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CR-Sub Manifolds of the Six Dimensional Sphere

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ABSTRACT
In this paper a proper CR-submanifolds of the 6-dimensional sphere was considered, S^6. some characterization theorems of their submanifolds were provided. In particular, 4- dimensional CR-submanifolds were introduced and their theorems were illustrated with examples.

المستخلص

فى هذه الورقة اعتبرنا عديدة الطيات الجزئية الفعلية لكوشى ريمان للكرة ذات الابعاد الست . استطعنا ان نقدم بعض نظريات التصنيف لهذه الفضاءات. على وجة الخصوص قدمنا عديدات الطيات ذات الاربعة ابعاد لكوشى ريمان وبينا النظريات التى توصلنا اليها من خلال بعض الامثلة.

***KEYWORDS*:** CR-submanifolds, submanifolds Kahlerian structure , holomorphic distribution, real submanifolds.

INTRODUCTION

The six-dimensional sphere S^6 is the most typical example of nearly Kaehlerian manifolds. The existence of such a nearly Kaehlerian structure for the 6-sphere was proved by Fukami and Ishihara[(1)] by making use of the properties of the Cayley division algebra. The almost complex submanifolds of the 6-dimensional sphere were studied by Gray and Sekigawa. A. Gray proved that with respect to the Canonical nearly Kaehlerian structure [(2)]. S^6 has no 4-dimensional almost complex submanifolds. On the other hand Sekigawa studied the 2-dimensional almost complex submanifolds of S^6 [(3)]. He proved that, among other things, a 2- dimensional almost complex sub manifold of S^6 with Gaussian curvature $K<1$ is either diffeomorphic to a 2-dimensional torus or a 2-dimensional sphere. Bashir M. A also have many results in this article [(4 - 7)].

The six–dimensional space is any space that has six dimensions, that is six degrees of freedom, and that needs six pieces of data, or coordinates.

Formally six–dimensional Euclidean space S^6, is generated by considering all real 6-tuples as 6–vectors in this space. As such it has the properties of all Euclidean spaces, so it is linear, has a metric and a full set of vector operations. In particular, the dot product between two 6-vectors is readily defined, and can be used to calculate the metric. 6x6 metrics can be used to describe transformations such as rotations that keep the origin fixed.

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More generally, any space that can be described with six coordinates not necessarily Euclidean ones, is six dimensional. One example is surface of 6-sphere S^6. This is the set of all points in seven dimensional Euclidean space R^7 that is equidistant from origin. This constraint reduces the number of coordinates need to describe a point on 6-sphere by one, so it has six dimensions they have for more applications.

The 6–sphere, or hypersphere is in seven dimensions, is the six dimensional surface equidistant from a point, e.g. the origin. It has symbol S^6, with formal definition for 6-sphere with radius r is:

$$S^6 = \{x \in R^7 : \|x\| = r\}$$

Let C be the set of all purely imaginary Cayley numbers. C can be viewed as a 7-dimensional linear subspace R^7 of R^8. Consider the unit hypersurface which is centered at the origin

$$S^6(1) = \{x \in C :< x, x >= 1\}$$

The tangent space $T_x S^6$ of S^6 at a point X may be identified with the affine subspace of C which is orthogonal to X. $A(1,1)$ tensor field J on S^6 is defined by: $J_x U = X \times U$

where the above product is defined for $x \in S^6$ and $U \in T_x S^6$. The tensor field J determines an almost complex structure (i.e. $J^2 = -id$) on S^6 If $\overline{\nabla}$ is the Riemannian connection on S^6, then $(\overline{\nabla}_x J)X = 0$ for any $X \in \mathcal{X}(S^6)$, i.e. S^6 is nearly Kaehler. J is orthogonal with respect to indusced metric g and they are related by the form

$$\forall X, Y \in TM, g(X,Y) = g(JX,Y)$$

A 2p + q –dimensional submanifold M on S^6 is called a CR-Submanifold if there exists a pair of orthogonal complementary distributions D and $D^{\perp}$ such that JD = D and $JD^{\perp} \in \nu$ where ν is the normal bundle of M. The distributions D and $D^{\perp}$ are called the holomorphic distribution and the totally real distribution respectively with dim D = 2p and dim $D^{\perp}$ p. The normal bundle ν splits as $\nu = JD^{\perp} \oplus \mu$ where μ is invariant sub-bundle of ν under J. The CR-submanifold is said to be proper if neither $D = \{0\}$ nor $D^{\perp} = \{0\}$. A proper CR-submanifold M of S^6 is said to be a CR-product submanifold if it is locally the Riemannian product of a holomorphic submanifold and a totally real submanifold of S^6. It is known that there does not exist any CR-product submanifolds in S^6 (8).

