Chapter 6
Functional Calculus and Estimates

We find a class Υ of functions in two real variables and construct a functional calculus on this class which extends functional calculi constructed earlier. It is proved that for this functional calculus the trace formula of Melton-Howe holds.

We show that if f is a nondecreasing continuous function on $\mathbb{R}$ that vanishes on $(-\infty, 0]$ and is concave on $[0, \infty)$, then its operator modulus of continuity Ω_f admits the estimate

$$\Omega_f(\delta) \leq \text{const}\int_e^{\infty} \frac{f(\delta t)dt}{t^2 \log t} \quad , \quad \delta > 0 .$$

We also study the problem of sharpness of estimates obtained in Aleksandrov and Peller.
We construct a C^{∞} function f on $\mathbb{R}$ such that $\|f\|_{L^{\infty}} \leq 1$, $\|f\|_{Lip} \leq 1$ and

$$\Omega_f(\delta) \geq \text{const}\,\delta\sqrt{\log\frac{2}{\delta}} \ , \ \delta \in (0, 1].$$

We obtain sharp estimates of $\|f(A) - f(B)\|$ in the case when the spectrum of A has n points. Moreover, we obtain a more general result in terms of the ε-entropy of the spectrum that also improves the estimate of the operator moduli of continuity of Lipschitz functions on finite intervals ,which was obtained in Aleksandrov and Peller .

Section (6.1): A pair of Almost Commuting Selfadjoint Operators.

For a pair (A, B) of commuting selfadjoint operators the spectral theorem allows one to construct a functional calculus on the class of Borel functions of two real variables. This functional calculus has natural properties. Namely, it is linear, multiplicative, and satisfies the estimate $\|\varphi(A, B)\| \leq \sup_{s,t\in\mathbb{R}} |\varphi(s, t)|$.

Suppose now that we have a pair (A, B) of almost commuting selfadjoint operators, i.e., $AB - BA \in \boldsymbol{S}_1$, where $\boldsymbol{S}_1$ denotes the trace class. We consider only hounded selfadjoint operators. In this case it is impossible to construct a functional calculus which is linear and multiplicative. But one can try instead to construct a functional calculus which is linear and multiplicative modulo the trace class. This problem is important in spectral theory and it has been treated in [240, 241, 242].

If $\varphi(s, t) = \sum_{n,k\geq 0} \varphi_{nk} s^n t^k$ is a polynomial in two variables then one can define the operator $\varphi(A, B)$ by

$$\varphi(A, B) = \sum_{n,k\geq 0} \varphi_{nk} A^n B^k . \tag{1}$$

It is well known that this polynomial functional calculus satisfies the following remarkable trace formula. To write it down we need the notion of Pincus principal function. To each pair of almost commuting selfadjoint operators (A, B) one can associate the so-called principal function g of two variables which has compact support and belongs to L^1 with respect to planar Lebesgue measure. Then for any polynomials φ and ψ the following formula holds

$$\operatorname{trace}(\varphi(A,B)\psi(A,B)-\psi(A,B)\varphi(A,B))$$
$$=\frac{1}{2\pi i}\int_C\int\left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)g(x,y)dxdy. \qquad (2)$$

The principal function was introduced by Pincus [243] in the case of a rank one commutator. In [240] it was proved that this formula holds with right hand side

$$\frac{1}{2\pi i}\int_C\int\left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial x}\right)dP(x,y),$$

where P is a complex measure. Later it was shown in [241] that P is absolutely continuous and coincides with the Pincus principal function (see also [249,245,246]). The functional calculus in (2) is defined for the polynomials by formula (1). In fact it is easy to see that if we change in (1) the order appearance of A and B, the left-hand side of (2) will be the same. Note that formula (2) and the principal function are of great importance in the study of hyponormal operators. Namely, if T is a purely hyponormal operator (i.e., has no reducing subspace on which it is normal), then the real and the imaginary parts of T are almost commuting selfadjoint operators and there are important relations between the properties of T and those of the corresponding principal function (see [249,245,246]).

We extend the functional calculus defined on the polynomials to a class of functions in two variables which should be as big as possible. We are going to find a big class $\mathfrak{L}$ of functions in two real variables and a mapping $\varphi\longmapsto\varphi(A,B)$, $\varphi\in\mathfrak{L}$, defined for any pair of almost commuting operators that satisfy the following natural properties:

(F_1) if $\varphi(s,t)f=(s)$, then $\varphi(A,B)=f(A)$; if $\varphi(s,t)=g(t)$, then $\varphi(A,B)=f(B)$;

(F_2) $(\varphi+\psi)(A,B)=\varphi(A,B)+\psi(A,B)$;

(F_3) $(\varphi\psi)(A,B)-\varphi(A,B)\psi(A,B)\in\boldsymbol{S}_1$;

(F_4) formula (2) holds for any almost commuting pair (A,B) and any $\varphi,\psi\in\mathfrak{L}$.

(F_2)-(F_3) mean that the functional calculus is linear and multiplicative modulo the trace class.

It is seen from (2) that the right-hand side is well-defined for any C^1 functions φ and ψ (and even for any Lipschitz functions). This might give a hope that it could be possible to construct such a functional calculus defined on the set of C^1 functions. However, it turns out that this is not the case. We prove that if we have a functional calculus on a class $\mathfrak{L}$ of functions in two variables which satisfies (F_1)-(F_3) and if φ is a function in $\mathfrak{L}$ depending only on one variable, then φ satisfies certain necessary conditions. These necessary conditions imply that it is impossible to construct such a functional calculus on the class C^1 of continuously differentiable functions.

In [240] the polynomial functional calculus has been extended to the class of infinitely smooth functions. In [242] the class of smooth functions has been enlarged to the class of Fourier transforms $\mathcal{F}\omega$ of measures ω on $\mathbb{R}^2$ that satisfy the condition

$$\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|x|)(1+|y|)d|\omega(x,y)|<\infty.$$

But this assumption is too restrictive. First of all the first partial derivatives of such $\mathcal{F}\omega$ must locally have absolutely convergent Fourier expansion, and secondly, such function $\mathcal{F}\omega$ must have continuous second derivative $\partial^2/\partial s\partial t$. We enlarge this class of functions and construct a functional calculus on the enlarged class $\mathfrak{L}$ that satisfies (F_1)-(F_4). Note that the functions in $\mathfrak{L}$ belong to C^1, but they do not necessarily have second derivative $\partial^2/\partial s\partial t$ and their first partial derivates do not necessarily have absolutely convergent Fourier expansions.

If we want the functional calculus to satisfy the additional property to be involutive modulo trace class, i.e.,

(F_5) $\bar{\varphi}(A,B)-\left(\varphi(A,B)\right)^* \in S_1$,

then we have to impose more restrictive conditions on functions. A smaller class $\mathfrak{L}_1$, of functions of two real variables and construct a functional calculus on $\mathfrak{L}_1$, satisfying (F_1)-(F_5).

Namely, let A be a selfadjoint operator (not necessarily bounded) and B be a trace class selfadjoint perturbation of it. M.G.Krein (see [247]) has associated with each such pair the so-called spectral shift ξ which is a function on $\mathbb{R}$ in L^1 with respect to Lebesgue measure and proved that

$$\operatorname{trace}\big((A)-(B)\big)=\int_{\mathbb{R}}\varphi'(t)\xi(t)dt \qquad (3)$$

for sufficiently smooth functions φ .Namely he has proved (3) under the assumption that φ' is the Fourier transform of a finite measure. Then some weaker sufficient conditions on φ were found by Birman and Solomyak [248]. It turned out that the technique of double operator integrals developed by Birman and Solomyak [249] allows one to reduce the problem of the validity of the trace formula (3) to the question of when the function in two real variables $(\varphi(s)-\varphi(t))/(s-t)$ is a Schur multiplier . In [250] for bounded A and [251] for unbounded A the previous results have been improved. Namely, it has been proved in [250,251]that if φ belongs of the Besov class $\boldsymbol{B}^1_{\infty 1}$, then $(\varphi(s)-\varphi(t))/(s-t)$ is a Schur multiplier, and so $\varphi(A)-\varphi(B)\in \boldsymbol{S}_1$ and (3) holds.

Arazy, Barton, and Friedman [252] have found another sufficient condition in order that the function $(\varphi(s)-\varphi(t))/(s-t)$ be a Schur multi-plier. If $\varphi\in\boldsymbol{B}^1_{\infty 1}$ then φ satisfies their condition.

As in the case of formula (2), the right-hand side of (3) is well-defined for any Lipschitz function φ. M. G. Krein has asked in [247] whether (3) is valid for any Lipschitz φ. It turned out that this is not the case. First, Farforovskaya in [253] has constructed an explicit example of operators A and B and a Lipschitz function φ such that $A-B\in\boldsymbol{S}_1$ but the operator on the left-hand side of (3) does not belong to $\boldsymbol{S}_1$. Then the author has found in [250,251]necessary conditions on φ in order that $\varphi(A)-\varphi(B)$ be in $\boldsymbol{S}_1$, whenever $A-B\in$

$\boldsymbol{S}_1$. To obtain such conditions a technique of Hankel operators has been used. Those necessary conditions imply that $\varphi(A) - \varphi(B)$ is not necessarily in $\boldsymbol{S}_1$for any C^1 function φ. The simplest necessary condition obtained in [250,251]is that φ must be locally in the Besov class $\boldsymbol{B}_1^1$. A stronger condition is that that Hankel operators H_φ and H_φ must act from the Hardy class H^1 and the Besov class $\boldsymbol{B}_1^1$. Recently, S. Semmes has found a nice description of such functions φ.

The above two problems are not only similar to each other but they can be attacked by using the same approach. The approach relies on the Hankel operators and the technique of double operator integrals developed by Birman and Solomyak .

An operator T acting from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$is called of trace class (notationally $T \in \boldsymbol{S}_1$)if

$$\|T\|_{\boldsymbol{S}_1} = \sum s_n(T) < \infty,$$

where

$$s_n(T) \stackrel{\text{def}}{=} \inf\{\|T - K\|: \text{rank } K \leq n\}.$$

If T is a trace class operator on a Hilbert space $\mathcal{H}$, then its trace is defined by

$$\text{trace } T = \sum_{n\geq 0} (Te_n, e_n),$$

where $\{e_n\}_{n\geq 0}$is an orthonormal basis in $\mathcal{H}$, the value of trace does not depend on the choice of $\{e_n\}_{n\geq 0}$.

The Hilbert-Schmidt class $\boldsymbol{S}_2$ is defined by the condition

$$\sum (s_n(T))^2 < \infty.$$

We refer to [254].

Let E and F be Borel spectral measures defined on separable metric spaces Λ and M and T be a bounded operator on Hilbert space. Given a bounded measurable function φ on $\Lambda \times M$, we can consider the double operator integral

$$\Phi T = \int_\Lambda \int_M \varphi(s,t) dF(s) T dE(t).$$

The theory of such integrals has been developed in [249] (see also [25241]). If $\varphi \in L^\infty(\Lambda \times M)$, then Φ is a bounded transformation on the Hilbert Schmidt class $\boldsymbol{S}_2$. If Φ is bounded on $\boldsymbol{S}_1$, we can define ΦT by duality for any bounded T. In this case φ is called a Schur multiplier of $\boldsymbol{S}_1$.

Let A,B be selfadjoint operators on Hilbert space with bounded $A - B$ and φ a function in $C^1(\mathbb{R})$ such that the function $\check{\varphi}$,

$$\check{\varphi}(s,t) = \frac{\varphi(s) - \varphi(t)}{s - t}$$

in two variables is a Schur multiplier. Then

$$\varphi(B) - \varphi(A) = \int_{\mathbb{R}} \int_{\mathbb{R}} \check{\varphi}(s,t) dE_B(s)(B - A) dE_A(t),$$

where E_A and E_B are the spectral measures of A and B (see [248]). This formula turned out to be very useful in the problem of validity of the trace formula (1.3).

In [255] Birman and Solomyak have established the formula

$$\varphi(A)B - B\varphi(A) = \int_{\mathbb{R}}\int_{\mathbb{R}} \check{\varphi}(s,t)dE_B(s)(AB - BA)dE_A(t) \tag{4}$$

for any selfadjoint A and bounded B with bounded $AB - BA$, whenever $\check{\varphi}$ is a Schur multiplier. This formula will be used for the construction of a functional calculus for a pair of almost commuting selfadjoint operators.

We deal with bounded selfadjoint operators, we can assume that the functions involved in functional calculus are periodic with a sufficiently large period. Therefore we consider here the Besov classes periodic functions or, which is equivalent, the Besov classes of functions on the unit circle. These classes admit many equivalent definitions (see [256]).

Let $n > 0$. The trigonometric polynomial W_n is defined as follows. The Fourier coefficients $\widehat{W}_n(k)$ vanish outside $(2^{n-1}, 2^{n+1})$, $\widehat{W}_n(2^n) = 1$, and $\widehat{W}_n$ is a linear function on $[2^{n+1}, 2^n]$ and $[2^n, 2^{n+1}]$. If $n < 0$ then $W_n \stackrel{\text{def}}{=} \overline{W_n}$, $W_0(z) \stackrel{\text{def}}{=} \bar{z} + 1 + z$. Clearly for any function f we have

$$f = \sum_{n\in\mathbb{Z}} f * W_n.$$

The Besov class $\boldsymbol{B}^s_{pq}$ of functions on the unit circle is defined by

$$f \in \boldsymbol{B}^s_{pq} \iff \left\{2^{|n|s}\|f * W_n\|_{L^p}\right\} \in \ell^q. \tag{5}$$

We use the notation $\boldsymbol{B}^s_p$ for $\boldsymbol{B}^s_{pq}$.

We deal with two Besov spaces $\boldsymbol{B}^1_{\infty 1}$ and $\boldsymbol{B}^1_1$. They admit the following description. Let $\varphi_{\mathcal{H}}$ be the harmonic extension of a function φ to the unit disc. Then

$$\varphi \in \boldsymbol{B}^1_1 \iff \int_{\boldsymbol{D}} |(\nabla^2 \varphi_{\mathcal{H}})(s)|\, dA(\zeta) < \infty, \tag{6}$$

where A is the area measure, and

$$\varphi \in \boldsymbol{B}^1_{\infty 1} \iff \int_0^1 \left(\sup_{|\zeta|=1} |(\nabla^2 \varphi_{\mathcal{H}})(r\zeta)|\right) dr < \infty. \tag{7}$$

$$\left(\text{Here} \quad \nabla^2 F = \left|\frac{\partial^2 F}{\partial x^2}\right| + \left|\frac{\partial^2 F}{\partial y^2}\right| + \left|\frac{\partial^2 F}{\partial x \partial y}\right|.\right)$$

In a similar way one can define Besov classes on the real line (see [256]).

Given a bounded function φ on the unit circle T, the Hankel operator H_φ acts from the Hardy class H^2 to the class $H^2 \stackrel{\text{def}}{=} L^2 \ominus H^2$ by the formula

$$H_\varphi f \stackrel{\text{def}}{=} \mathbb{P}_- \varphi f,$$

where $\mathbb{P}_-$ is the orthogonal projection onto H^2_-.

It has been found in [257] that $H_\varphi \in \boldsymbol{S}_1$ if and only if $\mathbb{P}_- \varphi \in \boldsymbol{B}^1_1$.

One can also consider Hankel operators defined on the Hardy class $H^2(C_+)$ of functions in the upper half-plane.

The role of Hankel operators in the problems considered above can be explained as follows. It is important for us to know when the function $\check{\varphi}$ is a Schur multiplier, or since we deal with bounded operators, we have to know when the function $\check{\varphi}$ restricted to $I \times I$ is a Schur multiplier for any bounded interval I.

If φ is a periodic function then the last property can be reformulated as follows. We can identify φ with a function ψ on the unit circle. Let

$$\check{\psi}(\tau) = \frac{\psi(\zeta) - \psi(\tau)}{1 - \bar{\zeta}\tau}, \quad \zeta, \tau \in \mathbb{T}.$$

Then the question is equivalent to the question of whether $\check{\psi}$ is a Schur multiplier. The last property is equivalent to the fact that the integral operator on $L^2(\mathbb{T})$ with kernel

$$k(\zeta, \tau)\check{\psi}(\zeta, \tau)$$

belongs to $\boldsymbol{S}_1$ whenever $k(\zeta, \tau)$ is the kernel of a trace class operator on $L^2(\mathbb{T})$ (see [249]). In particular, the integral operator with kernel $\check{\psi}(\zeta, \tau)$ must be in $\boldsymbol{S}_1$which is equivalent in turn to the fact that

$$H_\psi H_{\check{\psi}}^* \in \boldsymbol{S}_1$$

(see [250]). As mentioned the latter is equivalent to the fact that $\psi \in \boldsymbol{B}_1^1$.

A stronger necessary condition obtained in [250] with the help of Hankel operators is that both Hankel operators H_ψ, and $H_{\check{\psi}}$ map the Hardy class H^1 into the Besov class $\boldsymbol{B}_1^1$ (the class of such functions ψ is denoted in [250,251]by $\mathfrak{L}$).

Similar results hold for unbounded operators. We have to consider arbitrary functions φ on $\mathbb{R}$ for which $\check{\varphi}$ is a Schur multiplier on $L^2(\mathbb{R})$ (see [251]).

S. Semmes has found a nice characterization of the class $\mathcal{L}$ (private communication). Namely, he has proved that $\psi \in \mathfrak{L}$ if and only if $|(\nabla^2 \varphi_{\mathcal{H}})(r\zeta)| d\xi d\eta$ is a Carleson measure in the unit disc. It is easy to see from this description and from (6), (7) that $\boldsymbol{B}_{\infty 1}^1 \subset \mathfrak{L} \subset \boldsymbol{B}_1^1$.

It has been shown in [250,251]that if $\varphi \in \boldsymbol{B}_{\infty 1}^1$ then the function $\check{\varphi}$ is a Schur multiplier. In [252] another sufficient condition has been found. Namely, let $\mathcal{X}$ be the class of functions analytic in the unit disc that satisfies the property

$$\sup_{\zeta \in \mathbb{T}} \int_{\boldsymbol{D}} \frac{1 - |\tau|^2}{|1 - \bar{\tau}\zeta|^2} |f''(\tau)| dA(\tau) < \infty. \tag{8}$$

A function f on the unit circle is said to belong to the space $\mathcal{Y}$ if both functions $f_+ \stackrel{\text{def}}{=} (I - \mathbb{P}_-)f$ and $\bar{f} \stackrel{\text{def}}{=} \overline{\mathbb{P}_- f}$ belong to $\mathcal{X}$. It has been proved in [252] that if $\psi \in \mathcal{Y}$, then $\check{\psi}$ is a Schur multiplier. Moreover, it has been shown in [252] that $\boldsymbol{B}_{\infty 1}^1 \subset \mathcal{Y}$. S. A. Vinogradov has recently proved that the inclusion is proper (private communication).

We show that it is impossible to construct a functional calculus on the class of continuously differentiable functions. More precisely, we prove that if $\mathfrak{L}$ is a class of functions in two real variables on which there exists a functional calculus satisfying (F_1)-(F_3) and φ is a function in $\mathfrak{L}$ depending only on one variable, $\varphi(s, t) = f(s)$, then f locally belongs to the class $\mathcal{L}$. We also present here the Semmes characterization of the class $\mathcal{L}$.

Definition (6.1.1)[239]: Let X be a class of functions on the real line. A function f on $\mathbb{R}$ is said to belong to X locally if for any interval I there exists a function g in X such that $f|I = g|I$.

We say that a function f locally belongs to the class $\mathcal{L}$ if for any interval I,$f|I$ coincides with the restriction to I of a periodic function of class $\mathcal{L}$.

Corollary(6.1.2)[239]: Under the hypothesis of Theorem(6.1.2), f belongs locally to $\boldsymbol{B}_1^1$.

$\mathcal{L} \subset \boldsymbol{B}_1^1$ And so the corollary follows immediately from the theorem.

It is well known (see [250]) that $C^1 \not\subset \boldsymbol{B}_1^1$ and it follows that it is impossible to construct a functional calculus satisfying (F_1)-(F_3) on the class of continuously differentiable functions.

Before we proceed to the proof of Theorem (6.1.2) we prove here for the sake of completeness the Semmes theorem stated.

Let φ be a function on $\mathbb{T}$. Recall that $\varphi \in \mathcal{L}$ means that both Hankel operators H_φ and $H_{\breve{\varphi}}$ map H^1 into $\boldsymbol{B}_1^1$.

Definition (6.1.3)[239]: Let μ be a positive measure on the unit disc $\boldsymbol{D}$. It is called a Carleson measure if for any $\zeta \in \mathbb{T}$ and for any $\varepsilon > 0$

$$\mu\{\tau \in \boldsymbol{D}: |\zeta - \tau| \leq \varepsilon\} \leq \text{const}\ \varepsilon.$$

Let

$$\|\mu\|_{CM} \stackrel{\text{def}}{=} \sup_{\zeta \in \mathbb{T}, \varepsilon > 0} \frac{1}{\varepsilon} \mu\{\tau \in \boldsymbol{D}: |\zeta - \tau| \leq \varepsilon\}.$$

The following characterizations of the Carleson measures are well known (see [9, 14]). Let μ be a positive measure on $\boldsymbol{D}$. The following are equivalent:

(i) μ is a Carleson measure;

(ii) $\int_{\boldsymbol{D}} |f(\zeta)|^p d\mu(\zeta) \leq \text{const}\|f\|_{H^p}^p$ for some $p > 0$;

(iii) $\int_{\boldsymbol{D}} |f(\zeta)|^p d\mu(\zeta) \leq \text{const}\|f\|_{H^p}^p$ for any $p > 0$;

(iv) $\sup_{\zeta \in \boldsymbol{D}} \int_{\boldsymbol{D}} ((1 - |\zeta|^2)/|1 - \bar{\tau}\zeta|^2) d\mu(\tau) < \infty$.

Moreover the constants in (ii) and (iii) are equivalent to $\|\mu\|_{CM}$.

It follows from (iv) that if φ is a function analytic in $\boldsymbol{D}$, then $\varphi \in \mathcal{L}$ if and only if

$$\sup_{\zeta \in \boldsymbol{D}} \int_{\boldsymbol{D}} \frac{1 - |\zeta|^2}{|1 - \bar{\tau}\zeta|^2} |\varphi''(\tau)| dA(\tau) < \infty.$$

It is interesting to compare this condition with (8).

Theorem (6.1.4)[239]:(S. Semmes). Let φ be a function on the unit circle. Then $\varphi \in \mathcal{L}$ if and only if $|(\nabla^2 \varphi_{\mathcal{H}})| dA$ is a Carleson measure on $\boldsymbol{D}$.

First recall the definition of Carleson measures.

Proof . Fist of all it is sufficient to consider the case of a function φ analytic in $\boldsymbol{D}$ (otherwise we could represent φ as $\varphi = \varphi_+ + \varphi_-$, $\varphi_- = \mathbb{P}_-\varphi$, and prove the result separately for φ_+ and φ_-).

Suppose that φ is analytic in $\boldsymbol{D}$ and $|\varphi''(\zeta)|d\xi d\eta$ is a Carleson measure. Let us show that H_φ is a bounded operator from H^1 to $\boldsymbol{B}_1^1$. Without loss of generality we can assume that $\varphi(0) = \varphi'(0) = 0$.

Consider the following pairing defined on the set of polynomials

$$\langle f, g\rangle = \int f(\zeta)\, g(\zeta) d\sigma(\zeta),$$

where a is normalized Lebesgue measure on $\mathbb{T}$.

The dual space to the subspace $\mathbb{P}_-\boldsymbol{B}_1^1$ can be identified with respect to the above pairing with the space $\boldsymbol{B}_{\infty 1}^1$ of functions g analytic in the unit disc and satisfying

$$\sup_{\zeta\in\boldsymbol{D}}(1-|\zeta|^2)|g(\zeta)| < \infty.$$

We have to show that

$$\left|\langle H_\varphi f, g\rangle\right| \le \text{const}\|f\|_{H^1}\|g\|_{\boldsymbol{B}_{\infty 1}^1},$$

for any analytic polynomials f and g. By Green's formula the last inequality is equivalent to

$$\left|\int_{\boldsymbol{D}} \overline{\varphi''(\zeta)} f(\zeta) g(\zeta)(1-|\zeta|^2) dA(\zeta)\right| \le \text{const}\|f\|_{H^1}\|g\|_{\boldsymbol{B}_{\infty 1}^1}.$$

We have

$$\sup_{\zeta\in\boldsymbol{D}}|g(\zeta)|(1-|\zeta|^2) \le \text{const}\|g\|_{\boldsymbol{B}_{\infty 1}^1},$$

and so

$$\||\varphi''(\zeta)|(1-|\zeta|^2)dA(\zeta)\|_{CM} \le \text{const}\|g\|_{\boldsymbol{B}_{\infty 1}^1}\||\varphi''|dA\|_{CM}.$$

Therefore it follows from (iii) with $p = 1$ that

$$\left|\langle H_\varphi f, g\rangle\right| \le \text{const}\|f\|_{H^1}\|g\|_{\boldsymbol{B}_{\infty 1}^1}\||\varphi''|dA\|_{CM}$$

Now suppose that H_φ is a bounded operator from H^1 to $\boldsymbol{B}_1^1$. Let us show that $|\varphi''|dA$ is a Carleson measure. It is easy to see that $\varphi \in \boldsymbol{B}_1^1$ and so φ is a continuous function on $\mathbb{T}$.

Given $\zeta \in \mathbb{T}$, put

$$C_\zeta^{(\varepsilon)} = \{\tau \in \boldsymbol{D}: |\zeta - \tau| \le \varepsilon\}.$$

We have

$$\left|\langle H_\varphi f, g\rangle\right| \le \text{const}\|f\|_{H^1}\cdot\|g\|_{\boldsymbol{B}_{\infty 1}^1},$$

for any smooth f and g.

Let G be a function in L^∞such that supp $G \subset C_\zeta^{(\varepsilon)}$ and supp $G \cap \mathbb{T} = \emptyset$.

