

Applications on Symmetry Solutions of Partial Differential Equations

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ABSTRACT: This paper analyze a class of partial differential equations of the form

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + f(x) \frac{\partial u}{\partial x}$$

using Lie symmetry group method. It was shown that if the function f is a solution of a family of Ricatti equations, then symmetry techniques can be used to find the characteristic functions and fundamental solutions for partial differential equations (PDES).

KEYWORDS: *Airy function, Bessel function, Infinitesimal symmetries, Laplace transform, Lie algebra, Lie symmetry group and Ricatti equations.*

INTRODUCTION

The purpose of this paper was to show how symmetry group methods may be used to compute characteristic functions and fundamental solutions for partial differential equation (PDES), of the form (1).

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + f(x) \frac{\partial u}{\partial x} \tag{0.1}$$

when f is a solution of one of the following three families of Ricatti equations

$$x f' - f + \frac{1}{2} f^2 = A x + B \tag{0.2}$$

$$x f' - f + \frac{1}{2} f^2 = A x^2 + B x + C \tag{0.3}$$

$$x f' - f + \frac{1}{2} f^2 = A x^{\frac{3}{2}} + C x - \frac{3}{8} \tag{0.4}$$

A, B and C are arbitrary constants.

We will show that if f is a solution of (0.2) or (0.4), with $B = 0$, then we can obtain the characteristic function for

the PDE (0.1), from the solution $u = 1$, using symmetry group transformation.

The characteristic function $U_\lambda(x, t)$ of (0.1) is defined to be (2, 3):

$$U_\lambda(x, t) = \int_0^\infty e^{-\lambda y} p(t, x, y) dy \tag{0.5}$$

Where $p(t, x, y)$ the fundamental solution of equation (0.1) is, $U_\lambda(x, t)$ is the Laplace transform of $p(t, x, y)$.

The fundamental solution can be then obtained by taking the inverse Laplace transform of U_λ (4, 5).

When f is a solution of (0.3) we can still find the fundamental solution by

symmetry methods. Finally, we will consider the case when f satisfies (0.4) with $B \neq 0$.

Fundamental solutions and characteristic functions we determine are the infinitesimal symmetries for the PDE (0.1), and they show how these symmetries can be used to obtain characteristic functions and fundamental solutions (6 – 8).

Fundamental solutions and characteristic functions

We use a method for computing fundamental solutions, which involves taking the inverse Laplace transform of the characteristic functions.

Case 1: Theorem 1. 1. let f be a solution of the Riccati equation (9).

$$xf' - f + \frac{1}{2}f^2 = Ax + B \quad (1.1)$$

Then the characteristic function $U_\lambda(x, t)$ for the PDE (1.1) is given by

$$U_\lambda(x, t) = \exp \left\{ -\frac{\lambda(x + \frac{1}{2}At^2)}{1 + \lambda t} - \frac{1}{2} \left(F(x) - F\left(\frac{x}{(1 + \lambda t)^2}\right) \right) \right\} \quad (1.2)$$

Proof :

Clearly $U_\lambda(x, 0) = e^{-\lambda x}$, Now, since $xf' - f + \frac{1}{2}f^2 = Ax + B$, then, equation (0.1) has an infinitesimal symmetry of the form

$$\mathbf{v} = 8xt \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} - (4x + 4f(x)t + 2At^2)u \frac{\partial}{\partial u} \quad (1.3)$$

The exponentiation of \mathbf{v} shows that if u is a solution of (0.1) with $xf' - f + \frac{1}{2}f^2 = Ax + B$, then

$$\tilde{u}_\varepsilon(x, t) = \exp \left\{ -\frac{(4\varepsilon x + 2A\varepsilon^2 t^2)}{1 + 4\varepsilon t} - \frac{1}{2} \left(F(x) - F\left(\frac{x}{(1 + 4\varepsilon t)^2}\right) \right) \right\} \times u \left(\frac{x}{(1 + 4\varepsilon t)^2}, \frac{t}{1 + 4\varepsilon t} \right) \quad (1.4)$$

Let $u = 1$ and $\lambda = 4\varepsilon$, then we obtain (1.2).

