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On The Development and Recent Work of Cayley Transforms [II]

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ABSTRACT

This paper deals with the recent work of Cayley transforms of the symmetric bounds unitary operators and the multidimensional and tuples of unbounded operators, and representation transforms.

المستخلص

تهتم هذه الورقة بإظهار التطورات الحديثة لتحويلات كايلي للمؤثرات المتماثلة والمحدودة والمعيارية والمتعددة الأبعاد وتحويلات الإحداثيات للمؤثرات غير المحدودة وتحويلات التمثيل.

KEYWORDS**:** Cayley transforms, unitary operators, representation transforms.

INTRODUCTION

This paper gives the accretive and dissipative operators transform, and recent work of parameterizing representation, also crossaction and Cayley transformation of representations, important and necessary advances were given previously [1].

Quite recently, Nollau has extend the uniqueness theorem to operator in arbitrary Banach spaces [2]. The more recent work of Cayley transforms is on some notes on parameterizing representation.

This includes the base point issue which is the only interested at the moment to the experts [3].

Beside this we shall study the Cayley transforms of positive classes of matrices.

Lastly, Hackan Sam Uelsson in (2006) deals with Cayley transform of multidimensional and tuples of unbounded operators [4].

1. Accretive and dissipative operators and Cayley transform.

Definition 1: Let A_o be an operator, not necessarily bounded, but which domain:

$$D(A_o) = (D_{AO}) \text{ dense in } H.$$

A_o is called accretive if:

$$Re(A_o f f) \geq 0 \text{ for every } f \subset D(A_o)$$

and dissipative if [1]:

$$Im(A_o, f) \geq 0 \text{ for every } f \in D(A_o)$$

As dissipative operators can be derived from accretive ones by multiplication by i. all we shall say on accretive operators carries over immediately to dissipative operators

$$\|A_o f + f\|^2 \geq \|f\|^2 + \|f\|^2 \geq \|A_o f - f\|^2 \quad \text{....(1)}$$

And hence we see that $(A_0 + I) f = 0$. Thus $A_0 + -1$ is invertible, and furthermore it follows that the operative T_0 defined by $T_0 (A_0 + I) f = (A_0 - I) f \; (f € D(A_0)$ is a contraction of $(Au + I) D(Ao)$ onto $(Ao - I) D(Ao)$; we have

$$T_0 c (A_0 - I)(A_0 + I)^{-1} \quad \text{.........(2)}$$

Hence

$$T_0 = 1 - 2 (A_0 + I)^{-1}$$

$$1 - T_0 = 2 (A_0 + I)^{-1}$$

Thus $I - T_0$

is invertible and

$$A_0 = (I + T_0)(I - T_0)^{-1} \quad \text{.....(3)}$$

We call T_0 the Cayley transform of the accretive operator A_0.

Remark 1: we see that the definition of Cayley transform (2) is equivalent to that in previous study [5], regardless of the operator on which the transformation depend on.

Now from (1) and (2) it follows that an accretive operator A_1 is a proper extension of A_0 if and only if its Cayley transform T_1 is a proper extension of T_0.

An accretive operator which has no proper accretive extension is called maximal accretive. An obvious sufficient condition for the accretive operator A_0 to be maximal is that its Cayley transform to be defined on the whole space H, i-e., that $D(T_0) = (A_0 + I) D(A_0) = H$.

This condition turns out to be necessary [1]. Moreover we shall prove that every accretive operator has a maximal extension.

To this end let us suppose that $D(T_0)^1$ H. Let T be any extension of T_0 to a contraction defined every where on H.

Such a T exists since Ta extends to the subspace $\overline{D(T_0)}$ by continuity and then to H by linearity.

Setting, e.g.: $Th = 0$ for $h \wedge D(T_0)$,) there is no non-Zero invariant vector for T. In fact, if h is invariant for T then it is invariant for T* too [1] and we have for every $f € D(A_0)$:

$$[h, T_0(A_0 + I)f] = [T^* h, (A_0 + I)f] = [h, (A_0 + I)f] =$$

$$[h, T_0(A_0 + I)f] = [h, (A_0 - I)f]$$

Hence (h, f) = 0, and as $D(A_0)$ is dense in H this implies that h = 0.

