Chapter 5

Multiple operator integrals and operator Holder Zygmund

We improve earlier results by Sten'kin. In order to do this, we give a new approach to multiple operator integrals. This approach improves the earlier approach given by Sten'kin. We also consider a similar problem for unitary operators. We study moduli of continuity, for which $\|f(A) - f(B)\| \leq \text{const}\,\omega(\|A - B\|)$ for self-adjoint A and B, and for an arbitrary function f in Λ_ω. We obtain similar estimates for commutators $f(A)Q - Qf(A)$ and quasicommutators $f(A)Q - Qf(B)$. Finally,we estimate the norms of finite differences $\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} f(A + jK)$ for f in the class $\Lambda_{\omega,m}$ that is defined in terms of finite differences and a modulus continuity ω of order m. We also obtain similar results for unitary operators and for contractions.

Section (5.1): Higher Operator Derivative

If A is a bounded self-adjoint operator on Hilbert space, the spectral theorem allows one for a Borel function φ on the real line $\mathbb{R}$ to define the function $\varphi(A)$ of A. We are going to study smoothness property of the map $A \to \varphi(A)$. It is easy tosee that if this map is differentiable (in the sense of Gâteaux), then φ is continuously differentiable.

If K is another bounded self-adjoint operator, consider the function $t \to \varphi(A + tK)$, $t \in \mathbb{R}$. In [179] it was shown that if $\varphi \in C^2(\mathbb{R})$ (i.e., is twice continuously differentiable), then the map $t \to \varphi(A + tK)$ is norm differentiable and

$$\frac{d}{ds}\left(\varphi(A + tK)\right)\Big|_{s=0} = \int\int \frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} dE_A(\lambda) K dE_A(\mu) \qquad (1)$$

where E_A is the spectral measure of A. Note that in the case $\lambda = \mu$ we assume that

$$\frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} = \varphi'(\lambda).$$

The expression on the right-hand side of (1) is a double operator integral. Later Birman and Solomyak developed their beautiful theory of double operator integrals in [180-182](see also [183]).

If we integrate a function on $\mathbb{R}^d$(or$\mathbb{T}^d$) and the domain of integration is not specified, it is assumed that the domain of integration is $\mathbb{R}^d$(or$\mathbb{T}^d$).

Birman and Solomyak relaxed in [182] the assumptions on φ under which (1) holds. They also considered the case of an unbounded self-adjoint operator A. However, it turned out that the condition $\varphi \in C^1(\mathbb{R})$is not sufficient for the differentiability of the function $t \to \varphi(A + tK)$ even in the case of bounded A. This can be deducedfrom an explicit example constructed by Farforovskaya [185] (in fact, this can also be deduced from an example given in [184]).

In [186] a necessary condition on φ for the differentiability of the function$t \to \varphi(A + tK)$ for all A and K was found. That necessary condition was deducedfrom the nuclearity criterion for Hankel operators (see the monograph [187]) and it implies that the condition

$\varphi \in C^1(\mathbb{R})$ is not sufficient. We also refer to[188] where a necessary condition is given in the case of an unbounded self-adjoint operator A.

Sharp sufficient conditions on φ for the differentiability of the function $t \to \varphi(A + tK)$ were obtained in [186] in the case of bounded self-adjoint operators and in[188] in the case of an unbounded self-adjoint operator A. In particular, it follows fromthe results of [188] that if φ belongs to the homogeneous Besov space $\boldsymbol{B}_{\infty 1}^1(\mathbb{R})$,$A$ is a self-adjoint operator and K is a bounded self-adjoint operator, then the function$t \to \varphi(A + tK)$is differentiable and (1) holds. Inthe case of bounded self-adjoint operators formula (1) holds if φ belongs to $\boldsymbol{B}_{\infty 1}^1(\mathbb{R})$locally (see [186]).

A similar problem for unitary operators was considered in [182] and later in [186]. Let φ be a function on the unit circle $\mathbb{T}$. For a unitary operator U and a bounded self-adjoint operator A, consider the function $t \to \varphi(e^{itA}U)$. It was shown in [186] that ifφ belongs to the Besov space $\boldsymbol{B}_{\infty 1}^1$, then the function $t \to \varphi(e^{itA}U)$ is differentiable and

$$\frac{d}{ds}\varphi(e^{isA}U)\Big|_{s=0} = i\left(\iint \tau \frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} dE_U(\lambda) A dE_U(\mu)\right) U \qquad (2)$$

(earlier this formula was obtained in [182] under more restrictive assumptions on φ). We refer the reader to [186] and [188] for necessary conditions. We also mention herethe paper [189], which slightly improves the sufficient condition $\varphi \in \boldsymbol{B}_{\infty 1}^1$.

The problem of the existence of higher derivatives of the function $t \to \varphi(A + tK)$was studied by Sten'kin in [190]. He showed that under certain conditions on φ the function $t \to \varphi(A + tK)$ has m derivatives and

$$\frac{d^m}{ds^m}\left(\varphi(A_s)\right)\Big|_{s=0} = m! \underbrace{\int \dots \int}_{m+1} (\mathfrak{D}^m \varphi)(\lambda_1, \dots, \lambda_{m+1}) dE_A(\lambda_1) K \dots K dE_A(\lambda_{m+1}), \qquad (3)$$

where for a k times differentiable functionφ the divided differences $\mathfrak{D}^k\varphi$ of order k are defined inductively as follows:

$$\mathfrak{D}^0\varphi \stackrel{\text{def}}{=} \varphi;$$

if $k \geq 1$, then

$$(\mathfrak{D}^k\varphi)(\lambda_1, \dots, \lambda_{k+1}) \stackrel{\text{def}}{=} \begin{cases} \dfrac{(\mathfrak{D}^{k-1}\varphi)(\lambda_1, \dots, \lambda_{k-1}, \lambda_k) - (\mathfrak{D}^{k-1}\varphi)(\lambda_2, \dots, \lambda_{k-1}, \lambda_{k+1})}{\lambda_k - \lambda_{k+1}}, & \lambda_k \neq \lambda_{k+1}, \\ \dfrac{\partial}{\partial t}\left((\mathfrak{D}^{k-1}\varphi)(\lambda_1, \dots, \lambda_{k-1}, t)\right)\Big|_{t=\lambda_k}, & \lambda_k = \lambda_{k+1}, \end{cases}$$

(the definition does not depend on the order of the variables). We are also going to use the notation

$$\mathfrak{D}\varphi = \mathfrak{D}^1\varphi.$$

The Birman–Solomyak theory of double operator integrals does not generalize to the case of multiple operator integrals. In [191] multiple operator integrals

$$\underbrace{\int \dots \int}_{k} \psi(\lambda_1, \dots, \lambda_{m+1}) dE_1(\lambda_1) T_1 dE_2(\lambda_2) T_2 \dots T_{k-1} dE_k(\lambda_k),$$

were defined for bounded operators $T_1, T_2, \dots, T_{k-1}$ and sufficiently smooth functions ψ.
In [190], Sten'kin considered iterated integration and he defined multiple operator integrals for a certain class of functions ψ. However, the approaches of [191] and [190] inthe case $k = 2$ lead to a considerably smaller class of functions ψ than the Birman–Solomyak approach. In particular the functionψ identically equal to 1, is not integrablein the sense of the approach developed in [190], while it is very natural to assume that

$$\underbrace{\int \dots \int}_{k} dE_1(\lambda_1)T_1 dE_2(\lambda_2)T_2 \dots T_{k-1} dE_k(\lambda_k) = T_1T_2 \dots T_{k-1}.$$

We use a different approach to the definition of multiple operator integrals. The approach is based on integral projective tensor products. In the case$k = 2$ our approach produces the class of integrable functions that coincides with theclass of so-called Schur multipliers, which is the maximal possible class of integrable functions in the case $k = 2$.
We also mention here the paper by Solomyak and Sten'kin [192], in which the authors found sufficient conditions for the existence of multiple operator integrals in the case when

$$\psi(\lambda_1, \dots, \lambda_k) = (\mathfrak{D}^{k-1}\varphi)(\lambda_1, \dots, \lambda_k).$$

Our approach allows us to improve the results of [192] and Sten'kin's results on the existence of higher order derivatives of the function $t \to \varphi(A + tK)$. We prove that formula (3) holds for functions φ in the intersection $\boldsymbol{B}^m_{\infty 1}(\mathbb{R}) \cap \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$homogeneous Besov spaces.
Note that the Besov spaces $\boldsymbol{B}^1_{\infty 1}$and $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$ appear in a natural way when studying the applicability of the Lifshits–Krein trace formula for trace class perturbations(see [186] and [188]), while the Besov spaces $\boldsymbol{B}^2_{\infty 1}$ and $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ arise when studying the applicability of the Koplienko–Neidhardt trace formulae for Hilbert–Schmidtperturbations (see [193]).
It is also interesting to note that the Besov class $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ appears in a natural wayin perturbation theory in [194], where the following problem is studied: in which case

$$\varphi(T_f) - T_{\varphi \circ f} \in \boldsymbol{S}_1 ?$$

($T_g$is a Toeplitz operator with symbol g.)
We obtain similar results in the case of unitary operators and generalize formula(2) to the case of higher derivatives.
Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. The Besov class $\boldsymbol{B}^s_{pq}$of functions (or distributions)on $\mathbb{T}$can be defined in the following way. Let w be a C^∞function on $\mathbb{R}$such that

$$w \ge 0, \quad \operatorname{supp} w \subset \left[\frac{1}{2}, 2\right], \quad \text{and} \quad \sum_{-\infty}^{\infty} w(2^n x) = 1 \text{ for } \quad x > 0. \qquad (4)$$

Consider the trigonometric polynomialsW_n, and $W_n^\#$ defined by

$$W_n(z) = \sum_{k \in \mathbb{Z}} w\left(\frac{k}{2^n}\right) z^k, \quad n \ge 1, \quad W_0(z) = \bar{z} + 1 + z, \quad \text{and}$$

$$W_n^\#(z) = \overline{W_n(z)}, \quad n \ge 0.$$

Then for each distribution φon $\mathbb{T}$,

$$\varphi = \sum_{n\geq 0} \varphi * W_n + \sum_{n\geq 0} \varphi * W_n^{\#}.$$

The Besov class $\boldsymbol{B}_{pq}^s$consists of functions (in the case $s > 0$) or distributions φ on $\mathbb{T}$such that

$$\{\|2^{ns}\varphi * W_n\|_{L^p}\}_{n\geq 0} \in \ell^q \text{and} \{\|2^{ns}\varphi * W_n^{\#}\|_{L^p}\}_{n\geq 1} \in \ell^q$$

Besov classes admit many other descriptions. For $s > 0$, the space $\boldsymbol{B}_{pq}^s$admits the following characterization. A function$i$belongs to $\boldsymbol{B}_{pq}^s,\ s > 0$, if and only if

$$\int_{\mathbb{T}} \frac{\|\Delta_\tau^n\|_{L^p}^q}{|1-\tau|^{1+sq}} d\boldsymbol{m}(\tau) < \infty \quad \text{for} \quad q < \infty$$

and

$$\sup_{\tau\neq 1} \frac{\|\Delta_\tau^n\|_{L^p}}{|1-\tau|^s} < \infty \quad \text{for} \quad q = \infty,$$

Where $\boldsymbol{m}$ is normalized Lebesgue measure on $\mathbb{T}$, n is an integer greater than s and Δ_τ is the difference operator: $(\Delta_\tau f)(\tau\zeta) = f(\tau\zeta) - f(\zeta),\ \zeta \in \mathbb{T}$.

To define (homogeneous) Besov classes $\boldsymbol{B}_{pq}^s(\mathbb{R})$ on the real line, we consider the same function w as in (4) and define the functions W_n and $W_n^{\#}$ on $\mathbb{R}$ by

$$\mathcal{F}W_n(x) = w\left(\frac{x}{2^n}\right), \qquad \mathcal{F}W_n^{\#}(x) = \mathcal{F}W_n(-x),\ \ n \in \mathbb{Z},$$

where $\mathcal{F}$ is the Fourier transform. The Besov class $\boldsymbol{B}_{pq}^s(\mathbb{R})$ consists of distributions φ on $\mathbb{R}$ such that

$$\{\|2^{ns}\varphi * W_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}) \text{and} \{\|2^{ns}\varphi * W_n^{\#}\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}).$$

According to this definition, the space $\boldsymbol{B}_{pq}^s(\mathbb{R})$contains all polynomials However, it is not necessary to include all polynomials.

We need only Besov spaces $\boldsymbol{B}_{\infty 1}^d,\ d \in \mathbb{Z}_+$. In the case of functions onthe real line it is convenient to restrict the degree of polynomials in $\boldsymbol{B}_{\infty 1}^d(\mathbb{R})$by . Itis also convenient to consider the following seminorm on $\boldsymbol{B}_{\infty 1}^d(\mathbb{R})$:

$$\|\varphi\|_{\boldsymbol{B}_{\infty 1}^d(\mathbb{R})} = \sup_{x\in\mathbb{R}} \left|\varphi^{(d)}(t)\right| + \sum_{n\in\mathbb{Z}} 2^{nd}\, \|\varphi * W_n\|_{L^\infty} + \sum_{n\in\mathbb{Z}} 2^{nd}\, \|\varphi * W_n^{\#}\|_{L^\infty}.$$

The classes $\boldsymbol{B}_{\infty 1}^d(\mathbb{R})$ can be described as classes of function on $\mathbb{R}$ in the followingway:

$$\varphi \in \boldsymbol{B}_{\infty 1}^d(\mathbb{R}) \iff \sup_{x\in\mathbb{R}} \left|\varphi^{(d)}(t)\right| + \int_{\mathbb{R}} \frac{\left\|\Delta_t^{d+1}\varphi\right\|_{L^\infty}}{|\tau|^{d+1}} dt < \infty,$$

Where Δ_t is the difference operator defined by $(\Delta_t\varphi)(x) = \varphi(x+t) - \varphi(x)$.We refer to [195] for more detailed information on Besov classes. We define multiple operator integrals using integral projective tensor products of L^∞-spaces. However, we begin with a brief review of the theory of doubleoperator integrals that was developed by Birman and Solomyak in [BS1–BS3]. We state a description of the Schur multipliers associated with two spectral measures in terms of integral projective tensor products. This suggests the idea to define multiple operator integrals with the help of integral projective tensor products.

Double operator integrals. Let $(\mathcal{X}, E)$ and $(\mathcal{Y}, F)$ be spaces with spectral measures$E$ and F on a Hilbert space $\mathcal{H}$. Let us first define double operator integrals

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda)TdF(\mu), \tag{5}$$

for bounded measurable functions ψ and operators T of Hilbert Schmidt class $\boldsymbol{S}_2$. Consider the spectral measure $\mathcal{E}$whose values are orthogonal projections on the Hilbert space$\boldsymbol{S}_2$, which is defined by

$$\mathcal{E}(\Lambda \times \Delta)T = E(\Lambda)TF(\Delta), \quad T \in \boldsymbol{S}_2,$$

ΛandΔ being measurable subsets of $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathcal{E}$ extends to a spectral measure on $\mathcal{X} \times \mathcal{Y}$ and if ψ is a bounded measurable function on $\mathcal{X} \times \mathcal{Y}$, by definition,

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda)TdF(\mu) = \left(\int_{\mathcal{X}\times\mathcal{Y}} \psi d\mathcal{E}\right)T.$$

Clearly,

$$\left\|\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda)TdF(\mu)\right\|_{\boldsymbol{S}_2} \le \|\psi\|_{L^\infty}\|T\|_{\boldsymbol{S}_2}.$$

If

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda)TdF(\mu) \in \boldsymbol{S}_2$$

for every $T \in \boldsymbol{S}_2$, we say that ψ is a Schur multiplier (of $\boldsymbol{S}_1$) associated with thespectral measure E and F. In this case by duality the map

$$T \longmapsto \int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda)TdF(\mu), \qquad T \in \boldsymbol{S}_2, \tag{6}$$

extends to a bounded linear transformer on the space of bounded linear operators on $\mathcal{H}$. We denote by $\mathfrak{M}(E,F)$ the space of Schur multipliers of $\boldsymbol{S}_1$associated with the spectral measures E and F. The norm of ψ in $\mathfrak{M}(E,F)$ is, by definition, the norm of the transformer (6) on the space of bounded linear operators.

In [182] it was shown that if A is a self-adjoint operator (not necessarily bounded), K is a bounded self-adjoint operator and ifφ is a continuously differentiable function on $\mathbb{R}$ such that the divided difference $\mathfrak{D}_\varphi$ is a Schur multiplier of $\boldsymbol{S}_1$with respect tothe spectral measures of A and $A+K$, then

$$\varphi(A+K) - \varphi(A) = \iint \frac{\varphi(\lambda)-\varphi(\mu)}{\lambda-\mu}\, dE_{A+K}(\lambda)KdE_A(\mu) \tag{7}$$

and

$$\|\varphi(A+K)-\varphi(A)\| \le \mathrm{const}\|\varphi\|_{\mathfrak{M}(E_A,E_{A+K})}\|K\|,$$

i.e., φ is an operator Lipschitz function.

It is easy to see that if a function ψ on $\mathcal{X} \times \mathcal{Y}$ belongs to the projective tensor product $L^\infty(E)\,\widehat{\otimes}\,L^\infty(F)$ of $L^\infty(E)$ and $L^\infty(F)$ (i.e., ψ admits a representation)

$$\psi(\lambda,\mu) = \sum_{n\geq 0} f_n(\lambda)\, g_n(\mu),$$

where $f_n \in L^\infty(E)$, $g_n \in L^\infty(F)$, and

$$\sum_{n\geq 0} \|f_n\|_{L^\infty} \|g_n\|_{L^\infty} < \infty,$$

then $\psi \in \mathfrak{M}(E,F)$. For such functions ψ we have

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda) T dF(\mu) = \sum_{n\geq 0} \left(\int_{\mathcal{X}} f_n\, dE\right) T \left(\int_{\mathcal{Y}} g_n\, dF\right).$$

More generally, ψ is a Schur multiplier of $\boldsymbol{S}_1$ if ψ belongs to the integral projective tensor product $L^\infty(E)\,\widehat{\otimes}_i\, L^\infty(F)$ of $L^\infty(E)$ and $L^\infty(F)$ (i.e., ψ admits a representation

$$\psi(\lambda,\mu) = \int_Q f(\lambda,x) g(\mu,x) d\sigma(x), \qquad (8)$$

where(Q,σ) is a measure space, f is a measurable function on $\mathcal{X} \times Q$, g is a measurable function on $\mathcal{Y} \times Q$, and

$$\int_Q \|f(\cdot,x)\|_{L^\infty(E)} \|g(\cdot,x)\|_{L^\infty(F)} d\sigma(x) < \infty. \qquad (9)$$

If $\psi \in L^\infty(E)\,\widehat{\otimes}_i\, L^\infty(F)$, then

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \psi(\lambda,\mu)\, dE(\lambda) T dF(\mu) = \int_Q \left(\int_{\mathcal{X}} f(\lambda,x) dE(\lambda)\right) T \left(\int_{\mathcal{Y}} g(\mu,x) dF(\mu)\right) d\sigma(x).$$

Clearly, the function $x \mapsto \left(\int_{\mathcal{X}} f(\lambda,x) dE(\lambda)\right) T \left(\int_{\mathcal{Y}} g(\mu,x) dF(\mu)\right)$ is weakly mea-surable

$$\int_Q \left\| \left(\int_{\mathcal{X}} f(\lambda,x) dE(\lambda)\right) T \left(\int_{\mathcal{Y}} g(\mu,x) dF(\mu)\right) \right\| d\sigma(x) < \infty.$$

It turns out that all Schur multipliers can be obtained in this way. More precisely, the following result holds (see [186]):

Theorem on Schur multipliers Letψ be a measurable function on $\mathcal{X} \times \mathcal{Y}$. Thefollowing are equivalent:

(i) $\psi \in \mathfrak{M}(E,F)$;

(ii) $\psi \in L^\infty(E)\,\widehat{\otimes}_i\, L^\infty(F)$;

(iii) there exist measurable functions f on $\mathcal{X} \times Q$ and $g$on $\mathcal{Y} \times Q$ such that (8)holds and

$$\left\| \int_Q |f(\cdot,x)|^2 d\sigma(x) \right\|_{L^\infty(E)} \left\| \int_Q |g(\cdot,x)|^2 d\sigma(x) \right\|_{L^\infty(F)} < \infty. \qquad (10)$$

Note that the implication (iii)⇒(ii) was established in [182]. Note also that in thecase of matrix Schur multipliers (this corresponds to discrete spectral measures ofmultiplicity 1) the equivalence of (i) and (ii) was proved in [196].

It is interesting to observe that if f and g satisfy (9), then they also satisfy (10), but the converse is false. However, if ψ admits a representation of the form (8) with f and g

satisfying (10), then it also admits a (possibly different) representation of the form (8) with f and g satisfying (9).

In a similar way we can define the projective tensor product $A \widehat{\otimes} B$ andthe integral projective tensor product $A \widehat{\otimes}_i B$ of arbitrary Banach functions spaces $A$and B.

The equivalence of (i) and (ii) in the Theorem on Schur multipliers suggests an idea how to define multiple operator integrals.

Multiple operator integrals We can easily extend the definition of the projective tensor product and the integral projective tensor product to three ormore function spaces.

Consider first the case of triple operator integrals.

Let $(\mathcal{X}, E)$, $(\mathcal{Y}, F)$, and $(\mathcal{Z}, G)$ be spaces with spectral measures E,F, and G on a Hilbert space $\mathcal{H}$. Suppose that ψ belongs to the integral projective tensor product$L^\infty(E) \widehat{\otimes}_i L^\infty(F) \widehat{\otimes}_i L^\infty(G)$, i.e., ψ admits a representation

$$\psi(\lambda, \mu, \nu) = \int_Q f(\lambda, x) g(\mu, x) h(\nu, x)\, d\sigma(x), \tag{11}$$

where (Q, σ) is a measure space, f is a measurable function on $\mathcal{X} \times Q$, g is a measurable function on $\mathcal{Y} \times Q$, h is a measurable function on $\mathcal{Z} \times Q$, and

$$\int_Q \|f(\cdot, x)\|_{L^\infty(E)} \|g(\cdot, x)\|_{L^\infty(F)} \|h(\cdot, x)\|_{L^\infty(G)} d\sigma(x) < \infty. \tag{12}$$

We define the norm$\|\psi\|_{L^\infty \widehat{\otimes}_i L^\infty \widehat{\otimes}_i L^\infty}$in the space$L^\infty(E) \widehat{\otimes}_i L^\infty(F) \widehat{\otimes}_i L^\infty(G)$as theinfimum of the left-hand side of (12) over all representations (11).

Suppose now that T_1 and T_2 be bounded linear operators on $\mathcal{H}$. For a function ψ in$L^\infty(E) \widehat{\otimes}_i L^\infty(F) \widehat{\otimes}_i L^\infty(G)$of the form (11), we put

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Z}} \psi(\lambda, \mu, \nu) dE(\lambda)\, T_1 dF(\mu) T_2 dG(\nu) \stackrel{\text{def}}{=} \int_Q \left(\int_{\mathcal{X}} f(\lambda, x) dE(\lambda) \right) T_1 \left(\int_{\mathcal{Y}} g(\mu, x) dF(\mu) \right) \times T_2 \left(\int_{\mathcal{Z}} h(\nu, x) dG(\nu) \right) d\sigma(x). \tag{13}$$

The following lemma shows that the triple operator integral

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Z}} \psi(\lambda, \mu, \nu) dE(\lambda)\, T_1 dF(\mu) T_2 dG(\nu)$$

is well-defined.

Lemma(5.1.1)[178]. Suppose that $\psi \in L^\infty(E) \widehat{\otimes}_i L^\infty(F) \widehat{\otimes}_i L^\infty(G)$. Then the right-hand side of (13) does not depend on the choice of a representation (11) and

$$\left\| \int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Z}} \psi(\lambda, \mu, \nu) dE(\lambda)\, T_1 dF(\mu) T_2 dG(\nu) \right\| \le \|\psi\|_{L^\infty \widehat{\otimes}_i L^\infty \widehat{\otimes}_i L^\infty} \cdot \|T_1\| \cdot \|T_2\|. \tag{14}$$

Proof. To show that the right-hand side of (13) does not depend on the choice of a representation (11), it suffices to show that if the right-hand side of (11) is the zero function, then the right-hand side of (13) is the zero operator. Denote our Hilbert spaceby $\mathcal{H}$ and let $\zeta \in \mathcal{H}$. We have

$$\int_{\mathcal{Z}} \left(\int_Q f(\lambda, x) g(\mu, x) h(\nu, x)\, d\sigma(x) \right) dG(\nu) = 0 \quad \text{for almost } \lambda \text{ and } \mu,$$

and so for almost all λ and μ ,

$$\int_Q f(\lambda, x) g(\mu, x) T_2 \left(\int_{\mathcal{Z}} h(\nu, x)\, dG(\nu) \right) \zeta\, d\sigma(x)$$

$$= T_2 \int_{\mathcal{Z}} \left(\int_Q f(\lambda, x) g(\mu, x) h(\nu, x)\, d\sigma(x) \right) dG(\nu) \zeta = 0 .$$

Putting

$$\xi_x = T_2 \left(\int_{\mathcal{Z}} h(\nu, x)\, dG(\nu) \right) \zeta,$$

we obtain

$$\int_Q f(\lambda, x) g(\mu, x) \xi_x\, d\sigma(x) = 0 \quad \text{for almost } \lambda \text{ and } \mu.$$

We can realize the Hilbert space $\mathcal{H}$ as a space of vector functions so that integration with respect to the spectral measure F corresponds to multiplicat-on. It follows that

$$\int_Q f(\lambda, x) T_1 \left(\int_{\mathcal{Y}} g(\mu, x) dF(\mu) \right) \xi_x d\sigma(x) = T_1 \int_{\mathcal{Y}} \int_Q f(\lambda, x) g(\mu, x) \xi_x\, d\sigma(x) dF(\mu) = 0$$

for almost all λ. Let now

$$\eta_x = T_1 \left(\int_{\mathcal{Y}} g(\mu, x)\, dF(\mu) \right) \xi_x,$$

We have

$$\int_Q f(\lambda, s) \eta_x\, d\sigma(x) = 0 \quad \text{for almost all } \lambda.$$

Now we can realize $\mathcal{H}$ as a space of vector functions so that integration with respect to the spectral measure E corresponds to multiplication. It follows that

$$\int_Q \left(\int_{\mathcal{X}} f(\lambda, x) dE(\lambda) \right) \eta_x d\sigma(x) = \int_{\mathcal{X}} \int_Q f(\lambda, x) \eta_x\, d\sigma(x) dF(\lambda) = 0.$$

This exactly means that the right-hand side of (13) is the zero operator.
Inequality (14) follows immediately from (13).
In a similar way we can define multiple operator integrals

$$\underbrace{\int \cdots \int}_{m+1} \psi(\lambda_1, \ldots, \lambda_{m+1}) dE_1(\lambda_1) T_1 dE_2(\lambda_2) T_2 \ldots T_{k-1} dE_k(\lambda_k)$$

for functions ψ in the integral projective tensor product $\underbrace{L^\infty(E_1)\,\widehat{\otimes}_i \cdots \widehat{\otimes}_i\, L^\infty(E_k)}_{k}$ (the latter space is defined in the same way as in the case $k = 2$).

