Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 2)
sust-journal@hotmail.com

December/2012

Solution of A fractional Differential Equation Based on Hilfer's Derivative Operator

Maher Y. K. Hamada[1*], Shawgy Hussein[2], Mohsen Hassan Abdallah[3]
[*]1-Palestine Technical College, Dair Al-Balah, Gaza-Palestine. Email:maherhamada@hotmail.com
2-Department of Mathematics - College of Science, Sudan University of Science and Technology, Sudan.
3- Department of Mathematics, Faculty of mathematical Sciences, Khartoum University, Sudan.

ABSTRACT

In this paper, the solution of the fractional differential equation $x(D_{0^+}^{u,v}y)(x)=\lambda(\varepsilon_{\alpha,\beta,pw,0^+}^{\gamma,\delta,q})(x)$ with the initial condition $(I_{0^+}^{(1-v)(1-u)}y)(0_+)=c_1$ was investigated, based on the Hilfer's fractional derivative.

المستخلص

في هذة الورقة تمت مناقشة حل المعادلة التفاضلية الكسرية $x(D_{0^+}^{u,v}y)(x)=\lambda(\varepsilon_{\alpha,\beta,pw,0^+}^{\gamma,\delta,q})(x)$ طبقا للشروط الابتدائية $(I_{0^+}^{(1-v)(1-u)}y)(0_+)=c_1$ المعتمدة علي المشتقة الكسرية لهيلفر.

KEYWORDS: Fractional differential equation, Hilfer derivative operator, Mittag-Leffler function, Fractional calculus, Laplace convolution theorem.

INTRODUCTION

Mittag-Leffler function has been studied in the early 1900s(1-3), its importance is realized during the last two decades due to its involvement in problems of physics, chemistry and applied science. Further properties of generalization of Mittag-Leffler function associated with fractional calculus operators have been studied (4-6).
Kilbas made relevant references to analytical solution of initial and boundary value problems associated with fractional differential equations(7), one of them is Cauchy type problem

$$(D_{a^+}^{\lambda}y)(x)=\lambda(\varepsilon_{\alpha,\beta.w,a^+}^{\lambda,1})(x)+f(x) \qquad (1)$$

with the initial condition $(D_{a^+}^{\lambda-k}y)(0_+)=b_k$.

The homogeneous differential equation corresponding to (1) when $f(x)=0$ is a generalization of a certain first-order Volterra-type integral differential equation govering the unsaturated behavior of free electron laser (3,8).
Hence by using Laplace transform method (9), an explicit solution on $L(0,\infty)$ of a more complicated fractional differential equation than (1) can be given which contains the generalized Riemann-Liouville fractional derivative operator, (10-13)

$$D^{u}f(x)=D^{m}\left[D^{-v}f(x)\right] \quad \mu,t>0, \qquad v=m-\mu>0$$

The following definitions were used in this study,

Definition (1):-

(**i**)- Hilfer's fractional derivative (3) is defined as:

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$$(D_{a^+}^{u,v}\varphi)(x)=\left[I_{a^+}^{v(1-u)}\frac{d}{dx}\left(I_{a^+}^{(1-u)(1-v)}\varphi\right)\right](x)$$

and its Laplace transformation [9] is given by

$$L(D_{0^+}^{u,v}\varphi)(s)=s^u L[\varphi(x)](s)-s^{v(1-u)}\left(I_{0^+}^{(1-v)(1-u)}\varphi(x)\right)(0_+)$$

(**ii**)- An integral operator [13,14] is defined as

$$(\varepsilon_{\alpha,\beta,w,a^+}^{\gamma,\delta,q}\varphi)(x)=\int_0^x(x-t)^{\beta-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}\left[w(x-t)^{\alpha}\right]\varphi(t)dt$$

(**iii**)- First order differential equation [15- 17] $y'+p(x)y=Q(x)$ has the solution

$$y(x)=e^{-\int p(x)dx}\left[c_2+\int Q(x)e^{\int p(x)dx}dx\right]$$

(**iv**)- Laplace transformation is given by :[9]

$$L\{z^{a-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(xz^{\sigma})\ ;s\}=\frac{\Gamma\delta s^{-a}}{\Gamma\gamma}\,{}_3\psi_2\left[\begin{matrix}(\gamma,q),(a\sigma),(1,1)\\(\beta,\alpha),(\delta,\sigma)\end{matrix}\ ;\frac{x}{s^{\sigma}}\right]$$

