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Approximation of Symmetric Diffusion Contraction Semigroups and Kernels

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ABSTRACT: We give an $L^P -$ operator norm estimate of diffusion contraction Semigroups $(P_t)_{t\geq 0}$ on $\mathbb{R}^d$ $(d \geq 2)$with corresponding diffusion kernels $P_t(x, y)$ associated with uniformly elliptic operators with measurable coefficients and limitting Markov transition semigroup Q_t^h on the state space $\mathbb{R}_h^d$.

المستخلص :

طبقا لنواة الانتشار المقابلة$(d \geq 2)$ $\mathbb{R}^d$ على $(P_t)_{t\geq 0}$لشبه زمر انكماش الانتشار $- L^P$تم إعطاء تقدير لنظيم مؤثر Q_t^h المشاركة لـ P_1 (x,y) الناقصية المنتظمة مع المعاملات المقيسة ونهائية شبه زمر انتقال ماركوف

على فضاء الحالة $\mathbb{R}_h^d$.

KEYWORDS*: *Elliptic operator, Markov transition semigroup, diffusion kerned, Gaussian kernel, density kernel

INTRODUCTION

This paper follows the work of Zhen- Qing Chen, *et al* [(1)] in which they proved the minimal fundamental solution to the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right)\mathbf{u} = \mathbf{0}$ **on** $\mathbb{R}^d$,

with the Gaussian kernel

$$H(t, x, y) = (4\pi t)^{-\frac{d}{t}} e^{-\frac{|x-y|^2}{4t}}$$

which describes the heat propagation in the space $\mathbb{R}^d$ $(d \geq 2)$.

We state the needed statement of results for the matter of convenience. Let $L = \nabla.(A\nabla)$ and $\tilde{L} = \nabla.(\tilde{A}\nabla)$ be two uniformly elliptic operators of divergence form on $\mathbb{R}^d$ with measurable coefficients. Let $\lambda \geq 1$ be a constant such that

$$\lambda^{-1} I_{d\times d} \leq A(.) = (a_{ij}(.)) \leq \lambda I_{d\times d}$$

$$\lambda^{-1} I_{d\times d} \leq \tilde{A}(.) = (\tilde{a}_{ij}(.)) \leq \lambda I_{d\times d}$$

Let $P_t = e^{tL}$ *and* $\tilde{P}_t = e^{t\tilde{L}}$ be the diffusion semi groups of L and $\tilde{L}$, respectively. It is well known that P_t *and* $\tilde{P}_t$, have density kernels $P_t(x,y)$ *and* $\tilde{P}_t(x,y)$ with respect to the Lebesgue measure, called diffusion kernels[(2-5)] Furthermore, by Aronson's inequality and Nash's Hölder estimate for diffusion kernels, there are constants

$c_1 = c_1(d,\lambda) > 1$ *and* $\upsilon = \upsilon(d,\lambda) \in (0,1)$

such that

$$P_t(x,y) \le c_1 t^{-\frac{d}{2}} \exp^{-\frac{|x-y|^2}{c_1 t}}$$

$$|P_t(x,y) - P_t(x_1,y_1)| \le c_1 t^{-\frac{d+\upsilon}{2}} (|x-x_1| \vee |y-y_1|)^{\upsilon} \qquad (1)$$

for all $t > 0$ *and* $(x,y),(x_1,y_1) \in \mathbb{R}^d \times \mathbb{R}^d$

The Authors in their paper [(1)] established a quantitative upper bound estimate for:

$$|P_t(x,y) - \tilde{P}_t(x,y)| \ \text{as well as} \ \|P_t - \tilde{P}_t\|_P$$

For 1≤ P≤∞,

in terms of the local L^2- distance between A *and* $\tilde{A}$ defined below. Let Z^d be the integer lattice in $\mathbb{R}^d$, and for each $k \in Z^d$ let $D_k = \{x \in \mathbb{R}^d : |x-k| < 2\sqrt{d}\}$

For $q \ge 1$ define the local L^2 - norm distance between two matrices A *and* $\tilde{A}$ by

$$\|A - \tilde{A}\|_{L^q_{Loc}} = \sup_{k \in z^d} \sum_{i_1 j=1}^{d} \|a_{ij} - \tilde{a}_{ij}\|_{L^q}(D_k)$$

Note that A *and* $\tilde{A}$ are bounded, and therefore the topologies induced by $\|A-\tilde{A}\|_{L^q_{Loc}}$ *and* $\|A-\tilde{A}\|_{L^2_{Loc}}$ are equivalent for any $1 \le q < \infty$.