Example 1.1. if $f(x) = \alpha$ (constant). In this case we have $xf'' + ff' = 0$, so $xf' - f + \frac{1}{2}f^2 = B$. Consider

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial u}{\partial x} \quad (1.5)$$

A basis for the Lie algebra of symmetries of (1.5) is: (7, 8, 10)

$$\mathbf{v}_1 = \frac{\partial}{\partial t}$$

$$\mathbf{v}_2 = u \frac{\partial}{\partial u}$$

$$\mathbf{v}_3 = 2x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - \alpha u \frac{\partial}{\partial u}$$

$$\mathbf{v}_4 = 8xt \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} - (4x + 4\alpha t)u \frac{\partial}{\partial u}$$

$$\mathbf{v}_\beta = \beta(x, t) \frac{\partial}{\partial u}$$

We compute the action of the one parameter local Lie group generated by \mathbf{v}_4 we obtain

$$\rho(\exp(\varepsilon \mathbf{v}_4))u(x, t) = \exp \left\{ \frac{-4\varepsilon x}{1 + 4\varepsilon t} - \frac{\alpha}{2} \left(\ln(x) - \ln\left(\frac{x}{(1 + 4\varepsilon t)^2}\right) \right) \right\} \times u \left(\frac{x}{(1 + 4\varepsilon t)^2}, \frac{t}{1 + 4\varepsilon t} \right) \quad (1.6)$$

Thus if u is any solution of (1.5), then (1.6) is also a solution

Let $\lambda = 4\varepsilon$, consider the solution $u = 1$, then by symmetry

$$U_\lambda(x, t) = (1 + \lambda t)^{-\alpha} \exp \left\{ \frac{-\lambda x}{1 + \lambda t} \right\} \quad (1.7)$$

is also a solution of (1.5), and it is the characteristic function for (1.5), it is the Laplace transform of $p(t, x, y)$,

which can be inverted using the fundamental identity

$$L^{-1} \left(\frac{1}{\lambda^\mu} e^{\frac{k}{\lambda}} \right) = \left(\frac{y}{k} \right)^{\frac{\mu-1}{2}} I_{\mu-1} \left(2\sqrt{ky} \right) \tag{1.8}$$

where I_ν is a modified Bessel function of the first kind with order ν [1, 3, 11]). Thus we obtain

$$p(t, x, y) = L^{-1} \left((1+\lambda t)^{-\alpha} \exp\left\{ \frac{-\lambda x}{1+\lambda t} \right\} \right) = \frac{1}{t^2} \left(\frac{x}{y} \right)^{\frac{1-\alpha}{2}} I_{\alpha-1} \left(\frac{2\sqrt{xy}}{t} \right) \exp\left\{ \frac{-(x+y)}{t} \right\} \tag{1.9}$$

Example 1.2. Consider the function

$$f(x) = \frac{ax}{1+\frac{1}{2}ax}, a, x > 0$$

Since, $xf' - f + \frac{1}{2}f^2 = 0$, by Theorem 1.1 the characteristic function for the PDE

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \frac{ax}{1+\frac{1}{2}ax} \frac{\partial u}{\partial x} \tag{1.10}$$

$$U_\lambda(x, t) = \left(\frac{(1+\lambda t)^2 + \frac{1}{2}ax}{(1+\lambda t)^2 (1+\frac{1}{2}ax)} \right) \exp\left\{ \frac{-\lambda x}{1+\lambda t} \right\} \tag{1.11}$$

By the inversion of the Laplace transform, we have

$$p(t, x, y) = L^{-1} \left(\left(\frac{(1+\lambda t)^2 + \frac{1}{2}ax}{(1+\lambda t)^2 (1+\frac{1}{2}ax)} \right) \exp\left\{ \frac{-\lambda x}{1+\lambda t} \right\} \right) \tag{1.12}$$

After some calculations, we get

$$\begin{aligned} p(t, x, y) &= \frac{e^{-\frac{x+y}{t}}}{1+\frac{1}{2}ax} L^{-1} \left(\exp\left\{ \frac{x/t^2}{\lambda} \right\} + \frac{ax}{2t^2\lambda^2} \exp\left\{ \frac{x/t^2}{\lambda} \right\} \right) \\ &= \frac{e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t} \left[\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2} \right) I_1 \left(\frac{2\sqrt{xy}}{t} \right) + t \delta(y) \right] \end{aligned} \tag{1.13}$$

in which δ is the Dirac delta function. Consequently

$$\begin{aligned} u(x, t) &= \int_0^\infty \frac{\varphi(y)e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t} \left[\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2} \right) I_1 \left(\frac{2\sqrt{xy}}{t} \right) + t \delta(y) \right] dy \\ &= \frac{\varphi(0)e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} + \int_0^\infty \frac{\varphi(y)e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t} \left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2} \right) I_1 \left(\frac{2\sqrt{xy}}{t} \right) dy \end{aligned} \tag{1.14}$$

is a solution of the PDE (1.10), with initial data $u(x, 0) = \varphi(x)$, which satisfies

$$\lim_{t \rightarrow 0} u(x, t) = \varphi(x) \tag{1.15}$$

Therefore

$$\int_0^\infty \frac{e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t} \left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2} \right) I_1 \left(\frac{2\sqrt{xy}}{t} \right) dy = 1 - \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)}$$

and hence

$$\int_0^\infty p(t, x, y) dy = \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} + 1 - \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} = 1 \tag{1.16}$$