Put

$$\tilde{g}(\tau) = \int_{C_\zeta^{(\varepsilon)}} \frac{G(\omega)}{(1-\overline{\omega}\tau)^3} dA(\omega).$$

Clearly, $\tilde{g}$ is analytic in a neighbourhood of the closed unit disc.

The function g satisfies the following properties:

(α) $|\tilde{g}(\tau)| \le \text{const}\|G\|_{L^\infty},\big(1/(1-|\zeta|)\big)$ for any $\tau \in \boldsymbol{D}$**;**

(β) $|\tilde{g}(\tau)| \le \text{const}\ \varepsilon^2\|G\|_{L^\infty},(1/|1-(1-\varepsilon)\bar{\zeta}\tau|^3)$ for any $\tau \notin C_\zeta^{(2\varepsilon)}$**.**

Property (β) is obvious. Let us establish (α). Clearly, (α) is equivalent to the following inequality.

Let ψ be a function in the upper half-plane C_+defined by

$$\psi(z) = \int_{\mathfrak{D}_\varepsilon} \frac{dA(\omega)}{(2-\bar{\omega})^3},$$

where $\mathfrak{D}_\varepsilon = \{\omega: |\mathrm{Re}\,\omega| \le \varepsilon, 0 \le \mathrm{Im}\,\omega \le \varepsilon\}$. Then

$$|\psi(z)| \cdot |\mathrm{Im}\,z| \le \text{const.} \tag{9}$$

Indeed, it is sufficient to prove (9) for $x = iy, y \in \mathbb{R}_+$. We have

$$y|\psi(iy)| \le \int_{\mathfrak{D}_\varepsilon} \frac{y^3}{|iy-\bar{\omega}|^3} \frac{dA(\omega)}{y^2} \le \int_{C_+} \frac{1}{|i-\bar{\omega}|^3} dA(\omega) < \infty.$$

Let now

$$g(\tau) = \frac{1}{\varepsilon^2}(1-(1-\varepsilon)\bar{\zeta}\tau)^2 \tilde{g}(\tau).$$

It is easy to see from (α) and (β) that

$$|g(\tau)| \le \text{const}\|G\|_{L^\infty} \frac{1}{1-|\tau|}, \quad \tau \in \boldsymbol{D}.$$

Consider now the function

$$f(\tau) = \frac{\varepsilon\tau^2}{(1-(1-\varepsilon)\bar{\zeta}\tau)^2}.$$

It is easy to see that $f \in H^1$ and

$$\|f\|_{H^1} \le \text{const.}$$

We have

$$\langle H_\varphi f, g\rangle = \int_{\mathbb{T}} \bar{\varphi}\, f g d\sigma = \frac{1}{\varepsilon}\int_{\mathbb{T}} \bar{\varphi}\,(\tau)\tilde{g}(\tau)\tau^2 d\sigma(\tau) = \frac{1}{\varepsilon}\int_{\mathbb{T}} \bar{\varphi}\,(\tau)\left(\int_{C_\zeta^{(\varepsilon)}} \frac{G(\omega)}{(1-\bar{\omega}\tau)^3} dA(\omega)\right)\tau^2 d\sigma(\tau)$$

$$= \frac{1}{\varepsilon}\int_{C_\zeta^{(\varepsilon)}} \left(\int_{\mathbb{T}} \frac{\overline{\varphi(\tau)}\tau^2}{(1-\bar{\omega}\tau)^3} d\sigma(\tau)\right) G(\omega) dA(\omega) = \frac{1}{\varepsilon}\int_{C_\zeta^{(\varepsilon)}} \overline{\varphi''(\omega)}\, G(\omega) dA(\omega).$$

Let us now choose the function G. Put

$$C_\zeta^{(\varepsilon,\delta)} = \left\{\omega \in C_\zeta^{(\varepsilon)}: \mathrm{dist}\,(\omega,\mathbb{T}) \ge \delta\right\}.$$

Let us define G by

$$G(\omega) = \begin{cases} \dfrac{\varphi''(\omega)}{|\varphi''(\omega)|}, & \text{if } \omega \in C_\zeta^{(\varepsilon,\delta)} \text{ and } \varphi''(\omega) \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$\frac{1}{3}\int_{C_\zeta^{(\varepsilon)}} |\varphi''(\omega)|\, dA(\omega) \le \text{const.}$$

The result follows by making $\delta \to 0$.

Theorem (6.1.5)[239]: Let $\mathfrak{L}$ be a class of functions in two real variables and $\varphi \mapsto \varphi(A, B)$, $\varphi \in \mathfrak{L}$, a functional calculus for pairs of almost commuting selfadjoint operators that satisfies (F_1)-(F_3). Let $\varphi \in \mathfrak{L}$ be a function dependent only on one variable $\varphi(s, t) = f(s)$. Then f belongs locally to $\mathcal{L}$.

Proof . Let $\varphi(s, t) = f(s)$, $\varphi \in \mathfrak{L}$, $\psi(s, t) = t$. It follows from (F_3) that

$$\varphi(A, B)\psi(A, B) - \psi(A, B)\varphi(A, B) = f(A)B - Bf(A) \in \boldsymbol{S}_1$$

for any almost commuting pair of selfadjoint operators A and B. It follows that

$$f(A)B - Bf(A) \in \boldsymbol{S}_1$$

for any selfadjoint A and bounded B with $AB - BA \in \boldsymbol{S}_1$ (one can consider separately the real and the imaginary part of B).

It is easy to see that for any pair A, B with bounded B, bounded selfadjoint A, and $AB - BA \in \boldsymbol{S}_1$ the following inequality holds:

$$\|f(A)B - Bf(A)\|_{\boldsymbol{S}_1} \leq \text{const}\|AB - BA\|_{\boldsymbol{S}_1}.$$

Otherwise we could find sequences of operators $\{A_n\}, \{B_n\}$such that $\sup_n \|A_n\| < \infty$, $\sup_n \|B_n\| < \infty$the $A_n$are selfadjoint,

$$\sum_n \|A_n B_n - B_n A_n\|_{\boldsymbol{S}_1} < \infty$$

but

$$\sum_n \|f(A_n)B_n - B_n f(A_n)\|_{\boldsymbol{S}_1} = \infty.$$

Then we can put

$$A = \sum \oplus A_n, \qquad B = \sum \oplus B_n.$$

Obviously A is a bounded selfadjoint operator, B is a bounded operator, $AB - BA \in \boldsymbol{S}_1$but $f(A)B - Bf(A) \notin \boldsymbol{S}_1$.

Let I be an interval and p, q be arbitrary functions in $L^2(I)$. Let us show that the integral operator T defined by

$$(Th)(t) = \int_I \frac{f(t) - f(s)}{t - s} p(t)\overline{q(s)}h(s)ds \tag{10}$$

belongs to $\boldsymbol{S}_1$ and $\|T\|_{\boldsymbol{S}_1} \leq \text{const}\|p\|_{L^2}\|q\|_{L^2}$ .This would imply that $\check{f}$ is a Schur multiplier.

Consider the truncated functions p_n and q_n defined by

$$p_n(t) = \begin{cases} p(t), & |p(t)| \leq n \\ 0, & |p(t)| > n, \end{cases}$$

$$q_n(t) = \begin{cases} q(t), & |q(t)| \leq n \\ 0, & |q(t)| > n. \end{cases}$$

Define the operators A and B by

$$(Ah)(t) = th(t), \quad h \in L^2(I),$$

$$(B_n h)(t) = p_n(t)\int_I \frac{\overline{q_n(s)}h(s)}{t-s}ds, \quad h \in L^2(I).$$

Then the B_n are bounded, A is bounded and selfadjoint. It is easy to see that

$$\big((AB_n - B_nA)h\big)(t) = p_n(t)\int_I h(s)q_n(s)ds = (h, q_n)p_n(t), \quad h \in L^2(I)$$

and

$$\big((f(A)B_n - B_nf(A))h\big)(t) = \int_I \frac{f(t)-f(s)}{t-s} p_n(t)\overline{q_n(s)}h(s)ds, \quad h \in L^2(I)$$

Clearly

$$\|AB_n - B_nA\|_{\boldsymbol{S}_1} = \|p_n\|_{L^2}\|q_n\|_{L^2} \le \|p\|_{L^2}\|q\|_{L^2}$$

Therefore

$$\|f(A)B_n - B_nf(A)\|_{\boldsymbol{S}_1} \le \text{const}\|p\|_{L^2}\|q\|_{L^2}.$$

It is evident that $f(A)B_n - B_nf(A)$ converges in the weak operator topology to the operator T defined by (10) which implies the desired estimate for $\|T\|_{\boldsymbol{S}_1}$.

We define a class $\mathfrak{L}$ of functions in two real variables and construct a functional calculus on $\mathfrak{L}$ for pairs of almost commuting selfadjoint operators. This functional calculus satisfies properties (F_1)-(F_4) and extends the functional calculi constructed earlier. Then we give a description of the class $\mathfrak{L}$ similar to that of Besov classes (5).

We define another class of functions $\mathfrak{L}_1$ contained in $\mathfrak{L}$ and we prove that the restriction of the functional calculus on $\mathfrak{L}$ to $\mathfrak{L}_1$ in addition to (F_1)-(F_4) satisfies (F_5)

Definition (6.1.6)[239]: Let $\mathfrak{L}$ be the class of periodic functions φ (with a fixed period) in two real variables which admit a representation

$$\varphi(s,t) = \sum_{n\ge 0} f_n(s)g_n(t), \tag{11}$$

where f_n, g_n are periodic functions in one variable such that

$$\sum_{n\ge 0}\left(\|f_n\|_{\boldsymbol{B}^1_{\infty 1}}\|g_n\|_{L^\infty} + \|f_n\|_{L^\infty}\|g_n\|_{\boldsymbol{B}^1_{\infty 1}}\right) < \infty \tag{12}$$

Here we can choose the period to be an arbitrary number greater than $2(\|A\| + \|B\|)$. Note that this is not the projective tensor product of two function spaces in one variable.

If A, B are almost commuting selfadjoint operators, then we define the operator $\varphi(A,B)$ by

$$\varphi(A,B) = \sum_{n\ge 0} f_n(A)g_n(B). \tag{13}$$

Such a functional calculus can be extended by the same formula to the class of functions in two variables which belong to $\mathfrak{L}$ locally.

Let us show first that our functional calculus is well-defined by (13).

Lemma (6.1.7)[239]: Suppose that φ is a function in two variables that admits two representations (11) satisfying (12):

$$\varphi(s,t) = \sum_{n\geq 0} f_n^{(1)}(s)g_n^{(1)}(t) = \sum_{n\geq 0} f_n^{(2)}(s)g_n^{(2)}(t).$$

Then

$$\sum_{n\geq 0} f_n^{(1)}(A)g_n^{(1)}(B) = \sum_{n\geq 0} f_n^{(2)}(A)g_n^{(2)}(B).$$

Proof. It is easy to see that

$$\sum f_n^{(j)}(A)g_n^{(j)}(B) = \iint \varphi(s,t)dE_A(s)dE_B(t), \quad j = 1,2, \tag{14}$$

and the right-hand side of (14) does not depend on j. The integral in (14) can be understood in the sense of weak operator topology.

Let us now show that the functional calculus on $\mathfrak{L}$ defined by (13) satisfies (F_1)-(F_4).

Theorem (6.1.8)[239]: The functional calculus on the class $\mathfrak{L}$ defined by (13) satisfies properties (F_1)-(F_4).

Proof. Let f be a function in $\boldsymbol{B}_{\infty 1}^1, A, B$ almost commuting selfadjoint operators. Then it follows from the Birman-Solornyak formula (4) and from the fact that j is a Schur multiplier (see [250]) that

$$\|f(A)B - Bf(A)\|_{\boldsymbol{S}_1} \leq \|f\|_{\boldsymbol{B}_{\infty 1}^1}\|AB - BA\|_{\boldsymbol{S}_1}. \tag{15}$$

Let us now establish properties (F_1)-(F_4) with the help of (15). Properties (F_1) and (F_2) are obvious. Let us prove (F_3). Let

$$\varphi(s,t) = \sum_{n\geq 0} f_n(s)g_n(t), \quad \psi(s,t) = \sum_{n\geq 0} u_n(s)v_n(t)$$

be representations of φ and ψ satisfying (12).

It is sufficient to prove that for any $n, k \geq 0$

$$(f_n u_k)(A)(g_n v_k)(B) - f_n(A)g_n(B)u_k(A)v_k(B) \in \boldsymbol{S}_1$$

and

$$\|(f_n u_k)(A)(g_n v_k)(B) - f_n(A)g_n(B)u_k(A)v_k(B)\|_{\boldsymbol{S}_1} \leq \text{const}\|f\|_{L^\infty}\|g\|_{\boldsymbol{B}_{\infty 1}^1}\|u_k\|_{\boldsymbol{B}_{\infty 1}^1}\|v_k\|_{L^\infty}$$

We have

$$(f_n u_k)(A)(g_n v_k)(B) - f_n(A)g_n(B)u_k(A)v_k(B) = f_n(A)\big(u_k(A)g_n(B) - f_n(B)u_k(A)\big)v_k(B).$$

Now applying (15) twice, we obtain

$$\begin{aligned}\|(f_n u_k)(A)(g_n v_k)(B) - f_n(A)g_n(B)u_k(A)v_k(B)\|_{\boldsymbol{S}_1} \\ \leq \text{const}\|f_n\|_{L^\infty}\|v_k\|_{L^\infty}\|u_k\|_{\boldsymbol{B}_{\infty 1}^1}\|Ag_n(B) - g_n(B)A\|_{\boldsymbol{S}_1} \\ \leq \text{const}\|f_n\|_{L^\infty}\|v_k\|_{L^\infty}\|u_k\|_{\boldsymbol{B}_{\infty 1}^1}\|g_n\|_{\boldsymbol{B}_{\infty 1}^1}\|AB - BA\|_{\boldsymbol{S}_1}.\end{aligned}$$

Property (F_4) follows from the fact that formula (2) holds for smooth functions (see [240, 241]) and from the fact that the set of smooth functions is dense in $\mathfrak{L}$.

Now we are going to give a description of the class $\mathfrak{L}$. Since we deal with periodic functions, we can identify the functions in $\mathfrak{L}$ with functions on $\mathbb{T} \times \mathbb{T}$. Let us define trigometric polynomials $W_{n,k}$. $n, k \in \mathbb{Z}$, in two variables by

$$W_{n,k}(\zeta,\tau) = W_n(\zeta)W_k(\tau)$$

Note that for any function φ on $\mathbb{T} \times \mathbb{T}$ we have

$$\varphi = \sum_{n,k\in\mathbb{Z}} \varphi * W_{n,k}.$$

Recall that the tensor algebra $C(\mathbb{T})\hat{\otimes}C(\mathbb{T})$ is the set of functions φ on $\mathbb{T}^2$ of the form

$$\varphi(\zeta,\tau) = \sum_{n\geq 0} f_n(\zeta)g_n(\tau). \quad (16)$$

where

$$\sum_{n\geq 0} \|f_n\|_{C(\mathbb{T})}\|g_n\|_{C(\mathbb{T})} < \infty, \quad (17)$$

$C(\mathbb{T})$ being the space of continuous functions on $\mathbb{T}$. The norm $\|\varphi\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})}$ in $C(\mathbb{T})\hat{\otimes}C(\mathbb{T})$ is by definition the infimum of (17) over all representations (16).

The following description of $\mathfrak{L}$is similar to that of Besov spaces given.

Theorem (6.1.9)[239]: Let φ be a function $\mathbb{T}^2$. Then $\varphi \in \mathfrak{L}$ if and only if

$$\sum_{n,k\in\mathbb{Z}} \left(2^{|n|} + 2^{|k|}\right) \left\|\varphi * W_{n,k}\right\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})} < \infty. \quad (18)$$

Proof. Suppose that φ admits a representation (16) satisfying (17). Let us show that $\varphi \in \mathfrak{L}$. It is sufficient to prove that for any $n, k \in \mathbb{Z}$

$$\left\|\varphi * W_{n,k}\right\|_{\mathfrak{L}} \leq \text{const}\left(2^{|n|} + 2^{|k|}\right)\left\|\varphi * W_{n,k}\right\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})}.$$

Suppose, to be definite, that $n, k > 0$. Then it suffices to prove that for any polynomial ψ of the form

$$\psi(\zeta,\tau) = \sum_{i=2^{n-1}}^{2^{n+1}} \sum_{j=2^{k-1}}^{2^{k+1}} \hat{\varphi}(i,j)\zeta^i\tau^j \quad (19)$$

the following inequality holds:

$$\|\psi\|_{\mathfrak{L}} \leq \text{const}(2^n + 2^k)\|\psi\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})}. \quad (20)$$

Let

$$\psi(\zeta,\tau) = \sum f_m(\zeta)g_m(\tau),$$

where $f_m, g_m \in C(\mathbb{T})$ and

$$\sum_{m\geq 0} \|f_m\|_{L^\infty} \|g_m\|_{L^\infty} \leq 2\|\psi\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})}.$$

Let $V_n = W_{n+1} + W_n + W_{n+1}$. Then $\|V_n\|_{L^1} \leq$ const and $\hat{V}_n(j) = 1$ for

$$V_{n,k}(\zeta,\tau) = V_n(\zeta)V_k(\tau).$$

Since ψ has the form (19), it follows that and so

$$\psi * V_{n,k} = \psi,$$

and so

$$\psi(\zeta,\tau) = \sum F_m(\zeta)G_m(\tau),$$

where

$$F_m = f_m * V_n, \quad G_m = g_m * V_k.$$

Therefore

$$\|\psi\|_{\mathfrak{L}} \le \sum_{m\ge 0} \left(\|F_m\|_{\boldsymbol{B}^1_{\infty 1}} \|G_m\|_{L^\infty} + \|F_m\|_{L^\infty} \|G_m\|_{\boldsymbol{B}^1_{\infty 1}} \right).$$

It follows easily from the definition of $\boldsymbol{B}^1_{\infty 1}$ that

$$\|F_m\|_{\boldsymbol{B}^1_{\infty 1}} \le \text{const}\, 2^n \|F_m\|_{L^\infty}, \qquad \|G_m\|_{\boldsymbol{B}^1_{\infty 1}} \le \text{const}\, 2^k \|G_m\|_{L^\infty}$$

Then (20) follows from the obvious estimates

$$\|F_m\|_{L^\infty} \le \text{const}\, \|f_m\|_{L^\infty}, \qquad \|G_m\|_{L^\infty} \le \text{const}\, \|g_m\|_{L^\infty}.$$

Let us now prove that any function φ in $\mathfrak{L}$ satisfies (18). It is sufficient to consider the case

$$\varphi(\zeta,\tau) = f(\zeta)g(\tau)$$

and show that

$$\sum_{n,k\in\mathbb{Z}} \left(2^{|n|} + 2^{|k|}\right) \left\|\varphi * W_{n,k}\right\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})} \le \text{const} \left(\|f\|_{\boldsymbol{B}^1_{\infty 1}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\boldsymbol{B}^1_{\infty 1}} \right).$$

We have

$$\left(\varphi * W_{n,k}\right)(\zeta,\tau) = (f * W_n)(\zeta)\left(g * W_k(\tau)\right).$$

Therefore

$$\begin{aligned}
\sum_{n,k\in\mathbb{Z}} \left(2^{|n|} + 2^{|k|}\right) \left\|\varphi * W_{n,k}\right\|_{C\hat{\otimes}C} &\le \sum_{n,k\in\mathbb{Z}} \left(2^{|n|} + 2^{|k|}\right) \|f * W_n\|_{L^\infty} \|g * W_k\|_{L^\infty} \\
&\le 2\left(\sum_{|n|\ge|k|} 2^{|n|} \|f * W_n\|_{L^\infty} \right) \|g\|_{L^\infty} + 2\left(\sum_{|k|\ge|n|} 2^{|k|} \|f * W_k\|_{L^\infty} \right) \|f\|_{L^\infty} \\
&\le 2\left(\sum_{n\in\mathbb{Z}} 2^{|n|} \|f * W_n\|_{L^\infty} \right) \|g\|_{L^\infty} + 2\left(\sum_{n\in\mathbb{Z}} 2^{|k|} \|f * W_k\|_{L^\infty} \right) \|f\|_{L^\infty} \\
&\le \text{const}\left(\|f\|_{\boldsymbol{B}^1_{\infty 1}} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{\boldsymbol{B}^1_{\infty 1}} \right).
\end{aligned}$$

Recall that in [242] a functional calculus for almost commuting operators has been constructed on the class of functions $\varphi = \mathcal{F}\omega$ which are Fourier transforms of measures ω on $\mathbb{R}^2$ satisfying the condition

$$\int_{\mathbb{R}^2} (1 + |t|)(1 + |s|)d|\omega(t,s)| < \infty. \tag{21}$$

The functional calculus has been defined by the formula

$$\varphi(A,B) = \iint \varphi(x,y)dE_A(x)dE_B(y). \tag{22}$$

It is easy to see that the condition (21) is more restrictive than the condition (12). Indeed, the condition (21) implies that φ has continuous derivatives $\partial\varphi/\partial s$, $\partial\varphi/\partial t$, and $\partial^2\varphi/\partial s\partial t$ which have locally absolutely convergent Fourier expansions. However, the condition (12) does not imply the existence of continuous derivative $\partial^2\varphi/\partial s\partial t$ and the first partial derivatives $\partial\varphi/\partial s$, $\partial\varphi/\partial t$ do not necessarily have absolutely convergent Fourier expansions.

It is also easy to see that for φ satisfying (21) the definition (13) coincides with (22). Thus our functional calculus extends the functional calculus defined by (22).

Let us now consider the problem to construct a functional calculus satisfying property (F_5):

$$\bar{\varphi}(A.B) - (\varphi(A,B))^* \in \boldsymbol{S}_1.$$

In this case we have to impose a more restrictive assumption on φ.

Definition (6.1.10)[239]: Let $\mathfrak{L}_1$ be a class of periodic functions of the form

$$\varphi(s,t) = \sum_{n\geq 0} f_n(s) g_n(t),$$

where

$$\sum_{n\geq 0} \|f_n\|_{\boldsymbol{B}^1_{\infty 1}} \|g_n\|_{\boldsymbol{B}^1_{\infty 1}} < \infty.$$

(In other words $\mathfrak{L}_1$ is the projective tensor product of two spaces $\boldsymbol{B}^1_{\infty 1}$)

For φin $\mathfrak{L}_1$ and almost commuting selfadjoint operators A and B we can define $\varphi(A,B)$ by (13)

Theorem (6.1.11)[239]: The functional calculus on $\mathfrak{L}_1$ defined by (13) satisfies properties (F_1)-(F_5).

Proof Clearly we have to prove only (F_5). It is sufficient to consider only the functions φ of the form

$$\varphi(s,t) = f(s)g(t), \quad f,g \in \boldsymbol{B}^1_{\infty 1}.$$

We have by (15)

$$\left\|\bar{f}(A)\bar{g}(B) - \bar{g}(B)\bar{f}(A)\right\|_{\boldsymbol{S}_1} \leq \text{const } \|f\|_{\boldsymbol{B}^1_{\infty 1}} \|A\bar{g}(B) - \bar{g}(B)A\|_{\boldsymbol{S}_1}$$
$$\leq \text{const } \|f\|_{\boldsymbol{B}^1_{\infty 1}} \|g\|_{\boldsymbol{B}^1_{\infty 1}} \|AB - BA\|_{\boldsymbol{S}_1}.$$

The class $\mathfrak{L}_1$admits a description similar to (18).

Theorem (6.1.12) [239]: Let φ be a function on $\mathbb{T} \times \mathbb{T}$. Then $\varphi \in \mathfrak{L}_1$ if and only if

$$\sum_{n,k\in\mathbb{Z}} 2^{|n|+k} \left\|\varphi * W_{n,k}\right\|_{C\hat{\otimes}C} < \infty.$$

The proof is similar to the proof of Theorem (6.1.9).

Section (6 . 2): Operator Moduli of Continuity☆ .

We study operator moduli of continuity of functions on subsets of the real line. For a closed subset $\mathfrak{F}$ of the real line $\mathbb{R}$ and for a continuous function f on $\mathfrak{F}$, the operator modulus of continuity $\Omega_{f,\mathfrak{F}}$ is defined by

$$\Omega_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\| f(A) - f(B)\| \;, \quad \delta > 0,$$

where the supremum is taken over all self-adjoint operators A and B such that

$$\sigma(A) \subset \mathfrak{F}, \quad \sigma(B) \subset \mathfrak{F}, \text{ and } \|A - B\| \leq \delta.$$

If $\mathfrak{F} = \mathbb{R}$, we use the notation $\Omega_f \stackrel{\text{def}}{=} \Omega_{f,\mathbb{R}}$. Recall that a continuous function f on $\mathfrak{F}$ is called operator Lipschitz if $\Omega_{f,\mathfrak{F}}(\delta) \leq \text{const } \delta, \delta > 0$.

It turns out that a Lipschitz function f on $\mathbb{R}$, i.e., a function f satisfying

$$|f(x) - f(y)| \le \text{const}|x - y|, \quad x, y \in \mathbb{R},$$

does not have to be operator Lipschitz. This was established for the first time by Farforovskaya [9]. It was shown later in [262]that the function $x \longmapsto |x|$ on $\mathbb{R}$ is not operator Lipschitz.

The [262] followed the by [265], in which it was shown that the function $x \longmapsto |x|$ is not commutator Lipschitz. Note that nowadays it is well known that operator Lipschitzness is equivalent to commutator Lipsc-hitzness.

We would like to also mention that in [266]necessary conditions for operator Lipschitzness were found that also imply that Lipschitzness is not sufficient for operator Lipschitzness. On the other hand, it was shown in [266]and [28] that if f belongs to the Besov class $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, then f is operator Lipschitz (we see [267]and [268]).