Let U be a unitary operator and A a bounded self-adjoint on Hilbert space. For $t \in \mathbb{R}$, we put
$$U_t = e^{itA}U.$$
We obtain sharp conditions on the existence of higher operator derivatives of the function $t \mapsto \varphi(U_t)$.
Recall that it was proved in [186] that for a function φ in the Besov space $\boldsymbol{B}^1_{\infty 1}$the divided difference$\mathfrak{D}_\varphi$belongs to the projective tensor product $C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})$, andso for arbitrary unitary operators U and V the following formula holds:
$$\varphi(V) - \varphi(U) = \iint \frac{\varphi(\lambda) - \varphi(\mu)}{\lambda - \mu} dE_V(\lambda)(V - U)dE_U(\mu). \quad (15)$$
First we state the main results for second derivatives.
Theorem(5.1.2)[178]. If $\varphi \in \boldsymbol{B}^1_{\infty 1}$, then
$$(\mathfrak{D}^2\varphi) \in C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})$$
Proof . It is easy to see that
$$(\mathfrak{D}^2\varphi)(z_1, z_2, z_3) = \sum_{i,j,k\geq 0} \hat{\varphi}\,(i + j + k + 2)z_1^i z_2^j z_3^k + \sum_{i,j,k\leq 0} \hat{\varphi}\,(i + j + k - 2)z_1^i z_2^j z_3^k, \qquad (16)$$
where$\hat{\varphi}(n)$ is the nth Fourier coefficient of φ . We prove that
$$\sum_{i,j,k\geq 0} \hat{\varphi}\,(i + j + k + 2)z_1^i z_2^j z_3^k \in C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T}).$$
The fact that
$$\sum_{i,j,k\leq 0} \hat{\varphi}\,(i + j + k - 2)z_1^i z_2^j z_3^k \in C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T}).$$
can be proved in the same way. Clearly, we can assume that $\hat{\varphi}(j) = 0$ for $j < 0$.
We have
$$\sum_{i,j,k\geq 0} \hat{\varphi}\,(i + j + k + 2)z_1^i z_2^j z_3^k = \sum_{i,j,k\geq 0} \alpha_{ijk}\hat{\varphi}\,(i + j + k + 2)z_1^i z_2^j z_3^k$$
$$+ \sum_{i,j,k\geq 0} \beta_{ijk}\hat{\varphi}\,(i + j + k + 2)\,z_1^i z_2^j z_3^k + \sum_{i,j,k\geq 0} \gamma_{ijk}\hat{\varphi}\,(i + j + k + 2)z_1^i z_2^j z_3^k,$$
where
$$\alpha_{ijk} = \begin{cases} \dfrac{1}{3}, & i = j = k = 0, \\ \dfrac{i}{i + j + k}, & i + j + k \neq 0, \end{cases}$$
$$\beta_{ijk} = \begin{cases} \dfrac{1}{3}, & i = j = k = 0, \\ \dfrac{j}{i + j + k}, & i + j + k \neq 0, \end{cases}$$

and

$$\gamma_{ijk} = \begin{cases} \frac{1}{3}, & i = j = k = 0, \\ \frac{k}{i+j+k}, & i+j+k \neq 0. \end{cases}$$

Clearly, it suffices to show that

$$\sum_{i,j,k\geq 0} \alpha_{ijk} \hat{\varphi}\,(i+j+k+2) z_1^i z_2^j z_3^k \in C(\mathbb{T}) \hat{\otimes} C(\mathbb{T}) \hat{\otimes} C(\mathbb{T}). \qquad (17)$$

It is easy to see that

$$\sum_{i,j,k\geq 0} \alpha_{ijk} \hat{\varphi}\,(i+j+k+2) z_1^i z_2^j z_3^k$$

$$= \sum_{j,k\geq 0} \left(\left(\left((S^*)^{j+k+2} \varphi \right) * \sum_{i\geq 0} \alpha_{ijk} z^i \right) (z_1) \right) z_2^j z_3^k,$$

where S^* is backward shift, i.e., $(S^*)^k \varphi = \mathbb{P}_+ \bar{z}^k \varphi$ ($\mathbb{P}_+$ is the orthogonal projectionfrom L^2 onto the Hardy class H^2). Thus,

$$\left\| \sum_{i,j,k\geq 0} \alpha_{ijk} \hat{\varphi}\,(i+j+k+2) z_1^i z_2^j z_3^k \right\|_{L^\infty \hat{\otimes} L^\infty \hat{\otimes} L^\infty} \leq \sum_{j,k\geq 0} \left\| \left((S^*)^{j+k+2} \varphi \right) * \sum_{i\geq 0} \alpha_{ijk} z^i \right\|_{L^\infty}$$

Put

$$Q_m(z) = \sum_{i\geq m} \frac{i-m}{i} z^i, \quad m > 0 \quad \text{and} \quad Q_0(z) = \frac{1}{3} \sum_{i\geq 1} z^i.$$

Then it is easy to see that

$$\left\| \left((S^*)^{j+k+2} \varphi \right) * \sum_{i\geq 0} \alpha_{ijk} z^i \right\|_{L^\infty} = \left\| \psi * Q_{j+k} \right\|_{L^\infty},$$

where $\psi = (S^*)^2 \varphi$, and so

$$\left\| \sum_{i,j,k\geq 0} \alpha_{ijk} \hat{\varphi}\,(i+j+k+2) z_1^i z_2^j z_3^k \right\|_{L^\infty \hat{\otimes} L^\infty \hat{\otimes} L^\infty} \leq \sum_{j,k\geq 0} \left\| \psi * Q_{j+k} \right\|_{L^\infty} = \sum_{m\geq 0} (m+1) \| \psi * Q_m \|_{L^\infty}.$$

Consider the function r on $\mathbb{R}$ defined by

$$r(x) = \begin{cases} 1, & |x| \leq 1; \\ \frac{1}{|x|}, & |x| \geq 1. \end{cases}$$

It is easy to see that the Fourier transform $\mathcal{F}r$ of h belongs to $L^1(\mathbb{R})$. Define the functions $R_n, n \geq 1$, on $\mathbb{T}$ by

$$R_n(\zeta) = \sum_{k\in\mathbb{Z}} r\left(\frac{k}{n}\right) \zeta^k.$$

Lemma (5.1.3)[178].

$$\|R_n\|_{L^1} \le \text{const.}$$

Proof . For $N > 0$ consider the function ξ_N defined by

$$\xi_N(x) = \begin{cases} 1, & |x| \le N, \\ \frac{2N - |x|}{N}, & N \le |x| \le 2N, \\ 0, & |x| \ge 2N. \end{cases}$$

It is easy to see that $\mathcal{F}\xi_N \in L^1(\mathbb{R})$ and $\|\mathcal{F}\xi_N\|_{L^1(\mathbb{R})}$ does not depend on . Let

$$R_{N,n}(\zeta) = \sum_{k\in\mathbb{Z}} r\left(\frac{k}{n}\right)\xi_N\left(\frac{k}{n}\right)\zeta^k, \quad \zeta \in \mathbb{T}.$$

It was proved in Lemma 2 of [186] that $\left\|R_{N,n}\right\|_{L^1} \le \|\mathcal{F}(r\xi_N)\|_{L^1(\mathbb{R})}$. Since

$$\|\mathcal{F}(r\xi_N)\|_{L^1(\mathbb{R})} \le \|\mathcal{F}r\|_{L^1(\mathbb{R})}\|\mathcal{F}\xi_N\|_{L^1(\mathbb{R})} = \text{const},$$

it follows that the L^1-norms of $R_{N,n}$ are uniformly bounded. The result follows from the obvious fact that

$$\lim_{N\to\infty}\left\|R_n - R_{N,\_}\right\|_{L^2} = 0.$$

Let us complete the proof of Theorem (5.1.1).

For $f \in L^\infty$, we have

$$\|f * Q_m\|_{L^\infty} = \|f - f * R_m\|_{L^\infty} \le \|f\|_{L^\infty} + \|f * R_m\|_{L^\infty} \le \text{const}\|f\|_{L^\infty}.$$

Thus,

$$\sum_{m\ge0}(m+1)\|\psi * Q_m\|_{L^\infty} = \sum_{m\ge0}(m+1)\left\|\sum_{n\ge0}\psi * W_n * Q_m\right\|_{L^\infty} \le \sum_{m,n\ge0}(m+1)\|\psi * W_n * Q_m\|_{L^\infty}$$

$$= \sum_{n\ge0}\sum_{0\le m\le 2^{n+1}}(m+1)\|\psi * W_n * Q_m\|_{L^\infty} \le \text{const}\sum_{n\ge0}\sum_{0\le m\le 2^{n+1}}(m+1)\|\psi * W_n\|_{L^\infty}$$

$$\le \text{const}\sum_{n\ge0} 2^{2n}\,\|\psi * W_n\|_{L^\infty} \le \text{const}\|\psi\|_{\boldsymbol{B}^2_{\infty 1}},$$

where the W_n are defined .

This proves that

$$\sum_{i,j,k\ge0}\alpha_{ijk}\hat{\varphi}\,(i+j+k+2)z_1^i z_2^j z_3^k \in L^\infty(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})\,\widehat{\otimes}\,C(\mathbb{T})$$

and

$$\left\|\sum_{i,j,k\ge0}\alpha_{ijk}\hat{\varphi}\,(i+j+k+2)z_1^i z_2^j z_3^k\right\|_{L^\infty(\mathbb{T})\widehat{\otimes}C(\mathbb{T})\widehat{\otimes}C(\mathbb{T})} \le \text{const}\|\varphi\|_{\boldsymbol{B}^2_{\infty 1}} \qquad (18)$$

To prove (17), it suffices to represent φ as

$$\varphi = \sum_{n\ge0}\varphi * W_n.$$

Then we can apply the above reasoning to each polynomial $\varphi * W_n$. Since

$$\left(\left((S^*)^{j+k+2}\varphi * W_n\right) * \sum_{i\geq 0} \alpha_{i+j+k} z^i\right)$$

is obviously a polynomial, the above reasoning shows that

$$\sum_{i,j,k\geq 0} \alpha_{ijk} \widehat{\varphi * W_n}\,(i+j+k+2) z_1^i z_2^j z_3^k \in C(\mathbb{T}) \widehat{\otimes} C(\mathbb{T}) \widehat{\otimes} C(\mathbb{T})$$

and by (18),

$$\left\|\sum_{i,j,k\geq 0} \alpha_{ijk} \widehat{\varphi * W_n}\,(i+j+k+2) z_1^i z_2^j z_3^k\right\|_{C(\mathbb{T})\widehat{\otimes} C(\mathbb{T})\widehat{\otimes} C(\mathbb{T})} \leq \text{const}\|\varphi * W_n\|_{\boldsymbol{B}_{\infty 1}^2}$$

$$\leq \text{const} 2^{2n}\|\varphi * W_n\|_{L^\infty}.$$

The result follows now from the fact that

$\sum_{n\geq 0} 2^{2n}\|\varphi * W_n\|_{L^\infty} \leq \text{const}\|\varphi\|_{\boldsymbol{B}_{\infty 1}^2}$.

Theorem (5.1.4)[178]. Let φ be a function in the Besov class $B_{\infty 1}^1$, then the function $t \mapsto \varphi(U_t)$ has second derivative and

$$\frac{d}{ds}(\varphi(U_s))|_{s=0} = -2\left(\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)\, dE_U(\lambda) A dE_U(\mu) A dE_U(\nu)\right) U^2. \quad (19)$$

Note that by Theorem (5.1.2), the right-hand side of (19) makes sense and determines a bounded linear operator.

Proof . It follows from the definition of the second order divided difference that

$$(\mu-\nu)(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu) = (\mathfrak{D}\varphi)(\lambda,\mu) - (\mathfrak{D}\varphi)(\lambda,\nu). \quad (20)$$

By (15), we have

$$\frac{1}{t}\left(\frac{d}{ds}(\varphi(U_s))\Big|_{s=t} - \frac{d}{ds}(\varphi(U_s))\Big|_{s=0}\right)$$

$$= \frac{i}{t}\left(\iint (\mathfrak{D}\varphi)(\lambda,\nu) dE_{U_t}(\lambda) A\, dE_{U_t}(\nu) U_t - \iint (\mathfrak{D}\varphi)(\mu,\nu) dE_U(\mu) A\, dE_U(\nu) U\right)$$

$$= \frac{i}{t}\left(\iint (\mathfrak{D}\varphi)(\lambda,\nu) dE_{U_t}(\lambda) A\, dE_{U_t}(\nu) - \iint (\mathfrak{D}\varphi)(\mu,\nu) dE_U(\mu) A\, dE_U(\nu)\right) U_t$$

$$+ \frac{i}{t}\left(\iint (\mathfrak{D}\varphi)(\mu,\nu) dE_U(\mu) A\, dE_{U_t}(\nu) U_t - \iint (\mathfrak{D}\varphi)(\mu,\nu) dE_U(\mu) A\, dE_{U_t}(\nu) U\right)$$

$$+ \frac{i}{t}\left(\iint (\mathfrak{D}\varphi)(\lambda,\nu) dE_U(\lambda) A\, dE_{U_t}(\nu) - \iint (\mathfrak{D}\varphi)(\lambda,\nu) dE_U(\lambda) A\, dE_U(\mu)\right) U.$$

By (20), we have

$$\iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{U_t}(\lambda)AdE_{U_t}(\nu) - \iint (\mathfrak{D}\varphi)(\mu,\nu)dE_U(\mu)AdE_{U_t}(\nu)$$
$$= \iiint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{U_t}(\lambda)dE_U(\mu)AdE_{U_t}(\nu)$$
$$- \iiint (\mathfrak{D}\varphi)(\mu,\nu)dE_{U_t}(\lambda)dE_U(\mu)AdE_{U_t}(\nu)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)(\lambda-\mu)dE_{U_t}(\lambda)dE_U(\mu)AdE_{U_t}(\acute{\mathrm{G}})$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{U_t}(\lambda)U_t dE_U(\mu)AdE_{U_t}(\nu)$$
$$- \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{U_t}(\lambda)UdE_U(\mu)AdE_{U_t}(\nu)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{U_t}(\lambda)(e^{itA}-I)UdE_U(\mu)AdE_{U_t}(\nu).$$

Similarly,

$$\iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_U(\lambda)AdE_{U_t}(\nu) - \iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_U(\lambda)AdE_U(\mu)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_U(\lambda)AdE_U(\mu)(e^{itA}-I)UAdE_{U_t}(\nu).$$

Thus,

$$\frac{1}{t}\left(\frac{d}{ds}(\varphi(U_s))\Big|_{s=t} - \frac{d}{ds}(\varphi(U_s))\Big|_{s=0}\right)$$
$$= \frac{i}{t}\left(\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{U_t}(\lambda)(e^{itA}-I)UdE_U(\mu)AdE_{U_t}(\nu)\right)U_t$$
$$+ \frac{i}{t}\left(\iint (\mathfrak{D}\varphi)(\mu,\nu)dE_U(\mu)AdE_{U_t}(\nu)\,U_t - \iint (\mathfrak{D}\varphi)(\mu,\nu)dE_U(\mu)AdE_{U_t}(\nu)\,U\right)$$
$$+ \frac{i}{t}\left(\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_U(\lambda)AdE_U(\mu)\left(e^{it\text{짝}}-I\right)UdE_{U_t}(\nu)\right)U.$$

Since $\lim_{t\to 0}\|U_t - U\| = 0$, to complete the proof it suffices to show that

$$\lim_{t\to 0}\frac{1}{t}\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{U_t}(\lambda)(e^{itA}-I)UdE_U(\mu)AdE_{U_t}(\nu)$$
$$= i\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_U(\lambda)AdE_U(\mu)AdE_U(\nu)U, \quad (21)$$

$$\lim_{t\to 0}\iint (\mathfrak{D}^2\varphi)(\mu,\nu)UdE_U(\mu)AdE_{U_t}(\nu)$$
$$= \iint (\mathfrak{D}^2\varphi)(\mu,\nu)dE_U(\lambda)dE_U(\mu)AdE_U(\nu)U, \quad (22)$$

and

$$\lim_{t\to 0}\frac{1}{t}\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_U(\lambda)AdE_U(\mu)(e^{itA}-I)UdE_{U_t}(\nu)$$
$$= i\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_U(\lambda)AdE_U(\mu)AdE_U(\nu)U. \quad (23)$$

Let us prove (21). Since $\mathfrak{D}^2\varphi \in C(\mathbb{T}) \widehat{\otimes} C(\mathbb{T}) \widehat{\otimes} C(\mathbb{T})$, it suffices to show that for $f, g, h \in C(\mathbb{T})$,

$$\lim_{t\to 0}\frac{1}{t}\iiint f(\lambda)g(\mu)h(\nu)dE_{U_t}(\lambda)(e^{itA}-I)UdE_U(\mu)AdE_{U_t}(\nu)$$

$$= i\iiint f(\lambda)g(\mu)h(\nu)dE_U(\lambda)AdE_U(\mu)AdE_U(\nu)U. \qquad (24)$$

We have

$$\frac{1}{t}\iiint f(\lambda)g(\mu)h(\nu)dE_{U_t}(\lambda)(e^{itA}-I)UdE_U(\mu)AdE_{U_t}(\nu)$$

$$= f(U_t)\left(\frac{1}{t}(e^{itA}-I)U\right)g(U)Ah(U_t)$$

and

$$\iiint f(\lambda)g(\mu)h(\nu)dE_U(\lambda)AdE_U(\mu)AdE_U(\nu)U = f(U)Ag(u)Ah(U)U.$$

Since f and h are in $C(\mathbb{T})$, it follows that
(it suffices to prove this for trigonometric polynomials f and h which is evident). This together with the obvious fact

$$\lim_{t\to 0}\left(\frac{1}{t}(e^{itA}-I)\right) = iA$$

proves (24) which in turn implies (21).
The proof of (23) is similar. To prove (22), we observe that $\boldsymbol{B}^2_{\infty 1} \subset \boldsymbol{B}^1_{\infty 1}$ and usethe fact that $\mathfrak{D}^2\varphi \in C(\mathbb{T}) \widehat{\otimes} C(\mathbb{T})$ (this was proved in [186]). Again, it suffices to provethat for $f, g \in C(\mathbb{T})$,

$$\lim_{t\to 0}\iint f(\mu)g(\nu)dE_U(\mu)AdE_{U_t}(\nu) = \iint f(\mu)g(\nu)dE_U(\mu)AdE_U(\nu)$$

which follows from the obvious equality:

$$\lim_{t\to 0}\|g(U_t)-g(U)\| = 0.$$

The proofs of Theorems (5.1.1) and (5.1.2) given above generalize easily to the case of higher derivatives.

Theorem(5.1.5)[178]. Let m be a positive integer. If $\varphi \in \boldsymbol{B}^m_{\infty 1}$, then

$$\mathfrak{D}^2\varphi \in \underbrace{C(\mathbb{T}) \widehat{\otimes} \cdots \widehat{\otimes} C(\mathbb{T})}_{m+1}.$$

Theorem(5.1.6)[178]. Let m be a positive integer and let φ be a function in the Besov class$\boldsymbol{B}^m_{\infty 1}$, then the function $t \longmapsto \varphi(U_t)$ has mth derivative and

$$\frac{d^m}{ds^m}(\varphi(U_s))\bigg|_{s=0} = i^m m!\left(\underbrace{\int\ldots\int}_{m+1}(\mathfrak{D}^2\varphi)(\lambda_1,\ldots,\lambda_{m+1})dE_U(\lambda_1)A\ldots AdE_U(\lambda_{m+1})\right)U^m.$$

We consider the problem of the existence of higher derive- atives of the function

$$t \longmapsto \varphi(A_t) = \varphi(A+tK).$$

Here A is a self-adjoint operator (not necessarily bounded), K is a bounded self-adjoint operator, and $A_t \stackrel{\text{def}}{=} A + tK$.

In [188] it was shown that if $\varphi \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, then $\mathfrak{D}\varphi \in \mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R})$, where $\mathfrak{B}(\mathbb{R})$ is the space of bounded Borel functions on $\mathbb{R}$ equipped with the sup-norm, and so

$$\|\varphi(A + tK) - \varphi(A)\| \le \text{const}\|\varphi\|_{\boldsymbol{B}^1_{\infty 1}} \|K\|. \qquad (25)$$

In fact, the construction given in [188] shows that for $\varphi \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, the function $t \mapsto \varphi(A + tK)$ is differentiable and

$$\frac{d}{ds}(\varphi(A_s))\Big|_{s=0} = \iint (\mathfrak{D}\varphi)(\lambda, \mu) dE_A(\lambda) K dE_A(\mu). \qquad (26)$$

For completeness, we show briefly how to deduce (26) from the construction given in [188]. We are going to give a detailed proof in the case of higher derivatives.

We need the following.

Definition(5.1.7). A continuous function φ on $\mathbb{R}$ is called operator continuous if

$$\lim_{s \to 0} \|\varphi(A + tK) - \varphi(A)\|$$

for any self-adjoint operator A and any bounded self-adjoint operator K .

It follows from (25) that functions in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$ are operator continuous. It is alsoeasy to see that the product of two bounded operator continuous functions is operator continuous.

Proof of (26) . The construction given in [188] shows that if $\varphi \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, then $\mathfrak{D}\varphi$ admits a representation

$$(\mathfrak{D}\varphi)(\lambda, x) = \int_Q f(\lambda, x) g(\mu, x) d\sigma(x),$$

Where (Q, σ) is a measure space, f and g are measurable functions on $\mathbb{R} \times Q$ suchthat

$$\int_Q \|f_x\|_{\mathfrak{B}(\mathbb{R})} \|g_x\|_{\mathfrak{B}(\mathbb{R})} d\sigma(x) < \infty,$$

and for almost all $x \in Q$, and f_x and g_x are operator continuous functions where $f_x(\lambda) \stackrel{\text{def}}{=} f(\lambda, x)$ and $g_x(\mu) \stackrel{\text{def}}{=} f(\mu, x)$. Indeed, it is very easy to verify that thefunctions f_x and g_x constructed in [188] are products of bounded functions in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$.

By (7), we have

$$\frac{1}{s}\big(\varphi(A_s) - \varphi(A)\big) = \frac{1}{s}\iint (\mathfrak{D}\varphi)(\lambda, \mu) dE_{A_s}(\lambda) sK dE_A(\mu) = \int_Q f_x(A_s) K g_x(A) d\sigma(x).$$

Since f_x is operator continuous, we have

$$\lim_{s \to 0} \|f_x(A_s) - f_x(A)\| = 0.$$

It follows that

$$\left\| \int_Q f_x(A_s) K g_x(A) d\sigma(x) - \int_Q f_x(A) K g_x(A) d\sigma(x) \right\| \le \|K\| \int_Q \|f_x(A_s) - f_x(A)\| \cdot \|g_x(A)\| d\sigma(x)$$

$$\to 0 \quad \text{as} \to 0,$$

which implies (26).

Consider first the problem of the existence of the second operator derivative. Firstwe prove that if $f \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$, then$\mathfrak{D}^2\varphi \in \mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R})$.Actually, to provethe existence of the second derivative, we need the following slightly stronger result.

Theorem (5.1.8)[178]. Let $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$. Then there exist a measure space (Q, σ) and measurable functions f, g, and h on $\mathbb{R} \times Q$ such that

$$(\mathfrak{D}^2\varphi)(\lambda, \mu, \nu) = \int_Q f(\lambda, x) g(\mu, x) h(\nu, x) d\sigma(x), \tag{27}$$

f_x, g_x, and $h_x$are operator continuous functions for almost all $x \in Q$, and

$$\int_Q \|f_x\|_{\mathfrak{B}(\mathbb{R})} \|g_x\|_{\mathfrak{B}(\mathbb{R})} \|h_x\|_{\mathfrak{B}(\mathbb{R})} d\sigma(x) \le \text{const}\|\varphi\|_{\boldsymbol{B}^2_{\infty 1}(\mathbb{R})}. \tag{28}$$

As before, $f_x(\lambda) = f(\lambda, x)$, $g_x(\mu) = g(\mu, x)$, and $h_x(\nu) = g(\nu, x)$.

Theorem (5.1.8) will be used to prove the main result.

