Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
www.sustech.edu

June /2012

On the Development and Recent Work of Cayley Transforms [I]

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ABSTRACT: This paper deals with the recent work and developments of Cayely transforms which had been rapidly developed.

المستخلص :

تتعلق هذه الورقة العلمية بتقديم العمل الحديث والتطورات السريعة لتحويلات كمايلي.

INTRODUCTION

In this study we shall give the basic important transforms and historical background of the subject, then introduce the accretive operator of Cayley transform and a maximal accretive and dissipative operator, and relate this with Cayley Transform which yield the homography in function calculus. After this we will study the fractional power and maximal accretive operators; indeed all this had been developed by Sz.Nagy and C. Foias [1]. The method of extending and accretive (or dissipative) operator to a maximal one via Cayley transforms modeled on Von Neumann's theory on symmetric operators, is due to Philips [1] fractional powers of operators A in Hilbert space , or even in Banachh space, such as – A is infinitesimal generator of a conts one parameter sem–group of contractions, have been studied or constructed by different authors using different methods , we shall appear Sz. Nagy's method here. Many had been worked in the uniqueness theorem (in its form on dissipative operators) , MaCaev and Palant [in case of bounded operators] and Langer in the general case. The proof of Belasz – Nagy is slightly simplified variant of that of Langer [1, 2].

Cayley Transforms $V = (A - iI)(A+iI)^{-1}$

First we introduce the transformation $B = (I + T^*T)^{-1}$ and

$C = T(I + T^*T)^{-1}$ in order to study the Cayely transforms.

If the linear transform T is bounded , it is clear that the transformation B appearing above is also bounded , symmetric , and such that $0 \le B \le I$; [3].

C = TB is then bounded too. If T is a linear transformation with dense domain, we know that T*, and consequently also T*T, exist, but we know nothing of their domains of definition [4]. The proof of following theorem gives a rather surprising fact:

Theorem 1: If the linear transformation T is closed and if its domain is dense in H, the transformations $B = (I + T^*T)^{-1}$, $C = T (I + T^*T)^{-1}$ are defined every where and bounded, $\|B\| \le 1 \quad \|C\| \le 1$;

Moreover, B is symmetric and positive.

Proof: In order to prove this theorem, we use the graph of T [(4)] which in this case is a closed linear set.

Let h be arbitrary element of H. since G_r and $V_{G_{T*}}$ are complementary orthogonal subspaces of N [(4)], we can decompose the element {h , 0 } of N into the sum of an element of G_T and element of $V_{G_{T*}}$, and there is only one way.

$\{ h , 0\} = \{ f , Tf \} + \{T^* g , - g \}$ (1)

This means, passing to the component, that the system of equations :

$h = f + T^*g , 0 = Tf - g$

has a unique solution f in D_T and g in D_{T^*}.

writing f = Bh, g = Ch

we define two transformation of H into it self which are obviously linear.

The system of equations can then be written in the form :

$I = B + T^*C , O = TB – C$

From which :

$C = TB, I = B + T^*TB = (I + T^*T) B$ (2)

Now since the two terms in the second member of (1) are orthogonal, we have is obtained.

$$\|h\|^2=\|\{h,o\}\|^2=\|\{f,Tf\}\|+\|\{T^*g,-g\}\|^2=\|f\|^2+\|Tf\|^2+\|T^*g\|^2+\|g\|^2$$

From which we have

$$\|Bh\|^2+\|Ch\|^2=\|f\|^2+\|g\|^2\leq\|h\|^2,$$

Therefore:

$$\|B\|\leq 1, \|C\|\leq 1.$$

For any element u in the domain of T*T, we have ((I +T*T) u, u)=(u , u)+(Tu, Tu)≥ (u, u)

Hence (I + T*T) u = 0 implies u = 0

This assures that the inverse transformation $(I + T^*T)^{-1}$ exists. According to equation (2), it is defined every where and equal to B ;

$B = (I + T^*T)^{-1}$

The transformation B is symmetric and positive in fact

(Bu , v) = (Bu , (I + T*T) BV) = (Bu , Bv) + (Bu, T*TBv) =

= (Bu , Bv) + (T*TBu,Bv) = ((I + T*T) Bu , Bv) = (u , Bv) and (Bu , Bv) = (Bu , (I + T*T) Bu) = (Bu , Bu) + (TBu , TBu) ≥ 0.