In [269] and [261] we obtain the following upper estimate for continuous functions f on $\mathbb{R}$:

$$\Omega_f(\delta) \le \text{const}\, \delta \int_\delta^\infty \frac{\omega_f(t)}{t^2} dt = \text{const} \int_1^\infty \frac{\omega_f(t\delta)}{t^2} dt, \quad \delta > 0, \qquad (23)$$

where ω_f is the modulus of continuity of f , i.e.,

$$\omega_f(\delta) \stackrel{\text{def}}{=} \sup\{|f(x) - f(y)|: x, y \in \mathbb{R}, |x - y| \le \delta\}, \qquad \delta > 0.$$

We deduced from (23) in [261] that for a Lipschitz function f on $[a, b]$, the following estimate for the operator modulus of continuity $\Omega_{f,[a,b]}$ holds:

$$\Omega_{f,[a,b]}(\delta) \le \text{const}\, \delta \left(1 + \log\left(\frac{b - a}{\delta}\right)\right) \|f\|_{\text{Lip}},$$

where

$$\|f\|_{\text{Lip}} \stackrel{\text{def}}{=} \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|}.$$

A similar estimate was obtained earlier in [262]in the very special case $f(x) = |x|$. Namely, it was shown in [262]that for bounded self-adjoint operators A and B on Hilbert space, the following inequality holds:

$$\||A| - |B|\| \le \frac{2}{\pi} \|A - B\| \left(2 + \log \frac{\|A\| - \|B\|}{\|A - B\|}\right).$$

It turns out, however, that for the function $x \mapsto |x|$ the operator modulus of continuity admits a much better estimate. Namely, we show that under the same hypotheses

$$\||A| - |B|\| \le \text{const}\|A - B\| \left(2 + \log \frac{\|A\| - \|B\|}{\|A - B\|}\right).$$

We also prove that this estimate is sharp.

Note that in [270]an estimate slightly weaker than (23) was obtained by a different method.

We show that if f is a continuous non decreasing function on $\mathbb{R}$, such that $f(x) = 0$ for $x \le 0$ and the restriction of f to $[0, \infty)$ is a concave function, then estimate (23) can also be improved considerably:

$$\Omega_f(\delta) \le \text{const} \int_e^\infty \frac{f(\delta t)}{t^2 \log t}, \qquad \delta > 0.$$

We also obtain other estimates of operator moduli of continuity.

It is still unknown whether inequality (23) is sharp. It follows easily from (23) that if f is a function on $\mathbb{R}$ such that $\|f\|_{L^\infty} \le 1$, $\|f\|_{\text{Lip}} \le 1$, then

$$\Omega_f(\delta) \le \text{const}\delta\left(1 + \log\frac{1}{\delta}\right), \quad \delta \in (0,1].$$

We construct a C^∞ function f on $\mathbb{R}$ such that $\|f\|_{L^\infty} \le 1$, $\|f\|_{\text{Lip}} \le 1$, and

$$\Omega_f(\delta) \le \text{const}\delta\sqrt{\log\frac{2}{\delta}}, \quad \delta \in (0,1].$$

To construct such a function f , we use necessary conditions for operator Lipschitzness found in [27].

We obtain lower estimates in the case of functions on the unit circle and unitary operators.

Finally, we obtain the following sharp estimate for the norms $\|f(A) - f(B)\|$ for Lipschitz functions f and self-adjoint operators A and B on Hilbert space such that the spectrum $\sigma(A)$ of A has n points:

$$\|f(A) - f(B)\| \le C(1 + \log n)\|f\|_{\text{Lip}} \|A - B\| . \tag{24}$$

We obtain an upper estimate in the general case (see Theorem (6.2.85) in terms of the ε-entropy of the spectrum of A, where $\varepsilon = \|A - B\|$. It includes inequalities (23) and (24) as special cases. Note that (24) improve earlier estimates in [264]and [271].

We give a brief introduction to Schur multipliers, we collect auxiliary estimates of certain functions in the space of functions with absolutely converging Fourier integrals. To obtain upper estimates for operator moduli of continuity of concave functions, we estimate the operator modulus of continuity of a very special piecewise continuous function on $\mathbb{R}$.

We define Schur multipliers and discuss their properties. Note that the notion of a Schur multiplier can be defined in the case of two spectral measures (see, e.g., [27]). We define Schur multipliers in the case of two scalar measures. This corresponds to the case of spectral measures of multiplicity 1.

Let $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ be σ-finite measure spaces. Let $k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu)$. Then k induces the integral operator $\mathfrak{I}_k = \mathfrak{I}_k^{\mu,\nu}$ from $L^2(\mathcal{Y}, \nu)$ to $L^2(\mathcal{X}, \mu)$ defined by

$$(\mathfrak{I}_k)(x) = \int_{\mathcal{Y}} k(x, y) f(y) d\nu(y), \qquad f \in L^2(\mathcal{Y}, \nu).$$

Denote by $\|k\|_{\mathfrak{B}} = \|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$ the operator norm of $\mathfrak{I}_k$. Let Φ be a $\mu \otimes \nu$-measu-rable function defined almost everywhere on $\mathcal{X} \times \mathcal{Y}$. We say that Φ is a Schur multiplier with respect to μ and ν if

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \stackrel{\text{def}}{=} \{\|\Phi k\|_{\mathfrak{B}} : k, \Phi k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu), \|k\|_{\mathfrak{B}} \le 1\} < \infty.$$

We denote by $\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}$ the space of Schur multipliers with respect to μ and ν. It can be shown easily that $\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu} \subset L^{\infty}(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu)$ and $\|\Phi\|_{L^{\infty}(\mathcal{X}\times\mathcal{Y},\mu\otimes\nu)}$. Thus if $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}$, then

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \sup\{\|\Phi k\|_{\mathfrak{B}}: k, \Phi k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu), \|k\|_{\mathfrak{B}} \le 1\}$$

Note that $\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}$ is Banach algebra:

$$\|\Phi_1\Phi_2\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \le \|\Phi_1\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \|\Phi_2\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}.$$

It is easy to see that $\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \|\Psi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$ for $\Psi(y,x) = \Phi(x,y)$.

If $\mathcal{X}_0$ is a μ-measurable subset of $\mathcal{X}$, then we denote by $(\mathcal{X}_0, \mu)$ the corresponding measure space on the σ-algebra of μ-measurable subsets of $\mathcal{X}_0$.

Let $\mathcal{X} = \bigcup_{n=0}^{\infty} \mathcal{X}_n$ and $\mathcal{Y} = \bigcup_{n=0}^{\infty} \mathcal{Y}_n$, where the $\mathcal{X}_n$ are μ-measurable subsets of $\mathcal{X}$, and the $\mathcal{Y}_n$ are ν-measurable subsets of $\mathcal{Y}$. It is easy to see that

$$\sup_{m,n\ge 1} \|k\|^2_{\mathfrak{B}_{\mathcal{X}_m,\mathcal{Y}_n}^{\mu,\nu}} \le \|k\|^2_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \|k\|^2_{\mathfrak{B}_{\mathcal{X}_m,\mathcal{Y}_n}^{\mu,\nu}}$$

for every $k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu)$, and

$$\sup_{m,n\ge 1} \|\Phi\|^2_{\mathfrak{M}_{\mathcal{X}_m,\mathcal{Y}_n}^{\mu,\nu}} \le \|\Phi\|^2_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \|\Phi\|^2_{\mathfrak{M}_{\mathcal{X}_m,\mathcal{Y}_n}^{\mu,\nu}} \quad (25)$$

for every $\Phi \in L^{\infty}(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu)$.

We state the following elementary theorem:

Theorem (6.2.1)[260]. Let $(\mathcal{X}, \mu)$, $(\mathcal{Y}, \mu_0)$, $(\mathcal{Y}, \nu)$ and $(\mathcal{Y}, \nu_0)$ be σ-finite measure spaces. Suppose that μ_0 is absolutely continuous with respect to μ and $\nu_0$is absolutely continuous with respect to ν.

Let $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}$. Then $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}$ and$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$.

Proof. By the Radon–Nikodym theorem, $d\mu_0 = \varphi d\mu$ and $d\nu_0 = \psi d\nu$ for non-negative measurable functions φ and ψ on $\mathcal{X}$ and $\mathcal{Y}$. Let $k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu_0 \otimes \nu_0)$. Put

$$(Tk)(x,y) \stackrel{\text{def}}{=} k(x,y)\sqrt{\varphi(x)\psi(y)}.$$

Clearly, T is an isometric embedding from $L^2(\mathcal{X} \times \mathcal{Y}, \mu_0 \otimes \nu_0)$ in $L^2(\mathcal{X} \times \mathcal{Y}, \mu \otimes \nu)$. Moreover,$\|Tk\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}}$. We have

$$\|\Phi k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}} = \|T(\Phi k)\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \|\Phi Tk\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \|Tk\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}}$$

for every $k \in L^2(\mathcal{X} \times \mathcal{Y}, \mu_0 \otimes \nu_0)$. Hence, $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}$ and$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$.

Note that if $\mathcal{X}$ and $\mathcal{Y}$ coincide with the set $\mathbb{Z}_+$ of nonnegative integers and μ and ν are the counting measure, the above definition coincides with the definition of Schur multipliers on the space of matrices: a matrix $A = \{a_{jk}\}_{j,k\ge 0}$ is called a Schur multiplier on the space of bounded matrices if

$A \star B$ is a matrix of a bounded operator, whenever B is.

Here we use the notation

$$A \star B = \{a_{jk}b_{jk}\}_{j,k\ge 0}. \quad (26)$$

for the Schur–Hadamard product of the matrices $A = \{a_{jk}\}_{j,k\ge 0}$ and $B = \{b_{jk}\}_{j,k\ge 0}$.
Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$. We denote by $\mathfrak{M}_{\mathcal{X},\mathcal{Y}}$ the space of Borel Schur multipliers on $\mathcal{X} \times \mathcal{Y}$, i.e., the space of Borel functions Φ defined everywhere on $\mathcal{X} \times \mathcal{Y}$ such that

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} \stackrel{\text{def}}{=} \sup\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} < \infty,$$

where the supremum is taken over all regular positive Borel measures μ and ν on $\mathcal{X}$ and $\mathcal{Y}$. It can be shown easily that

$$\sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} |\Phi(x,y)| \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}}.$$

It is also easy to verify that if $\Phi_n \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}$, Φ is a bounded Borel function on $\mathcal{X} \times \mathcal{Y}$, and $\Phi_n(x,y) \to \Phi(x,y)$ for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$, then

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} \le \liminf_{n\to\infty}\|\Phi_n\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}}.$$

In particular, $\Phi \in \mathfrak{M}_{\mathcal{X},\mathcal{Y}}$ if $\liminf_{n\to\infty}\|\Phi_n\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} < \infty$.

We are going to deal with functions f on $\mathcal{X} \times \mathcal{Y}$ that are continuous in each variable. It must be a well-known fact that such a function f has to be a Borel function. Indeed, one can construct an increasing sequence $\{\mathcal{Y}_n\}_{n=1}^{\infty}$of discrete closed subsets of $\mathcal{Y}$ such that $\cup_{n=1}^{\infty}\mathcal{Y}_n$ is dense in $\mathcal{Y}$. Let us consider the function $f_n: \mathcal{X} \times \mathbb{R} \to \mathbb{C}$ such that $f|(\mathcal{X} \times \mathcal{Y}_n) = f_n|(\mathcal{X} \times \mathcal{Y}_n)$ and $f_n(x,\cdot)$ is a piecewise linear function with nodes in $\mathcal{Y}_n$for all $x \in \mathcal{X}$. Clearly, the function
f_n is defined uniquely if we require that $f_n(x,\cdot)$ is constant on each unbounded complimentary interval of $\mathcal{Y}_n$. It is easy to see that f_n is continuous on $\mathcal{X} \times \mathbb{R}$ and $\lim_{n\to\infty} f_n(x,y) = f(x,y)$for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$. Thus, f belongs to the first Baire class, and so it is Borel.

Lemma (6.2.2)[260]. Let $\mathcal{X}$ and $\mathcal{Y}$ be compact subsets of $\mathbb{R}$ and let μ and ν be finite positive Borel measures on $\mathcal{X}$ and $\mathcal{Y}$. Suppose that $\{\nu_j\}_{j=1}^{\infty}$is a sequence of finite positive Borel measures on $\mathcal{Y}$ that converges to ν in the weak-$*$ topology $\sigma((C(\mathcal{Y}))^*, C(\mathcal{Y}))$. If k is a bounded Borel function on $\mathcal{X} \times \mathcal{Y}$ such that $k(x,\cdot) \in C(\mathcal{Y})$ for every $x \in \mathcal{X}$, then

$$\lim_{j\to\infty}\left\|\mathfrak{T}_k^{\mu,\nu_j}\right\|_{\mathfrak{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} = \left\|\mathfrak{T}_k^{\mu,\nu}\right\|_{\mathfrak{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

Proof. Clearly, $\mathfrak{T}_k^{\mu,\nu_j}(\mathfrak{T}_k^{\mu,\nu_j})^*$ is an integral operator on $L^2(\mathcal{X},\mu)$ with kernel $l_j(x,y) = \int_{\mathcal{Y}} k(x,t)\overline{k(y,t)}d\nu_j(t)$. Besides, the sequence $\{l_j\}$ converges in $L^2(\mathcal{X} \times \mathcal{X}, \mu \otimes \mu)$ to the function l defined by $l(x,y) = \int_{\mathcal{Y}} k(x,t)\overline{k(y,t)}d\nu(t)$, which is the kernel of the integral operator$\mathfrak{T}_k^{\mu,\nu}(\mathfrak{T}_k^{\mu,\nu})^*$. Hence,

$$\lim_{j\to\infty}\left\|\mathfrak{T}_k^{\mu,\nu_j}\right\|^2_{\mathfrak{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} = \lim_{j\to\infty}\left\|\mathfrak{T}_k^{\mu,\nu_j}(\mathfrak{T}_k^{\mu,\nu_j})^*\right\|_{\mathfrak{B}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}} = \left\|\mathfrak{T}_k^{\mu,\nu}(\mathfrak{T}_k^{\mu,\nu})^*\right\|_{\mathfrak{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} = \left\|\mathfrak{T}_k^{\mu,\nu}\right\|^2_{\mathfrak{B}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}}.$$

Corollary (6.2.3)[260]. Let X and $\mathcal{Y}$ be compact subsets of R, and let μ and ν be finite positive Borel measures on $\mathcal{X}$ and $\mathcal{Y}$. Suppose that $\{\nu_j\}_{j=1}^{\infty}$ is a sequence of finite positive Borel measures on $\mathcal{Y}$ that converges to ν in $\sigma((C(\mathcal{Y}))^*, C(\mathcal{Y}))$. If Φ is a Borel function on $\mathcal{X} \times \mathcal{X}$ such that $\Phi(x,\cdot) \in C(\mathcal{Y})$ for all $x \in \mathcal{X}$, then $\|\Phi\|_{\mathfrak{M}^{\mu,\nu}_{\mathcal{X},\mathcal{Y}}} \le \liminf_{j\to\infty} \le \|\Phi\|_{\mathfrak{M}^{\mu,\nu_j}_{\mathcal{X},\mathcal{Y}}}$.

Proof. It is easy to see that

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \sup\left\{\|\Phi k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} : k \in C(\mathcal{X} \times \mathcal{Y}), \|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} \le 1\right\}.$$

Let $k \in C(\mathcal{X} \times \mathcal{Y})$ with $\|k\|_{L^2(\mu\otimes\nu)} > 0$. Then

$$\|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}} = \lim_{j\to\infty}\|\Phi k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}} \le \liminf_{j\to\infty}\left(\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}}\|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}}\right) = \lim_{j\to\infty}\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}}\liminf_{j\to\infty}\|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}}$$
$$= \|k\|_{\mathfrak{B}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}\liminf_{j\to\infty}\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}}$$

which implies the result.

For a measure μ and an integrable function φ, we write $\nu = \varphi\mu$ if ν is the (complex) measure defined by $d\nu = \varphi d\mu$.

The following fact can be proved very easily.

Lemma (6.2.4)[260]. Let ν and ν_0 be finite Borel measures on $\mathbb{R}$ with compact supports. Suppose that supp $\nu_0 \subset$ supp ν . Then there exists a sequence $\{\varphi_j\}_{j=0}^{\infty}$ in $C(\mathbb{R})$ such that $\varphi_j \ge 0$ everywhere on $\mathbb{R}$ for all j and $\nu_0 = \lim_{j\to\infty}\varphi_j\nu$ in $\sigma((C(\text{supp}\,\nu))^*, C(\text{supp}\,\nu))$.

Theorem (6.2.5)[260]. Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$ and let Φ be a function on $\mathcal{X} \times \mathcal{Y}$ that is continuous in each variables. Suppose that μ and μ_0 are positive regular Borel measures on $\mathcal{X}$, and ν and ν_0 are positive regular Borel measures on $\mathcal{Y}$. If supp $\mu_0 \subset$ supp μ and $\nu_0 \subset$ supp ν , then $\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$.

We need two lemmata.

Proof . Put $\mathcal{X}_n \overset{\text{def}}{=} [-n, n] \cap \mathcal{X}$ and $\mathcal{Y}_n \overset{\text{def}}{=} [-n, n] \cap \mathcal{Y}$. Clearly, $\{\|\Phi\|_{\mathfrak{M}_{\mathcal{X}_n,\mathcal{Y}_n}^{\mu,\nu}}\}$ is a nondecreasing sequence and

$$\lim_{n\to\infty}\|\Phi\|_{\mathfrak{M}_{\mathcal{X}_n,\mathcal{Y}_n}^{\mu,\nu}} = \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}.$$

This allows us to reduce the general case to the case when $\mathcal{X}$ and $\mathcal{Y}$ are compact. Besides, it suffices to consider the case where $\mu_0 = \mu$. Indeed, the case $\nu_0 = \nu$ can be reduced to the case $\mu_0 = \mu$, and we have

$$\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu_0,\nu_0}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_0}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be compact, and $\mu_0 = \mu$. Applying Lemma (6.2.5), we can take a sequence $\{\varphi_j\}_{j=0}^{\infty}$ of nonnegative functions in $C(\mathbb{R})$ such that $\nu_0 = \lim_{j\to\infty}\varphi_j\nu$ in the weak topology $\sigma((C(\mathcal{Y}))^*, C(\mathcal{Y}))$. Put $\nu_j \overset{\text{def}}{=} \varphi_j\nu$. By Theorem (6.2.1), $\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu_j}} \le \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$ for every $j \ge 1$. It remains to apply Corollary (6.2.4).

Theorem (6.2.2) implies the following fact:

Theorem (6.2.6)[260]. Let $\mathcal{X}$ and $\mathcal{Y}$ be closed subsets of $\mathbb{R}$ and let Φ be a function on $\mathcal{X} \times \mathcal{Y}$ that is continuous in each variable. Suppose that μ and ν are positive regular Borel measures on $\mathcal{X}$ and $\mathcal{Y}$ such that supp $\mu = \mathcal{X}$ and supp $\nu = \mathcal{Y}$. Then $\|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}} = \|\Phi\|_{\mathfrak{M}_{\mathcal{X},\mathcal{Y}}^{\mu,\nu}}$.

The following result is well known.

Let $f \in C(\mathbb{R})$. Put $\Phi(x, y) \overset{\text{def}}{=} f(x - y)$. Then $\Phi \in \mathfrak{M}_{\mathbb{R},\mathbb{R}}$ if and only if f is the Fourier transform of a complex measure on $\mathbb{R}$. Moreover, $\|\Phi\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} = |\mu|(\mathbb{R})$.

A similar statement holds for any locally compact abelian group. In particular, it is true for the group $\mathbb{Z}$:
Let f be a function defined on $\mathbb{Z}$. Put $\Phi(m,n) \stackrel{\text{def}}{=} f(m-n)$. Then $\Phi \in \mathfrak{M}_{\mathbb{Z},\mathbb{Z}}$ if and only if $\{f(n)\}_{n\in\mathbb{Z}}$ are the Fourier coefficients of a complex Borel measure μ on the unit circle $\mathbb{T}$. Moreover, $\|\Phi\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = |\mu|(\mathbb{T})$.
We need the following well-known fact.
Lemma (6.2.7)[260]: Let

$$H(m,n) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{m-n}, & \text{if } m,n \in \mathbb{Z}, m \neq n, \\ 0, & \text{if } \ m = n \in \mathbb{Z}. \end{cases}$$

Then $\|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = \frac{\pi}{2}$.
Proof. It suffices to observe that

$$H(n,0) = \frac{1}{2\pi}\int_0^{2\pi} i(\pi - t)\, e^{-int} dt \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} |\pi - t|\, dt = \frac{\pi}{2}.$$

We collect elementary estimates of certain functions in the space of absolutely convergent Fourier integrals.

We are going to deal with the space

$$\hat{L}^1 = \hat{L}^1(\mathbb{R}) \stackrel{\text{def}}{=} \mathcal{F}\left(\hat{L}^1(\mathbb{R})\right), \quad \|f\|_{\hat{L}^1} = \|f\|_{\hat{L}^1(\mathbb{R})} \stackrel{\text{def}}{=} \|\mathcal{F}^{-1}f\|_{L^1}.$$

Here we use the notation $\mathcal{F}$ for Fourier transform:

$$(\mathcal{F}f)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}} f(t)e^{-ixt}\, dt, \quad f \in L^1(\mathbb{R}).$$

Unless otherwise stated, an interval means a closed non degenerate (not necessarily finite) interval. For such an interval J, we consider the class $\hat{L}^1(J)$ defined by $\hat{L}^1(J) \stackrel{\text{def}}{=} \{f|J : f \in \hat{L}^1\}$, we put

$$\|\varphi\|_{\hat{L}^1(J)} \stackrel{\text{def}}{=} \inf \{\|f\|_{\hat{L}^1} : f|J = \varphi\}.$$

For $\in C(\mathbb{R})$, we put $\|\varphi\|_{\hat{L}^1(J)} \stackrel{\text{def}}{=} \|\varphi|J\|_{\hat{L}^1(J)}$. Clearly, $\|\varphi\|_{L^\infty(J)} \le \|\varphi\|_{\hat{L}^1(J)}$.

For an interval J, we use the notation $|J|$ for its length.
It is easy to see that the constant functions belong to the space $\hat{L}^1(J)$ for bounded intervals J and $\|\mathbf{1}\|_{\hat{L}^1(J)} = 1$. Moreover,
$\hat{L}^1(J) = \{(\mathcal{F}\mu)|J : \mu \in \mathcal{M}(\mathbb{R})\}$ and $\|f\|_{\hat{L}^1(J)} \stackrel{\text{def}}{=} \inf \{\|\mu\|_{\mathcal{M}} : (\mathcal{F}\mu)|J = f\}$
for every bounded interval J, where $\mathcal{M}(\mathbb{R})$ denotes the space of (complex) Borel measures on $\mathbb{R}$.
We are going to discuss (mostly known) estimates for $\|\cdot\|_{\hat{L}^1(J)}$.

First, we recall the Pólya theorem, see [272].

Let f be an even continuous function such that $f|[0,\infty)$ is a decreasing convex function vanishing at the infinity. Then $f \in \hat{L}^1$ and $\|f\|_{\hat{L}^1} = f(0)$.

This theorem readily implies the following fact.

Lemma (6.2.8)[260]. Let f be a continuous function on a closed ray J that vanishes at infinity. Suppose that f is monotone and convex (or concave). Then $f \in \hat{L}^1(J)$ and $\|f\|_{\hat{L}^1} = \max_J |f|$.

In what follows by a locally absolutely continuous function on $\mathbb{R}$ we mean a function whose restriction to any compact interval is absolutely continuous.

Lemma (6.2.9)[260]. Let f be a locally absolutely continuous function in $L^2(\mathbb{R})$ such that $f' \in L^2(\mathbb{R})$.Then $f \in \hat{L}^1(\mathbb{R})$ and $\|f\|_{\hat{L}^1}^2 \le \|f\|_{L^2}\|f'\|_{L^2}$.

Proof. Put $a = \|f\|_{L^2}$, $b = \|f'\|_{L^2}$. By Plancherel's theorem,

$$\|\mathcal{F}^{-1}f\|_{L^2}^2 = \frac{a^2}{2\pi} \quad \text{and} \quad \|x\mathcal{F}^{-1}f\|_{L^2}^2 = \frac{b^2}{2\pi}.$$

Hence,

$$\left\|\sqrt{b^2 + a^2x^2}\mathcal{F}^{-1}f\right\|_{L^2}^2 = \frac{a^2b^2}{2\pi}.$$

and by the Cauchy–Bunyakovsky inequality,

$$\|\mathcal{F}^{-1}f\|_{L^1} \le \frac{ab}{\sqrt{\pi}}\left\|\frac{1}{\sqrt{a^2x^2 + b^2}}\right\|_{L^2} = \sqrt{ab}.$$

Corollary (6.2.10)[260]. Let $a > 0$. Put

$$f_a(x) \stackrel{\text{def}}{=} \begin{cases} a^{-2}x, & \text{if } |x| \le a, \\ x^{-1}, & \text{if } |x| \ge a. \end{cases}$$

Then $f_a \in \hat{L}^1(\mathbb{R})$ and $\|f_a\|_{\hat{L}^1} \le \frac{a}{2}$.

Proof. It suffices to observe that $\|f_a\|_{L^2}^2 = \frac{8}{3a^3}$, $\|f_a'\|_{L^2}^2 = \frac{8}{3a^3}$ and $\sqrt{\frac{8}{3}} \le 2$.

Lemma (6.2.11)[260]. Let J be a bounded interval and let f be a Lipschitz function on $\mathbb{R}$ such that supp$f \subset J$. Then $f \in \hat{L}^1$ and

$$\|f\|_{\hat{L}^1} \le \frac{1}{\sqrt[4]{12}}|J| \cdot \|f'\|_{L^\infty}.$$

Proof. Let $J = [-a, a]$. Clearly, $|f(x)| \le (a - |x|)\|f'\|_{L^\infty}$ for all $\in J$. Hence,

$$\|f\|_{L^2}^2 \le 2\|f'\|_{L^\infty}^2 \int_0^a (a - t)^2\, dt = \frac{1}{12}\|f'\|_{L^\infty}^2 |J|^3.$$

Using the obvious inequality $\|f'\|_{L^2}^2 \le 2\|f'\|_{L^\infty}^2 |J|$, we get the desired estimate.