Now every contraction T on H has non-zero invariant vector. Thus A is accretive and it is obvious that its Cayley transform equals T.As D(T) = H, we have that A is maximal accretive [1].

Let us also observe that whenever T is a contraction on H not having the Eigen value I, then so is T^* Hence.

$I - T^*$ is invertible;

$$(I - T^*)^{-1} \acute{\text{e}} (1 - T)^{-1} \grave{\text{u}}^* \quad \text{and}$$

$$(1 + T^*)(I - T^*)^{-1} = -1 + 2(1 - T^*)^{-1}$$

$$= [-I + 21 - T^{-1*} = [(1 + T)(1 - T)^{-1}]]^*$$

Thus the maximal accretive operators corresponding to 'I' and T^* are adjoins of one another. Now we can state the following:

Theorem 1: Every accretive operator in H has a maximal accretive extension. For an operator A in H the following conditions are equivalent.

a) A is maximal accretive;
b) A is accretive and (I + A) D(A) = H;
c) $A = (I + T)(I - T)^{-1}$ with a contraction I' on H not having the eigen value I;
d) –A is generator of a continuous semi-group (1,6) of contractions $\{T_s\} s \geq 0$.

If A is maximal accretive then So is A*, and the corresponding Cayley transforms are adjoint of one another (1).

Let us note that if A is maximal accretive, then the seml group of contraction generated by – A has its cogeamator equal to the Cayley transform of 1. Let us also observe that a normal operator A (bounded or not) is maximal accretive if and only if its spectrum is situated in the half – plane $\operatorname{Re} s \geq 0 (= \operatorname{Re} 1 \geq 0)$, of the plane of complex numbers s (1).

Proposition 2: Let A_0, B_0 be two accretive operators in H, S.t

$$(A_o f, g) = (f, B_o g)$$

$$for\, f \in D(A)\, and\, g \in D(B) \ldots.(4)$$

Proof [1]: Equation (4) implies for $f \in D(A_o)$ and $g \in D(B_0)$ that $((A_0 + I)f, (B_0 - I)g =$

$$(A_o f, B_o g)(A_o f, g) + (f B_o g)(f, g) =$$

$$(A_o f, B_o g) - (A_o f, g) + (f B_o g) - (f, g)$$

$$(A_o f, B_o g) + (A_o f, g) - (f B_o g) - (f, g) =$$

$$((A_o - I)f, (B_o + I)g)$$

Hence it follows that for the Cayley transforms T_o of A_0 and S_o of B_0 we have

$$(T_o j, y) = (j, S_0, y)(j \in D(T_0), y \in D(S_o)).$$

If we construct extensions of T_o and S_o to contraction T and S defined on the whole space, s.t

$$(Th, k) = (h, Sk) \;\; for\, h, k \in H \ldots\ldots\ldots(5)$$

Then the corresponding maximal accretive operators A and B will satisfy eqn. (3). This is in fact a ((I+T)h, (I – S)k) = (h, k) + (Th, k) – (h, Sk) – (Th, Sk) = (h, k) + (h, Sk) – (Th, k) – (Th, Sk) = ((I–T) h, (I + S)k) (k, $h \in H$).

To obtain T and S we first extend T_o and S_o by continunity to the coloures of their domains; these extensions will be denoted by T_o and S_0; they satisfy eqn. (4). Since D (S_0) is a subspace of H, we can set K = H – D (S_0).

For and fixed h in H, $(h, S_o y)$ defines a conjugate linear form on D(S_0) S.t $\left|(h, S_o y) \leq \|h\| - \|y\|\right.$.

Thus there exists a (unique):

$$h^* \in D(S_o) S.t (h, S_o y) = (h^* y)$$

$and \|h^*\| \leq \|h\| setting\, h^* = P_o h$ We have defined a contraction P_o of H into D (S_o) S.t.

$$(P_o h y) = (h, S_o)(h \in H, y D(S_o)) \ldots.(6)$$

In particular if:
$h = f \in D(T_o)$ then $(Pof, y) = (f, So\, y) = (To\, f, y)$;
hence it follows that
$P_o f$ is the orthogonal projection of $T_o f$ into $D(S_0)$
Hence it follows that for the Cayley transforms T_o of A_0 and S_o of B_o we have $(Tof, y) = (f, S_o y)(f \in D(To)(y \in D S_0)$.

