Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
www.sustech.edu

June /2012

Structures of Inner Functions on Hardy Spaces via the Least Harmonic Majorants

Emad Aldeen A. A. Rahim

Sudan University of Science and Technology - College of Science-Department of Mathematics

ABSTRACT: In this work we showed that the inner functions on Hardy space can be written in the canonical factorization form as exponential of least harmonic majorants.

المستخلص:

في هذا العمل أوضحنا أن الدوال الداخلية على فضاء هاردى يمكن أن تكتب بصيغة المعامل القانونى كدالة أسية لأقل مقياس توافقى.

KEYWORDS: *Hardy space, Inner functions, Harmonic and least Harmonic majorant.*

Definition of H^p

For $1 \le p < \infty$ the Hardy space H^p is defined as the space of all analytic functions φ in the unit disk $D = \{z \in \mathbb{C}: |z| < 1\}$

for which the norm

$$\|\varphi\|_p = \sup_{r<1}\left[\frac{1}{2\pi}\int_0^{2\pi} \left|\varphi\left(re^{it}\right)\right|^p\right]^{\frac{1}{p}} \quad (1)$$

is finite. The space H^∞ (Banach space) consist of all bounded analytic functions φ on the disk, and the norm is now

$$\|\varphi\| = \sup_{|z|<1} |\varphi(z)| \quad (2)$$

For function φ in H^p, for $1 \le p < \infty$, the radial limit

$$\tilde{\varphi}(e^{it}) = \lim_{r\to 1} \varphi(e^{it}) \quad (3)$$

exists almost everywhere in t (Fatou's Theorem), and needed , $\tilde{\varphi} \in L^p(T)$, where T denotes the unit circle which we equip with normalized Lebesque measure; moreover:
$\|\varphi H = \tilde{\varphi}\| L^p$.

We normally identify φ with $\tilde{\varphi}$, and can thus regard H^p as a closed subspace of $L^p(T)$. It is also possible to start by defining H^p directly as the subspace of those $L^p(T)$ functions for which the negative Fourier coefficients vanish, that is:

$$\frac{1}{2\pi}\int_0^{2\pi} \tilde{\varphi}(e^{it})e^{-int}\, dt = 0 \quad (4)$$

for all $n < 0$.

It is classical that $\varphi \in H^p$ has the factorizations

$$\varphi = B\,S\,O \tag{5}$$

where B is a Blaschke product, S is a singular function, and O is an outer function. Specifically, these factors are:[1]

$$B(z) = z^m \prod \mu_k . \frac{z_k - z}{1 - z\bar{z}_k} \;\; , \mu_k = \frac{|z_k|}{z_k} \tag{6}$$

where m is the order of the zeros of φ at the origin and $z_1, z_2, \ldots$ are the zeros of φ in $D \setminus \{0\}$;

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} d\nu(t) \right\} \tag{7}$$

where ν is a non-negative measure singular with respect to Lebesgue measure, and

$$O(z) = \lambda . \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} k\left(e^{it}\right) dt \right\} \tag{8}$$

where λ is a unimodular constant and k is real-valued integrable function. Also φ has the factorization (canonical factorization)

$$\varphi = I\,O \tag{9}$$

where I is an inner function (has a unit modulus a.e on D), and O as in Eq.(5).[1, 2]

It is a well-known that if φ is an inner function, then $\frac{w - \varphi(z)}{1 - \bar{w}\,\varphi(z)}$ is a Blaschke product for all $w \in D$ with the exception at most of a set of capacity zero[3].

Theorem A[3] :

Let N_* denote the set of all analytic functions f on the unit disk such that the functions $\log^+|f_r|$ have uniformly absolutely continuous integrals, and let $\varphi \in N_*$, then the set of points w for which $\varphi(z) - w$ has non-trivial singular inner factor has logarithmic capacity zero. Conversely, given any compact set E of logarithmic capacity zero, there is a bounded analytic function φ such that $\varphi(z) - w$ has a non-trivial singular inner factor if and only if $w \in E$.

The converse statement is well- known. Let E be a compact set of capacity zero in D , the covering map F of the domain $D \setminus E$ is an inner function since E has capacity zero. For each $w \in E$, $\frac{F(z) - w}{1 - \bar{w}F(z)}$ is a non-vanishing inner function and so is singular. Thus since $1 - \bar{w}F(z)$ is an outer function, $F(z) - w$ is a function with non-trivial singular inner factor for all w in E.

Note that for mutually prime inner functions u and v which have no zero in common and that there is singular inner function S with $u = Su_1$ and $v = Sv_1$ for inner functions u_1 and v_1, and $\rho > 0$, the function $u(z) + \rho e^{it} v(z)$ has a trivial singular inner factor for almost all (w.r.to Lebesque measure) real t.

Journal of Science and Technology Vol. 13
ISSN -1605 – 427X
Natural and Medical Sciences (NMS No. 1)
www.sustech.edu
June /2012

The generalization of the above concept is given in the following Theorem.

Theorem B: Let $f, g \in H^p, 0 < p < \infty$, have mutually prime singular inner factors. Then the set of points w for which $f(z) - wg(z)$ has a non-trivial singular inner factor has logarithmic capacity zero[3].

In Theorem B above, we see that if g is an outer function, then the lack of a singular factor in $f(z) - wg(z)$ is equivalent to the lack of a singular factor in the decomposition of the function $\frac{f(z)}{g(z)} - w$ in N_*, and is thus covered in Theorem A.