Corollary (6.2.12)[260]: Let f be a Lipschitz function on $\mathbb{R}$ such that $f(0) = 0$. Then

$$\|f\|_{\hat{L}^1(J)} \le \frac{1}{\sqrt[4]{12}}|J| \cdot \|f'\|_{L^\infty}.$$

for every bounded interval J that contains 0.

Proof. Put $2J \stackrel{\text{def}}{=} \{2x: x \in J\}$. Clearly, there exists a function f_J in $C(\mathbb{R})$ such that $f_J = f$ on J , supp$f_J \subset 2J$, and $\left\|f_J'\right\|_{L^\infty} \le \|f'\|_{L^\infty}$.

Lemma (6.2.13)[260]. Let f be a locally absolutely continuous function on $\mathbb{R}$ such that$(1 + |x|)f'(x) \in L^2(\mathbb{R})$. Suppose that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Then

$$\|f\|_{\hat{L}^1(-\infty,a]} \le \frac{1}{\sqrt{\pi}}\|f'\|_{L^2} + \sqrt{\frac{2}{\pi}}\|xf'\|_{L^2} + \frac{7}{2\pi} + \frac{2}{\pi}\log a$$

for every $a \ge 2$.

Proof. Put

$$f_a(x) \stackrel{\text{def}}{=} f(x) - a^{-1}\int_{-\infty}^{x}\chi_{[a,2a]}(t)dt\,.$$

Clearly, $\|f\|_{\hat{L}^1(-\infty,a]} \le \|f_a\|_{\hat{L}^1}$.

We have

$$-ix\mathcal{F}^{-1}f_a = \mathcal{F}^{-1}(f_a') = \mathcal{F}^{-1}(f') - \frac{e^{2aix}-e^{aix}}{2\pi aix}.$$

Put $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}(f')$. Then

$$\|f_a\|_{\hat{L}^1} = \int_{\mathbb{R}}\left|h(x) - \frac{e^{2aix}-e^{aix}}{2\pi aix}\right|\cdot\frac{dx}{|x|}$$

$$\le \int_{-1}^{1}\frac{|h(x)-h(0)|}{|x|}dx + \frac{1}{2\pi}\int_{-1}^{1}\left|\frac{e^{2aix}-e^{aix}}{aix} - 1\right|\cdot\frac{dx}{|x|} + \int_{\{|x|\ge1\}}\frac{|h(x)|}{|x|}dx$$

$$+ \frac{1}{2\pi a}\int_{\{|x|\ge1\}}\frac{\left|e^{aix}-1\right|}{x^2}dx.$$

We have

$$\int_{0}^{1}\frac{|h(x)-h(0)|}{x}dx \le \int_{0}^{1}\frac{1}{x}\left(\int_{0}^{x}|h'(t)|dt\right)dx = \int_{0}^{1}|h'(t)|\cdot|\log t|dt.$$

Hence,

$$\int_{-1}^{1}\frac{|h(x)-h(0)|}{|x|}dx \le \int_{-1}^{1}|h'(t)|\cdot|\log|t||dt \le \|h'\|_{L^2}\left(\int_{-1}^{1}\log^2|t|\,dt\right)^{1/2} = \sqrt{\frac{2}{\pi}}\|xf'(x)\|_{L^2}$$

because $h' = \mathcal{F}^{-1}(ixf')$.

By Taylor's formula for the function $e^{2ix}-e^{ix}$, we have

$$\left|e^{2ix}-e^{ix}-ix\right| \le \frac{5}{2}x^2.$$

Thus

$$\frac{1}{2\pi}\int_{-1}^{1}\left|\frac{e^{2aix}-e^{aix}}{aix}-1\right|\cdot\frac{dx}{|x|} = \frac{1}{2\pi}\int_{-a}^{a}\left|\frac{e^{2ix}-e^{ix}}{ix}-1\right|\cdot\frac{dx}{|x|} \le \frac{1}{2\pi}\int_{-a}^{a}\min\left\{\frac{5}{2},\frac{2}{|x|}\right\}dx$$

$$\le \frac{1}{2\pi}(5 + 4\log a).$$

Finally,

$$\int_{|x|\ge 1} \frac{|h(x)|}{|x|}dx \le \sqrt{2}\|h\|_{L^2} = \frac{1}{\sqrt{\pi}}\|f'\|_{L^2}$$

by the Cauchy–Bunyakovsky inequality and

$$\frac{1}{2\pi a}\int_{\{|x|\ge 1\}} \frac{|e^{aix}-1|}{x^2}dx = \frac{1}{2\pi a}\int_{\{|x|\ge a\}} \frac{|e^{ix}-1|}{x^2}dx \le \frac{2}{\pi a} \le \frac{1}{\pi}$$

for $a \ge 2$. This implies the desired inequality.

Theorem (6.2.14)[260]. Let J be a bounded interval containing 0. Then

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\hat{L}^1(J)} \le \frac{1}{\sqrt[4]{12}}|J| \le \frac{3}{5}|J|. \tag{27}$$

Proof. It suffices to observe that $\left\|\left(\frac{e^x-1}{e^x+1}\right)'\right\|_{L^\infty} = \frac{1}{2}$ and apply Corollary (6.2.12).

Theorem (6.2.14) gives a sufficiently sharp estimate of the $\hat{L}^1$-norm for little intervals J . For big intervals J , this estimate will be improved in Corollary (6.2.17).

Theorem (6.2.15)[260]. Let $a \ge 2$. Then

$$\left\|\frac{e^x}{1+e^x}\right\|_{\hat{L}^1(-\infty,a]} \le 2 + \frac{2}{\pi}\log a.$$

Proof. We have

$$\left\|\left(\frac{e^x}{e^x+1}\right)'\right\|_{L^2}^2 = \int_{\mathbb{R}} \frac{e^{2x}dx}{(e^x+1)^4} = \int_0^\infty \frac{tdt}{(t+1)^4} = \frac{1}{6},$$

and

$$\left\|x\left(\frac{e^x}{e^x+1}\right)'\right\|_{L^2}^2 = 2\int_0^\infty \frac{x^2e^{2x}dx}{(e^x+1)^4} \le 2\int_0^\infty x^2e^{-2x}dx = \frac{1}{2},$$

whence for $a \ge 2$,

$$\left\|\frac{e^x}{1+e^x}\right\|_{\hat{L}^1(-\infty,a]} \le \frac{1}{\sqrt{6\pi}} + \frac{1}{\sqrt{\pi}} + \frac{7}{2\pi} + \frac{2}{\pi}\log a \le 2 + \frac{2}{\pi}\log a$$

by Lemma (6.2.13).

Lemma (6.2.8) implies that

$$\left\|\frac{e^x}{1+e^x}\right\|_{\hat{L}^1(-\infty,a]} \le \frac{e^a}{1+e^a} \le e^a$$

for $a \le 0$ but we do not need this inequality.

Corollary (6.2.16)[260]. Let J be a bounded interval containing 0. Then

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\hat{L}^1(J)} \le 5 + \frac{4}{\pi}\log\left(\frac{1}{2}|J|\right)$$

if $|J| \ge 4$.

Proof. We may assume that the center of J is nonpositive. Then $J \subset (-\infty, \frac{1}{2}|J|]$. We have

$$\left\|\frac{e^x-1}{e^x+1}\right\|_{\hat{L}^1(J)} \le 1+2\left\|\frac{e^x}{e^x+1}\right\|_{\hat{L}^1(J)} \le 5+\frac{4}{\pi}\log a = 5+\frac{4}{\pi}\log\left(\frac{1}{2}|J|\right).$$

We are going to obtain sharp estimates for the Schur multi-plier norms

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}} = \left\|\frac{e^{x-y}-1}{e^{x-y}+1}\right\|_{\mathfrak{M}_{J_1,J_2}} \quad (28)$$

for all intervals J_1 and J_2. First, we consider two special cases. In the first case $J_1 = J_2$ while in the second case J_1 and J_2 do not overlap, i.e., their intersection has at most one point.

Theorem (6.2.17)[260]. Let J_1 and J_2 be nonoverlapping intervals. Then

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}} \le 2.$$

Proof. Clearly, either $J_1-J_2 \subset (-\infty,0]$ or $J_1-J_2 \subset [0,\infty)$. It suffices to consider the case when $J_1-J_2 \subset (-\infty,0]$. Then

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}} \le 1+2\left\|\frac{e^{x-y}}{e^{x-y}+1}\right\|_{\mathfrak{M}_{J_1,J_2}} \le 1+2\left\|\frac{e^x}{e^x+1}\right\|_{\hat{L}^1(-\infty,0]} = 2$$

by the Pólya theorem [272], see also Lemma (6.2.8).

Theorem (6.2.18)[260]. Let J be a bounded interval. Then

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}} \le \min\left\{\frac{6}{5}|J|, 5+\frac{4}{\pi}\log_+|J|\right\}$$

and so

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}} \le 4\log(1+|J|).$$

Proof. We have

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J,J}} \le \left\|\frac{e^x-1}{e^x+1}\right\|_{\hat{L}^1(J-J)}.$$

Note that $|J-J| = 2|J|$ and $0 \in J-J$. The result follows now from (6.2.14) and Corollary (6.2.16).

Theorem (6.2.19)[260]. Let J_1 and J_2 be nonoverlapping intervals and let J be the convex hull of $J_1 \cup J_2$. Then

$$\frac{e-1}{e+1}\min\{1,|J|\} \le \left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}} \le \min\left\{2,\frac{6}{5}|J|\right\}.$$

Proof. The upper estimate follows readily from Theorems (6.2.17) and (6.2.18). Let us prove the lower estimate. We have

$$\left\|\frac{e^x-e^y}{e^x+e^y}\right\|_{\mathfrak{M}_{J_1,J_2}} \ge \sup_{x\in J_1, y\in J_2}\left|\frac{e^x-e^y}{e^x+e^y}\right| \ge \frac{e^{|J|}-1}{e^{|J|}+1} \ge \frac{e-1}{e+1}\min\{1,|J|\}$$

because the function $t \mapsto \frac{e^t-1}{t(e^t+1)}$ decreases on $[0,\infty)$, while the function $t \mapsto \frac{e^t-1}{e^t+1}$ increases.

Theorem (6.2.20)[260]. Let J be a bounded interval. Then

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}} \geq \frac{1}{7}\min\{|J|, 1 + \log_+|J|\}.$$

Proof. Put $Q_\varepsilon(t) \stackrel{\text{def}}{=} \frac{1}{\pi}\frac{t}{t^2+\varepsilon^2}$, where $\varepsilon > 0$. Let us consider the convolution operator C_{Q_ε} on $L^2(\mathbb{R})$, $C_{Q_\varepsilon} f \stackrel{\text{def}}{=} f * Q_\varepsilon$. Clearly, $\|C_{Q_\varepsilon}\| = \|\mathcal{F}Q_\varepsilon\|_{L^\infty} = 1$, see, for example, [273, Chapter III,§1]. Note that C_{Q_ε} is an integral operator with kernel $Q_\varepsilon(x-y)$. We can define the integral operator $X_{J,\varepsilon}$ on $L^2(J)$ with kernel

$$\frac{1}{\pi}\frac{x-y}{(x-y)^2+\varepsilon^2}\frac{e^x - e^y}{e^x + e^y}.$$

We have

$$|J| \cdot \|X_{J,\varepsilon}\| \geq (X_{J,\varepsilon}\chi_J, \chi_J) = \frac{1}{\pi}\iint\limits_{J\times J}\frac{x-y}{(x-y)^2+\varepsilon^2}\frac{e^x - e^y}{e^x + e^y}dxdy$$

$$= \frac{1}{\pi}\int_0^{|J|}\frac{t}{t^2+\varepsilon^2}\frac{e^t-1}{e^t+1}(|J|-t)dt$$

and

$$\|X_{J,\varepsilon}\| \geq \|C_{Q_\varepsilon}\| \cdot \left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}} = \left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}}.$$

Hence,

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}} \geq \frac{2}{\pi}\cdot\frac{1}{|J|}\int_0^{|J|}\frac{t}{t^2+\varepsilon^2}\frac{e^t-1}{e^t+1}(|J|-t)dt$$

for every $\varepsilon > 0$, whence

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}} \geq \frac{2}{\pi}\int_0^{|J|}\frac{e^t-1}{t(e^t+1)}\left(1-\frac{t}{|J|}\right)dt \geq \frac{1}{\pi}\int_0^{|J|}\frac{e^t-1}{t(e^t+1)}dt$$

because the function $t \mapsto \frac{e^t-1}{t(e^t+1)}$ decreases on $(0,\infty)$. It follows that

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{J,J}} \geq \frac{1}{\pi}\cdot\frac{e-1}{e+1}\int_0^{|J|}\min\{1, t^{-1}\}\,dt.$$

This implies the desired estimate.

Theorem (6.2.21)[260]. There exists a positive number C such that

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{[a,\infty),(-\infty,b]}} \leq C\log(2 + (b-a)_+)$$

for all $a, b \in \mathbb{R}$.

Proof. The result follows from Theorems (6.2.17) if $a \geq b$. If $a < b$, then

$$[a,\infty) \times (-\infty, b] = ([a,b] \times [a,b]) \cup ([a,b] \times (-\infty, a]) \cup ([b,\infty) \times (-\infty, b]) ,$$

and we can apply Theorem (6.2.18)to the first rectangle and Theorem (6.2.17) to the remaining rectangles.

Theorem (6.2.22)[260]. There exists a positive number C such that

$$\left\|\frac{e^x - e^y}{e^x + e^y}\right\|_{\mathfrak{M}_{\mathbb{R}[a,b]}} \leq C \log(2 + b - a)$$

for all $a, b \in \mathbb{R}$ satisfying $a < b$.

Proof. We have

$$\mathbb{R} \times [a, b] = ([a, b] \times [a, b]) \cup \big((-\infty, a] \times [a, b]\big) \cup ([b, \infty) \times [a, b]).$$

It remains to apply Theorem (6.2.18) to the first rectangle and Theorem (6.2.17) to the remaining rectangles.

Theorem (6.2.23)[260]. There exists a positive number c such that

$$\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[a,\infty),[0,b]}} \leq c \log\left(2 + \log_+ \frac{b}{a}\right)$$

for all $a, b \in (0, \infty)$.

Proof. Theorem (6.2.21) with the help of the change of variables $x \mapsto \log x$ and $y \mapsto \log y$ yields

$$\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[a,\infty),[\varepsilon,b+\varepsilon]}} \leq c \log\left(2 + \log_+ \frac{b + \varepsilon}{a}\right)$$

for every $\varepsilon > 0$, whence

$$\left\|\frac{x - y - \varepsilon}{x + y + \varepsilon}\right\|_{\mathfrak{M}_{[a,\infty),[0,b]}} \leq c \log\left(2 + \log_+ \frac{b + \varepsilon}{a}\right)$$

for every $\varepsilon > 0$. It remains to pass to the limit as $\varepsilon \to 0$.

Theorem (6.2.24)[260]. There exists a positive number c such that

$$\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[a,b],[0,\infty)}} \leq c \log\left(2 + \log \frac{b}{a}\right)$$

whenever $a, b \in (0, \infty)$ and $a < b$.

Proof. The result follows from Theorem (6.2.22) in the same way as Theorem (6.2.23) follows from Theorem (6.2.21).

Theorem (6.2.25)[260]. There exists a positive number c such that

$$\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[a,b],[a,b]}} \leq c \log\left(1 + \log \frac{b}{a}\right)$$

whenever $a, b \in (0, \infty)$ and $a < b$.

Proof. The result follows from Theorem (6.2.20) with the help of the change of variables $x \mapsto \log x$ and $y \mapsto \log y$.

We study operator Lipschitz functions on closed subsets of the real line. It is well known that a function f on $\mathbb{R}$ is operator Lipschitz if and only if it is commutator Lipschitz, i.e.,

$$\|f(A)R - Rf(A)\| \leq \text{const}\|AR - RA\|$$

for an arbitrary bounded operator R and an arbitrary self-adjoint operator A.

The same is true for functions on closed subsets of $\mathbb{R}$; moreover the operator Lipschitz constant coincides with the commutator Lipschitz constant. The following theorem was proved in [2, Theorem 10.1] in the case $\mathfrak{F} = \mathbb{R}$. The general case is analogous to the case $\mathfrak{F} = \mathbb{R}$. See also [274] where similar results for symmetric ideal norms are considered.

Theorem (6.2.26)[260]. Let f be a continuous function defined on a closed subset $\mathfrak{F}$ of $\mathbb{R}$ and let $C \geq 0$. The following are equivalent:

(i) $\|f(A) - f(B)\| \leq C\|A - B\|$ for arbitrary self-adjoint operators A and B with spectra in $\mathfrak{F}$;

(ii) $\|f(A)R - Rf(A)\| \leq C\|AR - RA\|$ for all self-adjoint operators A with $\sigma(A) \subset \mathfrak{F}$ and all bounded operators R;

(iii) $\|f(A)R - Rf(B)\| \leq C\|AR - RB\|$ for arbitrary self-adjoint operators A and B with spectra in $\mathfrak{F}$ and for an arbitrary bounded operator R.

A function $f \in C(\mathfrak{F})$ is said to be operator Lipschitz if it satisfies the equivalent statements of Theorem (6.2.26). We denote the set of operator Lipschitz functions on $\mathfrak{F}$ by OL($\mathfrak{F}$). For $f \in$ OL($\mathfrak{F}$), we define $\|f\|_{\mathrm{OL}(\mathfrak{F})}$ to be the smallest constant satisfying the equivalent statements of Theorem (6.2.26). Put $\|f\|_{\mathrm{OL}(\mathfrak{F})} = \infty$ if $f \notin$ OL($\mathfrak{F}$).

It is well known that every f in OL($\mathfrak{F}$) is differentiable at every nonisolated point of $\mathfrak{F}$, see [275]. Moreover, the same argument gives differentiability at ∞ in the following sense: there exists a finite limit $\lim_{|x|\to+\infty} x^{-1} f(x)$ provided $\mathfrak{F}$ is unbounded.

Let $f \in$ OL($\mathfrak{F}$). Suppose that $\mathfrak{F}$ has no isolated points. Put

$$(\mathfrak{D}f)(x,y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(x) - f(y)}{x - y}, & \text{if } x, y \in \mathfrak{F}, x \neq y, \\ f'(x), & \text{if } x \in \mathfrak{F}, x = y. \end{cases}$$

The following equality holds:

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} = \|\mathfrak{D}f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}}. \tag{29}$$

The inequality $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leq \|\mathfrak{D}f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}}$ is an immediate consequence of the formula

$$f(A) - f(B) = \iint (\mathfrak{D}f)(x,y)dE_A(x)(A - B)dE_B(y), \tag{30}$$

where A and B are self-adjoint operators with bounded $A - B$ whose spectra are in $\mathfrak{F}$, and E_A and E_B are the spectral measures of A and B. The expression on the right is called a double operator integral. We refer the reader to [276-278] for the theory of double operator integrals elaborated by Birman and Solomyak. The validity of formula (30) under the assumption $\mathfrak{D}f \in \mathfrak{M}_{\mathfrak{F},\mathfrak{F}}$ and the inequality

$$\left\|\iint (\mathfrak{D}f)(x,y)dE_A(x)(A - B)dE_B(y)\right\| \leq \|\mathfrak{D}\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}}\|A - B\|$$

was proved in [6]. The opposite inequality in (29) is going to be proved in Corollary (6.2.29).

In the general case for $f \in$ OL($\mathfrak{F}$) we can define the function

$$(\mathfrak{D}_0 f)(x,y) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(x)-f(y)}{x-y}, & \text{if } x,y \in \mathfrak{F}, x \neq y, \\ 0, & \text{if } x \in \mathfrak{F}, x = y. \end{cases}$$

The following inequalities hold:

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \le \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \le 2\|f\|_{\mathrm{OL}(\mathfrak{F})}. \quad (31)$$

The first inequality in (31) follows from the formula

$$f(A)-f(B) = \iint (\mathfrak{D}_0 f)\,(x,y)dE_A(x)(A-B)dE_B(y), \quad (32)$$

whose validity can be verified in the same way as the validity of (30). The second inequality in (31) is going to be verified in Corollary (6.2.30).

Let f be a continuous function on a closed set $\mathfrak{F}, \mathfrak{F} \subset \mathbb{R}$. We define the operator modulus of continuity $\Omega_{f,\mathfrak{F}}$ as follows

$$\Omega_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A)-f(B)\|: A = A^*, B = B^*, \sigma(A), \sigma(B) \subset \mathfrak{F}, \|A-B\| \le \delta\},$$

and the commutator modulus of continuity as follows

$$\Omega^{\flat}_{f,\mathfrak{F}}(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A)-f(B)\|: A = A^*, B = B^*, \sigma(A), \sigma(B) \subset \mathfrak{F}, \|A-B\| \le \delta\},$$

One can prove that we get the same right-hand side if we require in addition that R is self-adjoint.

On the other hand, $\|f(A)R - Rf(B)\| \le \Omega^{\flat}_{f,\mathfrak{F}}(\|AR-RB\|)$ for all self-adjoint operators with $\sigma(A), \sigma(B) \subset \mathfrak{F}$ and for every bounded operator R with $\|R\| \le 1$. Also, $\Omega_{f,\mathfrak{F}} \le \Omega^{\flat}_{f,\mathfrak{F}} \le 2\Omega_{f,\mathfrak{F}}$.

These results were obtained in [261] in the case $\mathfrak{F} = \mathbb{R}$. The same reasoning works in the general case.

Lemma (6.2.27)[260]. Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ and let μ and ν be regular positive Borel measures on $\mathfrak{F}$. Suppose that k is a function in $L^2(\mathfrak{F} \times \mathfrak{F}, \mu \otimes \nu)$ such that $k = 0$ on the diagonal $\Delta_{\mathfrak{F}} \stackrel{\text{def}}{=} \{(x,x): x \in \mathfrak{F}\}$almost everywhere with respect to $\mu \otimes \nu$. Then

$$\|k\mathfrak{D}_0 f\|_{\mathfrak{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathfrak{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}}$$

for every continuous function f on $\mathfrak{F}$.

Proof. Let $\mathfrak{F}_n \stackrel{\text{def}}{=} \mathfrak{F} \cap [-n,n]$, and let μ_n and ν_n be the restrictions of μ and ν to $\mathfrak{F}_n$. Clearly,

$$\lim_{n\to\infty} \|k\|_{\mathfrak{B}^{\mu,\nu}_{\mathfrak{F}_n,\mathfrak{F}_n}} = \|k\|_{\mathfrak{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}} \quad \text{for every } k \in L^2(\mathfrak{F} \times \mathfrak{F}, \mu \otimes \nu)$$

and

$$\lim_{n\to\infty} \|f\|_{\mathrm{OL}(\mathfrak{F}_n)} = \|f\|_{\mathrm{OL}(\mathfrak{F})} \quad \text{for every } f \in C(\mathfrak{F}).$$

Thus we may assume that $\mathfrak{F}$ is compact. It suffices to consider the case when k vanishes in a neighborhood of the diagonal $\Delta_{\mathfrak{F}}$. Put$l(x,y) \stackrel{\text{def}}{=} (x-y)^{-1}1k(x,y)$. Denote by A and B multiplications by the independent variable on $L^2(\mathfrak{F},\mu)$ and $L^2(\mathfrak{F},\nu)$. Then $\mathfrak{T}^{\mu,\nu}_k = A\mathfrak{T}^{\mu,\nu}_l - \mathfrak{T}^{\mu,\nu}_l B$ and$\mathfrak{T}^{\mu,\nu}_{k\mathfrak{D}_0 f} = f(A)\mathfrak{T}^{\mu,\nu}_l - \mathfrak{T}^{\mu,\nu}_l f(B)$. It remains to observe that

$$\left\|f(A)\mathfrak{T}^{\mu,\nu}_l - \mathfrak{T}^{\mu,\nu}_l f(B)\right\| \le \|f\|_{\mathrm{OL}(\mathfrak{F})}\left\|A\mathfrak{T}^{\mu,\nu}_l - \mathfrak{T}^{\mu,\nu}_l B\right\|,$$

$$\left\|A\mathfrak{T}^{\mu,\nu}_l - \mathfrak{T}^{\mu,\nu}_l B\right\| = \|k\|_{\mathfrak{B}^{\mu,\nu}_{\mathfrak{F},\mathfrak{F}}},$$

and

$$\left\|f(A)\mathfrak{T}_l^{\mu,\nu}-\mathfrak{T}_l^{\mu,\nu}f(B)\right\| = \|k\mathfrak{D}_0 f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}.$$

Corollary (6.2.28)[260]. Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ with no isolated points, and let μ and ν be finite positive Borel measures on $\mathfrak{F}$. Suppose that f is a differentiable function on $\mathfrak{F}$ and $k \in L^2(\mathfrak{F}\times\mathfrak{F},\mu\otimes\nu)$ If k vanishes $\mu\otimes\nu$-almost every-where on the diagonal $\Delta_{\mathfrak{F}} \overset{\text{def}}{=} \{(x,x): x\in\mathfrak{F}\}$, then

$$\|k\mathfrak{D}f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}.$$

Corollary (6.2.29)[260]. Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$ with no isolated points, and let μ and ν be finite positive Borel measures on $\mathfrak{F}$. If f is a differentiable function on $\mathfrak{F}$, then

$$\|\mathfrak{D}f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}.$$

Proof. Let μ be a regular Borel measure on $\mathfrak{F}$ with no atoms and such that supp $\mu=\mathfrak{F}$. Then $(\mu\otimes\mu)(\Delta_{\mathfrak{F}})=0$ and Corollary (6.2.28) implies that

$$\|k\mathfrak{D}f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}}$$

for all $k\in L^2(\mathfrak{F}\times\mathfrak{F},\mu\otimes\mu)$. Hence, $\|\mathfrak{D}f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\mu}} \le \|f\|_{\mathrm{OL}(\mathfrak{F})}$. It remains to apply Theorem (6.2.6).