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If we construct extension of T_o and So to contractions T and S defined on the whole space H, S.t

$$P_0 f = P'_{D(s0)} T_0 F$$

Thus $\|P_{k'} T_{0F}\|^2 = \|T_0 F\|^2 - \|p_0 F\|^2 \leq \|F\|^2 - \|p_0 F\|^2$ For Φ€D (To).

We conclude, by an argument due M.G Krein [1], that there exists an extension of P' K' T_o to an operator P_I.

Defined on the whole space H, with values in K, and S.t.

$\|p_I h\|^2 \leq \|h\|^2 - \|P_0 h\|^2$ for every h € H; now defining the operator T by

$$Th = P_0 h + P_1 h \text{ for } h \text{ €} D(S_0).$$

will thus be a contraction and we shall have

$$Tf = P_0 f P_1 f =$$

$$P_D(S_0) T_0 f + P_k T_0 f \; for \in D(T_0) i.e. TT_0$$

Moreover:

$$(Th, y) = (P_0 h + P_1 h, y) = (P_0 h, y) =$$

$$(h, S_0 y) \; for h \text{€} H, y \text{€} D(S_0)$$

This shows that $T^* \supseteq S_o$; setting S = T* the pair [T,S] satisfies $T \supseteq To, S \supseteq S_o$, and (5), so the proof is complete.

Corollary 3

For every accretive Ao with D (Ao) ⊂ (A^*_0), there exists a maximal accretive A so that Ao ⊂ A ⊂ B^*_0 with $B_0 = A^*_0$ D (A_0).

In fact, we have (A_0f, g) = (f, A^*_0g) = (f, Bog) for f, g € D (A_0) = D (B_0), and in particular Re(f, B_of) = Re(A_of,f) ≥o. Thus B_0 is accretive and there exists a maximal accretive extension A of A_0 S.t. (Af,g)=(f, B_0g)=(f,B_0g) for f€A⊂ B^*_0€D(and in particular Re (f, B_0f) = Re(A_0f,f) ≥0. Thus B_0 is accretive and there exists a maximal accretive extension A of A_0 s.t: $(A,f,g) = (f, B_0 g) B_0^* = (f, B_0 g) for f \text{€} D(A)$

$$and\ g \in DB_0, hence\ A'IB_0^*$$

Let H be a maximal accretive in H and let 1' e its Cayley transform is a contraction defined on the whole space H, and

$$T = (A - I)(A + I)^{-1}, A = (I + T)(I = T)^{-1} \dots \dots (7)$$

Now we shall give the notions of antiadjoint operator and purely maximal accretive. By virtue of the these relation, a subspace M of H reduces A (I.e P A ⊂ AP, P being the orthogonal projection of H onto M) if and only if it reduces T; it also follows that T is unitary if and only if A has the form iN with some self – adjoint [7].

The operators A = iN are characterized by the property A* = – A, so we call them antiadjoint operators. Another is completely non-adjoint if no non-zero subspace reduces it to an anti-adjoint operator, A maximal accretive, completely non-anti-adjoint operator will be called purely maximal accretive.

The canonical decomposition of the Cayley transform T of a maximal accretive A generates a decomposition of A.

Proposition 4: For every maximal accretive operator A on H there exists a decomposition H = Ho ⊕ HI, reducing A, and s.t the part of A in Ho is anti adjoint and the part of A in HI is purely maximal accretive. This decomposition is unique; Ho or HI may equal {0}[1].

Propositional 5: Suppose u and v satisfies the following conditions:

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u, v are continuous on D, holmorphic on D, and have no common zeros in D. moreover, let $v€K^{¥}_{T}$ when T is a contraction on H, then;

$v(T)^{-1}u(T) = u(T)v(T)^{-1}$

And for $j = u/v, j(T)^{*} = j \sim (T^{*})$.

Proof: see reference (1)

Now it follows form proposition 5, that T is a contraction on H, then every subspace HO = $\{h : h€H, Th = ah\}$ with $|a| = 1$ reduce T. This implies that if A is maximal accretive in H, then every subspace $\{f : f€H, Af = bf\}$, with a purely imaginary number b, reduces A. In particular, the subspace:

$R_A = \{f : Af = 0\}$.

reduces A, or what amounts to the same, $Af = 0$ implies A*f = 0, and conversely. If A is purely maximal accretive operator, then necessarily: $R_A = \{0\}$.

