Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

SUST
Journal of natural and Medical Sciences

Journal homepage: http://journals.sustech.edu/

Verification of the Sharp Inequalities for Higher order Riesz-Laguerre Transforms

Shawgy Hussein*[1] and Ammar Elobied[2]

1-Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan.
2-Sinnar University, Faculty of Education, Dept of Physics and Mathematics, Sudan.

*Corresponding Auther: Shawgy Hussein, Email address: Shawgy2020@gmail.com

ABSTRACT:

The weak-type (1, 1) boundedness of the higher order Riesz–Laguerre transforms associated with the Laguerre polynomials and the boundedness for the Riesz-Laguerre transforms of order 2 are considered. We show the sharp polynomial weight w that makes the Riesz–Laguerre transforms of order greater than or equal to 2 continuous from $L^1(wd\mu_\alpha)$ into $L^{1,\infty}(d\mu_\alpha)$,under specific value α ,where μ_α is the Laguerre measure.

المستخلص

تم اعتبار محدودية النوع الضعيف (1,1) لتحويلات ريس–لاقري المشاركة طبقا لكثيرات حدود لاقري و المحدودية لاجل تحويلات ريس–لاقري من رتبة 2.تم توضيح مرجحة كثيرة الحدود القاطعة w التي تجعل تحويلات ريس–لاقري من رتبة اكبر من او تساوى 2 مستمرة من $L^1(wd\mu_\alpha)$ الى $L^{1,\infty}(d\mu_\alpha)$

تحت قيمة معينة α حيث μ_α قياس لاقري

KEYWORDS: *Riesz-Laguerre transform, Polynomial expansion, Weak-type, Stein complex interpolation Theorem, Calderon-Zygmund-type, Riesz-Gauss transform.*

INTRODUCTION

The aim of this paper is to study ,by considering the two components of the Claim raised by Liliana Forzani, Emanuela Sasso and Roberto Scotto[1] ,following the same notations appear there, the weak type(1,1) boundedness of $\mathcal{R}_\propto^{m+1}$, the$(m+1)$th Riesz-Laguerre transform with $m \in Z_{\geq 0}^d$ associated with the multidimensional Laguerre operator $\mathcal{L}_\propto$,where $\propto = (\propto_1, \ldots, \propto_d)$ is a multi-indexwith $\propto_i \geq 0, i = 1, \ldots, d$.

The Laguerre operator $\mathcal{L}_\propto$, is a self-adjoint "Laplacian" on $L^2(d\mu_\alpha)$,where μ_α is the Laguerre measure of type $\propto = (\propto_1, \ldots, \propto_d)$ with $\propto_i > -1, i = 1, \ldots, d$; defined on $R_+^d = \{x \in R^d : x_i > 0, \text{for each } i = 1, \ldots, d\}$, by

$$d\mu_\alpha(x) = \prod_{i=1}^d \frac{x_i^{\propto_i} e^{-x_i}}{\Gamma(\propto_i + 1)} dx.$$

It is well known that the spectral resolution of $\mathcal{L}_\propto$ is

$$\mathcal{L}_\propto = \sum_{n=0}^\infty n \mathcal{P}_n^\propto,$$

where $\mathcal{P}_n^{\propto}$ is the orthogonal projection on the space spanned by Laguerre polynomials of total degree n and type $\propto$ in d variables [2,3]. The operator $\mathcal{L}_{\propto}$ is the infinitesimal generator of a "heat" semigroup,

called the Laguerre semigroup, $\{e^{-t\mathcal{L}_{\propto}}: t \geq 0\}$, defined in the spectral sense as

$$e^{-t\mathcal{L}_{\propto}} = \sum_{n=0}^{\infty} e^{-nt}\mathcal{P}_n^{\propto}.$$

For any multi-index $m = (a_1, \dots, a_d) \in Z_{\geq 0}^d$, the Riesz-Laguerre transforms $\mathcal{R}_{\propto}^{m+1}$ of order $|m| = a_1 + \cdots + a_d$ are defined by

$$\mathcal{R}_{\propto}^{m+1} = \nabla_{\alpha}^{m+1}(\mathcal{L}_{\alpha})^{-|m+1|/2}\mathcal{P}_0^{\propto\perp},$$

where $\nabla_{\propto}$ is associated to $\mathcal{L}_{\propto}$ defined as $\nabla_{\propto} = \left(\sqrt{x_1}\partial_{x_1}, \dots, \sqrt{x_d}\partial_{x_d}\right)$, and $\mathcal{P}_0^{\propto\perp}$, denotes the orthogonal projection onto the orthogonal complement of the eigenspace corresponding to the eigenvalue 0 of $\mathcal{L}_{\propto}$.

Some preliminaries

In order to use the well-known relationship with the Ornstein-Uhlenbeck context, but not too much exploited in the weak-type inequalities, we are going to perform a change of coordinates in R_+^d. If $x = (x_1, \dots, x_d)$ is a vector R_+^d, then x^2 will denote the vector $(x_1^2, \dots, x_d^2)$. Let $\Psi: R_+^d \to R_+^d$ be define as $\Psi(x) = x^2$ and let $d\tilde{\mu}_{\propto} = d\mu_{\propto} o\, \Psi^{-1}$ be the pull-back measure from $d\mu_{\propto}$. Then the modified Laguerre measure $d\tilde{\mu}_{\propto}$ is the probability measure

$$d\tilde{\mu}_{\propto}(x) = 2^d \prod_{i=1}^{d} \frac{x_i^{2\propto_i+1} e^{-x_i^2}}{\Gamma(\propto_i + 1)} dx$$

$$= 2^d \prod_{i=1}^{d} \frac{x_i^{2\propto_i+1}}{\Gamma(\propto_i + 1)} e^{-|x|^2} dx, \qquad (1)$$

on R_+^d.

The map $f \to \mathcal{U}_{\Psi} f = f o\, \Psi$ *is* an isometry from $L^q(d\mu_{\propto})$ onto $L^q(d\tilde{\mu}_{\propto})$ and from $L^{q,\infty}(d\mu_{\alpha})$ onto $L^{q,\infty}(d\tilde{\mu}_{\propto})$, for every q in $[1, \infty]$. So we may reduce the problem of studying the weak-type boundedness of $\mathcal{R}_{\propto}^{m+1}$ to the study of the same boundedness for the modified Riesz-Laguerre transforms $\tilde{\mathcal{R}}_{\propto}^{m+1} = \mathcal{U}_{\Psi}\mathcal{R}_{\propto}^{m+1}\mathcal{U}_{\Psi}^{-1}$ with respect to the measure $d\tilde{\mu}_{\propto}$.