RESULTS:

Theorem 1: Let $A(.)$ be $\beta-$Hölder continuous in L^p_{Loc}.Then there are positive finite constants $\alpha = \alpha(P,\lambda,\beta,d)$, $c = c(d,\lambda,P,\beta)$ and Markov transition semi group Q^h_t on the state space $\mathbb{R}^d_h$ such that for any $\beta-$Hölder continuous f *in* L^∞

$$\|Q^h_t f - P_t f\|_{h,\infty} \le ct^{-\frac{1}{8}} \left(\frac{1}{L\bar{o}gh}\right)^{\alpha} (1 \vee \|f\|_\infty), 0 < t \le 1 \qquad (2)$$

for $x > 0$, define $L\bar{o}g\,x = \max\{-L og\,x, 0\}$.

Theorem 2: Suppose that D is a bounded $C'-$smooth domain in $\mathbb{R}^d$, then there is a constant $q_0 = q_0(D,\lambda) > 1$ such that for $q > q_0$ *and* $\alpha \ge 2$ there is a constant $c(\alpha) = c(D,\lambda,q,\alpha) > 1$ *so that*

$$\|p^D_t - \tilde{p}^D_t\|^{\alpha}_{\alpha} \le c(\alpha) t^{-\left(d+\frac{\cup}{2}\right)+\frac{d}{\alpha}} \sum_{i,j=1}^{d} \|a_{ij} - \bar{a}_{ij}\|_{L^{2q}(D)}$$

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when $1<\alpha<2$, by duality we have $\left\|P_t^D-\tilde{P}_t^D\right\|_\alpha=\left\|P_t-\tilde{P}_t\right\|_{\alpha'}$, *where* $\alpha'>2$ is the conjugate number for α.

Theorem 3 : There are constants $c_3=c_3(d,\lambda)>1$ *and* $q_0=q_0(d,\lambda)>1$ such that for any $q>q_0$ *and* $\alpha\geq 2$, there is a constant $c_4=c_4(d,\lambda,q)>0$ such that

$$\left\|P_t-\tilde{P}_t\right\|_\alpha^\alpha\leq c_3 e^{-\left\{\frac{1}{c_3 t}\right\}+c_4 t-\left(\frac{d+1}{2}\right)+\frac{d}{\alpha}}\left\|A-\tilde{A}\right\|_{L^{2q}_{Loc}} \quad (3)$$

when $1<\alpha<2$, by duality we have

$\left\|P_t-\tilde{P}_t\right\|_\alpha=\left\|P_t-\tilde{P}_t\right\|_{\alpha'}$, where $\alpha'>2$ is the conjugate number for α.

The right hand side of (3) is not a good estimate for Large t raised by Zhen-Quing[1], Fuku Shima[6] and Hassan[7] however we have:

Theorem 4: If P_t *and* $\tilde{P}_t$ are contractions in $L^\propto(\mathbb{R}^d)$ then for each $\alpha\geq 1$ and $n\geq 1$ we have $\left\|P_{(n+1)t}-\tilde{P}_{(n+1)t}\right\|_\alpha\leq(n+1)\left\|p_t-\tilde{p}_t\right\|$.

