Definition (6.2.4) we refer to [270] but point out that it constitutes a generalization of the so-called curvature-dimension inequality $CD(\rho_1, n)$ from Riemannian geometry. We recall that the latter is said to hold on a n-dimensional Riemannian manifold M with Laplacian Δ if there exists $\rho_1 \in R$ such that for every $f \in C^\infty(M)$ one has

$$\Gamma_2(f) \ge \frac{1}{n}(\Delta f)^2 + \rho_1|\nabla f|^2 \tag{30}$$

where

$$\Gamma_2(f) = \frac{1}{2}(\Delta|\nabla f|^2 - 2\langle\nabla f, \nabla(\Delta f)\rangle).$$

To see that (29) contains (30) it is enough to take $L = \Delta, \Gamma^Z = 0, \kappa = 0$, and $d = n$, and notice that (25) gives $\Gamma(f) = |\nabla f|^2$ (also note that in this context the distance (27) is simply the Riemannian distance on M). It is worth emphasizing at this moment that, remarkably, on a complete Riemannian manifold the inequality (30) is equivalent to the lower bound $Ric \ge \rho_1$.

The essential new aspect of the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with respect to the Riemannian inequality $CD(\rho_1, n)$ in (30) is the presence of the a priori non-intrinsic forms Γ^Z and Γ_2^Z. In the non-Riemannian framework the form Γ plays the role of the square of the length of a gradient along the (horizontal) directions canonically associated with the operator L, whereas the form Γ^Z should be thought of as the square of the length of a gradient in the missing (vertical) directions.

In Definition (6.2.4) the parameter ρ_1 plays a special role. For the results such parameter represents the lower bound on a sub-Riemannian generalization of the Ricci tensor. The case when $\rho_1 \ge 0$ is, in our framework, the counterpart of the Riemannian $Ric \ge 0$. For this reason, when we say that M satisfies the curvature dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ with $\rho_1 \ge 0$, we will routinely avoid repeating at each occurrence the sentence "for some $\rho_2 > 0, \kappa \ge 0$ and $d > 0$".

Before stating our main result we need to introduce further technical assumptions on the forms Γ and Γ^Z:

Hypothesis (6.2.5) [290]. There exists an increasing sequence $h_k \in C_0^\infty(M)$ such that $h_k \nearrow 1$ on M, and

$$\left\|\Gamma(h_k)\right\|_\infty + \left\|\Gamma^Z(h_k)\right\|_\infty \to 0, \qquad as \ k \to \infty.$$

Hypothesis (6.2.6) [290]. For any $f \in C^\infty(M)$ one has

$$\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).$$

Hypothesis (6.2.7) [290]. The heat semigroup generated by L, which will be denoted by P_t throughout the section, is stochastically complete that is, for $t \geq 0, P_t 1 = 1$, and for every $f \in C_0^\infty(M)$ and $T \geq 0$, one has

$$\sup_{t \in [0,T]} \|\Gamma(P_t f)\|_\infty \ \|\Gamma^Z(P_t f)\|_\infty < +\infty.$$

The hypothesis (6.2.5) and hypothesis (6.2.6) will be in force throughout. Let us notice explicitly that when M is a complete Riemannian manifold with $L = \Delta$, then hypothesis (6.2.5) and hypothesis (6.2.6) are fulfilled. In fact, hypothesis (6.2.6) is trivially satisfied since we can take $\Gamma^Z \equiv 0$, whereas hypothesis (6.2.5) follows from (and it is in fact equivalent to) the completeness of (M, d). Actually, in the geometric examples encompassed by the framework (as we have said before, for a detailed discussion of these examples should consult the preceding section [270]), hypothesis (6.3.5) is equivalent to assuming that (M, d) be a complete metric space. The reason is that in those examples $\Gamma + \Gamma^Z$ is the carr´e du champ of the Laplace-Beltrami of a Riemannian structure whose completeness is equivalent to the completeness of (M, d). The hypothesis (6.2.7) has been shown in [270] to be a consequence of the curvature-dimension inequality CD(ρ_1, ρ_2,κ,d) only in many examples. We mention the following results from [270].

Theorem (6.2.8) [290]. Let (M, θ) be a complete CR manifold with real dimension $2n + 1$ and vanishing Tanaka-Webster torsion, i.e., a Sasakian manifold. If for every $x \in M$ the Tanaka-Webster Ricci tensor satisfies the bound

$$Ric_x(v, v) \geq \rho_1 |v|^2,$$

for every horizontal vector $v \in H_x$, then, for the CR sub-Laplacian of M the curvature-dimension inequality $CD(\rho_1, d \ 4, 1, d)$ holds, with $d = 2n$. Furthermore, the Hypothesis (6.2.5), (6.2.6) and (6.2.7) are satisfied.

Theorem (6.2.9) [290]. Let G be a Carnot group of step two, with d being the dimension of the horizontal layer of its Lie algebra. Then, G satisfies the generalized curvature-dimension inequality $CD(0, \rho_2, \kappa, d)$ (with respect to any sub-Laplacian L on G), where ρ_2 and κ are appropriately (and explicitly) determined in terms of the group constants. Moreover, the Hypothesis (6.2.5), (6.2.6) and (6.2.7) are satisfied.

Theorem (6.2.9) says, in particular, that in our framework, every Carnot group of step two is a sub- Riemannian manifold with nonnegative Ricci curvature. CR Sasakian manifolds and Carnot groups of step two are included in, but do not exhaust, the class of sub-Riemannian manifolds with transverse symmetries of Yang-Mills type. Such wide class was extensively analyzed, and we refer to that source for the relevant notions. In view of Theorems (6.2.8) and Theorem (6.2.9) it should be clear that our approach allows for the first time to extend the Li-Yau program, and many of its fundamental consequences, to situations which are genuinely non-Riemannian. As a further comment, we mention that, if we assume that the generalized curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$ is satisfied, then the assumption (6.2.7) should not be seen as restrictive. As we mentioned above, it was shown in [270] that (6.2.7) is fulfilled for all sub- Riemannian manifolds with transverse symmetries of Yang-Mills type.

Theorem (6.2.10) [290]. Suppose that the generalized curvature-dimension inequality hold for some $\rho_1 \in R$. Then, there exist constants $C_1, C_2 > 0$, depending only on $\rho_1, \rho_2, \kappa, d$, for which one has for every $x, y \in M$ and every $r > 0$:

$$\mu\big(B(x, 2r)\big) \leq C_1 \exp(C_2 r^2) \, \mu\big(B(x, r)\big); \tag{31}$$

Theorem (6.2.11) [290]. Suppose that the generalized curvature-dimension inequality hold for some $\rho_1 \in R$. Let $\tau \geq 0$. Then, there exists a constant $C(\tau) > 0$, depending only on $\rho_1, \rho_2, \kappa, d$ and τ for which one has for every $x, y \in M$:

$$d(x, y) \leq C(\tau) \max\{\sqrt{d_\tau(x, y)}, d_\tau(x, y)\}. \tag{32}$$

In the sequel, we assume that besides the assumptions specified in the previous section the generalized curvature dimension of Definition (6.2.4) is satisfied for some parameters $\rho_1, \rho_2, \kappa, d$. We will denote

$$D = d\left(1 + \frac{3\kappa}{2\rho_2}\right). \tag{33}$$

The main tool to prove the fore mentioned theorems, is the heat semigroup $P_t = e^{tL}$, which is defined using the spectral theorem. Thanks to the hypoellipticity of L, for $f \in L^P(M), 1 \leq p \leq \infty$, the function $(t, x) \rightarrow P_t f(x)$ is smooth on $M \times (0, \infty)$ and

$$P_t f(x) = \int_M p(x, y, t) f(y) d\mu(y)$$

where $p(x, y, t) = p(y, x, t) > 0$ is the so-called heat kernel associated to P_t. It was proved in [270] that the generalized curvature dimension inequality implies a Li-Yau type estimate for the heat semigroup. Let $f \geq 0$, be a non zero smooth and compactly supported function then the following inequality holds for $t > 0$:

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{3} t \Gamma^z(\ln P_t f) \leq \left(\frac{D}{d} + \frac{2\bar\rho_1}{3} t\right) \frac{L P_t f}{P_t f} + \frac{d(\bar\rho_1)^2}{6} t + \frac{\bar\rho_1 D}{2} + \frac{D^2}{2dt}, \tag{34}$$

where $\bar{\rho}_1 = max(-\rho_1, 0)$. A consequence of the Li-Yau inequality is the parabolic Harnack inequality for the heat semigroup as it was established in [270]. The distance used in [270] to control the heat kernel is the sub-Riemannian distance d. Purpose is to take advantage of the upper bound on Γ^Z ($\ln P_t f$) that is also provided by the Li-Yau inequality in order to deduce a control of the heat kernel by a family of Riemannian distances. This control of the heat kernel in terms of Riemannian distances is the key point to prove the distance comparison theorem.

As a first step, we observe that as a straightforward consequence of (34) we obtain that for every $\tau \geq 0$ and $t > 0$,

$$\left(1 + \frac{3\tau^2}{2\rho_2 t}\right)^{-1} (\Gamma(\ln P_t f) + \tau^2 \Gamma^Z (\ln P_t f)) \leq \left(\frac{D}{d} + \frac{2\bar{\rho}_1}{3}t\right)\frac{LP_t f}{P_t f} + \frac{d(\bar{\rho}_1)^2}{6}t + \frac{\bar{\rho}_1 D}{2} + \frac{D^2}{2dt}. \quad (35)$$

Theorem (6.2.12) [290]. (Harnack inequality). Let $f C_b^\infty(M)$ be such that $f \geq 0$, and consider $v(x, t) = P_t f(x)$. For every $(x, s), (y, t) \in M \times (0, \infty)$ with $s < t$ one has with D as in (33)

$$\frac{v(x, s)}{v(y, t)} \leq \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{d\bar{\rho}_1(t - s)}{4}\right) \exp\left(\frac{d_\tau(x, y)^2}{4(t - s)}\left(\frac{D}{d} + \tau^2 \frac{2\bar{\rho}_1}{\rho_2}\right) + \frac{\bar{\rho}_1}{3}(t + s) + \frac{3\tau^2 D}{2(t - s)\rho_2 d}\ln\left(\frac{t}{s}\right)\right)$$

Proof. We can assume $\rho_1 \leq 0$. Otherwise, if $\rho_1 > 0$ then $CD(0, \rho_2, \kappa, d)$ anyhow also holds. We can rewrite the Li-Yau type inequality in the form

$$\Gamma(\ln P_u f) + \tau^2 \Gamma^Z(\ln P_u f) \leq a_\tau(u)\frac{LP_u f}{P_u f} + b_\tau(u) \quad (36)$$

