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# The Geometric Formulation of Electromagnetic Field

**Tagreed Ahmed[1*] and Mohamed A. B.[2]**

*1. Khartoum-Sudan, P.O.Box:3045. Tel+249 183 779241, Email: twithyou1@gmail.com
2. Mathematical Faculty of Sciences Technology EL-Neilain University. Khartoum-Sudan

## ABSTRACT

This paper gives a geometrical interpretation of Maxwell's equations describing the electromagnetic field. The geometrical interpretation is not only based on 4 -dimensional Minkowski space, but extended to $U(1)$ fiber bundle, describing the electromagnetic field. The gauge potential is described by the connection of this fiber bundle. A differential form of electromagnetic theory was also provided.

**المستخلص**

في هذه الورقة تم إعطاء دلالات هندسية لمعادلات ماكسويل لوصف الحقل المغنطيسي الكهربي. الدلالات الهندسية ليس لها أساس فقط على فضاء منكو سكاي ذو البعد -4. ولكن يمدد إلى طرفة الليفية (1) u الواصفة للحقل المغنطيسي الكهربي. الجهد القياسي يوصف بواسطة ربط هذه الحزمة الليفية. أيضاً تم إشتراط صيغة تفاضلية للنظرية المغنطيسية الكهربية.

***KEYWORDS***: Geometric Formulation, Minkowski space-time, continuity equation.

## INTRODUCTION

Principal fiber bundles (PFBs) will be defined and some nontrivial examples will be given (e.i., double covering of the circle and the frame bundle of a manifold).The connection of PFBs with group $U(1)$ over space-time will physically be identified as the four-dimensional vector potential of electromagnetism.

The geometrization of electrical forces requires another dimension (the charge dimension) [(1)]. The resulting five dimensional spaces are actually PFBs with group $U(1)$ . A connection (or gauge potential) en-dows the five component of a geodestic in five –space. If the charge component of a certain geodestices$q$, then the projection of this geodestic onto space-time is the non-geodestic path of an object charge $q$ of subject to the force of gauge potential [(2,3)]. The invariance of the gauge potential under $U(1)$ action on the PFBs implies the conservation of charge. There are many other types of charge that respond to various other kind of force (e.g., isospin geometrization, of the force requires PFBs with larger non–abilian groups) and so five dimensions are not nearly enough. At the end of this paper we have provided a form of Maxwell's equations using the language of exterior derivative [(3)].

Principal Fiber bundle (PFB) consists of a manifold $P$ (called the total space) [(4)], a Lie group G, a base manifold $M$, and a projection map $\pi : p \to M$ such that the following hold.

For each $x \in M$ there is an open set $U$ with $x \in U$ and diffeomorphism $T_u : \pi^{-1}(U) \to U \times G$ of the form $T_u(P) = \big(\pi(P)\big)$ , $s_u(p)$ where $s_u : \pi^{-1}(U) \to G$ has property $s_u(pg) = s_u\,(p)g$ for all $g \in G$ ,$p \in \pi^{-1}(U)$. The map $T_u$ is called a local trivialization (LT), or (in physical language) a choice of gauge.

**Definition:** Let $T_u : \pi^{-1}(U) \to U \times G$ and $T_v : \pi^{-1}(V) \to V \times G$ be two LTs of a PFB

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with a group. The transitions function from $T_u$ to $T_v$ be the map $g_{uv} : U \cap V \to G$ defined for $x = \pi(P) \in U \cap V$, by $s_u(P)\ s_v(P)^{-1}$. Note that $g_{uv}(x)$ is independent of the choice of $P \in \pi^{-1}(x)$.

Because $s_u(Pg)\ s_v(P\,g)^{-1} =$
$s_u(P)g\ s_v\big((P)g\big)^{-1} =$
$s_u(P)gg^{-1}\ s_v(P)^{-1} = s_u(P)\ s_v(P)^{-1}$

We have

(i) $g_{uu}(y) = e$ for all $y \in U$ ;

(ii) $g_{vu}(y) = g_{uv}(y)^{-1}$ for all $y \in U \cap V$;

(iii) $g_{vu}(y)\, g_{vw}(y)\, g_{wu}(y) =$ e for all $y \in U \cap V \cap W$.

The transition functions describe how various product $U \times G, V \times G, \ldots..$ glue together to form the total space P [4]. Indeed $P$ may be considered as the space obtained from disjoint union $(U \times G) \cup (V \times G) \cup \ldots..$ by identifying the point $(x, g) \in U \times G$ with $(x, \acute{g}) \in V \times G$ if $g = g_{uv}(x)\acute{g}$. Because of (i), (ii) and (iii), this identification is equivalence relation.

**Definition:** define a local section of a PFB $\pi : P \to M$ with group $G$ to be a map $\sigma : U \to P(U \subset M, U\ open)$ such that $\pi\ {}^\circ \sigma = I_u \equiv the\ identity\ function\ on\ U(x \to x)$.

**Theorem:** There is a natural correspondence between local section and local trivialization, thus[5]:

$\mathcal{G}\sigma_{g_{uv}}(x)^{-1}\omega_u(Y_x) =$
$\sigma_{g_{uv}}(x)^{-1}\omega_u(Y_x)\ g_{uv(x)}$.

Consequently the transform rule from $\omega_u$ to $\omega_v$ can be expressed as:

$\omega_v = g_{uv}^{-1}\, d\, g_{uv} + g_{uv}^{-1}\, \omega_v g_{uv}$ .