Proof: inductively we have $\left\|P_{(n+1)t}-\tilde{P}_{(n+1)t}\right\|_\alpha=\left\|P_{nt+t}-\tilde{P}_{nt+t}\right\|_\alpha=\left\|P_{nt}\left(P_t-\tilde{P}_t\right)+\left(P_{nt}-\tilde{P}_{nt}\right)\tilde{P}_t\right\|_\alpha$
$=\left\|P_{nt}P_t-\tilde{P}_{nt}\tilde{P}_t\right\|_\alpha=\left\|P_t^{(n+1)}-\tilde{P}_t^{(n+1)}\right\|_\alpha$.

$\leq(n+1)\left\|P_t-\tilde{P}_t\right\|_\alpha$.

Which can be used to get upper bounds for large t .We can set $\left\|P_t-\tilde{P}_t\right\|\leq 2^\alpha$ which leads to that.

$$\left\|A-\tilde{A}\right\|_{L^{2q}_{Loc}}\leq\left(c_5-c_6 e^{-\frac{1}{c_3 t}}\right)t^{\frac{d+1}{2}+\frac{d}{k}}$$

where
$c_5=c_5(d,\lambda,q,\alpha)>0$ *and* $c_6=c_6(d,\lambda,q)>0$.

Theorem 5: There is a constant $c=c(d,\lambda)>0$ such that

$$\sup_{x,y\in\mathbb{R}^d}\left|p_t(x,y)-\tilde{p}_t(x,y)\right|\leq ct^{\frac{d}{2}}\left\|p_t-\tilde{p}_t\right\|\leq ct^{\frac{d}{2}}2^{\frac{v}{d+v}}$$

for all $t>0$.

Proof: Note that

$$\frac{1}{\vee(r)^2}\left|\left(P_t I_{B_x(r)}\ ,\ I_{B_y(r)}\right)-\left(\tilde{P}_t I_{B_x(r)},I_{B_y(r)}\right)\right|$$
$$\frac{1}{\vee(r)^2}\left|\left(\left(p_t-\tilde{p}_t\right)I_{B_x(r)},I_{B_x(r)}\right)\right|$$
$$\leq\frac{1}{\vee(r)^2}\left\|P_t-\tilde{P}_t\right\|_2\leq\frac{2}{\vee(r)}$$
$$=\frac{2}{\omega d^r}.$$

However, by (1),

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$$\left|p_t(x,y)-\frac{1}{\vee(r)^2}\left(p_t I_{B_x(r)}, I_{B_y(r)}\right)\right|$$
$$\le \frac{1}{\vee(r)^2}\int_{B_x(r)\times B_y(r)}\left|P_t(x,y)-P_t(z,v)\right|dz\ \ dv\ |$$

$$\le c_1 t^{-\frac{(d+v)}{2}} r^{v}.$$

This proves theorem 5 after choosing r so that :

$$t^{-\left(\frac{d+v}{2}\right)} r^{v}=\frac{1}{r^d}\left\|P_t-\tilde{P}_t\right\|_2 \le \frac{2}{r^d}\ ,\ that\ is\ ,$$

$$r \le t^{\frac{1}{2}}\ 2^{\frac{1}{(d+v)}} \bullet$$

Now we prove the following estimate [(7)] :

Theorem 6: There is a constant $c=c(d,\lambda)>0,\ P_t\ and\ \tilde{P}_t$ are contractions in $L^{\alpha}(\mathbb{R}^d)$ for each $\alpha \ge 1$ such that**:**

$$\sup_{x,y\in\mathbb{R}^d} |p_t(x,y)\quad \tilde{p}_t(x,y)| < c_\alpha t^{-\frac{d}{2}}$$

Proof: Since P_t and similarly $\tilde{P}_t$ are contractions, set $\alpha=\frac{v}{d+v}\ge 1$,*we have*

$$\sup_{x,y\in\mathbb{R}^d} |p_t(x,y)-\tilde{p}_t(x,y)| < c_\alpha t^{-\frac{d}{2}} \text{ for } t>0\ and\ \ \alpha\ge 1$$

