A NEW DERIVATION OF THE GENERALIZED FIELD EQUATION WITH A SOURCE TERM

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ABSTRACT

A simple derivation of the generalized field equation with a source term is presented by restricting ourselves to a locally inertial frame. It reduces to Einstein's field equation when the Lagrangian is linear. Assuming the metric to be Minkowskian, simple solutions for the scalar curvature show the existence of a short range field and the emission of gravitational waves by objects which have strong fields.

ملخص

تم عمل استنباط مبسط لمعادلة الحقل المعمم عندما قصر البحث في إحداثيات القصور المحلية. حيث تؤول هذه المعادلة لمعادلات انشتاين الحقلية عندما تكون دالة 'لاجرانج' خطية. وعندما تؤول الدالة المترية لدوال 'منكوسكي' يمكن الحصول على حلول مبسطة لدالة الانحناء القياسية التي تبين وجود قوى ذات مدى قصير وتبين كذلك انبعاث موجات التثاقل من الأجسام ذات المجالات القوية.

INTRODUCTION

Einstein's theory of General Relativity (GR) is seen as one of the big scientific achievements that have caused considerable changes in concepts of space, time and matter. It gives a correct description of a number of gravitational phenomena, which agrees with astronomical observations to a high degree of precision. In spite of these successes GR suffers from noticeable set - backs. For instance, the cosmological model based on it suffers from certain cosmological problems [(1)]. In additions, the nature of some astronomical exotica including quasars and pulsars, where the strong field is presumed to be predominant, is difficult to be understood in terms of GR[(2)].

Many attempts were made to go beyond GR [(3)]. Among these, the generalized Field Equation (GFE), which is based on a more general form of fourth order differential equation, is the best candidate. This is due to

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the successes of higher order theories in making these problems controllable [4,5,6,7].

In this paper a new simple derivation of the GFE with a source term is done by adopting a locally inertial frame in section 2. The GFE reduces to GR when the Lagrangian is linear. Considering the metric to be that of flat space, simple solutions for the scalar curvature are obtained. These solutions indicate existence of a short range field and the possibility of emitting gravitational waves when the field is strong, i.e when the Lagrangian consists of a quadratic term. These aspects are discussed in sections 3 and 4 respectively.

2 THE GENERALIZED FIELD EQATION IN PRESENCE OF A SOURCE

The GRE was first obtained by Lanczos [8] and alternatively derived by Ali El-Tahir[9]. In Ali ELtahir's approach the field equations are derived from the principle of least action by taking the field variable to the metric tensor $g_{\mu\nu}$. The action integral in this theory is invariant and depends on a function representing the Lagrangian density and depending in the scalar curvature R [9].

$$f = f(x^\lambda, g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \partial_{\gamma\lambda} g_{\mu\nu}) = \sqrt{g}\ L(R)$$

Where L is the general lagrangian density with :-

$$\partial_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda}$$

The action integral I is given by :-

$$I = \int f(x^\lambda, g_{\mu\nu}, \partial_\lambda g_{\mu\nu}, \partial_{\gamma\lambda} g_{\mu\nu}) d^4x$$

Confining ourselves to a locally inertial frame and subjecting this action to variation so that $\delta I=0$, and integrating by parts the following equation is obtained

$$\frac{\partial f}{\partial g_{\mu\nu}} = \frac{\partial}{\partial x^\lambda}\left[\frac{\partial f}{\partial \partial_\lambda g_{\mu\nu}}\right] - \frac{\partial^2}{\partial x^\lambda \partial x^\gamma}\left[\frac{\partial f}{\partial \partial_{\gamma\lambda} g_{\mu\nu}}\right] \qquad (1)$$

The Lagrangian L can be written as a sum of two parts, the first one, L representing the gravitational field and the second term, γ, stands for matter fields including ordinary matter γ_m and vacuum energy density

γ_v. By a suitable choice of γ, the GFE can be reduced to GR with a source term. For example let

$$\gamma = -\frac{1}{2}(\rho + p)g_{\rho\sigma}U^{\rho}U^{\sigma} - \frac{1}{2}(\rho + 3p) + \gamma_v = -p + \gamma_v \qquad (2)$$

If confined to a locally inertial frame, the scalar curvature becomes

$$R = g^{\delta\eta}g\lambda\gamma(\partial_{\gamma\lambda}g_{\delta\eta} - \partial_{\gamma\delta}\, g_{\lambda\eta} - \partial_{\eta\lambda}g_{\delta\gamma} + \partial_{\delta\eta}\, g_{\lambda\gamma})$$