Example 1.3. Consider the function

$$f(x) = \frac{(1+3\sqrt{x})}{(1+\sqrt{x})}$$

For this we have $xf' - f + \frac{1}{2}f^2 = -\frac{3}{8}$. Thus, by Theorem 1.1 the characteristic function for the PDE

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \frac{(1+3\sqrt{x})}{2(1+\sqrt{x})} \frac{\partial u}{\partial x} \tag{1.17}$$

is

$$U_\lambda(x, t) = \left(\left(\frac{x}{(1+\lambda t)^2} \right)^{\frac{1}{4}} + \left(\frac{x}{(1+\lambda t)^2} \right)^{\frac{3}{4}} \right) \frac{\exp\left\{ -\frac{\lambda x}{(1+\lambda t)} \right\}}{(1+\sqrt{x})x^{\frac{1}{4}}} \tag{1.18}$$

Inverting the Laplace transform gives the fundamental solution

$$p(t, x, y) = \frac{e^{-\frac{(x+y)t}{\sqrt{\pi y t}(1+\sqrt{x})}}}{\sqrt{\pi y t}(1+\sqrt{x})} \left(\cosh\left(\frac{2\sqrt{xy}}{t} \right) + \sqrt{y} \sinh\left(\frac{2\sqrt{xy}}{t} \right) \right) \tag{1.19}$$

which can be integrable at $y = 0$, and

$$\int_0^\infty p(t, x, y) dy = 1 \tag{1.20}$$

As an example, let us compute a solution of (1,17) with initial data $u(x, 0) = x$, which is continuous at the origin, we have

$$u(x, t) = \int_0^\infty y p(t, x, y) dy = x + \frac{t(1+3\sqrt{x})}{2(1+\sqrt{x})} \tag{1.21}$$

then u is a solution of (1,17) and

$$\lim_{t \rightarrow 0} u(x, t) = x,$$

Example 1.4. Consider the three separate problems arising from

$$xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x - \frac{3}{8} \tag{1.22}$$

We exhibit three different solutions to this Ricatti equation these are,

$$f^1(x) = \frac{1}{2} + \sqrt{x} \tag{1.23}$$

$$f^2(x) = \frac{1}{2} + \sqrt{x} \tanh(\sqrt{x}) \tag{1.24}$$

$$f^3(x) = \frac{1}{2} + \sqrt{x} \coth(\sqrt{x}) \tag{1.25}$$

First, the equation arising from f^1 is

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x} \right) \frac{\partial u}{\partial x} \tag{1.26}$$

By Theorem 1.1 the characteristic function for (1.26) is

$$U_\lambda^1(x, t) = \frac{1}{\sqrt{1+\lambda t}} \exp \left\{ -\frac{\lambda(t+2\sqrt{x})^2}{4(1+\lambda t)} \right\} \tag{1.27}$$

Inverting the Laplace transform gives

$$p^1(t, x, y) = \frac{1}{\sqrt{\pi y t}} e^{-\sqrt{x}} \cosh \left(\frac{(t+2\sqrt{x})\sqrt{y}}{t} \right) \exp \left\{ -\frac{(x+y)}{t} - \frac{1}{4}t \right\} \tag{1.28}$$

Next the equation arising from f^2

is

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x} \tanh(\sqrt{x}) \right) \frac{\partial u}{\partial x} \tag{1.29}$$

By Theorem 1.1, the characteristic function for (1.29) is

$$U_{\lambda}^2(x, t) = \frac{1}{\cosh(\sqrt{x})\sqrt{1+\lambda t}} \cosh\left(\frac{\sqrt{x}}{\sqrt{1+\lambda t}}\right) \exp\left\{-\frac{\lambda(x + \frac{1}{4}t^2)}{1+\lambda t}\right\} \tag{1.30}$$

Inverting the Laplace transform leads to the fundamental solution

$$p^2(t, x, y) = \frac{1}{\sqrt{\pi y t}} \frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})} \cosh\left(\frac{2\sqrt{xy}}{t}\right) \exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\} \tag{1.31}$$

Finally, the equation arising from f^3 is

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x} \coth(\sqrt{x}) \right) \frac{\partial u}{\partial x} \tag{1.32}$$

