LARGE DEFORMATION FINITE ELEMENT ANALYSIS OF SHELL STRUCTURES

By:

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ABSTRACT:

A finite element formulation is presented for conducting large deformation analysis of shells. The element adopted herein is a degenerated three-dimensional 8-nodes isoparametric finite element. Derivations of the nonlinear geometric element stiffness matrices were made on the basis of total lagrangian formulation by using both geometric strains (Engineering Strains) and Green's stains. The formulations were implemented into a nonlinear finite element program. The nonlinear equilibrium equations are solved by combined incremental load and Newton-Raphson method. Examples are presented for the analysis of cylindrical shells. Agreement with existing solutions is generally good.

ملخص:

عرضت طريقة العناصر المحددة لتحليل سلوك القشريات ذات الإنحراف الكبير. تمت صياغة العنصر المحدد بمعالجة عنصر آيزوبارامتري ثلاثي الأبعاد ذي ثمانية عقد. مصفوفة الجساءة اللاخطية تم إستنتاجها على أساس علاقة اللاقرانج الكلي وذلك بإستخدام إنفعال قرين **(Green's Strains)** والإنفعال الهندسي **(Engineering Strains)**. تم عمل برنامج حاسوب يتضمن تحليل العنصر المحدد اللاخطى. أستخدم الحمل التكراري مع طريقة نيوتن – رابسون لحل معادلات الإتزان اللاخطية. الأمثلة الواردة في هذا البحث هي لتحليل نماذج من القشريات الأسطوانية. تبين الدراسة أن التوافق مع الحلول المتوفرة هو بشكل عام جيد.

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Introduction:

A formulation of three dimensional large rotation elasto-plastic theory for thin curved eccentric beams elements has been developed by AbdelRahman Elzubier [1], which involves the use of geometric strains in the formulation. He shows that there was a big difference in stresses obtained when using the geometric strains and Green's strains in the case of large rotations.

In this paper A.Elzubier formulation using geometric strains [1], is extended using shell finite element to enable large deformation finite element analysis of shell structures to be examined using geometric stains and Green's strains in geometrical nonlinear formulation.

The concept of treating a shell element as special case of three dimensional analysis was used by Ahmed, Irons, and Zienkiewicz [2] and Pawsey [3],and it seems to provide asimple and efficient .strategy development of isoparametric shell elements.

Curved elements based on exact or appropriate shapes of shells began to appear in the late 1960s, Bogner, Fox, and Schmit [4] describe a cylindrical shell element, which used interpolation functions defined the shell coordinates.Aformulation of geometrically nonlinear formulation for the axisymmteric shell element is given by Surana [5], using total lagrangian formulation with the concept that the displacements are nonlinear functions of nodal rotations.

The formulation by Zienkiewicz [6],involves the use of Green's strains tensor with large displacement formulation,as an application on plates and shells.

Atotal lagrangian formulation based on geometric strains with geometric stresses and Green's strains with 2^{nd} Piola Kirchhoff stresses is adopted in this paper. Results obtained using geometric strains and Green's strains compared with previous solutions are presented numerically.

Geometric Definition of the element: -

A typical curved shell element, with eight nodes on mid-surface is shown in Fig.(1). Each node has five degrees of freedom, three translations and two rotations as shown in Fig.(2). By assuming the lines joining the top and bottom nodes to be straight, the shape of the element is defined by the eight nodal values as:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} N_i \left[\begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \zeta \frac{t_i}{2} v_{3i} \right] \tag{1}$$

Where x_i , y_i , z_i are the global coordinates of the mid-surface node i. , t_i It is the shell thickness at node i , v_{3i} is aunit vector in the direction normal to the middle plane, and N_i is the shape function at node i

The displacement vector can be written as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{8} N_i \left[\begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} + \zeta \frac{t_i}{2} [v_{1i}, v_{2i}] \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} \right] \tag{2}$$

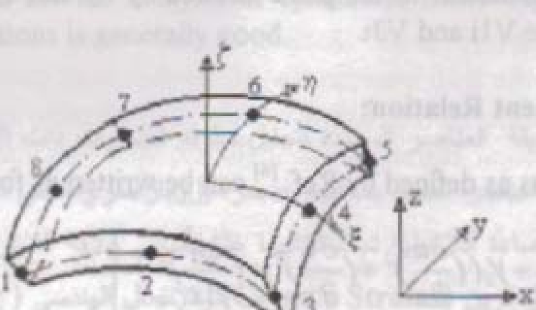
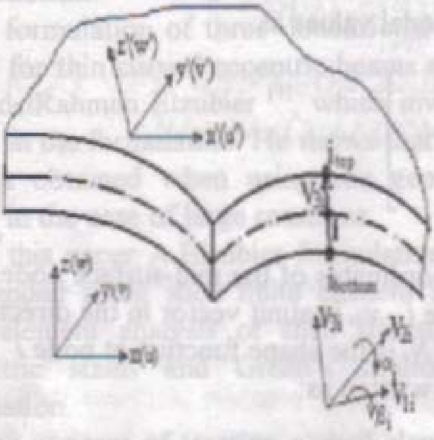