Observe that $\tilde{\mathcal{R}}_{\propto}^{m+1}$ coincides, up to a multiplicative constant, with $\nabla^{m+1}\left(\tilde{\mathcal{L}}_{\propto}\right)^{-|m+1|}\tilde{\mathcal{P}}_0^{\propto\perp}$, being $\tilde{\mathcal{L}}_{\propto} = \mathcal{U}_{\Psi}\mathcal{L}_{\propto}\mathcal{U}_{\Psi}^{-1}$, $\tilde{\mathcal{P}}_0^{\propto\perp} = \mathcal{U}_{\Psi}\mathcal{P}_0^{\propto\perp}\mathcal{U}_{\Psi}^{-1}$ and ∇ the gradient of R^d associated to the Laplacian operator [4].

For the sequel it is convenient to express the kernel of $\tilde{\mathcal{R}}_{\propto}^{m+1}$ with respect to the Polynomial measure $(m+1)_{\propto}$ defined on R_+^d as

$$d(m+1)_{\propto}(x) = e^{|x|^2} d\tilde{\mu}_{\propto}(x). \qquad (2)$$

According to [5,6], for $\propto_i > -1/2, i = 1, \dots, d;$, the kernel of the modified Riesz–Laguerre transforms of order $|m+1|$ with respect to the polynomial measure $(m+1)_{\propto}$ is defined, off the diagonal, as

$$\mathcal{K}^{m+1}(x,s) = \int_{[-1,1]^d} \mathcal{K}^{m+1}(x,s) \prod\nolimits_{\alpha}(s) ds$$

with

$$\mathcal{K}^{m+1}(x,s) = \int_0^1 (\sqrt{r})^{|m+1|-2} \left(-\frac{logr}{1-r}\right)^{\frac{|m+1|-2}{2}} \prod_{i=1}^{d} H_{a_i}\left(\frac{x_i(\sqrt{r} - s_i)}{\sqrt{1-r}}\right) \frac{e^{-\frac{q-(rx^2,x^2,s)}{1-r}}}{(1-r)^{|\alpha|+d+1}} dr \quad (3)$$

where H_{a_i} is the Hermite polynomial of degree a_i and

$$q_{\pm}(x,s) = \sum_{i=1}^{d} 2x_i(1 \pm s_i),$$

$$\prod\nolimits_{\alpha}(s) = \prod_{i=1}^{d} \frac{\Gamma(\alpha_i+1)}{\Gamma\left(\alpha_i+\frac{1}{2}\right)\sqrt{\pi}} (1-s_i^2)^{\alpha_i-1/2},$$

$$\cos\theta = \cos\theta(x,s) = \frac{\sum_{i=1}^{d} x_i s_i}{|x|^2},$$

$$\sin\theta = \sin\theta(x,s) = (1-cos^2\theta)^{1/2} = \left(1 - \left(\frac{\sum_{i=1}^{d} x_i s_i}{|x|^2}\right)^2\right)^{1/2}.$$

The symbol $a \lesssim b$ means $a \le Cb$ where C is a constant that may be different on each occurrence. And we write a ~ b whenever $a \lesssim b$ and $b \lesssim a$.
We state the global region G and an upper bound for $|\mathcal{K}^{m+1}(x,s)|$ on G,
with respect to the proposed Claim 3 [1].

$$G = R_+^d \times [-1,1]^d \backslash R_0 = R_1 \cup R_2 \cup R_3 \cup R_4.$$

With

$$R_0 = \left\{(x,s) \in R_+^d \times [-1,1]^d : (2|x|^2(1-cos\theta))^{1/2} \le \frac{C}{1+|x|}\right\},$$

$$R_1 = \{(x,s) \notin R_0 : cos\theta < 0\},$$

$$R_2 = R_3 = R_4 = \{(x,s) \notin R_0 : cos\theta \ge 0, for\ any|x|\},$$

And

$$\mathcal{K}^*(x,s) = \begin{cases} e^{-|x|^2}, & (x,s) \in R_1 \quad (4) \\ |x|^{2(|\propto|+d)} e^{-C\left(|x|^4(1-cos\theta)\right)^{1/2}}, & (x,s) \in R_2 \quad (5) \\ |x|^{2(|\propto|+d)} \left(1 \wedge \frac{e^{-\frac{C|x|^4 sin^2\theta}{sin\theta|x|^2}}}{(sin\theta|x|^2)^{\frac{2(|\propto}{}}} \ 27 \right), & (x,s) \in R_3 \quad (6) \\ (1+|x|)e^{-sin^2\theta|x|^2}, & (x,s) \in R_4 \quad (7) \end{cases}$$

Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

RESULTS:

The following result was proved recently [(2)] only for half-integer that includes $\propto_i = -1$. The corresponding proof is based on the technique of transference to the Hermite setting. This technique firstly appears for the Riesz–Laguerre transform of order one in[(6)] .The method seems to be inapplicable for any other value of α [(2)] .

Theorem1:

The second order Riesz-Laguerre transforms map $L^1(d\mu_\alpha)$continuously into $L^{1,\infty}(d\mu_\alpha)$.

Proof: The result follows by splitting the modified Riesz–Laguerre transforms of second order into a local operator and a global one. Let us observe that for a simple covering Lemma, we may pass from estimates with respect to the measure $(m+1)_\propto$ on the local part R_0 to estimates with respect to the modified Laguerre measure $\tilde{\mu}_\propto$. Therefore the local operator is equivalent to T_0^{m+1} for

$|m+1| = 2$ .The global operators bounded weak type (1,1) and therefore so are the second order modified Riesz-Laguerre transforms.

From [(7)]and [(8)] ,it is known that an upper bound for $|\mathcal{K}^{m+1}(x,s)|$ on G is

$$\tilde{\mathcal{K}}^{m+1}(x,s) = \begin{cases} (|2x|^2)^{\frac{|m+1|-2}{2}} e^{-|x|^2} & ,\cos\theta < 0 \\ \frac{1}{2}(4|x|^4 \sin^2\theta)^{\frac{|m+1|-2}{4}} \left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{|\alpha|-d}{2}} (1+(4|x|^4\sin^2\theta)^{\frac{1}{4}} e^{-u_0}, & \cos\theta \geq 0 \end{cases} \tag{8}$$

With

$$u_0 = \frac{(q_+(x^2,s)q_-(x^2,s))^{1/2}}{2}$$

We improve the following properties shown by[(1)] .