Where

$$a_\tau(u) = \left(1 + \frac{3\tau^2}{2\rho_2 u}\right)\left(\frac{D}{d} + \frac{2\bar{\rho}_1}{3}u\right)$$

and

$$b_\tau(u) = \left(1 + \frac{3\tau^2}{2\rho_2 u}\right)\frac{d(\bar{\rho}_1)^2}{6}u + \frac{\bar{\rho}_1 D}{2} + \frac{D^2}{2du}.$$

Let now $x, y \in M$ and let $\sigma: [0, T] \to M$ be a subunit curve for $\Gamma + \tau^2 \Gamma^Z$ such that $\sigma(0) = x, \sigma(T) = y$. For $s \leq u \leq t$, we denote

$$\gamma(u) = \sigma\left(\frac{u - s}{t - s}T\right).$$

Let us now consider

$$\varphi(u) = \ln P_u(f)(\gamma(u)).$$

We compute

$$\varphi'(u) = \frac{1}{P_u f(\gamma(u))}\left(LP_u f(\gamma(u)) + \frac{d}{du}(P_u f(\gamma(u)))\right).$$

Since σ is subunit for $\Gamma + \tau^2 \Gamma^Z$, we have

$$\frac{d}{du}\left(P_u f(\gamma(u))\right) \geq -\frac{T}{t-s}\sqrt{\Gamma(P_u f)(\gamma(u)) + \tau^2 \Gamma^Z(P_u f)(\gamma(u))}$$

Now, for every $\lambda > 0$, we have

$$\sqrt{\Gamma(P_u f)(\gamma(u)) + \tau^2 \Gamma^Z(P_u f)(\gamma(u))} \leq \frac{1}{2\lambda} + \frac{\lambda}{2}\left(\Gamma(P_u f)(\gamma(u)) + \tau^2 \Gamma^Z(P_u f)(\gamma(u))\right).$$

Therefore we obtain

$$\varphi'(u) \geq \frac{1}{P_u f(\gamma(u))}\left(LP_u f(\gamma(u)) - \frac{T}{t-s}\left(\frac{1}{2\lambda} + \frac{\lambda}{2}\left(\Gamma(P_u f)(\gamma(u)) + \tau^2 \Gamma^Z(P_u f)(\gamma(u))\right)\right)\right)$$

$$\geq \frac{1}{P_u f(\gamma(u))}LP_u f(\gamma(u)) - \frac{T}{t-s}\left(\frac{1}{2\lambda} + \frac{\lambda}{2}\left(a_\tau(u)(LP_u f)(\gamma(u))(P_u f)(\gamma(u)) + b_\tau(u)(P_u f)^2(\gamma(u))\right)\right)$$

Choosing $\lambda = \frac{2(t-s)}{Ta_\tau(u)P_u f(\gamma(u))}$ yields

$$\varphi'(u) \geq -\frac{a_\tau(u)T^2}{4(t-s)^2} - \frac{b_\tau(u)}{a_\tau(u)}.$$

By integrating this inequality from s to t we infer

$$\ln P_t f(y) - \ln P_s f(x) \geq -\frac{\int_s^t a_\tau(u)\,du}{4(t-s)^2}T^2 - \int_s^t \frac{b_\tau(u)}{a_\tau(u)}\,du,$$

Minimizing over sub-unit curves gives

$$\ln P_t f(y) - \ln P_s f(x) \geq -\frac{\int_s^t a_\tau(u)\,du}{4(t-s)^2}d_\tau(x,y)^2 - \int_s^t \frac{b_\tau(u)}{a_\tau(u)}\,du,$$

which is the claimed result after tedious computations.

Corollary (6.2.13) [290]. Let $p(x, y, t)$ be the heat kernel on M. For every $x, y, z \in M$, every $0 \leq s \leq t < \infty$ and every $\tau \geq 0$, one has

$$\frac{p(x,y,s)}{p(x,z,t)} \leq \left(\frac{t}{s}\right)^{\frac{D}{2}} \exp\left(\frac{d\bar{\rho}_1(t-s)}{4}\right)\exp\left(\frac{d_\tau(x,y)^2}{4(t-s)}\left(\frac{D}{d} + \tau^2\frac{2\bar{\rho}_1}{\rho_2}\right) + \frac{\bar{\rho}_1}{3}(t+s) + \frac{3\tau^2 D}{2(t-s)\rho_2 d}\ln\left(\frac{t}{s}\right)\right)$$

The following proposition provides a pointwise estimate of the volume of the balls, for the proof see [282].

Proposition (6.2.14) [290]. There exists a constant $C(d, \kappa, \rho_2) > 0$ such that, given $R_0 > 0$, for every $x \in M$ and every $R \geq R_0$ one has

$$\mu(B(x,R)) \leq C(d,\kappa,\rho_2)\frac{\exp(2d\bar{\rho}_1 R_0^2)}{R_0^2 p(x,x,R_0^2)}R^D \exp(2d\bar{\rho}_1 R^2).$$

We now turn to the proof of Theorem (6.2.10). Though, some new ideas and more careful estimates are required, the proof mainly follows the lines of [269] where the results is proved when $\rho_1 = 0$. Therefore, several results are stated without proof and we only justify the statements involving these new ideas and careful estimates. The results given without justification may be proved as in [269] by keeping track of the term $\bar{\rho}_1$.

Henceforth in the sequel we denote

$$C_b^\infty(M) \ = \ C^\infty(M) \cap L^\infty(M).$$

For $\varepsilon > 0$ we also denote by A_ε the set of functions $f \in C_b^\infty(M)$ such that

$$f \ = \ g + \varepsilon,$$

for some $\varepsilon > 0$ and some $g \in C_b^\infty(M), g \geq 0$, such that $g, \sqrt{\Gamma(g)}, \sqrt{\Gamma^Z(g)} \in L^2(M)$. As shown in[270], this set is stable under the action of P_t, i.e., if $f \in A_\varepsilon$, then $P_t f \in A_\varepsilon$. The first ingredient to prove the doubling property is the following reverse log-Sobolev inequality.

Theorem (6.2.15) [290]. Let $\varepsilon > 0$ and $f \in A_\varepsilon$, then for every $C \geq 0$, one has for $x \in M, t > 0$,

$$\frac{t}{\rho_2} P_t f(x) \Gamma(\ln P_t f)(x) + t^2 P_t f(x) \Gamma^Z(\ln P_t f)(x)$$

$$\leq \frac{1}{\rho_2} \left(1 + \frac{2\kappa}{\rho_2} + \frac{4C}{d} + 2t\bar{\rho}_1\right) [P_t (f \ln f)(x) - P_t f(x) \ln P_t f(x)] - \frac{4C}{d\rho_2} \frac{t}{1+\delta} L P_t f(x)$$

$$+ \frac{2C^2}{d\rho_2} \ln\left(1 + \frac{1}{\delta}\right) P_t f(x).$$

This inequality admits the following corollary.

Proposition (6.2.16) [290]. Let $\varepsilon > 0$, $f \in A_\varepsilon$ such that $\varepsilon \leq f \leq 1$ and consider the function $u(x, t) = \sqrt{-\ln P_t f(x)}$. Then,

$$2t u_t + \left(u + \left(1 + \sqrt{\frac{D^*}{2}}\right) u^{1/3} + \sqrt{\frac{D^*}{2}} u^{-1/3}\right)\left(1 + \sqrt{d\bar{\rho}_1\, t}\right) \geq 0,$$

Where

$$D^* \ = \ d\left(1 + \frac{2\kappa}{\rho_2}\right).$$

Introduce the function $g: (0, \infty) \to (0, \infty)$ defined by

$$g(v) \ = \ \frac{1}{u + \left(1 + \sqrt{\dfrac{D^*}{2}}\right) v^{1/3} + \sqrt{\dfrac{D^*}{2}} v^{-1/3}}$$

Note that g verifies

$$\lim_{v \to 0^+} \sqrt{\frac{D^*}{2}}\, v^{-1/3} g(v) \ = \ 1, \qquad \lim_{v \to +\infty} v\, g(v) \ = \ 1.$$

Therefore, we have $g \in L^1(0, A)$ for every $A > 0$, but $g \notin L^1(0, +\infty)$. If we set

$$G(u) \ = \ \int_0^u g(v) dv,$$

then $G'(u) = g(u) > 0$, and thus $G: (0, \infty) \to (0, \infty)$ is invertible. Furthermore, we can write

$$G(u) = \ln(u) + C_0 + R(u), \quad u > 0 \tag{37}$$

where C_0 is a constant and $R: (0, +\infty) \to R$ a function such that $\lim\limits_{u \to \infty} R(u) = 0$. Proposition (6.2.16) can be re-written in terms of g as follows

$$2tu_t + \frac{1 + \sqrt{td\bar{\rho}_1}}{g(u)} \geq 0.$$

Since $g(u) = G'(u)$, we conclude

$$\frac{dG(u)}{dt} = G'(u)u_t \geq -\frac{1}{2t} - \frac{1}{2}\sqrt{\frac{d\bar{\rho}_1}{t}} \tag{38}$$

Integrating this differential inequality leads to the following result:

Corollary (6.2.17) [290]. Let $f \in L^\infty(M), 0 \leq f \leq 1$, then for any $x \in M$ and $0 < s < t$,

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq G\sqrt{-\ln P_s f(x)} - \frac{1}{2}\ln\left(\frac{t}{s}\right) - \sqrt{d\bar{\rho}_1}(\sqrt{t} - \sqrt{s}).$$

The second ingredient in our proof is the following small time asymptotics.

Proposition (6.2.18) [290]. Given $x \in M$ and $r > 0$, let $f = 1_{B(x,r)^c}$. One has,

$$\liminf_{s \to 0^+} (-s \ln P_s f(x)) \geq \frac{r^2}{4}$$

We are now ready for the following estimate.