We conclude that in this case A^{-1} exists and its domain is dense in H.

The relation (7) between a maximal accretive operator A and its Cayley transform 1' yield a method for constructing a function calculus for A based on the functional calculus for T.

To this end let us consider the homography.

$$1 ® +d = \frac{1+1}{1-1} = w(1) \ldots\ldots(8)$$

which maps the unit disc D onto the right half – plane

$D = \{d : Re\, d > 0\}$

The inverse map is

$$d ® 1 = \frac{d-1}{d+1} \ldots\ldots(9)$$

We define then the functions of A by the formula

$$f(A) = f_0 w€(T) \ldots\ldots(10)$$

when ever $f_0\, w € N_T$

Example:

As $e^{-1}d = \exp \left(t \frac{1+1}{1-1} \right) = e_t(1), e^{-tA}$

makes sense for $t \geq 0$ and equals e_1 (T).

For fixed Z (Rez < 0) and M ($0 < M < ¥$) the trgral $f_M(z;d) = \int_0^M e^{tz} e^{-td}\, dt$ is the limit, by bounded convergence on T, of the corresponding Reimann sums (1), hence follows that;

$f_M(Z;A) = \int_0^M e^{tz} e^{-tA}\, dt$, as $M ® ¥$, $f_M(z;d)$

Tends to $(d - Z)^{-1}$ uniformly on $\bar{D}$; thus on account of theorem III 2.3 (CI) of 11 I 2.3 (c_1) and theorem 1.1 (iv) of ref. (1) we have;

$fM(Z;A) \quad (A - ZI)^{-1}$

So we arrive within the frame work of our functional calculus to the relation

$$(A - ZI)^{-1} = Q_0^{x} e^{tz} e^{-tA}\, dt \ldots\ldots(11)$$

valid for any maximal accretive operator A and for any complex with Re Z < 0.

2- Cross action and Cayley transform of representation:

It might be useful to collect some objects here which are defined later. Our group is G, fixed hand B.

The parameter set is Z, and (x,y) is a typical elements of Z.

An element x defines a real form Hx (IR) of H. A typical character of a cover of Hx (IR) is written Λ. We parameterize characters Λ as;

$\Lambda(x, l, v)$, we define the cross action

$W \times A = A \, m$

with $m = wl - 1$. This may also be written

$w \times \Lambda(x, A, v) = \Lambda(x, l + \mu, v + \mu)$

Then the Cayley transform is [2]:

$C^a \Lambda(X, l, v) = \{\Lambda(s_a x, l, v), \Lambda(s_a X, l, v + a)\}$

We parameterize characters A in terms of maps of W R into the dual group as

$$\Lambda[x, y, l, Y_0] \ldots \ldots \ldots (12)$$

We then compute the cross and Cayley transforms in these coordinates: $w \times \Lambda(x, y, l, y_o) = \Lambda(x, y, w\, l, y_o)$

$C^a \Lambda(x, y, l, y_o) = \Lambda(s_{aX}, s^a y_0, l, s^a y_0)$.

A standard module is I $(x, y_R +, \Lambda)$[2,8]. Here x specifies the strong real for, y_R^+ is a set of positive real roots with respect to qx and Λ is a genuine character of a cover Hx (IR.).

We may write y_R^+, x to indicate that the real roots are defined by x. the differential of Λ is Λ, which is fixed, and the infinitesimal character. Usually we can take $y_R^+ = Y^+_R$ and the standard module is $I(x, y_R^+, \Lambda)$ and (Y^+_R, Λ) are in good position [2,3].

The cross action is given by:

$w \times I(x, Y^+_R \Lambda) = I(x, Y^+_R, w^{-1} x \Lambda)$ and its Cayley transform is given by:

$C^a I(x, Y^+_R, \Lambda) = I(s^a x, y_R^+ s^a x, C^a \Lambda)$.

Finally we get to the parameterization [2,8]. So a standard module is $I(x, y, \Lambda)$ and the actions are

$$wXI(x, y, l) = I(w.x, w.y, l)$$

$$and\ C_z l(x, y, l) = I(s_z x, s^z xyl)$$

Now we turn to Cayley transforms of A and cross action, with propositions and examples.