This completes the proof of the theorem. The transformations B and C, which play an essential role in the above discussion, are obviously the symmetric components of the normal transformations.

$C + iB = (A + iI) (I + A^2)^{-1} = (A – iA)^{-1}$

This transformation and its adjoint, $C – iB = (A –iI)^{-1}$ are more generally the transformations $R_z=(A-zI)^{-1}$, where z is a real or complex parameter , also play an essential role in other proofs of the theorem.

Now the existence of

$R_{\pm i} = (A \pm iI)^{-1}$

can be proved directly from the relation

$$\|(A \pm iI)f\|^2 = (Af, Af) \pm i(f, Af) \mp i(Af, f) + (f, f) = \|Af\|^2 + \|f\|^2 \quad (3)$$

In fact, it show that neither of equations

(A-iI) f = 0 , (A + iI) f = 0

is possible unless f = 0 , which suffices for the existence of the inverse . Furthermore, we see that $\|(A \mp iI)f\| \geq \|f\|,$

which implies that

$$\|g\| \geq \|R_{\pm i}g\|. \quad (4)$$

for all elements g in the domain of R_i, R_{-i} , respectively.

Now these domains coincide with the entire space, this will follow from the fact that these domains are:

a) closed, and b) everywhere dense in H.

Proposition a) follows from the fact that the transformations R_i and R_{-i} are continuous (consequence of (4) and closed (since A and A ± iI are closed). Proposition b) is proved , for example for R_i ; in the following manner. If the domain of R_i, which is a linear set, not every where dense in H , there would be an element h≠0, orthogonal to the domain R_i , that is to all elements of the form (A-iI) f. But it then follows from the equation ((A-iI)f,h) = 0 that is the domain of (A-iI)* = A + iI and that (A + iI) h = 0 .

Hence h = 0, which contractdicts the hypothesis that h ≠ 0.

The transformations $R_{\pm i}$ are therefore defined every where and bounded. The same is true for $R_z = R_{x+iy}$ when $y \neq 0$, since $(A-(x+iy)I)^{-1} = \frac{1}{y}\left(\frac{A-xI}{y} - iI\right)^{-1}$. Of course R_z can exist and be bounded even for certain real values of the parameter Z.

Now returning to relation (3). It showed that $\|(A-iI)f\| = \|A+iI)f\|,$

That is, $\|A-iI)(A+iI)^{-1}g\| = \|g\|.$ The transformation

V = (A-iI) $(A+iI)^{-1,}$ ……………….(4a).

called the Cayley transformation of A and it is therefore isometric[(4)]. It is defined for element of the form:

g = (A + iI) f (5)

by Vg = (A – iI) f (6)

where f runs through D_A. Then g and Vg each run through the entire space H. Hence V is also unitary.

We give another equivalent definition.

Definition 1.

Let A ε M_n (¢) s.t , I + A is invertible.

The Cayley transform of A , denoted by C (A) is defined to be [3]:

$$C(A) = (I + A)^{-1} (I – A) \qquad (7)$$

The Cayley transform , not surprisingly , was defined in 1846 by Cayley[3]. He proved that if A is skew Hermitain, then C(A) is unitary and the conversely, provided of course that C(A) is exist. This feature is useful e.g, in solving matrix equations subject to the solution being unitary by transforming them into equation for skew –Hermitian matrices, later we shall discuss this point deeply. Now it is easy to recover A starting with V. It follows from (5) and (6), by addition and subtraction that

$(I + V) g = 2Af$, $(I-V)g = 2if$,

From that we see that $(I – V) g = 0$ implies that $f = 0$ and consequently , by (5), $g = 0$ also hence $(I – V)^{-1}$ exist and $2Af=(I+V)(I-V)^{-1}2if$, that is,

$$A = i (I + V) \qquad (I – V)^{1} \quad (8)$$

Example 1 :

Let V =

$$\int_0^{2\Pi} e^{i\phi} dF_\phi \qquad (F_0 = o, F_{2\Pi} = I)$$

be the spectral decomposition[4] of the unitary transformation V, using relation (8), we can deduce the spectral decomposition of A from that of V in the following manner:

We begin by observing that F_Φ is a continuous function of Φ not only at the point $\Phi = 0$, but also at the point $\Phi=2\pi$.

If not, V would have the characteristic value I; hence $(I-V)^{-1}$ would not exist, contradicting (8).