Proposition 2: For $|m+1| = 2$,

$$|\mathcal{K}^{m+1}(x,s)| \leq \mathcal{K}^*(x,s)$$

Proof: In this Proposition , $|m+1| = 2$ .If cos θ < 0, it is immediate that$|\mathcal{K}^{m+1}(x,s)| \leq \mathcal{K}^*(x,s)$.

Let us then assume that $\cos\theta \geq 0$.

(i) First let us consider $|x| \lesssim |x|$.

Since $\cos\theta \geq 0$, $q_+^{1/2} \geq |x|$ and since $|x| \gtrsim |x|$, then $q_+^{1/2} \leq 2|x|$. Therefore $q_+^{1/2} \sim |x|$. On the other hand, since $q_-^{1/2} \geq \frac{c}{1+|x|}$ then$|x| \geq c$. Thus

$$|\mathcal{K}^{m+1}(x,s)| \lesssim \left[\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{\frac{|\alpha|+d}{2}} (1+ \text{[illegible] 28 [illegible]} \sin^2\theta)^{1/4}\right] e^{-u_0}$$

$$\lesssim \left[|x|^{|\alpha|+d}(1+|x|)^{|\alpha|+d} + \frac{(2|x|^2(1+\cos\theta))^{\frac{|\alpha|+d}{2}+1}}{(2|x|^2(1-\cos\theta))^{\frac{|\alpha|+d}{2}-1}}\right] e^{-u_0}$$

$$\lesssim |x|^{2(|\alpha|+d)} e^{-u_0} \lesssim |x|^{2(|\alpha|+d)} e^{-|x|^2 \sin\theta}$$

$$= \mathcal{K}^*(x,s).$$

(ii) Now let us assume for any$|x|$ and rewrite u_0 in the following way:

$$u_0 = \frac{(q_+(x^2,s)q_-(x^2,s))^{1/2}}{2}$$

$$= \frac{(q_+q_-)^{1/2}}{2}$$

$$= \frac{2\sin^2\theta|x|^2 - |x|^2}{(q_+q_-)^{\frac{1}{2}}} \qquad (9)$$

Hence

$$q_+q_- = 4|x|^4\sin^2\theta$$

and taking into account that $\sin\theta$ is non-negative, we obtain that

$$2|x|^2\sin\theta \sim |x|^2\sin\theta \geq |x|^2\sin\theta. \qquad (10)$$

Thus, from (9) together with (10) we get

$$u_0 \geq C|x|^2\sin\theta. \qquad (11)$$

We proposed the following(see[1]).

Claim 3: We choose $|x|^2 = |y|^2$and $|x|^2\sin\theta \geq 1$.

Proof: $sin\theta \geq \frac{1}{|x|^2}$, then the inequality is immediate.

Hence $|x|^2 \gtrsim |x|^2 + 1$ .This inequality is immediate when $|x| \leq 1$ by adjusting conveniently the constant C in the definition of the global zone and it is also immediate for d =1 and $|x| > 1$. Now let us assume that $d \geq 2$ and $|x| > 1$.

$$\frac{(C/2)^2}{|x|^2} \leq \frac{C^2}{(1+|x|)^2} \leq 2|x|^2(1-cos\theta)$$

$$\leq 2|x|^2\left(1-\sqrt{1-\frac{1}{|x|^4}}\right).$$

Hence

$$-2|x|\sqrt{1-\frac{1}{|x|^4}}|x| - \frac{(C/2)^2}{|x|^2} \geq 0$$

for all $|x|$, then

$$|x| \geq |x|\sqrt{1-\frac{1}{|x|^4}} + \frac{\sqrt{(C/2)^2-1}}{|x|}$$

which implies that

$$|x|^2 \geq |x|^2 + 2\sqrt{(C/2)^2-1}\sqrt{1-\frac{1}{|x|^4}} + \frac{(C/2)^2-2}{|x|^2} \gtrsim |x|^2+1\,.$$

Therefore by applying this Claim to inequality (10) we obtain that $q_+q_- \geq c$ in this context. If $|x| \leq 2|x|$, we get

$$u_0 \geq \frac{c|x|^2 \sin^2\theta}{\sin\theta} = c\,|x|^2 \sin\theta\,. \qquad (12)$$

Then

$$|\mathcal{K}^{m+1}(x,s)| \leq \left(\frac{1+cos\theta}{1-cos\theta}\right)^{\frac{|\alpha|+d}{2}} (4|x|^4\sin^2\theta)^{1/4} e^{-u_0}$$

$$\leq \left(2|x|^2(1-cos\theta)\right)^{-\left(\frac{|\alpha|+d}{2}\right)} (4|x|^4\sin^2\theta)^{1/4} e^{-u_0}$$

$$\lesssim \frac{|x|^{2(|\propto|+d)}}{[2|x|^2\sin\theta]^{\frac{2(|\propto|+d)-1}{2}}} e^{-u_0}$$

$$\lesssim \frac{|x|^{2(|\propto|+d)}}{(|x|^2\sin\theta)^{\frac{2(|\propto|+d)-1}{2}}} e^{-C|x|^2\sin\theta}.$$

To get the last inequality we have used (10) and (12). On the other hand, since $4|x|^4\sin^2\theta \geq c$ it is immediate the following inequality

$$|\mathcal{K}^{m+1}(x,s)| \lesssim |x|^{2(|\propto|+d)}.$$

Thus

$$|\mathcal{K}^{m+1}(x,s)| \lesssim K^*(x,s)\ .$$

Now $(4|x|^4\sin^2\theta)^{1/4} \leq (2|x|^2)^{1/2} \lesssim |x|$, and thus

$$\left(1+\frac{(4|x|^4\sin^2\theta)^{1/4}}{2}\right) \lesssim (1+|x|).$$

Besides $q_- \geq C_1|x|^2$ and $q_+ \leq C_2|x|^2$ therefore $\frac{q_+}{q_-} \leq C$. We get

$$u_0 \geq \frac{sin^2\theta|x|^4}{|x|^2 sin\theta} \geq \frac{sin^2\theta}{2}|x|^2 \geq |x|^2.$$

Therefore

$$|\mathcal{K}^{m+1}(x,s)| \lesssim \mathcal{K}^*(x,s)\ .$$

Now we can easily show that

$$q_+ = 2|x|^2(1+cos\theta)$$

Proposition 4: The operator $\mathcal{K}^*$ defined as

$$\mathcal{K}^*f(x) = e^{|x|^2}\int_{R_+^d}\int_{[-1,1]^d}\chi_G(x,s)\,\mathcal{K}^*(x,s)\Pi_\propto(s)|f(x)|d\tilde{\mu}_\propto(x),$$

is of weak type (1,1) with respect to the measure $\tilde{\mu}_\propto$.