Fig.(1): 8-Nodes Element



$$V_{3i} = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{top} - \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}_{bottom}$$

$$V_{1i} = i \times V_{3i}$$

$$V_{2i} = V_{1i} \times V_{3i}$$

Fig.(2): Geometry of the element

Where V3i is a vector in the direction normal to the middle plane, V_{1i} is perpendicular to the plane defined by V_{3i} and x axis and V_{2i} is normal to the V1i and V3i.

Strain-displacement Relation:

Green's Strains:

The Green's strains as defined by Ref. [6] can be written as follows:

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right) \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right) \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial w}{\partial y} \end{bmatrix} \quad \text{Or } e = e_o + e_L$$

(3)

Where:

$e_o = B_o\, a$

(4)

where a is the element nodal displacements vector.

$e_L = \frac{1}{2} A\, q$

(5)

e_o are the strains for infinitesimal displacements

e_L are the strains due to large displacements

B_o is a (5x40) matrix contains shape functions derivatives.

A and q are given as follows:

$$A = \begin{bmatrix} \theta_x & 0 & 0 \\ 0 & \theta_y & 0 \\ \theta_y & \theta_x & 0 \\ \theta_z & 0 & \theta_x \\ 0 & \theta_z & \theta_y \end{bmatrix}$$

(6)

$$\theta = \{\theta_x \quad \theta_y \quad \theta_z\}^T = G\, a$$

(7)

where:

$$\theta_x = \left\{\frac{\partial u}{\partial x} \quad \frac{\partial v}{\partial x} \quad \frac{\partial w}{\partial x}\right\}; \quad \theta_y = \left\{\frac{\partial u}{\partial y} \quad \frac{\partial v}{\partial y} \quad \frac{\partial w}{\partial y}\right\}; \quad \theta_z = \left\{\frac{\partial u}{\partial z} \quad \frac{\partial v}{\partial z} \quad \frac{\partial w}{\partial z}\right\}$$

G is a (9x40) matrix containing shape function derivatives.

Geometric strains:

The geometric strains ε'_x, ε'_y are defined by the change in length per unit initial length of line elements originally oriented parallel to the x,y axes respectively. The sheer strain is defined by the change in the right angle [1].

The geometric strains are given as follows:

$$\varepsilon' = \begin{Bmatrix} \varepsilon'_x \\ \varepsilon'_y \\ \gamma'_{xy} \\ \gamma'_{xz} \\ \gamma'_{yz} \end{Bmatrix} = \begin{Bmatrix} e_x^{1/2} - 1 \\ e_y^{1/2} - 1 \\ \dfrac{\gamma_{xy}}{e_x^{1/2}.e_y^{1/2}} \\ \dfrac{\gamma_{xz}}{e_x^{1/2}} \\ \dfrac{\gamma_{yz}}{e_y^{1/2}} \end{Bmatrix}$$

(8)

where $e_x = (1+2e_x)$; $e_y = (1+2e_y)$

Variations of strains:

By taking the variation of Eq.(8),we have:

$$\delta\varepsilon' = \begin{Bmatrix} \delta\varepsilon_x' \\ \delta\varepsilon_y' \\ \delta\gamma_{xy}' \\ \delta\gamma_{xz}' \\ \delta\gamma_{yz}' \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{e_x^{1/2}} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{e_y^{1/2}} & 0 & 0 & 0 \\ \dfrac{-\gamma_{xy}}{e_x^{1/2}.e_y^{1/2}} & \dfrac{-\gamma_{xy}}{e_x^{1/2}.e_y^{1/2}} & \dfrac{1}{e_x^{1/2}.e_y^{1/2}} & 0 & 0 \\ \dfrac{-\gamma_{xz}}{e_x^{1/2}} & 0 & 0 & \dfrac{1}{e_x^{1/2}} & 0 \\ 0 & \dfrac{-\gamma_{yz}}{e_y^{1/2}} & 0 & 0 & \dfrac{1}{e_y^{1/2}} \end{bmatrix} \begin{Bmatrix} \delta\varepsilon_x \\ \delta\varepsilon_y \\ \delta\gamma_{xy} \\ \delta\gamma_{xz} \\ \delta\gamma_{yz} \end{Bmatrix} \text{ or}$$