Let us decompose the interval $(0 , 2\Pi)$ by means of an infinite number of points having the two end points for limit points , say by menas of the points ϕ_m for which -cot ϕ_m = m (m = 0 , ± 1 , ± 2 ,)

The projections $P_m = F_{\phi m} - F_{\phi m-1}$

Are then pair wise orthogonal[4] and

$$\sum_{-\infty}^{\infty} P_m = \lim_{\phi\to 2\Pi} F_\phi - \lim_{\phi\to 0} F_\phi = I - 0 = I.$$

The projection P_m, being permutable (= commutant) with V, is also commutable with A ; the subspace L_m corresponding to p_m therefore reduces the transformations V and A. Since the function $(1- e^{i\emptyset})^{-1}$ is bounded in the interval $\emptyset_{m-1} \le \emptyset \le \emptyset_m$ we have, for f in L_m:

$$Af = AP_m f = i (I + V) (I – V)^{-1} P_m f$$

$$\int_{\phi_{m-1}}^{\phi_m} i(1+e^{i\phi})(1-e^{i\phi})^{-1} dF_\phi f \qquad = \int_{m-1}^{\phi m} (-Cot\frac{\phi}{2}) dF_\phi f.$$

or $Af = \int_{m-1}^{m} \lambda \, dE_\lambda \, f$

where we have set $E_\lambda = F_{-2\mathrm{arccot}\lambda;}$

$\{E_\lambda\}$ obviously is a spectral family over $(-\infty , \infty)$ for spectral family, see SZ – Nagy[1].

Let us denote the spectral family of A, considered as a transformation in L_n , by $\{E_{\lambda,n}\}$; it is a spectral family over some finite segment of the λ- axis determined by the bounds of A in L_n.

According to lemma [4], there exists a self – adjoint transformation E_λ of H which reduces in each L_n to $E_{\lambda,n}$. It is easy to see that E_λ is also a projection, and that moreover it possesses the following properties.

a) $E_\lambda \leq E\mu$ for $\lambda < \mu$,
b) $E_{\lambda+o} = E_\lambda$
c) $E_\lambda \to 0$ for $\lambda \to -\infty$ and $E_\lambda \to I$ for $\lambda \to \infty$

It is therefore a spectral family over the entire line (- ∞ , ∞) ,

Now we establish the formula

$$A = \int_{-\infty}^{\infty} \lambda dE\lambda \quad , \qquad .(9)$$

But since neither the domain of integration nor the function under the integral sign is bounded, it is first necessary to make precise the meaning of an integral of this type.

Now denoting the integral in right hand side of eq (9) by J. Then this definition will be valid for an arbitrary spectral family (this means that in the definition we shall only make use of properties a) and c) of the family of projections E_λ.
Let us consider the projections

$E_m - E_{m-1}$ (m = 0 , ± 1 , ± 2 ,)

and the corresponding subspace K_m into itself. Making use of lemma [4], we define the integral J as the uniquely determined self – adjoint transformation in H which reduces to the transformation J_m in each subspace K_m.

Setting $f_m = (E_m - E_{m-1})f$, the domain of definition of J therefore consists of the element f for which the series [4].

$$\sum_{-\infty}^{\infty} \|J_m f_m\|^2 = \sum_{-\infty}^{\infty} \|(J_m^2 f_m, f_m)\| = \sum_{-\infty}^{\infty} \int_{m-1}^{m} \lambda^2 d\|E_\lambda f_m\|^2$$

Converges; for equivalently , since $E_\lambda f_m = E_\lambda f - E_{m-1} f$

In the interval $m-1 \leq \lambda \leq m$, those for which the integral

$$\left[\left[\int_{-\infty}^{\infty} \lambda^2 d \|E_\lambda f\|^2 \qquad (10)\right]\right]$$

Converges for these f ,

$$Jf == \sum_{-\infty}^{\infty} J_m f_m = \sum_{-\infty}^{\infty} \int_{m-1}^{m} \lambda dE_\lambda f_m = \sum_{m-1}^{m} \lambda dE_\lambda f.$$

It is clear that if f belongs to the domain of f, the same is true of E_μ f and we have

$$JE_\mu f = \sum_{-\infty}^{\infty} J_m (E_\mu f)_m = \sum_{-\infty}^{\infty} J_m E_\mu f_m = E_\mu \sum_{-\infty}^{\infty} J_m f_m = E_\mu Jf$$

hence $E_\mu J = JE_\mu$ (10a)