Proof: The method of proof we use here [(1)] is an adaptation to our context of the techniques developed in[(9,10,11)] which allows us to get rid of the classical one called "forbidden regions technique.

The kernels(4) and (5)define strong type(1,1) operators. Indeed,

$$e^{|x|^2}\int_{R_+^d}\int_{[-1,1]^d}\chi_{R_1}(x,s)e^{-|x|^2}\,\Pi_\propto(s)|f(x)|d\tilde{\mu}_\propto(x)\le C\|f\|_1.,$$

Moreover, for semi-integer values of the parameter α, by [5]

$$|x|^{2(|\propto|+d)}e^{-C|x|^2(2(1-cos\theta))^{1/2}} \tag{13}$$

is in $L^1(d\mu_\alpha)$uniformly in x and s and so the operator is of strong type with respect to $\tilde{\mu}_\propto$ on R_2 . Finally the result for the other values of $\propto$ is obtained via the multidimensional Stein's complex interpolation Theorem. So to get the weak-type(1,1) inequality for the operator $\mathcal{K}^*$ it suffices to prove that the operators

$$S_if(x)=e^{|x|^2}\int_{R_+^d}\int_{[-1,1]^d}\chi_{R_{i+3}}(x,s)\mathcal{K}^*(x,s)\,\Pi_\propto(s)ds|f(x)|d\tilde{\mu}_\propto(x)\,,i=0,1$$

map $L^1(d\tilde{\mu}_\propto)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\propto)$. Without loss of generality, we may assume that$f\ge 0$. Fix$\lambda>0$ and let

$E_i=\{x\in R_+^d: S_if(x)>\lambda\}$,

for $i=0,1$. We must prove that $\tilde{\mu}_\propto(E_i)\le C\frac{\|f\|_1}{\lambda}$. Let r_0 and r_1 be the positive roots of the equations

$$r_0^{2(|\propto|+d)}e^{r_0^2}\|f\|_1=\lambda \; and \; r_1e^{r_1^2}\|f\|_1=\lambda\,.$$

We may observe that indeed, if $E_i\cap\{x\in R_+^d:|x|<r_i\}=\emptyset$: indeed, if $|x|<r_i$, we have

$$S_0f(x)\le|x|^{2(|\propto|+d)}e^{|x|^2}\|f\|_1<\lambda,$$
$$S_1f(x)\le|x|e^{|x|^2}\|f\|_1<\lambda.$$

Now we deduce the possible roots of the equations mentioned above specifically in the following Remark

Remark 5:(i). (a) if $r_0=r_1$then we have

$$r_0^{2|\alpha|+2d-1}=1,$$

and

$$2|\alpha|+2d-1=0,$$

which implies that

$|\alpha|=\frac{1}{2}(1-2d)$,

(b) if $r_0\ne r_1$we have the quadratic equation

$(2|\alpha|+2d)\ln r_0=\ln r_1+r_1^2-r_0^2$

we assume ,for simplicity, that $r_0=e^n$ and $r_1=e^{2n}$we can find

$$(2|\alpha|+2d)\ln e^n=\ln e^{2n}+e^{4n}-e^{2n}$$
$$(e^{2n})^2-e^{2n}+2n(1-|\alpha|-d)=0$$

so that

$$e^{2n}=\frac{1\pm\sqrt{1-8n(1-|\alpha|-d)}}{2},$$

where $n\ge 1$,we can easily find r_0.

(ii) S_0 and S_1 are monotone.

(iii)Since $\frac{|x|^{2(|\propto|+d)}}{|x|} < 1$,then $|x|^{2(|\propto|+d)} \leq |x| < C$.

On the other hand, we may take $\lambda > K\|f\|$ in[1] ,and by choosing K large enough we may assume that both r_0 and r_1 are larger that one. Hence

$$\tilde{\mu}_{\propto}\{x \in R_+^d : |x| < 2r_i\} \leq \int_{|x|< 2r_i} \prod_{j=1}^{d} x_j^{2\propto_j+1} e^{-|x|^2}\, dx \lesssim r_i^{2|\alpha|} e^{-4r_i^2} \leq C\frac{\|f\|_1}{\lambda}.$$

Thus we only need to estimate $\tilde{\mu}_{\propto}\{x \in R_+^d : r_i \leq |x| \leq 2r_i\}$.

We let E_i' denote the set of $x' \in S^{d-1}$ for which there exists a $\rho \in [r_i, 2r_i]$ with $\rho x' \in E$ .For each $x' \in E_i'$ we let $\rho(x')$ be the smallest such ρ. Observe that $sin\theta(x,s) = sin\theta(x',s) = sin\theta$.

Then $S_i f(\rho(x')x') = \lambda$, by continuity. This implies for $i = 0$ and $x' \in E_0'$,
$\lambda = S_0 f(\rho_0(x')x') =$

$$= \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_3} e^{|x|^2} x^{2(|\alpha|+d)} \left(1 \wedge \frac{e^{-\frac{c|x|^2 \sin^2\theta}{\sin\theta |x|^2}}}{(\sin\theta\, |x|^2)^{\frac{2(|\alpha|+d)-1}{2}}}\right)$$

$$\times \prod\nolimits_\alpha (s)ds\, f(x) d_{\tilde{\mu}_\alpha}(x), \quad \lesssim e^{p_0^2(\acute{x})^2} r_0^{2(|\alpha|+d)} \quad \times \prod\nolimits_\alpha (s)ds\, f(x) d_{\tilde{\mu}_\alpha}(x),$$

$$\int_{R_+^d} \int_{[-1,1]^d \{|x|\geq r_0\}} \chi(x) \left(1 \wedge \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{(|x|^2 - r_0^2 + \sin\theta r_0^2)^{\frac{2(|\alpha|+d)-1}{2}}}\right) \tag{14}$$

and for $i = 1$ and$x' \in E_1'$,

$$\lambda = S_1 f(p_i(x')x')$$

$$= \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_5}(x) e^{|x|^2} (1 + |x|) e^{-\sin^2\theta |x|^2} \prod\nolimits_\alpha (s)ds\, f(x) d_{\tilde{\mu}_\alpha}(x)$$

$$\lesssim e^{p_0^2(\acute{x})^2} r_1 \int_{R_+^d} \int_{[-1,1]^d \{|x|>r_1\}} \chi(x) e^{-c \sin^2\theta r_1^2} \prod\nolimits_\alpha (s)ds\, f(x) d_{\tilde{\mu}_\alpha}(x) \ . \tag{15}$$