$\delta\varepsilon' = H\,\delta\varepsilon$ (9)

Taking the varition of Eq.(4), we have:

$\delta\varepsilon_o = B_o\,\delta a$

(10)

By using Eq.(7) and taking the vaiation of Eq.(5), we have:

$\delta\varepsilon_L = \frac{1}{2}\,\delta A\,\theta + \frac{1}{2}\,A\,\delta\theta = A\,\delta\theta = A\,G\,\delta a = B_L\,\delta a$

(11)

$$\delta\varepsilon = \delta\varepsilon_o + \delta\varepsilon_L = (B_o + B_L)\,\delta a = B\,\delta a$$
(12)

Substituting Eq.(12) in Eq.(9), we have:

$$\delta\varepsilon' = H\,B\,\delta a = B^*\,\delta a \quad , \quad B^* = H\,B$$
(13)

Stresses and Strains Relations:

By considering the material to be linear elastic, the engineering stresses corresponding to the geometric strains are defined by:

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = D\,\varepsilon'$$
(14)

Where D is the elasticity matrix for isoparametric material and is given by Ref.[6].

Tangent Stiffness Matrix:

The tangent stiffness matrix K_T is obtained by differentiating the residual force vector ψ with respect to the displacement vector a. Where ψ is given by:

$$\psi = \int_v B^{*T}\,\sigma\,dv - f$$
(15)

where f is the external force vector.

Taking the variation of Eq.(14), we have:

$$K_T = K_o + K_L + K_s + K_s^*$$
(16)

Where:

$$K_o = \int_v B_o^T\,H^T\,D\,H\,B_o\,dv$$
(17)

$$K_L = \int_v (B_L^T H^T DHB_L + B_L^T H^T DHB_s + B_L^T H^T DHB_s)d$$

(18)a

$$K_e + K_L = \int_v B^{*T} DB^* dv = \int_v B^T H^T DH Bdv$$

(18)b

K_s is the initial stress stiffness matrix given by:

$$K_s = \int_v G^T P^* G\, dv$$

(19)a

P^* is the initial stress matrix given by:

$$P^* = \begin{bmatrix} \sigma_x^*[I] & \tau_{xy}^*[I] & \tau_{xz}^*[I] \\ & \sigma_y^*[I] & \tau_{yz}^*[I] \\ \text{Symmetric} & & o[I] \end{bmatrix} ; \quad I \text{ is a } 3\times3 \text{ identity matrix}$$

(19)b

$$\sigma^* = \begin{Bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \\ \tau_{xz}^* \\ \tau_{yz}^* \end{Bmatrix} = H^T \sigma$$

(19)c

K_s^* is the additional geometric stiffness matrix and is given by:

$$K_s^* = \int_v B^T P B dv$$

(20)a

P, is the additional initial stress matrix and is given in terms of the initial stress and strains by:

$$P=\begin{bmatrix} \left(\frac{-\sigma_x}{e_x^{k/2}}+\frac{3\gamma_{xy}\tau_{xy}}{e_x^{k/2}e_y^{k/2}}+\frac{3\gamma_{xz}\tau_{xz}}{e_x^{k/2}}\right) & \frac{\gamma_{xy}\tau_{xy}}{e_x^{k/2}e_y^{k/2}} & \frac{-\tau_{xy}}{e_x^{k/2}e_y^{k/2}} & \frac{-\tau_{xz}}{e_x^{k/2}} & 0 \\ & \left(\frac{-\sigma_y}{e_y^{k/2}}+\frac{3\gamma_{xy}\tau_{xy}}{e_x^{k/2}e_y^{k/2}}+\frac{3\gamma_{yz}\tau_{yz}}{e_y^{k/2}}\right) & \frac{-\tau_{xy}}{e_x^{k/2}e_y^{k/2}} & 0 & \frac{-\tau_{yz}}{e_y^{k/2}} \\ & & 0 & 0 & 0 \\ & \text{Symmetric} & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

(20)b

The solution is based on the Newton-Raphson method with incremental loading.

Numerical Examples:

1- A glass-epoxy thin-walled cylinder:

The cylinder is clamped at both ends and subjected to internal pressure Fig.(3). By use of symmetry, only one-eight of the cylinder was modeled by 4x4 mesh elements. The material is assumed to be linearly elastic and isotropic. The pressure load was applied in 10-equal increments. The incremental solution converges with an average of 3 iteration cycles. The load deformation response is compared with those obtained by Chang and Sawamiphakdi [7] Good agreement can be seen from Fig.(3).