In the area of number theory, the Euler numbers are a sequence E_n of integers defined by the following Taylor series expansion:

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n$$

The Euler numbers appear as a special value of the Euler polynomials.

Given a real vector bundle E over M, its k-th Pontryagin class $p_k(E)$ is defined as

$$p_k(E) = p_k(E, ¢) = (-1)^k c_{2k}(E Ä £) Î H^{4k}(M, ¢)$$

Here $c_{2k}(E Ä £)$ denotes the 2k-th Chern class of the complexification $E Ä £ = E Ä iE$ of E and $H^{4k}(M, ¢)$, the Z^{4k-} homology group of M with integer coefficients.

The rational Pontryagin class $p_k(E, ¤)$ is defined to be the image of $p_k(E)$ in $H^{4k}(M, ¤)$, the 4k-cohomology group of M with rational coefficients.

The Pontryagin classes of a smooth manifold are defined to be the Pontryagin classes of its tangent bundle.

Novikov proved in (1966) that if manifolds are homeomorphic then their rational Pontryagin classes: $p_k(M,\mathbb{Q}) \in H^{4k}(M,\mathbb{Q})$ are the same.

If the dimensions are at least five, there are at most finitely many different smooth manifolds with given homotopy type and Pontryagin classes.

G_2 manifold is a seven-dimensional Riemannian manifold with holonomy group G_2. The group G_2 is one of the five exceptional simple Lie groups.

A G_2-structure is an important type of M-structure that can be defined on a smooth manifold. If M is a smooth manifold of dimension seven, then a G_2-structure is a reduction of structure group of the frame bundle of M to the compact, exceptional Lie group G_2.

We denote by D the Levi-Civita connection of S^6. Then we have

$$(D_x J)Y = -X \times Y + < X \times Y, x > x \quad , (1)$$

for $X, Y \in T_x S^6, x \in S^6$. Thus we see that the almost Hermitian structure J on S^6 is a nearly Kaehler structure $((D_x J)X = 0)$ which is not Kaehler one.

Now, we prepare fundamental formula for Riemannian submanifolds of S^6. Let (M,ϕ) be a submanifolds of S^6 with the isometric immersion $\phi : M \to S^6$. We set $x = \iota \circ \phi$ and consider x as the corresponding position vector to the image of ϕ in $\operatorname{Im} C$, where ι denote the inclusion map from S^6 to $\operatorname{Im} C$. We denote by ∇ and $\nabla^{\perp}$ the Riemannian connections on M and the normal bundle $T^{\perp}M$ induced by the Riemannian connection D on S^6, respectively. Then, the Gauss and Weingarten formulas are given respectively by

$$D_X Y = \nabla_X Y + \sigma(X,Y) \quad (2)$$

$$D_X \xi = A_\xi X + \nabla^{\perp}_x \xi \quad ....(3)$$

where σ and A_ξ are the second fundamental form and the shape operator (with respect to the normal vector field ξ), respectively, and $X, Y \in \mathfrak{X}(M)$ where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth tangent vector fields on M. The second fundamental form σ and the shape operator A_ξ are related by

$$< \sigma(X,Y), \xi > = < A_\xi X, Y > .$$

The Gauss, Codazzi and Ricci equations are given respectively by

$$< R(X,Y)Z, W > = < X,W >< Y,Z > - < X,Z >< Y,W >$$
$$+ < \sigma(X,W), \sigma(Y,Z) > - < \sigma(X,Z), \sigma(Y,W) > \quad (4)$$

$$(\nabla'_X \sigma)(Y,Z) = (\nabla'_Y \sigma)(X,Z) \quad (5)$$

$$< R^{\perp}(X,Y)\xi, \eta > = < [A_\xi, A_\eta]X, Y > \quad (6)$$

where

$$(\nabla'_X \sigma)(Y,Z) = \nabla^{\perp}_X \sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$$

and

$$R^{\perp}(X,Y)\xi = [\nabla^{\perp}_X, \nabla^{\perp}_X]\xi - \nabla^{\perp}_{[X,Y]}\xi$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ and ξ, η are vector fields normal to M.