Now instead of starting with the sequence of integers:

m = 0 , ± 1 , ± 2 , …………. we start with

another sequence of real number which goes to infinity in both directions, we arrive at the same definition of integral J. This being the case , inorder to establish formula (9) – that is , the given self – adjoint transformation A is equal to the

Integral J formed strarting with the spectral family of A. It suffices by virture of lemma[4] , to everify that the two self – adjoint transformations A and J coincide in each of the orthogonal subspaces L_n (n = 1,2 , …) but for an element f of L_n we have , by definition ,

$E_\lambda f = E_{\lambda,n} f$

Since { $E_{\lambda,n}$} is spectral family over the finite interval [a , b] , E_λ f is constant for $\lambda < a$ and $\lambda \geq b$, and consequently the integral (10) converges ; hence f belongs to the domain of J and we have

$$Jf = \sum_{-\infty}^{\infty} \int_{m-1}^{m} \lambda dE_\lambda f = \int_{a-o}^{b} \lambda dE_{\lambda,n} f = Af$$

by the definition of { $E_{\lambda,n}$} a spectral family corresponding to A in the subspaces L_n[4]. This completes the proof of the fundamental formula (9).

Now since we have defined that the integral :

$$\int_{-\infty}^{\infty} \lambda dE_\lambda$$

is self – adjoint transformation , and which reduces in each of the subspaces L_m =

$$(F_{\phi m} - F_{\phi m_{-1}})H = (E_m - E_{m-1})H$$

to the bounded self – adjoint transformation

$$\int_{m-1}^{m} \lambda \, dE_\lambda \qquad (m = 0, \pm 1, \pm 2, ...)$$

we have thus arrived to a new formula

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

It is in this manner that J. Von Neu Mann in (1929), first proved the spectral composition of unbounded self – adjoint transformation [4].

Definition 2:

A symmetric transformation S is said to be lower semi bounded if there exist a real quantity.

$(Sf, f) \geq c (f,f)$

For all f in D_S it is said to be upper semi – bounded if the opposite in equality is valid. If , in particular , $(Sf,f) \geq 0$.

We shall say, following the definition set down for bounded transformation, that S is positive [4].

Since every semi – bounded symmetric transformation obtained from a positive transformation T by one or other of the formula : S = T + cI, S = -T + cI ,

it suffices to consider positive transformations in the sequel, namely the positive classes of matrices in special case an n – by – n matrices A ($A \varepsilon M_n$ (¢) is called a positive matrix if every principal minor of A is positive). Among the positive matrices we consider : an (invertible) M – matrix is a real non – positive an inverse M – matrix and hence a positive matrix it self ; a (Hermitian) positive definite matrix is simply a Hermitian P – matrix[3]. Our interest here and later lies in considering the Cayley transform of matrices in positivity classes above.

But also our interest is in general positive transofroms which relate to Cayley transform, therefore for a positive self – adjoint transformation A, the spectral decomposition can be deduced very simply from that for abounded self – adjoint transformation. This is done with the aid of a linear transformation of semi– axis $\lambda \geq 0$ into a finite segement of μ– axis.

For example, the transformation

$$\mu = \frac{\lambda - 1}{\lambda + 1} \qquad (10b)$$

which carries the semi – axis $\lambda \geq 0$ into the segement $-1 \leq \mu \leq 1$. This is the analogue of the linear transformation

$$\mu = \frac{\lambda - i}{\lambda + i},$$

which maps the circumference of the unit circle in the plane of complex number onto the entire λ–axis–the transformation which led to the idea of the "Cayley transformation[3,4].

One other important analogue or feature of the Cayley transform is that it can be viewed as an extension to matrices of the conformal mapping[3].

$$T(Z) = \frac{1 - z}{1 + z}$$

from the complex plane into itself. In this regard Stein and Tans sky [3] both considered the Cayley transform , for the most part indirectly, when they provided connections between matrices stable matrices (i,e matrices for which Re (A) < 0 for all eigenvalues λ) and convergent matrices (i.e. those matrices A for which $\lim_{m \to \infty} A^m = 0$

In both of these papers the Key connection came via Lyapunov's equation AC+GA* =-I, and the Cayley transforms[3]. The use of the Cayley transform for stable matrices was recently made explicit in the paper of Haynes [3]. He proved that a matrix B is convergent if and only if there exists a astable matrix A such that:

$B = C(-A)$.

Since we now in the coming deal only with the semi – axis $\lambda \geq 0$, it is not necessary to use imaginary numbers in order to transform it into a bounded curve.