Corollary (6.2.30)[260]. Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$. Then

$$\|\mathfrak{D}_0 f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}} \le 2\|f\|_{\mathrm{OL}(\mathfrak{F})}$$

for every $f\in C(\mathfrak{F})$.

Proof. Let μ and ν be regular Borel measures on $\mathfrak{F}$. We have to verify that

$$\|k\mathfrak{D}_0 f\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \le 2\|f\|_{\mathrm{OL}(\mathfrak{F})}\|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}$$

for every $k\in L^2(\mathfrak{F}\times\mathfrak{F},\mu\otimes\nu)$. Put $k_0 \overset{\text{def}}{=} \chi_{\Delta_{\mathfrak{F}}}k$ and $k_1 \overset{\text{def}}{=} k-k_0$. We have

$$\|k_0\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \le \|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}.$$

This inequality can be verified easily. We leave the verification to the reader.

It follows that$\|k_1\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \le \|k_1\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} + \|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}} \le 2\|k\|_{\mathfrak{B}_{\mathfrak{F},\mathfrak{F}}^{\mu,\nu}}$. It remains to observe that

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be closed subsets of $\mathbb{R}$. We define the space $\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)$ as the space of functions f in $C(\mathfrak{F}_1\cup\mathfrak{F}_2)$ such that

$$\|f(A)R-Rf(B)\| \le C\|AR-RB\| \tag{33}$$

for all bounded operator R and all self-adjoint operators A and B such that $\sigma(A)\subset\mathfrak{F}_1$ and $\sigma(B)\subset\mathfrak{F}_2$ with some positive number C . Denote by $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)}$ the minimal constant satisfying (33). Clearly, $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} = \|f\|_{\mathrm{OL}(\mathfrak{F}_2,\mathfrak{F}_1)}$ and $\|f\|_{\mathrm{OL}(\mathfrak{F},\mathfrak{F})} = \|f\|_{\mathrm{OL}(\mathfrak{F})}$. As in the case $\mathfrak{F}_1=\mathfrak{F}_2$, we can prove that

$$\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \le \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}} \le 2\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \tag{34}$$

In the case when $\mathfrak{F}_1=\mathfrak{F}_2$ we cannot claim that the inequality

$$\|f(A)-f(B)\| \le C\|A-B\| \tag{35}$$

for all self-adjoint A and B such that $\sigma(A)\subset\mathfrak{F}_1$ and $\sigma(B)\subset\mathfrak{F}_2$ implies (33).

Indeed, in the case $f(t)=|t|$, $\mathfrak{F}_1=(-\infty,0]$, and $\mathfrak{F}_2=[0,\infty)$, inequality (35) holds with $C=1$ because

$$\|A - B\| \le \|A + B\|$$

for positive self-adjoint operators A and B . However, inequality (33) does not hold with any positive C. Indeed,

$$\left\| \frac{|x| - |y|}{x - y} \right\|_{\mathfrak{M}_{(-\infty,1],[1,\infty)}} = \left\| \frac{x - y}{x + y} \right\|_{\mathfrak{M}_{[1,\infty),[1,\infty)}} = \infty$$

by Theorem (6.2.25).

Theorem (6.2.31)[260]. Suppose that inequality (33) holds for every bounded operator R and arbitrary self-adjoint operators A and B with simple spectra such that σ $\sigma(A) \subset \mathfrak{F}_1$ and $(B) \subset \mathfrak{F}_2$. Then $f \in \mathrm{OL}(\mathfrak{F}_1, \mathfrak{F}_2)$ and $\|f\|_{\mathrm{OL}(\mathfrak{F}_1,\mathfrak{F}_2)} \le C$.

Proof. We have to prove inequality (33) for arbitrary self-adjoint operators A and B with $\sigma(A) \subset \mathfrak{F}_1$ and $(B) \subset \mathfrak{F}_2$. It is convenient to think that the operators A and B act in different Hilbert spaces. Let A act in $\mathcal{H}_1$ and B in $\mathcal{H}_2$. Then R acts from $\mathcal{H}_1$ into $\mathcal{H}_2$. We are going to verify that

$$|(f(A)Ru, v) - (Rf(B)u, v)| = \left|\left(Ru, \bar{f}(A)v\right) - (f(B)u, R^* v)\right| \le C\|AR - RB\|$$

for all unit vectors $u \in \mathcal{H}_2$ and $v \in \mathcal{H}_1$. Denote by $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ the invariant subspaces of A and B generated by v and u. Clearly, $A_0 \stackrel{\text{def}}{=} A|\mathcal{H}_1^0$ and $B_0 \stackrel{\text{def}}{=} B|\mathcal{H}_2^0$ are self-adjoint operators with simple spectra. Consider the operator $R_0 : \mathcal{H}_2^0 \to \mathcal{H}_1^0$, $R_0 h \stackrel{\text{def}}{=} PRh$ for $h \in \mathcal{H}_2$, where P is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}_1^0$. Note that for $h \in \mathcal{H}_2^0$, we have $A_0 R_0 h = APRh = PARh$ and $R_0 B_0 h = PRBh$. Clearly, $\|A_0 R_0 - A_0 B_0\| \le \|AR - RB\|$. Applying (33) to the operators A_0,B_0, and R_0, we obtain

$$\begin{aligned} |(f(A)Ru, v) - (Rf(B)u, v)| &= \left|\left(Ru, \bar{f}(A)v\right) - (Rf(B)u, v)\right| \\ &= \left|\left(R_0 u, \bar{f}(A_0)v\right) - (R_0 f(B_0)u, v)\right| = |(f(A_0)R_0 u, v) - (R_0 f(B_0)u, v)| \\ &\le \|A_0 R_0 - A_0 B_0\| \le C\|AR - RB\|. \end{aligned}$$

Theorem (6.2.32)[260]. Let f be a function defined on $\mathbb{Z}$. Then

$$\Omega^{\mathfrak{b}}_{f,\mathbb{Z}}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{Z})}$$

for $\delta \in (0, \frac{2}{\pi}]$.

Proof. The inequality

$$\Omega^{\mathfrak{b}}_{f,\mathbb{Z}}(\delta) \le \delta\|f\|_{\mathrm{OL}(\mathbb{Z})} , \quad \delta > 0,$$

is a consequence of Theorem (6.2.26). Let us prove the opposite inequality for $\delta \in (0, \frac{2}{\pi}]$. Fix $\varepsilon > 0$. There exists a self-adjoint operator A and a bounded operator R such that $\|AR - RA\| = 1$,$\sigma(A) \subset \mathbb{Z}$, and $\|f(A)R - Rf(A)\| \ge \|f\|_{\mathrm{OL}(\mathbb{Z})} - \varepsilon$. Put

$$R_A \stackrel{\text{def}}{=} \sum_{j \ne k} E_A(\{j\})\, R E_A(\{k\}) = R - \sum_{j \in \mathbb{Z}} E_A(\{j\})\, R E_A(\{j\}).$$

Clearly, $AR - RA = AR_A - R_A A$ and $f(A)R - Rf(A) = f(A)R_A - R_A f(A)$. Thus we may assume that $R = R_A$. Note that

$$AR - RA = \sum_{j \ne k} (j - k) E_A(\{j\}) R E_A(\{k\}).$$

Since

$$R = R_A = \sum_{j \neq k} \frac{1}{j-k}(j-k)E_A(\{j\})RE_A(\{k\}),$$

we have $R = H \star (AR - RA)$, where

$$H(j,k) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{j-k}, & \text{if } j \neq k \\ 0, & \text{if } j = k, \end{cases}$$

where $\star$ denotes Schur–Hadamard multiplication, see (26). It follows that

$$\|R\| \leq \|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}\|AR - RA\| = \|H\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = \frac{\pi}{2}$$

by Lemma (6.2.7).

Let $\delta \in (0,2]$. Then $\|A(\delta R) - (\delta R)A\| = \delta$ and $\|\delta R\| \leq 1$. Hence,

$$\Omega^{\mathfrak{b}}_{f,\mathbb{Z}}(\delta) \geq \|f(A)R - Rf(A)\| \geq \delta\left(\|f\|_{\mathrm{OL}(\mathbb{Z})} - \varepsilon\right).$$

Passing to the limit as $\varepsilon \to 0$, we obtain the desired result.

Let $\omega_{f,\mathfrak{F}}$ denote the usual scalar modulus of continuity of a continuous function f defined on $\mathfrak{F}$. Clearly, $\omega_{f,\mathfrak{F}} \leq \Omega_{f,\mathfrak{F}}$. Put $\omega_f \stackrel{\text{def}}{=} \omega_{f,\mathbb{R}}$ and $\Omega_f \stackrel{\text{def}}{=} \Omega_{f,\mathbb{R}}$. We are going to get some estimates for the commutator modulus of conti-nuity $\Omega^{\mathfrak{b}}_{f,\mathbb{Z}}$. We consider first the case when $\mathfrak{F} = \mathbb{R}$. The following theorem is contained implicitly in [270].

Theorem (6.2.33)[260]. Let f be a continuous function on $\mathbb{R}$. Then

$$\Omega^{\mathfrak{b}}_{f}(\delta) \leq 2\omega_f(\delta/2) + 2\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})}.$$

Proof. Let $\|AR - RA\| \leq \delta$ with $\|R\| = 1$. We can take a self-adjoint operator A_δ such that $A_\delta A = AA_\delta$, $\|A - A_\delta\| \leq \delta/2$ and $\sigma(A_\delta) \subset \delta\mathbb{Z}$. Then $\|f(A) - f(A_\delta)\| \leq \omega_f(\delta/2)$ and

$$\|A_\delta R - RA_\delta\| \leq \|A_\delta R - AR\| + \|AR - AR\| + \|RA - RA_\delta\| \leq 2\delta.$$

Hence,

$$\begin{aligned}\|f(A)R - Rf(A)\| &\leq \|f(A)R - f(A_\delta)R\| + \|f(A_\delta)R - Rf(A_\delta)\| + \|Rf(A_\delta) - Rf(A)\| \\ &\leq 2\omega_f\left(\frac{\delta}{2}\right) + \|A_\delta R - RA_\delta\| \cdot \|f\|_{\mathrm{OL}(\delta\mathbb{Z})} \leq 2\omega_f\left(\frac{\delta}{2}\right) + 2\delta\|f\|_{\mathrm{OL}(\delta\mathbb{Z})} \\ &= 2\omega_f\left(\frac{\delta}{2}\right) + 2\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})}.\end{aligned}$$

Theorem (6.2.34)[260]. Let f be a continuous function on $\mathbb{R}$. Then

$$\Omega^{\mathfrak{b}}_{f}(\delta) \geq \max\left\{\omega_f(\delta), \frac{2}{\pi}\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})}\right\}$$

for all $\delta > 0$.

Proof. Clearly, $\omega_f \leq \Omega_f \leq \Omega^{\mathfrak{b}}_f$. It remains to prove that $\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})} \leq \frac{2}{\pi}\Omega^{\mathfrak{b}}_f(\delta)$. We have

$$\Omega^{\mathfrak{b}}_{f}(\delta) \geq \Omega^{\mathfrak{b}}_{f,\delta\mathbb{Z}}(\delta) = \Omega^{\mathfrak{b}}_{f(\delta x),\mathbb{Z}}(1) \geq \Omega^{\mathfrak{b}}_{f(\delta x),\mathbb{Z}}\left(\frac{2}{\pi}\right) = \frac{2}{\pi}\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})}$$

We consider now similar estimates of $\Omega^{\mathfrak{b}}_{f,\mathfrak{F}}$ for an arbitrary closed subset $\mathfrak{F}$ of $\mathbb{R}$. Recall that a subset Λ of $\mathbb{R}$ is called a δ-net for $\mathfrak{F}$ if $\mathfrak{F} \subset \cup_{t\in\Lambda}[t - \delta, t + \delta]$.

Theorem (6.2.35)[260]. Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $\mathfrak{F}_\delta$ is a subset of F that forms a $(\delta/2)$-net of $\mathfrak{F}$. Then

$$\Omega^{\mathrm{b}}_{f,\mathfrak{F}}(\delta) \le 2\omega_{f,\mathfrak{F}}(\delta/2) + 2\delta\|f\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

Theorem (6.2.36)[260]. Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$ and let $\delta > 0$. Suppose that Λ and M are closed subsets of $\mathfrak{F}$ such that $(\Lambda - M) \cap (-\delta,\delta) \subset \{0\}$. Then

$$\Omega^{\mathrm{b}}_{f,\mathfrak{F}}(\delta) \ge \max\left\{\omega_{f,\mathfrak{F}}(\delta), \frac{\delta}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}}\right\}.$$

Proof. Clearly, $\omega_{f,\mathfrak{F}} \le \Omega_{f,\mathfrak{F}} \le \Omega^{\mathrm{b}}_{f,\mathfrak{F}}$. Note that

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}} = \sup_{a>0}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda\cap[-a,a],M\cap[-a,a]}}.$$

Thus it suffices to prove that

$$\Omega^{\mathrm{b}}_{f,\mathfrak{F}}(\delta) \ge \frac{\delta}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}}$$

in the case when Λ and M are bounded.

Let $\varepsilon > 0$. There exist positive regular Borel measures λ on Λ, μ on M, and a function k in $L^2(\Lambda \times M, \lambda \otimes \mu)$ such that $\|k\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} = 1$ and $\|k\mathfrak{D}_0 f\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \ge \|k\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}} - \varepsilon$. We define the function k_0 in $L^2(\Lambda \times M, \lambda \otimes \mu)$ by

$$k_0(x,y) \overset{\mathrm{def}}{=} \begin{cases} k(x,y), & \text{if } x \ne y \\ 0, & \text{if } x = y. \end{cases}$$

Then $k\mathfrak{D}_0 f = k_0\mathfrak{D}_0 f$ and $\|k_0\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le 2$. Put $\Phi(x,y) \overset{\mathrm{def}}{=} f_\delta(x-y)$ where f_δ denotes the same as in Corollary (6.2.10). We define the self-adjoint operators $A: L^2(\Lambda,\lambda) \to L^2(\Lambda,\lambda)$ and $B: L^2(M,\mu) \to L^2(M,\mu)$ by $(Af)(x) \overset{\mathrm{def}}{=} xf(x)$ and $(Bg)(y) \overset{\mathrm{def}}{=} yg(y)$. Put

$$h(x,y) \overset{\mathrm{def}}{=} \Phi(x,y)k(x,y) = \Phi(x,y)k_0(x,y).$$

Clearly,

$$\|h\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le \|\Phi\|_{\mathfrak{M}^{\lambda,\mu}_{\Lambda,M}}\|k\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le \|\Phi\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \le \frac{2}{\delta}$$

by Corollary (6.2.10).

Clearly, $A\mathfrak{T}_h - \mathfrak{T}_h B = \mathfrak{T}_{k_0}$ and $(A)\mathfrak{T}_h - \mathfrak{T}_h f(B) = \mathfrak{T}_{k_0\mathfrak{D}_0 f}$. (Recall that $\mathfrak{T}_\varphi$ is the integral operator from $L^2(M,\mu)$ into $L^2(\Lambda,\lambda)$ with kernel $\varphi \in L^2(\Lambda \times M, \lambda \otimes \nu)$.) Then

$$\left\|\frac{\delta}{2}\mathfrak{T}_h\right\| = \frac{\delta}{2}\|h\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le 1,$$

$$\left\|A\left(\frac{\delta}{2}\mathfrak{T}_h\right) - \left(\frac{\delta}{2}\mathfrak{T}_h\right)B\right\| = \frac{\delta}{2}\|k_0\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le \delta,$$

and

$$\left\|f(A)\left(\frac{\delta}{2}\mathfrak{T}_h\right) - \left(\frac{\delta}{2}\mathfrak{T}_h\right)f(B)\right\| = \frac{\delta}{2}\|k_0\mathfrak{D}_0 f\|_{\mathfrak{B}^{\lambda,\mu}_{\Lambda,M}} \le \frac{\delta}{2}\left(\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}} - \varepsilon\right).$$

Hence, $\Omega^{\mathrm{b}}_{f,\mathfrak{F}}(\delta) \ge \frac{\delta}{2}\left(\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,M}} - \varepsilon\right)$ for every $\varepsilon > 0$.

Theorem (6.2.36) allows us to obtain another proof of Theorem 4.17 in [263].

Theorem (6.2.37)[260]. Let f be a continuous function on an unbounded closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $\Omega_{f,\mathfrak{F}}(\delta) < \infty$ for $\delta > 0$. Then the function $t \mapsto t^{-1}f(t)$ has a finite limit as $|t| \to \infty, t \in \mathfrak{F}$.

Proof. Assume the contrary. Then there exists a sequence $\{\lambda\}_{n=1}^{\infty}$in $\mathfrak{F}$ such that $|\lambda_{n+1}| - |\lambda_n| > 1$ for all$n \geq 1$, $\lim_{n\to\infty} |\lambda_n| = \infty$ and the sequence $\{\lambda_n^{-1} f(\lambda n)\}_{n=1}^{\infty}$has no finite limit. Denoten by Λ the image of the sequence $\{\lambda\}_{n=1}^{\infty}$. Then $\|f\|_{\mathrm{OL}(\Lambda)} = \infty$. This fact is contained implicitly in [275]. Indeed, Theorem 4.1 in [275] implies that every operator Lipschitz function f is differentiable at every nonisolated point. It is well known that the same argument gives us the differentiability at ∞ in the following sense: the function $t \mapsto t^{-1}f(t)$ has a finite limit as $|t| \to \infty$, provided the domain of f is unbounded. Applying Theorem (6.2.36) for $M = \Lambda$ and $\delta = 1$, we find that $\Omega_{f,\mathfrak{F}}(1) = \infty$.

We need the following known result, see [279].

Theorem (6.2.38)[260]. Let f be a bounded continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Suppose that $f \in \mathrm{OL}\big((-\infty,1] \cap \mathfrak{F}\big)andf \in \mathrm{OL}([-1,\infty) \cap \mathfrak{F})$.Then $f \in \mathrm{OL}(\mathfrak{F})$ and

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leq C\left(\|f\|_{\mathrm{OL}((-\infty,1]\cap\mathfrak{F})} + \|f\|_{\mathrm{OL}([-1,\infty)\cap\mathfrak{F})} + \sup_{\mathfrak{F}}|f|\right),$$

where C is a numerical constant.

Proof. Put $\mathfrak{F}_1 \stackrel{\text{def}}{=} \mathfrak{F} \cap (-\infty,-1]$, $\mathfrak{F}_2 \stackrel{\text{def}}{=} \mathfrak{F} \cap [-1,1]$, and $\mathfrak{F}_3 \stackrel{\text{def}}{=} \mathfrak{F} \cap [1,\infty)$. We have

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leq \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F},\mathfrak{F}}} \leq \sum_{j=1}^{3}\sum_{k=1}^{3}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_k}}$$

$$= \sum_{j=1}^{3}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_j}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_2,\mathfrak{F}_3}} + 2\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}.$$

Each term $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_j,\mathfrak{F}_k}}$ except $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}$ can be estimated in terms of $2\|f\|_{\mathrm{OL}(\mathfrak{F}_1\cup\mathfrak{F}_2)}$ or $2\|f\|_{\mathrm{OL}(\mathfrak{F}_2\cup\mathfrak{F}_3)}$.

Let us estimate $\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}$. We have

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} = \left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \leq \left(\sup_{\mathfrak{F}_1}|f|\right)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} + \left(\sup_{\mathfrak{F}_3}|f|\right)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}}$$

$$\leq 2\left(\sup_{\mathfrak{F}}|f|\right)\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \leq 2\sup_{\mathfrak{F}}|f|$$

because by Corollary (6.2.10),

$$\left\|\frac{1}{x-y}\right\|_{\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_3}} \leq \|f_2(x-y)\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \leq 1,$$

where $f_2$means the same as in Corollary (6.2.10).

Thus

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leq 6\|f\|_{\mathrm{OL}(\mathfrak{F}_1\cup\mathfrak{F}_2)} + 4\|f\|_{\mathrm{OL}(\mathfrak{F}_2\cup\mathfrak{F}_3)} + 4\sup_{\mathfrak{F}}|f|$$

We obtain sharp estimates of the operator modulus of continuity of the function $x \mapsto |x|$ on certain subsets of the real line. This allows us to obtain sharp estimates of $\||S| - |T|\|$ for arbitrary bounded linear operators S and T . Note that our estimates considerably improve earlier results of [18].

Put Abs$(x) \stackrel{\text{def}}{=} |x|$. For $J \subset [0,\infty)$, we put $\log(J) \stackrel{\text{def}}{=} \{\log t : t \in J, t > 0\}$.

Theorem (6.2.39)[260]. There exist positive numbers C_1 and C_2 such that

$$C_1 \log(2 + |\log(J_1 \cap J_2)|) \le \|\text{Abs}\|_{\text{OL}((-J_1)\cup J_2)} \le C_2 \log(2 + |\log(J_1 \cap J_2)|)$$

for all intervals J_1 and J_2 in $(0,\infty)$.

Proof. Put $J = J_1 \cap J_2$. Let us first establish the lower estimate. Note that $\|\text{Abs}\|_{\text{OL}((-J_1)\cup J_2)} \ge \|\text{Abs}\|_{\text{OL}(J_2)} = 1$. This proves the lower estimate in the case $|\log(J)| \le 1$. In the case $|\log(J)| > 1$ we have

$$\|\text{Abs}\|_{\text{OL}((-J_1)\cup J_2)} \ge \|\text{Abs}\|_{\text{OL}((-J)\cup J)} \ge \left\|\frac{|x| - |y|}{x - y}\right\|_{\mathfrak{M}_{-J,J}} = \left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{-J,J}} \ge c_1 \log(1 + |\log(J)|)$$
$$\ge c_2 \log(2 + |\log(J)|)$$

by Theorem (6.2.25).

We proceed now to the upper estimate. We consider first the case when $J = J_1$. Then

$$\|\text{Abs}\|_{\text{OL}((-J_1)\cup J_2)} \le \|\text{Abs}\|_{\text{OL}((-J_1)\cup[0,\infty))} \le 2 + 2\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{J,[0,\infty)}}$$

and we can apply Theorem (6.2.24). The case $J = J_2$ is similar. Suppose that $J \ne J_1$ and $J \ne J_2$ .Then $\inf J_1 \ne \inf J_2$. Let $\inf J_1 > \inf J_2$. Put $a \stackrel{\text{def}}{=} \inf J_1$ and $b \stackrel{\text{def}}{=} \inf J_2$. Then

$$\|\text{Abs}\|_{\text{OL}((-J_1)\cup J_2)} \le \|\text{Abs}\|_{\text{OL}((-\infty,a]\cup[0,b))} \le 2 + 2\left\|\frac{x - y}{x + y}\right\|_{\mathfrak{M}_{[a,\infty),[0,b)}}$$

and the result follows from Theorem (6.2.23).

Let us state two special cases of Theorem (6.2.39).

Theorem (6.2.40)[260]. There exist positive constants C_1 and C_2 such that

$$C_1 \log(2 + \log(ba^{-1})) \le \|\text{Abs}\|_{\text{OL}((-\infty,0]\cup[a,b])} \le C_2 \log(2 + \log(ba^{-1}))$$

for all $a, b \in (0,\infty)$ with $a < b$.

Theorem (6.2.41)[260]. There exist positive constants C_1 and C_2 such that

$$C_1 \log(2 + \log_+(ba^{-1})) \le \|\text{Abs}\|_{\text{OL}((-b,0]\cup[a,\infty))} \le C_2 \log(2 + \log_+(ba^{-1}))$$

for all $a, b \in (0,\infty)$.

Theorem (6.2.42)[260]. Let $\xi_a = \text{Abs}|[-a,\infty)$ and $\eta_a = \text{Abs}|[-a,a]$, where $a > 0$. Then there exist positive numbers C_1 and C_2 such that

$$C_1\delta \log(2 + \log(a\delta^{-1})) \le \Omega_{\eta_a}(\delta) \le \Omega_{\xi_a}(\delta) \le C_2\delta \log(2 + \log(a\delta^{-1}))$$

for $\delta \in (0, a]$,

$$C_1\delta \le \Omega_{\xi_a}(\delta) \le C_2\delta$$

for $\delta \in [a,\infty)$, and

$$C_1 a \le \Omega_{\eta_a}(\delta) \le C_2 a$$

for $\delta \in [a,\infty)$.

Proof. Put $\mathfrak{F}_\delta \overset{\text{def}}{=} [-a,\infty)\backslash(0,\delta)$. Clearly, $\mathfrak{F}_\delta$is a $\delta/2$-net of $(-\infty,a]$. Hence, by Theorem (6.2.35) we have

$$\Omega_{\xi_a}(\delta) \le \Omega^{\mathfrak{b}}_{\xi_a}(\delta) \le \delta + 2\delta\|\xi_a\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

Applying Theorem (6.2.41), we obtain the desired upper estimate for Ω_{ξ_a}. Clearly, $\Omega_{\eta_a} \le 2a$ every- where because $0 \le \eta_a \le a$.

To obtain the lower estimates, we use Theorem (6.2.36). We consider first the case $\delta \in (0,\frac{a}{2})$. Put $\Lambda = [-a,0]$ and $\mathrm{M} = [\delta,a]$. By Theorem (6.2.36),

$$\Omega_{\eta_a}(\delta) \le \frac{1}{2}\Omega^{\mathfrak{b}}_{\eta_a}(\delta) \ge \frac{\delta}{4}\|\mathfrak{D}_0\eta_a\|_{\mathfrak{M}_{\Lambda,\mathrm{M}}}.$$

Theorem (6.2.25) implies now that $\Omega_{\eta_a}(\delta) \ge \mathrm{const}\,\delta\log(2+\log(a\delta^{-1}))$. The lower estimates in the case $\delta \in [\frac{a}{2},\infty)$ are trivial because $\Omega_{\eta_a} \ge \omega_{\eta_a}$ and $\Omega_{\xi_a} \ge \omega_{\xi_a}$.