CR-submanifolds of the six dimensional sphere:

There are many theorems of CR-submanifolds of S^6 have been studied by several mathematician.

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For instance Sekigawa [8] proved that S^6 does not contain any CR-product submanifold. Gray has shown that S^6 does not admit a 4-dimensional complex manifold[2]. Ejiri N. proved the following[9]**:**

Theorem [1]: A 3-dimensional totally real submanifold of S^6 is orientable and minimal.

Proof: Let (M,g) be a 3-dimensional totally real submanifold of $(\overset{6}{S},J,g)$. First of all, we shall prove the following.

Lemma 4.3.1. G(X,Y) is normal to M for X,Y tangent to M.

Proof: From the second fundamental form of the immersion given by

$$(X,Y)=\bar{\nabla}_X Y-\nabla_X Y \tag{7}$$

For vector fields X,Y on M. For a normal vector fields ξ, we denote by $-A_\xi X$ and $\nabla^\perp_X\xi$ the tangential and normal components of $\bar{\nabla}_X\xi$ respectively so that

$$\bar{\nabla}\xi_X Y=-A_\xi X+\nabla^\perp_X\xi \tag{8}$$

We have

$$g((\bar{\nabla}_x J)Y,Z)=g(J\sigma(X,Z),Y)-g(J\sigma(X,Y),Z),$$
$$g((\bar{\nabla}_z J)X,Y)=g(J\sigma(Z,Y),X)-g(J\sigma(Z,X),Y),$$
$$g((\bar{\nabla}_y J)Z,X)=g(J\sigma(Y,X),Z)-g(J\sigma(Y,Z),X),$$

for X,Y,Z tangent to M. Since $\bar{g}$ is Hermitian with respect to J, $\bar{\nabla}_x J$ is skew-symmetric with respect to $\bar{g}$. This, together with the fact $(\bar{\nabla}_X J)X=0$, holds for all vector fields X and S^6 implies that the left-hand sides of the above three equations are equal to each other. Therefore we have

$$g((\bar{\nabla}_X J)Y,Z)=0,$$

which means $G(X,Y)$ is orthogonal to M. By

$$G(X,JY)=-JG(X,Y) \tag{9}$$

we obtain

$$(\bar{\nabla}_x G)(JY,JZ)=\bar{\nabla}_x G(JY,JZ)-G(\bar{\nabla}_x JY,JZ)-G(JY,\bar{\nabla}_x JZ)=$$
$$-\bar{\nabla}_x G(Y,Z)-G((\bar{\nabla}_x J)Y,JZ)-G(J\bar{\nabla}_x Y,JZ)$$
$$-G(JY,(\bar{\nabla}_x J)Z)-G(JY,J\bar{\nabla}_x Z)$$
$$-\bar{\nabla}_x G(Y,Z)+JG(G(X,Y),Z)$$
$$+G((\bar{\nabla}_x Y,Z)+JG(Y,G(X,Z))+G(Y,\bar{\nabla}_x Z)$$
$$=-(\bar{\nabla}_x G)(Y,Z)+JG(G(X,Y),Z)+JG(Y,G(X,Z)$$

for X Y Z tangent to M.

This, combined with the fact

$$(\bar{\nabla}_x G)(Y,Z)=\bar{g}(Y,JZ)X+\bar{g}(X,Z)JY-\bar{g}(X,Y)JZ \tag{10}$$

holds for all vector fields X, Y, Z on S^6, implies

$$G((Y,G(Z,X)+G(Z,G(X,Y))=g(X,Y)Z-g(X,Z)Y$$

and hence

$$G(X,G(Y,Z))=g(X,Z)Y-g(X,Y)Z$$

or equivalently

$$JG(X,JG(Y,Z)=g(X,Z)Y-g(X,Y)Z \tag{11}$$

for X, Y, Z tangent to M. Since $JG(X,Y)$ is tangent to M by Lemma [1] we see from (11) that

$$g(JG(X,Y),Y)X-g(JG(X,Y),X)Y=JG(JG(X,Y),JG(X,Y))=0$$

Thus $JG(X,Y)$ is orthogonal to X and Y if X and Y are linearly independent. This property, together with (11), implies that M is orientable, because the orientation can be defined by regarding JG(X, Y) as the vector product of X and Y at each point of M. Next, we shall prove that M