Clearly, since r_0 and r_1 are greater than one, we have
$\tilde{\mu}_\alpha\{x \in E_i : r_i \leq |x| \leq 2r_i\} \leq \int_{E_i'} d\sigma(x') \int_{\rho_{i(x')}}^{2r_i} e^{-\rho^2} \rho^{2(|\propto|+d)-1} d\rho$

$$\lesssim \int_{E_i'} e^{-\rho_i (x')^2} r_i^{2(|\propto|+d-1)} d\sigma(x')$$

combining this estimate for $i = 0$ with (14), we get

$$\tilde{\mu}_\alpha\{x \in E_i : r_i \leq |x| \leq 2r_i\} \leq \frac{C}{\lambda} \int_{\acute{E}_0} r_0^{2((2|\propto|+2d)-1)} d\sigma(\acute{x}) \, (I_0 + II_0), \qquad (16)$$

with

$$I_0 = \int_{[-1,1]^d} \int_{\{\sin\theta r_0^2 \leq c\}} \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{(|x|^2 - r_0^2 + \sin\theta r_0^2)^{\frac{2(|\propto|+d)-1}{2}}} f(x) d_{\tilde{\mu}_\alpha}(x) \prod_\alpha (s) ds \,,$$

and

$$II_0 = \int_{[-1,1]^d} \int_{\{|x| \geq r_1 \sin\theta r_0^2 \leq c\}} f(x) d_{\tilde{\mu}_\alpha}(x) \prod_\alpha (s) ds$$

Similarly for $i = 1$ with (15), we obtain

$$\tilde{\mu}_\alpha\{x \in E_1 : r_1 \leq |x| \leq 2r_1\} \leq \frac{C}{\lambda} \int_{E_1'} r_1^{2(|\propto|+d)-1} d\sigma(\acute{x}) \, (I_1 + II_1), \qquad (17)$$

with

$$I_1 = \int_{[-1,1]^d} \int_{\{|x| \geq r_1 \sin\theta r_1^2 \geq c\}} e^{-c\sin^2\theta r_1^2} f(x) d_{\tilde{\mu}_\alpha}(x) \prod_\alpha (s) ds,$$

and

$$II_1 = \int_{[-1,1]^d} \int_{\{\sin\theta r_1^2 \leq c\}} f(x) d_{\tilde{\mu}_\alpha}(x) \prod_\alpha (s) ds \,.$$

It is immediate to verify that

$$r_0^{2(2(|\propto|+d)-1)} \int_{[-1,1]^d} \int_{\{\acute{x}: \sin\theta r_0^2 \leq c\}} \sigma(\acute{x}) \prod_\alpha (s) ds \leq C$$

and

$$r_1^{2(|\propto|+d)-1} \int_{[-1,1]^d} \int_{\{\acute{x}: \sin\theta r_0^2 \leq c\}} d\sigma(\acute{x}) \prod_\alpha (s) ds \leq C$$

Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

Which give, after changing the order of integration in (16) and(17), the desired estimate for the terms involving II_0 and II_1 , respectively as in[4].Now let us prove that for $|x| \geq r_0$

$$r_0^{2(2(|\propto|+d)-1)} \int_{[-1,1]^d} \int_{\{\acute{x}:\, \sin\theta r_0^2 \leq c\}} \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{(|x|^2 - r_0^2 + \sin\theta r_0^2)^{\frac{2(|\propto|+d)-1}{2}}} d\sigma(\acute{x}) \prod_\alpha (s) ds \leq C$$

and for $|x| > r_1$

$$r_1^{2(|\propto|+d)-1} \int_{[-1,1]^d} \int_{\{\acute{x}:\, \sin\theta r_1^2 \geq c\}} e^{-c \sin^2\theta r_1^2} d\sigma(\acute{x}) \prod_\alpha (s) ds \leq C\,.$$

Firstly, one considers the case where $\propto = \left(\frac{n_1}{2}, -1 \ldots, \frac{n_d}{2} - 1\right)$with $n_i \in N$ and $n_i > 1$ for each $i = 1, \ldots, d$.In this case the inner integrals can be interpreted as integrals over $S^{|n|-1}$ with respect to the Lebesgue measure, expressed in poly radial coordinates in[10] .The same estimates are obtained also for $\propto \in \frac{N^d}{2} - 1 + iR^d$. Finally the result for the other values of $\propto$ are obtained via the multidimensional Stein's complex interpolation Theorem. Indeed, let $F: C^d \to C$ the function defined by

$$F(\xi) = r_0^{2(2\xi+2d-1)} \int_{\{\sin\theta r_0^2 \leq c\}} \frac{e^{-\frac{c r_0^4 \sin^2\theta}{|x|^2 - r_0^2 + \sin\theta r_0^2}}}{(|x|^2 - r_0^2 + \sin\theta r_0^2)^{\frac{2(|\xi|+d)-1}{2}}} \prod_\xi (s) ds.$$

We have seen that $\left|F\left(\frac{n}{2} - 1\right)\right| \leq C$ and i 34 ;y to prove that $\left|F\left(\frac{n}{2} - 1 + i\zeta\right)\right| \leq \left|F\left(\frac{n}{2} - 1\right)\right|$, whenever n is a integer vector and $\zeta \in R^d$.

Proposition 6: For all m, the operator

$$T_0^{m+1} f(x) = p.v. \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_0}(x,s)\, \mathcal{K}^{m+1}(x,s) \Pi_\propto(s) ds f(x) d(m+1)_\propto(x),$$

which is the modified Riesz–Laguerre transform restricted to the local region R_0 ,
is of weak type (1,1) with respect to the measure $\tilde{\mu}_\propto$

Proof: The proof of this result follows the same steps like the proof of the weak-type boundedness on the local zone of the first order Riesz–Laguerre transforms done in[3,8] .For the former we have the Calderon–Zygmund-type estimates for the kernel K^{m+1}.

Lemma 7: There exists a constant C such that

$$|\mathcal{K}^{m+1}(x,s)| \varphi(x,s) \leq C(2|x|^2(1 - cos\theta))^{-(|\propto|+d)},$$

$$\left|\nabla_{(x,x)}\left(\mathcal{K}^{m+1}(x,s)\varphi(x,s)\right)\right| \leq C(2|x|^2(1 - cos\theta))^{-(|\propto|+d+1/2)}$$

being $\varphi(x,s)$ a cut-off function defined in[2] and $(x,s) \in R_0$.