Theorem (6.2.43)[260]. There exists a positive number C such that

$$\||A|-|B|\| \le C\|A-B\|\log\left(2+\log\frac{\|A\|+\|B\|}{\|A-B\|}\right)$$

for all bounded self-adjoint operators A and B .

Proof. This is a special case of Theorem (6.2.42) that corresponds to $a = \|A\| + \|B\|$.

Theorem (6.2.42) also allows us to prove that the upper estimate in Theorem (6.2.43) is sharp.

Theorem (6.2.44): Let $a > 0$. There is a positive number c such that for every$\delta \in (0,a)$, there exist self-adjoint operators A and B such that$\|A\| + \|B\| \le a$,$\|A-B\| \le \delta$, but

$$\||A|+|B|\| \ge c\delta\log\left(2+\log\frac{a}{\delta}\right).$$

We proceed now to the case of arbitrary (not necessarily self-adjoint) oper-ators. Recall that for a bounded operator S on Hilbert space, its modulus $|S|$ is defined by

$$|S| \overset{\text{def}}{=} (S^*S)^{1/2}.$$

Theorem (6.2.45)[260]. There exists a positive number C such that

$$\||S|-|T|\| \le C\|S-T\|\log\left(2+\log\frac{\|S\|+\|T\|}{\|S-T\|}\right)$$

for all bounded operators S and T .

Proof. Put

$$A = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

Clearly, A and B are self-adjoint operators with

$$|A| = \begin{pmatrix} |S| & 0 \\ 0 & |S^*| \end{pmatrix} \quad \text{and} \quad |B| = \begin{pmatrix} |T| & 0 \\ 0 & |T^*| \end{pmatrix}$$

Hence,

$$\||S| - |T|\| \le \||A| - |B|\| \le C\|A - B\| \log\left(2 + \log\frac{\|A\| + \|B\|}{\|A - B\|}\right)$$
$$= C\|S - T\| \log\left(2 + \log\frac{\|S\| + \|T\|}{\|S - T\|}\right).$$

Theorem (6.2.45) significantly improves Kato's inequality obtained in [18]:

$$\||S| - |T|\| \le \frac{1}{\pi}\|S - T\| \log\left(2 + \log\frac{\|S\| + \|T\|}{\|S - T\|}\right).$$

We obtain a sharp estimate for the operator modulus of conti-nuity of the piece- wise linear function $\mathscr{x}$ defined by

$$\mathscr{x} \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } t \ge 1, \\ t, & \text{if } -1 < t \le 1 \\ -1, & \text{if } t > 1. \end{cases}$$

It is easy to see that $\mathscr{x}(t) = \frac{1}{2}(|1 + t| - |1 - t|)$.

Theorem (6.2.46)[260]. There exist positive numbers C_1 and C_2 such that

$$C_1 \log|\log\delta| \le \|\mathscr{x}\|_{\mathrm{OL}((-\infty,-1-\delta]\cup[-1,1]\cup[1+\delta,\infty))} \le C_2 \log|\log\delta|$$

for every $\delta \in (0, \frac{1}{2})$.

Proof. Put $\mathscr{x}_1 = \mathscr{x}|((-\infty, -1 - \delta] \cup [-1,1])$ and $\mathscr{x}_2 = \mathscr{x}|([-1,1] \cup [1 + \delta, \infty))$. Note that

$$\mathscr{x}_1(t) = \frac{1}{2}(|1 + t| - 1 + t) \quad \text{and} \quad \mathscr{x}_1(t) = \frac{1}{2}(1 + t - |t - 1|).$$

It follows from Theorem (6.2.41) that

$$C_1 \log|\log\delta| \le \|\mathscr{x}_1\|_{\mathrm{OL}} \le C_2 \log|\log\delta|$$

and

$$C_1 \log|\log\delta| \le \|\mathscr{x}_2\|_{\mathrm{OL}} \le C_2 \log|\log\delta|.$$

Thus the desired lower estimate is evident and the desired upper estimate follows from Theorem (6.2.38).

Theorem (6.2.47)[260]. There exist positive numbers c_1 and c_2 such that

$$c_1\delta \log(1 + \log(1 + \delta^{-1})) \le \Omega_{\mathscr{x}}(\delta) \le c_2\delta \log(1 + \log(1 + \delta^{-1}))$$

for every $\delta > 0$.

Proof. Note that $\lim_{t\to\infty} \log(1 + \log(1 + \delta^{-1})) = 1$. Thus it suffices to consider the case when $0 < \delta \le 1$. Put$\mathfrak{F}_\delta \stackrel{\text{def}}{=} (-\infty, -1 - \delta] \cup [-1,1] \cup [1 + \delta, \infty)$. Clearly, $\mathfrak{F}_\delta$is a δ-net for $\mathbb{R}$. Hence, by Theorem (6.2.35), we have

$$\Omega_{\mathscr{x}}(\delta) \le \Omega_{\mathscr{x}}^{\mathfrak{b}}(\delta) \le \delta + 2\delta\|\mathscr{x}\|_{\mathrm{OL}(\mathfrak{F}_\delta)}.$$

The desired upper estimate follows now from Theorem (6.2.46).

To obtain the lower estimate we can apply Theorem (6.2.42) because $(t) = \frac{1}{2}(|1 + t| - 1 + t)$ for $t \ge 1$.

In [261] we proved that if f is a continuous function on $\mathbb{R}$, then its operator modulus of continuity Ω_f admits the estimate

$$\Omega_f(\delta) \le \text{const}\ \delta \int_{\delta}^{\infty} \frac{\omega_f(t)}{t^2} dt = \text{const} \int_{1}^{\infty} \frac{\omega_f(t\delta)}{t^2} dt, \quad \delta > 0.$$

We show that if f vanishes on $(-\infty, 0]$ and is a concave nondecreasing function on$[0, \infty)$, then the above estimate can be considerably improved.

We also obtain several other estimates of operator moduli of continuity.

Theorem (6.2.48)[260]. Suppose that $f'' = \mu \in \mathcal{M}(\mathbb{R})$ (in the distributional sense), $\mu(\mathbb{R}) = 0$, and

$$\int_{\mathbb{R}} \log(\log(|t| + 3)) d|\mu|(t) < \infty.$$

Then

$$\Omega_f(\delta) \le c\|\mu\|_{\mathcal{M}(\mathbb{R})} \delta \log(\log(\delta^{-1} + 3)),$$

where c is a numerical constant.

Proof. Put

$$\varphi_s(t) \stackrel{\text{def}}{=} \frac{1}{2}(|t| + |s|) - \frac{|t - s|}{2}, \ \ s, t \in \mathbb{R}. \tag{36}$$

It is easy to see that

$$\varphi_s(t) \stackrel{\text{def}}{=} \frac{|s|}{2} \varkappa\left(\frac{2t}{s} - 1\right) + \frac{|s|}{2}, \quad \text{for } s \ne 0.$$

Clearly,

$$\varphi_s'' = \delta_0 - \delta_s \text{ and } \varphi_s(0) = 0. \tag{37}$$

Theorem (6.2.47) implies that

$$\begin{aligned}\Omega_{\varphi_s}(t) &\le \text{const}\ t \log\left(1 + \log\left(1 + \frac{|s|}{2t}\right)\right) \\ &\le \text{const}\ t \log\left(1 + \log\left(1 + \frac{|s|}{t}\right)\right), \ \ t > 0.\end{aligned} \tag{38}$$

It is easy to see that

$$t \log(1 + \log(1 + t^{-1}|s|)) \le \text{const}\ (\log(\log(|s| + 3)))t \log(\log(t^{-1} + 3)).$$

To complete the proof, it suffices to observe that

$$f(t) = at + b - \int_{\mathbb{R}} \varphi_s(t) d\mu(s)\ , \text{for some } a, b \in \mathbb{C},$$

which follows easily from (37).

The assumption that $\mu(\mathbb{R}) = 0$ in the hypotheses of Theorem (6.2.48) is essential. Moreover, the following result holds.

Theorem (6.2.49)[260]. Suppose that $f'' = \mu \in \mathcal{M}(\mathbb{R})$ and $\mu(\mathbb{R}) \ne 0$. Then $\Omega_f(t) = \infty$ for every $t > 0$.

Proof. Indeed, it is easy to see that there exists $c \in \mathbb{R}$ such that $f'(t) = c + \mu((-\infty, t))$ for almost all $t \in \mathbb{R}$. Hence,

$$\lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to\infty} f'(t) = c + \mu(\mathbb{R}) \quad \text{and} \quad \lim_{t\to\infty}\frac{f(t)}{t} = \lim_{t\to-\infty} f'(t) = c.$$

The result follows from Theorem (6.2.37).

Let G be an open subset of $\mathbb{R}$. Denote by $\mathcal{M}_{\text{loc}}(G)$ the set of all distributions on G that are locally (complex) measures.

Theorem (6.2.50)[260]. Let $f \in C(\mathbb{R})$. Put $\mu \overset{\text{def}}{=} f''$ in the sense of distributions. Suppose that $\lim_{|t|\to\infty} t^{-1} f(t) = 0$, $\mu(\mathbb{R}\backslash\{0\}) \in \mathcal{M}_{\text{loc}}(\mathbb{R}\backslash\{0\})$ and

$$\int_{\mathbb{R}\backslash\{0\}} \log(1 + \log(1 + |s|))\, d|\mu|(s) < \infty.$$

Then

$$\Omega_f(\delta) \leq \text{const}\,\delta \int_{\mathbb{R}\backslash\{0\}} \log(1 + \log(1 + |s|\delta^{-1}))\, d|\mu|(s).$$

Proof. Put

$$g(t) = -\int_{\mathbb{R}\backslash\{0\}} \varphi_s(t) d\mu(s),$$

where $\varphi_s$is defined by (36). Inequality (40) implies that

$$\Omega_g(\delta) \leq \text{const}\,\delta \int_{\mathbb{R}\backslash\{0\}} \log(1 + \log(1 + |s|\delta^{-1}))\, d|\mu|(s). \tag{39}$$

In particular, g is continuous on $\mathbb{R}$. Clearly, $g'' = f''$ on $\mathbb{R}\backslash\{0\}$. Hence, $f(x) - g(x) = a|x| + bx + c$ for some $a, b, c \in \mathbb{C}$. It follows from (39) that

$$\lim_{|t|\to\infty}\left|\frac{g(t)}{t}\right| \leq \lim_{t\to\infty}\frac{\omega_g(t)}{t} \leq \lim_{t\to\infty}\frac{\Omega_g(t)}{t} = 0 = \lim_{|t|\to\infty}\frac{f(t)}{t}$$

which implies that $f - g = \text{const}$.

Corollary (6.2.51)[260]. Let $a > 0$ and let f be a continuous function on $\mathbb{R}$ that is constant on $\mathbb{R}\backslash(-a, a)$. Put $\mu \overset{\text{def}}{=} f''$ in the sense of distributions. Suppose that $\mu|(\mathbb{R}\backslash\{0\}) \in \mathcal{M}_{\text{loc}}(\mathbb{R}\backslash\{0\})$ and

$$C \overset{\text{def}}{=} \sup_{s>0}|\mu|([s, 2s] \cup [-2s, -s]) < \infty. \tag{40}$$

Then

$$\Omega_f(\delta) \leq C\text{const}\delta\left(\log\frac{a}{\delta}\right)\log\left(\log\frac{a}{\delta}\right) \quad \text{for} \quad \delta \in \left(0, \frac{a}{3}\right).$$

Proof. By Theorem (6.2.50),

$$\Omega_f(\delta) \le \text{const}\,\delta\left(\int_0^a \log(1+\log(1+s\delta^{-1}))\,d|\mu(s)| + \int_0^a \log(1+\log(1+s\delta^{-1}))\,d|\mu(-s)|\right)$$
$$= \text{const}\,\delta\sum_{n\ge 0}\int_{2^{-n-1}a}^{2^{-n}a} \log(1+\log(1+s\delta^{-1}))\,d|\mu|(s)$$
$$+ \text{const}\,\delta\sum_{n\ge 0}\int_{2^{-n-1}a}^{2^{-n}a} \log(1+\log(1+s\delta^{-1}))\,d|\mu|(-s).$$

It follows now from (40) and the inequality

$$\log(1+\log(1+\alpha x)) \le 2\log(1+\log(1+x)), \quad 0 < x < \infty, \qquad 1 < \alpha \le 2,$$

that

$$\Omega_f(\delta) \le \text{const}\,\delta\sum_{n\ge 0}\int_{2^{-n-1}a}^{2^{-n}a} \log(1+\log(1+s\delta^{-1}))\frac{ds}{s}$$
$$= \text{const}\,\delta\int_0^a \log(1+\log(1+s\delta^{-1}))\frac{ds}{s}$$
$$= \text{const}\,\delta\int_0^{a/\delta} \log(1+\log(1+s))\frac{ds}{s}$$
$$\le \text{const}\,\delta + \text{const}\,\delta\int_1^{a/\delta} \log(1+\log(1+s))\frac{ds}{s}$$
$$= \text{const}\,\delta\left(1 + (\log(1+\log(1+s))\log s)\big|_1^{a/\delta} - \int_1^{a/\delta} \frac{\log s\,ds}{(1+s)(\log 1+\log(1+s))}\right)$$
$$\le \text{const}\,\delta + \text{const}\,\delta(\log(1+\log(1+s))\log s)\big|_1^{a/\delta} \le \text{const}\,\delta\left(\log\frac{a}{\delta}\right)\log\left(\log\frac{a}{\delta}\right)$$

for sufficiently small δ.

Corollary (6.2.52)[260]: Let f be a continuous function on $\mathbb{R}$ that is constant on $\mathbb{R}\backslash(-a,a)$. Suppose that f is twice differentiable on $\mathbb{R}\backslash\{0\}$ and

$$C \stackrel{\text{def}}{=} \sup_{s\ne 0}|sf''(s)| < \infty.$$

Then

$$\Omega_f(\delta) \le \text{const}\,C\delta\left(\log\frac{a}{\delta}\right)\log\left(\log\frac{a}{\delta}\right) \quad \text{for } s \in \left(0,\frac{a}{3}\right).$$

The following result shows that in a sense Theorem (6.2.48) cannot be improved.

We need the following lemma, in which φ_s is the function defined by (36).

Lemma (6.2.53)[260]. There is a positive number c such that for every $s \ge 10$, there exist self-adjoint operators A and B satisfying the conditions:

$$\sigma(A),\sigma(B)\subset\left(\frac{s}{2},\frac{3s}{2}\right),\quad \|A-B\|\le 1, \text{and } \|\varphi_s(A)-\varphi_s(B)\|\ge c\log\log s.$$

Proof. Clearly, it suffices to prove the lemma for sufficiently large s . By Theorem (6.2.47), there exist self-adjoint operators $A_0$and B_0such that $\|A_0\|$, $\|B_0\| < 1$, $\|A_0-B_0\|\le 2/s$, and $\||A_0|-|B_0|\|\ge \text{const } s^{-1}\log(2+\log s)$. Put $A\stackrel{\text{def}}{=} sI+\frac{s}{2}A_0$ and$B\stackrel{\text{def}}{=} sI+\frac{s}{2}B_0$. Then $\sigma(A),\sigma(B)\subset(\frac{s}{2},\frac{3s}{2})$ and $\|A-B\|\le 1$. Let us estimate $\|\varphi_s(A)-\varphi_s(B)\|$. Clearly,

$$\varphi_s(A)-\varphi_s(B) = \frac{s}{4}(A_0-B_0)-\frac{s}{4}(|A_0|-|B_0|).$$

Hence,

$$\|\varphi_s(A)-\varphi_s(B)\|\ge\frac{s}{4}\||A_0|-|B_0|\|-\frac{s}{4}\|A_0-B_0\| \quad\ge \text{const}\log\log s-\frac{1}{2}\text{const}\log\log s$$

for sufficiently large s.

Theorem (6.2.54)[260]. Let h be a positive continuous function on $\mathbb{R}$. Suppose that for every $f\in C(\mathbb{R})$ such that

$$f''=\mu\in\mathcal{M}(\mathbb{R}),\mu(\mathbb{R})=0,\qquad \text{and } \int_{\mathbb{R}} h(t)d|\mu|(t)<\infty,$$

we have $\Omega_f(\delta)<\infty, \delta>0$. Then for some positive number c,

$h(t)\ge c\log(\log(|t|+3))$, $t\in\mathbb{R}$.

Proof .

Assume the contrary. Then there exists a sequence $\{s_n\}$ of real numbers such that and $\lim_{n\to\infty}|s_n|=\infty$ $\quad\lim_{n\to\infty}(\log(\log(|s_n|)))^{-1}\ h(s_n)=0$.Passing to a subsequence, we can reduce the situation to the case when $s_n>0$ for all n or $s_n<0$ for all n. Without loss of generality we may assume that $s_n>0$ for all n. Moreover, we may also assume that $s_1\ge 10, s_{n+1}\ge 2s_n$ and $\log\log s_n\ge n^3\big(1+h(s_n)\big)$ for every $n\ge 1$. Put $\alpha_n\stackrel{\text{def}}{=} n(\log\log s_n)^{-1}$ for $n\ge 1$ and$f(t)\stackrel{\text{def}}{=}\sum_{n\ge1}\alpha_n\varphi_{s_n}(t)$. Note that the series converges for every t because $\sigma\stackrel{\text{def}}{=}\sum_{n\ge1}\alpha_n<\infty$. Moreover,

$$f''=\sigma\delta_0-\sum_{n\ge1}\alpha_n\delta_{s_n}\quad\text{and}\quad\sigma h(0)+\sum_{n\ge1}\alpha_n h(s_n)<\infty.$$

By Lemma (6.2.54), there exist two sequences $\{A_n\}_{n\ge1}$ and $\{B_n\}_{n\ge1}$ of self-adjoint operators such that

$$\sigma(A_n),\sigma(B_n)\subset\left(\frac{s_n}{2},\frac{3s_n}{2}\right),\quad \|A_n-B_n\|\le 1$$

and

$$\left\|\varphi_{s_n}(A_n)-\varphi_{s_n}(B_n)\right\|\ge c\log\log s_n.$$

Note that $\varphi_{s_k}(A_n)=\varphi_{s_k}(B_n)=s_kI$ for $k<n$. Also, $\varphi_{s_k}(A_n)=A_n$ and $\varphi_{s_k}(B_n)=B_n$ for $k>n$. Hence,

$$f(A_n)-f(B_n)=\alpha_n\big(\varphi_{s_n}(A_n)-\varphi_{s_n}(B_n)\big)+\sum_{k>n}\alpha_k(A_n-B_n),$$

and so

$$\|f(A_n) - f(B_n)\| \geq \alpha_n \|\varphi_{s_n}(A_n) - \varphi_{s_n}(B_n)\| - \sum_{k>n} \alpha_k \|A_n - B_n\|$$
$$\geq C\alpha_n \log\log s_n - \sum_{k>n} \alpha_k \to \infty \quad \text{as } n \to \infty.$$

Thus $\Omega_f(1) = \infty$ and we get a contradiction.

In [261] it was proved that

$$\Omega_f(\delta) \leq \int_1^\infty \frac{\omega_f(\delta s)}{s^2} ds$$

for every $f \in C(\mathbb{R})$. The following theorem shows that this estimate can be improved essentially for functions f concave on a ray.

Theorem (6.2.55)[260]. Let f be a continuous nondecreasing function such that $f(t) = 0$ for $t \leq 0$, $\lim_{t\to\infty} t^{-1} f(t) = 0$, and f is concave on $[0, \infty)$. Then

$$\Omega_f(\delta) \leq c \int_e^\infty \frac{f(\delta s) ds}{s^2 \log s},$$

where c is a numerical constant.

Proof. Let $\mu = -f''$ (in the distributional sense). Clearly, $\mu = 0$ on $(-\infty, 0)$ and μ is a positive regular measure on $(0, \infty)$ because f is concave on $(0, \infty)$. Hence, $\mu \in \mathcal{M}_{\text{loc}}(\mathbb{R} \setminus \{0\})$. By Theorem (6.2.50), we have

$$\Omega_f(\delta) \leq \text{const}\, \delta \int_0^\infty \log(1 + \log(1 + s\delta^{-1})) d\mu(s).$$

To estimate this integral, we use the equality $f'(t) = \mu(t, \infty)$ for almost all $t > 0$ and apply the Tonelli theorem twice.

$$\delta\int_0^\infty \log(1+\log(1+s\delta^{-1}))d\mu(s) = \int_0^\infty \left(\int_0^s \frac{dt}{(1+\log(1+t\delta^{-1}))(1+t\delta^{-1})}\right)d\mu(s)$$
$$= \int_0^\infty \frac{f'(t)dt}{(1+\log(1+t\delta^{-1}))(1+t\delta^{-1})}$$
$$= \delta^{-1}\int_0^\infty \left(\int_0^\infty \frac{(2+\log(1+s\delta^{-1}))ds}{(1+\log(1+s\delta^{-1}))^2(1+s\delta^{-1})^2}\right)f'(t)dt$$
$$= \delta^{-1}\int_0^\infty \frac{2+\log(1+s\delta^{-1})}{(1+\log(1+s\delta^{-1}))^2(1+s\delta^{-1})^2}f(s)ds$$
$$= \int_0^\infty \frac{2+\log(1+s)}{(1+\log(1+s))^2(1+s)^2}f(s\delta)ds$$
$$\le 2\int_0^\infty \frac{1}{(1+\log(1+s))(1+s)^2}f(s\delta)ds.$$

It remains to observe that

$$\int_0^e \frac{1}{(1+\log(1+s))(1+s)^2}f(s\delta)ds \le f(e\delta)\int_0^e \frac{1}{(1+\log(1+s))(1+s)^2}ds$$
$$\le f(e\delta)\int_0^\infty \frac{1}{(1+s)^2}ds = f(e\delta) \le \text{const}\int_e^\infty \frac{f(s\delta)ds}{s^2\log s}$$

and

$$\int_e^\infty \frac{1}{(1+\log(1+s))(1+s)^2}f(s\delta)ds \le \int_e^\infty \frac{f(s\delta)ds}{s^2\log s}.$$

Corollary (6.2.56)[260]. Suppose that under the hypotheses of Theorem (6.2.61), the function f is bounded and has finite right derivative at 0. Then

$$\Omega_f(\delta) \le \text{const}\,a\delta\log\left(\log\frac{M}{a\delta}\right) \text{ for } \delta \in \left(0, \frac{M}{3a}\right),$$

where $a = f'_+(0)$ and $M = \sup f$.

Proof. Since $f(t) \le \min\{at, M\}$, $t > 0$, the result follows from Theorem(6.2.61) and the following obvious facts:

$$\int_e^{\frac{M}{3a}} \frac{a\delta\,ds}{s\log s} = a\delta\log\left(\log\frac{M}{a\delta}\right) \text{ and } \int_{\frac{M}{3a}}^\infty \frac{Mds}{s^2\log s} \le \int_{\frac{M}{3a}}^\infty \frac{Mds}{s^2} = a\delta.$$

In [261] we proved that if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$, $0 \le \alpha < 1$, then

$$\Omega_f(\delta) \le \text{const}(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha}\delta^\alpha, \quad \delta > 0, \qquad (41)$$

where

$$\| f\|_{\Lambda_\alpha} \stackrel{\text{def}}{=} \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|}.$$

The next result shows that if in addition to this f satisfies the hypotheses of Theorem (6.2.55), then the factor $(1-\alpha)^{-1}$ on the right-hand side of (41) can considerably be improved.

Corollary (6.2.57)[260]. Suppose that under the hypotheses of Theorem (6.2.55), the function f belongs to $\Lambda_\alpha(\mathbb{R})$, $0 \le \alpha < 1$. Then

$$\Omega_f(\delta) \le \text{const}\left(\log\frac{2}{1-\alpha}\right)\| f\|_{\Lambda_\alpha}\delta^\alpha$$

for every $\delta > 0$.

Proof. Indeed,

$$\int_e^\infty \frac{ds}{s^{2-\alpha}\log s} = \int_1^\infty e^{(\alpha-1)t}\frac{dt}{t} = \int_{1-\alpha}^\infty \frac{e^{-t}dt}{t} \le \text{const}\ \frac{2}{1-\alpha}.$$

The following theorem is a symmetrized version of Theorem (6.2.55).

Theorem (6.2.58)[260]. Let f be a continuous function on $\mathbb{R}$ such that f is convex or concave on each of two rays $(-\infty,0]$ and $[0,\infty)$. Suppose that there exists a finite limit $\lim_{|t|\to\infty} t^{-1} f(t) \stackrel{\text{def}}{=} a$. Then

$$\Omega_f(\delta) \le a\delta + c\int_e^\infty \frac{|f(\delta s)-f(0)-\delta as| + |f(-\delta s)-f(0)+\delta as|}{s^2\log s}ds,$$

where c is a numerical constant.

Proof. It suffices to consider the case where $f(0) = a = 0$. We assume first that $f(t) = 0$ for $t \le 0$. To be definite, suppose that f is concave on $[0,\infty)$. Then f is a nondecreasing function because $\lim_{|t|\to\infty} t^{-1} f(t) = 0$, and so the result reduces to Theorem (6.2.55). The case $f(t) = 0$ for $t \ge 0$ follows from the considered case with the help of the change of variables $t \mapsto -t$. It remains to observe that each function f with $a = f(0) = 0$ can be represented in the form $f = g + h$ in such way that $g(t) = 0$ for $t \le 0$, $h(t) = 0$ for $t \ge 0$, and the cases of the function g and h have been treated above.

Theorem (6.2.59)[260]. Let f be a nonnegative continuous function on $\mathbb{R}$ such that $f(x) = 0$ for all $x \le 0$ and the function $x \mapsto x^{-1}f(x)$ is nonincreasing on $(0,\infty)$. Suppose that $\Omega_f(\delta) < \infty$ for $\delta > 0$. Then

$$f(x) \le \text{const}\frac{x}{\log\log x}$$

for every $x \ge 4$.