Proof . Suppose thatsupp $\mathcal{F}\varphi \subset [M/2, 2M]$. Let us show that each summand on the right-hand side of (30) admits a desired representation. Clearly, it suffices to do it for the first summand. Put

$$\psi(\lambda, \mu, \nu) = \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \big((S^*_{t+u}\varphi) * q_{t+u}\big)(\lambda) e^{it\mu} e^{iu\nu}\, dtdu = \iint_{\mathbb{R}_+ \times \mathbb{R}_+} f_{t+u}(\lambda) g_t(\mu) h_u(\nu) \,\square\, tdu,$$

where

$$f_v(\lambda) = \big((S^*_v\varphi) * q_v\big)(\lambda), g_t(\mu) = e^{it\mu} \text{and} \quad h_u(\nu) = e^{iu\nu}.$$

Clearly, $\|g_t\|_{\mathfrak{B}(\mathbb{R})} = 1$ and$\|h_u\|_{\mathfrak{B}(\mathbb{R})} = 1$. Since

$$\|f_v\|_{\mathfrak{B}(\mathbb{R})} = \|f_v\|_{L^\infty} = \|\varphi - \varphi * r_v\|_{L^\infty} \le \begin{cases} (1 + \|r_v\|_{L^1})\|\varphi\|_{L^\infty}, & v \le 2M, \\ 0, & v > 2M, \end{cases}$$

we have

$$\|\psi\|_{\mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R})} \le \text{const}\|\varphi\|_{L^\infty} \iint_{t,u>0, t+u \le 2M} dtdu \le \text{const} \cdot M^2 \|\varphi\|_{L^\infty}.$$

In the same way we can treat the case whensupp $F\varphi \subset [-2M, -M/2]$. If$\varphi$ is apolynomial of degree at most 2, the result is trivial.

Let now $\varphi \in B^2_{\infty 1}(\mathbb{R})$ and

$$\varphi = \sum_{n \in \mathbb{Z}} \varphi * W_n + \sum_{n \in \mathbb{Z}} \varphi * W_n^{\#}.$$

It follows from the above estimate that $\|\mathfrak{D}^2\varphi\|_{\mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R}) \widehat{\otimes}_i \mathfrak{B}(\mathbb{R})} \le \text{const}(\sum_{n \in \mathbb{Z}} 2^{2n} \|\varphi * W_n\|_{L^\infty} + \sum_{n \in \mathbb{Z}} 2^{2n} \|\varphi * W_n^{\#}\|_{L^\infty})$. To complete the proof of Theorem (5.1.8), we observe that the functions $\lambda \mapsto e^{it\lambda}$ areoperator continuous, because they belong to$B^1_{\infty 1}(\mathbb{R})$. On the other hand, it is easy tosee that ifsupp $\varphi \subset [-2M, -M/2]$, then the function $(S^*_v\varphi) * q_v$ is the product of $e^{itv}$and a bounded function in $B^1_{\infty 1}(\mathbb{R})$.

Theorem(5.1.9)[178]. Suppose that A is a self-adjoint operator, K is a bounded self-adjoint operator. If $\varphi \in \boldsymbol{B}_{\infty 1}^{2}(\mathbb{R}) \cap \boldsymbol{B}_{\infty 1}^{1}(\mathbb{R})$, then the function $s \longmapsto (A_s)$ has second derivativethat is a bounded operator and

$$\frac{d^2}{ds^2}(\varphi(A_s))\bigg|_{s=0} = 2\iint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_A(\lambda)KdE_A(\mu)KdE_A(\nu). \qquad (29)$$

Note that by Theorem (5.1.8), the right-hand side of (29) makes sense and is a bounded linear operator.

For $t > 0$ and a function f , we define $\boldsymbol{S}_t^* f$ by

$$\left(\mathcal{F}(\boldsymbol{S}_t^* f)\right)(s) = \begin{cases} (\mathcal{F}f)(s-t), & t \le s, \\ 0, & t > s, \end{cases}$$

We also define the distributions $q_t$and $r_t, t > 0$, by

$$(\mathcal{F}q_t)(s) = \begin{cases} \dfrac{s}{s+t}, & s \ge 0, \\ 0, & s < 0, \end{cases}$$

and

$$(\mathcal{F}r_t)(s) = \begin{cases} 1, & |s| \le t, \\ \dfrac{1}{s}, & |s| > t, \end{cases}$$

It is easy to see that $r_t \in L^1(\mathbb{R})$ and $\|r_t\|_{L^1(\mathbb{R})}$ does not depend on t.

Proof . It follows from Lemma (5.1.11) that

$$\frac{1}{t}\left(\iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A_t}(\lambda)KdE_{A_t}(\mu) - \iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{A_t}(\lambda)KdE_A(\nu)\right)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_{A_t}(\mu)KdE_A(\nu).$$

Similarly

$$\frac{1}{t}\left(\iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{A_t}(\lambda)KdE_A(\nu) - \iint (\mathfrak{D}\varphi)(\mu,\nu)dE_A(\mu)KdE_A(\nu)\right)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_A(\mu)KdE_{A_t}(\nu).$$

Thus,

$$\frac{1}{t}\left(\frac{d}{ds}(\varphi(A_s))\bigg|_{s=t} - \frac{d}{ds}(\varphi(A_s))\bigg|_{s=0}\right)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_{A_t}(\mu)KdE_A(\nu)$$
$$+ \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_A(\mu)KdE_A(\nu).$$

The fact that

$$\lim_{t\to 0}\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_{A_t}(\mu)KdE_A(\nu) = \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_A(\lambda)KdE_A(\mu)KdE_A(\nu)$$

follows immediately from (27) and (28) and from the fact that the functions f_x, g_x, and h_x in (27) are operator continuous.

Similarly,

$$\lim_{t\to 0}\iiint(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A_t}(\lambda)KdE_A(\mu)KdE_A(\nu)=\iiint(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_A(\lambda)KdE_A(\mu)KdE_A(\nu),$$

which completes the proof.

Lemma (5.1.10)[178]. Let $M>0$ and let φbe a bounded function on $\mathbb{R}$ such thatsupp$\mathcal{F}\varphi\subset[M/2,2M]$. Then

$$\begin{aligned}(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)\\ &=-\iint_{\mathbb{R}_+\times\mathbb{R}_+}\big((S^*_{t+u}\varphi)*q_{t+u}\big)(\lambda)e^{it\mu}e^{iu\nu}\,dtdu\\ &-\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Big((\boldsymbol{S}^*_{s+u}\varphi)*q_{s+u}\Big)(\mu)e^{is\lambda}e^{iu\nu}\,dsdu\\ &-\iint_{\mathbb{R}_+\times\mathbb{R}_+}\Big((\boldsymbol{S}^*_{s+t}\varphi)*q_{s+t}\Big)(\nu)e^{is\lambda}e^{it\mu}\,dsdt.\end{aligned}\tag{30}$$

Proof. Let us first assume that $\mathcal{F}\varphi\in L^1(\mathbb{R})$. We have

$$\begin{aligned}&\iint_{\mathbb{R}_+\times\mathbb{R}_+}((\boldsymbol{S}^*_{t+u}\varphi)*q_{t+u})(\lambda)e^{it\mu}e^{iu\nu}\,d\mu d\nu\\ &=\iiint_{\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+}(\mathcal{F}\varphi)(s+t+u)\frac{s}{s+t+u}e^{is\lambda}e^{it\mu}e^{iu\nu}dsdtdu.\end{aligned}$$

We can write similar representations for the other two terms on the right-hand side of(30), take their sum and reduce (30) to the verification of the following identity:

$$(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)=\iiint_{\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+}(\mathcal{F}\varphi)(s+t+u)\,e^{is\lambda}e^{it\mu}e^{iu\nu}dsdtdu$$

This identity can be verified elementarily by making the substitution $a=s+t+u$,$b=t+u$, and $c=u$.

Consider now the general case, i.e., $\varphi\in L^\infty(\mathbb{R})$ andsupp $\mathcal{F}\varphi\subset[M/2,2M]$. Consider a smooth function ωon $\mathbb{R}$ such that $\omega\geq 0$, supp $\omega\subset[-1,1]$, and $\|\omega\|_{L^1(\mathbb{R})}=1$.For $\varepsilon>0$ we put $\omega_\varepsilon(x)=\omega(x/\varepsilon)/\varepsilon$ and define the function φ_ε by $\mathcal{F}\varphi_\varepsilon=(\mathcal{F}\varphi)*\omega_\varepsilon$ .Clearly,

$$\mathcal{F}\varphi_\varepsilon\in L^1(\mathbb{R}),\quad \lim_{\varepsilon\to 0}\|\varphi_\varepsilon\|_{L^\infty(\mathbb{R})}=\|\varphi\|_{L^\infty(\mathbb{R})},$$

and

$$\lim_{\varepsilon\to 0}\varphi_\varepsilon(x)=\varphi(x)\text{for almost all}\quad x\in\mathbb{R}.$$

Since we have already proved that (30) holds for φ_ε in place of φ, the result follows by passing to the limit as $\varepsilon\to\infty$.

To prove (26), we need the following lemma.

Lemma (5.1.11)[178]. Let A be a self-adjoint operator and let K be a bounded self-adjoint operator. Suppose that $\varphi$is a function on $\mathbb{R}$ such that $\mathfrak{D}\varphi\in L^\infty(\mathbb{R})\widehat{\otimes}_i L^\infty(\mathbb{R})$and $\mathfrak{D}^2\varphi\in L^\infty(\mathbb{R})\widehat{\otimes}_i L^\infty(\mathbb{R})\widehat{\otimes}_i L^\infty(\mathbb{R})$. Then

$$\iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A+K}(\mu) - \iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{A+K}(\lambda)KdE_{A}(\nu)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)KdE_{A}(\nu).$$

Proof. Put
$P_n = E_A([-n,n])$, $Q_n = E_{A+K}([-n,n])$, $A_{[n]} = P_nA$ and $B_{[n]} = Q_n(A+K)$.
We have

$$\iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A+K}(\mu) - \iint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{A+K}(\lambda)KdE_{A}(\nu)$$
$$= \iiint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)K^{\vee}E_{A}(\nu)$$
$$- \iiint (\mathfrak{D}\varphi)(\lambda,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)KdE_{A}(\nu).$$

Thus,

$$Q_n\left(\iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A+K}(\mu) - \iint (\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A}(\mu)\right)P_n$$
$$= \int_{-n}^{n}\int_{-n}^{n}\int_{-n}^{n}(\mathfrak{D}\varphi)(\lambda,\mu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)\, dE_{A}(\nu)$$
$$- \int_{-n}^{n}\int_{-n}^{n}\int_{-n}^{n}(\mathfrak{D}\varphi)(\lambda,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)\, d\_{A}(\nu)$$
$$= \iiint (\mu-\nu)(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)KdE_{A_{[n]}}(\nu),$$

since

$$(\mathfrak{D}\varphi)(\lambda,\mu) - (\mathfrak{D}\varphi)(\lambda,\nu) = (\mu-\nu)(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu).$$

On the other hand,

$$Q_n\left(\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)KdE_{A}(\nu)\right)P_n$$
$$= \int_{-n}^{n}\int_{-n}^{n}\int_{-n}^{n}(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)\big((A+K)-A\big)\, dE_{A}(\nu)$$
$$= \int_{-n}^{n}\int_{-n}^{n}\int_{-n}^{n}(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu)\, Q_n \times \big((A+K)-A\big)P_nE_{A}(\nu)$$
$$= \int_{-n}^{n}\int_{-n}^{n}\int_{-n}^{n}(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{A+K}(\lambda)KdE_{A+K}(\mu) \times \big(B_{[n]}P_n - Q_nA_{[n]}\big)dE_{A}(\nu)$$
$$= \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)(B_{[n]}P_n - Q_nA_{[n]})KdE_{A_{[n]}}(\nu).$$

It is easy to see that this is equal to

$$\iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)B_{[n]}P_n dE_{A_{[n]}}(\nu)$$
$$- \iiint (\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)Q_n A_{[n]}dE_{A_{[n]}}(\nu)$$
$$= \iiint \mu(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)dE_{A_{[n]}}(\nu)$$
$$- \iiint \nu(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)dE_{A_{[n]}}(\nu)$$
$$= \iiint (\mu-\nu)(\mathfrak{D}^2\varphi)(\lambda,\mu,\nu)dE_{B_{[n]}}(\lambda)KdE_{B_{[n]}}(\mu)dE_{A_{[n]}}(\nu).$$

The result follows now from the fact that
$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} Q_n = I$$
in the strong operator topology.

Theorem (5.1.12)[178]. Let m be a positive integer and let $\varphi \in \boldsymbol{B}^m_{\infty 1}(\mathbb{R})$. Then there exist a measure space (Q,σ) and measurable functions $f_1,\dots,f_{m+1}$ on $\mathbb{R}\times Q$ such that
$$(\mathfrak{D}^m\varphi)(\lambda_1,\dots,\lambda_{m+1}) = \int_Q f_1(\lambda_1,x)f_2(\lambda_2,x)\cdots f_{m+1}(\lambda_{m+1},x)d\sigma(x),$$
the functions $f_1(\cdot,x),\dots,f_{m+1}(\cdot,x)$ are operator continuous for almost all $x\in Q$,and
$$\int_Q \|f_1(\cdot,x)\|_{\mathfrak{B}(\mathbb{R})}\cdots\|f_{m+1}(\cdot,x)\|_{\mathfrak{B}(\mathbb{R})}d\sigma(x) \le \text{const}\|f\|_{\boldsymbol{B}^m_{\infty 1}(\mathbb{R})}.$$

Theorem(5.1.13)[178]. Let $m$be a positive integer. Suppose that A is a self-adjoint operatorand K is a bounded self-adjoint operator. If $\varphi \in \boldsymbol{B}^m_{\infty 1}(\mathbb{R}) \cap \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, then the function$s \to \varphi(A_s)$ has mth derivative that is a bounded operator and
$$\frac{d^m}{ds^m}(\varphi(A_s))\Big|_{s=0} = m!\underbrace{\int\dots\int}_{m+1}(\mathfrak{D}^m\varphi)(\lambda_1,\dots,\lambda_{m+1})dE_A(\lambda_1)K\dots KdE_A(\lambda_{m+1}).$$

Section (5.2): Operator Hölder–Zygmund Functions

It is well known that a Lipschitz function on the real line is not necessarily operator Lipschitz, i.e., the condition
$$\|f(x)-f(y)\| \le \text{const}|x-y|, \quad x,y\in\mathbb{R},$$
does not imply that for self-adjoint operators A and B on Hilbert space,
$$\|f(x)-f(y)\| \le \text{const}\|x-y\|.$$
The existence of such functions was proved in [198]. Later in [199] necessary conditions were found for a function f to be operator Lipschitz. Those necessary conditions also imply that Lipschitz functions do not have to be operator Lipschitz. It was shown in [199] that an operator Lipschitz function must belong locally to the Besovspace $\boldsymbol{B}^1_1(\mathbb{R})$. Note that in [199] and [200] a stronger necessary condition was also obtained.

It is also well known that a continuously differentiable function does not have to be operator differentiable. Moreover, the fact that f is continuously differentiable does not imply that for bounded self-adjoint operators A and K the function

$$t \longmapsto f(A + tK)$$

is differentiable. For f to be operator differentiable it must satisfy the same necessary conditions [31, 33].(Note that Widom posed in [201] a problem entitled "When are differentiable functions differentiable?".)

On the other hand it was proved in [199] and [200] that the condition that a function belongs to the Besov space $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$ is sufficient for operator Lipschitzness (as well as for operator differentiability). We also mention here [202,203,204-206]and [207] that study operator Lipschitz functions.

Many mathematicians working on such problems in perturbation theory believed that a similar situation occurs when considering Hölder classes of order α and operator Hölder classes of order α, $0 < \alpha < 1$. In particular, Farforovskaya obtained in [198] the following estimate

$$\|f(A) - f(B)\| \leq \text{const}\|f\|_{\Lambda_\alpha(\mathbb{R})}\left(\log_2^2 \frac{b-a}{\|A-B\|} + 1\right)^\alpha \|A - B\|^\alpha$$

for self-adjoint operators A and B with spectra in [a, b] and for an arbitrary function f in $\Lambda_\alpha(\mathbb{R})$,$0 < \alpha < 1$. She also obtained the same inequality for $\alpha = 1$ and a Lipschitz function f (seealso [298]).

However, we show that the situation changes dramatically if we consider Hölder classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case Hölder functions are necessarily operator Hölder, i.e., the condition

$$|f(x) - f(y)| \leq \text{const}\,|x - y|^\alpha,\ x, y \in \mathbb{R}, \tag{31}$$

implies that for self-adjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \leq \text{const}\|A - B\|^\alpha. \tag{32}$$

The constant in (32) must depend not only on the constant in (31), but also on α and must tend to infinity as the constant in (31) is fixed and α goes to 1.

Our method gives the following estimate:

$$\|f(A) - f(B)\| \leq \text{const}(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^\alpha,\ \ 0 < \alpha < 1, \tag{33}$$

where

$$\|f\|_{\Lambda_\alpha(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We consider the same problem for the Zygmund class$\Lambda_1(\mathbb{R})$, i.e., the problem of whether a function f in the Zygmund class Λ_1(i.e., f is continuous and satisfies the inequality

$$|f(x + t) - 2f(x) + f(x - t)| \leq \text{const}|t|, \quad x, t \in \mathbb{R}$$

implies that f is operator Zygmund, i.e., for arbitrary self-adjoint operators A and K ,

$$\|f(A + K) - 2f(A) + f(A - K)\| \leq \text{const}\|K\|.$$

This problem was posed in [209].
We show that the situation is the same as in the case of Hölder classes$\Lambda_\alpha(\mathbb{R})$,$0 < \alpha < 1$. Namely we prove that a Zygmund function must necessarily be operator Zygmund.
We also obtain similar results for the whole scale of Hölder–Zygmundclasses $\Lambda_\alpha(\mathbb{R})$, $0 < \alpha < \infty$, of continuous functions f satisfying

$$\left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt)\right| \le |t|^\alpha (\text{ here } m-1 \le \alpha < m).$$

There are many natural equivalent (semi)norms on$\Lambda_\alpha(\mathbb{R})$, for example,

$$\|f\|_{\Lambda_\alpha(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x\neq y}|t|^\alpha \left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x+kt)\right|. \qquad (34)$$

Analogs of these results for unitary operators and for contractions.
We estimate $\|f(A)-f(B)\|$ in terms of $\|A-B\|$ for functions f of classΛ_ω , (i.e., $|f(x)-f(y)| \le \text{const}\,\omega(|x-y|)$) for arbitrary moduli of continuity ω. In particular, we study those moduli of continuity, for which the fact that $f \in \Lambda_\omega$ implies that

$$\|f(A)-f(B)\| \le \text{const}\,\omega(\|x-y\|)$$

for arbitrary self-adjoint operators A and B . We compare this class of moduli of continuity with the class of moduli of continuity ω, for which the Hilbert transform acts onΛ_ω .
We study the class of operator continuous functions and for a uniformly continuous function f we introduce the operator modulus of continuity Ω_f . The material is closely related. We construct a universal family $\{A_t\}_{t\ge 0}$ of self-adjoint operators in the sense that to compute $\Omega_f$for arbitrary f , it suffices to consider the family $\{A_t\}_{t\ge 0}$.
We compare the operator modulus of continuity with several othermoduli of continuity defined in terms of commutators and quasicommutators.
We obtain norm estimates for finite differences

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jK) \qquad (35)$$

where$f$ belongs to the class $\Lambda_{\omega,m}$ that is defined in terms of finite differences and ω is a modulus of continuity of order m.
We collect necessary information on Besov classes (and in particular, the Hölder– Zygmund classes), and spaces $\Lambda_\omega$and$\Lambda_{\omega,m}$. We give a brief introduction into double and multiple operator integrals.
In [211] we are going to study the problem of the behavior of functions of operators under perturbations of Schatten–von Neumann class$\boldsymbol{S}_p$. We are going to study properties of functions of perturbed dissipative operators in [236], where we improve results of [212].
Finally, we would like mention that Farforovskaya and Nikolskaya have informed us recently that they had found another proof of the fact that a Hölder function of order α, $0 < \alpha < 1$, must be operator Hölder of order α. However, their method gives the estimate

$$\|f(A)-f(B)\| \le \text{const}(1-\alpha)^{-2}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A-B\|^\alpha, \qquad 0<\alpha<1$$

(compare with (33)).

We are extremely grateful to the referee for his numerous remarks to improve the text.

The purpose of this point is gave a brief introduction to the Besov spaces that play an important role in problems of perturbation theory. We start with Besov spaces on the unit circle.

Let$1 \le p, q \le \infty$ and $s \in \mathbb{R}$. The Besov class $\boldsymbol{B}_{pq}^s$of functions (or distributions) on $\mathbb{T}$can bedefined in the following way. Let w be an infinitely differentiable function on $\mathbb{R}$such that

$$w \ge 0, \quad \text{supp}\, w \subset \left[\frac{1}{2}, 2\right] \;, \text{ and } \; w(x) = 1 - w\left(\frac{x}{2}\right) \text{for } x \in [1,2]. \qquad (36)$$

Consider the trigonometric polynomialsW_n , and $W_n^\#$ defined by

$$W_n(z) = \sum_{k\in\mathbb{Z}} w\left(\frac{k}{2^n}\right) z^k, \;\; n \ge 1, W_0(z) = \bar{z} + 1 + z, \text{and} W_n^\#(z) = \overline{W_n(z)}, n \ge 1$$

Then for each distribution f on $\mathbb{T}$,

$$f = \sum_{n\ge 0} f * W_n + \sum_{n\ge 1} f * W_n^\#.$$

The Besov class $\boldsymbol{B}_{pq}^s$consists of functions (in the case $s > 0$) or distributions f on $\mathbb{T}$such that

$$\{\|2^{ns} f * W_n\|_{L^\infty}\}_{n\ge 1} \in \ell^q \text{and} \{\|2^{ns} f * W_n^\#\|_{L^p}\}_{n\ge 1} \in \ell^q. \qquad (37)$$

To define a regularized de la ValléePoussin type kernel V_n, we define the C^∞ function v on $\mathbb{R}$by

$$v(x) = 1 \text{for} x \in [-1,1] \text{ and } v(x) = w(|x|) \text{ if } |x| \ge 1, \qquad (38)$$

where$w$ is a function described in (36). Then the trigonometric polynomial $V_n$is defined by

$$V_n(z) = \sum_{k\in\mathbb{Z}} v\left(\frac{k}{2^n}\right) z^k, \qquad n \ge 1.$$

Besov classes admit many other descriptions. In particular, for $s > 0$, the space $\boldsymbol{B}_{pq}^s$admitsthe following characterization. A function $f \in L^p$ belongs to $\boldsymbol{B}_{pq}^s$, $s > 0$, if and only if

$$\int_{\mathbb{T}} \frac{\|\Delta_\tau^m f\|_{L^p}^q}{|1-\tau|^{1+sq}} d\boldsymbol{m}(\tau) < \infty \;\text{ for }\; q < \infty$$

and

$$\sup_{\tau \ne 1} \frac{\|\Delta_\tau^m f\|_{L^p}}{|1-\tau|^s} < \infty \;\text{ for }\; q = \infty, \qquad (39)$$

where$m$ is normalized Lebesgue measure on $\mathbb{T}$, m is an integer greater than s , and $\Delta_\tau, \tau \in \mathbb{T}$, isthe difference operator:

$$(\Delta_\tau f)(\zeta) = f(\tau\zeta) - f(\zeta), \;\; \zeta \in \mathbb{T}.$$

We use the notation $\boldsymbol{B}_p^s$for $\boldsymbol{B}_{pq}^s$.

The spaces $\Lambda_\alpha \stackrel{\text{def}}{=} \boldsymbol{B}_\infty^\alpha$form the Hölder–Zygmund scale. If $0 < \alpha < 1$, then $f \in \Lambda_\alpha$ if and onlyif

$$|f(\zeta)-f(\tau)| \le \text{const}|\zeta-\tau|^{\alpha}, \quad \zeta,\tau \in \mathbb{T},$$

while $f \in \Lambda_1$ if and only if f is continuous and

$$|f(\zeta\tau)-2f(\zeta)+f(\zeta\bar{\tau})| \le \text{const}|1-\tau|, \quad \zeta,\tau \in \mathbb{T}.$$

By (39), for $\alpha > 0$, $f \in \Lambda_\alpha$ if and only if f is continuous and

$$|(\Delta_\tau^m f)(\zeta)| \le \text{const}|\zeta-\tau|^{\alpha},$$

where m is a positive integer such that $m > \alpha$.

By analogy with (34) we can define the natural (semi)norm on Λ_α in terms of finite differences. Note that the seminorm of a function f in Λ_α is equivalent to

$$\left\|(f-\hat{f}(0)) * W_0\right\|_{L^\infty} + \sup_{n\ge1} 2^{n\alpha}\left(\|f * W_n\|_{L^\infty} + \|f * W_n^{\#}\|_{L^\infty}\right),$$

where for a function or a distribution f on $\mathbb{T}$, $\hat{f}(n)$ is the nth Fourier coefficient of f .

We denote by λ_α the closure of the set of trigonometric polynomials in Λ_α . It is easy to see that f belongs to λ_α if and only if

$$\lim_{n\to\infty} 2^{n\alpha}\|f * W_n\|_{L^\infty} = \lim_{n\to\infty} 2^{n\alpha}\|f * W_n^{\#}\|_{L^\infty} = 0.$$

If $\alpha > 0$, this is equivalent to the fact that

$$\lim_{\tau\to1}\frac{\|\Delta_\tau^m f\|_{L^\infty}}{|1-\tau|^{\alpha}} = 0, \quad m > \alpha.$$

It is well known that the dual space $(\lambda_\alpha)^*$ can be identified naturally with the Besov space $\boldsymbol{B}_1^{-\alpha}$ with respect to the following pairing:

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} \hat{f}(n)\hat{g}(n)$$

in the case when g is a trigonometric polynomial. It is also well known that the dual space $(\boldsymbol{B}_1^{-\alpha})^*$ can be identified naturally with the space Λ_α with respect to the same pairing.

It is easy to see from the definition of Besov classes that the Rieszprojection $\mathbb{P}_+$,

$$\mathbb{P}_+ f = \sum_{n\in\mathbb{Z}} \hat{f}(n) z^n,$$

is bounded on $\boldsymbol{B}_{pq}^s$. Functions (or distributions) in $\left(\boldsymbol{B}_{pq}^s\right)_+ \overset{\text{def}}{=} \mathbb{P}_+\boldsymbol{B}_{pq}^s$ admit a natural extensionto analytic functions in the unit disk following description:

$\mathbb{D}$. It is well known that the functions in $\left(\boldsymbol{B}_{pq}^s\right)_+$ admit the

$$f \in \left(\boldsymbol{B}_{pq}^s\right)_+ \iff \int_0^1 (1-r)^{q(m-s)-1}\left\|f_r^{(m)}\right\|_p^q dr < \infty, q < \infty,$$

and

$$f \in \left(\boldsymbol{B}_{pq}^s\right)_+ \iff \sup_{0<r<1}(1-r)^{m-s}\left\|f_r^{(m)}\right\|_p < \infty,$$

where $f_r(\zeta) \overset{\text{def}}{=} f(r\zeta)$ and m is a nonnegative integer greater than s.