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is minimal. It follows immediately from eqns. (7), (8) and Lemma [1] that

$$\nabla_X^{\perp} JY = G(X,Y) + J\nabla_x Y) \quad (12)$$

and

$$A_{JX} = -J\sigma(X,Y) \quad (13)$$

hold for X, Y tangent to M. By (7), (8), (12), (13) and (9), we obtain

$$(\bar{\nabla}_x G)(Y,Z) = \bar{\nabla}_x G(Y,Z) - G(\bar{\nabla}_x Y,Z) - G(Y,\bar{\nabla}_x Z)$$
$$= -A_{G(Y,Z)} X + \nabla_X^{\perp} G(Y,Z) - G(\bar{\nabla}_x Y,Z) - G(Y,\bar{\nabla}_x Z)$$
$$= J\sigma(JG(Y,Z),X) + JG(X,G(Y,Z)) - J(\nabla_x JG(Y,Z)$$
$$-G(\sigma(X,Y),Z) - G(Y,\sigma(X,Z))$$

for X, Y, Z tangent to M. This, combined with (10), implies

$$(\nabla_x JG)(Y,Z) = g(X,Y)Z - g(X,Z)Y + G(X,G(Y,Z)) + \sigma(X,JG(Y,Z))$$
$$+ JG(\sigma(X,Y),Z) + JG(Y,\sigma(Z,X))$$

Taking the normal component, we have

$$\sigma(X,JG(Y,Z)) + JG(\sigma(X,Y),Z) + JG(Y,\sigma(Z,X)) = 0 \quad (14)$$

for X, Y, Z tangent to M. Let e_1, e_2, e_3 be a local field of orthonormal frames on M. Then we may assume without loss of generality that $JG(e_1, e_2) = e_3$, $JG(e_2, e_3) = e_1$ and Hence we have from eqn. (14) that the trace of $\sigma = 0$, which implies that M is minimal.

Theorem [2]: Let M be a 3–dimensional totally real submanifold of constant curvature C in S^6, then either C = 1 (i.e. M is totally geodesic) or $c = 1/16$.

Proof: Let M be a 3-dimensional totally real submanifold of constant curvature c in S^6. Then the equation of Gauss reduces to

$$(1-c)\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\}$$
$$+\bar{g}(\sigma(X,Z),\sigma(Y,W) - \bar{g}(\sigma(X,W),\sigma(Y,Z)) = 0 \quad (15)$$

If C = 1, then M is totally geodesic. Therefore it is sufficient to consider the case C< 1.

Consider a cubic function $f(x) = \bar{g}(\sigma(X,X),JX)$ defined on $\{X \in T_x; \|X\| = 1\}$. If f attains its maximum at x, then $\bar{g}(\sigma(X,X),JX) = 0$ for Y orthogonal to x and hence $\sigma(X,X)$ is proportional to JX. Therefore, if f is constant, $\sigma(X,X) = 0$ for all X, since M is minimal. Thus f is not constant, since we are considering the case where M is not totally geodesic.

Choose e_1 to be the maximum point of f at each point $x \in M$. By the similar argument to the above, we see that f restricted to $\{X \in T_x M; \|X\| = 1\}$ and $g(X,e_1) = 0$} is not constant. Choose e_2 to be the maximum point of f restricted to $\{X \in T_x M; \|X\| = 1$ and $g(X,e_1) = 0$} and choose e_3 so that e_1, e_2, e_3 form an orthonormal frame field. Then we easily see that

$$\bar{g}(\sigma(e_2,e_2),Je_3) = 0 \quad (16)$$

Put $a_i = \bar{g}(\sigma(e_i,e_i),Je_1) = 0$. Then we have $a_1 + a_2 +, a_3 = 0$, since M is minimal. We see that $a_1 > 0$, because a_1 is the maximum value for the cubic function f and M is not totally geodesic. Moreover, from (15) we have $1 - c + a_1 a_2 - a_3^2 = 0$ and $1 - c + a_1 a_3 - a_3^2 = 0$, since (13) implies that $\bar{g}(\sigma(X,Y),JZ)$ is symmetric in X,Y,Z. Therefore we get