Therefore

$$C^{\mu\upsilon\rho} = \frac{\partial R}{\partial\partial_{\rho}g_{\mu\upsilon}} = 0 \qquad (3)$$

$$C^{\mu\upsilon\rho\sigma} = \frac{\partial R}{\partial\partial_{\rho\sigma}g_{\mu\upsilon}} =$$

$$= g^{\lambda\gamma}g^{\delta\eta}[\delta^{\mu}_{\delta}\delta^{\upsilon}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\lambda} - \delta^{\mu}_{\lambda}\delta^{\nu}_{\eta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta} - \delta^{\mu}_{\delta}\delta^{\nu}_{\gamma}\delta^{\rho}_{\eta}\delta^{\sigma}_{\lambda} + \delta^{\mu}_{\lambda}\delta^{\nu}_{\gamma}\delta^{\rho}_{\delta}\delta^{\sigma}_{\eta}]$$

$$= g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}, \qquad (4)$$

Using equations (2) and (3), the left-hand side of equation (1) gives

$$\frac{df}{\partial g_{\mu\nu}} = \frac{\partial\sqrt{g}\,L}{\partial g_{\mu\nu}} + \frac{\partial\sqrt{g}\,\gamma}{\partial g_{\mu\nu}} = \frac{\partial\sqrt{g}\,L}{\partial g_{\mu\nu}} + \gamma\frac{\partial\sqrt{g}}{\partial g_{\mu\nu}} + \sqrt{g}\frac{\partial\gamma}{\partial g_{\mu\nu}}$$

$$= \sqrt{g}\left(\frac{1}{2}[g^{\mu\nu}L - T^{\mu\nu(m)}] - L'R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}\gamma_v\right) \qquad (5)$$

with m denoting matter. By virtue of equation (3) the first term on the right-hand side of equation (1) reads

$$\partial\frac{\partial f}{\partial\partial_{\lambda}g_{\mu\nu}} = \partial_{\lambda}[\frac{\sqrt{g}\partial L}{\partial R}\frac{\partial R}{\partial_{\lambda}g^{\mu\nu}}] = 0 \qquad (6)$$

By putting (4) in the second term on the right-hand side of equation (1) one gets

$$\partial_{\lambda\gamma}[\frac{\partial f}{\partial g_{\mu\nu}}] = \partial\lambda\gamma[\sqrt{g}(\frac{\partial L}{\partial R}\frac{\partial R}{\partial\partial_{\lambda\gamma}g_{\mu\nu}} + \frac{\partial\gamma}{\partial\partial_{\lambda\gamma}g_{\mu\nu}})]$$

$$= \partial_{\lambda\gamma}[\sqrt{g}L'c^{\mu\nu\lambda\gamma}] = \sqrt{g}c^{\mu\nu\lambda\gamma}\partial_{\lambda}\frac{\partial L'}{\partial R}\partial_{\gamma}R$$

$$= \sqrt{g}c^{\mu\nu\lambda\gamma}[R_{;\lambda;\gamma}L''+R_{;\lambda}R_{;\gamma}L''']$$

$$= \sqrt{g}(g^{\mu\nu}g^{\lambda\gamma} - g^{\mu\lambda}g^{\nu\gamma})(R_{;\lambda;\gamma}L''+R_{;\lambda}R_{;\gamma}L''')$$

$$= \sqrt{g}[L''(g^{\mu\nu}g^{\lambda\gamma}R_{;\lambda;\gamma} - g^{\mu\rho}g^{\nu\sigma}R_{;\rho}R_{;\sigma})$$
$$+ L'''(g^{\mu\nu}g^{\lambda\gamma}R_{;\lambda}R_{;\gamma} - R_{;}^{\mu}R_{;}^{\nu})] \qquad (7)$$

where the local inertial frame secures the replacement of the ordinary derivative by the covariant one and taking $\sqrt{g}$ and $c^{\mu\nu\lambda\rho}$ outside the differential operator. The relation

$g^{\mu\nu}R_{;\mu;\nu} = \Box^2 R$

is the invariant D' Alembert operator. In view of equations (5), (6), and (7) equation (1) is given by

$$L'''(g_{\mu\nu}g^{\lambda\gamma}R_{;\lambda}R_{;\gamma} - R_{;\mu}R_{;\nu}) + L''(g_{\mu\nu}\Box^2 R - R_{;\mu;\nu})$$
$$- L'R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}L - \frac{1}{2}(T_{\mu\nu(m)} - g_{\mu\nu}\gamma_{\nu}) = 0 \qquad (8)$$

which is the ordinary GFE with a source term.

If we consider linear lagrangian $L = \beta R$ and with $\beta = \frac{1}{16\pi G}$, $\gamma_{\nu} = 0$, equation (8) reduces to Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu(m)} \qquad (9)$$

3 SHORT RANGE FIELD SOLUTION

Contracting the GFE (8) by $g_{\mu\nu}$ to see how the field of a certain star looks like within the framework of the GFE yields.

$$\Box^2 R = \frac{L'R - 2L - L'''R_{;\rho}R_{;}^{\rho}}{3L''} + \frac{1}{6L''}T^{u}_{u(m)} - \frac{2}{3L''}\gamma_{\nu} \qquad (10)$$