Let us proceed now to Besov spaces on the real line. We consider homogeneous Besov spaces $\boldsymbol{B}_{pq}^s(\mathbb{R})$ of functions (distributions) on $\mathbb{R}$. We use the same function w as in (36) and define the functions W_n and $W_n^{\#}$ on $\mathbb{R}$ by

$$\mathcal{F}W_n(x) = w\left(\frac{x}{2^n}\right), \quad \mathcal{F}W_n^{\#}(x) = \mathcal{F}W_n(-x), \; n \in \mathbb{Z},$$

where $\mathcal{F}$ is the Fourier transform:

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}} f(x)e^{-ixt}\,dx, \quad f \in L^1.$$

With every tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ we associate a sequence $\{f_n\}_{n\in\mathbb{Z}}$,

$$f_n \stackrel{\text{def}}{=} f * W_n + f * W_n^{\#}.$$

Initially we define the (homogeneous) Besov class $\dot{\boldsymbol{B}}_{pq}^s(\mathbb{R})$ as the set of all $f \in \mathcal{S}'(\mathbb{R})$ suchthat

$$\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}). \tag{40}$$

According to this definition, the space $\dot{\boldsymbol{B}}_{pq}^s(\mathbb{R})$contains all polynomials. Moreover, the distribution f is defined by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ uniquely up to a polynomial. It is easy to see that theseries $\sum_{n\geq 0} f_n$converges in$\mathcal{S}'(\mathbb{R})$. However, the series $\sum_{n<0} f_n$can diverge in general. It iseasy to prove that the series $\sum_{n<0} f_n^{(r)}$ converges on uniformly $\mathbb{R}$for each nonnegative integer$r > s - 1/p$. Note that in the case $q = 1$ the series $\sum_{n<0} f_n^{(r)}$converges uniformly, whenever $r \geq s - 1/p$.

Now we can define the modified (homogeneous) Besov class$\boldsymbol{B}_{pq}^s(\mathbb{R})$. We say that a distribution f belongs to $\boldsymbol{B}_{pq}^s(\mathbb{R})$ if $\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$ and $f^{(r)} = \sum_{n\in\mathbb{Z}} f_n^{(r)}$ in the space$\mathcal{S}'(\mathbb{R})$,where r is the minimal nonnegative integer such that $r > s - 1/p$ ($r \geq s - 1/p$if $q = 1$). Nowthe function f is determined uniquely by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ up to a polynomial of degree lessthan r, and a polynomial φbelongs to $\boldsymbol{B}_{pq}^s(\mathbb{R})$ if and only if $\deg \varphi < r$.

We can also define de la ValléePoussin type functionsV_n,$n \in \mathbb{Z}$ by

$$\mathcal{F}V_n(x) = v\left(\frac{x}{2^n}\right),$$

where$v$ is a function given by (38). We put $V \stackrel{\text{def}}{=} V_0$.

We use the same notation V_n , W_n and $W_n^{\#}$ for functions on $\mathbb{T}$ and on $\mathbb{R}$. This will not lead to confusion. For positive n we can easily obtain the function $V_n$on the circle from the corresponding function $V_n$on the line. It suffices to consider the 2π-periodic function

$$\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} V_n\,(x + 2j\pi)$$

and identify it with a function on $\mathbb{T}$. The same can be done with the functions $W_n$and $W_n^{\#}$.

Besov spaces $\boldsymbol{B}_{pq}^s(\mathbb{R})$admit equivalent definitions that are similar to those discussed above inthe case of Besov spaces of functions on$\mathbb{T}$. In particular, the Hölder–Zygmund classes$\Lambda_\alpha(\mathbb{R}) \stackrel{\text{def}}{=} \boldsymbol{B}_\infty^\alpha(\mathbb{R})$, $\alpha > 0$, can be described as the classes of continuous functions f on $\mathbb{R}$ such that

$$|(\Delta_t^m)(x)| \leq \text{const}|t|^\alpha, \; t \in \mathbb{R},$$

where the difference operator Δ_t is defined by

$$(\Delta_t f)(x) = f(x+t) - f(x), \quad x \in \mathbb{R},$$

and$m$ is the integer such that $m-1 \le \alpha < m$.
As in the case of functions on the unit circle, we can introduce the following equivalent semi-norm on $\Lambda_\alpha(\mathbb{R})$ that is equivalent to the seminorm (34):

$$\sup_{n\in\mathbb{Z}} 2^{n\alpha} \left(\|f * W_n\|_{L^\infty} + \|f * W_n^{\#}\|_{L^\infty}\right), \quad f \in \Lambda_\alpha(\mathbb{R}).$$

Consider now the class$\lambda_\alpha(\mathbb{R})$, which is defined as the closure of the Schwartz class $\mathcal{S}(\mathbb{R})$in $\Lambda_\alpha(\mathbb{R})$. The following result gives a description of $\lambda_\alpha(\mathbb{R})for\alpha > 0$. We use the following notation: $C_0(\mathbb{R})$ stands for the space of continuous functions f on $\mathbb{R}$such that$\lim_{|x|\to\infty} f(x) = 0$.

Theorem (5.2.1)[197]. Let $\alpha > 0$ and let m be the integer such that $m-1 \le \alpha < m$. Suppose that $f \in \Lambda_\alpha(\mathbb{R})$. The following are equivalent:
(i) $f \in \lambda_\alpha(\mathbb{R})$;
(ii) $f_n \in C_0(\mathbb{R})$for every $n \in \mathbb{Z}$ and

$$\lim_{|n|\to\infty} 2^{ns} \|f_n\|_{L^\infty} = 0;$$

(iii) the following equalities hold:

$$\lim_{t\to 0} |t|^{-\alpha} (\Delta_t^m f)(x) = 0 \text{ uniformly in } x \in \mathbb{R},$$
$$\lim_{|t|\to 0} |t|^{-\alpha} (\Delta_t^m f)(x) = 0 \text{ uniformly in } x \in \mathbb{R},$$

and

$$\lim_{|x|\to 0} |t|^{-\alpha} (\Delta_t^m f)(x) = 0 \text{ uniformly in } t \in \mathbb{R}\backslash\{0\}.$$

Proof. (ii) ⇒ (i). It follows from the definition of $\Lambda_\alpha(\mathbb{R})$ in terms of convolutions with $W_n$and $W_n^{\#}$ that

$$\lim \left\| f - \sum_{n=-N}^{N} f_n \right\|_{\Lambda_\alpha(\mathbb{R})} = 0.$$

Thus it suffices to prove that $f_n \in \lambda_\alpha(\mathbb{R})$. However, this is a consequence of the following easily verifiable fact:

$$\lim_{\varepsilon\to 0} \sup_{x\in\mathbb{R}} \left| \left(e^{-\varepsilon^2 x^2} f_n(x)\right)^{(j)} - f_n^{(j)}(x) \right| = 0 \text{ for all } j \ge 0. \quad (41)$$

Indeed, (41) is obvious if we observe that all derivatives of $f$belong to $C_0(\mathbb{R})$.
The implication (i) ⇒ (iii) follows very easily from the fact that (iii) holds for all functions in$\mathcal{S}(\mathbb{R})$ which can easily be established.
It remains to show that (iii) implies (ii). Consider the function Q_n defined by

$$Q_n(t) = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \frac{1}{k} V_n\left(\frac{t}{k}\right). \quad (42)$$

It is easy to see that

$$f(x) - (f * Q_n)(x) = f(x) - \int_{\mathbb{R}} f(x-t) \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} \frac{1}{k} V_n\left(\frac{t}{k}\right) dt$$
$$= f(x) + \int_{\mathbb{R}} \sum_{k=1}^{m} (-1)^k \binom{m}{k} f(x-kt) V_n(t) dt$$
$$= \int_{\mathbb{R}} (\Delta_{-t}^m f)(x)\, V_n(t) dt. \quad (43)$$

Hence,

$$2^{\alpha n} \|f - f * Q_n\|_{L^\infty} = \sup_{x \in \mathbb{R}} 2^{\alpha n} \left| \int_{\mathbb{R}} (\Delta_{-t}^m f)(x)\, V_n(t) dt \right| = \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} \frac{\left(\Delta_{-2^{-n}t}^m f\right)(x)}{|t|^\alpha 2^{-\alpha n}} V(t) |t|^\alpha dt \right| \to 0 \text{ as } |n| \to \infty$$

by the Lebesgue dominated convergence theorem.

Let us observe now that supp$\mathcal{F} Q_n \subset [-2^{n+1}, 2^{n+1}]$, and so

$$\|f - f * V_n\|_{L^\infty} = \|f - f * Q_{n-1} - (f - f * Q_{n-1}) * V_n\|_{L^\infty}$$
$$\leq \|f - f * Q_{n-1}\|_{L^\infty} + \|(f - f * Q_{n-1}) * V_n\|_{L^\infty} \leq \text{const}\, \|f - f * Q_{n-1}\|_{L^\infty}$$

which immediately implies that

$$\lim_{|n| \to \infty} 2^{ns} \|f_n\|_{L^\infty} = 0.$$

Similarly, we can prove that $f - f * Q_n \in C_0(\mathbb{R})$and $f_n \in C_0(\mathbb{R})$.

The dual space $(\lambda_\alpha(\mathbb{R}))^*$to $\lambda_\alpha(\mathbb{R})$ can be identified in a natural way with $\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$ with respect to the pairing

$$\langle f, g \rangle \stackrel{\text{def}}{=} \lim_{N \to \infty} \sum_{n=-N}^{N} \int_{\mathbb{R}} \big(\mathcal{F}(f_n)\big)(t)\, (\mathcal{F} f)(t) dt, \qquad f \in \lambda_\alpha(\mathbb{R}), g \in \boldsymbol{B}_1^{-\alpha}(\mathbb{R}).$$

The dual space $(\boldsymbol{B}_1^{-\alpha}(\mathbb{R}))^*$ to $\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$ can be identified with $\Lambda_\alpha(\mathbb{R})$ with respect to the same pairing.

We refer to [213] and [214] .

Lemma(5.2.2)[197]. Let $\alpha > 0$ and let P be a polynomial whose degree is at most α. Then for an arbitrary $\varepsilon > 0$ there exists a function $f \in \Lambda_\alpha(\mathbb{R})$ with compact support such that

$$f|[0,1] = P|[0,1] \text{and} \|f\|_{\Lambda_\alpha(\mathbb{R})} < \varepsilon$$

Proof. It suffices to consider the case when $P(x) = x^n$with $n \leq \alpha$. Assume first that $n < \alpha$. Let g be an arbitrary function in $\Lambda_\alpha(\mathbb{R})$ with compact support and such that $g(x) = x^n$ for $x \in [0,1]$. For $t \in (0,1)$, we define the function g_t by

$$g_t(x) = t^{-n} g(tx).$$

It is easy to see that $g_t(x) = x^n$ for $x \in [0,1]$ and

$$\|g_t\|_{\Lambda_\alpha(\mathbb{R})} = t^{\alpha - n} \|g\|_{\Lambda_\alpha(\mathbb{R})} \to 0 \quad \text{as } t \to 0.$$

Suppose now that α is an integer and $n = \alpha$. It is well known that the function h defined by $h(x) = x^n \log|x|$ belongs to$\Lambda_n(\mathbb{R})$. Multiplying it by a suitable function in $\Lambda_n(\mathbb{R})$ with compact support, we obtain a function $g \in \Lambda_n(\mathbb{R})$ with compact support such that $g(x) = x^n \log|x|$ for $x \in [0,1]$. For $t \in (0,1)$, we define the function $g_t$by

$$g_t(x) = (t^{-n} g(tx) - g(x)) / \log t.$$

Then $g_t(x) = x^n$ for $x \in [0,1]$ and

$$\|g_t\|_{\Lambda_n(\mathbb{R})} \le 2|\log t|^{-1}\|g\|_{\Lambda_n(\mathbb{R})} \to 0 \quad \text{as } t \to 0.$$

Theorem(5.2.3)[197]. Let $\alpha > 0$. Then for each $\varepsilon > 0$ and each function $f \in \Lambda_\alpha(\mathbb{R})$ there exists a function $g \in \Lambda_\alpha(\mathbb{R})$ with compact support such that $f(t) = g(t)$ for $t \in [0,1]$ and

$$\|g\|_{\Lambda_\alpha} \le \text{const}\|f\|_{\Lambda_\alpha} + \varepsilon,$$

where the constant can depend only on α.

To prove Theorem (5.2.2), we use the well-known fact that if φ and f are functions in $\Lambda_\alpha(\mathbb{R})$ and φ has compact support, then $\varphi f \in \Lambda_\alpha(\mathbb{R})$. We refer to [215], Section 4.5.2 for the proof.

Proof . Let φ be a function in $\Lambda_\alpha(\mathbb{R})$ with compact support. We fix a subset Δ of [0,1] that has m elements, where m is the largest integer such that $m \le \alpha + 1$. It follows from the closed graph theorem that $\|\varphi f\|_{\Lambda_\alpha} \le C(\varphi, \alpha, \Delta)\|g\|_{\Lambda_\alpha}$ for every $f \in \Lambda_\alpha$ that vanisheson Δ. It remains to observe that an arbitrary function in Λ_α can be represented as the sum of a polynomial of degree at most α and a function Λ_α vanishing on Δ.

Let ω be a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0) = 0, \omega(x) > 0$ for $x > 0$, and

$$\omega(x + y) \le \omega(x) + \omega(y), \quad x, y \in [0,\infty).$$

We denote by $\Lambda_\omega(\mathbb{R})$ the space of functions on $\mathbb{R}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x \ne y} \frac{|f(x) - f(y)|}{\omega(|x - y|)} < \infty.$$

We also consider the spaces Λ_ω of functions on the unit circle and $(\Lambda_\omega)_+$ of func- tions analytic in the unit disc that can be defined in a similar way.

Theorem (5.2.4)[197]. There exists a constant $c > 0$ such that for an arbitrary modulus of continuity ω and for an arbitrary function f in $\Lambda_\omega(\mathbb{R})$, the following inequality holds:

$$\|f - f * V_n\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}, \quad n \in \mathbb{Z}. \tag{44}$$

Proof. We have

$$|f(x) - (f * V_n)(x)| = 2^n \left|\int_{\mathbb{R}} (f(x) - f(x - y))V(2^n y)dy\right| \le 2^n\|f\|_{\Lambda_\omega(\mathbb{R})} \int_{\mathbb{R}} \omega(|y|)\,|V(2^n y)|dy$$

$$= 2^n\|f\|_{\Lambda_\omega(\mathbb{R})} \int_{-2^{-n}}^{2^{-n}} \omega(|y|)\,|V(2^n y)|dy + 2^{n+1}\|f\|_{\Lambda_\omega(\mathbb{R})} \int_{2^{-n}}^{\infty} \omega(y)\,|V(2^n y)|dy.$$

Clearly,

$$2^n \int_{-2^{-n}}^{2^{-n}} \omega(|y|)\,|V(2^n y)|dy \le \omega(2^{-n})\|V\|_{L^1}.$$

On the other hand, keeping in mind the obvious inequality $2^{-n}\,\omega(y) \le 2y\omega\,(2^{-n})$ for $y \ge 2^{-n}$, we obtain

$$2^{n+1} \int_{2^{-n}}^{\infty} \omega(y)\,|V(2^n y)|dy \le 4 \cdot 2^{2n}\omega(2^{-n}) \int_{2^{-n}}^{\infty} y\,|V(2^n y)|dy$$

$$= 4\omega(2^{-n})\int_1^{\infty} y\,|V(y)|dy \le \text{const}\;\omega(2^{-n}).$$

This proves (44).

Corollary(5.2.5)[197]. There exists $c > 0$ such that for every modulus of continuity ω and for every $f \in \Lambda_\omega(\mathbb{R})$, the following inequalities hold:

$$\|f * W_n\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}, \quad n \in \mathbb{Z},$$

and

$$\|f * W_n^{\#}\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}, \quad n \in \mathbb{Z}.$$

We proceed now to moduli of continuity of higher order. For a continuous function f on $\mathbb{R}$, we define the mth modulus of continuity $\omega_{f,m}$ of f by

$$\omega_{f,m}(x) = \sup_{\{h:0\le h\le x\}} \|\Delta_h^m f\|_{L^\infty} = \sup_{\{h:0\le |h|\le x\}} \|\Delta_h^m f\|_{L^\infty}, \;\; x > 0.$$

The following elementary formula can easily be verified by induction:

$$(\Delta_{2h}^m f)(x) = \sum_{j=0}^{m} \binom{m}{j} (\Delta_h^m f)(x + jh). \tag{45}$$

It follows from (15) that $\omega_{f,m}(2x) \le 2^m \omega_{f,m}(x), x > 0$.

Suppose now that ω is a nondecreasing function on $(0,\infty)$ such that

$$\lim_{x\to 0} \omega(x) = 0 \text{ and } \;\omega(2x) \le 2^m\omega(x) \;\text{ for }\; x > 0. \tag{46}$$

It is easy to see that in this case

$$\omega(tx) \le 2^m t^m \omega(x), \;\text{ for all }\; x > 0 \;\text{ and }\; t > 1. \tag{47}$$

$$\|f\|_{\Lambda_{\omega,m}(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{t>0} \frac{\|\Delta_t^m f\|_{L^\infty}}{\omega(t)} < +\infty.$$

Theorem(5.2.6)[197]. There exists $c > 0$ such that for an arbitrary nondecreasing function ω on $(0,\infty)$satisfying (46) and for an arbitrary function $f \in \Lambda_{\omega,m}(\mathbb{R})$, the following inequality holds:

$$\|f - f * V_n\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}, \quad n \in \mathbb{Z}.$$

Proof. Consider the function Q_n defined by (42). Applying formula (43), we obtain

$$|f(x) - (f * Q_n)(x)| = \left|\int_{\mathbb{R}} (\Delta_{-t}^m f)(x) V_n(t)dt\right| \le \|f\|_{\Lambda_{\omega,m}(\mathbb{R})} \int_{\mathbb{R}} \omega(|t|)\,|V_n(t)|dt.$$

It follows from (47) that

$$\int_{\mathbb{R}} \omega(|t|)\,|V_n(t)|dt = \int_{-2^{-n}}^{2^{-n}} \omega(|t|)\,|V_n(t)|dt + 2^{n+1}\int_{2^{-n}}^{\infty} \omega(t)\,|V(2^n t)|dt$$

$$\le \|V_n\|_{L^1}\omega(2^{-n}) + 2^{n+1}\cdot 2^{m(n+1)}\omega(2^{-n})\int_{2^{-n}}^{\infty} t^m\,|V(2^n t)|dt$$

$$= \|V\|_{L^1}\omega(2^{-n}) + 2^{m+1}\omega(2^{-n})\int_1^{\infty} t^m\,|V(t)|dt \le \text{const}\,\omega(2^{-n}).$$

Summarizing the above estimates, we obtain

$$\|f - f * Q_n\|_{L^\infty} \le \text{const}\,\omega(2^{-n})\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}$$

As in the proof of Theorem 2.1, we have

$$\begin{aligned}\|f - f * V_n\|_{L^\infty} &= \|f - f * Q_{n-1} - (f - f * Q_{n-1}) * V_n\|_{L^\infty} \\ &\le \|f - f * Q_{n-1}\|_{L^\infty} + \|(f - f * Q_{n-1}) * V_n\|_{L^\infty} \le \text{const}\|f - f * Q_{n-1}\|_{L^\infty} \\ &\le \text{const}\,\omega(2^{-n})\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}.\end{aligned}$$

Corollary(5.2.7)[197]. Let $m$be a positive integer. Then there exists a positive number c_m such that for every ω satisfying (46) and for every $f \in \Lambda_{\omega,m}(\mathbb{R})$, the following inequalities hold:

$$\|f * W_n\|_{L^\infty} \le c_m \omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}, \quad n \in \mathbb{Z},$$

and

$$\|f * W_n^\#\|_{L^\infty} \le c_m \omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R})}, \quad n \in \mathbb{Z}.$$

As in the case $m = 1$, a similar result holds for the space $\Lambda_{\omega,m}$ of functions on the unitcircle, which consists of continuous f functions such that

$$\|f\|_{\Lambda_{\omega,m}} \stackrel{\text{def}}{=} \sup_{\tau \ne 1} \frac{|(\Delta_\tau^m f)(\zeta)|}{\omega(|1-\tau|)} < \infty.$$

Again, identifying a function f in $\Lambda_{\omega,m}$ with a 2π-periodic function on $\mathbb{R}$, we can see that

$$\|f - f * V_n\|_{L^\infty} \le \text{const}\,\omega(2^{-n})\|f\|_{\Lambda_{\omega,m}}, \quad n > 0.$$

We give a brief introduction in the theory of double operator integrals developed by Birman and Solomyak in [9,10] and [218], see also their survey [219].

Let $(\mathcal{X}, E_1)$ and $(\mathcal{Y}, E_2)$ be spaces with spectral measures $E_1$and E_2 on Hilbert spaces $\mathcal{H}_1$and$\mathcal{H}_2$. Let us first define double operator integrals

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y) E_1(x) Q dE_2(y), \tag{48}$$

for bounded measurable functions Φ and operators $Q: \mathcal{H}_2 \to \mathcal{H}_1$ of Hilbert–Schmidt class $\boldsymbol{S}_2$.Consider the set function F whose values are orthogonal projections on the Hilbert space$\boldsymbol{S}_2(\mathcal{H}_2, \mathcal{H}_1)$ of Hilbert–Schmidt operators from $\mathcal{H}_2$to$\mathcal{H}_1$, which is defined on measurable rectangles by

$$F(\Delta_1 \times \Delta_2)Q = E_1(\Delta_1) Q E_2(\Delta_1), \qquad Q \in \boldsymbol{S}_2(\mathcal{H}_2, \mathcal{H}_1),$$

$\Delta_1$andΔ_2 being measurable subsets of $\mathcal{X}$and $\mathcal{Y}$. Note that left multiplication by $E_1(\Delta_1)$ obviously commutes with right multiplication by$E_2(\Delta_1)$.

It was shown in [237] that Fextends to a spectral measure on $\mathcal{X} \times \mathcal{Y}$. If Φ is a bounded measurable function on $\mathcal{X} \times \mathcal{Y}$, we define

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y) E_1(x) Q dE_2(y) = \left(\int_{\mathcal{X}\times\mathcal{Y}} \Phi dF \right) Q.$$

Clearly,

$$\left\| \int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y) E_1(x) Q dE_2(y) \right\|_{S_2} \leq \|\Phi\|_{L^\infty} \|Q\|_{S_2}.$$

If the transformer

$$Q \mapsto \int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y) E_1(x) Q dE_2(y)$$

maps the trace class $\boldsymbol{S}_1$into itself, we say that Φ is a Schur multiplier of $\boldsymbol{S}_1$associated with the spectral measures E_1 and E_2. In this case the transformer

$$Q \mapsto \int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y) E_1(x) Q dE_2(y)\,, \quad Q \in \boldsymbol{S}_2(\mathcal{H}_2, \mathcal{H}_1), \tag{49}$$

extends by duality to a bounded linear transformer on the space of bounded linear operators from$\mathcal{H}_1$to $\mathcal{H}_2$ and we say that the function Ψ on $\mathcal{X} \times \mathcal{Y}$ defined by

$$\Psi(x,y) = \Phi(x,y)$$

is a Schur multiplier of the space of bounded linear operators associated with $E_2$and E_1. We denote the space of such Schur multipliers by $\mathfrak{M}(E_2, E_1)$.

To state a very important formula by Birman and Solomyak, we consider for a continuously differential function f on $\mathbb{R}$, the divided difference $\mathfrak{D}f$,

$$(\mathfrak{D}f)(x,y) \stackrel{\text{def}}{=} \frac{f(x) - f(y)}{x - y}, \quad x \neq y, \quad (\mathfrak{D}f)(x,x) \stackrel{\text{def}}{=} f'(x), \qquad x, y \in \mathbb{R}.$$

Birman in Solomyak proved in [218] that if A is a self-adjoint operator (not necessarily bounded),K is a bounded self-adjoint operator, and f is a continuously differentiable function on $\mathbb{R}$such that $\mathfrak{D}f \in \mathfrak{M}(E_{A+K}, E_A)$, then

$$f(A+K) - f(A) = \iint\limits_{\mathbb{R}\times\mathbb{R}} (\mathfrak{D}f)(x,y)\, dE_{A+K}(x) K dE_A(y) \tag{50}$$

and

$$\|f(A+K) - f(A)\| \leq \text{const}\|\mathfrak{D}f\|_{\mathfrak{M}} \|K\|,$$

where$\|\mathfrak{D}f\|_{\mathfrak{M}}$ is the norm of $\mathfrak{D}f$in $\mathfrak{M}(E_{A+K}, E_A)$.

A similar formula and similar results also hold for unitary operators, in which case we have tointegrate the divided difference $\mathfrak{D}f$of a function f on the unit circle with respect to the spectral measures of the corresponding operator integrals.

It is easy to see that if a function Φ on $\mathcal{X} \times \mathcal{Y}$ belongs to the projective tensor product$L^\infty(E_1) \widehat{\otimes} L^\infty(E_2)$ of $L^\infty(E_1)$ and $L^\infty(E_2)$ (i.e., Φ admits a representation

$$\Phi(x,y) = \sum_{n\geq 0} \varphi_n(x) \psi_n(y)\,, \tag{51}$$

where $\varphi_n \in L^\infty(E_1)$, $\psi_n \in L^\infty(E_2)$, and

$$\sum_{n\geq 0} \|\varphi_n\|_{L^\infty} \|\psi_n\|_{L^\infty} < \infty, \tag{52}$$

then $\Phi \in \mathfrak{M}(E_1, E_2)$, i.e., Φ is a Schur multiplier of the space of bounded linear operators. Forsuch functions Φ we have

$$\int_{\mathcal{X}}\int_{\mathcal{Y}}\Phi(x,y)E_1(x)QdE_2(y) = \sum_{n\geq 0}\left(\int_{\mathcal{X}}\varphi_n dE_1\right)Q\left(\int_{\mathcal{Y}}\psi_n dE_2\right).$$

Note that if Φ belongs to the projective tensor product $L^\infty(E_1)\,\widehat{\otimes}\,L^\infty(E_2)$, its norm in $L^\infty(E_1)\,\widehat{\otimes}\,L^\infty(E_2)$ is, by definition, the infimum of the left-hand side of (52) over all representations (51).