Proposition (6.2.19) [290]. Let $x \in M$ and $r > 0$ be arbitrarily fixed. There exists a constant $C_0^* \in R$ independent of x and r, such that for any $t > 0$,

$$G\left(\sqrt{-\ln P_t 1_{B(x,r)^c}(x)}\right) \geq \ln\frac{r}{\sqrt{t}} + C_0^* - \sqrt{d\bar{\rho}_1 t}$$

Proof. Let $f = 1_{B(x,r)^c}$. Corollary (6.2.17) and (38) give

$$G\left(\sqrt{-\ln P_t f(x)}\right) \geq G\sqrt{-\ln P_s f(x)} - \frac{1}{2}\ln\sqrt{t} - \sqrt{d\bar{\rho}_1}(\sqrt{t} - \sqrt{s})$$

$$= \ln\sqrt{-s \ln P_s f(x)} + C_0 + R\left(\sqrt{-\ln P_s f(x)}\right) - \ln\sqrt{t} - \sqrt{d\bar{\rho}_1 t} + \sqrt{d\bar{\rho}_1 s}.$$

Since $\lim\limits_{s \to 0^+} (-\ln P_s f(x)) = \infty$, we infer $\lim\limits_{s \to 0^+} R(\sqrt{-\ln P_s f(x)}) = 0$. Letting $s \to 0^+$, Proposition (6.2.18) yields we obtain

$$G\left(\sqrt{-\ln P_s f(x)}\right) \geq \ln\frac{r}{2} - \ln\sqrt{t} + C_0 - \sqrt{d\bar{\rho}_1 t} = \ln\frac{r}{\sqrt{t}} - \sqrt{d\bar{\rho}_1 t} + C_0^*,$$

with $C_0^* = C_0 - \ln 2$.

The following uniform lower bound on the heat content of balls, which is already interesting in itself, will imply the volume doubling property.

Theorem (6.2.20) [290]. Set $C_0^{**} = G(\sqrt{\ln 2}) - C_0^*$ and for $R \geq 0$, define $U(R) = \Psi_R^{-1}(C_0^{**})$ where Ψ_R^{-1} is the inverse function of

$$\Psi_R(u) = \ln\left(\frac{1}{u}\right) - \sqrt{d\bar{\rho}_1 t}\, Ru, \qquad u \in (0, \infty).$$

Then for every $x \in M$ and every function $A: [0, +\infty) \to (0, \infty)$ such that $\sqrt{A(R)} \leq U(R)$, we have for $r > 0$,

$$P_{A(r)r^2}\left(1_{B(x,r)}\right)(x) \geq \frac{1}{2} .$$

Proof. By the stochastic completeness of M

$$P_{A(r)r^2}\left(1_{B(x,r)}\right)(x) = 1 - P_{A(r)r^2}\left(1_{B(x,r)^c}\right)(x)$$

The desired estimate is equivalent to prove

$$\sqrt{\ln 2} \leq \sqrt{-P_{A(r)r^2}\left(1_{B(x,r)^c}\right)(x)}$$

or equivalently,

$$G(\sqrt{\ln 2} \leq G\left(\sqrt{-P_{A(r)r^2}\left(1_{B(x,r)^c}\right)(x)}\right). \tag{39}$$

At this point Proposition (6.2.19) gives

$$G\left(\sqrt{-P_{A(r)r^2}\left(1_{B(x,r)^c}\right)(x)}\right) \geq \ln\left(\frac{1}{\sqrt{A(r)}}\right) + C_0^* - \sqrt{d\bar{\rho}_1 A(r)} r$$

$$\geq \ln\left(\frac{1}{U(r)}\right) + C_0^* - \sqrt{d\bar{\rho}_1} r U(r) = G(\sqrt{\ln 2}).$$

We now give some estimates for the function $U(R)$ appearing in Theorem (6.2.20).

Proposition (6.2.21) [290]. The function U is non-increasing and satisfies, for $R \geq 0$,

$$U(R) \geq \frac{1}{\sqrt{d\bar{\rho}_1} R + e^{C_0^{**}}} .$$

Proof. First notice that $U(0) = e^{-C_0^{**}}$ and U is positive. Since $\Psi_R(U(R))$ is constant, taking derivative yields:

$$U'(R) = -\frac{\sqrt{d\bar{\rho}_1} U(R)}{\sqrt{d\bar{\rho}_1} R + \frac{1}{U(R)}} \geq -\sqrt{d\bar{\rho}_1} U(R)^2.$$

Therefore U is non-increasing and integrating the above inequality we infer that

$$U(R) \geq \frac{1}{\sqrt{d\bar{\rho}_1} R + U^{-1}(0)}$$

Henceforth, in the sequel, for $r \geq 0$, we set

$$A(r) = \min(U(r)^2, 1) \geq \min\left(\left(\frac{1}{\sqrt{d\bar{\rho}_1} R + e^{C_0^{**}}}\right)^2, 1\right). \tag{40}$$

There exists a constant $C_1 > 0$ such that, for all $R \geq 0$,

$$\frac{C_1}{1 + d\bar{\rho}_1 R^2} \leq A(R) \leq 1. \tag{41}$$

A first consequence of the uniform estimate we obtained are the following lower bounds for the heat kernel. Observe, and this is another main novelty with respect to [269] that these bounds are written with respect to the distance d_τ (we recall that d_0 is the sub-Riemannian distance).

Theorem (6.2.22) [290]. Set $C_2 = \frac{C_1}{4}$. For $t > 0$ and $x \in M$, then

$$p(x,x,t) \geq \frac{\left(A\left(\sqrt{\frac{t}{2}}\right)\right)^{\frac{D}{2}} \exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)}{4\mu\left(B\left(x,\sqrt{\frac{t}{2}}\right)\right)} \geq \frac{C_2}{4\mu\left(B\left(x,\sqrt{\frac{t}{2}}\right)\right)} \frac{\exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)}{\left(1 + \frac{d\bar{\rho}_1 t}{2}\right)^{\frac{D}{2}}}. \tag{42}$$

As a consequence, for $x, y \in M, t > 0$ and $\tau \geq 0$,

$$p(x,x,t) \geq \frac{\left(A\left(\sqrt{\frac{t}{2}}\right)\right)^{\frac{D}{2}} 2^{-\frac{D}{2}} \exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)}{4\mu\left(B\left(x,\sqrt{\frac{t}{2}}\right)\right)} \exp\left(-\frac{d_\tau(x,y)^2}{2t}\left(\frac{D}{d} + \frac{\bar{\rho}_1}{2}t + \frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2} + \frac{3D}{2\rho_2 d}\ln(2)\right)\right)\right)$$

$$\geq \frac{2^{-\frac{D}{2}}C_2}{4\mu\left(B\left(x,\sqrt{\frac{t}{2}}\right)\right)\left(1 + \frac{d\bar{\rho}_1 t}{2}\right)^{\frac{D}{2}}} \frac{\exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)}{} \exp\left(-\frac{d_\tau(y,x)^2}{2t}\left(\frac{D}{d} + \frac{\bar{\rho}_1}{2}t + \frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2} + \frac{3D}{2\rho_2 d}\ln(2)\right)\right)\right). \tag{43}$$

Proof. With the same notations as in Theorem (6.2.20), for $R > 0$,

$$P_{A(R)R^2}(1_{B(x,R)})(x) \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{4} \leq P_{A(R)R^2}(1_{B(x,R)})(x)^2$$

$$= \left(\int_M p(x,y,A(R)R^2)(1_{B(x,R)}d\mu(y)\right)^2$$

$$\leq \int_M p(x,y,A(R)R^2)2d\mu(y) \int_M 1_{B(x,R)}d\mu(y)$$

$$= p(x,x,2A(R)R^2)\mu(B(x,R)).$$

Now since $0 < A(R) \leq 1$, Harnack inequality in Corollary (6.2.13) gives

$$p(x,x,2A(R)R^2 \leq p(x,x,2R^2)(A(R))^{-\frac{D}{2}} \exp\left(\frac{1}{2}d\bar{\rho}_1 R^2\right). \tag{44}$$

Therefore, we proved

$$p(x,x,2R^2) \geq \frac{A(R)^{\frac{D}{2}}\exp\left(-\frac{1}{2}d\bar{\rho}_1 R^2\right)}{4\,\mu(B(x,R))}. \tag{45}$$

The first point follows by setting $t = 2R^2$.

For the second point, we recall that Harnack inequality with the distance d_τ reads, for $t > 0, s = \frac{t}{2}$,

$$P\left(x, x, \frac{t}{2}\right) \leq p(x, y, t) 2^{\frac{D}{2}} \exp\left(\frac{d\bar{\rho}_1 t}{8}\right) \exp\left(-\frac{d_\tau(y, x)^2}{2t}\left(\frac{D}{d} + \frac{\bar{\rho}_1}{2}t + \frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2} + \frac{3D}{2\rho_2 d}\ln(2)\right)\right)\right)$$

Using the first point the conclusion follows directly.

Theorem (6.2.23) [290]. There exist constants $C_3 > 0$ and $C_4 > 0$ depending only on d, κ and ρ_2 such that for all $x \in M$ and all $R > 0$, we have

$$\mu(B(x, 2R)) \leq C_3 \exp(C_4 \bar{\rho}_1 R^2) \mu(B(x, R)) \tag{46}$$

Proof. Due to Proposition (6.2.14), we have for $R > 0$

$$\mu(B(x, 2R)) \leq C(d, \kappa, \rho_2) 2^{D/2} \frac{\exp(12\bar{\rho}_1 R^2)}{p(x, x, 2R^2)}$$

Combining this inequality with the lower bound for the heat kernel (45) gives the desired result:

$$\mu(B(x, 2R)) \leq C(d, \kappa, \rho_2) 2^{\frac{D}{2}+2} \exp\left(\frac{25}{2}\bar{\rho}_1 R^2\right) A(R)^{-D/2} \mu(B(x, R)).$$

We prove a global version of the celebrated Nagel-Stein-Wainger estimate, Theorem (6.2.1). We need the following optimal upper bound for the heat kernel $p(x, y, t)$.