Given a connection on1-form $\omega$ on a PFB $\pi : P \to M$ with group G, we can write any$X \in T_P\, P$ as $X = X^V + X^H$ where $X^V$ is vertical $(i.e\,, \pi_*(X^V) = 0)$ and $X^H$ is horizontal$(i.e\,, \omega(X^H) = 0)$.

**Definition:** The exterior derivative of $\varphi \in \Lambda^k(P, \mathcal{G})$ is $D^\omega \varphi \equiv (d\varphi)^H \in \Lambda^{k+1}(P, \mathcal{G})$ where $d\varphi$ is the usual exterior derivative of $\varphi$ , although the operator $D^\omega$ depends on $\omega$. Because we consider functional on the space of connections and other situations where more than one connection is involved, we will usually not observe this custom.

**Definition:** The curvature of the $\varphi \in \Lambda^1(P, \mathcal{G})$ is $\omega \in \Lambda^1(P, \mathcal{G})$. That is $\Omega^\omega \equiv D^\omega \omega \in \Lambda^2(P, \mathcal{G})$. When $\omega$ is regarded as a potential, $\Omega^\omega$ is the field strength of $\omega$ [6].

**Theorem (The Structural Equation):** The curvature form is given by $\Omega^\omega = d\,\omega + \frac{1}{2}[\omega\,, \omega]$ (i.e, $D^\omega = d\,\omega + \frac{1}{2}[\omega\,, \omega]$ ).

**Maxwell's equations in Minkowski space-time:**

Let M = R4 with coordinate $(X^\circ, X^1, X^2, X^3) = (t\,, X\,, Y\,, Z)$ and metric $g$ such that $g(\partial_\circ\,, \partial_\circ) = 1$ , $g(\partial_i\,, \partial_i) = -1$ for $i = 1,2,3$ and $g(\partial_i\,, \partial_j) = 0$ for $i \neq j$ .

then $(M, g)$ is called Minkowski space.

**Hodge star operator and wedge product:**

On an oriented $n$-dimensional Riemannian manifold, the Hodge star is a linear function which converts alternating differential $k$ −forms to alternating $(n - k)$-forms. If $\omega$ is an alternating $k -$ form, its Hodge star is given by [7]:

$\omega(v_{k+1}\ ,........, v_n) = (*_\omega)(v_1\ ,........, v_n)$ where $v_1\ ,........, v_n$ is an oriented ortho-normal basis.

Then the wedge product is the product in an exterior algebra. If $\alpha$ and $\beta$ are differential $k$ −forms of degrees $p$ and $q$ respectively, then

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \qquad (1)$$

It is not (in general) commutative, but it is associative,

$$(\alpha \wedge \beta) \wedge u = \alpha \wedge (\beta \wedge u) \qquad (2)$$

**Exterior differential form of Maxwell's equations:**

Consider the 2-form:

$$F = E_1 dx\Lambda dt + E_2 dy\Lambda \mathrm{dt} + E_3\, dz\Lambda dt + B_1 dy\Lambda dz + B_2 dz\Lambda \mathrm{dx} + B_3\, dx\Lambda dy.$$

For $dr = (dx, dy, dz)$ and $d\sigma = (dy\Lambda dz\,, dz\Lambda dx\,, dx\Lambda dy)$ (8), we employ the shorthand $F = E\,.\,dr\Lambda dt + B.\,d\sigma$. By simple computation, we obtain

$$dF = \left(\nabla \times E + \frac{\partial B}{\partial t}\right).\,d\sigma\Lambda dt + (\nabla\,.\,B)d\tau$$

where $d\tau = dx\,\Lambda\, dy\Lambda\, dz$. Thus $dF = 0$ if $\nabla \times E + \frac{\partial B}{\partial t} = 0$ and $\nabla\,.\,B = 0$, which are two Maxwell's four equations (where $E$ is the electric field and $B$ is the magnetic field). Now $* F = E.\,d\sigma - B.\,dr\Lambda dt$ and so

$$d * F = (\nabla\,.\,E)d\tau - \left(\nabla.\,B + \frac{\partial E}{\partial t}\right).\,d\sigma\Lambda dt$$

Now $\delta = -(-1)^g(-1)^{4(k+1)} * d * = * d *$ on $\Lambda^k(R^4)$. Thus

$$\delta F = * d * F = (\nabla\,.\,E)dt - \left(\nabla \times B + \frac{\partial E}{\partial t}\right).\,dr$$

$\left(j \in \Lambda^1(R^4)\right)$ is defined by $j = \rho dt - j.\,dr$

Then

$$\delta F = j$$

is equivalent to the other. Let the maps $\rho: R^4 \to R$ and $j : R^4 \to R^3$ be the charge density and the current density (9). The source 1- form two (unhomogeneous) Maxwell's equation $\nabla\,.\,E = \rho$ and $\left(\nabla \times B - \frac{\partial E}{\partial t}\right) = j$. Thus, the four Maxwell equations are summarized by $dF = 0$ and $\delta F = j$ , and we can obtain:

$$0 = \delta^2 F == \delta * j = * d(\rho d\tau - j.\,d\sigma\Lambda dt) - * \left(\frac{\partial \rho}{\partial t}\, dt\Lambda d\tau - \nabla.\,j\, d\tau\Lambda dt\right) = -\left(\frac{\partial \rho}{\partial t} - \nabla.\,j\right).$$

Thus, we obtain the so-called *continunity equation*

$$\frac{\partial \rho}{\partial t} - \nabla.\,j = 0\,.$$

which says that charge is conserved.

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