The model is different from GR if the Lagrangian terms of higher order are added. The simplest lagrangian is the one which consists of a quadratic term beside the linear term, i.e

$$L = -\alpha R^2 + \beta R \tag{11}$$

The contracted equation thus becomes

$$\Box^2 R = \frac{\beta}{6\alpha} R + \frac{\rho - 3p}{12\alpha} + \frac{\gamma_v}{3\alpha} \tag{12}$$

The field of any isolated star can be described by a static isotropic metric [2] where the non-vanishing components are given by

$$g_{rr} = A(r)\ ,\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 \sin^2\theta,\ g_{tt} = -B(r) \tag{13}$$

Then by using the relationship (13) in (12) and neglecting the pressure, one gets

$$\Box^2 R = \frac{1}{A}[R - R(\frac{A}{2A} - \frac{B}{2B} - \frac{2}{r})] = \frac{\beta}{6\alpha} R + \frac{\rho}{12\alpha} + \frac{\gamma_v}{3\alpha} \tag{14}$$

In the case when $R = R_0 = \sigma$ constant, equation (14) yields

$$R_0 = -\frac{\rho}{2\beta} - \frac{2}{\beta}\gamma_v \tag{15}$$

This means that R feels the existence of both matter and vacuum, and one can consider matter and vacuum here as a frozen gravitational field. In other words, the constant background gravitational field manifests itself in the form of matter and vacuum.

Equation (14) is very complex and highly non-linear, but it can be simplified by assuming the metric to be flat [10], i.e

$A \to 1, B \to 1$

Therefore in the region outside the source, equation (14) reduces to

$$R + \frac{2}{r} R = \frac{\beta}{6\alpha} R + \frac{\gamma_v}{3\alpha} \tag{16}$$

To solve equation (16), consider a solution of the form

$$R = \frac{c_1}{r} \exp c_2 r + R_0 \tag{17}$$

Putting equation (17) in (16) yields

$$\frac{c_1 c_2^2}{r} \exp C_2 r = R_0 + \frac{\beta c_1}{6\alpha r} \exp C_2 r + \frac{\gamma}{3\alpha}$$

$$c_2 = \pm\sqrt{\frac{\beta}{6\alpha}} \quad , R_0 = \frac{\gamma_\nu}{3\alpha} \quad , \quad R = \frac{c_1}{r} \exp - \sqrt{\frac{\beta}{6\alpha}} r - \frac{\gamma_\nu}{3\alpha} \qquad (18)$$

where c_2 with the plus sign is excluded by the condition $R \to 0$ as $r \to \infty$. To express R in terms of the potential ϕ one uses quasi Minkowskian approximation [10] where

$$g^{ii} \approx \eta^{ii} \ , \ g^{00} = \eta 00 \eta^{00} g_{00} = -(1+2\phi)$$

$$R = g^{00} g^{ii} R_{0i0i} = g^{00} g^{ii} \frac{1}{2} \nabla^2 g_{00}$$

$$R = 8\pi G_{\rho\phi} + 4\pi G_\rho \qquad (19)$$

in view of equations (18) and (19) one gets

$$\phi = \frac{c_1}{8\pi G_{\rho r}} \exp - \sqrt{\frac{\beta}{6\alpha}} r \qquad (20)$$

This expression for the potential resembles the Yukawa potential and therefore shows the existence of a short range gravitational field or a possible link between gravitational and strong nuclear force.

4 GRAVITATIONAL WAVES

Consider the possibility of emitting gravitational waves by a certain source in empty space. In free space, equation (12) becomes

$$\Box^2 R = \frac{\beta}{6\alpha} R \qquad (21)$$

Taking into account the coordinate condition in empty space equation (21) reads [10]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{c \partial t^2}\right) R = \frac{\beta}{6\alpha} R \qquad (22)$$

and one of the possible solutions is

$$R = R_m \sin(\omega t - k.r) \qquad (23)$$

which leads to

$$k^2 - \frac{\omega^2}{c^2} = \frac{\beta}{6\alpha} \quad , v = \frac{\omega}{k} = c\sqrt{1 - \frac{\beta}{6\alpha k^2}} \qquad (24)$$

The non vanishing frequency mode shows the possibility of emitting gravitational waves by a certain star where the strong field presumably dominates via the contribution of the quadratic Lagrangian. From equations (19) and (23) the potential is given by equation (25)

$$\phi = \frac{R}{8\pi G_p} - \frac{1}{2} \approx \phi_m \sin(\omega t - k.r) \qquad (25)$$

The travelling wave solution agrees with the recently observed decline in the orbit period of binary pulsars which was assumed to occur because system emitted gravitational waves.

5 Concluding Remarks

Equation (15) shows how the curvature of space indicates the existence of matter. The fact that the GFE with a source term reduces to GR in a weak field limit indicates that it shares with GR all its successes [1] in this limit. The solutions of the GFE differ from those of GR in many respects. First of all the scalar curvature does not vanish outside the source. Secondly, the expression for the potential shows the existence of a short range field or presumably a possible link with strong nuclear force. On the other hand, the travelling wave solution is in conformity with the recently observed declining in orbit period of the binary pulsars.

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