(**v**)- Also another Laplace transformation see[7,18-20] is given by

$$L\left\{\frac{1}{\Gamma u}\int_0^x t^{u-1}(x-t)^{\beta-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}\left[w(x-t)^{\alpha}\right]dt;s\right\}=\frac{1}{s^u}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}w^n}{(\delta)_{pn}\Gamma(\alpha n+\beta)}\frac{1}{s^{\alpha n+\beta}}$$

(vi)- The two –parameter function of Mittag-Leffler [1-3] is defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\qquad \alpha>0\quad \beta>0$$

(vii)- Generalized Mittag-leffler function [1-3] defined as

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{pn}}$$

We need the theorem proved by [18]

Theorem (2):-

In the space $L(0,\infty)$ if the fractional differential equation is considered

$$(D_{0^+}^{u,v}y)(x)=\lambda(\varepsilon_{\alpha,\beta,pw,0^+}^{\gamma,\delta,q})(x)+f(x)$$

$0<u<1$, $0\le v\le 1$, $\gamma,\alpha,\beta,w\in C$, with min $\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\gamma),\mathrm{Re}(\delta)\}>0$ and $p,q>0$, and with initial condition $(I_{0^+}^{(1-v)(1-u)}\varphi)(0_+)=c$, then we have the solution

$$y(x)=c\frac{x^{u-v(1-u)-1}}{\Gamma(u-v+uv)}+\lambda x^{\beta+u}E_{\alpha,\beta+u+1,p}^{\gamma,\delta,q}(wx^{\alpha})$$

$$+\frac{1}{\Gamma u}\int_0^x(x-t)^{u-1}\varphi(t)dt \qquad (3)$$

Proof:-

Using Definition (1) (i) and (ii), also making use of (iv) we can get

$$L(D_{0^+}^{u,v}(y))=L\left[\lambda(\varepsilon_{\alpha,\beta,p,w,0^+}^{\gamma,\beta,q})\right](x)+L(f(x)).$$

Hence

$$s^uL(y)-s^{v(1-u)}\left(I_{0^+}^{(1-v)(1-u)}(y)\right)(0_+)=$$

$$\lambda L\left[\int_0^x(x-t)^{\beta-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(x-t)^{\alpha}dt\right]+L(f(x))$$

According to Laplace convolution theorem and by applying the initial condition we have

$$s^uY(s)-cs^{v(1-u)}=\lambda L\left[x^{\beta-1}E_{\alpha,\beta,p}^{\gamma,\delta,q}(wx^{\alpha}\right](s)+L(1)(s)+F(s)$$

$$=\frac{\lambda s^{-\beta}\Gamma\delta}{\Gamma\gamma}\,{}_3\psi_2\left[\begin{matrix}(\gamma,q),(\beta,\alpha),(1,1)\\(\beta,\alpha),(\delta,p)\end{matrix}\ ;\frac{w}{s^{\alpha}}\right]\frac{1}{s}+F(s)$$

$$=\frac{\lambda s^{-\beta-1}\Gamma\delta}{\Gamma\gamma}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1)\\(\delta,p)\end{matrix}\ ;\frac{w}{s^{\alpha}}\right]+F(s).$$

Divided the above equation by s^u we get

Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
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$$Y(s)=cs^{v(1-u)-u}+\frac{\lambda s^{-\beta-u-1}\Gamma\delta}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(1+n)}{\Gamma(\delta+pn)n!}\left(\frac{w}{s^{\alpha}}\right)+s^{-u}F(s).$$

Taking Laplace inverse of both sides of the last equation, we get

$$y(x)=cL^{-1}\left[s^{v(1-u)-u}\right](x)+\frac{\lambda\Gamma\delta}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(1+n)}{n!\Gamma(\delta+pn)}w^{n}L^{-1}\left[s^{-\alpha n-\beta-u-1}\right]$$

$$+L^{-1}L\left(\frac{x^{u-1}}{\Gamma u}f(x)\right),$$

where

$$s^{-u}F(s)=L\left(\frac{x^{u-1}}{\Gamma u}f(x)\right).$$

Again applying Laplace convolution theorem, we get

$$y(x)=c\frac{x^{u-v(1-u)}}{\Gamma u-v+uv}+\lambda x^{\beta+u}E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha})$$

$$+\frac{1}{\Gamma u}\int_{0}^{x}(x-t)^{u-1}f(t)dt,$$

which gives the solution.