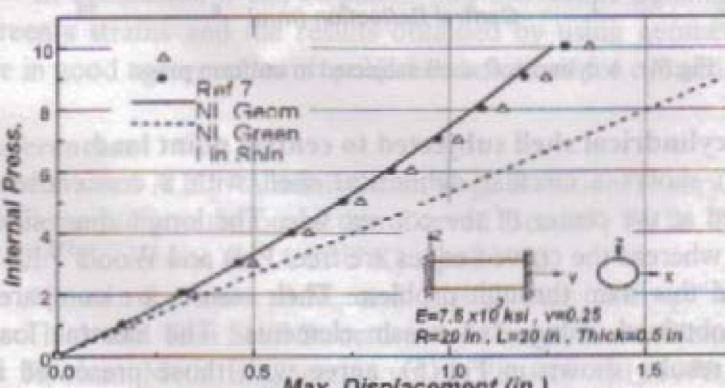


Fig. (3): Large deflection analysis of thin wall cylinder

2-Circular cylindrical shell:

The circular cylindrical shell as shown in Fig.(4) is clamped along all four edges and subjected to a uniform surface pressure up to 3 kN/m^2. The material property is linearly elastic and isotropic. One quarter of the shell was modeled by 4x4 mesh elements. In the analysis the pressure load increment was varied at three stages: 0.25 kN/m^2 for the softening part, 0.0625 kN/m^2 near the snap through deformation and 0.25 kN/m^2 for the stiffing part. The incremental solution at the initial and final stages converges with an average of 2 iteration cycles and for the softening part 1 iteration cycle. The load deformation response is compared with those obtained by Chang and Sawamiphakdi [7]. Again, good agreement can be seen from Fig.(4).

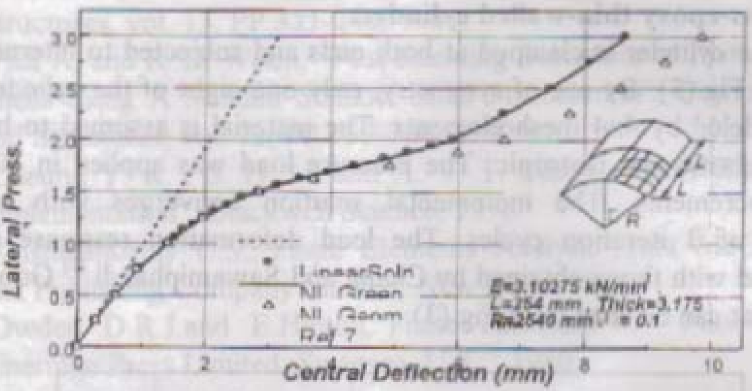


Fig.(4): A cylindrical shell subjected to uniform press.

3-Hinged cylindrical shell subjected to central point load:

Fig.(5) shows a circular cylindrical shell with a concentrated load applied at the center of the convex side. The longitudinal sides are hinged whereas the curved edges are free. Pica and Wood [8] have investigated this snap through problem. Their results are compared to those obtained using 4x4 mesh elements. The central load deflection results shown in Fig.(5), agree with those presented in Ref.[8] up to a load of about 2.1 kN, after that the solution diverges.

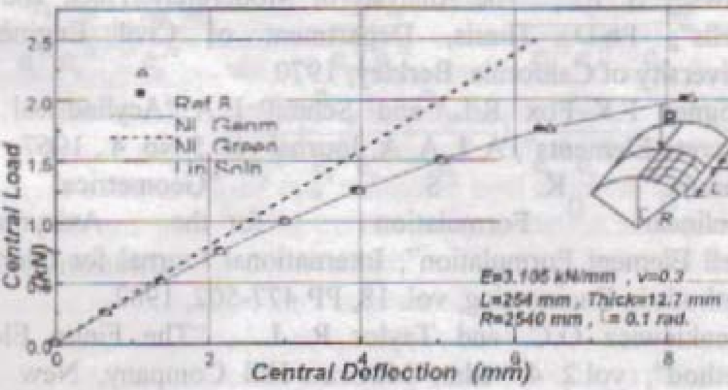


Fig. (5): Hinge cylindrical shell

Conclusions:

This paper has presented a geometrically nonlinear analysis of shells using the total lagrangian formulation based on geometric strains. The nonlinear equilibrium equations are solved by combined incremental and Newton-Raphson method. It is demonstrated that the displacements obtained by using geometric strains are nearly the same as those obtained by using Green's strains in the case of small rotations and further investigation is necessary for the case of large rotation.

In conclusion it can be stated that the results obtained by using Green's strains and the results obtained by using geometric strains are in good agreement with published solutions for cylindrical shells.

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