More generally, Φ is a Schur multiplier if Φ belongs to the integral projective tensor product $L^\infty(E_1)\,\widehat{\otimes}_i\,L^\infty(E_2)$ of $L^\infty(E_1)$ and $L^\infty(E_2)$, i.e., Φ admits a representation

$$\Phi(x,y) = \int_{\Omega}\varphi(x,\omega)\psi(y,\omega)d\sigma(\omega),$$

where (Ω,σ) is a σ-finite measure space, φ is a measurable function on $\mathcal{X}\times\Omega$, ψ is a measurable function on $\mathcal{Y}\times\Omega$, and

$$\int_{\Omega}\|\varphi(\cdot,\omega)\|_{L^\infty(E_1)}\|\psi(\cdot,\omega)\|_{L^\infty(E_2)}d\sigma(\omega) < \infty.$$

If $\Phi\in L^\infty(E_1)\,\widehat{\otimes}_i\,L^\infty(E_2)$, then

$$\iint\limits_{\mathcal{X}\times\mathcal{Y}}\Phi(x,y)E_1(x)QdE_2(y) = \int_{\Omega}\left(\int\limits_{\mathcal{X}}\varphi(x,\omega)dE_1(x)\right)Q\left(\int\limits_{\mathcal{Y}}\psi(y,\omega)dE_2(y)\right)d\sigma(\omega). \qquad (53)$$

Clearly, the function

$$\omega\longmapsto\left(\int\limits_{\mathcal{X}}\varphi(x,\omega)dE_1(x)\right)Q\left(\int\limits_{\mathcal{Y}}\psi(y,\omega)dE_2(y)\right)$$

is weakly measurable and

$$\int_{\Omega}\left\|\left(\int\limits_{\mathcal{X}}\varphi(x,\omega)dE_1(x)\right)Q\left(\int\limits_{\mathcal{Y}}\psi(y,\omega)dE_2(y)\right)\right\|d\sigma(\omega) < \infty.$$

It turns out that all Schur multipliers of the space of bounded linear operators can be obtained in this way (see [199]).

In connection with the Birman–Solomyak formula, it is important to obtain sharp estimates of divided differences in integral projective tensor products of L^∞ spaces. It was shown in [199] that if f is a trigonometric polynomial of degree d, then

$$\|\mathfrak{D}f\|_{C(\mathbb{T})\hat{\otimes}C(\mathbb{T})}\leq \text{const}\, d\|f\|_{L^\infty}. \qquad (54)$$

On the other hand, it was shown in [200] that if f is a bounded function on $\mathbb{R}$ whose Fourier transform is supported on $[-\sigma,\sigma]$ (in other words, f is an entire function of exponential type at most σ that is bounded on $\mathbb{R}$), then $\mathfrak{D}f\in L^\infty\,\widehat{\otimes}_i\,L^\infty$ and

$$\|\mathfrak{D}f\|_{L^\infty\widehat{\otimes}_i L^\infty}\leq \text{const}\,\sigma\|f\|_{L^\infty(\mathbb{R})}. \qquad (55)$$

Note that inequalities (54) and (55) were proved in [199] and [200] under the assumption that the Fourier transform of f is supported on $\mathbb{Z}_+$ (or $\mathbb{R}_+$); however it is very easy to deduce the general results from those partial cases.

Inequalities (54) and (55) led in [199] and [200] to the fact that functions in $\boldsymbol{B}_{\infty 1}^{1}$and $\boldsymbol{B}_{\infty 1}^{1}(\mathbb{R})$are operator Lipschitz.

It was observed in [200] that it follows from (50) and (55) that if f is an entire function of exponential type at most σ that is bounded on $\mathbb{R}$, and A and B are self-adjoint operators withbounded $A-B$, then

$$\|f(A)-f(B)\| \leq \text{const}\, \sigma\|f\|_{L^{\infty}}\|A-B\|.$$

Actually, it turns out that the last inequality holds with constant equal to 1. This will be proved in [220].

The approach by Birman and Solomyak to double operator integrals does not generalize to the case of multiple operator integrals. However, formula (53) suggests an approach to multiple operator integrals that is based on integral projective tensor products. This approach was given in [221].

To simplify the notation, we consider here the case of triple operator integrals; the case of arbitrary multiple operator integrals can be treated in the same way.

Let $(\mathcal{X}, E_1)$, $(\mathcal{Y}, E_2)$, and $(\mathcal{Z}, E_3)$ be spaces with spectral measures E_1, E_2 , and E_3 onHilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$. Suppose that Φ belongs to the integral projective tensor product$L^{\infty}(E_1) \widehat{\otimes}_i L^{\infty}(E_2) \widehat{\otimes}_i L^{\infty}(E_3)$, i.e., Φ admits a representation

$$\Phi(x, y, z)=\int_{\Omega} \varphi(x, \omega) \psi(y, \omega) \chi(z, \omega) d\sigma(\omega), \tag{56}$$

where(Ω, σ) is a σ -finite measure space, $\varphi$is a measurable function on $\chi \times \Omega$, ψ is a measurable function on $\mathcal{Y} \times \Omega$, χ is a measurable function on $\mathcal{Z} \times \Omega$, and

$$\int_{\Omega}\|\varphi(\cdot, \omega)\|_{L^{\infty}(E_1)}\|\psi(\cdot, \omega)\|_{L^{\infty}(F)}\|\chi(\cdot, \omega)\|_{L^{\infty}(G)} d\sigma(\omega)<\infty.$$

Suppose now that T_1 is a bounded linear operator from $\mathcal{H}_2$to $\mathcal{H}_1$ and $T_2$is a bounded linear operator from $\mathcal{H}_3$to$\mathcal{H}_2$. For a function Φ in $L^{\infty}(E_1) \widehat{\otimes}_i L^{\infty}(E_2) \widehat{\otimes}_i L^{\infty}(E_3)$ of the form (56), we put

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Z}} \Phi(x, y, z) dE_1(x) T_1 dE_2(y) T_2 dE_3(z)$$

$$\stackrel{\text{def}}{=} \int_{\Omega}\left(\int_{\mathcal{X}} \varphi(x, \omega)\, dE_1(x)\right) T_1\left(\int_{\mathcal{Y}} \psi(y, \omega)\, dE_2(y)\right) T_2\left(\int_{\mathcal{Z}} \chi(z, \omega)\, dE_3(z)\right) d\sigma(\omega) \tag{57}$$

It was shown in [221] (see also [222] for a different proof) that the above definition does not depend on the choice of a representation (56).

It is easy to see that the following inequality holds

$$\left\|\int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Z}} \Phi(x, y, z) dE_1(x) T_1 dE_2(y) T_2 dE_3(z)\right\| \leq\|\Phi\|_{L^{\infty} \widehat{\otimes}_i L^{\infty} \widehat{\otimes}_i L^{\infty}} \cdot\|T_1\| \cdot\|T_2\|.$$

In particular, the triple operator integral on the left-hand side of (57) can be defined if Φbelongs to the projective tensorproduct$L^\infty(E_1)\,\widehat{\otimes}_i\, L^\infty(E_2)\,\widehat{\otimes}_i\, L^\infty(E_3)$i.e., Φ admits a representation

$$\Phi(x,y,z) = \sum_{n\ge 1}\varphi_n(x)\psi_n(y)\chi_n(z)\,,$$

where $\varphi_n \in L^\infty(E_1)$, $\psi_n \in L^\infty(E_2)$,$\chi_n \in L^\infty(E_3)$ and

$$\sum_{n\ge 1}\|\varphi_n\|_{L^\infty(E_1)}\|\psi_n\|_{L^\infty(E_2)}\|\chi_n\|_{L^\infty(E_3)} < \infty.$$

In a similar way one can define multiple operator integrals, see [221].

Recall that multiple operator integrals were considered earlier in [223] and [224]. However, in those papers the class of functions Φ for which the left-hand side of (57) was defined is much narrower than in the definition given above.

Multiple operator integrals are used in [221] in connection with the problem of evaluating higher order operator derivatives. To obtain formulae for higher operator derivatives, one has to integrate divided differences of higher orders (see [221]).

We are going to integrate divided differences of higher orders to estimate the norms of higher order differences (35).

For a function f on the circle the divided differences $\mathfrak{D}^k$of order k are defined inductivelyas follows:

$$\mathfrak{D}^0 f \stackrel{\text{def}}{=} f;$$

if $k \ge 1$, then in the case when $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$ are distinct points in $\mathbb{T}$,

$$(\mathfrak{D}^k f)(\lambda_1,\lambda_2,\ldots,\lambda_{k+1}) \stackrel{\text{def}}{=} \frac{(\mathfrak{D}^{k-1}f)(\lambda_1,\ldots,\lambda_{k-1},\lambda_k) - (\mathfrak{D}^{k-1}f)(\lambda_1,\ldots,\lambda_{k-1},\lambda_{k+1})}{\lambda_k - \lambda_{k+1}}$$

(the definition does not depend on the order of the variables). Clearly,

$$\mathfrak{D}f = \mathfrak{D}^1 f;$$

If $f \in C^k(\mathbb{T})$, then $\mathfrak{D}^k f$ extends by continuity to a function defined for all points $\lambda_1, \lambda_2, \ldots, \lambda_{k+1}$.

It can be shown that

$$(\mathfrak{D}^n f)(\lambda_1,\lambda_2,\ldots,\lambda_{n+1}) = \sum_{k=1}^{n+1} f(\lambda_k)\prod_{j=0}^{k-1}(\lambda_k-\lambda_j)^{-1}\prod_{j=k+1}^{n+1}(\lambda_k-\lambda_j)^{-1}.$$

Similarly, one can define the divided difference of order k for functions on the real line. It was shown in [221] that if f is a trigonometric polynomial of degree d, then

$$\|\mathfrak{D}^k f\|_{C(\mathbb{T})\hat{\otimes}\cdots\hat{\otimes}C(\mathbb{T})} \le \text{const}\; d^k\|f\|_{L^\infty}. \tag{58}$$

If f is an entire function of exponential type at most σ that is bounded on $\mathbb{R}$, then

$$\|\mathfrak{D}^k f\|_{L^\infty\hat{\otimes}_i\cdots\hat{\otimes}_i L^\infty} \le \text{const}\; d^k\|f\|_{L^\infty(\mathbb{R})}. \tag{59}$$

In [225] Haagerup tensor products were used to define multiple operator integrals. However, it is not clear whether this can lead to stronger results in perturbation theory.

Let $\mathcal{H}$ be a Hilbert space and let $(\mathcal{X}, \mathfrak{B})$ be a measurable space. A map $\mathcal{E}$ from $\mathfrak{B}$ to the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is called a semi-spectral measure if

$$\mathcal{E}(\Delta) \geq \mathbf{0}, \quad \Delta \in \mathfrak{B},$$

$$\mathcal{E}(\emptyset) = \mathbf{0} \text{ and } \mathcal{E}(\mathcal{X}) = I,$$

and for a sequence $\{\Delta_j\}_{j\geq 1}$ of disjoint sets in $\mathfrak{B}$,

$$\mathcal{E}\left(\bigcup_{j=1}^{\infty} \Delta_j\right) = \lim_{N\to\infty} \mathcal{E}(\Delta_j) \text{ in the weak operator topology.}$$

If $\mathcal{K}$ is a Hilbert space, $(\mathcal{X}, \mathfrak{B})$ is a measurable space, $E: \mathfrak{B} \to \mathcal{B}(\mathcal{K})$ is a spectral measure, and $\mathcal{H}$ is a subspace of $\mathcal{K}$, then it is easy to see that the map $\mathcal{E}: \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ defined by

$$\mathcal{E}(\Delta) = P_{\mathcal{H}} E(\Delta)|\mathcal{H}, \quad \Delta \in \mathfrak{B}, \tag{60}$$

is a semi-spectral measure. Here $P_{\mathcal{H}}$ stands for the orthogonal projection onto $\mathcal{H}$.

Naimark proved in [226] that all semi-spectral measures can be obtained in this way, i.e., asemi-spectral measure is always a compression of a spectral measure. A spectral measure E satisfying (60) is called a spectral dilation of the semi-spectral measure $\mathcal{E}$.

A spectral dilation E of a semi-spectral measure $\mathcal{E}$ is called minimal if

$$\mathcal{K} = \text{clos span}\,\{E(\Delta)\mathcal{H}: \Delta \in \mathfrak{B}\}.$$

It was shown in [227] that if E is a minimal spectral dilation of a semi-spectral measure $\mathcal{E}$, then E and $\mathcal{E}$ are mutually absolutely continuous and all minimal spectral dilations of a semi-spectral measure are isomorphic in the natural sense.

If φ is a bounded complex-valued measurable function on $\mathcal{X}$ and $\mathcal{E}: \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ is a semispectral measure, then the integral

$$\int_{\mathcal{X}} \varphi(x)\, d\mathcal{E}(x) \tag{61}$$

can be defined as

$$\int_{\mathcal{X}} \varphi(x)\, d\mathcal{E}(x) = P_{\mathcal{H}}\left(\int_{\mathcal{X}} \varphi(x)\, dE(x)\right)\Bigg|\mathcal{H}, \tag{62}$$

where E is a spectral dilation of $\mathcal{E}$. It is easy to see that the right-hand side of (62) does not depend on the choice of a spectral dilation. The integral (61) can also be computed as the limit of sums

$$\sum \varphi(x_\alpha)\mathcal{E}(\Delta_\alpha), \quad x_\alpha \in \Delta_\alpha,$$

over all finite measurable partitions $\{\Delta_\alpha\}_\alpha$ of $\mathcal{X}$.

If T is a contraction on a Hilbert space $\mathcal{H}$, then by the Sz.-Nagy dilation theorem (see [228]), T has a unitary dilation, i.e., there exist a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$ and a unitary operator U on $\mathcal{K}$ such that

$$T^n = P_{\mathcal{H}} U^n|\mathcal{H}, \quad n \geq 0, \tag{63}$$

where$P_{\mathcal{H}}$is the orthogonal projection onto $\mathcal{H}$. Let E_U be the spectral measure of U. Consider the operator set function $\mathcal{E}$defined on the Borel subsets of the unit circle $\mathbb{T}$ by

$$\mathcal{E}(\Delta) = P_{\mathcal{H}} E_U(\Delta)|\mathcal{H}, \quad \Delta \subset \mathbb{T}.$$

Then $\mathcal{E}$is a semi-spectral measure. It follows immediately from (63) that

$$T^n = \int_{\mathbb{T}} \zeta^n d\mathcal{E}(\zeta) = P_{\mathcal{H}} \int_{\mathbb{T}} \zeta^n E_U(\zeta)|\mathcal{H}, \quad n \geq 0. \tag{64}$$

Such a semi-spectral measure $\mathcal{E}$is called a semi-spectral measure of T. Note that it is not unique. To have uniqueness, we can consider a minimal unitary dilation U of T, which is unique up to an isomorphism (see [228]).

It follows easily from (64) that

$$f(T) = P_{\mathcal{H}} \int_{\mathbb{T}} f(\zeta) E_U(\zeta)|\mathcal{H}$$

for an arbitrary function φin the disk-algebra C_A.

In [129] and [230] double operator integrals and multiple operator integrals with respect to semi- spectral measures were introduced.

Suppose that $(\mathcal{X}, \mathfrak{B}_1)$ and $(\mathcal{Y}, \mathfrak{B}_2)$ are measurable spaces, and $\mathcal{E}_1: \mathfrak{B}_1 \to \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{E}_2: \mathfrak{B}_2 \to \mathcal{B}(\mathcal{H}_2)$ are semi-spectral measures. Then double operator integrals

$$\iint_{\mathcal{X}\times\mathcal{Y}} \Phi(x,y) d\mathcal{E}_1(x) Q d\mathcal{E}_2(y)$$

were defined in [230] in the case when $Q \in \boldsymbol{S}_2$ and Φ is a bounded measurable function. Doubleoperator integrals were also defined in [230] in the case when Q is a bounded linear operator andΦ belongs to the integral projective tensor product of the spaces $L^\infty(\mathcal{E}_1)$ and $L^\infty(\mathcal{E}_2)$.

In particular, the following analog of the Birman–Solomyak formula holds:

$$f(R) - f(T) = \iint_{\mathbb{T}\times\mathbb{T}} (\mathfrak{D}f)(\zeta,\tau) d\mathcal{E}_R(\zeta)(R-T) d\mathcal{E}_T(\tau) \tag{65}$$

Here T and R contractions on Hilbert space, $\mathcal{E}_T$and $\mathcal{E}_R$ are their semi-spectral measures, and $f$is an analytic function in $\mathfrak{D}$of class $(\boldsymbol{B}^1_{\infty 1})_+$

Similarly, multiple operator integrals with respect to semi-spectral measures were definedin [230] for functions that belong to the integral projective tensor product of the correspondingL^∞ spaces.

We also mention here [231], in which another approach is used to study perturbations of functions of contractions.

We show that Hölder functions on $\mathbb{R}$ of order α, $0 < \alpha < 1$, must also be operator Hölder of order α. We also obtain similar results for all Hölder–Zygmund classes$\Lambda_\alpha(\mathbb{R})$,$0 < \alpha < \infty$. For simplicity, we give complete proofs in the case of bounded self-adjoint operatorsand explain without details that similar inequalities also hold for unbounded self-adjoint operators. We are going to give a detailed treatment of the case of unbounded operators in

[220]. In the case of first order differences, the corresponding estimates for unbounded operators also follow from Theorem (5.2.37).

If A and B are self-adjoint operators, we say that the operator $A - B$ is bounded if $B = A + K$ for some bounded self-adjoint operator K. In particular, this implies that the domains of A and B coincide. We say that $\|A - B\| = \infty$ if there is no such a bounded operator K that $B = A + K$.

Theorem(5.2.8)[197]. Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R})$ and for arbitrary self-adjoint operators A and B on Hilbert space the following inequality holds:

$$\|f(A) - f(B)\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R})} \cdot \| A - B\|^\alpha.$$

Proof. If A and B are bounded operators, it follows from Theorem 2.2 that we may assume that $f \in L^\infty(\mathbb{R})$ and we have to obtain an estimate for $\|f(A) - f(B)\|$ that does not dependon $\|f\|_{L^\infty}$.

Put

$$f_n = f * W_n + f * W_n^\#.$$

Let us show that

$$f(A) - f(B) = \sum_{n=-\infty}^{\infty} \big(f_n(A) - f_n(B)\big) \tag{66}$$

and the series on the right converges absolutely in the operator norm.

For $N \in \mathbb{Z}$, we put

$$g_N = f * V_N$$

Clearly,

$$f = f * V_N + \sum_{n>N} f_n$$

and the series on the right converges absolutely in the L^∞ norm. Thus

$$f(A) = (f * V_N)(A) + \sum_{n>N} f_n(A) \text{ and } f(B) = (f * V_N)(B) + \sum_{n>N} f_n(B)$$

and the series converge absolutely in the operator norm. We have

$$f(A) - f(B) - \sum_{n>N} (f_n(A) - f_n(B)) = \left(f(A) - \sum_{n>N} f_n(A)\right) - \left(f(B) - \sum_{n>N} f_n(B)\right)$$
$$= g_N(A) - g_N(B).$$

Since $g_N \in L^\infty(\mathbb{R})$ and g_N is an entire function of exponential type at most 2^{N+1} , it follows from (50) and (55) that

$$\|g_N(A) - g_N(B)\| \le \text{const} 2^N \|f * V_N\|_{L^\infty} \|A - B\|$$
$$\le \text{const} 2^N \|f\|_{L^\infty} \|A - B\| \to 0$$

as $N \to -\infty$. This proves (66).

Let now N be the integer such that

$$2^{-N} < \|A - B\| \le 2^{-N+1}. \tag{67}$$

We have

$$f(A) - f(B) - \sum_{n \le N} (f_n(A) - f_n(B)) + \sum_{n > N} (f_n(A) - f_n(B)).$$

It follows from Corollary(5.2.5) and from (67) that

$$\left\| \sum_{n \le N} (f_n(A) - f_n(B)) \right\| + \sum_{n \le N} \|f_n(A) - f_n(B)\| \le \text{const} \sum_{n \le N} 2^n \, \|f\|_{L^\infty} \|A - B\|$$

$$\le \text{const} \sum_{n \le N} 2^n 2^{-n\alpha} \, \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\| \le \text{const} 2^{N(1-\alpha)} \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\|$$
$$\le \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\|^\alpha.$$

On the other hand,

$$\left\| \sum_{n > N} (f_n(A) - f_n(B)) \right\| + \sum_{n > N} (\|f_n(A)\| + \|f_n(B)\|) \le 2 \sum_{n > N} \|f\|_{L^\infty} \le \text{const} \sum_{n \le N} 2^{-N\alpha} \, \|f\|_{\Lambda_\alpha(\mathbb{R})}$$
$$\le \text{const } 2^{-N\alpha} \|f\|_{\Lambda_\alpha(\mathbb{R})} \le \text{const} \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\|^\alpha$$

by(67). This completes the proof in the case of bounded self-adjoint operators.

In the case of unbounded self-adjoint operators the same reasoning holds if by $f(A) - f(B)$we understand the series

$$\sum_{n \in \mathbb{Z}} (f_n(A) - f_n(B)).$$

which converges absolutely.

Note that. (i)The proof of Theorem(5.2.8) allows us to obtain the following estimate

$$\| f(A) - f(B) \| \le \text{ const}(1 -)^{-1} \|f\|_{\Lambda_\alpha(\mathbb{R})} \|A - B\|^\alpha, \quad 0 < \alpha < 1.$$

We do not know whether this estimate can be improved.

(ii)Birman, Koplienko, and Solomyak obtained in [232] the following result: if$A$ and B are positive self-adjoint operators and $0 < \alpha < 1$, then

$$\|A^\alpha - B^\alpha\| \le \|A - B\|^\alpha.$$

It follows from Theorem(5.2.8) that under the same assumptions

$$\|A^\alpha - B^\alpha\| \le \text{const } \|A - B\|^\alpha.$$

Indeed, it suffices to apply Theorem(5.2.8) to the operators A, B and the function $f \in \Lambda_\alpha(\mathbb{R})$defined by $f(t) = |t|^\alpha, t \in \mathbb{R}$.

We state the result for arbitrary Hölder–Zygmund classes$\Lambda_\alpha(\mathbb{R})$.

Lemma (5.2.9)[197]. Let m be a positive integer and let f be a bounded function of class $\boldsymbol{B}^m_{\infty 1}(\mathbb{R})$. If $A$and K are self-adjoint operators on Hilbert space, then

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(A + jK)$$
$$= m! \underbrace{\int \cdots \int}_{m+1} (\mathfrak{D}f)(x_1, \dots, x_{m+1}) dE_A(x_1) K dE_{A+K}(x_2) K \cdots K dE_{A+mK}(x_{m+1}).$$

Proof. In the case $m = 2$ we have to establish the following formula for $\in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$:

$$f(A+K)-2f(A)+f(A-K)=2\iiint(\mathfrak{D}^2f)(x,y,z)dE_{A+K}(x)KdE_A(y)KdE_{A-K}(z).$$

Put $T=f(A+K)-2f(A)+f(A-K)$. By (50),

$$\begin{aligned}T &= f(A+K)-f(A)-\big(f(A)-f(A-K)\big)\\ &= \iint(\mathfrak{D}f)(x,y)dE_{A+K}(x)KdE_A(y)-\iint(\mathfrak{D}f)(x,y)dE_A(x)KdE_{A-K}(y)\\ &= \iint(\mathfrak{D}f)(x,y)dE_{A+K}(x)KdE_A(y)-\iint(\mathfrak{D}f)(x,y)dE_{A+K}(x)KdE_{A-K}(y)\\ &+\iint(\mathfrak{D}f)(x,y)dE_{A+K}(x)KdE_{A-K}(y)-\iint(\mathfrak{D}f)(x,y)dE_A(x)KdE_{A-K}(y).\end{aligned}$$

We have

$$\begin{aligned}&\iint(\mathfrak{D}f)\,(x,y)dE_{A+K}(x)KdE_A(y)-\iint(\mathfrak{D}f)\,(x,y)dE_{A+K}(x)KdE_{A-K}(y)\\ &= \iint(\mathfrak{D}f)\,(x,y)dE_{A+K}(x)KdE_A(y)-\iint(\mathfrak{D}f)\,(x,z)dE_{A+K}(x)KdE_{A-K}(z)\\ &= \iiint(\mathfrak{D}f)(x,y)dE_{A+K}(x)KdE_A(y)dE_{A-K}(z)\\ &- \iiint(\mathfrak{D}f)(x,z)dE_{A+K}(x)KdE_A(y)dE_{A-K}(z)\\ &= \iiint(y-z)(\mathfrak{D}^2f)(x,y,z)dE_{A+K}(x)KdE_A(y)dE_{A-K}(z)\\ &= \iiint(\mathfrak{D}^2f)(x,y,z)dE_{A+K}(x)KdE_A(y)KdE_{A-K}(z).\end{aligned}$$

Similarly,

$$\begin{aligned}&\iint(\mathfrak{D}f)\,(x,y)dE_{A+K}(x)KdE_{A-K}(y)-\iint(\mathfrak{D}f)\,(x,y)dE_A(x)KdE_{A-K}(y)\\ &= \iiint(\mathfrak{D}^2f)(x,y,z)dE_{A+K}(x)KdE_A(y)KdE_{A-K}(z).\end{aligned}$$

Thus

$T=2\iiint(\mathfrak{D}^2f)(x,y,z)dE_{A+K}(x)KdE_A(y)KdE_{A-K}(z)$.