Now let $\varphi \in H^2$, then there exist a harmonic function h in D, such that

$$\left|\varphi(z)\right|^2 \le h(z) \qquad (10)$$

and h is called the harmonic majorant of φ[4-6]. It is well known that if φ has harmonic majorant in *D*, then there exist a least harmonic majorant , hence there exist a harmonic function h_φ in *D*, such that:

$$\left|\varphi(z)\right|^2 \le h_\varphi(z) \qquad (11)$$

and such that $h_\varphi \le h$. Also if $\varphi \in H^2$, then for a fixed $z_0 \in D$ there is a norm on H^2 defined by: $\|\varphi\| = \inf \left\{ h(z)^{\frac{1}{2}} : h \text{ is a harmonic majorant of } |\varphi|^2 \right\}$

Now, one can make the following definitions:

Definition 1: for $\varphi \in H^2$, we say that h is the harmonic majorant of φ in D if h is harmonic function in D, such that $\varphi \le h$.

Definition 2: let $\varphi \in H^2$, and h is the harmonic majorant of φ in D, we say that h_φ is least harmonic majorant of φ in D if h_φ is harmonic function in D, such that $\varphi \le h_\varphi$,and such that $h_\varphi \le h$.

Now let φ and φ' be two functions in H^2. We say that φ divides φ' (or $\varphi | \varphi'$), if φ' can be written as $\varphi' = \varphi u$, for some $u \in H^2$.

Now we need the following lemma.

Lemma C[3]**:**

If φ_1 and φ_2 are inner functions without a common factor, then:

$$\lim_{r \to 1} \int_T \log\left(\max\left\{ \left|\varphi_1\left(re^{i\theta}\right)\right|, \left|\varphi_2\left(re^{i\theta}\right)\right| \right\} \right) d\sigma(\theta) = 0$$

where $d\sigma$ is the normalized Lebesgue measure on the unit circle T .

proof: The limit on the left side is the value at the origin of the least harmonic majorant in D of the subharmonic function $\max\{\log|\varphi_1|, \log|\varphi_2|\}$ (2,8). So it remains to show that this least harmonic majorant is the constant function 0. Let h denote this least harmonic majorant. Then $\log|\varphi_1| \le h \le 0$. This implies that h has radial limits 0 almost everywhere on T . So, if h is not identically zero, then h is the Poisson integral of a negative singular measure on T. Hence $\varphi = e^{h+i\tilde{h}}$ is a singular inner function (here $\tilde{h}$ denotes the harmonic conjugate of h in D (9). Since $|\varphi_1| \le |e^{h+i\tilde{h}}|$, the inner function φ divides φ_1. But $|\varphi_2| \le |e^{h+i\tilde{h}}|$ implies that φ also divides φ_2, contradicting our assumption about φ_1 and φ_2. Thus $h \equiv 0$. This completes the proof of the lemma.

RESULTS

Now we arrive to the following theorem which gives the principle result of this paper.

Theorem 1: If φ_1 and φ_2 are inner functions with a common factor, say φ, then $\varphi = e^{h+i\tilde{h}}$, where $\tilde{h}$ denotes the harmonic conjugate of h in D , and h is the least harmonic majorant of the subharmonic function $\max\{\log|\varphi_1|, \log|\varphi_2|\}$.

Proof:

By lemma C above, if h denotes the least harmonic majorant of the subharmonic function $\max\{\log|\varphi_1|, \log|\varphi_2|\}$, then $\log|\varphi_1| \le h \le 0$. This implies that h has radial limits 0 almost everywhere on T . So, if h is not identically zero, then h is the Poisson integral of a negative singular measure on T. Hence $\varphi = e^{h+i\tilde{h}}$ is a singular inner function. Since $|\varphi_1| \le |e^{h+i\tilde{h}}|$, the inner function φ divides φ_1. But $|\varphi_2| \le |e^{h+i\tilde{h}}|$ implies that φ also divides φ_2 , moreover this representation of inner function φ is similar to that one in Eq. (9), i.e. $\varphi = e^{h+i\tilde{h}} = e^{h}.e^{i\tilde{h}}$, where $e^{i\tilde{h}}$ determine the inner factor, and e^{h} determine the outer factor, and the theorem is proved.

REFERNCES:

1. Jonathan, R. P., (2008). *Hardy spaces, inner and outer functions, shift-invariant subspaces, Toeplitz and Hankel operators*,http://www1.maths.leeds.ac.uk/~pmt6jrp/yorkleeds.pdf (retrieved June 2011).
2. Duren, P. (1970). *Theory of H^p Spaces*, Academic Press, New York and London.
3. Stephen D. Fisher and Jonathan E. Shapiro,(1991). Singular Factors are Rare, Mathematics Subject Classification, Primary 30H05; Secondary 46E20.

4. Walter, R., (2010). Analytic Functions of class H^p ,http://www.ams.org/journals/tran/1955-078-01/S0002-9947-1955-0067993-3/S0002-9947-1955-0067993-3.pdf (retrieved June 2011).

5. Sawai, N., (1973). On Least Harmonic majorants in Half-spaces, *London Math. Soc.* **27**: 243-260.

6. Matts, E., (1987). Harmonic majorization and Thinness, *The Czech Mathematics Library,* http://project.dml.cz. (retrieved June 2011).

7. Yun-Su, K., (2008). Quasi-inner functions of the Generalized Beurling's Theorem, *2000 Mathematics Subject Classification.* 47A15, 47A56, 47B37, 47B38.

8. Andreas, H., (2006). Interpolation and Harmonic majorants in big Hardy-Orlicz spaces, *1991 Mathematical Subject Classification*. 30E05, 31A05.

9. Aini, J., (2009). Properties of Harmonic functions which are Starlike of Complex Order with respect to Conjugate point, Int. *J. Contemp. Math. Sciences*. **4**: 1353-1359.