Remark (3):

If we put $f(t)=t^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wt^{\alpha})$ in (2) we get the following particular case of (3) which is stated in the next lemma.

Lemma (4):

Under the same conditions of theorem (2), the following fractional differential equation

$$(D^{u,v}_{0^+}y)(x)=\lambda(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,pw,0^+})(x)+x^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wx^{\alpha}),$$

with the initial condition $(I^{(1-v)(1-u)}_{0^+}\varphi)=c$ has the solution

$$y(x)=c\frac{x^{u-v(1-u)}}{\Gamma(u-v(1-u))}+(\lambda+1)E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha}).$$

Proof:-

Putting $f(x)=x^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha})$ in (3) we get

$$y(x)=c\frac{x^{u-v(1-u)-1}}{\Gamma u-v+uv}+\lambda x^{\beta+u}E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha})$$

$$+\frac{1}{\Gamma u}\int_{0}^{x}(x-t)^{u-1}t^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wt^{\alpha})dt.$$

But

$$\frac{1}{\Gamma u}\int_{0}^{x}(x-t)^{u-1}t^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wt^{\alpha})dt=$$

$$L^{-1}L\left[\frac{1}{\Gamma u}\int_{0}^{x}(x-t)^{u-1}t^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wt^{\alpha})dt\right]$$

$$=L^{-1}\left\{L\left(\frac{x^{u-1}}{\Gamma u}\right)(s)L(x^{\beta+1-1}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wx^{\alpha}))(s)\right\}.$$

From Definition (1) (iv) we get

$$\frac{1}{\Gamma u}\int_{0}^{x}(x-t)^{u-1}t^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(wt^{\alpha})dt$$

$$=L^{-1}\left\{\frac{1}{s^{u}}\frac{\Gamma\delta}{\Gamma\gamma}s^{-\beta-1}{}_{3}\psi_{2}\left[\begin{matrix}(\gamma,q),(\beta+1,\alpha),(1,1)\\(\beta+1),(\delta,p)\end{matrix};\frac{w}{s^{\alpha}}\right]\right\}(s)$$

$$=L^{-1}\left\{\frac{\Gamma\delta}{\Gamma\gamma}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(1+n)}{n!\Gamma(\delta+pn)}\left(\frac{w}{s^{\alpha}}\right)^{n}s^{u-\beta-1}\right\}(x)\quad(4)$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{(\delta)_{pn}}w^{n}L^{-1}\left\{s^{-\alpha n-\beta-u-1}\right\}(x)$$

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$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{(\delta)_{pn}} w^{n} \frac{s^{(\alpha n+\beta+u+1)}}{\Gamma(\alpha n+\beta+u+1)}$$

$$= x^{\beta+u} E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha}),$$

by applying Laplace's convolution theorem. Hence the solution is given by

$$y(x) = c\frac{x^{u-v(1-u)-1}}{\Gamma(u-v(1-u))} + (\lambda+1)E^{\gamma,\delta,q}_{\alpha,\beta+u+1,p}(wx^{\alpha}). \quad (5)$$

RESULTS

Now we give our main results

Theorem (5):

Under the same conditions of Theorem (2), the following fractional differential equation

$$x(D^{u,v}_{0^+}y)(x) = \lambda(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,pw,0^+})(x) \quad (6)$$

with the initial condition $(I^{(1-v)(1-u)}_{0^+}y)(0_+) = c_1$ has the following solution

$$y(x) = c_2\frac{x^{u-1}}{\Gamma u} + c_1\frac{x^{u-v(1-u)-1}}{\Gamma(u-v(1-u))} + \frac{\lambda}{\Gamma u}\int_0^x t^{u-1}(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}w(w-t)^{\alpha}dt \quad (7)$$

where c_1, c_2 are constants.

Proof:

We will use the same procedures as in Theorem (2) in addition, we can use

$$\frac{\partial^n}{\partial s^n}[Ly(x)](s) = (-1)^n L[x^n y(x)](s) \quad (8)$$

Now take the Laplace transform for both sides of eqn. (6)

$$L[x(D^{u,v}_{0^+}y)](s) = \lambda L[\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,pw,0^+}](s).$$

Now using (8) with $n=1$ we get

$$\frac{\partial}{\partial s}[L(D^{u,v}_{0^+}y)](s) = (-1)L[xD^{u,v}_{0^+}y](s)$$