From Theorem 1.1, the characteristic function for (1.32) is

$$U_{\lambda}^3(x, t) = \frac{1}{\sinh(\sqrt{x})\sqrt{1+\lambda t}} \sinh\left(\frac{\sqrt{x}}{\sqrt{1+\lambda t}}\right) \exp\left\{-\frac{\lambda(x + \frac{1}{4}t^2)}{1+\lambda t}\right\} \tag{1.33}$$

Inversion of the Laplace transform leads to

$$p^3(t, x, y) = \frac{1}{\sqrt{\pi y t}} \frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})} \sinh\left(\frac{2\sqrt{xy}}{t}\right) \exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\} \tag{1.34}$$

For each of these cases, $\int_0^{\infty} p^i(t, x, y) dy = 1, i = 1, 2, 3$

Example 1.5. We consider now an example of a function which possesses discontinuities. The equation

$$x f' - f + \frac{1}{2} f^2 = -1$$

has a solution

$$f(x) = 1 + \cot(\ln \sqrt{x})$$

This solution is discontinuous at points of the form $x = e^{4n\pi}, n \in \{0, 1, 2, \dots\}$.

obtain a characteristic function and fundamental solution for the PDE

By applying Theorem 1.1, we can

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(1 + \cot(\ln \sqrt{x}) \right) \frac{\partial u}{\partial x} \tag{1.35}$$

Applying equation (1.2), we have the characteristic function

$$U_{\lambda}(x, t) = \operatorname{cosec}(\ln \sqrt{x}) \left[x^{\frac{i}{2}} (1 + \lambda t)^{-i} - x^{-\frac{i}{2}} (1 + \lambda t)^i \right] \frac{\exp\left\{-\frac{\lambda x}{1+\lambda t}\right\}}{2i(1+\lambda t)} \tag{1.36}$$

where $i = \sqrt{-1}$. Inversion of the Laplace transform gives the fundamental solution

$$p(t, x, y) = \operatorname{cosec}(\ln \sqrt{x}) \frac{e^{\frac{(x+y)}{t}}}{2it} \left(y^{\frac{i}{2}} I_i\left(\frac{2\sqrt{xy}}{t}\right) - y^{-\frac{i}{2}} I_{-i}\left(\frac{2\sqrt{xy}}{t}\right) \right) \tag{1.37}$$

See [11] and [3] we obtain

$$\int_0^{\infty} p(t, x, y) dy = \frac{\left(-\left(\frac{1}{t}\right)^{2i} + \left(\frac{x}{t^2}\right)^i \right) \operatorname{cosec}(\ln \sqrt{x})}{2i \left(\frac{1}{t}\right)^i \left(\frac{x}{t^2}\right)^{\frac{i}{2}}}$$

$$= \frac{\left(-\left(\frac{1}{t}\right)^{2i} + \left(\frac{x}{t^2}\right)^i \right)}{2i\left(\frac{1}{t}\right)^i\left(\frac{x}{t^2}\right)^{\frac{i}{2}}} \frac{2ix^{\frac{i}{2}}}{x^{i-1}} = 1 \tag{1.38}$$

Case 2:

The Riccati Equation

$$xf' - f + \frac{1}{2}f^2 = Ax^2 + Bx + C$$

Next we consider the case when the function satisfies the Riccati equation (0.3). Equation (0.1) has infinitesimal symmetries of the form

$$\mathbf{v}_3 = x\sqrt{A}e^{\sqrt{A}t} \frac{\partial}{\partial x} + e^{\sqrt{A}t} \frac{\partial}{\partial t} - \frac{1}{2}\left(Ax + \sqrt{A}f(x) + B\right)e^{\sqrt{A}t} u \frac{\partial}{\partial u}$$

$$\mathbf{v}_4 = -x\sqrt{A}e^{-\sqrt{A}t} \frac{\partial}{\partial x} + e^{-\sqrt{A}t} \frac{\partial}{\partial t} - \frac{1}{2}\left(Ax - \sqrt{A}f(x) + B\right)e^{-\sqrt{A}t} u \frac{\partial}{\partial u}$$

In order to compute fundamental solutions, we require the corresponding group actions. This is given by our next result.

