Technical Notes

NOTE ON THE SOLUTIONS OF COUPLED ELLIPTIC EQUATIONS

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ABSTRACT

A theorem is firstly proved which governs separation of a system of two coupled equations. This separation leads to study boundary value problems for elliptic equations. The existence and uniqueness of the generalized solution is proved Sovolev space.

ملخص

تهتم الدراسة بمنظومة معادلتين ثنائيتين حيث تثبت نظرية تؤدي إلى فصل المعادلتين إلى مسألتين لقيم الحدية وذلك ضمن شروط معينة على المعاملات. ثم تثبت نظرية الوجود والوحدانية في محيط سوبولوف.

INTRODUCTION

Coupled partial differential equations are frequent in many different problems, such as in the study of temperature distribution within a composite heat conduction [1], in diffusion problems [2], biochemistry [3], armament models [4], analysis of pollutant migration through soil, elastic and inelastic contact problems of solids [5], etc.

Let Ω be a bounded domain in R^n of $x=(x_1, x_2, \dots, x_n)$, with sufficiently smooth boundary $\Gamma = \partial\Omega$ Considering the following system of coupled partial differential equations:

$$(\mathrm{P}) \begin{cases} -\Delta u_1 + a_1(x)u_1 = b(x)u_2 + g_1(x) \\ -\Delta u_2 + a_2(x)u_2 = b(x)u_1 + g_2(x), \\ u_1 = u_2 = 0, on \quad \Gamma = \partial\Omega \end{cases}$$

where $a_i(x)$, $b(x)$ are assumed to be analytic functions and $g_i \in L_2(\Omega)$, $(i=1,2)$.and Δ denotes the Laplacian operator in R^n

The theorem on the separation of the equation of this system states the following:

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Theorem 1

The system (P) can always be decoupled without increase of the order of the differential equation if and only if b(x) is proportional to the difference $\alpha_1(x)-\alpha_2(x)$.

Proof

In matrix notation let

$(Z+A)\,U = BU + G$ (1-1)

where $U=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $B=\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$, $G=\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$,

$$Z + A = \begin{pmatrix} -\Delta + a_1 & 0 \\ 0 & -\Delta + a_2 \end{pmatrix}$$

Considering a transformation N defined by

$$N = \begin{pmatrix} 1-\lambda & 1+\lambda \\ -(1+\lambda) & 1-\lambda \end{pmatrix} \quad (1\text{-}2)$$

Where λ may be any function of x. Note that

$N = X_1 X_2$, where $X_1 = X_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 & \lambda \\ -\lambda & 1 \end{pmatrix}$

The form (1-2) my be used, to diagonals the matrix equation

$N(Z+)N^{-1}\,NU = NBN^{-1}\,NU + NG$ (1-3)

It may be shown that equation (1-3) is separated if and only if, the following conditions are satisfied simultaneously:

(i) $[Z, \lambda] = 0$, ([ ] is a commutator)

(ii)$-(\alpha_1 . \alpha_2)\lambda^2 - 4b\lambda - (\alpha_1 . \alpha_2) = 0$,

Condition (i) means that λ must be independent of, x, while (ii) connects this quantity with b(x), $\alpha_1(x)$, $\alpha_2(x)$.

Solving this last equation, are comes to the conclusion that the ratio $\dfrac{b}{a_1 - a_2}$ Must be independent of x.

when defining $V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then V=NU,

And (1-3) is separated in the forms:

$$(P_1)\begin{cases}-\Delta v_1+\Phi_1(x)v_1=f_1(x)\\ v_1=0\ ,on\ \ \Gamma=\partial\Omega\end{cases}\quad ,\ (P_2)\begin{cases}-\Delta v_1+\Phi_1(x)v_2=f_2(x)\\ v_2=0\ ,on\ \ \Gamma=\partial\Omega\end{cases}$$

Where $\Phi_1(x)=-\frac{1}{2}(a_1(x)+a_2(x))+\frac{1}{2}\left[(a_1(x)-a_2(x))^2+4b^2\right]^{\frac{1}{2}}$

$$\Phi_2(x)=-\frac{1}{2}(a_1(x)+a_2(x))-\frac{1}{2}\left[(a_1(x)-a_2(x))^2+4b^2\right]^{\frac{1}{2}}\ ,\ \begin{pmatrix}f_1\\ f_2\end{pmatrix}=2(1+\lambda^2)NG$$

EXISENCE AND UNIQUENESS THEOREM

Here the existence and uniqueness of solutions of problems (P_1) (*resp.*(P_2)) is proved

Theorem 2

If the conditions of theorem 1 are satisfied and supposing that Φ_i (x) ?ki ? 0 On ? .Then, for all functions f_i єL₂ (?),i =1,2, there is a unique solution $v_1\in H_0^1(\Omega)[resp. v_2\in H_0^1(\Omega)]$of the problem$(P_1)(resp.(P_2))$.

Proof

Form the theorem of Lax–Milgram follows immediately existence and uniqueness of solution of problem

(P_1)(resp.(P_2)) in $H_0^1(\Omega)$.

COROLLARY

Under the Hypothesis of theorem 1 and 2. Then problem (P) has a unique generalized solution $\{u_1,u_2\}$ in $H_0^1(\Omega).\times H_0^1(\Omega)$.

REFERENCES

1. Cannon J.R and. Klein, R.E, (1973). On the observability of the temperature distribution in a composite heat conductor, SIAMJ. Appl. Math .24, 596-602.
2. Morimoto, H. (1962). Stability in the wave equation coupled with heat flow, Numerishe Math. 4, , 136-145.
3. King and A.I Chou, C.C. ,(1976). Mathematical modeling, simulation and experimental testing of biochemical systems crash response, .Biomech.9301-317.
4. Golpasamy, K. ,(1977). An arms race with deteriorating armaments, Math Biosci. 37, 191-203.
5. Lewis, R.W. Hinton, E. Bettess P. and Schrefler, B.A. (1987).Numerical Methods in Transient and Coupled Problems, John While &Sons, New York.