Hence we form the transformation

$B = (A - I)(A + I)^{-1}$,

instead of the Cayley transform (4a) or which is (7).

Since $((A + I) f, f) \geq (f,f)$, the transformation

$C = (A + I)^{-1}$ exists and $(g, (Cg) \geq (Cg, Cg)$ for all g in D_C. It follows that:

Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
www.sustech.edu

June /2012

(C g , g) ≥ 0 and $\|Cg\| \leq \|g\|$

Since C is inverse of a self - adjoint transformation, it is also self – adjoint [(4)], and since it is bounded in its domain D_C, this domain necessarily coincides with the entire space H, thus the transformation

I – 2C = (A + I) C – 2 C = (A – I) C = B

is also self – adjoint and bounded , and since

O≤ C ≤ I , we have $\|B\| \leq 1$.

Let B = $\int_{-1-o}^{1} \mu \, dF_{\mu}$

be spectral composition of B. Since the transformation I – B = 2C possesses an inverse (namely , ½ (A+I), the value 1 is not a characteristic value or eigen value of B , hence Fμ is a continuous function of μ at the point μ = 1 , that is , $F_{1-0} = F_1 = I$.

consequently we have ,

$$\int_{-1-o}^{1} \frac{1+\mu}{1-\mu} dF_{\mu} = \int_{-o}^{\infty} \lambda \, dE_{\lambda} \quad ,.....(11)$$

A = (I +B) $(I-B)^{-1}$ =

where $E_\lambda = F_\mu$ f or $\mu = \frac{\lambda - 1}{\lambda + 1}$ (11a)

{ E_λ} is obviously a spectral family over semi-axis ≥ 0. For a rigorous proof of (11), we can use a decomposition of the segement – 1 ≤ μ < 1 by means of an infinite number of points which tend to 1.

Since $E_\lambda = F_\mu$ is the limit of polynomials in B, it is obviously commutant with A and with all the bounded transformations which permute (= commute) with A.

Now due to equations (4a) and (7), we give two important lemmas, theorems and some examples on positivity matrices.

Lemma l : Let A Є M_n (¢) s.t – 1 ∉ σ (A) and B = C (A) , then ,

A = C (F) = $(I + F)^{-1}(I – F)$

Proof [2] : As F = C (A) , we have (I + A) F = I – A or A (I + F) = I – F .

Now notice that if Fx = -x, then x = 0, that is -1 ∉ C (F).

Thus by (12) and since $(I + F)^{-1}$ and (I + F) commute it follows that A = C (F).

Lemma 2: Let A Є M_n (¢) s.t -1 ∉ σ (A) and let F = C (A) , then ,

$$I + F = 2 (I + A)^{-1} \qquad (13)$$

If, in addition, A is invertible, then

$$I – F = 2 (I + A)^{-1} \qquad (14)$$

Proof [3]: As F = C (A) , we have

$I + F = I + (I + A)^{-1} (I – A) = (I + A)^{-1}$
$(I + A + I – A) = 2 I (A + I)^{-1}$

Similarly, I – F = 2 $(I + A)^{-1}$ A . So if A is invertible,

$I – F = 2 (A^{-1} (I + A))^{-1} = 2 (I + A^{-1})^{-1}$

as claimed .

Finally notice that if F = C (A) , then

$\lambda \in \sigma(A) \Leftrightarrow \lambda = \frac{1-\mu}{1+\mu}$; for some $\sigma\mu$ Є (F) ,

which is (10b) and (11a).

Now for a matrix A in each of the a forementation positivity classes, we examine properties of its Cayley transform F = C(A) , specially since A can be factored into $A=(I + F)^{-1}(I-F)$, we investigate whether the factors $(I + F)^{-1}$ and (I – F) belong to same positivity classes A and , conversely under what conditions does A belong to one of these positivity classes. Indeed Fallat and Tsatsomeros interested in factorization of the form $A = X^{-1}Y$, where X and Y have certain properties such as diagonal dominance and stability(3). They obtained result of this type by using the fact that the Cayley transform is an involution and by employing the factorization of A interms of its Cayley transform then the following theorem gives us the relation of P – matrices and Cayley transforms.

Theorem 2 : Let A Є M_n (C) be a P – matrix. Then F = C (A) is well – defined and both I – F and I + F are P – matrices. In particular , $A = G + F^{-1}$ (I – F) is a factionization of a P - matrix into (commuting) P – matrix.

