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Group Invariant Solutions of Partial Differential Equations

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ABSTRACT

In this paper we utilize the symmetry group of differential equations in reducing the number of variables, thus reducing the order of differential equations. The reduction procedure is discussed thorouly. We also illustrate the procedure with some examples.

المستخلص

استخدمنا في هذه الورقة تماثل المعادلات التفاضلية الجزئية في تخفيض عدد المتغيرات، وبذا تخفيض رتبة المعادلة التفاضلية. في هذا البحث ناقشنا بإتقان وتفصيل هذه المسألة ووضحناها ببعض الأمثلة.

***KEYWORDS*:** symmetry group, invariant solutions, partial differential equations.

INTRODUCTION

The methods used to find group-invariant solutions generalizing the well known techniques for finding similarity solutions, provide a systematic computational method for determining large classes or special solutions. These group-invariant solutions are characterized by their invariance under some symmetry group of the system can all be found by solving a system of differential equations involving r fewer independent variables than the original system [1,2].

Consider a system of partial differential equations Δ defined over an open subset $M \subset X \times U \approx R^p \times R^q$ of the space of independent and dependent variables. Let G be a local group of transformations acting on M. A solution u = f (x) of the system is said to be G -invariant if it remains unchanged by all the group transformations in G, i.e. for each $g \in G$ [3].

The functions f and g . f agree on their common domains of definition. For example, the fundamental solution u = log $(x^2 + y^2)$ for the tow-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ is invariant under the one-parameter rotation group:

SO(2) : $(x,y,u) \rightarrow (x\cos\Theta - y\sin\Theta,\ x\sin\Theta + y\cos\Theta, u)$ acting on the independent variables x,y. We can define a G -invariant solution of a system of partial differential equations as solution u = f (x) whose graph $S \equiv \{(x, f(x))\} \subset M$ is a locally G – invariant subset of M.

If G is a symmetry group of a system of partial differential equations Δ then, under some additional regularity assumptions on the action of G, we can find all the G – invariant solutions to Δ by solving a reduced system of differential equations, denoted by $\Delta/_{G}$, which will involve fewer independent variables than the original system Δ [4].

Method of reducing the variables in partial differential equations

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i) Find all the infinitesimal generators υ of symmetry group of the system using the basic prolongation methods.

ii) Decide on the "degree of symmetry" s of the invariant solutions. Here $1 \leq s \leq p$ will correspond to the dimension of the orbits of some subgroup of the full symmetry group. The reduced systems of differential equations for the invariant solutions will depend on p – s independent variables [5]. Thus to reduce the system of partial differential equations to a system of ordinary differential equation, we need to choose: s = p -1.

iii) Find all s –dimensional subgroups G the full symmetry group found in part (i). This is equivalent to finding all s –dimenional subalgebra of the full Lie algebra of infinitesimal symmetries υ. To each subgroup or sub algebra there will correspond a set of group-invariant solutions reflecting the symmetries inherent in G itself.

iv) Fixing the symmetry group G we construct a complete set functionally independent invariants, which we divide into two classes

$$y^1 = \eta^1(x,u), \dots\dots\dots\dots, y^{p-s} = \eta^{p-s}(x,u)$$
$$\upsilon^1 = \xi^1(x,u), \dots\dots\dots\dots, \upsilon^q = \xi^q(x,u) \quad \dots. (1)$$

corresponding to the new independent and dependent variables, respectively. If G acts projectably, the choice of independent and dependent variables is prescribed by requiring the $\eta^{i,s}$ to be independent of u; in the more general ease, there is quite a bit of freedom in this choice [6], and different choices lead to seemingly different reduced systems, all of which are related by some form of "hodograph" transformation.

v) Provided G acts transversally, we can solve eqn. (1) for p – s of the x' s, which we denote by $\bar{x}$, and all of the u's in term of y, υ and the remaining s- parametric variables $\bar{x}$.

$$\bar{x} = \gamma(\bar{x}, y, \upsilon), \quad u = \delta(\bar{x}, y, \upsilon) \dots.. (2)$$

Furthermore, considering υ as a function of y , we can use (1) , (2) and chain rule to differentiate and thereby find expressions for the x- derivatives of any G-invariant u in term of y, υ, y –derivatives of υ and the parametric variables $\bar{x}$,

$$u^{(n)} = \delta^{(n)}\left(\bar{x}, y, \upsilon^{(n)}\right) \dots\dots (3)$$

vi) Substitute the expressions (2) and (3) into the system $\Delta(x,u^{(n)}) = 0$, the resulting system of equations will always be equivalent to a system of differential equations for υ = h(y) independent of the parametric variables $\bar{x}$.

$$\Delta/_{G}(y, \upsilon^{(n)}) = 0 \dots.. (4)$$

vii) Solve the reduced system (4), for each solution υ = h(y) of $\Delta/_{G}$ there corresponds a G - invariant solution u = f(x) of the original system, which is given implicitly by the relation:

$$\xi(x,u) = h[\eta(x,u)] \dots\dots (5)$$

Repeating (iv) through (vii) for each symmetry group G determined in step (iii) will yield a complete set of group – invariant solutions for our systems.