Proof. Since $\left|x_i(\sqrt{r}-s_i)\right| \le q_-^{1/2}(rx^2,x^2,s)$,

Then

$$\left|\prod_{i=1}^{d} H_{a_i}\left(\frac{x_i(\sqrt{r}-s_i)}{\sqrt{1-r}}\right)\right| e^{-\frac{q_-(rx^2,x^2,s)}{1-r}}$$

$$\lesssim \sum_{k=o}^{|m+1|}\left(\frac{q_-^{1/2}(rx^2,x^2,s)}{1-r}\right)^{k/2} e^{-\frac{q_-(rx^2,x^2,s)}{1-r}} \lesssim e^{-\frac{q_-(rx^2,x^2,s)}{2(1-r)}} \lesssim e^{-c\frac{q_-(rx^2,x^2,s)}{1-r}}$$

where last inequality follows from this one:
$q_-(rx^2,x^2,s) \ge (2|x|^2(1-cos\theta))^{1/2} - 2C(1-r^{1/2})$
when $(x,s) \in R_0$ in[5].Thus on R_0

$$|\mathcal{K}^{m+1}(x,s)|\varphi(x,s) \lesssim \int_{1/2}^{1}(\sqrt{r})^{|m+1|-1}\left(-\frac{log\, r}{1-r}\right)^{\frac{|m+1|-2}{2}} \frac{e^{-c\frac{q_-(rx^2,x^2,s)}{1-r}}}{(1-r)^{|\propto|+d+1}}dr$$

$$\lesssim \int_{0}^{1/2}(\sqrt{r})^{|m+1|-2}(-log\, r)^{\frac{|m+1|-2}{2}}dr + \int_{1/2}^{1}\frac{e^{-c\frac{q_-(rx^2,x^2,s)}{1-r}}}{(1-r)^{|\propto|+d+1}}dr$$

$$\lesssim 1 + (2|x|^2(1-cos\theta))^{-(|\propto|+d)}.$$

In computing the gradient of the kernel with respect to x we are going to have integrals such as $\mathcal{K}^{m+1}(x,s)\partial_{x_j}(x,s)$,

$$\int_{0}^{1}(\sqrt{r})^{|m+1|-1}\left(-\frac{\log r}{1-r}\right)^{\frac{|m+1|-2}{2}}\prod_{i\neq j} H_{a_i}\left(\frac{x_i(\sqrt{r}-s_i)}{\sqrt{1-r}}\right)\times H_{a_i-1}\left(\frac{x_i(\sqrt{r}-s_i)}{\sqrt{1-r}}\right)\frac{e^{-\frac{q_-(rx^2,x^2,s)}{1-r}}}{(1-r)^{|\propto|+d+3/2}}dr,$$

The gradient with respect to x is treated similarly.
For the latter we have the following Theorem regarding the $L^p d\tilde{\mu}_\propto$-boundedness for $1 < p < \infty$ of the modified Riesz–Laguerre transform of any order on G.

Theorem 8: The operator
$\mathcal{R}_g^{m+1}f(x) = \int_{R_+^d}\int_{[-1,1]^d}\chi_G(x,s)\mathcal{K}^{m+1}(x,s)\,\Pi_\propto(s)ds f(x)d\mu_\propto(x)$

Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

Proof :

The proof of this result in [1]is an adaptation to our context of the same result for the higher order Riesz–Gauss transforms done in[5] .Taking into account that on G, $q_{+}(x^2,s)\, q_{-}(x^2,s) \geq c$ when $cos\theta \geq 0$,an upper bound for$|\mathcal{K}^{m+1}(x,s)|$ is

$$\tilde{\tilde{\mathcal{K}}}^{m+1}(x,s) = \begin{cases} (2|x|^2)^{\frac{|m+1|-2}{2}} e^{-|x|^2} & \text{if} \cos\theta < 0 \\ (2|x|^2(1+cos\theta))^{|\alpha|+d}(2|x|^2 \sin\theta)^{\frac{|m+1|-1}{2}} e^{-|x|^2 sin\theta} & \text{if} \cos\theta \geq 0 \end{cases}$$

Thus

$$\int_{R_+^d} \left| \int_{G\cap\{cos\theta<0\}} \mathcal{K}^{m+1}(x,s)\Pi_{\propto}(s)ds f(x) d\mu_{\propto}(x) \right|^p d\tilde{\mu}_{\propto}$$

$$\lesssim \int_{R_+^d} \left(\int_{G\cap\{cos\theta<0\}} \tilde{\tilde{\mathcal{K}}}^{m+1}(x,s)\Pi_{\propto}(s)ds |f(x)| d\mu_{\propto}(x) \right)^p d\tilde{\mu}_{\propto}$$

$$\lesssim \int_{R_+^d} \left(\int_{R_+^d} (2|x|^2)^{\frac{p'(|m+1|-2)}{2}} d\tilde{\mu}_{\propto} \right)^{p-1} d\tilde{\mu}_{\propto}(x) \|f\|^p_{L^p(d\tilde{\mu}_{\propto})}.$$

For the region $G \cap \{cos\theta \geq 0\}$ we are going to use the following estimates:

$$2x^2 \leq q_+ \leq |2x|^2, q_- \geq 0, (q_+q_-)^{\frac{1}{2}} \geq 0,$$

$$0 \leq \left|\frac{1}{p} - \frac{1}{2}\right| |x|^2 sin\theta < |x|^2 sin\theta \quad , since \;\; p > 1,$$

$$\int_{R_+^d} \left| \int_{G\cap\{cos\theta<0\}} \mathcal{K}^{m+1}(x,s)\Pi_{\propto}(s)ds f(x) d\mu_{\propto}(x) \right|^p d\tilde{\mu}_{\propto}$$

$$\lesssim \int_{R_+^d} \left(\int_{G\cap\{cos\theta<0\}} \tilde{\tilde{\mathcal{K}}}^{m+1}(x,s)\Pi_{\propto}(s)ds |f(x)| d\mu_{\propto}(x) \right)^p d\tilde{\mu}_{\propto}$$

$$\lesssim \int_{R_+^d} \left(\int_{G\cap\{cos\theta<0\}} |2x|^{2(|\propto|+d)} (2|x|^2 \sin\theta)^{\frac{(m+1)-1}{2}} e^{-\frac{2|x|^2 sin\theta}{2}} \Pi_{\propto}(s)ds \times |f(x)| e^{-\frac{|x|^2}{p}} d\mu_{\propto}(x) \right)^p d\mu_{\propto}(x)$$

$$\lesssim \int_{R_+^d} \left(\int_{G\cap\{cos\theta<0\}} |2x|^{2(|\propto|+d)} (2|x|^2 \sin\theta)^{\frac{(m+1)-1}{2}} e^{-\left(\frac{1}{2} - \left|\frac{1}{p}-\frac{1}{2}\right|\right)|x|^2 sin\theta} \Pi_{\propto}(s)ds \times |f(x)| e^{-\frac{|x|^2}{p}} d\mu_{\propto}(x) \right)^p d\mu_{\propto}(x)$$

Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

$\lesssim$

$$\int_{R_+^d}\left(\int_{G\cap\{cos\theta<0\}}|2x|^{2(|\propto|+d)}(2|x|^2\sin\theta)^{\frac{(m+1)-1}{2}}e^{-c|x|^2 sin\theta}\Pi_\propto(s)ds\times |f(x)|e^{-\frac{|x|^2}{p}}d\mu_\propto(x)\right)^p d\mu_\propto(x) \qquad 36$$

To finish the proof we just need to check that the kernel

$$H(x,s):=|\,2x\,|^{2(|\propto|+d)}e^{-c_p 2|x|^2 sin\theta}\chi\mathfrak{G}\cap\{cos\theta\geq 0\}$$

$$for\ \mathfrak{G}=\left\{(x,s): q_-^{\frac{1}{2}}(x^2,s)\geq\frac{c}{1+2|x|}\right\} is\ inL^1\big(d(m+1)_\propto(x)\big)$$

and independently of the remaining va 34 Due to the symmetry of the kernel we are going to check only the first Claim given in[1].

$$\int_{R_+^d}H(x,s)d(m+1)_\propto(x)\lesssim$$

$$\int_{0\leq 1}|\,x\,|^{2(|\propto|+d)}e^{-c_p|x|\left(2|x|^2(1-cos\theta)\right)^{1/2}}d(m+1)_\propto(x)$$

$$+\int_{0>1}2|\,x\,|^{2(|\propto|+d)}e^{-\tilde{c}_p(2|x|)}d(m+1)_\propto(x).$$

It is clear that the second integral is bounded independently of x and s, for the first one see (13) for any x.

It is known that the first order Riesz-Laguerre transforms are weak-type (1,1). Furthermore, we also know from that the Riesz–Laguerre transforms of order higher than 2 need not be weak-type (1,1) with respect to μ_α . However, we can prove the following result that has to do with certain kind of weights we can add on the domain of these transforms to make them satisfy a weak-type inequality.

In particular, in order to exploiting the well-known relationship with the Ornstein-Uhlenbeck context, we introduce the "modified" Riesz-Laguerre transforms related to the "modified" Laguerre measure[1] .

Let us mention that in the Gaussian context something quite similar occur with the higher order Riesz-Gauss transforms. Perez proved that for $|m+1|>2$, the Riesz-Gauss transforms of order $|m+1|$ associated to the Ornstein-Uhlenbeck semigroup, map$L^1\big((1+|x|^{|m+1|-2})d\gamma\big)$ continuously into

$L^{1,\infty}(d\gamma)$,with $d\gamma(x)=e^{-|x|^2}dx$.

Regarding the weights for the Riesz– Laguerre transforms of order higher than 2 ,then[1] proved the following

Theorem 9: The Riesz–Laguerre transforms of order $|m+1|$ with $|m+1|>2$, map $L^1(wd\mu_\alpha)$ continuously into $L^{1,\infty}(d\mu_\alpha)$. where

$$w(x)=\left(1+\sqrt{|x|}\right)^{|m+1|-2}$$

Proof: As we mention in the preliminaries to prove this theorem is equivalent to prove this Theorem is equivalent to prove that the modified Riesz–Laguerre transforms of order higher than 2 map $L^1(\tilde{w}_\varepsilon d\tilde{\mu}_\propto)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\propto)$

Journal of Science and Technology vol. 14
ISSN 1858-6805
ESSN 1858-6813
Natural and Medical Sciences (NMs No.2)
mena@sustech.edu

,with $\widetilde{w}(x) = (1+|x|)^{|m+1|-2}$.For each $x\epsilon R_+^d$ Let us write

$$R_+^d \times [-1,1]^d = \bigcup_{i=0}^{4} R_i.$$

Therefore, in order to get the result, it will be enough to prove that each of the following operators

$$T_i^{m+1} f(x) = \int_{R_+^d} \int_{[-1,1]^d} \chi_{R_i}(x,s) \mathcal{K}^{m+1}(x,s)\, \Pi_{\propto}(s) ds f(x) d\mu_{\propto}(x),$$

For
$i = 0,..,4$ maps $L^1(\widetilde{w}_\varepsilon d\widetilde{\mu}_\propto)$ continuously into $L^{1,\infty}(d\widetilde{\mu}_\propto)$

Observe that for all $m+1$ the operator $T_0^{m+1}$is weak-type (1,1)with respect to $\widetilde{\mu}_\propto$ .On the other hand, for the 'global parts': R_1 , R_2 , R_3 , and R_4 , we have the following estimate for the kernel $\mathcal{K}^{m+1}$

$$|\mathcal{K}^{m+1}(x,s)| \lesssim \begin{cases} (2|x|^2)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s), & \text{if } \cos\theta < 0, \\ (2|x|^2 \sin\theta)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s), & \cos\theta \geq 0 \end{cases}$$

If $(x,s) \in R_{i1}$, $|k^{m+1}(x,s)|$ is controlled by $C(1+\{|x|\})^{|m+1|-2}\, e^{-|x|^2}$and there for it is immediate to prove that T_1^{m+1} maps $L^1(\widetilde{w}_\varepsilon d\widetilde{\mu}_\propto)$ into $L^1(d\widetilde{\mu}_\propto)$.