Theorem(5.2.10)[197]. Let $0<\alpha<m$ and let $f\in\Lambda_\alpha(\mathbb{R})$. Then there exists a constant $c>0$ such that for every self-adjoint operators A and K on Hilbert space the following inequality holds:

$$\left\|\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(A+jK)\right\|\le\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha.$$

We need the following lemma.

Proof . By Theorem (5.2.2), we may assume that f is a bounded function.

We are going to use the same notation f_n and g_N as in the proof of Theorem(5.2.8). In the case when A and K are bounded self-adjoint operators we show that

$$f(A+K)-2f(A)+f(A-K)=\sum_{n=-\infty}^{\infty}\big(f_n(A+K)-2f_n(A)+f_n(A-K)\big),\qquad(68)$$

and the series converges absolutely in the operator norm. As in the proof of Theorem(5.2.8), we can easily see that

$$f(A+K)=(f*V_N)(A+K)+\sum_{n>N}f_n(A+K),$$

$$f(A)=(f*V_N)(A)+\sum_{n>N}f_n(A),$$

and

$$f(A-K)=(f*V_N)(A-K)+\sum_{n>N}f_n(A-K),$$

and the series converge absolutely in the operator norm. It follows that

$$\begin{aligned}
&f(A+K)-2f(A)+f(A-K)-\sum_{n>N}\big(f_n(A+K)-2f_n(A)+f_n(A-K)\big)\\
&\quad=\left(f(A+K)-\sum_{n>N}f_n(A+K)\right)-2\left(f(A)-\sum_{n>N}f_n(A)\right)+\left(f(A-K)-\sum_{n>N}f_n(A-K)\right)\\
&\quad=g_N(A+K)-2g_N(A)+g_N(A-K).
\end{aligned}$$

Since $g_N\in L^\infty(\mathbb{R})$ and g_N is an entire function of exponential type at most 2^{N+1}, it follows from Lemma (5.2.9)and from (59) that

$$\|g_N(A+K)-2g_N(A)+g_N(A-K)\|\le \text{const}\,2^{2N}\|g_N\|_{L^\infty}\|K\|\le \text{const}\,2^{2N}\|f\|_{L^\infty}\|K\|\to 0 \text{ as } N\to-\infty.$$

This implies that the series on the right-hand side of (68) converges absolutely in the operator norm.

As in the proof of Theorem(5.2.8), we consider the integer N satisfying

$$2^{-N}<\|K\|\le 2^{-N+1}. \tag{69}$$

Put now

$$T_1\stackrel{\text{def}}{=}\sum_{n\le N}\big(f_n(A+K)-2f_n(A)+f_n(A-K)\big)$$

and

$$T_2\stackrel{\text{def}}{=}\sum_{n>N}\big(f_n(A+K)-2f_n(A)+f_n(A-K)\big).$$

It follows now from Corollary (5.2.7), Lemma (5.2.9), from (69), and (59) that

$$\begin{aligned}
\|T_1\|&\le\sum_{n\le N}\|f_n(A+K)-2f_n(A)+f_n(A-K)\|\\
&=2\sum_{n\le N}\left\|\iiint\big(\mathfrak{D}^2f_n\big)(x,y,z)\,dE_{A+K}(x)K\,dE_A(y)K\,dE_{A-K}(z)\right\|\\
&\le \text{const}\sum_{n\le N}2^{2n}\left\|f_n\right\|_{L^\infty}\|K\|^2\le \text{const}\sum_{n\le N}2^{n(2-\alpha)}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^2\\
&\le \text{const}\,2^{N(2-\alpha)}\|K\|^2\|f\|_{\Lambda_\alpha(\mathbb{R})}\le \text{const}\,\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha.
\end{aligned}$$

On the other hand, by (69),

$$\|T_2\| \le \sum_{n>N} \|f_n(A+K) - 2f_n(A) + f_n(A-K)\| \le 4\sum_{n>N} \|f_n\|_{L^\infty} \le \text{const} \sum_{n>N} 2^{-n\alpha} \|f\|_{\Lambda_\alpha(\mathbb{R})}$$
$$\le \text{const } 2^{-N\alpha} \|f\|_{\Lambda_\alpha(\mathbb{R})} \le \text{const} \|K\|^\alpha.$$
As in the case $\alpha < 1$, for unbounded self-adjoint operators we understand by $f(A+K) - 2f(A) + f(A-K)$ the sum of the following series
$$\sum_{n\in\mathbb{Z}} \big(f_n(A+K) - 2f_n(A) + f_n(A-K)\big),$$
which converges absolutely. We refer to [220] where the case of unbounded self-adjoint operators will be considered in more detail.

Corollary (5.2.11)[197]. There exists a constant $c > 0$ such that for an arbitrary function f in the Zygmund class $\Lambda_1(\mathbb{R})$ and arbitrary self-adjoint operators A and K , the following inequality holds:
$$\|f(A+K) - 2f(A) + f(A-K)\| \le c\|f\|_{\Lambda_1(\mathbb{R})} \|K\|.$$
Note that. We can interpret Theorem(5.2.9) in the following way. Consider the measure ν on $\mathbb{R}$defined by
$$\nu \stackrel{\text{def}}{=} \Delta_1^m \delta_0 = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \delta_{-j},$$
where for $a \in \mathbb{R}$, δ_a is the unit point mass at a. Then
$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(A+jK) = \int_{\mathbb{R}} f(A-tK)\, d\nu(t).$$
Clearly, ν determines a continuous linear functional on $\lambda_\alpha(\mathbb{R})$ defined by
$$f \longmapsto \int_{\mathbb{R}} f(t)\, d\nu(t).$$
In other words, $\nu \in \boldsymbol{B}_1^{-\alpha}$. We are going to generalize Theorem(5.2.9) to the case of an arbitrary distribution in$\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$.

For simplicity, we consider here the case of bounded self-adjoint operators A. In [220] we will consider the case of an arbitrary (not necessarily bounded) self-adjoint operator A.

It follows from Theorem(5.2.9) that for arbitrary vectors u and v in our Hilbert space $\mathcal{H}$and for an arbitrary function f in$\Lambda_\alpha(\mathbb{R})$, the function
$$t \longmapsto f_{A,K}^{u,v}(t) \stackrel{\text{def}}{=} (f(A-tK)u, v)$$
belongs to $\Lambda_\alpha(\mathbb{R})$. Identifying the space $\Lambda_\alpha(\mathbb{R})$ with the dual space to $\boldsymbol{B}_1^{-\alpha}$, wecan consider for every distribution g in $\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$ the value $\langle f_{A,K}^{u,v}, g\rangle$of $f_{A,K}^{u,v} \in (\boldsymbol{B}_1^{-\alpha}(\mathbb{R}))^*$ at g.We define now the operator $\mathcal{Q}_{A,K}^{g}: \Lambda_\alpha(\mathbb{R}) \longrightarrow \mathcal{B}(\mathcal{H})$ by
$$\big((\mathcal{Q}_{A,K}^{g} f)u, v\big) = \langle f_{A,K}^{u,v}, g\rangle, \quad f \in \Lambda_\alpha(\mathbb{R}), \qquad u, v \in \mathcal{H}.$$
Theorem (5.2.12)[197]. Let $\alpha > 0$. Then there exists $c > 0$ such that for every self-adjoint operators $A$and , for every $f \in \Lambda_\alpha(\mathbb{R})$, and for every $g \in \boldsymbol{B}_1^{-\alpha}$,
$$\big\|\mathcal{Q}_{A,K}^{g} f\big\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R})} \|g\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})} \|K\|^\alpha. \tag{70}$$

Proof. Let m be the smallest integer greater than α. By Theorem(5.2.9), inequality (70) holds for= $\Delta_1^m\delta_0$. Hence, the result also holds for $g = \Delta_h^m\delta_a$ for arbitrary $h, a \in \mathbb{R}$.
To complete the proof, it suffices to use the following fact (see [1, Theorem 3.1]): if $g \in \boldsymbol{B}_1^{-\alpha}(\mathbb{R})$, then g admits a representation in the form of a norm convergent series

$$g = \sum_{j\geq 1} \lambda_j \, \Delta_{h_j}^m \delta_{a_j}, \qquad h_j, a_j \in \mathbb{R},$$

such that

$$\sum_{j\geq 1} |\lambda_j| \cdot \left\| \Delta_{h_j}^m \delta_{a_j} \right\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})} \leq \text{const} \|g\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})}.$$

We also obtain an estimate for $\|f(U) - f(V)\|$ for a function f in the Zygmund class$\Lambda_1$and unitary operators $U$and .

Theorem (5.2.13)[197]. Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U) - f(V)\| \leq c\|f\|_{\Lambda_\alpha} \|U - V\|^\alpha.$$

Proof.Let $\in \Lambda_\alpha$. We have

$$f = \mathbb{P}_+ f + \mathbb{P}_- f = f_+ + f_-.$$

We estimate $\|f_+(U) - f_+(V)\|$. The norm of $f_-(U) - f_-(V)$ can be obtained in the same way. Thus we assume that $f = f_+$. Let

$$f_n \stackrel{\text{def}}{=} f * W_n.$$

Then

$$f = \sum_{n\geq 0} f_n. \qquad (71)$$

Clearly, we may assume that $U \neq V$. Let N be the nonnegative integer such that

$$2^{-N} < \|U - V\| \leq 2^{-N+1}. \qquad (72)$$

We have

$$f(U) - f(V) = \sum_{n\leq N} \big(f_n(U) - f_n(V)\big) + \sum_{n>N} \big(f_n(U) - f_n(V)\big).$$

By the Birman–Solomyak formula for unitary operators and by (54),

$$\left\| \sum_{n\leq N} \big(f_n(U) - f_n(V)\big) \right\| \leq \sum_{n>N} \|f_n(U) - f_n(V)\|$$
$$\leq \text{const} \textstyle\sum_{n>N} 2^n \|U - V\| \cdot \|f_n\|_{L^\infty} \leq \text{const}\|U - V\| \textstyle\sum_{n>N} 2^n \, 2^{-n\alpha} \|f\|_{\Lambda_\alpha}$$
$$\leq \text{const}\|U - V\| 2^{N(1-\alpha)} \|f\|_{\Lambda_\alpha} \leq \text{const}\|U - V\|^\alpha \|f\|_{\Lambda_\alpha},$$

the last inequality being a consequence of (72).
On the other hand,

$$\left\| \sum_{n>N} \big(f_n(U) - f_n(V)\big) \right\| \leq \sum_{n>N} 2\|f_n\|_{L^\infty} \leq \text{const} \sum_{n>N} 2^{-n\alpha} \, \|f\|_{\Lambda_\alpha}$$
$$\leq \text{const} 2^{-N\alpha} \|f\|_{\Lambda_\alpha} \leq \text{const}\|U - V\|^\alpha \|f\|_{\Lambda_\alpha}$$

To obtain an analog of Theorem(5.2.9) for unitary operator, we are going to represent a finite difference

$$\sum_{j=1}^{N}(-1)^{j-1}\binom{N-1}{j-1}f(U_j)$$

for unitary operators $U_1,\dots,U_N$ as a linear combination of multiple operator integrals.

Note that algebraic formulae in the case of unitary operators are more complicated than inthe case of self-adjoint operators. That is why we consider the case of unitary operators in more detail.

We first illustrate the idea in the special case $N = 3$.

Let us show that for unitary operators U_1, $U_2$and U_3 and for $\in \boldsymbol{B}_{\infty 1}^2$,

$$\begin{aligned} f(U_1) - 2f(U_2) + f(U_3) & \\ &= 2\iiint (\mathfrak{D}^2 f)\,(\zeta,\tau,\upsilon)dE_1(\zeta)(U_1-U_2)dE_2(\tau)(U_2-U_3)dE_3(\upsilon) \\ &+ \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-2U_2+U_3)dE_3(\tau), \qquad (73) \end{aligned}$$

whereE_j is the spectral measure of U_j , $1 \le j \le 3$.

Indeed, let $T = f(U_1) - 2f(U_2) + f(U_3)$. Then

$$\begin{aligned} T &= f(U_1) - f(U_2) - \big(f(U_2) - f(U_3)\big) \\ &= \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_2(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_2(\zeta)(U_2-U_3)dE_3(\tau) \\ &= \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_2(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_3(\tau) \\ &+ \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_3(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_2-U_3)dE_3(\tau) \\ &+ \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_2-U_3)dE_3(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_2(\zeta)(U_2-U_3)dE_3(\tau). \end{aligned}$$

We have

$$\begin{aligned} &\iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_2(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_3(\tau) \\ &= \iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_2(\tau) - \iint (\mathfrak{D}f)(\zeta,\upsilon)dE_1(\zeta)(U_1-U_2)dE_3(\upsilon) \\ &= \iiint (\mathfrak{D}f)\,(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_2(\tau)dE_3(\upsilon) - \iiint (\mathfrak{D}f)\,(\zeta,\upsilon)dE_1(\zeta)(U_1-U_2)dE_2(\tau)dE_3(\upsilon) \\ &= \iiint (\tau-\upsilon)(\mathfrak{D}^2 f)\,(\zeta,\tau,\upsilon)dE_1(\zeta)(U_1-U_2)dE_2(\tau)dE_3(\upsilon) \\ &= \iiint (\mathfrak{D}^2 f)\,(\zeta,\tau,\upsilon)dE_1(\zeta)(U_1-U_2)dE_2(\tau)(U_2-U_3)dE_3(\upsilon). \end{aligned}$$

Similarly,

$$\begin{aligned} &\iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_2-U_3)dE_3(\tau) - \iint (\mathfrak{D}f)(\zeta,\tau)dE_2(\zeta)(U_2-U_3)dE_3(\tau) \\ &= \iiint (\mathfrak{D}^2 f)\,(\zeta,\tau,\upsilon)dE_1(\zeta)(U_1-U_2)dE_2(\tau)(U_2-U_3)dE_3(\upsilon). \end{aligned}$$

Finally,

$$\iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-U_2)dE_3(\tau)-\iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_2-U_3)dE_3(\tau)$$
$$=\iint (\mathfrak{D}f)(\zeta,\tau)dE_1(\zeta)(U_1-2U_2+U_3)dE_3(\tau).$$

Consider now the general case. Suppose that $\mathcal{U}=\{U_j\}_1^N$ is a finite family of unitary operators.

Denote by E_j the spectral measure of U_j. For $1\le j<k\le N$, we put

$$T(j,k)=\sum_{s=0}^{k-j}(-1)^s\binom{k-j}{s}U_{j+s}. \tag{74}$$

Note that

$$T(j,k)-T(j+1,k+1)=T(j,k+1), 1\le j<k\le N-1. \tag{75}$$

We would like to mention that formula (75) is purely algebraic and it is valid for arbitrary operators U_j in (74).

Let J be a nonempty subset of $\{1,2,\dots,N\}$. We denote by $d=d_J$ the number of elementsof J. Suppose that $J=\{j_1,j_2,\dots,j_d\}$, where$j_1<j_2<\cdots<j_d$. For $f\in \boldsymbol{B}_{\infty 1}^{d-1}$, we put

$$\mathfrak{T}_J(\mathcal{U},f)\overset{\text{def}}{=}\underbrace{\int\cdots\int}_{d}(\mathfrak{D}^{d-1}f)(\zeta_1,\dots,\zeta_d)dE_{j_1}(\zeta_1)\prod_{s=2}^{d}T(j_{s-1},j_s)\,dE_{j_s}(\zeta_s).$$

Though, we need the case,$d_J\ge 2$, but we still can assume that $d_J=1$, in which case we put

$$\mathfrak{T}_J(\mathcal{U},f)\overset{\text{def}}{=}\int f(\zeta)\,dE_j(\zeta),\quad \text{where } J=\{j\}.$$

We denote by $\mathfrak{A}$the collection of all finite subsets of the set of positive integers and by $\mathfrak{A}_N$the collection of all subsets $J\in\mathfrak{A}$ such that the maximal element of J is N.

If$J_1,J_2\in\mathfrak{A}$, we say that $J_1$is an ancestor of J_2if J_2can be partitioned in nonempty subsetsJ_2' and J_2''such that $\max J_2'<\max J_2''$ and $J_1=J_2'\cup(J_2''-1)$ (by $\Lambda-1$ we mean the left translateof a subset Λ of $\mathbb{Z}$ by 1). Each such partition is called an evidence of the fact that $J_1$is an ancestor of J_2. We denote by $\#(J_1,J_2)$ the number of such evidences and we put $\#(J_1,J_2)=0$ if $J_1$is notan ancestor ofJ_2. Note that the property of being an ancestor is not transitive.

If $\#(J_1,J_2)\ge 1$, then $\max J_2=1+\max J_1$ and $0\le d_{J_2}-d_{J_1}\le 1$. It is also easy to see that if$d_{J_1}-d_{J_2}$, then $\#(J_1,J_2)=0=1$.

Let us construct now the family x_J of integers by induction. Put$x_{\{1\}}=1$. Suppose that thenumbers $x_J$are defined for $J\in\mathfrak{A}_{N-1}$. Let $J\in\mathfrak{A}_N$. Put

$$x_J=\sum_{I\in\mathfrak{A}_{N-1}}\#(I,J)x_I.$$

Clearly,x_J is a positive integer for every $J\in\mathfrak{A}_N$. We leave for the reader the verification of the fact that for $\{j_1,j_2,\dots,j_d\}\in\mathfrak{A}$,

$$x_J=\frac{(j_d-j_1)!}{\prod_{s=2}^{d}T(j_s-j_{s-1})!}.$$

Theorem (5.2.14)[197]. Let Nbe a positive integer and let $\mathcal{U} = \{U_j\}_1^N$be unitary operators on Hilbertspace. Suppose that f is a function in the Besov space $\boldsymbol{B}_{\infty 1}^{N-1}$. Then

$$\sum_{j=0}^{N}(-1)^{j-1}\binom{N-1}{j-1}f(U_j) = \sum_{I\in\mathfrak{A}_N} x_J\,\mathfrak{T}_J(\mathcal{U},f).$$

We need one more lemma. To state it, we introduce some more notations. For $J \in \mathfrak{A}$, we denoteby $\mathfrak{L}(J)$ the collection of nonempty proper subsets of J such that

$$\max \Lambda < \min(J\backslash\Lambda).$$

For $\Lambda \in \mathfrak{L}(J)$, we put

$$\Lambda_J^\circ \stackrel{\text{def}}{=} J\backslash\Lambda \text{ and} \Lambda_J^\bullet \stackrel{\text{def}}{=} \Lambda_J^\circ \cup \{\max \Lambda\}$$

If J is specified, we write Λ° and $\Lambda^\bullet$ instead of $\Lambda_J^\circ$and$\Lambda_J^\bullet$.

Proof. We argue by induction on N. For $N = 1$, we have

$$f(U_1) = \int f(\zeta_1)\,dE(\zeta_1).$$

Suppose that the result holds for $\mathrm{N}-1$ unitary operators. Put $\mathcal{U}^- \stackrel{\text{def}}{=} \{U_{j+1}\}_{j=1}^{N-1}$. We have

$$\sum_{j=1}^{N-1}(-1)^{j-1}\binom{N-2}{j-1}f(U_j) = \sum_{J\in\mathfrak{A}_{N-1}} x_J\,\mathfrak{T}_J(\mathcal{U},f)$$

and

$$\sum_{j=1}^{N-1}(-1)^{j-1}\binom{N-2}{j-1}f(U_{j+1}) = \sum_{J\in\mathfrak{A}_{N-1}} x_J\,\mathfrak{T}_J(\mathcal{U}^-,f) = \sum_{J\in\mathfrak{A}_{N-1}} x_J\,\mathfrak{T}_{J+1}(\mathcal{U},f).$$

It follows now from Lemma (5.2.15) and from (75) with $f(U_j)$ in place of $U_j$that

$$\sum_{j=1}^{N}(-1)^{j-1}\binom{N-1}{j-1}f(U_j) = \sum_{J\in\mathfrak{A}_{N-1}} x_J\,\mathfrak{T}_J\Big((\mathcal{U},f) - \mathfrak{T}_{J+1}(\mathcal{U},f)\Big)$$

$$= \sum_{J\in\mathfrak{A}_{N-1}} x_J\left(\sum_{\Lambda\in\mathfrak{L}(J)}\mathfrak{T}_{\Lambda\cup(\Lambda^\circ+1)}(\mathcal{U},f) + \sum_{\Lambda\in\mathfrak{L}(J)}\mathfrak{T}_{\Lambda\cup(\Lambda^\bullet+1)}(\mathcal{U},f) + \mathfrak{T}_{J\cup\{N\}}(\mathcal{U},f)\right).$$

It remains to observe that a set J in $\mathfrak{A}_{N-1}$ is an ancestor of a set J_0in $\mathfrak{A}_N$if and only if $J_0 = \Lambda \cup (\Lambda^\circ + 1)$ for some $\Lambda \in \mathfrak{L}(J)$ or$J_0 = \Lambda \cup (\Lambda^\bullet + 1)$ for some $\Lambda \in \mathfrak{L}(J)$ or$J_0 = J \cup \{N\}$.

Lemma (5.2.15)[197]. Let $\in \mathfrak{A}_{N-1}$. Then

$$\mathfrak{T}_J(\mathcal{U},f) - \mathfrak{T}_{J+1}(\mathcal{U},f) = \sum_{\Lambda\in\mathfrak{L}(J)}\mathfrak{T}_{\Lambda\cup(\Lambda^\circ+1)}(\mathcal{U},f) + \sum_{\Lambda\in\mathfrak{L}(J)}\mathfrak{T}_{\Lambda\cup(\Lambda^\bullet+1)}(\mathcal{U},f) + \mathfrak{T}_{J\cup\{N\}}(\mathcal{U},f).$$

Proof. The above identity can be verified straightforwardly if we observe that the multiple operator integral

$$\underbrace{\int\cdots\int}_{d}(\mathfrak{D}^{d-1}f)(\zeta_1,\dots,\zeta_d)dF_1(\zeta_1)\prod_{s=2}^{d}Q_{s-1}\,dF_s(\zeta_s)$$

is a multilinear function in the operators $Q_s$and use the following easily verifiable identity:

$$\iint (\mathfrak{D}f)(\zeta_1,\zeta_2)dE_1(\zeta_1)(U_1-U_2)dE_2(\zeta_1) = \int f(\zeta)\,dE_1(\zeta) - \int f(\zeta)\,dE_2(\zeta).$$

Theorem (5.2.16)[197']. Let m be a positive integer and $0 < \alpha < m$. Then there exists a constant $c > 0$such that for every $f \in \Lambda_\alpha$ and for an arbitrary unitary operator U and an arbitrary boundedself-adjoint operator A on Hilbert space, the following inequality holds:

$$\left\|\sum_{k=0}^{m}(-1)^k\binom{m}{k}f(e^{ikA}U)\right\| \le c\|f\|_{\Lambda_\alpha}\|A\|^\alpha.$$

Proof. For simplicity we give a proof for $m = 2$. The general case can be treated in the same way. We have to show that for $0 < \alpha < 2$, there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha$and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(\mathcal{V}U) - 2f(U) + f(\mathcal{V}^*U)\| \le c\|f\|_{\Lambda_\alpha}\|I-\mathcal{V}\|^\alpha.$$

As in the proof of Theorem (5.2.13), we assume that $f = f_+$ and consider the expansion

$$f = \sum_{n\ge 0} f_n.$$

Let N be the nonnegative integer such that

$$2^{-N} < \|I-\mathcal{V}\| \le 2^{-N+1}. \quad (76)$$

We have

$$\begin{aligned} &f(\mathcal{V}U) - 2f(U) + f(\mathcal{V}^*U) \\ &\qquad = \sum_{n\le N}\big(f_n(\mathcal{V}U) - 2f_n(U) + f_n(\mathcal{V}^*U)\big) + \sum_{n>N}\big(f_n(\mathcal{V}U) - 2f_n(U) + f_n(\mathcal{V}^*U)\big). \end{aligned}$$

Let$T_n = f_n(\mathcal{V}U) - 2f_n(U) + f_n(\mathcal{V}^*U)$. It follows from (73) that

$$\begin{aligned} T_n = 2&\iiint (\mathfrak{D}^2 f_n)(\zeta,\tau,\upsilon)dE_{\mathcal{V}U}(\zeta)(\mathcal{V}-I)UdE_U(\tau)(I-\mathcal{V}^*)UdE_{\mathcal{V}^*U}(\upsilon) \\ &+ \iint (\mathfrak{D}f_n)(\zeta,\tau)dE_{\mathcal{V}U}(\zeta)(\mathcal{V}-2I+\mathcal{V}^*)UdE_{\mathcal{V}^*U}(\tau). \end{aligned}$$

By (58), we have

$$\begin{aligned} &\left\|\iiint (\mathfrak{D}^2 f_n)(\zeta,\tau,\upsilon)dE_{\mathcal{V}U}(\zeta)(\mathcal{V}-I)UdE_U(\tau)(I-\mathcal{V}^*)UdE_{\mathcal{V}^*U}(\upsilon)\right\| \\ &\qquad \le \text{const } 2^{2n}\|I-\mathcal{V}\|^2\|f_n\|_{L^\infty}. \end{aligned}$$

On the other hand, by (54),

$$\begin{aligned} \left\|\iint (\mathfrak{D}f_n)(\zeta,\tau)dE_{\mathcal{V}U}(\zeta)(\mathcal{V}-2I+\mathcal{V}^*)UdE_{\mathcal{V}^*U}(\tau)\right\| &\le \text{const } 2^n\|\mathcal{V}-2I+\mathcal{V}^*\|\|f_n\|_{L^\infty} \\ &\le \text{const } 2^n\|I-\mathcal{V}\|^2\|f_n\|_{L^\infty} \end{aligned}$$

Thus

$$\begin{aligned} \left\|\sum_{n\le N}\big(f_n(\mathcal{V}U) - 2f_n(U) + f_n(\mathcal{V}^*U)\big)\right\| &\le \text{const}\|I-\mathcal{V}\|^2\sum_{n\le N}2^{2n}\,\|f_n\|_{L^\infty} \\ &\le \text{const}\|I-\mathcal{V}\|^2\sum_{n\le N}2^{2n}2^{-n\alpha}\,\|f\|_{\Lambda_\alpha} \le \text{const}\|I-\mathcal{V}\|^2 2^{N(2-\alpha)}\|f\|_{\Lambda_\alpha} \\ &\le \text{const}\|f\|_{\Lambda_\alpha}\|I-\mathcal{V}\|^\alpha \end{aligned}$$

by(76).