Applications of the method of reducing the variants in partial differential equations:

Example (1)

Consider the one-parameter scaling group

$$(x,t,u) \to (\lambda x, \lambda t, u), \lambda \in R^+,$$

acting on $X \times U \approx R^3$. On the upper half space M ≡ {t > 0}, the action is regular, with global independent invariants y = x/t and υ = u. If we treat υ as a function of y, we can compute formulae for the derivatives of u with respect to x and t in terms of y, x and the derivatives of υ with respect to y, along with a single parametric variable which we designate to be t, so that x will be the corresponding principal variable [7], by using the chain rule, that if $u = \upsilon = h(y) = h\left(\frac{x}{t}\right)$ then:

$u_x = t^{-1}\upsilon_y$, $u_t = -t^{-2}x\upsilon_y = -t^{-1}y\upsilon_y$

Further differentiations yield the second order formula:

$u_{xx} = t^{-2}\upsilon_{yy}$, $u_{xt} = -t^{-2}(y\upsilon_{yy} + \upsilon_y)$

$u_u = -t^{-2}(y^2\upsilon_{yy} + 2y\upsilon_y)$ ……(6)

One the relevant formulae relating derivatives of u with respect to x to those of υ with respect to y, have been determined, the reduced system of differential equations for the G –invariant solutions to the system Δ is found by substituting these expressions into the system wherever they occur. In general this leads to a system of equations of the form:

$$\overline{\Delta_v}(\bar{x}, y, u^{(n)}) = 0, v = 1, \dots\dots.. I,$$

still involving parametric variables $\bar{x}$. If G is actually a symmetry group for Δ the resulting system is equivalent to a system of equations, denoted:

$$(\frac{\Delta}{G})(y, u^{(n)}) = 0, \qquad v = 1, \dots\dots.. I$$

which are independent of the parametric variables, and thus constitute a genuine system of differential equations for Δ as a function of y.

Example (2):

The one-dimensional wave equation $u_{tt} - u_{xx} = 0$ is invariant under the scaling group. To construct the corresponding scale-invariant solutions, we need only substitute the derivative formula (6) into the wave equation, and solve the resulting ordinary differential equation, we find:

$t^{-2}(y^2\upsilon_{yy} + 2y\upsilon_y - \upsilon_{yy}) = 0$

This equation is equivalent to an equation

$(y^2 - 1)\upsilon_{yy} + 2y\upsilon_y = 0$

in which the parametric variable t no longer appears. This latter ordinary differential equation is the reduced equation for the scale-invariant solutions to the wave equation. It is easily integrated, with general solution:

$$\upsilon = c \log\left|\frac{(y-1)}{(y+1)}\right| + \acute{c},$$

where c, $\acute{c}$ are arbitrary constants. Replacing the variables y and υ in the solution by their expressions in terms of x, t, u we deduce the general scale-invariant solution to the wave equation to be:

$u = c \log\left|\frac{(x-1)}{(x+1)}\right| + \acute{c}.$

Example (3): (The heat equation)

The symmetry group of the heat equation consists of a six-parameter group of symmetries particular to the equation itself plus an infinite-dimensional subgroup stemming from the linearity of the equation. For each one-parameter subgroup-invariant solutions, which will be determined from a reduced ordinary differential equation, whose from will in general depend on the particular subgroup under investigation [8].

a) **Travelling Wave Solutions:** in general, travelling wave solutions to a partial differential equation arise as special group-invariant solutions in which the group under consideration is a translation group on the space of independent variables. Consider the translation group:

$$(x, t, u) \to (x + c\varepsilon, t + \varepsilon, u), \varepsilon \in R$$

generated by $\partial_t + c\partial_x$ in which c is a fixed constant, which will determine the speed of the waves. Global invariants of this group are:

$y = x - ct$, $\upsilon = u$ ……….(7)

so that a group-invariant solution υ = h(y) takes the familiar form u = h (x-ct) determining a wave of unchanging profile moving at the constant velocity c. Solving for the derivatives of u with respect to x and in terms of those of υ with respect to y we find:

$$u_t = -c\upsilon_y, \quad u_x = \upsilon_y, \quad u_{xx} = \upsilon_{yy}$$

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Substituting these expressions into the heat equation, we find the reduced ordinary differential equation for the travelling wave solutions to be:

$$\upsilon_{yy} + c\upsilon_y = 0$$

The general solution of this linear, constant coefficient equation is

$$\upsilon(y) = k e^{-cy} + I,$$

for k,I arbitrary constants. Substituting back according to (7), we find the most general travelling wave solution to the heat equation to be an exponential of the form:

$$u(x,t) = k e^{-c(x-ct)} + I,$$

b) **Scale-invariant solutions:** There are two one-parameter groups of scaling symmetries of the heat equation, and we consider a linear combination [9]:

$$x\,\partial_x + 2\,t\,\partial_t + 2\,au\,\partial_u,\ a \in R,$$

of their infinitesimal generators, which corresponds to the group:

$$(x,t,u) \to (\lambda x + \lambda^2 t + \lambda^{2a} u), \lambda \in R^+$$

On the half space {(x,t,u) : t > 0}, global invariants of this one-parameter group are provided by the functions:

$$y = \frac{x}{\sqrt{t}}, \quad \upsilon = t^{-a} u$$

Solving for the derivatives of u in terms of υ, we find:

$$u = t^a u \upsilon,\ u_x = t^{a-1/2}\upsilon_y$$

$$u_{xx} = t^{a-1}\upsilon_y$$

$$u_t = -\tfrac{1}{2}xt^{a-3/2}\upsilon_y + at^{a-1}\upsilon =$$

$$t^{a-1}(-\tfrac{1}{2}y\upsilon_y + a\upsilon).$$

Substituting these expressions into the heat equation, we find:

$$t^{a-1}\upsilon_{yy} = t^{a-1}\left(-\tfrac{1}{2}y\upsilon_y + a\upsilon\right),$$

This equation is equivalent to one in which the parametric variable t does not occur, namely:

$$\upsilon_{yy} + \frac{1}{2}y\upsilon_y + a\upsilon = 0,$$

which forms the reduced equation for the scale-invariant solutions. If we set $w = \left[\upsilon exp.(\frac{1}{8}y^2)\right]$ then w satisfies a scaled from of Webers differential equation,

$$w_{yy} = \left[(a + \frac{1}{4}) + \frac{1}{16}y^2\right]w$$

The general solution of this equation is

$$w(y) = KU\left(2a + \frac{1}{2}, \frac{y}{\sqrt{2}}\right) + \bar{K}V(2a + \frac{1}{2}, \frac{y}{\sqrt{2}})$$

Thus the general scale-invariant solution to the heat equation takes the form:

$$u(x,t) = t^a e^{x^2/8t}\{KU\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right) + \bar{K}V\left(2a + \frac{1}{2}, \frac{x}{\sqrt{2t}}\right)\}$$

If a = 0, we obtain the probability solution:

$$u(x,t) = k^* \operatorname{erf}\left(\frac{x}{\sqrt{2t}}\right) + \overline{k^*}$$

where erf is the error function.

c) **Galilean-Invariant Solutions:** The one-parameter group of Galilean boosts, generated by $\upsilon_s = 2t\partial_x - xu\partial_u$ has global invariants y = t, υ = u exp $(x^2/4t)$ on the upper half space {t > 0} [9], we find

$$u_t = (\upsilon_y + \frac{x^2}{4t^2}\upsilon)e^{-x^2/4t}$$

$$u_{xx} = (\frac{x^2}{4t^2} - \frac{1}{2t})\upsilon\, e^{-x^2/4t}$$

Therefore, for the heat equation the reduced equation for Galilean- invariant solutions is

a first order ordinary differential equation $2y\upsilon_y + \upsilon = 0$. The solution is $\upsilon(y) = \frac{k}{\sqrt{y}}$.
Hence the most general Galilean- invariant solutions is scalar multiple of the source solution,

$$u(x,t) = (\frac{k}{\sqrt{t}})\, e^{-x^2/4t}$$

which we earlier found as a scale- invariant solution.

CONCLUSIONS

There is a one-to-one correspondence between G -invariant functions u = f(x) on M and arbitrary functions $\upsilon = h(y)$ involving the new variables. To explain this correspondence, we begin by invoking the implicit function theorem to solve the system $y = \eta(x)$ for p – s of the independent variables, say $\bar{x} = (x^{i_l}, \dots\dots.. x^{i_{p-1}})$, in terms of the new variables, $y^1, \dots\dots\dots y^{p-s}$ and the remaining s old independent variables, denoted as $\bar{x} = (x^{j_1}, \dots\dots.. x^{ij_s})$. Thus we have the solution $\bar{x} = \gamma(\bar{x}, y)$, for some well –defined function y. Then we solve the reduced system $\Delta/_G\,(y, \upsilon^{(n)}) = 0$. For each solution $\upsilon = h(y)$ of $\Delta/_G$, there corresponds a G -invariant solution u = f(x) of the original system, which is given implicitly by the relation:
$\xi(x,u) = h\,\{\eta(x,u)\}$.

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