3. Cayley transform of A

Suppose that:
$\Lambda = \Lambda(x, y, l, v)$ of course (l, v) satisfy the following lemma:

Lemma 6: Fix $g€X^*(H) \, C$ and q staisfying $q\,g - g€X^*(H)$. Write Hg(IR) for the corresponding real form of Hg. Suppose (l, v) satisfy $l\ €g+$

$X^*(H)\mathring{A}C, v€ + X^*(H)$
$and\ l + ql = v + qv \ldots \ldots (13)$

Then we obtain a genuine Hg(IR)- module $\Lambda(l, v)$ with differential 1. Further more

$$\Lambda = (l, v) = \Lambda(\bar{\nabla} v', v')\ if\ and\ only\ if\ l = l'$$

$$and\ \bar{\nabla} v' €(1 - (c))X^*(H) \ldots (13a)$$

We recover representations of H [IR] [2,8].

And now $q(c) = qx(c)x$ suppose CI is a non-compact imaginary root for q.

Definition 2

$c^a \Lambda(x, l, v) = [\Lambda(s_a x, l, v), \Lambda(s_a x, l, v + a)] \ldots\ldots (14)$

Here s_a is the multi valued Cayley transform as in reference (2). We note that the character on the right hand side is well defined. We have

$q_{ca\,x} = S_a q$

$(l - ts_a ql) - (v + s_a qv) =$

$\left(l + ql - (ql, a^v)\right) - (v + qv - \langle qv, v^a \rangle)a =$

$(l + ql) - (v + qv) + \langle q(v-1)), a^v =$

$= \langle v - l, a^v \rangle\ by\ (13)$

Now we need to show this is 0. By (13) we have $< (1 - v) + (ql - qv), a^{\geq} > = 0$ and since $qa^V = a^v$ and conclude that

$2 < 1 - V, a^V > 0.$

Note that the two characters on the right hand side of (14) might be same. Note also that

$\Lambda(s_a x, 1, v + a) = \Lambda(s_a x, 1, v)$ sasgn( ). And they are different if $a : H_{ca,x}(IR) ® X^*(H)$ is subjective i.e. a (now thought of a real root) is type 1. Equivalently by (13a) this held if $a €(1 - q)X^*(H)$.

Example 2: If G = SL (2) and a is a real root, then $a = (1 - q)a/2$, so they are the same. If G = PGL (2), then $a/2€X^*(H)$ and these characters are distinct.

We next need to compute C^a in $\Lambda[X, y, 1, y_0]$ coordinates.

Lemma 7
$(w\,x\,\Lambda[x, y, 1, y_0] = \Lambda[x, y, w, 1, y_0].$

Proof: [see references 2 and 4]

Note that

$w^{-1}(wx\,\Lambda[x_1 y, 1, Y_0]) = \Lambda[w^{-1}x, w^{-1}y, 1, w^{-1}y_0].$

Lemma 8:

Suppose that a is a non compact imaginary root, then;

$C^a \Lambda[x, y, 1, y_0] = \Lambda[(s^a x, s^a y, 1, s^a y_0)]$

Here s^a is the possibly multi valued inverse Cayley transform that is:

$$s^a y = \{y' : s_a y' = y\}$$

Proof

Let
$q = q_x$ write $y = e^{2P\,it} y_0$, SO $\Lambda[x, y, 1, y_0] = \Lambda(x, 1, v)$ with

$v = l - (t + q^v t)\ then C^a \Lambda[x, y, l, y_0] =$

$C^a \Lambda(X, l, v) = \Lambda(a_a x, l, v)$

On the other hand we have $s_a y = s_a e^{2P\,it} y_0 = e^{2P\,is_a t} s_a x y_0$

So
$\Lambda\left[s_a x, s^a y, 1, s^a y_0\right] = \Lambda(s_a x, 1, v')$

with
$v' = 1 - (S_a t + S_a q^V S_a t) = 1 - (S_a t + q^V t)$

Therefore

$v' - V = l - (S_a t + q^v t) - \Lambda + (t + q^v) =$

$$t - S_a t = \langle t, a^v \rangle a^v$$

We are on the less compact Cartan here, i.e., is a real root here, or in other words $S_a q_a - a$.