Now if $(x,s) \in R_i$, with $i = 2,3,4$, we Claim that

$$|\mathcal{K}^{m+1}(x,s)| \lesssim \widetilde{w}(x)\mathcal{K}^*(x,s)$$

If $(x,s) \in R_2$ since

$$q_+ \leq (2|x|)^2 \lesssim |x|^2,$$

then

$$|\mathcal{K}^{m+1}(x,s)| \lesssim (2|x|^2 \sin\theta)^{\frac{|m+1|-2}{2}} e^{-C(|x|^4(1-cos\theta))^{1/2}}$$

$$\lesssim \widetilde{w}(x) e^{-C(|x|^4(1-cos\theta))^{1/2}}.$$

Also

$$q_+ q_- = 4|x|^4 sin^2\theta.$$

Thus

$$|\mathcal{K}^{m+1}(x,s)| \lesssim (2|x|^2 \sin\theta)^{\frac{|m+1|-2}{2}} \mathcal{K}^*(x,s) \lesssim \widetilde{w}(x) \ \ \mathcal{K}^*(x,s).$$

and this concludes the proof of the Theorem.

It should be noted that there is another proof of Theorem 9 for multi-indices of half-integer type by taking f_w as the function f in[2,6] .

Now we give a sharp estimate for w that is

Corollary10: The Riesz–Laguerre transforms of order $|m+1|$ with

$|m+1| > 2,$ map $L^1(wd\mu_\alpha)$ continuously into $L^{1,\infty}(d\mu_\alpha)$. where

$$|\propto| = \frac{8n(1-d) - 1 - (2e^{2n} - 1)^2}{8n}$$

and

$$w(x) \le \left(1 + \sqrt{|C|}\right)^{|m+1|-2} = K^{|m+1|-2}$$

Proof: From Theorem 9 and Remark 5: We can directly see that

$$w(x) = \left(1 + \sqrt{|x|}\right)^{|m+1|-2}$$

$$\le \left(1 + \sqrt{|C|}\right)^{|m+1|-2}$$

$$\le K^{|m+1|-2} \text{ ,where } m \ge 2.$$

Theorem 11: The weight w is the optimal polynomial weight needed to get the weak type (1,1) inequality for the Riesz– Laguerre transforms of order $|\mathrm{m}+1|$.

Proof: This proof follows essentially in[8] .With the notation of that Theorem[1] one takes $\eta \in \mathbb{R}_+^d$ with $|\eta|$ sufficiently large, away from the axis and obtains the following lower bound for $\mathcal{K}^{m+1}(x,\eta)$

$$\mathcal{K}^{m+1}(x,\eta) = C \int_{[-1,1]^d} \mathcal{K}^{m+1}(x,\eta,s)\Pi_\propto(s)ds \ge C\,|\eta|^{|m+1|-2|\propto|-d-1} e^{\xi^2-|\eta|^2}. \quad (18)$$

For $x \epsilon J = \left\{\xi \frac{\eta}{|\eta|} + v : v \perp \eta . |v| < 1, \frac{1}{2}|\eta| < \xi < \frac{3}{2}|\eta|\right\}$

Now if we assume that the Riesz–Laguerre transforms of order

$|m+1| > 2$ map $L^1(\widetilde{w}_\varepsilon d\tilde{\mu}_\propto)$ continuously into $L^{1,\infty}(d\tilde{\mu}_\propto)$ with

$\widetilde{w}_\varepsilon = (1+|x|)^\varepsilon$ and $0 < \varepsilon < |m+1| - 2$ then by taking $f \geq 0$ in $L^1(\widetilde{w}_\varepsilon d\tilde{\mu}_\propto)$ close to an approximation of a point mass at η, with

$\|f\|_{L^1(\widetilde{w}_\varepsilon d\tilde{\mu}_\propto)} = 1$ we have that $\mathcal{R}_\propto^{m+1} f(x)$ is close to $e^{|\eta|^2}\mathcal{K}^{m+1}(x,\eta)|\eta|^{-\varepsilon}$ and by applying inequality (18) we get that $e|n|k^{m+1}(x,\eta)|\eta|^{-\epsilon} \geq |\eta|^{|m+1|-2|\alpha}$ 39 $e^{-(\frac{|n|}{2})^2}$.Therefore setting

$$\lambda = |\eta|^{|m+1|-2|\alpha|-d-1-\epsilon} e^{-\left(\frac{|n|}{2}\right)^2}$$

we obtain

$$e^{-(\frac{|\eta|}{2})^2}|\eta|^{2|\alpha|+d-1} \lesssim \tilde{\mu}_\alpha(J)$$

$$\leq \tilde{\mu}_\alpha\{x \in R_+^d : R_\alpha^{m+1} f(x) > \lambda\}$$

$$\lesssim \frac{1}{\lambda} = C|\eta|^{2|\alpha|+d-|m+1|+1+\epsilon} e^{-(\frac{|\eta|}{2})^2}.$$

Hence $|\eta|^{|m+1|-2-\epsilon}$ must be bounded which is a contradiction. Therefore the conclusion of Theorem (11) holds.

REFERENCES:

1- Liliana Forzani, Emanuela Sasso and Roberto Scotto(2009): Weak-type inequalities for Higher order Riesz-Laguerre transforms, Journal of Functional Analysis, 256,258-274.

2- P. Graczyk ,J. Loeb ,I .Lopez, A. Nowak and W. Urbina (2005):Higher order Riesz transforms, fractional derivatives and Sobolev spaces for Laguerre expansions, J. Math. Pures Appl. (9) 84 (3), 375–405.

3- G. Szego(1959): Orthogonal Polynomials, Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI.

4- E. Sasso(2005): Maximal operators for the holomorphic Laguerre semigroup, Math. Scand. 97 (2) ,235–265.

5- E. Sasso(2004): Spectral multipliers of Laplace transform type for the Laguerre operator, Bull. Austral. Math. Soc. 69 (2) ,255–266.

6-C. Gutierrez, A. Incognito and J.L. Torrea(2001): Riesz transforms, g-functions and multipliers for the Laguerre semigroup, Houston J. Math. 27 (3) ,579–592.

7- S. Perez and F. Soria(2000): Operators associated with the Ornstein–Uhlenbeck semigroup, J. London Math. Soc. (2) 61 (3) ,857–871.

8- E. Sasso(2007): Weak type estimates for the Riesz–Laguerre transforms, Bull. Austral. Math. Soc.
75 (3) ,397–408.

9- L. Forzani, E. Harboure and R. Scotto(2012): Weak type inequality for a family of singular integral operators related with the Gaussian measure, Potential Analysis 31(2) ,103-116.

10- L. Forzani, E. Harboure and R. Scotto(2009): Harmonic analysis related to Hermite expansions, Applied and Numerical Harmonic Analysis 978,73

11- J. Garcia-Cuerva, G. Mauceri, S. Meda, P. Sjogren and J.L. Torrea(2003): Maximal operators for the holomorphic Ornstein– Uhlenbeck semigroup, J. London Math. Soc . (2) 67 (1) ,219–234.