Proposition 2.1. [8] Let f be a solution of (0.3) and u be a solution of (0.1). Then, for ε sufficiently small, the following functions are also solutions of (0.1).

$$\rho(\exp(\varepsilon \mathbf{v}_3))u(x, t) = \left(1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}\right)^{\frac{B}{2\sqrt{A}}} u\left(\frac{x}{1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}}, \frac{1}{\sqrt{A}} \ln\left(\frac{e^{\sqrt{A}t}}{1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right) \times \exp\left\{\frac{-\varepsilon A e^{\sqrt{A}t} x}{2(1 + \varepsilon\sqrt{A}e^{\sqrt{A}t})} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right)\right\} \tag{2.1}$$

and

$$\rho(\exp(\varepsilon \mathbf{v}_4))u(x, t) = e^{-\frac{B}{2}t} \left(e^{\sqrt{A}t} - \varepsilon\sqrt{A}\right)^{\frac{B}{2\sqrt{A}}} u\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t} - \varepsilon\sqrt{A}}, \frac{\ln(e^{\sqrt{A}t} - \varepsilon\sqrt{A})}{\sqrt{A}}\right) \times \exp\left\{\frac{-\varepsilon A x}{2(e^{\sqrt{A}t} - \varepsilon\sqrt{A})} - \frac{1}{2}\left(F(x) - F\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t} - \varepsilon\sqrt{A}}\right)\right)\right\} \tag{2.2}$$

Since $u = 1$ is a solution of equation (0.1), then by Proposition (2.1), so are,

$$U_\varepsilon^1(x, t) = \left(1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}\right)^{\frac{B}{2\sqrt{A}}} \times \exp\left\{\frac{-\varepsilon A e^{\sqrt{A}t} x}{2(1 + \varepsilon\sqrt{A}e^{\sqrt{A}t})} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right)\right\} \tag{2.3}$$

and

$$U_\varepsilon^2(x, t) = e^{-\frac{B}{2}t} \left(e^{\sqrt{A}t} - \varepsilon\sqrt{A}\right)^{\frac{B}{2\sqrt{A}}} \times \exp\left\{\frac{-\varepsilon A x}{2(e^{\sqrt{A}t} - \varepsilon\sqrt{A})} - \frac{1}{2}\left(F(x) - F\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t} - \varepsilon\sqrt{A}}\right)\right)\right\} \tag{2.4}$$

Neither of these two solutions can be immediately identified with the characteristic function of (0.1). However it is often possible to derive

the fundamental solution from them. We illustrate the method with examples.

Example 2.1. consider the PDE

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(\frac{3}{2} - x\right) \frac{\partial u}{\partial x} \tag{2.5}$$

Applying equation (2.4), and setting $\varepsilon = \lambda / (1 + \lambda)$, we see that

$$\tilde{U}_\lambda(x, t) = \left(\frac{(1+\lambda)e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\} \tag{2.6}$$

is a solution of (2.5). Next, we use the fact that multiplication of solutions of (0.1) yields a new solution. We multiply \tilde{U}_λ by $1/(1+\lambda)^{\frac{3}{2}}$ to obtain

$$U_\lambda(x, t) = \left(\frac{e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\} \tag{2.7}$$

which is the characteristic function for (2.5). See for example, [9]. Inverting the Laplace transform, we obtain the fundamental solution for (2.5). It is

$$p(t, x, y) = \left(\frac{e^t}{e^t - 1}\right)^{\frac{3}{2}} \exp\left\{-\frac{(x+y)}{e^t - 1}\right\} I_{\frac{1}{2}}\left(\frac{2\sqrt{xye^t}}{e^t - 1}\right) \tag{2.8}$$

where

$$I_{\nu - \frac{1}{2}}(z) = 2^{\nu - \frac{1}{2}} \Gamma(\nu + \frac{1}{2}) Z^{-\nu + \frac{1}{2}} I_{\nu - \frac{1}{2}}(z) \tag{2.9}$$

Example 2.2. Consider the function $f(x) = x \coth(\frac{x}{2})$. Here

$$xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x^2.$$

By (2.4) of Proposition (2.1) the equation

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + x \coth\left(\frac{x}{2}\right) \frac{\partial u}{\partial x} \tag{2.10}$$

has a solution

$$u_\varepsilon(x, t) = \frac{\sinh\left(\frac{xe^t}{2(e^t - \varepsilon)}\right)}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-\varepsilon x}{2(e^t - \varepsilon)}\right\} \tag{2.11}$$

From this we can derive the fundamental solution $p(t, x, y)$ of (2.10).