Theorem (6.2.24) [290]. For all $\varepsilon > 0$, there exist some constants $C_5, C_6 > 0$, depending only on d, ρ_2, κ and $\varepsilon > 0$, such that for $t > 0$ and $x, y \in M$, one has

$$p(x, y, t) \leq \frac{C_5(\varepsilon)}{\mu(B(x, \sqrt{t}))^{1/2} \mu(B(y, \sqrt{t}))^{1/2}} \exp(C_6(\varepsilon)\bar{\rho}_1 t) \exp\left(-\frac{d^2(x, y)}{(4+\varepsilon)t}\right). \tag{47}$$

Proof. This proof follows the lines in Cao-Yau [274] and Baudoin-Garofalo [270]. Let $\alpha > 0$, then by the Harnack inequality in Corollary (6.2.13) with $\tau = 0$,

$$p(x, y, t)^2 \leq \frac{(1+\alpha)^D \exp\left(\frac{D}{2\alpha d}\right)}{\mu(B(y, \sqrt{t}))} \exp\left(\bar{\rho}_1 t\left(\frac{2+\alpha}{6\alpha} + \frac{d\alpha}{2}\right)\right) \int_{B(y,\sqrt{t})} p(x, z, (1+\alpha)t)^2 d\mu(z)$$

$$= \left(\frac{(1+\alpha)^D \exp\left(\frac{D}{2\alpha d}\right)}{\mu\left(B(y, \sqrt{t})\right)} \exp\left(\bar{\rho}_1 t\left(\frac{2+\alpha}{6\alpha} + \frac{d\alpha}{2}\right)\right) P_{(1+\alpha)t}\left(p(x, \cdot, (1+\alpha)t)1_{B(y,\sqrt{t})}\right)(x).$$

Applying the Harnack inequality in Theorem (6.2.12) once again, we have

$$P_{(1+\alpha)t}\left(p(x, \cdot, (1+\alpha)t)1_{B(y,\sqrt{t})}\right)(x)^2 = P_{(1+\alpha)t}(F_t)(x)^2$$

$$\leq \frac{(1+\alpha)^D \exp\left(\frac{D}{2\alpha(\alpha+1)d}\right)}{\mu(B(x, \sqrt{t}))} \exp\left(\bar{\rho}_1 t\left(\frac{2+\alpha}{6\alpha} + \frac{d\alpha(\alpha+1)}{2}\right)\right) \int_{B(x,\sqrt{t})} P_{(1+\alpha)^2 t}(F_t)(z)^2 d\mu(z)$$

with $F_t(.) = p(x, \cdot, (1+\alpha)t)1_{B(y,\sqrt{t})}(\cdot)$.

By using now an argument of the proof of Theorem 8.1 in [270] and the fact that for $z \in B(y, \sqrt{t})$,

$$d^2(x, z) \geq \frac{d^2(x, y)}{1 + \alpha} - \frac{t}{\alpha},$$

we have for $0 < (1 + \alpha)^2 t < T$,

$$\int_{B(x, \sqrt{t})} P_{(1+\alpha)^2 t}(F_t)(z)^2 d\mu(z)$$

$$\leq \exp\left(\frac{t}{2(T - (1 + \alpha)^2 t)}\right) e^{g((1+\alpha)^2 t, z)} P_{(1+\alpha)^2 t}(F_t)(z)^2 d\mu(z)$$

$$= \exp\left(\frac{t}{2(T - (1 + \alpha)^2 t)}\right) e^{g(0, z)}(F_t)(z)^2 d\mu(z)$$

$$\leq \exp\left(\frac{t}{2(T - (1 + \alpha)^2 t)}\right) \int_{B(x, \sqrt{t})} \exp\left(-\frac{d^2(x, z)}{2T}\right) P_{(1+\alpha)t}(x, z)^2 d\mu(z)$$

$$\leq \exp\left(\frac{t}{2(T - (1 + \alpha)^2 t)} + \frac{t}{2\alpha T} - \frac{d^2(x, z)}{2(1 + \alpha)T}\right) \int_{B(x, \sqrt{t})} P_{(1+\alpha)t}(x, z)^2 d\mu(z)$$

$$= \exp\left(\frac{t}{2(T - (1 + \alpha)^2 t)} + \frac{t}{2\alpha T} - \frac{d^2(x, z)}{2(1 + \alpha)T}\right) P_{(1+\alpha)t}(F_t)(x)$$

where for $0 \leq s < T$ and $z \in M$, the function g is defined by

$$g(s, z) = -\frac{d^2(x, z)}{2(T - s)}.$$

Finally,

$$p(x, y, t) \leq \frac{(1 + \alpha)^D \exp\left(\frac{D(\alpha + 2)}{4\alpha(\alpha + 1)d}\right)}{\mu(B(x, \sqrt{t}))^{1/2} \mu(B(y, \sqrt{t}))^{1/2}} \exp\left(\bar{\rho}_1(2 + \alpha)\left(\frac{1}{6\alpha} + \frac{d\alpha}{4}\right)\right)$$

$$\exp\left(\frac{t}{4(T - (1 + \alpha)^2 t)} + \frac{t}{4\alpha T} - \frac{d^2(x, y)}{4(1 + \alpha)T}\right).$$

Hence the result follows by choosing T $= (1 + \alpha)^3$t.

Theorem (6.2.25) [290]. There exists a constant $C_7 > 0$ which depends only on d, κ and ρ_2 such that for all x and y in M and all $0 < \tau \leq 1$,

$$d(x, y) \leq C_7\left(1 + \sqrt{\bar{\rho}_1}\right) \max\{\sqrt{d_\tau(x, y)}, d_\tau(x, y)\}.$$

Proof. Using the symmetry of the heat kernel, combining the lower estimate (43) for the heat kernel with the distance d_τ and the upper estimate (47) for the sub-elliptic distance d gives

$$\frac{\left(A\left(\sqrt{\frac{t}{2}}\right)\right)^{\frac{D}{2}} 2^{-\frac{D}{2}}\exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)}{4\mu(B(x,\sqrt{t}))^{1/2}\, 4\mu(B(y,\sqrt{t}))^{1/2}}\exp\left(-\frac{d_\tau(x,y)^2}{2t}\left(\frac{D}{d}+\frac{\bar{\rho}_1}{2}t+\frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2}+\frac{3D}{2\rho_2 d}\ln(2)\right)\right)\right)$$

$$\le \frac{C_5(\varepsilon)}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}}\, exp\big(C_6(\varepsilon)\bar{\rho}_1 t\big)\exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t}\right)$$

Therefore we have

$$A\left(\sqrt{\frac{t}{2}}\right)^{\frac{D}{2}} 2^{-\frac{D}{2}-2}\exp\left(-\frac{d\bar{\rho}_1 t}{4}\right)\exp\left(-\frac{d_\tau(x,y)^2}{2t}\left(\frac{D}{d}+\frac{\bar{\rho}_1}{2}t+\frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2}+\frac{3D}{2\rho_2 d}\ln(2)\right)\right)\right)$$

$$\le C_5(\varepsilon)\exp(C_6(\varepsilon)\bar{\rho}t)\exp\left(-\frac{d^2(x,y)}{(4+\varepsilon)t}\right).$$

Thus for all $t > 0$:

$$0 \le -\frac{D}{2}\ln A\left(\frac{\sqrt{t}}{2}\right)+\left(\frac{D}{2}+2\right)\ln 2+\ln C_5(\varepsilon)+C_6(\varepsilon)(1+\bar{\rho}_1)t+\frac{d\bar{\rho}_1 t}{4}$$

$$-\frac{d^2(x,y)}{(4+\varepsilon)t}+\frac{d_\tau(x,y)^2}{2t}\left(\frac{D}{d}+\frac{\bar{\rho}_1}{2}t+\frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2}+\frac{3D}{2\rho_2 d}\ln(2)\right)\right).$$

Since $-\ln A\left(\frac{\sqrt{t}}{2}\right)\le \ln\left(1+\frac{d\bar{\rho}_1 t}{4}\right)-\ln C_1 \le \frac{d\bar{\rho}_1 t}{4}-\ln C_1$, fixing $\varepsilon = 1$, there exist some constants E_1, E_2 which only depend on d, κ and ρ_2 such that for all $t > 0$, we have for all $x,y \in M, t > 0$ and $\tau > 0$

$$0 \le E_1 + E_2\bar{\rho}_1 t-\frac{d^2(x,y)}{(4+\varepsilon)t}+\frac{d_\tau(x,y)^2}{2t}\left(\frac{D}{d}+\frac{\bar{\rho}_1}{2}t+\frac{2\tau^2}{t}\left(\frac{\bar{\rho}_1}{\rho_2}+\frac{3D}{2\rho_2 d}\ln(2)\right)\right).$$

Therefore, for some positive constants $A_i, 1\le i \le 3$ which only depend on d, κ and ρ_2,

$$d(x,y)^2 \le A_1(1+\bar{\rho}_1 t)t+A_2(1+\bar{\rho}_1 t)d_\tau(x,y)^2+A_3(1+\bar{\rho}_1)\frac{\tau^2 d_\tau(x,y)^2}{t}.$$

Since $\tau \le 1$, if $d_\tau(x,y)\le 1$, choosing $t = \tau d_\tau(x,y)\le 1$ yields

$$d(x,y)^2 \le (1+\bar{\rho}_1 t)(A_1+A_3)\tau d_\tau(x,y)+A_2 d_\tau(x,y)^2 \le (A_1+A_2+A_3)(1+\bar{\rho}_1 t)d_\tau(x,y).$$

If $d_\tau(x,y)\ge 1$, choosing $t = \tau \le 1$, we infer

$$d(x,y)^2 \le (1+\bar{\rho}_1 t)A_1\tau+A_2 d_\tau(x,y)^2+A_3\tau d_\tau(x,y)^2 \le (A_1+A_2+A_3)(1+\bar{\rho}_1 t)d_\tau(x,y)^2.$$

We investigate some consequences of the curvature-dimension inequality. We show that if an operator L is an Hormander type operator and if it satisfies some curvature-dimension inequality $CD(\rho_1, \rho_2, \kappa, d)$, then it is necessarily a rank 2 operator. The proof is based on the distance comparison theorem (Theorem 6.2.25). Actually, only a local distance comparison is needed. The notions of Hormander type operator and rank 2 operator are explained below.

First, the comparison principle of Fefferman and Phong between sub-elliptic and elliptic balls (see [277]) implies the following local sub-elliptic estimate:

Theorem (6.2.26) [290]. Assume the operator L satisfies the condition $CD(\rho_1, \rho_2, \kappa, d)$ for some $\rho_1 \in R$, $\rho_2 > 0, \kappa \geq 0$, and $d \geq 2$. Assume moreover that the metric associated to d_τ is a Riemannian metric g_τ on M for some τ. Let Ω be a bounded domain in M and a chart $\varphi: U \subset R^m \to \Omega$. Then there exist some constants $c = c(\Omega, \varphi) > 0$ and $C = C(\Omega, \varphi) > 0$ such that

$$\|Lu\| + C\|u\| \geq c\|u\|_{(1)}, u \in C_0^\infty(\varphi(U)), \tag{48}$$

where $\|\cdot\|$ denotes the usual $L^2(\mu)$ norm in Ω and where $\|\cdot\|_{(s)}$ is the standard Sobolev norm.

Proof. By Theorem (6.2.25), there exists a constant A such that $d(x, y) \leq A\sqrt{d_\tau(x, y)}$ for all $x, y \in \Omega$. Therefore, $B_\tau(x, R) \subset B(x, A\sqrt{R})$. Since $\overline{\Omega}$ is a compact set, the metric g_τ is comparable with geucl the metric obtained from the Euclidean one in U by the map φ. If we pull back the result in U, we thus have, $B_{eucl}(x, R) \subset B(x, A'\sqrt{R})$ for some constant $A' > 0$.