To complete the proof, we observe that

$$\left\|\sum_{n>N}\big(f_n(\mathcal{V}U)-2f_n(U)+f_n(\mathcal{V}^*U)\big)\right\|\le\sum_{n>N}\big\|\big(f_n(\mathcal{V}U)-2f_n(U)+f_n(\mathcal{V}^*U)\big)\big\|\le\sum_{n>N}4\,\|f_n\|_{L^\infty}$$
$$\le \mathrm{const}\|f\|_{\Lambda_\alpha}\sum_{n>N}2^{-n\alpha}\le \mathrm{const}\|f\|_{\Lambda_\alpha}2^{-N\alpha}\le \mathrm{const}\|I-\mathcal{V}\|^\alpha$$

by(76).

The following result gives an estimate for $\|f(U)-f(V)\|$ for functions f in the ZygmundclassΛ_1.

Theorem (5.2.17)[197]. There exists a constant $c>0$ such that for every function $f\in\Lambda_1$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U)-f(V)\|\le c\|f\|_{\Lambda_1}\left(2+\log_2\frac{1}{\|U-V\|}\right)\|U-V\|.$$

Proof. Again, as in the proof of Theorem (5.2.13), we assume that $f=f_+$ and N is the nonnegative integer satisfying (72). Using the notation introduced in the proof of Theorem (5.2.13), we obtain

$$\left\|\sum_{n\le N}\big(f_n(U)-f_n(V)\big)\right\|\le\sum_{n\le N}\|f_n(U)-f_n(V)\|\le \mathrm{const}\sum_{n\le N}2^n\,\|U-V\|\cdot\|f_n\|_{L^\infty}$$
$$\le \mathrm{const}(1+N)\|f\|_{\Lambda_\alpha}\|U-V\|\le \mathrm{const}\|f\|_{\Lambda_\alpha}\left(2+\log_2\frac{1}{\|U-V\|}\right)\|U-V\|.$$

On the other hand,

$$\left\|\sum_{n>N}\big(f_n(U)-f_n(V)\big)\right\|\le\sum_{n>N}2\,\|f_n\|_{L^\infty}\le \mathrm{const}\sum_{n>N}2^{-n}\,\|f\|_{\Lambda_\alpha}\le \mathrm{const}\;2^{-N}\|f\|_{\Lambda_\alpha}$$
$$\le \mathrm{const}\;2^{-N}\|f\|_{\Lambda_\alpha}\|U-V\|.$$

In a similar way we can obtain an estimate for differences of order n and functions in $\Lambda_n$for an arbitrary positive integer n.

Let us obtain now an analog of Theorem (5.2.12) for unitary operators. Let U be a unitary operator and let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Suppose that $f\in\Lambda_\alpha$. By Theorem (5.2.16), for every $u,v\in\mathcal{H}$, the function

$$f\longmapsto f_{A,K}^{u,v}(t)\stackrel{\mathrm{def}}{=}(f(e^{itA}U)u,v)$$

on$\mathbb{R}$ belongs to the space $\Lambda_\alpha(\mathbb{R})$. Thus for every $g\in\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$, we can define the operator $\mathcal{R}_{U,A}^g:\Lambda_\alpha\to\mathcal{B}(\mathcal{H})$ such that

$$\left((\mathcal{R}_{U,A}^g f)u,v\right)=\langle f_{A,K}^{u,v},g\rangle$$

(here we identify the dual space $(\boldsymbol{B}_1^{-\alpha}(\mathbb{R}))^*$ with $\Lambda_\alpha(\mathbb{R})$).

Theorem (5.2.18)[197]. Let $\alpha>0$. Then there exists $c>0$ such that for arbitrary unitary operator U and a boundary self-adjoint operator A, and for every $g\in\boldsymbol{B}_1^{-\alpha}(\mathbb{R})$,

$$\left\|\mathcal{R}_{U,A}^g\right\|\le c\|g\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})}\|A\|^\alpha.$$

Proof. Clearly,

$$\left|\left(\left(\mathcal{R}_{U,A}^{g}f\right)u,v\right)\right| \le \mathrm{const}\left\|f_{A,K}^{u,v}\right\|_{\Lambda_\alpha(\mathbb{R})}\|g\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})} \le \mathrm{const}\|u\|\cdot\|v\|\cdot\|f\|_{\Lambda_\alpha}\|g\|_{\boldsymbol{B}_1^{-\alpha}(\mathbb{R})}\|A\|^\alpha.$$

Recall that if $T$is a contraction on Hilbert space, it follows from von Neumann's inequality that the polynomial functional calculus $f \longmapsto f(T)$ extends to the disk-algebra C_A and$\|f(T)\| \le \|f\|_{C_A}$, $f \in C_A$.

Theorem (5.2.19)[197]. Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space the following inequality holds:

$$\|f(T) - f(R)\| \le c\|f\|_{\Lambda_\alpha}\|T - R\|^\alpha.$$

Proof. The proof of Theorem (5.2.19) is almost the same as the proof of Theorem (5.2.13). For $f \in (\Lambda_\alpha)_+$, we use expansion (71) and choose N such that

$$2^{-N} < \|T - R\| \le 2^{-N+1}.$$

Then as in the proof of Theorem (5.2.13), for $\le N$, we estimate $\|f_n(T) - f_n(R)\|$ in terms of const $2^{-n}\|T - R\|$ (see (65) and (54)), while for $n > N$ we use von Neumann's inequality to estimate $\|f_n(T) - f_n(R)\|$ in terms of $2\|f_n\|_{L^\infty}$. The rest of the proof is the same.

Corollary (5.2.20)[197]. Let f be a function in the disk algebra and $0 < \alpha < 1$. Then the following two statements are equivalent:

(i) $\|f(T) - f(R)\| \le \mathrm{const}\|T - R\|^\alpha$for all contractions T and R,

(ii) $\|f(U) - f(V)\| \le \mathrm{const}\|U - V\|^\alpha$for all unitary operators U and V .

Remark. This corollary is also true for $\alpha = 1$. This was proved by Kissin and Shulman [231].

The following result is an analog of Theorem (5.2.16) for contractions.

Theorem (5.2.21)[197]. Let m be a positive integer and $0 < \alpha < m$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space the followinginequality holds:

$$\left\|\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\left(T + \frac{k}{m}(R - T)\right)\right\| \le c\|f\|_{\Lambda_\alpha}\|T - R\|^\alpha.$$

To prove Theorem (5.2.21), we use the following analog of Lemma 4.3.

Lemma (5.2.22)[197]. Let m be a positive integer and let f be a function of class $(\boldsymbol{B}_{\infty 1}^m)_+$. If T and R arecontractions on Hilbert space, then

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\left(T + \frac{k}{m}(R - T)\right)$$
$$= \frac{m!}{m^m}\underbrace{\int\cdots\int}_{m+1}(\mathfrak{D}^m f)(\zeta_1,\ldots,\zeta_{m+1})d\mathcal{E}_1(\zeta_1)(R - T)\cdots(R - T)d\mathcal{E}_{m+1}(\zeta_{m+1})$$

where$\mathcal{E}_k$is a semi-spectral measure of $T + \frac{k}{m}(R - T)$.

Theorem (5.2.23)[197]. There exists a constant $c > 0$ such that for every function $f \in (\Lambda_1)_+$ and for arbitrary contractions $T$and R on Hilbert space the following inequality holds:

$$\|f(T)-f(R)\| \le c\|f\|_{\Lambda_1}\left(2+\log_2\frac{1}{\|T-R\|}\right)\|T-R\|.$$

We consider the problem of estimating $\|f(A)-f(B)\|$ for self-adjoint operators A and B and functions f in the space Λ_ω, where ω is an arbitrary modulus of continuity. We give complete proofs for bounded self-adjoint operators. The same estimates also hold for unbounded self-adjoint operators. This will follow from Theorem (5.2.37). We also obtain similar results for unitary operators and for contractions.

We have mentioned in the introduction that a Lipschitz function does not have to be operator Lipschitz and a continuously differentiable function does not have to be operator differentiable. On the other hand, we have proved that a Hölder function of order $\alpha \in (0,1)$ must beoperator Hölder of order α as well as a Zygmund function must be operator Zygmund. Moreover, the same is true for all classes Λ_α with $\alpha > 0$. This suggests an idea that the situation is similar with continuity properties of the Hilbert transform.We consider the problem forwhich moduli of continuity ω the fact that $f \in \Lambda_\omega$ implies that f belongs to the "operator spaceΛ_ω", i.e.,

$$\|f(A)-f(B)\| \le \text{const}\ \omega(\|A-B\|).$$

We are going to compare this property with the fact that the Hilbert transform acts onΛ_ω.

Given a modulus of continuity ω, we define the function ω_* by

$$\omega_*(x) = x\int_x^\infty \frac{\omega(t)}{t^2}\,dt, \qquad x > 0.$$

Theorem (5.2.24)[197]. There exists a constant $c > 0$ such that for every modulus of continuity ω, every $f \in \Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators $A$and B, the following inequalityholds

$$\|f(A)-f(B)\| \le c\|f\|_{\Lambda_\omega(\mathbb{R})}\,\omega_*(\|A-B\|).$$

Proof. Since A and B are bounded operators and their spectra are contained in $[a,b]$, we can replace a function $f \in \Lambda_\omega(\mathbb{R})$ with the bounded function $f_\flat$ defined by

$$f_\flat(x) = \begin{cases} f(b), & x > b, \\ f(x), & x \in [a,b], \\ f(a), & x < a. \end{cases} \quad (77)$$

Clearly, $\|f_\flat\|_{\Lambda_\omega(\mathbb{R})} \le \|f\|_{\Lambda_\omega(\mathbb{R})}$. Thus we may assume that f is bounded.

Let N be an integer. We claim that

$$f(A)-f(B) = \sum_{n=-\infty}^{N}\big(f_n(A)-f_n(B)\big) + \big((f-f*V_N)(A)-(f-f*V_N)(B)\big) \quad (78)$$

andthe series converges absolutely in the operator norm. Here $f_n = f*W_n + f*W_n^\#$. Suppose that $M < N$. Indeed, it is

$$f(A)-f(B)-\left(\sum_{n=M+1}^{N}\big(f_n(A)-f_n(B)\big)+\big((f-f*V_N)(A)-(f-f*V_N)(B)\big)\right)$$
$$=(f*V_M)(A)-(f*V_M)(B).$$

Clearly, $f*V_N$ is an entire function of exponential type at most 2^{M+1}. Thus it follows from (55) that

$$\|(f*V_M)(A)-(f*V_M)(B)\|\le \text{const}\,2^M\|f\|_{L^\infty}\|A-B\|\to 0 \text{ as } M\to-\infty.$$

Suppose now that N is the integer satisfying (67). It follows from (2.2.4) that

$$\|(f-f*V_N)(A)-(f-f*V_N)(B)\|\le 2^M\|f-f*V_N\|_{L^\infty}$$
$$\le \text{const}\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(2^{-N})$$
$$\le \text{const}\|f\|_{\Lambda_\omega(\mathbb{R})}\omega(\|A-B\|).$$

On the other hand, it follows from Corollary(5.2.5) and from (55) that

$$\sum_{n=-\infty}^{N}\|f_n(A)-f_n(B)\|\le \text{const}\sum_{n=-\infty}^{N}2^n\|f\|_{L^\infty}\|A-B\|\le \text{const}\sum_{n=-\infty}^{N}2^n\omega(2^{-n})\,\|f\|_{\Lambda_\omega(\mathbb{R})}\|A-B\|$$
$$=\text{const}\sum_{k\ge 0}^{N}2^{N-k}\omega(2^{-N+k})\,\|f\|_{\Lambda_\omega(\mathbb{R})}\|A-B\|$$
$$\le \text{const}\left(\int_{2^{-N}}^{\infty}\frac{\omega(t)}{t^2}\,dt\right)\|f\|_{\Lambda_\omega(\mathbb{R})}\|A-B\|\le \text{const}\,2^N\omega_*(2^{-N})\|f\|_{\Lambda_\omega(\mathbb{R})}\|A-B\|$$
$$\le \text{const}\|f\|_{\Lambda_\omega(\mathbb{R})}\omega_*(\|A-B\|).$$

The result follows now from the obvious inequality $\omega(x)\le\omega_*(x), x>0$.

Obviously, if $\omega_*(x)<\infty$ for some $x>0$, then $\omega_*(x)<\infty$ for every $x>0$. It follows easily from l'Hôpital's rule that in this case

$$\lim_{x\to 0}\omega_*(x)=0.$$

Moreover, in this case ω_* is also a modulus of continuity. Indeed, it is easy to see that

$$\omega_*(x)=\int_1^\infty\frac{\omega(sx)}{s^2}\,ds$$

which implies that

$$\omega_*(x+y)\le\omega_*(x)+\omega_*(y),\qquad x,y\ge 0$$

and

$$\omega_*(x)\le\omega_*(y),\ \ 0\le x\le y.$$

Note that if the modulus of continuity ω is bounded, then obviously, $\omega_*(x)<\infty$ for every $x>0$. In the case when A and B are bounded self-adjoint operators and their spectra are contained in $[a,b]$, we can replace f with the function f_b defined by (77) redefine the function ω on $[b-a,\infty)$ by putting $\omega(x)=\omega(b-a)$. Clearly, the modified modulus of continuity is bounded.

Corollary (5.2.25)[197]. Let ω be a modulus of continuity such that

$$\omega_*(x)\le \text{const}\,\omega(x),\quad x>0.$$

Then for an arbitrary function $f \in \Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B onHilbert space the following inequality holds:

$$\|f(A) - f(B)\| \le \text{const}\|f\|_{\Lambda_\omega(\mathbb{R})}\, \omega(\|A - B\|). \tag{79}$$

In the next result we do not pretend for maximal generality.

Corollary (5.2.26)[197]. Let ω be a modulus of continuity such that $\omega(2x) \le x\omega(x)$ for some $x < 2$ and all $x > 0$. Then $\omega_*(x) \le \text{const}\, \omega(x)$ and

$$\|f(A) - f(B)\| \le \text{const}\|f\|_{\Lambda_\omega(\mathbb{R})}\, \omega(\|A - B\|)$$

for arbitrary self-adjoint operators A and B .

Proof. It is easy to see that

$$\omega(t) \le x\left(\frac{t}{x}\right)^{\log_2 x} \omega(x),$$

whenever $0 < x \le t$. Thus

$$\omega_*(x) = x\int_x^\infty \frac{\omega(t)}{t^2}dt \le x x^{1-\log_2 x}\omega(x)\int_x^\infty t^{\log_2 x - 2}dt \le \frac{x}{1 - \log_2 x}\omega(x)$$

In [233] it was proved that if A and B are self-adjoint operators on Hilbert space whose spectra are contained in $[a, b]$ and f is a continuous function on $[a, b]$, then

$$\|f(A) - f(B)\| \le 4\left(\log\left(\frac{b - a}{\|A - B\|}\right) + 1\right)^2 \omega_f(\|A - B\|),$$

where

$$\omega_f(\delta) = \sup\{|f(x) - f(y)|: x, y \in [a, b], |x - y| < \delta\}.$$

The following corollary improves the result of Farforovskaya and Nikolskaya.

Corollary (5.2.27)[197]. Supposethat A and B be self-adjoint operators with spectra in an interval $[a, b]$. Then for a continuous function f on $[a, b]$ the following inequality holds:

$$\|f(A) - f(B)\| \le \text{const}\, \log\left(e\frac{b - a}{\|A - B\|}\right)\omega_f(\|A - B\|).$$

Proof. Put $\omega = \omega_f$. Clearly, we may assume that $\omega(x) = \omega(b - a)$ for $x > a$. Using the obviousinequality

$$\frac{\omega(t)}{t} \le 2\frac{\omega(x)}{x},\ \ x \le t,$$

we obtain

$$\omega_*(x) = x\int_x^\infty \frac{\omega(t)}{t^2}dt = x\int_x^{b-a} \frac{\omega(t)}{t^2}dt + \int_{b-a}^\infty \frac{\omega(t)}{t^2}dt \le 2\omega(x)\int_x^{b-a}\frac{dt}{t} + x\frac{\omega(b - a)}{b - a}$$
$$\le 2\omega(x)\log\frac{b - a}{x} + 2\omega(x) =\le 2\omega(x)\log\left(e\frac{b - a}{x}\right).$$

The result follows now from Theorem (5.2.24).

Corollary (5.2.28)[197]. Let $f$be a Lipschitz function on $\mathbb{R}$. Then for self-adjoint operators A and B with spectra in an interval $[a, b]$, the following inequality holds

$$\|f(A) - f(B)\| \le \text{const}\, \|f\|_{\text{Lip}} \log\left(e\frac{b - a}{\|A - B\|}\right)\|A - B\|. \tag{80}$$

Note that a similar estimate can be obtained for bounded functions f in the Zygmund class$\Lambda_1(\mathbb{R})$.

Inequality (80) improves the estimate

$$\|f(A)-f(B)\| \leq \text{const}\,\|f\|_{\text{Lip}}\left(\log\left(e\frac{b-a}{\|A-B\|}+1\right)+1\right)^2\|A-B\|$$

obtained in [198] (see also [298]).

We state analogs of Theorem (5.2.24) for unitary operators and for contractions.

Theorem (5.2.29)[197]. There exists a constant $c>0$ such that for every modulus of continuity ω, for every $f\in\Lambda_\omega$, and for arbitrary unitary operators $U$and , the following inequality holds

$$\|f(U)-f(V)\| \leq c\|f\|_{\Lambda_\omega}\omega_*(\|U-V\|).$$

Theorem (5.2.30)[197]. There exists a constant $c>0$ such that for every modulus of continuity ω, for every $f\in(\Lambda_\omega)_+$, and for arbitrary contractions T and R, the following inequality holds

$$\|f(T)-f(R)\| \leq c\|f\|_{\Lambda_\omega}\omega_*(\|T-R\|).$$

The proofs of Theorems (5.2.29)and (5.2.30) are similar to the proof of Theorem (5.2.24). Actually, they are even simpler, since we do not have to deal with convolutions with W_n and $W_n^\#$ with negative$n$ which makes analogs of formula (78) trivial.

We introduce notions of operator continuous functions and uniformly operator continuous functions. We also define for a given continuous function on $\mathbb{R}$ the operator modulus of continuity associated with the function. We prove that a function is operator continuous if and only if it is uniformly operator continuous.

Definition1(5.2.31)[197]. For a continuous function f on $\mathbb{R}$, we consider the map

$$A\mapsto f(A) \quad (81)$$

defined on the set of (not necessarily bounded) self-adjoint operators. We say that f is operator continuous if the map (81) is continuous at every (bounded or unbounded) self-adjoint operator A.

This means that if A is a (not necessarily bounded) self-adjoint operator, then for an arbitrary$\varepsilon>0$ there exists $\delta>0$ such that$\|f(A+K)-f(A)\|<\varepsilon$, whenever K is a self-adjoint operatorwhose norm is less than δ.

It is easy to see that if f is a continuous function on $\mathbb{R}$, then the map (81) is continuous at every bounded self-adjoint operator A. Indeed, this is obvious for polynomials f. The result for arbitrary continuous functions follows from the Weierstrass theorem.

Definition2 (5.2.32)[197]. Let f be a Borel function on $\mathbb{R}$. It is called uniformly operator continuous if for every $\varepsilon>0$ there exists $\delta>0$ such that$\|f(A)-f(B)\|<\varepsilon$, whenever A and B are bounded self-adjoint operators such that$\|A-B\|<\delta$.

Theorem (5.2.33)[197]. Let f be a bounded uniformly continuous function on $\mathbb{R}$. Then f is uniformly operator continuous.

Proof. Let $= \omega_f$. Then ω is a bounded modulus of continuity, and so $\omega_*(x) < \infty, x > 0$. The result follows now from Theorem (5.2.24)and the remark following that theorem.

Definition (5.2.34)[197]. Let f be a continuous function on $\mathbb{R}$. Put

$$\Omega_f(\delta) \overset{\text{def}}{=} \sup\|f(A) - f(B)\|, \quad \delta > 0,$$

where the supremum is taken over all bounded self-adjoint operators A and B such that$\|A - B\| \le \delta$. We say that $\Omega_f$is the operator modulus of continuity of f .

It suffices to consider only operators A and B that are unitary equivalent to each other. Indeed, if A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, we can define on thespace $\mathcal{H} \oplus \mathcal{H}$ the self-adjoint operators $\mathcal{A}$and $\mathcal{B}$ by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{and} \mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}$$

Obviously,

$$\|\mathcal{A} - \mathcal{B}\| = \|A - B\| \text{and} \|f(\mathcal{A}) - f(\mathcal{B})\| = \|f(A) - f(B)\| .$$

We have by Theorem (5.2.24),

$$\omega_f(\delta) \le \Omega_f(\delta) \le \text{const}(\omega_f)_*(\delta), \quad \delta > 0.$$

Theorem (5.2.35)[197]. Let f be an operator continuous function. Then

$$\lim_{\delta \to 0} \Omega_f(\delta) = 0,$$

and so f is uniformly operator continuous.

Proof. Suppose that

$$\lim_{\delta \to 0} \Omega_f(\delta) > \sigma > 0,$$

Then there are sequences of self-adjoint operators $\{A_j\}_{j\ge 0}$ and $\{K_j\}_{j\ge 0}$ on Hilbert space $\mathcal{H}$such that $\|K_j\| < 1/j$ and $\|f(A_j + K_j) - f(A_j)\| > \sigma$. We define the operators A and R_n on$\ell^2(\mathcal{H})$ by

$$A\begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} A_0 h_0 \\ A_1 h_1 \\ A_2 h_2 \\ \vdots \end{pmatrix} \text{ and } R_n = \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ K_n h_n \\ K_{n+1} h_{n+1} \\ \vdots \end{pmatrix}.$$

Clearly, $\|R_n\| \to 0$ as $n \to 0$, while $\|f(A + R_n) - f(A)\| > \sigma$for $n \ge 0$, and so the map (81)is not continuous at A.

Example (5.2.36)[197]. Consider the function g defined by $g(t) = |t|, t \in \mathbb{R}$. It was proved in [234] that the function g is not operator Lipschitz. It was observed in [233] that the function g is not operator continuous. Let us show that

$$\Omega_g(\delta) = \infty \text{for every } \delta > 0,$$

which will also imply that g is not operator continuous. Indeed, suppose that $\Omega_g(\delta_0) < \infty$ for some$\delta_0 > 0$. Since g is homogeneous,it follows that$\Omega_g(\delta) = \delta\delta_0^{-1}\Omega_g(\delta_0) = \text{const}\delta$. However,this implies that g is an operator Lipschitz function which contradicts the result of [234].

Theorem (5.2.37)[197]. Let A and B be a pair of (not necessarily bounded) self-adjoint operators such that $A-B$ is bounded. Then

$$\|f(A)-f(B)\| \le \Omega_f(\|A-B\|) \tag{82}$$

for every continuous function f on $\mathbb{R}$.

Proof . Clearly, if $\Omega_f(\delta)<\infty$ for some $\delta>0$, it follows that f satisfies the hypotheses of Lemma (5.2.39). Let $K=B-A$. Then K is a bounded self-adjoint operator. Put

$$A_j \stackrel{\text{def}}{=} E_A([-j,j])A.$$

Clearly, (83) holds. It follows easily from Lemma (5.2.39) that

$$\|f(A+K)u-f(A)u\| \le \limsup_{j\to\infty}\left\|f(A_j+K)u-f(A_j)u\right\| \le \Omega_f(\|K\|)\|u\|, u \in \mathfrak{D}_A$$

To complete the proof, it suffices to observe that if f satisfies the hypotheses of Lemma (5.2.39), then $f(A)$ is the closure of its restriction to $\mathfrak{D}_A$. The same is true for $f(A+K)$. This implies (82).

Lemma (5.2.38)[197]. Let f be a bounded continuous function on $\mathbb{R}$. Suppose that A is a self-adjoint operator (not necessarily bounded) and $\{A_j\}_{j\ge0}$ is a sequence of bounded self-adjoint operatorssuch that

$$\lim_{j\to\infty}\left\|A_ju-Au\right\| = 0 \text{ for every } u \text{ in the domain of } A \tag{83}$$

Then

$$\lim_{j\to\infty} f(A_j) = f(A) \text{ in the strong operator topology.} \tag{84}$$

Proof. We consider first the special case when $(t)=(\lambda-t)^{-1}$, $\lambda\in\mathbb{C}\backslash\mathbb{R}$. Let u be a vector in $\mathfrak{D}_A$, where $\mathfrak{D}_A$denotes the domain of A. Put $u_\lambda \stackrel{\text{def}}{=} (\lambda I-A)^{-1}u$. Clearly, $u_\lambda\in\mathfrak{D}_A$ and

$$(\lambda I-A_j)^{-1}u = (\lambda I-A_j)^{-1}(\lambda I-A)u_\lambda$$
$$= u_\lambda + (\lambda I-A_j)^{-1}(A_ju-Au) \to u_\lambda \text{as } j\to\infty.$$

Since the linear combinations of such rational fractions are dense in the space $C_0(\mathbb{R})$ of continuous functions on $\mathbb{R}$ vanishing at infinity, it follows that (84) holds for an arbitrary function $f$in $C_0(\mathbb{R})$.

Suppose now that f is an arbitrary bounded continuous function on $\mathbb{R}$. By subtracting from fa continuous function with compact support, we may assume that f vanishes on $[-1,1]$. Then there exists a function g in $C_0(\mathbb{R})$ such that $f(t)=tg(t)$, $t\in\mathbb{R}$. Let $\in\mathfrak{D}_A$. We have

$$f(A_j)u = g(A_j)A_ju = g(A_j)Au + g(A_j)(A_ju-Au)$$
$$\to g(A)Au = f(A)u \quad \text{as } j\to\infty. \tag{85}$$

Lemma (5.2.39)[197]. Let f be a continuous function on $\mathbb{R}$ such that $|f(t)|\le \text{const}(1+|t|)$, $t\in\mathbb{R}$, and let A and $\{A_j\}_{j\ge0}$ be as in Lemma (5.2.38). Then

$$\lim_{j\to\infty}\left\|f(A_j)u-f(A)u\right\| = 0 \text{ for every } u\in\mathfrak{D}_A.$$

Proof. As in the proof of Lemma (5.2.38), we may assume that f vanishes on $[-1,1]$ and define the continuous function g by$f(t)=tg(t)$, $t\in\mathbb{R}$. It follows now from Lemma (5.2.38) that (85) holds for every $u\in\mathfrak{D}_A$.