Observe that

$$u_\varepsilon(x, 0) = \frac{1}{2} \left(\frac{e^{\frac{x}{2}}}{\sinh(\frac{x}{2})} - \frac{1}{\sinh(\frac{x}{2})} \exp\left\{\frac{-(1+\varepsilon)x}{2(1-\varepsilon)}\right\} \right) \tag{2.12}$$

we note that $g(x) = \frac{e^{\frac{x}{2}}}{\sinh(\frac{x}{2})}$ is a stationary solution of the equation (2.10). We

therefore look for a fundamental solution $p(t, x, y)$ with the property that

$$\int_0^\infty \frac{e^{\frac{y}{2}}}{\sinh(\frac{y}{2})} p(t, x, y) dy = \frac{e^{\frac{x}{2}}}{\sinh(\frac{x}{2})} \tag{2.13}$$

Use the new parameter $\lambda = \frac{1+\varepsilon}{2(1-\varepsilon)}$. The solution u_ε becomes

$$u_\lambda(x, t) = \frac{\sinh\left(\frac{(2\lambda+1)xe^t}{2((2\lambda+1)e^t - (2\lambda-1))}\right)}{\sinh(\frac{x}{2})} \exp\left\{\frac{-(2\lambda-1)x}{2((2\lambda+1)e^t - (2\lambda-1))}\right\} \tag{2.14}$$

By (2.13). we may write (2.14) as

$$u_{\lambda}(x, t) = \frac{1}{2} \int_0^{\infty} \left(\frac{e^{\frac{y}{2}}}{\sinh(\frac{y}{2})} - \frac{1}{\sinh(\frac{y}{2})} e^{-\lambda y} \right) p(t, x, y) dy$$

$$= \frac{e^{\frac{x}{2}}}{2 \sinh(\frac{x}{2})} - \frac{1}{2} L \left(\frac{1}{\sinh(\frac{y}{2})} p(t, x, y) \right) \tag{2.15}$$

Rearranging equation (2.15), we get the fundamental solution

$$p(t, x, y) = \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})} L^{-1} \left(\exp \left\{ \frac{-(2\lambda(1+e^t)+e^t-1)x}{2((2\lambda+1)e^t-(2\lambda-1))} \right\} \right)$$

$$= \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})} \exp \left\{ \frac{(e^t+1)(x+y)}{2(e^t-1)} \right\} \left[\frac{e^{\frac{t}{2}}}{e^t-1} \sqrt{\frac{x}{y}} I_1 \left(\frac{2\sqrt{xy}e^t}{e^t-1} \right) + \mathcal{D}(y) \right] \tag{2.16}$$

Case 3:

The Riccati equation

$$xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx - \frac{3}{8}$$

The last case which we must consider is when the function f is a solution of the third Riccati equation (0.4). There are two subcases here. $B = 0$, and $B \neq 0$, In the case $B = 0$, we can

obtain the characteristic function by symmetry directly as we did in Theorem 1.1. Recall that when f is a solution of (0.4), and $B = 0$, the PDE (0.1) has an infinitesimal symmetry of the form [11].

$$v_6 = \left(8xt + \frac{2A}{3} \sqrt{x} t^3 \right) \frac{\partial}{\partial x} + 4t^2 \frac{\partial}{\partial t} - \left(4x + 2Ct^2 + 4f(x)t + \frac{A^2}{36} t^4 + 2A \sqrt{x} t^2 - \frac{A(\frac{1}{2}f(x))}{3\sqrt{x}} t^3 \right) u \frac{\partial}{\partial u} \tag{3.1}$$

The group action generated by this symmetry allows us to determine the characteristic function for (0.1). Thus we have the following result.

Theorem 3.1. let f be a solution of the Riccati equation [9].

$$xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Cx - \frac{3}{8} \tag{3.2}$$

Then the characteristic function $U_{\lambda}(x, t)$ for the corresponding PDE (0.1) is given by

$$U_{\lambda}(x, t) = \sqrt{\frac{\sqrt{x}(1+\lambda t)}{\sqrt{x}(1+\lambda t) - \frac{A\lambda}{12}t^3}}$$

$$\times \exp \left\{ \frac{1}{2} \left(F(x) - F \left(\left(\frac{\sqrt{x}}{1+\lambda t} - \frac{A\lambda^4}{12(1+\lambda t)^2} \right)^2 \right) \right) \right\}$$

$$\times \exp \left\{ - \frac{\lambda(x + \frac{1}{2}Ct^2)}{1+\lambda t} - \frac{\frac{2}{3}A t^2 \sqrt{x}(3+\lambda t)}{(1+\lambda t)^2} + \frac{A^2 t^4 (2\lambda t (3 + \frac{1}{2}\lambda t) - 3)}{108(1+\lambda t)^3} \right\} \tag{3.3}$$