Proof. By Theorem (6.2.36),

$$\Omega_f^{\flat}(1) \ge \frac{1}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{[1,\infty],(-\infty,0]}}$$

Making the change of variables $y \mapsto -y$ we get

$$\left\|\frac{f(x)}{x+y}\right\|_{\mathfrak{M}_{[1,\infty],[0,\infty]}} \le 2\Omega_f^{\flat}(1).$$

Thus for every $a > 1$

$$\left\|\frac{x}{x+y}\right\|_{\mathfrak{M}_{[1,a],[1,a]}} \le \max_{[1,a]}\left|\frac{x}{f(x)}\right| \cdot \left\|\frac{f(x)}{x+y}\right\|_{\mathfrak{M}_{[1,a],[1,a]}}$$

$$\le \frac{a}{f(a)}\left\|\frac{f(x)}{x+y}\right\|_{\mathfrak{M}_{[1,a],[1,a]}} \le \frac{2a\Omega_f^{\flat}(1)}{f(a)}.$$

It remains to apply Theorem (6.2.25).

Let $x_0 > e$ and let g_α be a continuous function such that

$$g_\alpha(x) = \begin{cases} \dfrac{x}{\log^\alpha(\log x)}, & \text{if } x \ge x_0 > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

Then $\Omega_{g_\alpha}(\delta) < \infty$ for $\alpha > 1$. Indeed, in this case $g_\alpha$coincides with a function satisfying Theorem (6.2.55) outside a compact subset of $\mathbb{R}$. On the other hand, $\Omega_{g_\alpha}(\delta) = \infty$ for $\alpha < 1$. This follows from Theorem (6.2.59). Indeed, outside a compact subset of $\mathbb{R}$ the function $g_\alpha$coincides with a function f , for which the function $x \mapsto x^{-1}f(x)$ is nonincreasing on $(0,\infty)$. The case $\alpha = 1$ is an open problem.

Recall that it follows from (23) that if f is a function on $\mathbb{R}$ such that $\|f\|_{L^\infty} \le 1$, $\|f\|_{\text{Lip}} \le 1$, then

$$\Omega_f(\delta) \le \text{const}\, \delta\left(1 + \log\frac{1}{\delta}\right), \quad \delta \in (0,1].$$

It is still unknown whether this estimate is sharp. In particular, the question whether one can replace the factor $\left(1+\log\frac{1}{\delta}\right)$ on the right-hand side with $\left(1+\log\frac{1}{\delta}\right)^s$ for some $s < 1$ is still open.

We established a lower estimate for the operator modulus of continuity of the function $x \mapsto |x|$ on finite intervals.

A C^∞ function f on $\mathbb{R}$ such that $\|f\|_{L^\infty} \le 1$, $\|f\|_{\text{Lip}} \le 1$, and

$$\Omega_f(\delta) \le \text{const}\, \delta\sqrt{\log\frac{2}{\delta}}, \quad \delta \in (0,1].$$

Let $\sigma > 0$. Denote by $\mathcal{E}_\sigma$ the set of entire functions of exponential type at most σ . Let $F \in \mathcal{E}_\sigma \cap L^2(\mathbb{R})$. Then

$$F(z) = \sum_{n\in\mathbb{Z}} \frac{\sin(\sigma z - \pi n)}{\sigma z - \pi n} F\left(\frac{\pi n}{\sigma}\right),$$

see, e.g., [280, Lecture 20.2, Theorem 1]. Let $f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R})$. Then

$$f(z)\frac{\sin(\sigma(z-a))}{\sigma(z-a)} \in \mathcal{E}_{2\sigma} \cap L^2(\mathbb{R}).$$

Hence,

$$f(z)\frac{\sin(\sigma(z-a))}{\sigma(z-a)} = \sum_{n\in\mathbb{Z}}\frac{\sin(2\sigma z-\pi n)}{2\sigma z-\pi n}\cdot\frac{\sin\left(\sigma\left(\frac{\pi n}{2\sigma}-a\right)\right)}{\sigma\left(\frac{\pi n}{2\sigma}-a\right)}f\left(\frac{\pi n}{2\sigma}\right)$$
$$= 2\sum_{n\in\mathbb{Z}}\frac{\sin(2\sigma z-\pi n)\sin\left(\sigma a-\frac{\pi n}{2}\right)}{(2\sigma z-\pi n)(2\sigma a-\pi n)}f\left(\frac{\pi n}{2\sigma}\right).$$

Substituting $z = a$, we obtain

$$f(z) = 2\sum_{n\in\mathbb{Z}}\frac{\sin^2(2\sigma z-\pi n)\sin\left(\sigma z-\frac{\pi n}{2}\right)}{(2\sigma z-\pi n)^2}f\left(\frac{\pi n}{2\sigma}\right)$$
$$= \sum_{n\in\mathbb{Z}}\frac{\sin^2\left(\sigma z-\frac{\pi n}{2}\right)\cos\left(\sigma z-\frac{\pi n}{2}\right)}{\left(\sigma z-\frac{\pi n}{2}\right)^2}f\left(\frac{\pi n}{2\sigma}\right). \qquad (42)$$

for $f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R})$.

Denote by $\mathcal{E}_\sigma(\mathbb{C}^2)$ the set of all entire functions f on $\mathbb{C}^2$such that the functions $z \mapsto f(z,\xi)$ and $z \mapsto f(\xi,z)$ belong to $\mathcal{E}_\sigma$for every $\xi \in \mathbb{R}$ (or, which is the same, for all $\xi \in \mathbb{C}$). Equality (42) implies the following identity:

$$f(z,w) = \sum_{(m,n)\in\mathbb{Z}}\frac{\sin^2\left(\sigma z-\frac{\pi m}{2}\right)\cos\left(\sigma a-\frac{\pi m}{2}\right)\sin^2\left(\sigma w-\frac{\pi n}{2}\right)\cos\left(\sigma w-\frac{\pi n}{2}\right)}{\left(\sigma z-\frac{\pi m}{2}\right)^2\left(\sigma w-\frac{\pi n}{2}\right)^2}f\left(\frac{\pi m}{2\sigma},\frac{\pi n}{2\sigma}\right) \qquad (43)$$

for every $f \in \mathcal{E}_\sigma(\mathbb{C}^2) \cap L^\infty(\mathbb{R}^2)$.

Theorem (6.2.60)[260]. Let $\sigma > 0$ and $\Phi \in \mathcal{E}_\sigma(\mathbb{C}^2)$. Suppose that $\Phi\left(\frac{\pi m}{2\sigma}+\alpha,\frac{\pi n}{2\sigma}+\beta\right) \in \mathfrak{M}_{\mathbb{Z},\mathbb{Z}}$ for some $\alpha,\beta \in \mathbb{R}$. Then $\Phi \in \mathfrak{M}_{\mathbb{R},\mathbb{R}}$ and

$$\|\Phi(x,y)\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \le 2\left\|\Phi\left(\frac{\pi m}{2\sigma}+\alpha,\frac{\pi n}{2\sigma}+\beta\right)\right\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}}.$$

Proof. Clearly, it suffices to consider the case when $\alpha = \beta = 0$, $\sigma = \pi/2$ and $\|\Phi(x,y)\|_{\mathfrak{M}_{\mathbb{Z},\mathbb{Z}}} = 1$. Then (see [281, Theorem 5.1]) there exist two sequences $\{\varphi_m\}_{m\in\mathbb{Z}}$ and $\{\psi_n\}_{n\in\mathbb{Z}}$ of vectors in the closed unit ball of a Hilbert space $\mathcal{H}$such that $(\varphi_m,\psi_n) = \Phi(m,n)$. Put

$$g_x \stackrel{\text{def}}{=} \frac{4}{\pi^2}\sum_{m\in\mathbb{Z}}\frac{\sin^2\left(\frac{\pi}{2}(x-m)\right)\cos\left(\frac{\pi}{2}(x-m)\right)}{(x-m)^2}\varphi_m$$

and

$$h_y \stackrel{\text{def}}{=} \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{\sin^2\left(\frac{\pi}{2}(y-n)\right)\cos\left(\frac{\pi}{2}(y-n)\right)}{(x-n)^2}\psi_n.$$

We have

$$\|g_x\|_{\mathcal{H}} \le \frac{4}{\pi^2}\sum_{m\in\mathbb{Z}} \frac{\sin^2\left(\frac{\pi}{2}(x-m)\right)\left|\cos\left(\frac{\pi}{2}(x-m)\right)\right|}{(x-m)^2}$$
$$= \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{\sin^2\left(\frac{\pi x}{2}\right)\left|\cos\left(\frac{\pi x}{2}\right)\right|}{(x-2n)^2} + \frac{4}{\pi^2}\sum_{n\in\mathbb{Z}} \frac{\sin^2\left(\frac{\pi x}{2}-\frac{\pi}{2}\right)\left|\cos\left(\frac{\pi x}{2}-\frac{\pi}{2}\right)\right|}{(x-2n-1)^2}$$
$$= \left|\cos\left(\frac{\pi x}{2}\right)\right| + \left|\sin\left(\frac{\pi x}{2}\right)\right| \le \sqrt{2}$$

In the same way, $\|h_y\|_{\mathcal{H}} \le \sqrt{2}$ for all $y \in \mathbb{R}$. Clearly $|\Phi| \le 1$ on $\mathbb{Z}^2$. The Cartwright theorem (see [280, Lecture 21, Theorem 4]) implies that Φ is bounded on $\mathbb{R}\times\mathbb{Z}$. Applying once more the Cartwright theorem, we find that $\Phi \in L^\infty(\mathbb{R}^2)$. Hence, we can apply formula (43) to the function Φ, whence $\Phi(x,y) = (g_x, h_y)$ for all $x, y \in \mathbb{R}$. It remains to observe that by Theorem 5.1 in [281],

$$\|\Phi(x,y)\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \le \sup_{x\in\mathbb{R}}\|g_x\|_{\mathcal{H}} \cdot \sup_{y\in\mathbb{R}}\|h_y\|_{\mathcal{H}} \le 2.$$

Theorem (6.2.61)[260]. Let $f \in \mathcal{E}_\sigma$. Then

$$\Omega_f^{\flat}(\delta) \ge \frac{\delta}{2}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for every $\delta \in (0, \frac{1}{2\sigma}]$.

Proof. The general case easily reduces to the case $\sigma = \pi/4$. By Theorem (6.2.60), we have

$$\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \le 2\left\|\frac{f(2m+1)-f(2n)}{2m-2n+1}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \le 2\|f\|_{\mathrm{OL}(\mathbb{Z})}.$$

Hence, by Theorem (6.2.32),

$$\Omega_f^{\flat}(\delta) \ge \Omega_{f,\mathbb{Z}}^{\flat}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{Z})} \ge \frac{\delta}{2}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for $\delta \in (0, \frac{2}{\pi}]$.

Theorem (6.2.62): Let $f \in \mathcal{E}_\sigma$. Then

$$\Omega_f(\delta) \ge \frac{\delta}{4}\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for every $\delta \in (0, \frac{1}{2\sigma}]$.

Proof. It suffices to observe that $\Omega_f^{\flat}(\delta) \le 2\Omega_f(\delta)$ by Theorem 10.2 in [261].

Lemma (6.2.63)[260]. For every positive integer n, there exists a trigonometric polynomial f of degree n such that $\|f\|_{L^\infty} \le 1$, $\|f'\|_{L^\infty} \le 1$, and

$$\left\|\frac{f(x)-f(y)}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \ge c\sqrt{\log n}.$$

Proof. It follows from the results of [266] that for every function h in $C^1(\mathbb{T})$,

$$\left\|\frac{h(e^{ix})-h(e^{iy})}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geq \text{const}\,\|h\|_{\boldsymbol{B}_1^1}, \tag{44}$$

where $\boldsymbol{B}_1^1$ is a Besov space (see [268]for the definition) of functions on $\mathbb{T}$. Note that this result was deduced in [266]from the nuclearity criterion for Hankel operators (see [282] and [268, Chapter 6]). It is easy to see from the definition of $\boldsymbol{B}_1^1(\mathbb{T})$ (see, e.g., [268]) that

$$\|h\|_{\boldsymbol{B}_1^1} \geq \text{const} \sum_{j\geq 0} 2^j\left|\hat{h}(2^j)\right|. \tag{45}$$

It is well known (see, for example, [283]) that for every positive integer n, there exists an analytic polynomial h such that

$$h(0) = 0, \qquad \deg h = n, \quad \|h'\|_{L^\infty(\mathbb{T})} = 1, \text{ and } \quad \sum_{j\geq 0} 2^j\left|\hat{h}(2^j)\right| \geq d\sqrt{\log n}\ ,$$

where d is a positive numerical constant. Then inequality (44) implies that

$$\left\|\frac{h(e^{ix})-h(e^{iy})}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \geq \text{const}\sqrt{\log n}.$$

Put $f(x) \stackrel{\text{def}}{=} h(e^{ix})$. It remains to observe that $\|h'\|_{L^\infty} = \|h'\|_{L^\infty(\mathbb{T})} = 1$ and $\|f\|_{L^\infty} = \|h\|_{L^\infty(\mathbb{T})} \leq 1$.

Lemma (6.2.65)[260]. Let $n \in \mathbb{Z}$. Then

$$\left\|\frac{x-y-2\pi n}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{J_1,J_2}} \leq \frac{3\sqrt{2}\pi}{4}$$

for all intervals J_1 and J_2 with $J_1 - J_2 \subset [(2n-\frac{3}{2})\pi, (2n+\frac{3}{2})\pi]$.

Proof. We can restrict ourselves to the case $n = 0$. We have

$$\left\|\frac{x-y}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{J_1,J_2}} = \left\|\frac{x-y}{e^{i(x-y)}-1}\right\|_{\mathfrak{M}_{J_1,J_2}} \leq \left\|\frac{t}{e^{it}-1}\right\|_{\hat{L}^1\left(\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]\right)} = \left\|\frac{t}{2\sin(t/2)}\right\|_{\hat{L}^1\left(\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]\right)}.$$

Consider the 3π-periodic function ξ such that $\xi(t) = \frac{t}{2\sin(t/2)}$ for $t \in \left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$. We can expand ξ in Fourier series

$$\xi(t) = \sum_{n\in\mathbb{Z}} a_n e^{\frac{2}{3}nit}.$$

Note that $a_n = a_{-n} \in \mathbb{R}$ for all $n \in \mathbb{Z}$ because ξ is even and real. Moreover, ξ is convex on $\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$. Hence, by Theorem 35 in [16], $(-1)^n a_n \geq 0$ for all $n \in \mathbb{Z}$. It follows that

$$\left\|\frac{t}{2\sin(t/2)}\right\|_{\hat{L}^1\left(\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]\right)} \leq \sum_{n\in\mathbb{Z}} |a_n| = \xi\left(\frac{3\pi}{2}\right) = \frac{3\sqrt{2}\pi}{4}.$$

Corollary (6.2.66)[260]: Let $J_1 = [\pi j, \pi j + \pi]$ and $J_2 = [\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2}]$, where $j, k \in \mathbb{Z}$. Then

$$\left\|\frac{x-y-2\pi n}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{J_1,J_2}} \leq \frac{3\sqrt{2}\pi}{4}$$

for some $n \in \mathbb{Z}$.

Proof. We have $J_1 - J_2 = [\pi(j-k) - \frac{\pi}{2}, \pi(j-k) + \frac{3\pi}{2}]$. If $j-k$ is even, then we can apply Lemma (6.2.65) with $n = \frac{1}{2}(j-k)$. If $j-k$ is odd, then we can apply Lemma (6.2.65) with $n = \frac{1}{2}(j-k+1)$.

Lemma (6.2.67)[260]: Let g be a 2π-periodic function in $C^1(\mathbb{R})$. Then

$$\left\|\frac{g(x)-g(y)}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \le 3\sqrt{2}\pi \left\|\frac{g(x)-g(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}.$$

Proof. Note that

$$\left\|\frac{g(x)-g(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} = \left\|\frac{g(x)-g(y)}{x-y-2\pi n}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}}$$

for all $n \in \mathbb{Z}$ and

$$\left\|\frac{g(x)-g(y)}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \le 3\sqrt{2}\pi \left\|\frac{g(x)-g(y)}{x-y}\right\|_{\mathfrak{M}_{[0,2\pi],\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]}}.$$

Now we can represent the square $[0,2\pi] \times \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ as the union of four squares with sides of length π , each of which satisfies the hypotheses of Corollary (6.2.66).

Theorem (6.2.68)[260]. For every $\delta \in (0,1]$, there exists an entire function $f \in \mathcal{E}_1/\delta$ such that $\|f\|_{L^\infty(\mathbb{R})} \le 1$, $\|f'\|_{L^\infty(\mathbb{R})} \le 1$ and $\Omega_f(\delta) \ge C\delta\sqrt{\log\frac{2}{\delta}}$, where C is a positive numerical constant.

Proof. It suffices to consider the case when $\delta \in (0, \frac{1}{2}]$. Then $\delta \in [\frac{1}{n+1}, \frac{1}{n}]$ for an integer $n \ge 2$. By Lemma (6.2.64), there exists a trigonometric polynomial f of degree n such that

$$\left\|\frac{f(x)-f(y)}{e^{ix}-e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \ge c\sqrt{\log n}.$$

Hence,

$$\left\|\frac{f(x)-f(y)}{x-y}\right\|_{\mathfrak{M}_{\mathbb{R},\mathbb{R}}} \ge c\sqrt{\log n}.$$

by Lemma (6.2.67). Clearly, $g \in \mathcal{E}_n \subset \mathcal{E}_1/\delta$. Applying Theorem (6.2.62), we obtain

$$\Omega_f(t) \ge \mathrm{const}\sqrt{\log n}\, t, \qquad 0 < t \le \frac{1}{2n}.$$

Hence,

$$\Omega_f(\delta) \ge \Omega_f\left(\frac{1}{2n}\right) \ge C_0 \frac{\sqrt{\log n}}{n} \le C\delta\sqrt{\log\left(\frac{2}{\delta}\right)}$$

for some positive numbers C_0 and C .

Theorem (6.2.68)[260]. There exist a positive number c and a function $f \in C^{\infty}(\mathbb{R})$ such that $\|f\|_{L^{\infty}} \le 1, \|f'\|_{L^{\infty}} \le 1$, and $\Omega_f(\delta) \ge C\delta\sqrt{\log\left(\frac{2}{\delta}\right)}$ for every $\delta \in (0,1]$.

Proof. Applying Theorem (6.2.63) for $\delta = 2^{-n}$, we can construct a sequence of functions $\{f_n\}_{n\ge1}$ and two sequences of bounded self-adjoint operators $\{A_n\}_{n\ge1}$ and $\{B_n\}_{n\ge1}$ such that $\|f_n\|_{L^{\infty}} \le 1, \|f_n'\|_{L^{\infty}} \le 1$, $\|A_n - B_n\| \le 2^{-n}$ and $\|f_n(A_n) - f_n(B_n)\| \ge C\sqrt{n}2^{-n}$ for all $n \ge 1$. Denote by Δ_n the convex hull of $\sigma(A_n) \cup \sigma(B_n)$. Using the translations $f_n \mapsto f_n(x - a_n)$, $A_n \mapsto A_n + a_nI$, $B_n \mapsto B_n + a_nI$ and $\Delta_n \mapsto an + \Delta_n$ for a suitable sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$, we can achieve the condition that the intervals Δ_n are disjoint and $\operatorname{dist}(\Delta_m, \Delta_n) > 2$ for $m \ne n$. We can construct a function $f \in C^{\infty}(\mathbb{R})$ such that $\|f\|_{L^{\infty}} \le 1, \|f'\|_{L^{\infty}} \le 1$ and $f|\Delta_n = f_n|\Delta_n$ for all $n \ge 1$. Clearly, $\Omega_f(2^{-n}) \ge C\sqrt{n}2^{-n}$ for all $n \ge 1$ and some positive C which easily implies the result.

To obtain the lower estimate in Theorem (6.2.68), we used the inequality

$$\left\|\frac{f(e^{ix}) - f(e^{iy})}{e^{ix} - e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \ge \text{const} \sum_{j\ge0} 2^j \left|\hat{f}(2^j)\right|, \tag{46}$$

which in turn implies that there exists a positive number C such that for every positive integer n there exists a polynomial f of degree n such that

$$\left\|\frac{f(e^{ix}) - f(e^{iy})}{e^{ix} - e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \ge C\sqrt{\log n}\, \|f\|_{\text{Lip}}. \tag{47}$$

We do not know whether Theorem (6.2.68) can be improved. It would certainly be natural to try to improve (47). The best known lower estimate for the norm of divided differences in the space of Schur multipliers was obtained in [266]. To state it, we need some definitions.

Let $f \in L^1(\mathbb{T})$. Denote by $\mathcal{P}f$ the Poisson integral of,

$$(\mathcal{P})f(z) \overset{\text{def}}{=} \int_{\mathbb{T}} \frac{(1 + |z|^2)}{|z - \zeta|^2} d\boldsymbol{m}(\zeta), \quad \zeta \in \mathbb{D},$$

where m is normalized Lebesgue measure on $\mathbb{T}$.

For $t \in \mathbb{R}$ and $\delta \in (0,1)$, we define the Carleson domain $Q(t, \delta)$ by

$$Q(t, \delta) \overset{\text{def}}{=} \{re^{is}: 0 < 1 - r < h, |s - t| < \delta\}.$$

A positive Borel measure on μ on $\mathbb{D}$ is said to be a Carleson measure if

$$\mathfrak{L}(\mu) \overset{\text{def}}{=} \mu(\mathbb{D}) + \sup\{\delta^{-1}\mu(t, Q): t \in \mathbb{R}, \delta \in (0,1)\} < \infty.$$

If ψ is a nonnegative measurable function on $\mathbb{D}$, we put

$$\mathfrak{L}(\psi) \overset{\text{def}}{=} \mathfrak{L}(\mu), \quad \text{where } d\mu \overset{\text{def}}{=} \psi d\boldsymbol{m}_2.$$

Here $\boldsymbol{m}_2$ is planar Lebesgue measure.

It follows from results of [266](see also [284]) that

$$\left\|\frac{f(e^{ix}) - f(e^{iy})}{e^{ix} - e^{iy}}\right\|_{\mathfrak{M}_{[0,2\pi],[0,2\pi]}} \ge \text{const}\|f\|_{\mathcal{L}}. \tag{48}$$

where

$$\|f\|_{\mathcal{L}} \stackrel{\text{def}}{=} \mathfrak{L}(\|\text{Hess }(\mathcal{P}f)\|)$$

where for a function φ of class C^2 , its Hessian Hess(φ) is the matrix of its second order partial derivatives.

It turns out, however, that for a trigonometric polynomial f of degree n,

$$\|f\|_{\mathcal{L}} \leq \text{const}\sqrt{\log(1+n)}\,\|f\|_{\text{Lip}}, \tag{49}$$

and so even if instead of inequality (46) we use inequality (48), we cannot improve Theorem (6.2.68).

Inequality (49) is an immediate consequence of the following fact:

Theorem (6.2.69)[260]. For a trigonometric polynomial f of degree n,$n \geq 2$, the following inequality holds:

$$\mathfrak{L}(|\nabla(\mathcal{P}f)|) \leq \text{const}\sqrt{\log n}\,\|f\|_{L^\infty}.$$

We are going to use the well-known fact that a function f in $L^1(\mathbb{T})$ belongs to the space BMO$(\mathbb{T})$ if and only if the measure μ defined by $d\mu = |\nabla(\mathcal{P}f)|^2(1-|z|)d\boldsymbol{m}_2$ is a Carleson measure. We refer to [273] for Carleson measures and the space BMO.

Proof . Suppose that $\|f\|_{L^\infty} = 1$. We have to prove that

$$\int_{Q(t,\delta)} |\nabla(\mathcal{P}f)|\,dxdy \leq \text{const}\,\delta\sqrt{\log n}. \tag{83}$$

Note that $|\nabla(\mathcal{P}f)| \leq 2n$ by Bernstein's inequality. Hence,

$$\int_{\{1-n^{-1}<|\zeta|<1\}\cap Q(t,\delta)} |\nabla(\mathcal{P}f)|\,d\boldsymbol{m}_2 \leq 2n\boldsymbol{m}_2(\{1-n^{-1}<|\zeta|<1\}\cap Q(t,\delta)) = 2n\delta(1-(1-n)^2)$$
$$\leq 4\delta.$$

This proves (50) in the case $\delta \geq 1-n^{-1}$. In the case $\delta < 1-n^{-1}$ it remains to estimate the integral over the set $\{1-n^{-1}<|\zeta|<1\}\cap Q(t,\delta)$. Note that $\|f\|_{\text{BMO}} \leq \text{const}\|f\|_{L^\infty}$. Hence, there exists a constant C such that

$$\int_{Q(t,\delta)} |\nabla(\mathcal{P}f)|^2\,(|\zeta|<1)d\boldsymbol{m}_2(\zeta) \leq C\delta.$$

Thus

$$\int_{\{|\zeta|<1-n^{-1}\}\cap Q(t,\delta)} |\nabla(\mathcal{P}f)|\,d\boldsymbol{m}_2$$
$$\leq \left(\int_{Q(t,\delta)} |\nabla(\mathcal{P}f)|^2\,(|\zeta|<1)d\boldsymbol{m}_2(\zeta)\right)^{1/2}\left(\int_{\{|\zeta|<1-n^{-1}\}\cap Q(t,\delta)} (|\zeta|<1)^{-1}d\boldsymbol{m}_2(\zeta)\right)^{1/2}$$
$$\leq \text{const}\,\delta(\log(n\delta))^{1/2} \leq \text{const}\,\delta(\log n)^{1/2}.$$

We define an operator modulus of continuity of a continuous function f on $\mathbb{T}$ by

$$\Omega_f(\delta) \stackrel{\text{def}}{=} \sup\{\|f(U)-f(V)\|: U \text{ and } V \text{ are unitary } \|U-V\| \leq \delta\}.$$

As in the case of self-adjoint operators (see [261]), one can prove that

$$\|f(U)R - Rf(V)\| \le 2\Omega_f(\|UR - RV\|)$$

for all unitary operators U , V and an operator R of norm 1. We define the space OL($\mathbb{T}$) as the set of $f \in C(\mathbb{T})$ such that

$$\|f\|_{\mathrm{OL}(\mathbb{T})} \stackrel{\text{def}}{=} \sup_{\delta>0} \delta^{-1}\Omega_f(\delta) < \infty.$$

Given a closed subset $\mathfrak{F}$of $\mathbb{T}$, we can also introduce the operator modulus of continuity $\Omega_{f,\mathfrak{F}}$ and define the space OL($\mathfrak{F}$) of operator Lipschitz functions on $\mathfrak{F}$.