Proof [4]: first , since A is a P – matrix, A is totally non negative matrix and has no negative real eigenvalues . Hence F = C (A) is well – defined by lemma 1 and 2 and as addition of positive diangonal matrices and inversion are operation precerves positive matrices , it follows that I – F and I + F are commuting P – matrices.

One consequence of the above result is that if A is a P – matrix , then the main diagonal entries of the matrix F = C (A) all have absolute value less than one (3).

The converse of theorem 2 is not true. Let us examine this and the above by some examples as follows:

Example 2:

$$Let\ F = \begin{bmatrix} 0 & 1 & 1.1 \\ -1 & 01 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Then

$$1 - F = \begin{bmatrix} 1 & -1 & -1.1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

And

$$(1+F)^{-1} = \begin{bmatrix} .4\,762 & -.5 & .0238 \\ 0 & .5 & .5 \\ .4762 & 0 & .4762 \end{bmatrix}$$

is not P – matrix.

To obtain a characterization of a P – matrix interms of Cayley transform , we need the following lemma.

Lemma 3: Let B Є M_n (¢) so that σ (B [α])=σ(B [α]) for all α Є { 1,2,…,n}.

Then B is a P – matrix if and only if every real eigenvalue of every principal submatrix of B is positive(3).

Remark 1 : Based on lemma3 , theorem 2 can be written as follows : If A is a P – matrix , then $(I + F)^{-1}$ is a P – matrix and its every real eigenvalue of every principal submatrix of $(I + F)^{-1}$ is positive.

Theorem 3:

Let A Є M_n (¢) s.t σ (A [α]) = σ(A [α]) for all α Є { 1,2,..n} and -1 ∉ σ (A). Let F = C(A) then A is a P – matrix if and only if every real eigenvalue of principal submatrix of $(I + F)^{-1}$ is greater than ½ .

Proof [5] : In view of lemma 1 and by reversing the roles of C (A) and F in lemma 2 , we obtain A = 2(I+F) –I.

Also from the results of lemma 3 and Remark 1 we obtain $2(I+F)^{-1} – I$.

Notice that in example 2 all 1-by-1 and 2-by -2 prime positive matrices of $(I + F)^{-1}$ fail to satisfy the condition in theorem 3 (3).

Theorem 4 : Let B , G Є M_n (R). The set matrices { BT + G (I-T) : T = diag $(t_1 , t_2,\ldots, t_n)$, t_i Є [0,1] (1 ≤ i ≤ n) }

contains only non singular matrices if and only if $G^{-1}B$ are non singular (3).

Now we shall give an example shows that the natural question arising here , wether the factorization = $(I – A)^{-1}$ (I + A) of a totally nonnegative matrix (3). , F has factors $(I - A)^{-1}$ and (I+A) are totally nonnegative or not ?

Example 3(3)

consider the totally nonnegative matrix

$$\hat{F} = \begin{bmatrix} 1 & 0.9 & 0.8 \\ 0.9 & 1 & 0.9 \\ 0 & 0.9 & 1 \end{bmatrix}$$

and consider A = - C ($\hat{F}$).

Then $\hat{F} = C (-A) = (I – A)^{-1} (I + A)$ neither

$$(1 - A)^{-1} = \begin{bmatrix} 1 & 0.45 & 0.4 \\ .45 & 1 & 0.45 \\ 0 & .45 & 1 \end{bmatrix}$$

nor I + A is totally non negative.

We have seen in above how the Cayley transform developed with P matrix in recent work of Fallat and Tsatsomers. Now we shall study the extension of symmetric transformations.

Cayley transforms and Deficiency Indices

Since , we have just seen , it is the self adjoint transformations which have spectral decomposition , it is important to know wether or not a given symmetric transformation possesses a self – adjoint extension. More generally, the problem arises of characterizing all the symmetric extension of a given symmetric transformations.

Cayley transformations have been used in the study of this problem since their introduction in (1929)(4). The Cayley transform of a symmetric transformations is defined just as for a self – adjoint transformation previously , namely by :

$$V = (S – i I) (S + iI)^{-1} ,$$

just as these , we show that V isometric and that we can recover S from V by means of the formula :

$$S = i (I + V) (I – V)^{-1}.$$

By using relations (3) , (5) , (6) , written for S instead of A , it is easy to see that if S is closed then V is also closed , and conversely, since every symmetric transformation S has the closed extension S** (its cloure) , we shall consider only closed symmetric transformations.