Combining theorem 5, with
$$|p_t(x,y)-\tilde{p}_t(x,y)| < \min\left\{c_1 t^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{c_1 t}}, c_\alpha t^{-\frac{d}{2}}\right\}$$
for $t \ge D$, $\alpha \ge 1$. Using the fact that $\min\{a \wedge b\} \le \sqrt{ab}\ \ for\ any\ \ a,b \ge 0$ we get that there is a constant:

$$c=c(d,\lambda)>0$$

such that :

$$\left|P_t(x,y)-\tilde{P}_t(x,y)\right| \le f_\alpha(t)e^{\varphi(t)},\ where\ \ f_\alpha(t)=\left(c_\alpha t^{-d}\right)^{\frac{1}{2}}$$

Afractional polynomial, where $\varphi(t)=-\frac{|x-y|^2}{c_1 t}$ for $t>0\ \ and\ \ \alpha\ge 1$.

To prove the required result we now let Q_t^h be the semi group associate with L^h .

Theorem 7: Let $A\ be\ \beta$-Hölder continuous in L^p_{Loc} then there are positive finite constants μ_α , c and a Markov transition semigroups $Q_t^h\ and\ Q_t$ on the state space $\mathbb{R}_h^d$ such that for any function f :

$$\left\|Q_t^h f - Q_t f\right\| \le \int_0^t \left\|\ell^h\ Q_s\ f - L\ Q_s^{\delta} f\right\|_{h,\infty} ds$$

Theorem 8: Suppose that A and B satisfy (1) let Q and Q_t be constructed as above. Assume that the function f has bounded derivative up to third order. Then there are constants $0<c<\infty\ \ and\ \ 0<\mu<\infty$, independent of $h, \alpha\ \ and\ \ f$ such that:

$$\sup_{0\le t\le 1, x\in\mathbb{R}_h^d} \left|\mathbb{Q}_t^h f(x)-\mathbb{Q}_t f(x)\right| < che^{\mu_\alpha}\|f\|_{c_3}$$

Now we prove our main result and find a sharp estimate for the bound.

Theorem 9: let $A\ be\ \beta-$ Hölder continuous in L^p_{Loc} then there are positive finite constants μ_α, c and a Markov transition semi groups $Q_t^h\ \ and\ \ Q_t$ on the state space $\mathbb{R}_h^d$ such that for any function f with bounded derivative

$$\|P_t f - Q_t f\|_{h,\infty} \leq ch\|f\|_\infty$$

For

$0 << c << \infty, 0 < \mu_\alpha < \infty$ *and* $h = h_t$

Proof: let Q_t^h be a semi group associated with L^h.For the convergence rate of $Q_t^h f$ toward Q_t we have (see eqn (1))

$$\|Q_t^h f - Q_t f\| \leq \int_0^t \|\ell^h Q_s f - L Q_s^h f\|_{h,\infty} ds.$$

Applying theorems 7 and 8 we have

$$\sup_{0\leq t\leq 1, x\in \mathbb{R}_h^d} |\mathbb{Q}_t^h f(x) - \mathbb{Q}_t f(x)| < che^{\mu_\alpha}\|f\|_{c_\beta} \qquad (4)$$

Hence

$$\|p_t f - Q_t f\|_{h,\infty} = \|P_t f - Q_t^h f + Q_t^h f - Q_t f\|_{h,\infty}$$

$$\leq \|p_t f - Q^h f\|_{h,\infty} + \|Q^h f - Qf\|_{h,\infty} \qquad (5)$$

Theorem 1 shows that

$$\|Q_t^h f - P_t f\|_{h,\infty} \leq \left(t^{-\frac{1}{8}}\left(\frac{1}{\log h}\right)\right)^\alpha \|f\|_\infty \qquad (6)$$

Similarly for $\alpha > 1$ (see the end of the proof of Theorem 2),

$$\|P_t f\|_\infty \leq \left(ct^{-\frac{d}{2}}\right)^{\frac{1}{\alpha}} \|f\|_\alpha$$

Substituting equations (4) and (6) in (5) where $h = h_t = t^{-\frac{1}{8}}$, $c_\alpha = e^{\mu_\alpha}$ gives

$$\|P_t f - Q_t f\|_{h,\infty} \leq ch\|f\|_\infty.$$

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