Proof. The idea of the proof is the same as for Theorem 1.1. First we observe

that $U_{\lambda}(x, 0) = e^{-\lambda x}$. In order to show that (3.3) is the characteristic

function we exponentiate the infinitesimal symmetry (3.1). The only difficulty here is solving the equation

$$\frac{d \tilde{x}}{d \tilde{t}} = 8 \tilde{x} \tilde{t} + \frac{2A}{3} \sqrt{\tilde{x} \tilde{t}}^3 \tag{3.4}$$

making the change of variables $\sqrt{\tilde{x}} = y$, under this change of variables, equation (3.4) becomes first order linear. This leads to

$$\sqrt{\tilde{x}} = \frac{\sqrt{x}}{1-4\epsilon t} + \frac{A \epsilon t^3}{3(1-4\epsilon t)^2}$$

By symmetry, if u is a solution of (0.1), then so is

$$U_\epsilon(x,t) = \sqrt{\frac{\sqrt{x(1+4\epsilon t)}}{\sqrt{x(1+4\epsilon t)} - \frac{A\epsilon t^3}{3}}} u\left(\left(\frac{\sqrt{x}}{1+4\epsilon t} - \frac{A\epsilon t^3}{3(1+4\epsilon t)^2}\right)^2, \frac{t}{1+4\epsilon t}\right) \times \exp\left\{\frac{1}{2}\left(F(x) - F\left(\left(\frac{\sqrt{x}}{1+4\epsilon t} - \frac{A\epsilon t^3}{3(1+4\epsilon t)^2}\right)^2\right)\right)\right\} \\ \times \exp\left\{-\frac{4\epsilon(x + \frac{1}{2}Ct^2)}{1+4\epsilon t} - \frac{\frac{3}{2}At^2\sqrt{x}(3+4\epsilon t)}{(1+4\epsilon t)^2} + \frac{A^2t^4(8\epsilon t(3+2\epsilon t)-3)}{108(1+4\epsilon t)^3}\right\} \tag{3.6}$$

Taking $u = 1$, and setting $\lambda = 4\epsilon$ gives the result.

If we take $A = 0$, in (3.3), then it reduces to equation (1.2). In order to apply Theorem 3.1 we need solutions

of (0.4). And we can transform (0.4) to the linear equation,

$$2x^2y''(x) - (Ax^{3/2} + Cx - \frac{3}{8})y(x) = 0 \tag{3.7}$$

The general solution of (3.7), is easily found to be

$$y(x) = x^{\frac{1}{4}} \left(a_1 Ai \left(3^{\frac{2}{3}} \frac{(2C + \frac{4A}{3}\sqrt{x})}{(2^4 A^2)^{\frac{1}{3}}} \right) + a_2 Bi \left(3^{\frac{2}{3}} \frac{(2C + \frac{4A}{3}\sqrt{x})}{(2^4 A^2)^{\frac{1}{3}}} \right) \right) \tag{3.8}$$

Where Ai and Bi are the first and second kind Airy functions [6], and a_1 and a_2 are arbitrary constants, taking $f = 2xy / y'$ gives solutions of (0.4).

Taking $A = \frac{4}{3}, C = 0, a_1 = 1$ and $a_2 = 0$, gives the solution

$$f(x) = \frac{1}{2} + \frac{\sqrt{x} Ai'(\sqrt{x})}{Ai(\sqrt{x})} \tag{3.9}$$

Since

$$F(x) = \int \frac{f(x)}{x} dx = \frac{1}{2} \left(\ln(x) + 4 \ln(Ai(\sqrt{x})) \right) \tag{3.10}$$

An application of Theorem 3.1 allows us to determine the characteristic function for

$$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \frac{\sqrt{x} Ai'(\sqrt{x})}{Ai(\sqrt{x})} \right) \frac{\partial u}{\partial x} \tag{3.11}$$

However at this stage we are unable to invert the Laplace transform.

It should be possible to invert the transform numerically. See [5] on the numerical inversion of Laplace transforms [5].