We know that if S is self – adjoint , its Cayley transform V is unitary ; we shall show that the converse is also true. Suppose that V is unitary ; let g be an element of D_{s*} and set g*=S*g . then (Sf , g) = (f , g*), for all elements f of D_s , and since these elements f are of the form f = (I – V) h , where h runs through D_V=H, we have :

(i (I + V) h , g) = ((I – V) h , g*), or

i (h,g) + i (Vh , g) = (h,g*) –(Vh,g*), for all elements h of H. Since V is unitary (hence defined every where and isometric , we can replace (h, g) by (Vh , Vg) and (h,g) by (Vh,Vg*) and obtain :

(Vh, - i Vg – Vg* + g*) = 0 ,

The values Vh of the unitary transformations V exhaust the space H; this implies that :

$$g = (1-V)\frac{g-ig^*}{2} \quad g^* = i(1+V)\frac{g-ig^*}{2}$$

Consequently g also belongs to the domain of S and Sg = g* . This prove that S* = S ; S is therefore a self e- adjoint transformation [4].

Now we are in a position to introduce the notion deficiency subspaces and their dimensions, and relations to Cayley transforms , which had been studied by F Riez and Bela Sz Nagy in 1955 [4] .

Then since in the case of an arbitrary closed asymmetric transformation S , the domain of definition D_V and the set of values $D'_V = VD_v$ do not in general coincide with the entire space H; but since V is isometric and closed , D_v and D'_V are closed sets , that is , subspaces of H, one or the other of which may coincide with H. the orthogonal complements

H–D_v and H–D'_V are called the deficiency subspaces, and their dimensions the deficiency indices of the symmetric transformation S (or also of the isometric transformation V).

Let us recall that D_v is the set of values of S + i I and D'_V is the set of values of S – i I.

It follows from what we have just proved that a closed symmetric transformation is self –adjoint if and only if its deficiencies are (0,0).

We pass to the problem of extension. It is clear that if S' is an extension of S (we suppose that both S and S' are symmetric and closed), now the Cayley transform of V' of S' will be an extension of the Cayley transform V of S. D_v will be a subspace of D'_V it follows that when we pass from S to S', the deficiency indices diminish by the same (finite or infinite) number.

We now show that conversely, every isometric extension of the Cayley transform V of S determines a symmetric extension S' of S whose Cayley transform V' equals U.

Firstly observe that $(I – U)^{-1}$ exist , that is, (I – U) h = 0 , implies h = 0 . In fact,

Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
www.sustech.edu

June /2012

if (I – U) h = 0 then for every element of the form f = (I – U) g :

(h , f) = (h , g) – (h , Ug) = (Uh, Ug) – (h, Ug)

= - ((I – U) h , Ug) = 0 ;

hence is orthogonal to the set values of I – U , and h is orthogonal to the set of values of I–U , and therefore the domain of definitions of S . Since this domain is dense in H , we necessarily have h = 0

Now let us form the transformation

$$S' = i (I + U) (I – U)^{-1}$$

which obviously is an extension of S, is symmetric ; in fact , if f and g are elements of DS' they are of the form f = (I – U) φ ,

g = g ((I – U) ψ, and we have

$S'f = (i (I + U) φ$, $S'g = i (I + U) ψ$;

Hence , in as much as (φ, ψ)=(U φ, U ψ) = (f , g) = (i (I + U) φ , (I – U) ψ)

= i [(U φ , ψ)- (φ,U ψ)] = ((I – U) φ , i (I + U) ψ) = (f, S' g).

Finally the relation f = (I – U) φ implies that

$S'f = i (I + U) φ$, $(S' + iI)f = 2 i φ$, $(S' – iI)f = 2iU φ$,

from which we see that the domain of the Cayley transform V' consists of elements of the form 2i φ, where φ runs through D_V, and that $V'(2i φ) = 2iU φ = U (2i φ)$.

This proves that $V' = U$ which has to be shown.

We note that if U is an arbitrary isometric transformation for which the set of values of I- U is dense in H, the same reasoning proves that :

$S' = i (I + U) (I – U)^{-1}$ is asymmetric transformation whose Cayley transforms equals U.

Now, with this, the problem of finding all the (closed) symmetric extensions of the closed symmetric transformations reduces to the problem of finding all the isometric extension of its Cayley transform V; this problem is obviously much simpler than the original problem. This had been done by F. Riez and B. Sz – Nagy [(4)].