For closed subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$of $\mathbb{T}$, the space $\mathfrak{M}_{\mathfrak{F}_1,\mathfrak{F}_2}$ of Schur multipliers can be defined by analogy with the self-adjoint case. Note that the analogues of (29) and (31) for functions on closed subsets of $\mathbb{T}$.Let$f \in C(\mathbb{T})$. We put $f_\spadesuit(t) \stackrel{\text{def}}{=} f(e^{it})$. It is clear that $\Omega_{f_\spadesuit} \le \Omega_f$. Hence, $\|f_\spadesuit\|_{\mathrm{OL}(\mathbb{R})} \le \|f\|_{\mathrm{OL}(\mathbb{T})}$. Lemma (6.2.67) implies that$\|f\|_{\mathrm{OL}(\mathbb{T})} \le 3\sqrt{2}\pi\|f_\spadesuit\|_{\mathrm{OL}(\mathbb{R})}$. One can prove that $\Omega_f \le \text{const}\, \Omega_{f_\spadesuit}$.

Recall that it follows from results of [266]that for $f \in C(\mathbb{T})$,

$$\|f\|_{\mathrm{OL}(\mathbb{T})} \ge \text{const}\, \|f\|_{B_1^1};$$

See inequality (44).

We would like to remind also that for each positive integer n, there exists an analytic polynomial f such that $\deg f = n$,$\|f'\|_{L^\infty(\mathbb{T})} = 1$, and $\|f\|_{\mathrm{OL}(\mathbb{T})} \ge \text{const}\sqrt{\log n}$; see Lemma (6.2.64).

Put

$$\partial_n(z) \stackrel{\text{def}}{=} \frac{1}{n}\frac{z^n - 1}{z - 1} = \frac{1}{n}\sum_{k-0}^{n-1} z^k.$$

It is easy to see that

$$\partial_n(\zeta z^{-1}) = z^{1-n}\frac{z^n - \zeta^n}{n(z-\zeta)} = z^{1-n}\zeta^{n-1}\partial_n(z\zeta^{-1}).$$

Denote by $\mathbb{T}_n$ the set of nth roots of 1, i.e., $\mathbb{T}_n \stackrel{\text{def}}{=} \{\zeta \in \mathbb{T}: \zeta^n = 1\}$. Let f be an analytic polynomial of degree less that n. Then

$$\sum_{\zeta\in\tau\mathbb{T}_n} f(\zeta)\partial_n(z\zeta^{-1}) \quad \text{for every } \tau \in \mathbb{T}.$$

If f is a trigonometric polynomial and $\deg f \le n$, then for every $\xi \in \mathbb{T}$, the function $z^n f(z)\partial_{2n}(z\xi^{-1})$ is an analytic polynomial of degree less than $4n$. Hence,

$$z^n f(z)\partial_{2n}(z\xi^{-1}) = \sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)\partial_{2n}(\zeta\xi^{-1})\partial_{4n}(z\xi^{-1}).$$

Substituting $\xi = z$ we get

$$f(z) = z^{-n}\sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)\partial_{2n}(\zeta z^{-1})\partial_{4n}(z\zeta^{-1}) = \sum_{\zeta\in\tau\mathbb{T}_{4n}} f(\zeta)F_n(z,\zeta) \qquad (51)$$

for every $\tau \in \mathbb{T}$, where

$$F_n(z,\zeta) \stackrel{\text{def}}{=} z^{1-3n}\zeta^{1-4n}\frac{(z^{2n} - \zeta^{2n})(z^{4n} - \zeta^{4n})}{8n^2(z-\zeta)^2}.$$

Denote by $\mathcal{P}_n(\mathbb{T}^2)$ the set of all trigonometric polynomial f on $\mathbb{T}^2$such that the functions $z \mapsto f(z,\xi)$ and $z \mapsto f(\xi,z)$ are trigonometric polynomials on $\mathbb{T}$ of degree at most n for every $\xi \in \mathbb{T}$. Equality (51) implies the following identity:

$$f(z,w) = \sum_{\zeta\in\tau_1\mathbb{T}_{4n}} \sum_{\xi\in\tau_2\mathbb{T}_{4n}} f(\zeta,\xi)F_n(z,\zeta)F_n(w,\xi) \tag{52}$$

for every $f \in \mathcal{P}_n(\mathbb{T}^2)$ and for arbitrary τ_1 and τ_2 in $\mathbb{T}$.

Theorem (6.2.70) [260]. Let $\Phi \in \mathcal{P}_n(\mathbb{T}^2)$. Then

$$\|\Phi\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \le 2\|\Phi\|_{\mathfrak{M}_{\tau_1\mathbb{T}_{4n},\tau_2\mathbb{T}_{4n}}}$$

for all $\tau_1,\tau_2 \in \mathbb{T}$.

Proof. Clearly, it suffices to consider the case when $\tau_1 = \tau_2 = 1$ and $2\|\Phi\|_{\mathfrak{M}_{\mathbb{T}_{4n},\mathbb{T}_{4n}}} = 1$. Then (see [281, Theorem 5.1]) there exist two sequences $\{\varphi_\zeta\}_{\zeta\in\mathbb{T}_{4n}}$ and $\{\psi_\xi\}_{\xi\in\mathbb{T}_{4n}}$ of vectors in the closed unit ball of a Hilbert space $\mathcal{H}$such that $(\varphi_\zeta,\psi_\xi) = \Phi(\zeta,\xi)$. Put

$$g_z \stackrel{\text{def}}{=} \sum_{\zeta\in\mathbb{T}_{4n}} F_n(z,\zeta)\varphi_\zeta \quad \text{and} \quad h_w \stackrel{\text{def}}{=} \sum_{\xi\in\mathbb{T}_{4n}} F_n(z,\xi)\psi_\xi.$$

Taking into account that for $z \in \mathbb{T}$,

$$\frac{1}{2n}\sum_{\zeta\in\mathbb{T}_n}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 = \frac{1}{2n}\sum_{\zeta\in\mathbb{T}_{4n}/\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 = \int_{\mathbb{T}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 d\boldsymbol{m}(\zeta) = 2n,$$

we obtain

$$\|g_z\|_{\mathcal{H}} \le \sum_{\zeta\in\mathbb{T}_{4n}}|F_n(z,\zeta)| \le \frac{|z^{2n}+1|}{8n^2}\sum_{\zeta\in\mathbb{T}_n}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2 + \frac{|z^{2n}-1|}{8n^2}\sum_{\zeta\in\mathbb{T}_{4n}/\mathbb{T}_{2n}}\left|\frac{z^{2n}-\zeta^{2n}}{z-\zeta}\right|^2$$
$$= \frac{|z^{2n}+1|+|z^{2n}-1|}{2} \le \sqrt{2}.$$

In the same way, $\|h_w\|_{\mathcal{H}} \le \sqrt{2}$ for every $w \in \mathbb{T}$. By (52), we have $\Phi(z,w) = (g_z,h_w)$ for all $z,w \in \mathbb{T}$. It remains to observe that by Theorem 5.1
in [281],

$$\|\Phi(z,w)\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \le \sup_{z\in\mathbb{T}}\|g_z\|_{\mathcal{H}} \cdot \sup_{w\in\mathbb{T}}\|h_w\|_{\mathcal{H}} \le 2.$$

Lemma (6.2.71)[260]. Let n be a positive integer. Then

$$\|\lambda(z-w)\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} = \begin{cases} \dfrac{n}{4}, & \text{if } n \text{ is even,} \\ \dfrac{n^2-1}{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It is easy to verify that

$$\sum_{k=1}^{n}\left(k-\frac{n+1}{2}\right)z^k = \frac{nz^n}{z-1} - \frac{z^n-1}{(z-1)^2} - \frac{n+1}{2}z\frac{z^n-1}{z-1} = n\lambda(z-1)$$

for $\in \mathbb{T}_n$. Hence,

$$\lambda(z-w) = w^{-1}\lambda(zw^{-1}-1) = \frac{1}{n}\sum_{k=1}^{n}\left(k-\frac{n+1}{2}\right)z^k w^{-k-1}. \qquad (53)$$

Thus

$$\|\lambda(z-w)\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} \le \frac{1}{n}\sum_{k=1}^{n}\left|k-\frac{n+1}{2}\right| = \begin{cases} \frac{n}{4}, & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

The opposite inequality is also true. It can be deduced from the observation that equality (53) means that the function $\lambda(z-1)$ on the group $\mathbb{T}_n$is the Fourier transform of the n-periodic sequence $\{a_k\}_{k\in\mathbb{Z}}$ defined by $a_k = k-\frac{n+1}{2}$ for $k = 1,2,\dots,n$. Here we identify the group dual to $\mathbb{T}_n$with the group $\mathbb{Z}/n\mathbb{Z}$. We omit details because we need only the upper estimate.

We need the following version of Theorem (6.2.32):

Theorem (6.2.72)[260]: Let f be a function on $\mathbb{T}_n$. Then

$$\Omega^{\flat}_{f,\mathbb{T}_n}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{T}_n)}$$

for every $\delta \in (0,\frac{4}{n}]$.

To prove Theorem (6.2.71), we need a lemma. Put

$$\lambda(z) \overset{\text{def}}{=} \begin{cases} z-1, & \text{if } z\in\mathbb{C}, z\neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Proof . The inequality

$$\Omega^{\flat}_{f,\mathbb{T}_n}(\delta) = \delta\|f\|_{\mathrm{OL}(\mathbb{T}_n)}, \quad \delta > 0,$$

is a consequence of a unitary version of Theorem (6.2.26), which can be proved in the same way as the self-adjoint version, see also [263, Theorem 4.13].

We prove the opposite inequality for $\delta \in (0,\frac{4}{n}]$. Fix $\varepsilon > 0$. There exists a unitary operator U and bounded operator R such that $\|UR-RU\| = 1$, $\sigma(U) \subset \mathbb{T}_n$, and $\|f(U)R-Rf(U)\| \ge \|f\|_{\mathrm{OL}(\mathbb{T}_n)} - \varepsilon$. Put

$$R_U \overset{\text{def}}{=} \sum_{\zeta,\xi\in\mathbb{T}_n,\zeta\neq\xi} E_U(\{\zeta\})RE_U(\{\xi\}) = R - \sum_{\zeta\in\mathbb{T}_n} E_U(\{\zeta\})RE_U(\{\xi\}).$$

Clearly, $UR-RU = UR_U - R_U U$ and$f(U)R-Rf(U) = f(U)R_U - R_U f(U)$. Thus we may assume that $R = R_U$. Note that

$$UR-RU = \sum_{\zeta,\xi\in\mathbb{T}_n,\zeta\neq\xi} (\zeta-\xi)E_U(\{\zeta\})RE_U(\{\xi\})$$

Since

$$U = R_U = \sum_{\zeta,\xi\in\mathbb{T}_n,\zeta\neq\xi} \lambda(\zeta-\xi)(\zeta-\xi)E_U(\{\zeta\})RE_U(\{\xi\})$$

we have $R = H_n \star (UR - RU)$, where $H_n(\zeta,\xi) = \lambda(\zeta - \xi)$, where $\zeta,\xi \in \mathbb{T}_n$. Thus by Lemma (6.2.71),

$$\|R\| \le \|H_n\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} \|UR - RU\| = \|H_n\|_{\mathfrak{M}_{\mathbb{T}_n,\mathbb{T}_n}} \le \frac{n}{4}.$$

Let $\delta \in (0,\frac{4}{n}]$. Then $\|U(\delta R) - (\delta R)U\| = \delta$ and $\|\delta R\| \le 1$. Hence,

$$\Omega^{\mathrm{b}}_{f,\mathbb{T}_n}(\delta) \ge \delta\|f(U)R - Rf(U)\| \ge \delta(\|f\|_{\mathrm{OL}(\mathbb{T}_n)} - \varepsilon).$$

Passing to the limit as $\varepsilon \to 0$, we obtain the desired result.

Theorem (6.2.73)[260]: Let f be a trigonometric polynomial of degree $n \ge 1$. Then

$$\Omega^{\mathrm{b}}_{f,\mathbb{T}_n}(\delta) \ge \frac{\delta}{2}\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for $\delta \in (0,\frac{1}{n}]$.

Proof. Applying Theorems (6.2.70) and (6.2.71), we obtain

$$\left\|\frac{f(z) - f(w)}{z - w}\right\|_{\mathfrak{M}_{\mathbb{T},\mathbb{T}}} \le 2\left\|\frac{f(z) - f(w)}{z - w}\right\|_{\mathfrak{M}_{\mathbb{T}_{4n},\mathbb{T}_{4n}}} = 2\delta^{-1}\Omega^{\mathrm{b}}_{f,\mathbb{T}_{4n}}(\delta) \le 2\delta^{-1}\Omega^{\mathrm{b}}_{f,\mathbb{T}}(\delta)$$

for $\delta \in (0,\frac{1}{n}]$.

Theorem (6.2.74)[260]: Let f be a trigonometric polynomial of degree $n \ge 1$. Then

$$\Omega_{f,\mathbb{T}}(\delta) \ge \frac{\delta}{4}\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for $\delta \in (0,\frac{1}{n}]$.

Theorem (6.2.75) [260]. Let $f \in C(\mathbb{T})$. Then

$$\Omega_f(2^{-n}) \ge C2^{-n}\sum_{k=0}^{n-1} 2^k\left(\left|\hat{f}(2^k)\right| + \left|\hat{f}(-2^k)\right|\right),$$

where C is a positive constant.

Proof. Applying the convolution with the de la Vallée Poussin kernel, we can find an analytic polynomial f_n such thatdeg $f_n < 2^n$, $\hat{f}_n(k) = \hat{f}(k)$ for $k \le 2^{n-1}$ and $\Omega_{f_n} \le 3\Omega_f$. Applying inequalities (44) and (45), we obtain

$$\|f_n\|_{\mathrm{OL}(\mathbb{T})} \ge \mathrm{const}\sum_{k=0}^{n-1} 2^k\left(\left|\hat{f}(2^k)\right| + \left|\hat{f}(-2^k)\right|\right)$$

It remains to apply Theorem (6.2.74) for $\delta = 2^{-n}$.

In the following theorem we use the notation C_A for the disk-algebra:

$$C_A \overset{\mathrm{def}}{=} \{f \in C(\mathbb{T}): \hat{f}(n) = 0 \text{ for } n < 0\}.$$

Theorem (6.2.76) [260]. Let $\omega: (0,2] \to \mathbb{R}$ be a positive continuous function. Suppose that $\omega(2t) \le \mathrm{const}\,\omega(t)$, the function $t \mapsto t(\log\frac{4}{t})^{-1}\omega(t)$ is non-decreasing, and

$$\int_0^2 \frac{\omega^2(t)dt}{t^3\log^2\frac{4}{t}} < \infty. \tag{54}$$

Then there exists a function $f \in C_A$ such that $f' \in C_A$ and $\Omega_f(\delta) \geq \omega(\delta)$ for all $\delta \in (0,2]$.

Proof. Note that the inequality $\Omega_f(\delta) \geq \omega(\delta)$ for $\delta = 2^{-n}$ implies that $\Omega_f(\delta) \geq \text{const}\, \omega(\delta)$ for all $\delta \in (0,2]$. Thus it suffices to obtain the desired estimate for $\delta = 2^{-n}$. Taking Theorem (6.2.75) into account, we can reduce the result to the problem to construct a function $g \in C_A$ such that

$$a_n \stackrel{\text{def}}{=} \frac{2^n \omega(2^{-n})}{n} \leq \frac{1}{n} \sum_{k=0}^{n-1} |\hat{g}(2^k)|$$

for all nonnegative integer n.

Indeed, in this case the function f defined by

$$f(z) = \int_0^z \frac{g(\zeta) - g(0)}{\zeta}\, d\zeta$$

satisfies the inequality

$$a_n \leq \frac{1}{n} \sum_{k=0}^{n-1} 2^k \left|\hat{f}(2^k)\right|$$

Condition (54) implies that $\{a_n\}_{n\geq 0} \in \ell^2$. Moreover, $\{a_n\}_{n\geq 0}$ is a nonincre-asing sequence because the function $t \mapsto t^{-1}(\log \frac{4}{t})^{-1}\omega(t)$ is nondecreasing.

We can find a function $g \in C_A$ such that $\hat{g}(2^k) = a_k$ for all $k \leq 0$, see, for example, [283]. Then

$$\frac{1}{n} \sum_{k=0}^{n-1} |\hat{g}(2^k)| = \frac{1}{n} \sum_{k=0}^{n-1} a_k \geq a_{n-1} \geq a_n.$$

We obtain sharp estimates of the quasicommutator norms $\|f(A)R - Rf(B)\|$ in the case when A has finite spectrum. This allows us to obtain sharp estimates of the operator Lipschitz norm in terms of the Lipschitz norm in the case of operators on finite-dimensional spaces in terms of the dimension.

Moreover, we obtain a more general result (see Theorem (6.2.85)) in terms of ε-entropy of the spectrum of A, where $\varepsilon = \|AR - RA\|$. This leads to an improvement of inequality (23). Note that the results improve cum results of [271] and [285].

Let $\mathfrak{F}$ be a closed subset of $\mathbb{R}$. Denote by $\text{Lip}(\mathfrak{F})$ the set of Lipschitz functions on $\mathfrak{F}$. Put

$$\|f\|_{\text{Lip}(\mathfrak{F})} \stackrel{\text{def}}{=} \inf\{C > 0: |f(x) - f(y)| \leq C|x - y| \forall x, y \in \mathfrak{F}\}.$$

Let $\{s_j(T)\}_{j=0}^{\infty}$ be the sequence of singular values of a bounded operator. We use the notation $\boldsymbol{S}_\omega$ for the Matsaev ideal,

$$\boldsymbol{S}_\omega \stackrel{\text{def}}{=} \left\{T: \|T\|_{\boldsymbol{S}_\omega} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} (1+j)^{-1} s_j(T) < \infty\right\}.$$

We need the following statement which is contained implicitly in [286].

Theorem (6.2.77) [260]. Let f be a Lipschitz function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Then for every nonempty finite subset Λ in $\mathfrak{F}$,

$$\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\Lambda,\mathfrak{F}}} \le C(1 + \log(\operatorname{card}(\Lambda)))\|f\|_{\mathrm{Lip}(\mathfrak{F})},$$

where C is a numerical constant.

Proof. Let $k \in L^2(\mu \otimes \nu)$, where μ and ν are Borel measures on Λ and $\mathfrak{F}$. Clearly, rank $\mathfrak{T}_k^{\mu,\nu} \le \operatorname{card}(\Lambda)$. Hence, $\left\|\mathfrak{T}_k^{\mu,\nu}\right\|_{S_\omega} \le (1 + \log(\operatorname{card}(\Lambda)))\left\|\mathfrak{T}_k^{\mu,\nu}\right\|$. Now Theorem 2.3 in [286] implies that

$$\left\|\mathfrak{T}_{k\mathfrak{D}_0 f}^{\mu,\nu}\right\| \le (1 + \log(\operatorname{card}(\Lambda)))\left\|\mathfrak{T}_k^{\mu,\nu}\right\|\|\cdot\|_{\mathrm{Lip}(\mathfrak{F})}$$

Theorem (6.2.78) [260]. Let A and B be self-adjoint operators. Suppose that $\sigma(A)$ is finite. Then

$$\|f(A)R - Rf(B)\| \le C(1 + \log(\operatorname{card}(\Lambda)))\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|AR - RB\|$$

for all bounded operators R and $f \in \mathrm{Lip}(\sigma(A) \cup \sigma(B))$, where C is a numerical constant.

Proof. The result follows from Theorem (6.2.77) if we take into account the following generalizations of (30) and (32) (see [287]):

$$f(A)R - Rf(B) = \iint\limits_{\sigma(A)\times\sigma(B)} (\mathfrak{D}_0 f)(x,y)\, dE_A(x)(AR - RB)dE_B(y)$$

and

$$\left\|\iint\limits_{\sigma(A)\times\sigma(B)} (\mathfrak{D}_0 f)(x,y)\, dE_A(x)(AR - RB)dE_B(y)\right\| \le \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{\sigma(A)\times\sigma(B)}}\|AR - RB\|$$

which proves the result.

Corollary (6.2.79) [260]. Let A,B be self-adjoint operators and let R be a linear operator on $\mathbb{C}^n$. Then

$$\|f(A)R - Rf(B)\| \le C(1 + \log n)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|AR - RB\| \qquad (55)$$

for every function f on $\sigma(A) \cup \sigma(B)$, where C is a numerical constant.

Remark (6.2.80). Note that in the special case $f(t) = |t|$ inequality (55) is well-known, see, e.g., [288]. This special case also follows from Matsaev's theorem, see [289, Chapter III, Theorem 4.2] (see also [290] where a finite-dimensional improvement of Matsaev's theorem was obtained).

Remark (6.2.81). We also would like to note that inequality (55) is sharp. Indeed, it follows immediately from Lemma 15 of [288] that for each positive integer n there exist $n \times n$ self-adjoint matrices A and R such that

$$\||A|R - R|A|\| \le \text{const} \log(1 + n)\|AR - RA\| \quad \text{and} \quad AR - RA \ne 0. \qquad (56)$$

We also refer to [265] where inequality (56) is essentially contained. Moreover, (56) can be deduced from the results of Matsaev and Gohberg mentioned above.

The following result is a special case of Corollary (6.2.79) that corresponds to $R = I$.

Theorem (6.2.82)[260]: Let A, B be self-adjoint operators on $\mathbb{C}^n$. Then

$$\|f(A) - f(B)\| \le C(1 + \log n)\|f\|_{\mathrm{Lip}(\sigma(A)\cup\sigma(B))}\|A - B\|$$

for every function f on $\sigma(A) \cup \sigma(B)$, where C is an absolute constant.

Remark (6.2.83). The estimate in Theorem (6.2.82) is also sharp. Indeed, for each positive integer n there exist $n \times n$ self-adjoint matrices A and B such that $A \neq B$ and

$$\||A| - |B|\| \geq \text{const} \log(1 + n)\|A - B\|.$$

This follows easily from (56), see the proof of Theorem 10.1 in [261].

Definition (6.2.84)[260]: Let $\mathfrak{F}$ be a nonempty compact subset of $\mathbb{R}$. Recall that for $\varepsilon > 0$, the ε-entropy $K_\varepsilon(\mathfrak{F})$ of $\mathfrak{F}$ is defined as

$$K_\varepsilon(\mathfrak{F}) \stackrel{\text{def}}{=} \inf \log(\operatorname{card}(\Lambda)),$$

where the infimum is taken over all $\Lambda \subset \mathbb{R}$ such that Λ is an ε-net of $\mathfrak{F}$. The following result is a generalization of Theorem (6.2.78). On the other hand, it improves inequality (23) obtained in [261].

Theorem (6.2.85)[260]: Let A and B be self-adjoint operators and let R be bounded operator with $\|R\| \leq 1$. Suppose that $\sigma(A) \subset \mathfrak{F}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{R}$. Then for every $f \in \operatorname{Lip}(\sigma(A) \cup \sigma(B))$,

$$\| f(A)R - Rf(B)\| \leq \text{const}\,(1 + K_\varepsilon(\mathfrak{F}))\|f\|_{\operatorname{Lip}(\sigma(A)\cup\sigma(B))}\, \|AR - RB\|\ ,$$

where $\varepsilon \stackrel{\text{def}}{=} \|AR - RB\|$.

Proof. We repeat the argument of the proof of Theorem (6.2.33). Clearly, f can be extended to a Lipschitz function on $\mathbb{R}$ with the same Lipschitz constant. We can find a self-adjoint operator A_ε such that $A_\varepsilon A = AA_\varepsilon$,$\|A - A_\varepsilon\| \leq \varepsilon$, and $\log(\operatorname{card}(\sigma(A_\varepsilon))) \leq K_\varepsilon(\mathfrak{F})$. Then

$$\begin{aligned}\|f(A_\varepsilon)R - Rf(B)\| &\leq \text{const}\,(1 + K_\varepsilon(\mathfrak{F}))\|f\|_{\operatorname{Lip}(\sigma(A)\cup\sigma(B))}\|A_\varepsilon R - RB\| \\ &\leq 2\text{const}\delta(1 + K_\varepsilon(\mathfrak{F}))\|f\|_{\operatorname{Lip}(\sigma(A)\cup\sigma(B))}\end{aligned}$$

by Theorem (6.2.79). It remains to observe that since A commutes wit A_ε, we have

$$\begin{aligned}\|f(A)R - Rf(B)\| &\leq \|f(A) - f(A_\varepsilon)\| + \|f(A_\varepsilon)R - Rf(B)\| \\ &\leq \|f\|_{\operatorname{Lip}(\sigma(A))} + \|f(A_\varepsilon)R - Rf(B)\|\end{aligned}$$

Corollary (6.2.86)[260]: Let A and B be self-adjoint operators and let $\sigma(A) \subset \mathfrak{F}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{R}$. Then for every $f \in \operatorname{Lip}(\sigma(A) \cup \sigma(B))$,

$$\|f(A) - f(B)\| \leq \text{const}\,(1 + K_\varepsilon(\mathfrak{F}))\|f\|_{\operatorname{Lip}(\sigma(A)\cup\sigma(B))}\|A - B\|,$$

where $\varepsilon \stackrel{\text{def}}{=} \|A - B\|$.

Proof. It suffices to put $R = I$.

If we apply Theorem (6.2.85) to the case $K = [a, b]$, we obtain the following estimate, which improves inequality (23) in the special case $R = I$.