We call an H¨ormander type operator an operator L which satisfies the general assumptions above and which can be written locally as $L = \sum_{i=1}^r X_j^* X_j$ for some C^∞ vector fields X_j.

We say it is an operator of rank k if the vector fields and their commutators up to order k:

$$X_1, \ldots, X_r, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [X_{i_2}, [\ldots, X_{i_k}]] \ldots], ij = 1 \ldots r$$

span the tangent space in each point of M.

The following theorem is a direct consequence of an important result of Rothschild and Stein [288] and of the local subelliptic estimate (48).

Theorem (6.2.27) [290]. In addition of the hypothesis of Theorem (6.2.26), assume that the operator L is an H¨ormander type operator. Then L is a rank 2 operator.

The goal is to establish a generalisation Gromov's precompactness theorem for our class of subriemannian manifolds. Initially, the Gromov's precompactness theorem states that the space of Riemannian manifolds with Ricci curvature bounded below by k, dimension bounded by N and diameter less D is precompact for the Gromov-Hausdorff convergence. Moreover the result can be extended for the (pointed) measured Gromov-Hausdorff convergence by endowing the Riemannian manifolds with their Riemannian volume. In our generality, contrary to the Riemannian case, the measure μ is only defined up to a positive constant. Here, we need to normalize the measure. Let M be a compact smooth manifold and μ be a smooth measure on M such that there exists a smooth second order sub-elliptic differential operator L which satisfies the general assumptions described in above. Let us assume that the measure satisfies the normalisation property $\mu(M) = 1$. We say the compact metric measured space $M = (M, \mu)$ belongs to $M_R(\rho_1, \rho_2, \kappa, d), R > 0$ if moreover L satisfies $CD(\rho_1, \rho_2, \kappa, d)$ and the (sub-Riemannian) diameter of M is bounded above by R.

Theorem (6.2.28) [290]. Let $\rho_1 \in R, \rho_2 > 0, \kappa \geq 0$, and $d \geq 2, R > 0$. The set of metric measured spaces $M_R(\rho_1, \rho_2, \kappa, d)$ is precompact for the measured Gromov-Hausdorff convergence.

Proof. The proof is an easy consequence of Theorem 27.31 in [289] and of the doubling property of Theorem (6.2.23).

Corollary (6.2.29) [291]. (Harnack inequality). Let $f C_b^\infty(M)$ be such that $f \geq 0$, and consider $v_{s-1}(x_s, 1 - \epsilon) = P_{1-\epsilon} f(x_s)$. For every $(x_s, s), (x_s + \epsilon, 1 - \epsilon) \in M \times (0, \infty)$ with $s < 1 - \epsilon$ one has with D

$$\frac{v_{s-1}(x_s, s)}{v_{s-1}(x_s + \epsilon, 1 - \epsilon)}$$

$$\leq \left(\frac{1-\epsilon}{s}\right)^{\frac{D}{2}} \exp\left(\frac{(1+\epsilon)\bar{\rho}_s((1-\epsilon) - s)}{4}\right) \exp\left(\frac{d_{1-\epsilon}(x_s, x_s + \epsilon)^2}{4((1-\epsilon) - s)}\left(\left(\frac{D}{(1+\epsilon)}\right.\right.\right.$$

$$\left.\left.\left. + (1+\epsilon)^2\frac{2\bar{\rho}_s}{\rho_{s+1}}\right) + \frac{\bar{\rho}_s}{3}((1-\epsilon) + s) + \frac{3(1-\epsilon)^2 D}{2((1-\epsilon) - s)\rho_{s+1}(1+\epsilon)}\ln\left(\frac{1-\epsilon}{s}\right)\right)\right) \quad (48)$$

Proof. We can assume $\rho_s \leq 0$. Otherwise, if $\rho_s > 0$ then $C_{s-2}D(0, \rho_{s+1}, 1 + \epsilon, 1 + \epsilon)$ any how also holds. We can rewrite the Li-Yau type inequality in the form

$$\Gamma\left(\ln P_{u_{s-1}} f\right) + (1+\epsilon)^2\Gamma^Z\left(\ln P_{u_{s-1}} f\right) \leq a_{1+\epsilon}(u_{s-1})\frac{L P_{u_{s-1}} f}{P_{u_{s-1}} f} + b_{1+\epsilon}(u_{s-1}) \quad (50)$$

Where

$$a_{1+\epsilon}(u_{s-1}) = \left(1 + \frac{3(1+\epsilon)^2}{2\rho_{s+1}u_{s-1}}\right)\left(\frac{D}{1+\epsilon} + \frac{2\bar{\rho}_s}{3}u_{s-1}\right)$$

and

$$b_{1+\epsilon}(u_{s-1}) = \left(1 + \frac{3(1+\epsilon)^2}{2\rho_{s+1}u_{s-1}}\right)\frac{(1+\epsilon)(\bar{\rho}_s)^2}{6}u_{s-1} + \frac{\bar{\rho}_s D}{2} + \frac{D^2}{2(1+\epsilon)u_{s-1}}.$$

Let now $x_s, (x_s + \epsilon) \in M$ and let $\sigma: [0, T] \to M$ be a subunit curve for $\Gamma + (1+\epsilon)^2\Gamma^Z$ such that $\sigma(0) = x_s, \sigma(T) = x_s + \epsilon$. For $s \leq u_{s-1} \leq 1 - \epsilon$, we denote

$$\gamma(u_{s-1}) = \sigma\left(\frac{u_{s-1} - s}{(1+\epsilon) - s} T\right).$$

Let us now consider

$$\varphi(u_{s-1}) = \ln P_{u_{s-1}}(f)(\gamma(u_{s-1})).$$

We compute

$$\varphi'(u_{s-1}) = \frac{1}{P_{u_{s-1}}f(\gamma(u_{s-1}))}\left(LP_{u_{s-1}}f(\gamma(u_{s-1})) + \frac{d}{du_{s-1}}(P_{u_{s-1}}f(\gamma(u_{s-1})))\right).$$

Since σ is subunit for $\Gamma + (1+\epsilon)^2\Gamma^Z$, we have

$$\frac{d}{du_{s-1}}(P_{u_{s-1}}f(\gamma(u_{s-1})))$$

$$\geq -\frac{T}{(1-\epsilon)-s}\sqrt{\Gamma(P_{u_{s-1}}f)(\gamma(u_{s-1})) + (1+\epsilon)^2\Gamma^Z(P_{u_{s-1}}f)(\gamma(u_{s-1}))}$$

Now, for every $\epsilon \geq 0$, we have

$$\sqrt{\Gamma(P_{u_{s-1}}f)(\gamma(u_{s-1})) + (1+\epsilon)^2\Gamma^Z(P_{u_{s-1}}f)(\gamma(u_{s-1}))}$$

$$\leq \frac{1+(1+\epsilon)^2}{2(1+\epsilon)}\left(\Gamma(P_{u_{s-1}}f)(\gamma(u_{s-1})) + (1+\epsilon)^2\Gamma^Z(P_{u_{s-1}}f)(\gamma(u_{s-1}))\right).$$

Therefore we obtain

$$\varphi'(u_{s-1}) \geq \frac{1}{u_{s-1}f(\gamma(u_{s-1}))}\left(LP_{u_{s-1}}f(\gamma(u_{s-1}))\right.$$

$$\left.-\frac{T}{(1-\epsilon)-s}\left(\frac{1+(1+\epsilon)^2}{2(1+\epsilon)}\left(\Gamma(P_{u_{s-1}}f)(\gamma(u_{s-1})) + (1-\epsilon)^2\Gamma^Z(P_{u_{s-1}}f)(\gamma(u_{s-1}))\right)\right)\right)$$

$$\geq \frac{1}{P_{u_{s-1}}f(\gamma(u_{s-1}))}LP_{u_{s-1}}f(\gamma(u_{s-1}))$$

$$-\frac{T}{(1-\epsilon)-s}\left(\frac{1+(1+\epsilon)^2}{2(1+\epsilon)}\left(a_{1+\epsilon}(u_{s-1})(LP_{u_{s-1}}f)(\gamma(u_{s-1}))(P_{u_{s-1}}f)(\gamma(u_{s-1}))\right.\right.$$

$$\left.\left.+ b_{1+\epsilon}(u_{s-1})(P_{u_{s-1}}f)^2(\gamma(u_{s-1}))\right)\right)$$

Choosing $(1+\epsilon) = \frac{2(1+\epsilon-s)}{Ta_{1+\epsilon}(u_{s-1})P_{u_{s-1}}f(\gamma(u_{s-1}))}$ yields

$$\varphi'(u_{s-1}) \geq -\frac{a_{1+\epsilon}(u_{s-1})T^2}{4((1+\epsilon)-s)^2} - \frac{b_{1+\epsilon}(u_{s-1})}{a_{1+\epsilon}(u_{s-1})}.$$

By integrating this inequality from s to $1+\epsilon$ we infer

$$\ln P_{1+\epsilon}f(x_s+\epsilon) - \ln P_s f(x_s)$$

$$\geq -\frac{\int_s^{1+\epsilon}a_{1\mp\epsilon}(u_{s-1})du_{s-1}}{4((1+\epsilon)-s)^2}T^2 - \int_s^{1+\epsilon}\frac{b_{1+\epsilon}(u_{s-1})}{a_{1+\epsilon}(u_{s-1})}du_{s-1},$$

Minimizing over sub-unit curves gives

$$\ln P_{1+\epsilon}f(x_s+\epsilon) - \ln P_s f(x_s)$$

$$\geq -\frac{\int_s^{1+\epsilon}a_{1+\epsilon}(u_{s-1})du_{s-1}}{4((1+\epsilon)-s)^2}d_{1+\epsilon}(x_s, x_s+\epsilon)^2 - \int_s^{s+\epsilon}\frac{b_{1+\epsilon}(u_{s-1})}{a_{1+\epsilon}(u_{s-1})}du_{s-1},$$

which is the claimed result after tedious computations.