The last case we have to consider is the case when the function is a solution of (0.4) and $B \neq 0$. Recall that when f was a solution of (0.4), for $B \neq 0$, then (0.1) has two infinitesimal symmetries of the form

$$\begin{aligned} \mathbf{v}_5 &= \left(\frac{2A}{3\sqrt{B}} \sqrt{x} + \sqrt{B} x \right) e^{\sqrt{B}t} \frac{\partial}{\partial x} + e^{\sqrt{B}t} \frac{\partial}{\partial t} \\ &\quad - \left(\frac{B}{2} x + \frac{2A}{3} \sqrt{x} + \frac{\sqrt{B}}{2} f(x) - \frac{A(\frac{1}{2}-f(x))}{3\sqrt{B}\sqrt{x}} + \alpha \right) e^{\sqrt{B}t} u \frac{\partial}{\partial u} \\ \mathbf{v}_6 &= - \left(\frac{2A}{3\sqrt{B}} \sqrt{x} + \sqrt{B} x \right) e^{-\sqrt{B}t} \frac{\partial}{\partial x} + e^{-\sqrt{B}t} \frac{\partial}{\partial t} \\ &\quad - \left(\frac{B}{2} x + \frac{2A}{3} \sqrt{x} - \frac{\sqrt{B}}{2} f(x) + \frac{A(\frac{1}{2}-f(x))}{3\sqrt{B}\sqrt{x}} + \alpha \right) e^{-\sqrt{B}t} u \frac{\partial}{\partial u} \end{aligned}$$

where $\alpha = \frac{2A^2+9BC}{18B}$. At present we are unable to determine any characteristic functions for (0.1) because we have not yet found any explicit solutions of (0.4) for $B \neq 0$. Nevertheless, for completeness, we

present the group symmetries which are generated by \mathbf{v}_5 and \mathbf{v}_6 .

Proposition 3.2. Let f be a solution of (0.4) and u be a solution of (0.1). Then for ε sufficiently small, the following functions are also solutions of (0.1).

$$\begin{aligned} &\rho(\exp(\varepsilon \mathbf{v}_5))u(x, t) \\ &= \left(1 + \varepsilon \sqrt{B} e^{\sqrt{B}t}\right)^{\frac{2A^2+9B\alpha}{9B^{3/2}}} \sqrt{\frac{3B\sqrt{x} + 2A(1 - \sqrt{1 + \varepsilon\sqrt{B}e^{\sqrt{B}t}})}{3B\sqrt{x}}} \\ &\times \exp\left\{-\frac{\varepsilon e^{\sqrt{B}t}(2A + 3B\sqrt{x})^2}{18B(1 + \varepsilon\sqrt{B}e^{\sqrt{B}t})}\right\} \\ &\times \exp\left\{-\frac{1}{2}\left(F(x) - F\left(\frac{\sqrt{x}}{1 + \varepsilon\sqrt{B}e^{\sqrt{B}t}} - D\right)^2\right)\right\} \\ &\times u\left(\frac{\sqrt{x}}{1 + \varepsilon\sqrt{B}e^{\sqrt{B}t}} - D, \frac{1}{\sqrt{B}} \ln\left(\frac{e^{\sqrt{B}t}}{1 + \varepsilon\sqrt{B}e^{\sqrt{B}t}}\right)\right) \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\rho(\exp(\varepsilon \mathbf{v}_6))u(x, t) \\ &= \left(e^{\sqrt{B}t} - \varepsilon\sqrt{B}\right)^{\frac{9B\varepsilon-2A^2}{9B^{3/2}}} \sqrt{\frac{3Be^{\frac{\sqrt{B}t}{2}} + 2A\left(e^{\frac{\sqrt{B}t}{2}} - \sqrt{e^{\sqrt{B}t} - \varepsilon\sqrt{B}}\right)}{3B\sqrt{x}e^{\frac{\sqrt{B}t}{2}}}} \\ &\times \exp\left\{-\frac{\varepsilon(2A + 3B\sqrt{x})^2}{18B(e^{\sqrt{B}t} - \varepsilon\sqrt{B})}\right\} \\ &\times \exp\left\{-\frac{1}{2}\left(F(x) - F\left(\frac{(\sqrt{x} + \frac{2A}{3B})e^{\frac{\sqrt{2B}t}}}{\sqrt{e^{\sqrt{B}t} - \varepsilon\sqrt{B}}} - D\right)^2\right)\right\} \\ &\times u\left(\frac{(\sqrt{x} + \frac{2A}{3B})e^{\frac{\sqrt{2B}t}}}{\sqrt{e^{\sqrt{B}t} - \varepsilon\sqrt{B}}} - D, \frac{\ln(e^{\sqrt{B}t} - \varepsilon\sqrt{B})}{\sqrt{B}}\right) \end{aligned} \tag{3.13}$$

where, $D = \frac{2A}{3B}$ and $\alpha = \frac{1}{9}A^2 + \frac{1}{2}BC$ 101
Taking $u = 1$ allows us to write down solutions of (0.1) for any f that is a solution of (0.4). Our experience with the previous cases strongly suggests

that if we can obtain solutions to (0.4) with $B \neq 0$, then we would be able to determine the corresponding characteristic functions and fundamental solutions.

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