Therefore $a^V = (I - S_a q)½ a^V$, so the result follows from (13a).

Now we are going to study notions of Fokko's notes [7] and that of lifting of characters which introduced in 1991 by Adams [8] and the parameters for representation of real groups also deduced by Adams [9], and relations to Cayley transform.

Fix x, and let $y^{+}_{R.x.}$ be the real roots, with respect to q_x, and choose a set of positive real roots $y^{+}_{R.x.}$

Suppose $I(X, y^{-}_{R.x,} A)$ is a standard module

Now suppose a is a non-compact imaginary root and

$y^{+}_{R.x,} C^{a} x$ is a set of positive real roots for $q_c{}^{a} x$. Then

$I(s_a x, y^{-}_{R,sa\ x,} C^{a} \Lambda)$ is defined.

Assume $(y^{+}_{R,x,} \Lambda)$ and $(y^{+}_{R,s\,a\,x^{x,}} C^{a} \Lambda)$ are in good position.

For example take;

Proposition 9

$$C^{a} \left|\left(x, y^{-}_{R,x}, \Lambda\right) = \right|\left(s_a\ x, y^{-}_{R,s_a x,} C^{a} \Lambda\right).$$

Note: this proposition could be proved or checked by back ward working from the usual definition, see also references (2,7).

Proposition 10: Suppose a is a non-compact imaginary root of y with respect to x. Then,

$$C^{a} I(x, y, l) = I(S_a x, S^{a} y, l) \ldots\ldots(15)$$

Proof [2]: Firstly, we reduce to the case a is simple for $y_R x$ or is not simple choose $w \text{€} W_{im}$ (the corresponding set is of positive roots) [2,6,7] so that $b = w_a$ is simple.

For a positive system of real roots use $y^{+}_{R,x}$ the roots in y^{+} which are real with respect to q_x. and finally by using the base points provided Fokko's algorithm [2,7]. He says that there last choices are related, and you can modify them both simultaneously by W_{im}).

Now
$C^{a} I(x, y, l), = w x C^{b} I(x, y, l) = w x I(s_b x, s^{b} y, l)$
(assuming the result for simple root).

$= I(w(s_b x, w.s^{b} y, l)$ [from the two following propositions]:

$= w\ x\ 1(x, y, l)$
$= I(x, w\ y, l)^{[2]}].$
$= I(s_a\ x, s^{a}\ y, l)$

So assume a is simple for $y^{+}_{im,x}$ choose $y^{1}_{0} = y_0[y]$ so $y^{+}_{R}[y_0] = y^{+}_{R,y}$

The left hand side is then

$C_a\ I(x, y, l) = C^{a} I(x, y^{+}_{R,x,} \Lambda[x, y, l, y_0])$
$= I(s_a x, y^{+}_{Rs_a x,} \Lambda [s_a x, s^{a} y, l, s^{a} y_0])$

The last equality uses the fact that $\left(y^{+}_{R,s\,¥\,x,} \Lambda\right)$

is in good position (this only depends on $l = d\Lambda$).

On the other hand choose $y'_0 = Y_0 [s^{a} y]$ so the right hand side is:

$I\left(s_a x, s^{a} y, l\right) = I\left(s_a x, y^{+}_{R,s_a x}, \Lambda [s_a x, s^{a} y, l, y_0]\right)$
So we are done provided $y'_0 = s^{a} y$, i.e $y_0 [s^{a} y] = s^{a}\ y_0 [y]$.

Writing this on the G (is a group) side, and with w = P(y) in place of y we need

$X_b[s_a w] = s_a X_b[w]$ ……….(16)

for a an imaginary non-compact root, simple for $y^+/_{im,w.}$

Then eqn. (16) is the Cayley transformation, i.e., left multiplication by s_a.

Fokko built a check of the existence of $X_b[w]$ into the atlas has a software by computer [7,9].

CONCLUSIONS

This study showed the necessary historical advances and important three directions of recent work, that is to say the Cayley transform of accretive and dissipative operators and purely maximal of these, the symmetric, bounded, adjoin and unitary operators, the second direction is a P-matrix transformation and the third is that for Cayley transform of cross actions representations, multidimensional and tuples of unbounded operators.

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