$$(a_1,a_2,a_3) = (2\sqrt{(1-c)/3}, -\sqrt{(1-c)/3}, -\sqrt{(1-c)/3}),$$

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which implies that

$$\sigma(e_1,e_1) = 2\sqrt{(1-c)/3}Je_1 \quad (17)$$

and

$$\bar{g}(\sigma(X,X), Je_1) = -\sqrt{(1-c)/3} \quad (18)$$

for a unit vector X orthogonal to e_1. In particular, putting $X = (e_2+e_3)/\sqrt{2}$, we obtain

$$\bar{g}(\sigma(e_2,e_3), Je_1) = 0. \quad (19)$$

In consideration of eqns. (16), (17), (18), (19) and minimality of M, we may put

$$\sigma(e_2,e_2) = -\sqrt{(1-c)/3}Je_1 + \lambda Je_2,$$
$$\sigma(e_3,e_3) = -\sqrt{(1-c)/3}Je_1 - \lambda e_2,$$
$$\sigma(e_2,e_3) = -\lambda Je_3,$$

. Putting $X = W = e_2$ and $Y = Z = e_3$ in (15), we obtain $\lambda = \sqrt{2(1-c)/3}$. Therefore we have

$$\sigma(e_2,e_2) = -\sqrt{(1-c)/3}Je_1 + \sqrt{2(1-c)/3}Je_2,$$
$$\sigma(e_3,e_3) = -\sqrt{(1-c)/3}Je_1 - \sqrt{2(1-c)/3}Je_2,$$
$$\sigma(e_2,e_3) = -\sqrt{2(1-c)/3}Je_3, \quad (20)$$

which, together with (17), (18) and (19), implies

$$\sigma(e_1,e_2) = -\sqrt{(1-c)/3}Je_2,$$
$$\sigma(e_1,e_3) = -\sqrt{(1-c)/3}Je_3 \quad (21)$$

Applying the Codazzi equation to (17), (20) and (21), we obtain $\nabla_{e_i} e_i = 0$, $\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = -\frac{1}{4}e_3,$ $\nabla_{e_1} e_3 = -\nabla_{e_3} e_1 = -\frac{1}{4}e_2,$ $\nabla_{e_2} e_3 = -\nabla_{e_3} e_2 = -\frac{1}{4}e_1$.

Therefore we have $R(e_1,e_2)e_1 = 1/16e_2$ and hence $c = 1/16$.

Bashir M.A. proved the following:

Theorem [3]: S^6 does not admit any compact proper CR-submanifold with non-negative sectional curvature and integrable holomorphic distribution[5].

Proof: Since D is integrable, then the integral submanifold of the distribution Dis a Kahler manifold. Since M is proper then $\dim D = 4$ is ruled out by a result of Gray[2], namely S^6 does not contain a 4-dimensional complex submanifold. Therefore dim D = 2. Since $v = J\overset{\perp}{D} \oplus \mu$ and M is a proper CR-submanifold of S6 we have $\dim \overset{\perp}{D} = 1$, i.e., M is 3-dimensional. Now let ω be a 2-form on the integral submanifold of D and let h be its dual. Since the integral submanifold of D is Kaehler, ω is harmonic. Using Poincare duality theorem, its dual h is also harmonic, i.e., $d\eta = \delta\eta = 0$.

Now from the hypothesis of the theorem, we get $Ric(z,z) \geq 0$. Using the integral formula

$$\int_M \{Ric(x,x) + \|\nabla X\|^2 - \frac{1}{2}\|d\eta\|^2 - (divX)^2\}dv = 0$$

and $Z \in \overset{\perp}{D}$ we have

$$\int_M \{Ric(z,z) - \frac{1}{2}\|d\eta\|^2 + \|\nabla Z\|^2 - (\delta\eta)^2\}dv = 0$$

from which we get $\nabla x Z = 0$ for all $X \in \mathcal{X}(M)$ and $Z \in \overset{\perp}{D}$, i.e., the distribution $\overset{\perp}{D}$ is parallel. Also $g(Y,Z) = 0$ for all $Y \in D$ gives $\nabla xY = 0$ for all $X \in \mathcal{X}(M)$ and $Y \in D$ This means that

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D is also parallel. D and $\overset{\perp}{D}$ being parallel implies that M is a CR-product[11], which is a contradiction to the fact that S^6 does not have any CR-product submanifold. Therefore our theorem is proven.