Corollary (5.2.40)[197]. Let f be a continuous function on $\mathbb{R}$. Then f is operator continuous if and only if it is uniformly operator continuous.

We conclude with an estimate for the operator modulus of continuity of a bounded function in the Zygmund class $\Lambda_1(\mathbb{R})$. The proof of the following theorem is similar to the proof of Theorem 3.4 of Ch. 2 of [238].

Theorem (5.2.41)[197]. Let f be a bounded function in $\Lambda_1(\mathbb{R})$. Then there exists $c > 0$ such that

$$\Omega_f(\delta) \leq c\delta \log\frac{2}{\delta} \text{ for } \delta \leq 1.$$

Proof. By Corollary (5.2.11), there is a constant c_1 such that

$$\|f(A + 2K) - 2f(A + K) + f(A)\| \leq c_1\|f\|_{\Lambda_1(\mathbb{R})}\|K\|.$$

It is easy to see that

$$\|f(A + K) - f(A)\| \leq \frac{1}{2}\|f(A + 2K) - 2f(A + K) + f(A)\| + \frac{1}{2}\|f(A + 2K) - f(A)\|.$$

It follows that

$$\Omega_f(\delta/2) \leq \frac{c_1}{4}\|f\|_{\Lambda_1(\mathbb{R})}\delta + \frac{1}{2}\Omega_f(\delta),$$

and so

$$2^{k-1}\Omega_f(2^{-k}\delta) - 2^{k-2}\Omega_f(2^{1-k}\delta) \leq \frac{c_1}{4}\|f\|_{\Lambda_1(\mathbb{R})}\delta, \qquad \text{whenever } k \geq 1.$$

Substituting $\delta = \delta_0 \overset{\text{def}}{=} \frac{4}{c_1}\|f\|_{\Lambda_1(\mathbb{R})}\|f\|_{L^\infty}$, and keeping in mind the trivial estimate $\Omega_f(\delta) \leq 2\|f\|_{L^\infty}, \delta > 0$, we obtain

$$2^{n-1}\Omega_f(2^{-n}\delta_0) \leq (n + 1)\|f\|_{L^\infty}.$$

Hence, for $= 2^{-n}\delta_0$, $n \geq 0$, we have

$$\Omega_f(\delta) \leq \frac{c_1}{2}\|f\|_{\Lambda_1(\mathbb{R})}\delta \log_2\left(\frac{8\|f\|_{L^\infty}}{c_1\|f\|_{\Lambda_1(\mathbb{R})}\delta}\right).$$

Therefore

$$\Omega_f(\delta) \leq c_1\|f\|_{\Lambda_1(\mathbb{R})}\delta \log_2\left(\frac{8\|f\|_{L^\infty}}{c_1\|f\|_{\Lambda_1(\mathbb{R})}\delta}\right) \text{for } \delta \leq \frac{\delta_0}{2}$$

and $\Omega_f(\delta) \leq 2\|f\|_{L^\infty}$ for $\delta \geq \delta_0/2$.

We construct a universal family of (unbounded) self-adjoint operators $\{A_t\}_{t\geq 0}$ such that the operators A_t have purely point spectra and

$$\Omega_f(t) = \|f(A_t) - f(A_0)\|, \quad t > 0,$$

for every continuous function f. In particular, $\|A_t - A_0\| = t$, $t \geq 0$. Moreover, the operators A_t, $t \geq 0$, are unitarily equivalent to each other.

Denote by $\mathfrak{K}$ the set of finite rank self-adjoint operators on Hilbert space and let $\mathfrak{K}_0$ be a countable dense subset of $\mathfrak{K}$.

Lemma(5.2.42)[197]. Suppose that $\{A_j\}$ is a sequence of bounded self-adjoint operators that converges to A in the strong operator topology. Then $f(A_j) \to f(A)$ strongly for an arbitrary continuousfunction f .

Proof. The conclusion of the lemma is trivial if f is a polynomial. It remains to approximate f by polynomials uniformly on $[-\sup_J j\|A_j\|, \sup_J \|A_j\|]$.

Corollary (5.2.43)[197]. Let $f \in C(\mathbb{R})$and $t > 0$. Then

$$\Omega_f(t) = \sup\{\|B - A\|: A, B \in \mathfrak{K}_0(\mathcal{H}), \|B - A\| < t\}.$$

Proof. Clearly, we have to verify that the left-hand side is less than or equal to the right-hand side. Let A and B be bounded self-adjoint operators such that $\|A - B\| < t$. Let $\{A_j\}$ and $\{K_j\}$ be sequences of operators in $\mathfrak{K}_0$such that $A_j \to A$, $K_j \to B - A$ in the strong operator topology, and $\|K_j\| \leq \|B - A\|$ for all j. By Lemma(5.2.43), $f(A_j) \to f(A)$ and $f(A_j + K_j) \to f(B)$strongly. Hence,

$$\|f(B) - f(A)\| \leq \liminf_{j\to\infty} \|f(A_j + K_j) \to f(A_j)\|$$

which implies the desired inequality.

Suppose that $\{R_j\}_{j=1}^{\infty}$is an enumeration of $\mathfrak{K}_0$. For given $j \geq 1$ and $t > 0$ we consider the set

$$\mathfrak{K}_{jt} \stackrel{\text{def}}{=} \{A \in \mathfrak{K}_0: \|A - R_j\| < t\}$$

and let $\{R_{jk}^{(t)}\}_{k=1}^{\infty}$ be an enumeration of $\mathfrak{K}_{jt}$. Put$R_{jk}^{(0)} \stackrel{\text{def}}{=} R_j$.

We can define now a universal family $\{A_t\}_{t\geq 0}$ by

$$A_t \stackrel{\text{def}}{=} \bigoplus_{j=1}^{\infty} \bigoplus_{k=1}^{\infty} R_{jk}^{(t)}. \tag{86}$$

Theorem (5.2.44)[197].The operators $A_t$are pairwise unitarily equivalent. Each operator $A_t$has purely point spectrum. Moreover, for every continuous function f on $\mathbb{R}$, we have

$$\|f(A_t) - f(A_0)\| = \Omega_f(t), \quad t > 0.$$

Proof. It is easy to see that each operator in $\mathfrak{K}_0$occurs in the orthogonal sum on the right of (86) infinitely many times and each operator in the orthogonal sum on the right of (86) belongs to$\mathfrak{K}_0$. Thus A_t is unitarily equivalent to A_0 for all $t > 0$.

We have

$$\|f(A_t) - f(A_0)\| = \sup_{j,k} \|f(R_{jk}^{(t)}) - f(R_{jk}^{(0)})\| = \Omega_f(t)$$

by Corollary (5.2.43).

We obtain estimates for the norm of quasicommutators $f(A)Q - Qf(B)$ in terms of $\|AQ - QB\|$ for self-adjoint operators A and B and a bounded operator Q. We assumefor simplicity that A and B are bounded. However, we obtain estimates that do not depend onthe norms of A and B. In [220] we will consider the case of not necessarily bounded operators A and B . In the special case $A = B$ this problem turns into the problem of estimating the norm of commutators $f(A)Q - Qf(A)$ in terms of $\|AQ - QA\|$. On the other

hand, in the special case $Q = I$ the problem turns into the problem of estimating $\|f(A) - f(B)\|$ in terms $\|A - B\|$.
Note that similar results can be obtained for unitary operators and for contractions.
Birman and Solomyak (see [219]) discovered the following formula

$$f(A)Q - Qf(B) = \iint \frac{f(x) - f(y)}{x - y} dE_A(x)(AQ - QB)dE_B(y),$$

whenever f is a function such that the divided difference $\mathfrak{D}f$ is a Schur multiplier with respect to the spectral measures E_A and E_B.
We could use this formula to obtain estimates of quasicommutators as we have done in the case of functions of perturbed operators. However, we are going to reduce estimates of quasicommutators to those of functions of perturbed operators. For this purpose we obtain estimates that compare different moduli of continuities (the operator modulus of continuity, the (quasi)commutator modulus of continuity, etc.).
We start with the case of operator Lipschitz functions.
The following theorem compares different operator Lipschitz norms and (quasi)commutator Lipschitz norms. The fact that they are equivalent is well-known, see [205]. The following theorem says that all those norms are equal.
Theorem (5.2.45)[197]. Let f be a continuous function on $\mathbb{R}$. The following are equivalent:
(i) $\|f(A) - f(B)\| \le \|A - B\|$ for arbitrary self-adjoint operators A and B ;
(ii) $\|f(A) - f(B)\| \le \|A - B\|$ for all pairs of unitarily equivalent self-adjoint operators A and B ;
(iii) $\|f(A)R - Rf(B)\| \le \|AR - RB\|$ for arbitrary self-adjoint operators A and R;
(iv) $\|f(A)R - Rf(B)\| \le \|AR - RB\|$ for all self-adjoint operators A and bounded operators R;
(v) $\|f(A)R - Rf(B)\| \le \|AR - RB\|$ for arbitrary self-adjoint operators A and B and anarbitrary bounded operator R.
Proof. The implication (i) $\Rightarrow$ (ii) is obvious.
Let us show that (ii) $\Rightarrow$ (iii). Put $B = \exp(-itR)A\exp(itR)$. Clearly, B is unitarily equivalentto A and $f(B) = \exp(-it R)f(A)\exp(itR)$. Thus
$\|f(A) - \exp(-it R)f(A)\exp(itR)\| \le \|A - \exp(-itR)A\exp(itR)\|$ forall $t \in \mathbb{R}$.
It remains to observe that

$$\lim_{t\to 0} \frac{\|f(A) - \exp(-it R)f(A)\exp(itR)\|}{|t|} = \|f(A)R - Rf(B)\|$$

and

$$\lim_{t\to 0} \frac{\|A - \exp(-it R)A\exp(itR)\|}{|t|} = \|AR - RB\|.$$

To prove that (iii) $\Rightarrow$ (iv), we consider the following self-adjoint operators

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{and} \mathcal{R} = \begin{pmatrix} 0 & R \\ R^* & 0 \end{pmatrix}.$$

It is easy to see that

$$f(\mathcal{A})\mathcal{R} = \begin{pmatrix} 0 & f(A)R \\ f(A)R^* & 0 \end{pmatrix} \text{and} \mathcal{R}f(\mathcal{A}) = \begin{pmatrix} 0 & Rf(A) \\ R^*f(A) & 0 \end{pmatrix}.$$

Hence,

$$\|f(\mathcal{A})\mathcal{R} - \mathcal{R}f(\mathcal{A})\| \max\{\|f(A)R - Rf(A)\|, \|f(A)R^* - R^*f(A)\|\}$$

and

$$\|\mathcal{A}\mathcal{R} - \mathcal{R}\mathcal{A}\| \max\{\|AR - RA\|, \|AR^* - R^*A\|\} = \|AR - RA\|.$$

It follows that

$$\|f(A)R - Rf(B)\| \le \|f(\mathcal{A})\mathcal{R} - \mathcal{R}f(\mathcal{A})\| \le \|\mathcal{A}\mathcal{R} - \mathcal{R}\mathcal{A}\| = \|AR - RA\|$$

The implication (v) ⇒ (i) is trivial; it suffices to put $R = I$.

To complete the proof, it remains to show that (iv) ⇒ (v). Let us first consider the special case

when$A$ and B are unitarily equivalent, i.e., $A = U^*BU$ for a unitary operator U and we provethat

$$\|U^*f(B)UR - Rf(B)\| \le \|U^*BUR - RB\|.$$

This is equivalent to the inequality

$$\|f(B)UR - URf(B)\| \le \|BUR - URB\|$$

which holds by (iv).

Now we consider the case of arbitrary self-adjoint operators $A$and . Put

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \text{ and } \mathcal{R} = \begin{pmatrix} R & 0 \\ 0 & R^* \end{pmatrix}.$$

Then $\mathcal{A}$and $\mathcal{B}$are unitarily equivalent. We have

$$f(\mathcal{A})\mathcal{R} = \begin{pmatrix} f(A)R & 0 \\ 0 & f(B)R^* \end{pmatrix} \text{and} \mathcal{R}f(\mathcal{B}) = \begin{pmatrix} Rf(A) & 0 \\ 0 & R^*f(A) \end{pmatrix}.$$

Hence,

$$\|f(\mathcal{A})\mathcal{R} - \mathcal{R}f(\mathcal{B})\| = \max\{\|f(A)R - Rf(B)\|, \|f(B)R^* - R^*f(A)\|\}$$

and

$$\|\mathcal{A}\mathcal{R} - \mathcal{R}\mathcal{B}\| = \max\{\|AR - RB\|, \|BR^* - R^*A\|\} = \|AR - RB\|.$$

It follows that

$$\|f(A)R - Rf(B))\| \le \|f(\mathcal{A})\mathcal{R} - \mathcal{R}f(\mathcal{B})\| \le \|\mathcal{A}\mathcal{R} - \mathcal{R}\mathcal{B}\| = \|AR - RB\|.$$

For a continuous function f on $\mathbb{R}$ we have defined the operator modulus of continuity Ω_f.We define here 3 other versions of moduli of continuity in terms of commutators and quasicommutators.

Let f be a continuous function on $\mathbb{R}$. For $\delta > 0$, put

$\Omega_f^{[1]}(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A)R - Rf(A)\|: A = A^*, R = R^*, \|R\| = 1, \|AR - RA\| < \delta\}$;

$\Omega_f^{[2]}(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A)R - Rf(A)\|: A = A^*, \|R\| = 1, \|AR - RA\| < \delta\}$;

$\Omega_f^{[3]}(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A)R - Rf(B)\|: A = A^*, B = B^*, \|R\| = 1, \|AR - RB\| < \delta\}$.

Obviously, $\Omega_f^{[1]} \le \Omega_f^{[2]} \le \Omega_f^{[3]}$and $\Omega_f \le \Omega_f^{[3]}$.

Theorem (5.2.46)[197]. Let f be a continuous function on $\mathbb{R}$. Then

$$\Omega_f \le \Omega_f^{[1]} = \Omega_f^{[2]} = \Omega_f^{[3]} = 2\Omega_f.$$

Proof. The inequality $\Omega_f^{[2]} \le \Omega_f^{[1]}$ can be proved in the same way as the implication (iii) ⇒ (iv)

in the proof of Theorem (5.2.45)The inequality $\Omega_f^{[3]} \le \Omega_f^{[2]}$ can be proved in the same way as theimplication (iv) ⇒ (v) in the proof of Theorem (5.2.45). It remains to prove that Ω [235] $\Omega_f^{[1]} \le 2\Omega_f$. We need two lemma.

Lemma (5.2.47)[197]. Let X and Y be bounded operators. Then

$$\|XY^n - Y^nX\| \le n\|Y\|^{n-1}\|XY - YX\|.$$

Proof. We have

$$\|XY^n - Y^nX\| \le \left\|\sum_{k=1}^{n} Y^{k-1}(XY - YX)Y^{n-k}\right\| \le n\|Y\|^{n-1}\|XY - YX\|.$$

Lemma (5.2.48)[197]. Let T be a self-adjoint operator such that $\|T\| < 1$ and let Xbe a bounded operator. Then

$$\left\|(I - T^2)^{1/2}X - X(I - T^2)^{1/2}\right\| \le \frac{\|T\| \cdot \|XT - TX\|}{(1 - \|T\|^2)^{1/2}}.$$

Proof.Let $a_n \overset{\text{def}}{=} (-1)^{n-1}\binom{1/2}{n}$. Then $a_n > 0$ and $(1 - t^2)^{1/2} = \sum_{n=1}^{\infty} a_n t^{2n}$. Thus

$$\left\|(I - T^2)^{1/2}X - X(I - T^2)^{1/2}\right\| = \left\|\sum_{n=1}^{\infty} a_n(XT^{2n} - T^{2n}X)\right\| \le \|XT - TX\| \sum_{n=1}^{\infty} 2na_n\|T\|^{2n-1}$$
$$= \frac{\|T\| \cdot \|XT - TX\|}{(1 - \|T\|^2)^{1/2}}.$$

by Lemma (5.2.47).

Let us complete the proof of Theorem (5.2.47). Let R be a self-adjoint contraction and $\tau \in (0,1)$.Consider the operators

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \text{and} \mathcal{U} = \begin{pmatrix} \tau R & (I - \tau^2R^2)^{1/2} \\ -(I - \tau^2R^2)^{1/2} & \tau R \end{pmatrix}.$$

Clearly, $\mathcal{U}$is a unitary operator. We have

$$f(\mathcal{A})\mathcal{U} = \begin{pmatrix} \tau f(A)R & (I - \tau^2R^2)^{1/2}f(A) \\ -f(A)(I - \tau^2R^2)^{1/2} & \tau f(A)R \end{pmatrix}.$$

and

$$\mathcal{U}f(\mathcal{A}) = \begin{pmatrix} \tau Rf(A) & (I - \tau^2R^2)^{1/2}f(A) \\ -(I - \tau^2R^2)^{1/2}f(A) & \tau Rf(A) \end{pmatrix}.$$

Clearly,

$$\|f(\mathcal{A})\mathcal{U} - \mathcal{U}f(\mathcal{A})\| \ge \tau\|f(A)R - Rf(A)\|$$

and

$$\|\mathcal{A}\mathcal{U} - \mathcal{U}\mathcal{A}\| \le \tau\|AR - RA\| + \left\|A(I - \tau^2R^2)^{1/2} - (I - \tau^2R^2)^{1/2}A\right\|$$
$$\le (\tau + \tau^2(1 - \tau^2)^{-1/2})\|AR - RA\|$$

by Lemma (5.2.48) with $X = A$ and $T = \tau R$. Hence,

$$\|f(A)R - Rf(A)\| \le \tau^{-1}\|f(\mathcal{A})\mathcal{U} - \mathcal{U}f(\mathcal{A})\| = \tau^{-1}\|\mathcal{U}^*f(\mathcal{A})\mathcal{U} - f(\mathcal{A})\|$$
$$\le \tau^{-1}\Omega_f(\|\mathcal{U}^*\mathcal{A}\mathcal{U} - \mathcal{A}\|) = \tau^{-1}\Omega_f(\|\mathcal{A}\mathcal{U} - \mathcal{U}\mathcal{A}\|)$$
$$\le \tau^{-1}\Omega_f(\tau + \tau^2(1-\tau^2)^{-1/2})\|AR - RA\|$$

Taking $\tau = 1/2$, we obtain

$$\|f(A)R - Rf(A)\| \le 2\Omega_f\left(\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right)\|AR - RA\|\right) \le 2\Omega_f(\|AR - RA\|).$$

Lemma (5.2.49)[197]. Let $0 < \alpha < 1$. Then there exists $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R})$, for arbitrary self-adjoint operators A and B and a bounded operator R the following inequality holds:

$$\|f(A)R - Rf(B)\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R})}\|AR - RB\|^\alpha\|R\|^{1-\alpha}.$$

Proof. Clearly, we may assume that $R \ne 0$. By Theorems (5.2.8)and (5.2.46),

$$\|f(A)R - Rf(B)\| = \|R\| \cdot \left\|f(A)\left(\frac{1}{\|R\|}R\right) - \left(\frac{1}{\|R\|}R\right)f(B)\right\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R})}\|R\|\left\|\frac{1}{\|R\|}(AR - RB)\right\|^\alpha$$
$$= \text{const}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|AR - RB\|^\alpha\|R\|^{1-\alpha}.$$

Lemma (5.2.50)[197]. There exists $c > 0$ such that for every modulus of continuity ω, for every $f \in \Lambda_\omega(\mathbb{R})$, for arbitrary self-adjoint operators A and B , and a bounded nonzero operator R thefollowing inequality holds:

$$\|f(A)R - Rf(B)\| \le c\|R\|\omega_*\left(\frac{\|AR - RB\|}{\|R\|}\right).$$

The proof of Lemma (5.2.50) is the same as the proof of Lemma (5.2.49).

We obtain norm estimates for finite differences

$$(\Delta_K^m f)(A) \stackrel{\text{def}}{=} \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(A + jK)$$

for functions $f \in \Lambda_{\omega,m}(\mathbb{R})$ and self-adjoint operators A and K . For simplicity, we give proofs in the case of bounded operators and bounded functions f . Note that our estimate will not depend on the L^∞ norm of f , nor on the operator norm of A. In [220] we consider the case ofan arbitrary (not necessarily bounded) self-adjoint operator A (though K still must be bounded)and an arbitrary function $f \in \Lambda_{\omega,m}(\mathbb{R})$.

We also obtain similar results for unitary operators and for contractions.

Let ω be a nondecreasing function on $(0, \infty)$ such that

$$\lim_{x \to 0}\omega(x) = 0 \text{ and } \omega(2x) \le 2^m\omega(x) \text{ for } x > 0. \tag{87}$$

Recall that $\Lambda_{\omega,m}(\mathbb{R})$ is the space of continuous functions f on $\mathbb{R}$ satisfying

$$\|f\|_{\Lambda_{\omega,m}(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{t>0}\frac{\|\Delta_t^m f\|_{L^\infty}}{\omega(t)} < +\infty.$$

Given a nondecreasing function ω satisfying (87), we define the function $\omega_{*,m}$ by

$$\omega_{*,m}(x) = x^m\int_x^\infty \frac{\omega(t)}{t^{m+1}}dt = \int_1^\infty \frac{\omega(sx)}{s^{m+1}}dx.$$

Lemma (5.2.51)[197]. Let m be a positive integer. Then there is a positive number c such that for an arbitrary nondecreasing function ω on $(0,\infty)$ satisfying (87), an arbitrary bounded function f in $\Lambda_{\omega,m}(\mathbb{R})$, and arbitrary bounded self-adjoint operators A and K on Hilbert space the following inequality holds:

$$\|(\Delta_K^m f)(A)\| \le c\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\omega_{*,m}(\|K\|).$$

Proof. As in the proof of Theorem (5.2.24), we can easily see that

$$(\Delta_K^m f)(A) = \sum_{n=-\infty}^{N}(\Delta_K^m f_n)(A) + \left(\Delta_K^m(f - f * V_N)\right)(A),$$

where as before, $f_n = f * W_n + f * W_n^{\#}$.

Suppose that N is the integer satisfying (69). By Theorem (5.2.6),

$$\left\|\left(\Delta_K^m(f - f * V_N)\right)(A)\right\| \le \text{const}\|f - f * V_N\|_{L^\infty}$$
$$\le \text{const}\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\omega(2^{-N}) \le \text{const}\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\omega_{*,m}(\|K\|).$$

On the other hand, it follows from Lemma (5.2.10), (59), and Corollary (5.2.7) that

$$\|(\Delta_K^m f_n)(A)\| \le \text{const}2^{mn}\|f_n\|_{L^\infty}\|K\|^m \le \text{const}\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}2^{mn}\omega(2^{-n})\|K\|^m.$$

Thus

$$\sum_{n=-\infty}^{N}\|(\Delta_K^m f_n)(A)\| \le \text{const}\sum_{n=-\infty}^{N}\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}2^{mn}\omega(2^{-n})\,\|K\|^m$$
$$= \sum_{k\ge 0}2^{(N-k)m}\omega(2^{N-k})\,\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\|K\|^m \le \text{const}\left(\int_x^\infty \frac{\omega(t)}{t^{m+1}}dt\right)\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\|K\|^m$$
$$= \text{const}\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\omega_{*,m}(\|K\|).$$

This completes the proof.

Corollary (5.2.52)[197]. Let ω be a positive nondecreasing function on $(0,\infty)$ such that $\lim_{x\to 0}\omega(x) = 0$and $\omega(2x) \le x\omega(x)$ for some $x < 2^m$ and all $x > 0$. Then for $x > 0$, we have $\omega_{*,m}(x)$const $\le \omega(x)$ and so

$$\|(\Delta_K^m f)(A)\| \le \text{const}\,\|f\|_{\Lambda_{\omega,m}(\mathbb{R})}\omega(\|K\|).$$

The proof of Corollary (5.2.52) is the similar to the proof of Corollary (5.2.26).

Corollary(5.2.53)[197]. Suppose that under the hypotheses of Theorem(5.2.51) $\|f\|_{L^\infty} \le M$. Then for the function $\omega_{m,M}$ defined by

$$\omega_{m,M}(x) = x^m\int_x^\infty \frac{\min(2M,\omega(t))}{t^{m+1}}dt,$$

the following inequality holds:

$$\|(\Delta_K^m f)(A)\| \le \text{const}\,\|f\|_{\Lambda_{\omega,m}}\omega_{m,M}(\|K\|).$$

The following analogs of Lemma (5.2.51) for unitary operators and for contractions can be proved in a similar way.

Theorem (5.2.54)[197]. Let $m$be a positive integer. Then there exists a constant $c > 0$ such that for every nondecreasing function ω on $(0,\infty)$ satisfying (87), for every $f \in \Lambda_{\omega,m}$, and for an arbitrary unitary operator U and an arbitrary bounded self-adjoint operator A on Hilbert space, the following inequality holds:

$$\left\|\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\left(e^{ikA}U\right)\right\| \le c\|f\|_{\Lambda_{\omega,m}}\omega_{*,m}(\|A\|).$$

Theorem (5.2.55)[197]. Let $m$be a positive integer. Then there exists a constant $c > 0$ such that for every nondecreasing function ω on $(0, \infty)$ satisfying (87), for every $f \in (\Lambda_{\omega,m})$, and for arbitrary contractions T and R on Hilbert space, the following inequality holds:

$$\left\|\sum_{k=0}^{m}(-1)^k\binom{m}{k}f\left(T+\frac{k}{m}(R-T)\right)\right\| \le c\|f\|_{\Lambda_{\omega,m}}\omega_{*,m}(\|T-R\|).$$