Corollary (6.2.30) [291]. Let $x_s \in M$ and $\epsilon > -1$ be arbitrarily fixed. There exists a constant $C^*_{s-1} \in R$ independent of x_s and $(1 + \epsilon)$, such that for any $\epsilon > 0$,

$$G\left(\sqrt{-\ln P_{1+\epsilon} 1_{B_{s-1}(x_s,1+\epsilon)^{c_{s-2}}}(x_s)}\right) \geq \ln \sqrt{1 + \epsilon} + C^*_{s-1} - \sqrt{\bar{\rho}_s(1 + \epsilon)^2}$$

Proof. Let $f = 1_{B_{s-1}(x_s,1+\epsilon)^{c_{s-2}}}$.

$$G\left(\sqrt{-\ln P_{1+\epsilon} f(x_s)}\right)$$

$$\geq G\sqrt{-\ln P_s f(x_s)} + \ln \sqrt{s} - \ln \sqrt{1 + \epsilon} - \sqrt{(1 + \epsilon)\bar{\rho}_s}(\sqrt{1 + \epsilon} - \sqrt{s})$$

$$= \ln \sqrt{-s \ln P_s f(x_s)} + C_{s-1} + R\left(\sqrt{-\ln P_s f(x_s)}\right) - \ln \sqrt{1 + \epsilon} - \sqrt{\bar{\rho}_s(1 + \epsilon)^2}$$

$$+ \sqrt{(1 + \epsilon)\bar{\rho}_s s}.$$

Since $\lim_{s\to 0^+}(-\ln P_s f(x_s)) = \infty$, we infer $\lim_{s\to 0^+} R\left(\sqrt{-\ln P_s f(x_s)}\right) = 0$. Letting s→0⁺, yields we obtain

$$G\left(\sqrt{-\ln P_{1+\epsilon} f(x_s)}\right) \geq \ln \frac{1 + \epsilon}{2} - \ln \sqrt{(1 + \epsilon)} + C_{s-1} - \sqrt{\bar{\rho}_s(1 + \epsilon)^2}$$

$$= \ln \sqrt{1 + \epsilon} - \sqrt{\bar{\rho}_s(1 + \epsilon)^2} + C^*_{s-1},$$

with $C^*_{s-1} = C_{s-1} - \ln 2$.

The following uniform lower bound on the heat content of balls, which is already interesting in itself, will imply the volume doubling property.

Corollary (6.2.31) [291]. Set $C^{**}_{s-1} = G(\sqrt{\ln 2}) - C^*_{s-1}$ and for $\epsilon \geq 0$, define $U_{s-1}(1 + \epsilon) = \Psi^{-1}_{1+\epsilon}(C^{**}_{s-1})$ where $\Psi^{-1}_{1+\epsilon}$ is the inverse function of

$$\Psi_{1+\epsilon}(u_{s-1}) = \ln\left(\frac{1}{u_{s-1}}\right) - \sqrt{(1 + \epsilon)\bar{\rho}_s}(1 + \epsilon)u_{s-1}, \qquad u_{s-1} \in (0, \infty).$$

Then for every $x_s \in M$ and every function $A_{s-1}: [0, +\infty) \to (0, \infty)$ such that $\sqrt{A_{s-1}(1 + \epsilon)} \leq U_{s-1}(1 + \epsilon)$, we have for $\epsilon > -1$,

$$P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)}\right)(x_s) \geq \frac{1}{2}.$$

Proof. By the stochastic completeness of M

$$P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)}\right)(x_s) = 1 - P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)^{c_{s-2}}}\right)(x_s)$$

The desired estimate is equivalent to prove

$$\sqrt{\ln 2} \leq \sqrt{-\ln P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)^{c_{s-2}}}\right)(x_s)}$$

or equivalently,

$$G(\sqrt{\ln 2} \leq G\left(\sqrt{-s P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)^{c_{s-2}}}\right)(x_s)}\right). \tag{51}$$

At this point Corollary (6.2.31) gives

197

$$G\left(\sqrt{-P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_s,1+\epsilon)^{c_s-2}}\right)(x_s)}\right) \geq \ln\left(\frac{1}{\sqrt{A_{s-1}(1+\epsilon)}}\right) + C^*_{s-1} -$$

$$\sqrt{(1+\epsilon)\bar{\rho}_s A_{s-1}((1+\epsilon))(1+\epsilon)}$$

$$\geq \ln\left(\frac{1}{U_{s-1}(1+\epsilon)}\right) + C^*_{s-1} - \sqrt{(1+\epsilon)\bar{\rho}_s}(1+\epsilon)U_{s-1}(1+$$

$$\epsilon) = G(\sqrt{\ln 2}).$$

We now give some estimates for the function $U_{s-1}(1+\epsilon)$ appearing in Corollary (6.2.30).

Corollary (6.2.32) [291]. The function U_{s-1} is non-increasing and satisfies, for $\epsilon \geq 0$,

$$U_{s-1}(1+\epsilon) \geq \frac{1}{\sqrt{(1+\epsilon)\bar{\rho}_s}(1+\epsilon) + e^{C^{**}_{s-1}}}.$$

Proof. First notice that $U_{s-1}(0) = e^{-C^{**}_{s-1}}$ and U_{s-1} is positive. Since $\Psi_{(1+\epsilon)}(U_{s-1}(1+\epsilon))$ is constant, taking derivative yields:

$$U'_{s-1}(1+\epsilon) = -\frac{\sqrt{(1+\epsilon)\bar{\rho}_s}\,U_{s-1}(1+\epsilon)}{\sqrt{(1+\epsilon)\bar{\rho}_s}(1+\epsilon) + \frac{1}{U_{s-1}(1+\epsilon)}} \geq -\sqrt{(1+\epsilon)\bar{\rho}_s}U_{s-1}(1+\epsilon)^2.$$

Therefore U_{s-1} is non-increasing and integrating the above inequality we infer that

$$U_{s-1}(1+\epsilon) \geq \frac{1}{\sqrt{(1+\epsilon)\bar{\rho}_s}(1+\epsilon) + U^{-1}_{s-1}(0)}$$

Henceforth, in the sequel, for $\epsilon \geq 0$, we set

$$A_{s-1}(1+\epsilon) = \min(U_{s-1}(1+\epsilon)^2, 1) \geq \min\left(\left(\frac{1}{\sqrt{(1+\epsilon)\bar{\rho}_s}(1+\epsilon) + e^{C^{**}_{s-1}}}\right)^2, 1\right). \tag{52}$$

There exists a constant $C_s > 0$ such that, for all $\epsilon \geq 0$,

$$\frac{C_s}{1 + \bar{\rho}_s(1+\epsilon)^3} \leq A_{s-1}(1+\epsilon) \leq 1. \tag{53}$$

A first consequence of the uniform estimate we obtained are the following lower bounds for the heat kernel. Observe, and this is another main novelty with respect that these bounds are written with respect to the distance $d_{1+\epsilon}$ (we recall that d_0 is the sub-Riemannian distance).

Corollary (6.2.33) [291]. Set $C_{s+1} = \frac{C_s}{4}$. For $\epsilon > 0$ and $x_s \in M$, then

$$p(x_s, x_s, 1 + \epsilon) \geq \frac{\left(A_{s-1}\left(\sqrt{\frac{1+\epsilon}{2}}\right)\right)^{\frac{D}{2}} \exp\left(-\frac{\bar{\rho}_s(1+\epsilon)^2}{4}\right)}{4\mu\left(B_{s-1}\left(x_{s'}\sqrt{\frac{1+\epsilon}{2}}\right)\right)}$$

$$\geq \frac{C_{s+1}}{4\mu\left(B_{s-1}\left(x_{s'}\sqrt{\frac{1+\epsilon}{2}}\right)\right)} \frac{\exp\left(-\frac{\bar{\rho}_s(1+\epsilon)^2}{4}\right)}{\left(1 + \frac{\bar{\rho}_s(1+\epsilon)^2}{2}\right)^{\frac{D}{2}}}. \quad (54)$$

As a consequence, for $x_{s'}(x_s + \epsilon) \in M, \epsilon > 0$ and $\epsilon \geq -1$,

$$p(x_{s'} x_s + \epsilon, 1 + \epsilon) \geq$$

$$\frac{\left(A_{s-1}\left(\sqrt{\frac{1+\epsilon}{2}}\right)\right)^{\frac{D}{2}} 2^{-\frac{D}{2}} \exp\left(-\frac{\bar{\rho}_s(1+\epsilon)^2}{4}\right)}{4\mu\left(B_{s-1}\left(x_{s'}\sqrt{\frac{1+\epsilon}{2}}\right)\right)} \exp\left(-\frac{d_{1+\epsilon}(x_s, x_s+\epsilon)^2}{2(1+\epsilon)}\left(\frac{D}{(1+\epsilon)} + \frac{\bar{\rho}_s}{2}(1+\epsilon) + 2(1+\epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1+\epsilon)}\ln(2)\right)\right)\right)$$

$$\geq \frac{2^{-\frac{D}{2}}C_{s+1}}{4\mu\left(B_{s-1}\left(x_{s'}\sqrt{\frac{1+\epsilon}{2}}\right)\right)} \frac{\exp\left(-\frac{\bar{\rho}_s(1+\epsilon)^2}{4}\right)}{\left(1 + \frac{\bar{\rho}_s(1+\epsilon)^2}{2}\right)^{\frac{D}{2}}} \exp\left(-\frac{d_{1+\epsilon}(x_s + \epsilon, x_s)^2}{2(1+\epsilon)}\left(\frac{D}{1+\epsilon} + \frac{\bar{\rho}_s}{2}(1+\epsilon)\right.\right.$$

$$+ 2(1$$

$$\left.\left.+ \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1+\epsilon)}\ln(2)\right)\right)\right). \quad (55)$$

Proof. With the same notations, for $\epsilon \geq 0$,

$$P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_{s'}(1+\epsilon))}\right)(x_s) \geq \frac{1}{2}.$$

Thus,

$$\frac{1}{4} \leq P_{A_{s-1}((1+\epsilon))(1+\epsilon)^2}\left(1_{B_{s-1}(x_{s'}(1+\epsilon))}\right)(x_s)^2$$

$$= \left(\int_M p(x_{s'} x_s + \epsilon, A_{s-1}(1 + \epsilon)(1 + \epsilon)^2)(1_{B_{s-1}(x_{s'}(1+\epsilon))}d\mu(x_s + \epsilon)\right)^2$$

$$\leq \int_M p(x_{s'} x_s + \epsilon, A_{s-1}((1 + \epsilon))(1 + \epsilon)^2 2d\mu(x_s + \epsilon) \int_M 1_{B_{s-1}(x_{s'}(1+\epsilon))}d\mu(x_s + \epsilon)$$

$$= p(x_{s'} x_s, 2A_{s-1}((1 + \epsilon))(1 + \epsilon)^2)\mu(B_{s-1}(x_{s'} 1 + \epsilon)).$$

Now since $0 < A_{s-1}(1 + \epsilon) \leq 1$, Harnack inequality in Corollary 1 gives

$$p(x_{s'} x_s, 2A_{s-1}((1 + \epsilon))(1 + \epsilon)^2 \leq p(x_{s'} x_s, 2(1 + \epsilon)^2)\left(A_{s-1}(1 + \epsilon)\right)^{-\frac{D}{2}}\exp\left(\frac{1}{2}\bar{\rho}_s(1 + \epsilon)^3\right). \quad (56)$$

Therefore, we proved

$$p(x_s, x_s, 2(1 + \epsilon)^2) \geq \frac{A_{s-1}(1 + \epsilon)^{\frac{D}{2}} \exp\left(-\frac{1}{2}\bar{\rho}_s(1 + \epsilon)^3\right)}{4 \, \mu(B_{s-1}(x_s, (1 + \epsilon)))}. \tag{57}$$

The first point follows by setting $\epsilon = -1/2$.