Theorem [4]: Let M be a complex totally real 2-dimensional submanifold of S^6. Then M is flat and minimal (see ref. (7)).

Chen B.Y. proved the following theorem for the case dimension $M \leq 3$, and Bashir M.A. proved it here for general.

Theorem [5]: Let M be a simply connected compact mixed foliate CR-submanifold of a hyperbolic complex space form $\overline{M}(-4)$. Then M is either a complex submanifold or a totally real submanifold of $\overline{M}$ [4].

4-Dimensional CR-Submanifold[10].

Concerning 4-dimensional CR-submanifolds of S^6, it is only known that there does not exist a 4-dimensional CR-product submanifold of S^6 [10,11]

First, we recall the following characterization for a 4-dimensional oriented submanifold of S^6 to be a CR-submanifolds [12,13].

Proposition [1]: Let $\varphi : M^4 \to S^6$ be an orientable 4-dimensional submanifold of S^6. Then (M^4, φ) is CR-submanifold of S^6 if and only if it satisfies one of the following conditions

(1) $\omega(T^{\perp}M^4) = 0$,

(2) $*\omega(TM^4) = 0$,

where ω denote the Kaehler form of S^6.

Remark: Let $h \in SO(7)/G_2$ be a 4-dimensional CR-submanifold. If $g \in G_2$, then $g \circ f$ is also. However, if $h \in SO(7)/G_2$, then $h \circ \phi$ is not a CR-submanifold, in general (where G_2 is the compact Lie group of all automorphisms of the octonions (known also as the Cayley division algebra).

Let $\varphi : M^4 \to S^6$ be a 4-dimensional submanifold of S^6 and discuss some fundamental properties concerning (M^4, φ). Especially we discuss a (local) orthonormal CR-adapted frame field along (M^4, φ). Let ξ_1, ξ_2 be a local orthonormal frame fields ξ_1, ξ_2 of $H^{\perp}$. Then we have $span_R(J\xi_1, J\xi_2) = T^{\perp}M^4$. Then the exterior product $\xi_1 \times \xi_2$ depends only on the given orientation of $H^{\perp}$ and is independent on the choice of the orthonormal frame fields. Also we have

$\xi_1 \times \xi_2, J(\xi_1 \times \xi_2) \in H$.

Therefore, the vector field $\xi_1 \times \xi_2$ is well defined whole on M^4. Hence H has an absolute parallelizability. We see that $\{\xi_1 \times \xi_2, J(\xi_1 \times \xi_2), \xi_1 \times \xi_2\}$ is a local orthonormal frame field of M^4. We obtain

Proposition [2]: Let $\varphi : M^4 \to S^6$ be an orientable compact 4-dimensional CR-submanifold of S^6, then the Euler number of M^4 vanishes.

By Proposition (2), we may immediately see that 4-dimensional sphere, product of two 2-dimensional spheres and complex 2-dimensional projective space cannot be realized as a CR-submanifold of S^6. On the other hand, since $\dim H^{\perp} = 2$, and $H^{\perp}$ is orientable, we can define two kinds of almost complex structures J_1, J_2 on M^4 such that

$J_1 = J_H \oplus J', J_2 = J_H \oplus (-J')$

where J_H is the restriction of the almost complex structure of S^6 to the holomorphic distribution H. hence we have the following decomposition

$$TS^6\big|_{\varphi(M^4)} = H \oplus H^{\perp} \oplus T^{\perp}M^4$$

If we set $V = H^{\perp} \oplus T^{\perp}M^4$, then V is a C^2-vector bundle over M^4. Concerning the characteristic classes of these vector bundles, we have the following

Theorem [6]: Let $\varphi : M^4 \to S^6$ be a 4-dimensional CR-submanifold of S^6. The first Pontryagin class of the tangent bundle vanishes i.e., $p_1(TM^4) = 0$.