In fact, inorder to extend V; we have only to map the deficiency subspaces $H–D_V$, or a subspace of the latter, isometrically into the deficiency subspace $H–D'_v$; this is a complished, for example with the aid of two orthonormal systems taken in $H–D_v$ and in $H-D'_V$. It is thus possible to exhaust the deficiency subspaces with the smaller domain ; the corresponding symmetric transformation S' will then be a maximal extension of S. If two deficiency subspace are of the same dimension, they can be exhausted simultaneously, and we obtain a unitary extension of V, and consequently a self -adjoint extension of S.

Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
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F. Ries and Bela Sz–Nagy [4] summarized the essential points of the above in the following theorem.

Theorems 5 : In order for the closed symmetric transformation S to be maximal it is necessary and sufficient that one or the other of its deficiency indices be equal to 0 ; inorder to admit a self adjoint transformation as an extension , it is necessary and sufficient that its deficiency indices be equal ; finally , inorder that it its self be self-adjoint , it is necessary and sufficient that its two deficiency indices be equal to 0.

Example 4: Let H be a Hilbert space whose dimension is denumerably infinite, of an isometric non – unitary transformation V_o ; let { g_n} be a complete orthonormal sequence in H and set

$$V_o \sum_1^\infty C_k g_k = \sum_1^\infty C_k g_{k+1}$$

Then the domain of V_o is the entire space, while the set of values V_o f has the orthogonal complement of dimension1 consisting of elements of the form Cg_1. It is easily shown that the set of values of $I - V_o$ is dense in H. Hence V_o is the Cayley transform of symmetric transform :

$S_o = i (I + V_o) (I - V_o)^{-1}$, S_o is defined by

$S_o \sum_1^\infty C_K g_K = i C_1g_1 + i (2C_1 + C_2g_2) + i(2C_1+2C_2)g_3 + \ldots$

for all elements $f = \sum C_K g_K$ for which $(I - V_o)f$ has a meaning , that is, for which for all elements:

$$|C_1|^2 + |C_1 + C_2|^2 + |C_1 + C_2 + C_3|^2 + \ldots$$

converges (for these f we have , in particular , $\sum_1^\infty C_k = o$).

Therefore the transformation S_o has (0, 1) for deficiency indices; it is called the elementary symmetric transformation. Now it can be shown that every symmetric transformations S_0 of a Hilbert space H of arbitrary dimension with the deficiency indices (0,m) (where m is an arbitrary finite or infinte cardinal number) is composed of m elementary symmetric transformations plus possibly a self –adjoint transformation , in the following sense ; there are m mutually orthogonal subspaces K_α with denumerably infinite dimension in H, each of which reduces S to an elementary symmetric transformation, s.t, in the subspace K' of elements orthogonal to all the K_α (a subspace which may consist of the single element 0), S reduces to a self – adjoint transformation [4].

Now the problem of maximal symmetric transformation whose deficiency indices are (m,0) presents nothing new, since , in the fact that the Cayley transform of – S is obviously equal to the invese of that of S.

Remark 2: the real symmetric transformations of the space L^2 always have equal deficiency indices , hence they are either self–adjoint or possess self–adjoint extensions.

The transformation S is said to be real if its domain contain with a function f(x) its conjugate f(x) , and if in addition S $\bar{f}$ (x) = $\overline{Sf}(x)$.

Our proposition is verified as follows: the domains D_v and the range D'_v of the Cayley transform of S consist of the functions:

u(x) = Sf(x) = if(x) and v(x) = S g(x) – ig(x), respectively , where f and g run through the domain of S. Setting g(x) = $\bar{f}$ (x) we have v(x) = $\bar{u}$ (x), hence D' consists of the conjugates of two orthogonal function are also orthogonal, H–D_v consists of the conjugates of H – D_v,

For further details see Frigyes and Sz. Nagy [(4)] for positive symmetric transformation, and all its extensions to positive self–adjoint transformation.

CONCLUSIONS

Now we see how the Cayley transforms had been rapidly developed in the more recent work.

The study showed the necessary historical advances and important two directions of recent work, that is to say the Cayley transform of accretive and dissipative recent work, that is to say the Cayley transform of accretive and dissipative operators and purely maximal of these , the symmetric , bounded , self adjoint and unitary operators , the second direction is a P-matrix tranformaition.

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