For the second point, we recall that Harnack inequality with the distance $d_{1+\epsilon}$ reads, for $\epsilon > -1, s = \frac{1+\epsilon}{2}$,

$$P\left(x_s, x_s, \frac{1 + \epsilon}{2}\right) \leq$$

$$p(x_s, x_s + \epsilon, 1 + \epsilon)2^{\frac{D}{2}} \exp\left(\frac{\bar{\rho}_s(1 + \epsilon)^2}{8}\right) \exp\left(-\frac{d_{1+\epsilon}(x_s + \epsilon, x_s)^2}{2(1 + \epsilon)}\left(\frac{D}{1 + \epsilon} + \frac{\bar{\rho}_s}{2}(1 + \epsilon) + 2(1\right.\right.$$

$$\left.\left. + \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1 + \epsilon)}\ln(2)\right)\right)\right)$$

Using the first point the conclusion follows directly.

Corollary (6.2.34) [291]. There exist constants $C_{s+2} > 0$ and $C_{s+3} > 0$ depending only on $1+\epsilon$ and ρ_{s+1} such that for all $x_s \in M$ and all $\epsilon \geq 0$, we have

$$\mu\big(B_{s-1}(x_s, 2(1 + \epsilon))\big) \leq C_{s+2} \exp\big(C_{s+3}\bar{\rho}_s(1 + $$

$\epsilon)^2\big) \, \mu(B_{s-1}(x_s, 1 + \epsilon))$ \hfill (58)

Proof. We have for $\epsilon \geq 0$

$$\mu(B_{s-1}(x_s, 2(1 + \epsilon))) \leq C_{s-2}(1 + \epsilon, 1 + \epsilon, \rho_s)2^{D/2}\frac{\exp(12\bar{\rho}_s(1 + \epsilon)^2)}{p(x_s, x_s, 2(1 + \epsilon)^2)}$$

Combining this inequality with the lower bound for the heat kernel gives the desired result:
$$\mu(B_{s-1}(x_s, 2(1 + \epsilon)))$$

$$\leq C_{s-2}(1 + \epsilon, 1 + \epsilon, \rho_{s+1})2^{\frac{D}{2}+2} \exp\left(\frac{25}{2}\bar{\rho}_s(1 + \epsilon)^2\right)A_{s-1}(1 + \epsilon)^{-D/2}\mu(B_{s-1}(x_s, 1$$

$$+ \epsilon)).$$

Corollary (6.2.35) [291]. For all $\epsilon > 0$, there exist some constants $C_{s+4}, C_{s+5} > 0$, depending only on $1 + \epsilon, \rho_{s+1}, 1 + \epsilon \; and \; \varepsilon > 0$, such that for $\epsilon > -1$ and $x_s, (x_s + \epsilon) \in M$, one has

$$p(x_s, x_s + \epsilon, 1 + \epsilon) \leq \frac{C_{s+4}(\varepsilon)}{\mu(B_{s-1}(x_s, \sqrt{1 + \epsilon}))^{1/2}\mu(B_{s-1}(x_s + \epsilon, \sqrt{1 + \epsilon}))^{1/2}} \, exp(C_{s+5}(\varepsilon)\bar{\rho}_s(1 + \epsilon)) \exp\left(-\frac{d^2(x_s, x_s + \epsilon)}{(4 + \varepsilon)(1 + \epsilon)}\right) \tag{59}$$

Proof. This proof follows the lines in Cao-Yau [182] and Baudoin-Garofalo [178]. Let $\epsilon > 0$, then by the Harnack inequality with $\epsilon = -1$,

$$p(x_s, x_s + \epsilon, 1 + \epsilon)^2$$

$$\leq \frac{(2+\epsilon)^D \exp\left(\frac{D}{2(1+\epsilon)^2}\right)}{\mu(B_{s-1}(x_s+\epsilon,\sqrt{1+\epsilon}))} \exp\left(\bar{\rho}_s(1+\epsilon)\left(\frac{(3+\epsilon)}{6(1+\epsilon)}+\frac{(1+\epsilon)^2}{2}\right)\right)\int_{B_{s-1}(x_s+\epsilon,\sqrt{1+\epsilon})} p\,(x_s,x_s$$
$$+2\epsilon,(2+\epsilon)(1+\epsilon))^2 d\mu(x_s+2\epsilon)$$
$$= \frac{(2+\epsilon)^D \exp\left(\frac{D}{2(1+\epsilon)^2}\right)}{\mu\left(B_{s-1}\left(x_s+\epsilon,\sqrt{1+\epsilon}\right)\right)} \exp\left(\bar{\rho}_s(1+\epsilon)\left(\frac{3+\epsilon}{6(1+\epsilon)}+\frac{(1+\epsilon)^2}{2}\right)\right) P_{(2+\epsilon)(1+\epsilon)}\big(p(x_s,$$
$$\cdot,(2+\epsilon)(1+\epsilon))\mathbf{1}_{B(x_s+\epsilon,\sqrt{1+\epsilon})}\big)(x_s).$$

Applying the Harnack inequality in Corollary (6.2.29) once again, we have
$$P_{(2+\epsilon)(1+\epsilon)}\big(p_{(2+\epsilon)(1+\epsilon)}(x_s,\cdot)\mathbf{1}_{B_{s-1}(x_s+\epsilon,\sqrt{1+\epsilon})}\big)(x_s)^2 \;=\; P_{(2+\epsilon)(1+\epsilon)}(F_{1+\epsilon})(x_s)^2$$
$$\leq \frac{(2+\epsilon)^D \exp\left(\frac{D}{2(1+\epsilon)^2(2+\epsilon)}\right)}{\mu(B(x_s,\sqrt{1+\epsilon}))}\exp\Big(\bar{\rho}_s(1$$
$$+\epsilon)\left(\frac{3+\epsilon}{6(1+\epsilon)}+\frac{(1+\epsilon)^2((2+\epsilon))}{2}\right)\Big)\int_{B_{s-1}(x_s,\sqrt{1+\epsilon})} P_{(2+\epsilon)^2(1+\epsilon)}(F_{1+\epsilon})(x_s+2\epsilon)^2 d\mu(x_s+2\epsilon)$$

with $F_{1+\epsilon}(\cdot) \;=\; p(x_s,\cdot,(2+\epsilon)(1+\epsilon))\mathbf{1}_{B(x_s+\epsilon,\sqrt{1+\epsilon})}(\cdot).$

By using now an argument of the proof of Theorem 8.1 in [178] and the fact that for $(x_s+2\epsilon)\in B_{s-1}(x_s+\epsilon,\sqrt{1+\epsilon})$, $d^2(x_s,x_s+2\epsilon)\geq \frac{d^2(x_s,x_s+\epsilon)}{2+\epsilon}-1$,
we have for $0<(2+\epsilon)^2(1+\epsilon)<T$,

$$\int_{B_{s-1}(x_s,\sqrt{1+\epsilon})} P_{(2+\epsilon)^2(1+\epsilon)}(F_{1+\epsilon})(x_s+2\epsilon)^2 d\mu(x_s+2\epsilon)$$
$$\leq \exp\left(\frac{1+\epsilon}{2(T-(2+\epsilon)^2(1+\epsilon))}\right) e^{(f-\epsilon)(2+\epsilon)^2(1+\epsilon),(x_s+2\epsilon)} P_{(2+\epsilon)^2(1+\epsilon)}(F_{1+\epsilon})(x_s$$
$$+2\epsilon)^2 d\mu(x_s+2\epsilon)$$
$$= \exp\left(\frac{1+\epsilon}{2(T-(2+\epsilon)^2(1+\epsilon))}\right) e^{(f-\epsilon)(0,x_s+2\epsilon)}(F_{1+\epsilon})(x_s+2\epsilon)^2 d\mu(x_s+2\epsilon)$$
$$\leq \exp\left(\frac{1+\epsilon}{2(T-(2+\epsilon)^2(1+\epsilon))}\right)\int_{B_{s-1}(x_s,\sqrt{1+\epsilon})} \exp\left(-\frac{d^2(x_s,x_s+2\epsilon)}{2T}\right) P_{(2+\epsilon)(1+\epsilon)}(x_s,x_s$$
$$+2\epsilon)^2 d\mu(x_s+2\epsilon)$$
$$\leq \exp\Big(\frac{1+\epsilon}{2(T-(2+\epsilon)^2(1+\epsilon))}+\frac{1}{2T}$$
$$-\frac{d^2(x_s,x_s+\epsilon)}{2(2+\epsilon)T}\Big)\int_{B_{s-1}(x_s+\epsilon,\sqrt{1+\epsilon})} P_{(2+\epsilon)(1+\epsilon)}(x_s,x_s+2\epsilon)^2 d\mu(x_s+2\epsilon)$$
$$= \exp\Big(\frac{1+\epsilon}{2(T-(2+\epsilon)^2(1+\epsilon))}+\frac{1}{2T}-\frac{d^2(x_s,x_s+\epsilon)}{2(2+\epsilon)T}\Big) P_{(2+\epsilon)(1+\epsilon)}(F_{1+\epsilon})(x_s)$$

where for $0 \le s < T$ and $(x_s + 2\epsilon) \in M$, the function $(f - \epsilon)$ is defined by

$$(f - \epsilon)(s, x_s + 2\epsilon) = -\frac{d^2(x_s, x_s + 2\epsilon)}{2(T - s)}.$$

Finally,

$p(x_s, x_s + \epsilon, 1 + \epsilon)$

$$\le \frac{(2 + \epsilon)^D \exp\left(\frac{D(\epsilon + 3)}{4(1 + \epsilon)^2(2 + \epsilon)}\right)}{\mu(B_{s-1}(x_s, \sqrt{1 + \epsilon}))^{1/2}\mu(B_{s-1}(x_s + \epsilon, \sqrt{1 + \epsilon}))^{1/2}} \exp\left(\bar{\rho}_s(3 + \epsilon)\left(\frac{1}{6(1 + \epsilon)} + \frac{(2 + \epsilon)^2}{4}\right)\right)$$

$$\exp\left(\frac{1 + \epsilon}{4(T - (2 + \epsilon)^2(1 + \epsilon))} + \frac{1}{4T} - \frac{d^2(x_s, x_s + \epsilon)}{4(2 + \epsilon)T}\right).$$

Hence the result follows by choosing $T = (2+\epsilon)^3(1 + \epsilon)$.