By taking account of the G_2-structure equation on S^6, we can also show that 2-dimensional totally real distribution $H^{\perp}$ is not integrable.

Examples

The above arguments assert that there exist many obstructions for the existence of 4-dimensional CR-submanifolds of S^6. However, contrary to this circumstances, we may construct several examples of such submanifolds. We herewith introduce two typical examples of 4-dimensional CR-submanifolds of S^6.

Example [1]: Let $\gamma : 1 \to S^2 \subset \operatorname{Im} H$ be any curve in the 2-dimensional sphere $S^2 \subset \operatorname{Im} H \simeq R^3$, and $(q \in) S^3 \subset H$ be the 3-dimensional sphere of the quaternion H. Then the product immersion $\psi : 1 \times S^3 \to S^6$ which is defined by

$$\psi(t,q) = a\gamma(t) + b\bar{q}\varepsilon$$

Gives a 4-dimensional submanifold of S^6, for any $a,b > 0$ with $a^2 + b^2 = 1$. Here, t denotes the arc length parameter of γ.

In fact, let $(1 \times S^3, \psi)$ be the submanifold of S^6 gives in the above example 1. Then, we may choose the orthonormal frame field $\{v_1, v_2\}$ of the normal bundle in such a way that

$$v_1 = \gamma(t) \times \gamma(t), v_2 = b\gamma(t) - a\bar{q}\varepsilon$$

Thus we have

$$J(v_2) = (b\gamma(t) - a\bar{q}\varepsilon) \times (a\gamma(t) + b\bar{q}\varepsilon) = \gamma(t) \times \bar{q}\varepsilon = (\bar{q}\gamma(t))\varepsilon \in H\varepsilon$$

therefore, $< v_1, J(v_2) >= 0$. Thus (1) of Proposition [1] we get the desired result. Further, we may obtain the corresponding CR-frame along $(1 \times S^3, \psi)$ in the following way. A local orthonormal frame field of $H^{\perp}$ is given by

$$\psi_*(\xi_1) = J(v_1) = v_1 \times \psi = -a\gamma(t) + b(\gamma(t) \times \gamma(t)) \cdot \bar{q}\varepsilon$$

$$\psi_*(\xi_2) = J(v_2) = \gamma(t) \times \bar{q}\varepsilon.$$

On the other hand, an orthonormal frame field of H is given by

$$\psi_*(e_1) = J(v_1) \times J(v_2) = b\gamma(t) + b(\gamma(t) \times \gamma(t)) \cdot \bar{q}\varepsilon,$$

$$\psi_*(J(e_1)) = (\gamma(t)) \cdot \bar{q}\varepsilon$$

Example [2]: The following immersion $\varphi : S^1 \times S^3 \to S^6$ is a 4-dimensional CR-submanifold of S^6;

$$\varphi(\theta, q) = a(qi\bar{q}) + b(\tau(\theta)\bar{q}) \cdot \varepsilon,$$

For $a,b > 0$ with $a^2 + b^2 = 1$, where

$$\tau(\theta) = t\{-\sin(\theta) + \cos(\theta)i\} + s\{\cos(\theta)j + \sin(\theta)k\}$$

is a great circle of $S^3 \subset H$ for $t,s > 0$ with $t^2 + s^2 = 1$.

Proposition [3]: Let $\phi : S^1 \times S^3 \to S^6$ be a 4-dimensional CR-submanifold of S^6 in Example (2).

Then the map ϕ is not an imbedding. In fact, we have

$$\phi(\theta + \pi, -q) = \phi(\theta, q).$$

The immersion ϕ is full.

The immersion $\phi : S^1 \times S^3 \rightarrow S^6$ is minimal if and only if $a = \sqrt{(3+\sqrt{57})/24}, t = 1/\sqrt{2}$. For the other (a,t) in example (2), the length of the mean curvature vector field is constant, but the mean curvature vector field is not parallel with respect to the normal connection. In particular, the second fundamental is not parallel for any immersion of this type.

The normal curvature of the immersion ϕ is not flat.

The Ricci eigenvalues of the induced metric of the immersion ψ_1 are constant, but the metric is not Einstein.

If $a = 1/\sqrt{3}$ and $t = 1/\sqrt{2}$, the holomorphic distribution H is integrable.

For more details see references (11, 14, 15).

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