From definition (1) (i) and theorem (2), we have

$$\frac{\partial}{\partial s}\left[s^{u}Y(s) - c_1 s^{v(1-u)}\right] = -\lambda\frac{s^{-\beta-1}\Gamma\delta}{\Gamma\gamma}\,{}_3\psi_2\left[\begin{matrix}(\gamma,q),(\beta,\alpha),(1,1) \\ (\delta,p),(\beta,\alpha)\end{matrix};\frac{w}{s^{\alpha}}\right]$$

$$s^{u}Y'(s) - us^{u-1}Y(s) - c_1 v(1-u)s^{v(1-u)-1} + \lambda\frac{s^{-\beta-1}\Gamma\delta}{\Gamma\gamma}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1) \\ (\delta,p)\end{matrix};\frac{w}{s^{\alpha}}\right] = 0$$

Dividing the above equation by s^{u}, yields

$$Y'(s) - \frac{u}{s}Y(s) - c_1 v(1-u)s^{v(1-u)-u-1} + \lambda\frac{s^{-\beta-u-1}\Gamma\delta}{\Gamma\gamma}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1) \\ (\delta,p)\end{matrix};\frac{w}{s^{\alpha}}\right] = 0 \quad (9)$$

Which is a first order ordinary differential equation then, the solution of (9) is

$$Y(s) = \exp\left(-\int\frac{u}{s}ds\right)\times\left[c_2 + \int\left(\begin{matrix}c_1 v(1-u)s^{v(1-u)-u-1} - \\ \lambda\frac{s^{-\beta-u-1}\Gamma\delta}{\Gamma\gamma}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1) \\ (\delta,p)\end{matrix};\frac{w}{s^{\alpha}}\right]\end{matrix}\right) \exp\left(\int\frac{u}{s}ds\right) ds.\right]$$

So that

$$Y(s) = s^{-u}\left[c_2 + \left(c_1\int v(1-u)s^{v(1-u)-u-1} - \lambda\frac{s^{-\beta-u-1}\Gamma\delta}{\Gamma\gamma}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1) \\ (\delta,p)\end{matrix};\frac{w}{s^{\alpha}}\right]\right)s^{u}ds\right]$$

$$= s^{-u}\left[c_2 + \left(c_1 \int v(1-u)s^{v(1-u)-1} - \lambda \frac{s^{-\beta-1}\Gamma\delta}{\Gamma\gamma}\, {}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1) \\ (\delta,p)\end{matrix} ;\frac{w}{s^{\alpha}}\right]\right)ds\right],$$

$$= s^{=u}\left[\begin{matrix} c_2 + c_1 \dfrac{v(1-u)s^{v(1-u)}}{v(1-u)} - \lambda \dfrac{s^{-\beta-1}\Gamma\delta}{\Gamma\gamma} \\ \displaystyle\int \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+qn)\Gamma(1+n)}{n!\,\Gamma(\delta+pn)}\left(\frac{w}{s^{\alpha}}\right)^n ds \end{matrix}\right]$$

$$= s^{-u}\left[c_2 + c_1 s^{v(1-u)} - \lambda \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{(\delta)_{pn}} w^n \int s^{-\alpha n-\beta-1} ds\right]$$

$$= s^{-u}\left[c_2 + c_1 s^{v(1-u)} - \lambda \sum_{n=0}^{\infty}\frac{(\gamma)_{qn} w^n}{(\delta)_{pn}} \frac{s^{-\alpha n-\beta}}{-(\alpha n+\beta)}\right]$$

$$= c_2 s^{-u} + c_1 s^{v(1-u)-u} + \frac{\lambda}{s^u}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn} w^n}{(\delta)_{pn}(\alpha n+\beta)}\frac{1}{s^{\alpha n+\beta}}$$

Taking the Laplace inverse of the last equation, gives

$$y(x) = c_2 \frac{x^{u-1}}{\Gamma u} + c_1 \frac{x^{u-v(1-u)-1}}{\Gamma(u - v(1-u))} +$$

$$L^{-1}\left[\frac{\lambda}{s^u}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}\, w^n}{(\delta)_{pn}(\alpha n+\beta)s^{\alpha n+\beta}}\right]$$

Now using Definition (1) (v), we can get the required solution

$$y(x) = c_2 \frac{x^{u-1}}{\Gamma u} + c_1 \frac{x^{u-v(1-u)-1}}{\Gamma(u - v(1-u))} +$$

$$\frac{\lambda}{\Gamma u}\int_0^x t^{u-1}(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\, w(x-t)^{\alpha}\, dt$$

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