Corollary (6.2.36) [291]. There exists a constant $C_{s+6} > 0$ which depends only on $1 + \epsilon$ and ρ_{s+1} such that for all x_s and $x_s + \epsilon$ in M and all $-1 < \epsilon \le 0$,

$$d(x_s, x_s + \epsilon) \le C_{s+6}(1 + \sqrt{\bar{\rho}_s}) \max\{\sqrt{d_{1+\epsilon}(x_s, x_s + \epsilon)}, d_{1+\epsilon}(x_s, x_s + \epsilon)\}.$$

Proof. Using the symmetry of the heat kernel, combining the lower estimate (55) for the heat kernel with the distance $d_{1+\epsilon}$ and the upper estimate (59) for the sequance of sub-elliptic distance d gives

$$\frac{\left(A_{s-1}\left(\sqrt{\frac{1 + \epsilon}{2}}\right)\right)^{\frac{D}{2}} 2^{-\frac{D}{2}} \exp\left(-\frac{\bar{\rho}_s(1 + \epsilon)^2}{4}\right)}{4\mu(B_{s-1}(x_s, \frac{\sqrt{1 + \epsilon}}{2}))^{1/2} 4\mu(B_{s-1}(x_s + \epsilon, \frac{\sqrt{1 + \epsilon}}{2}))^{1/2}} \exp\left(-\frac{d_{1+\epsilon}(x_s, x_s + \epsilon)^2}{2(1 + \epsilon)}\left(\frac{D}{1 + \epsilon} + \frac{\bar{\rho}_s}{2}(1 + \epsilon)\right.\right.$$

$$\left.\left. + 2(1 + \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1 + \epsilon)} \ln(2)\right)\right)\right)$$

$$\le \frac{C_{s+4}(\varepsilon)}{\mu(B_{s-1}(x_s, \sqrt{1 + \epsilon}))^{1/2}\mu(B_{s-1}(x_s + \epsilon, \sqrt{1 + \epsilon}))^{1/2}} \exp(C_{s+5}(\varepsilon)\bar{\rho}_s(1 + \epsilon)) \exp\left(-\frac{d^2(x_s, x_s + \epsilon)}{(4 + \varepsilon)(1 + \epsilon)}\right)$$

Therefore we have

$$A_{s-1}\left(\frac{\sqrt{1 + \epsilon}}{2}\right)^{\frac{D}{2}} 2^{-\frac{D}{2}-2} \exp\left(-\frac{\bar{\rho}_s(1 + \epsilon)^2}{4}\right) \exp\left(-\frac{d_{1+\epsilon}(x_s, x_s + \epsilon)^2}{2(1 + \epsilon)}\left(\frac{D}{1 + \epsilon} + \frac{\bar{\rho}_s}{2}(1 + \epsilon) + 2(1\right.\right.$$

$$\left.\left. + \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1 + \epsilon)} \ln(2)\right)\right)\right)$$

$$\le C_{s+4}(\varepsilon) \exp(C_{s+5}(\varepsilon)\bar{\rho}_s(1 + \epsilon)) \exp\left(-\frac{d^2(x_s, x_s + \epsilon)}{(4 + \varepsilon)(1 + \epsilon)}\right).$$

Thus for all $\epsilon > -1$:

$$0 \le -\frac{D}{2} \ln A_{s-1}\left(\frac{\sqrt{1 + \epsilon}}{2}\right) + \left(\frac{D}{2} + 2\right) \ln 2 + \ln C_{s+4}(\varepsilon) + C_{s+5}(\varepsilon)(1 + \bar{\rho}_s)(1 + \epsilon) + \frac{\bar{\rho}_s(1 + \epsilon)^2}{4}$$

$$-\frac{d^2(x_s, x_s + \epsilon)}{(4 + \varepsilon)(1 + \epsilon)} + \frac{d_{1+\epsilon}(x_s, x_s + \epsilon)^2}{2(1 + \epsilon)}\left(\frac{D}{1 + \epsilon} + \frac{\bar{\rho}_s}{2}(1 + \epsilon) + 2(1 + \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1 + \epsilon)}\ln(2)\right)\right).$$

Since $-\ln A_{s-1}\left(\frac{\sqrt{1+\epsilon}}{2}\right) \leq \ln\left(1 + \frac{\bar{\rho}_s(1+\epsilon)^2}{4}\right) - \ln C_s \leq \frac{\bar{\rho}_s(1+\epsilon)^2}{4} - \ln C_s$, fixing $\epsilon = 1$, there exist some constants E_s, E_{s+1} which only depend on $1 + \epsilon$ and ρ_{s+1} such that for all $\epsilon > -1$, we have for all $x_s, (x_s + \epsilon) \in M, \epsilon > -1$.

$$0 \leq E_s + E_{s+1}\bar{\rho}_s(1 + \epsilon) - \frac{d^2(x_s, x_s + \epsilon)}{(4 + \varepsilon)(1 + \epsilon)}$$
$$+ \frac{d_{\epsilon+1}(x_s, x_s + \epsilon)^2}{2(1 + \epsilon)}\left(\frac{D}{1 + \epsilon} + \frac{\bar{\rho}_s}{2}(1 + \epsilon) + 2(1 + \epsilon)\left(\frac{\bar{\rho}_s}{\rho_{s+1}} + \frac{3D}{2\rho_{s+1}(1 + \epsilon)}\ln(2)\right)\right).$$

Therefore, for some positive constants $(A_{s-1})_i, 1 \leq i \leq 3$ which only depend on $1 + \epsilon$ and ρ_{s+1},

$d(x_s, x_s + \epsilon)^2 \leq A_s(1 + \bar{\rho}_s(1 + \epsilon))(1 + \epsilon) + A_{s+1}(1 + \bar{\rho}_s(1 + \epsilon))d_{\epsilon+1}(x_s, x_s + \epsilon)^2 + A_{s+2}(1 + \bar{\rho}_s)(1 + \epsilon)d_{\epsilon+1}(x_s, x_s + \epsilon)^2$.

Since $\epsilon \leq 0$, if $d_{\epsilon+1}(x_s, x_s + \epsilon) \leq 1$, choosing $1 = d_{1+\epsilon}(x_s, x_s + \epsilon) \leq 1$ yields
$$d(x_s, x_s + \epsilon)^2 \leq (1 + \bar{\rho}_s(1 + \epsilon))(A_s + A_{s+2})(1 + \epsilon)d_{1+\epsilon}(x_s, x_s + \epsilon) + A_{s+1}d_{1+\epsilon}(x_s, x_s + \epsilon)^2$$
$$\leq (A_s + A_{s+1} + A_{s+2})(1 + \bar{\rho}_s(1 + \epsilon))d_{1+\epsilon}(x_s, x_s + \epsilon).$$
If $d_{1+\epsilon}(x_s, x_s + \epsilon) \geq 1$, choosing $\epsilon \leq 0$, we infer
$$d(x_s, x_s + \epsilon)^2$$
$$\leq (1 + \bar{\rho}_s(1 + \epsilon))A_s(1 + \epsilon) + A_{s+1}d_{1+\epsilon}(x_s, x_s + \epsilon)^2 + A_{s+2}(1 + \epsilon)d_{1+\epsilon}(x_s, x_s + \epsilon)^2 \leq (A_s + A_{s+1} + A_{s+2})(1 + \bar{\rho}_s(1 + \epsilon))d_{1+\epsilon}(x_s, x_s + \epsilon)^2.$$

Corollary (6.2.37) [291]. Assume the operator L satisfies the condition $C_{s-2}D(\rho_s, \rho_{s+1}, 2 + \epsilon, 1 + \epsilon)$ for some $\rho_s \in R, \rho_{s+1} > 0, \epsilon \geq -1$, and $\epsilon \geq 0$. Assume moreover that the metric associated to $d_{1+\epsilon}$ is a Riemannian metric $g_{1+\epsilon}$ on M for some $(1 + \epsilon)$. Let Ω be a bounded domain in M and a chart $\varphi: U_{s-1} \subset R^m \to \Omega$. Then there exist some constants $c_{s-2} = c_{s-2}(\Omega, \varphi) > 0$ and $C_{s-2} = C_{s-2}(\Omega, \varphi) > 0$ such that
$$\|Lu_{s-1}\| + C_{s-2}\|u_{s-1}\| \geq c_{s-2}\|u_{s-1}\|_{(1)}, u_{s-1} \in C_0^\infty(\varphi(U_{s-1})), \tag{60}$$
where $\|\cdot\|$ denotes the usual $L^2(\mu)$ norm in Ω and where $\|\cdot\|_{(s)}$ is the standard Sobolev norm.

Proof. there exists a constant A_{s-1} such that $d(x_s, x_s + \epsilon) \leq A_{s-1}\sqrt{d_{1+\epsilon}(x_s, x_s + \epsilon)}$ for all $x_s, (x_s + \epsilon) \in \Omega$. Therefore, $B_{1+\epsilon}(x_s, 1 + \epsilon) \subset B_{s-1}(x_s, A_{s-1}\sqrt{1 + \epsilon})$. Since $\bar{\Omega}$ is a compact set, the metric $g_{1+\epsilon}$ is comparable with g_{eucl} the metric obtained from the Euclidean one in U_{s-1} by the map φ. If we pull back the result in U_{s-1}, we thus have, $(B_{s-1})_{eucl}(x_s, R) \subset B_{s-1}(x_s, A'_{s-1}\sqrt{1 + \epsilon})$ for some constant $A'_{s-1} > 0$. The result then follows from Theorem 1 in [185].

We call a Hormander type operator an operator L which satisfies the general assumptions and which can be written locally as $L = \sum_{i=1}^{1+\epsilon} X_j^* X_j$ for some C^∞ vector fields X_j.

We say it is an operator of rank k if the vector fields and their commutators up to order k:

$$X_1, \ldots, X_{1+\epsilon}, [X_{i_1}, X_{i_2}], \ldots, [X_{i_1}, [X_{i_2}, [\ldots, X_{i_k}]] \ldots], i_j = 1 \ldots 1 + \epsilon$$

span the tangent space in each point of M.

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