Chapter 4

Functions of Perturbed and Normal of Operators

We study properties of the operators f (A)-f (B) for $f \in \Lambda_\alpha(\mathbb{R})$and selfadjoint operators A and B such that $A - B$ belongs to the Schatten–von Neumann class Sp. We consider the same problem for higher order differences. Similar results also hold for unitary operators and for contractions. We obtain a more general result for functions in the space$\Lambda_\omega(\mathbb{R}^2)$ $=\{f: |f(\zeta_1) - f(\zeta_2)| \le \text{const}\, \omega(|\zeta_1 - \zeta_2|)\}$ for an arbitrary modulus of continuity ω. We show that if f belongs to the Besov class $B^1_{\infty 1}(\mathbb{R}^2)$, then it is operator Lipschitz, i.e., $\|f(N_1) - f(N_2)\| \le \text{const}\|f\|_{B^1_{\infty 1}}\| N_1 - N_2 \|$. We also study properties of $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2)$ and $N_1 - N_2$ belongs to the Schatten-von Neuman class Sp.

Section (4.1): Perturbed Operators.

It is well known that a Lipschitz function on the real line is not necessarily operator Lipschitz, i.e., the condition,

$$|f(x) - f(y)| \le \text{const}|\, x - y|, \quad x, y \in \mathbb{R},$$

does not imply that for selfadjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \le \text{const}\|A - B\| .$$

The existence of such functions was proved in [132] (see also [133] and [134]). Later in [135]necessary conditions were found for a function f to be operator Lipschitz. Those necessary conditions imply that Lipschitz functions do not have to be operator Lipschitz. It is also well known that a continuously differentiable function does not have to be operator differentiable, see [135]and [136]. The necessary conditions obtained in [135]and [136]are based on the nuclearity criterion for Hankel operators, see [137].

We consider Holder classes $\Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily operator Holder of order α, i.e., the condition:

$$|f(x) - f(y)| \le \text{const}|\, x - y|^\alpha, \quad x, y \in \mathbb{R},$$

implies that for selfadjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \le \text{const}\|A - B\|^\alpha \ . \tag{1}$$

Moreover, a similar result holds for the Zygmund class $\Lambda_1(\mathbb{R})$, i.e., the fact that

$$|f(x + t) - 2f(x) + f(x - t)| \le \text{const}|t|, \quad x, t \in \mathbb{R},$$

and f is continuous implies that f is operator Zygmund, i.e., for selfadjoint operators A and K ,

$$\|f(A + K) - 2f(A) + f(A - K)\| \le \text{const}\|K\|. \tag{2}$$

We also obtain similar results for the whole scale of Holder-Zygmund classes $\Lambda_\alpha(\mathbb{R})$ for $0 < \alpha < \infty$. Recall that for $\alpha > 1$, the class $\Lambda_\alpha(\mathbb{R})$ consists of continuous functions f such that

$$\left|\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x + kt)\right| \le \text{const}|t|^\alpha \quad , where\ n - 1 \le \alpha < n.$$

The same problems can be considered for unitary operators and for functions on the unit circle, and for contractions and analytic functions in the unit disk.

To prove (1), we use a crucial estimate obtained for trigonometric polynomials and unitary operators in [135]and for entire functions of exponential type and selfadjoint operators in [136]. We state here the result for selfadjoint operators. It can be considered as an analog of Bernstein's inequality.

Let f be an entire function of exponential type σthat is bounded on the real line R. Then for selfadjoint operators A and B with bounded $A-B$ the following inequality holds:

$$\|f(A)-f(B)\| \leq \text{const } \sigma\|f\|_{L^\infty(\mathbb{R})}\|A-B\|. \tag{3}$$

Inequality (3) was proved by using double operator integrals and the Birman-Solomyak formula:

$$f(A)-f(B) = \iint \frac{f(x)-f(y)}{x-y} dE_A(x)(A-B)dE_B(y),$$

where $E_A$and $E_B$are the spectral measures of selfadjoint operators A and B ; we refer the reader to [138], [139] and [140] for the theory of double operator integrals. Note that A and B do not have to be bounded, but $A-B$ must be bounded.

To estimate the second difference (2), we use the corresponding analog of Bernstein's inequality which was obtained in [141] with the help of triple operator integrals. To estimate higher order differences, we need multiple operator integrals. We refer to [141] for definitions and basic results on multiple operator integrals.

We also consider the problem of the behavior of functions of operators $f(A)$ under perturbations of A by operators of Schatten-von Neumann class $\boldsymbol{S}_p$ in the case when $f \in \Lambda_\alpha(\mathbb{R})$.

We start with first order differences. We use the notation by Λ_α, $0 < \alpha < \infty$, for the scale of Holder-Zygmund classes on the unit circle T.

Theorem(4.1.1)[131]. Let $0 < \alpha < 1$. Then there is a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U)-f(V)\| \leq c\|f\|_{\Lambda_\alpha} \cdot \|U-V\|^\alpha.$$

Theorem(4.1.2)[131]. There exists a constant $c > 0$ such that for every function $f \in \Lambda_1$ and for arbitrary unitary operators U and V on Hilbert space the following inequality holds:

$$\|f(U)-f(V)\| \leq c\|f\|_{\Lambda_1}\left(2+\log_2 \frac{1}{\|U-V\|}\right)\|U-V\|.$$

This result improves an estimate obtained in [132]for Lipschitz functions in the case of bounded selfadjoint operators.

We proceed now to higher order differences.

Theorem (4.1.3)[131]. Let n be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in \Lambda_\alpha$ and for an arbitrary unitary operator U and an arbitrary bounded selfadjoint operator A on Hilbert space the following inequality holds:

$$\left\|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} f(e^{ikA}U)\right\| \leq c\|f\|_{\Lambda_\alpha}\|A\|^\alpha.$$

Let us consider now a more general problem. Suppose that ω is a modulus of continuity, i.e., ω is a nondecreasing continuous function on [0,0) such that $\omega(0) = 0$ and $\omega(x +$

$y) \le \omega(x) + w(y), x, y \ge 0$. The space Λ_ω consists of functions f on T such that
$$|f(\zeta) - f(\tau)| \le \text{const } \omega(|\zeta - \tau|), \quad \zeta, \tau \in \mathbb{T}$$
With a modulus of continuity ωwe associate the function ω^* defined by:
$$\omega^*(x) = x\int_x^\infty \frac{\omega(t)}{t^2}dt, \quad x \ge 0.$$
Theorem(4.1.4)[131]. Suppose that ω is a modulus of continuity and $f \in \Lambda_\omega$. If U and V are unitary operators, then
$$\|f(U) - f(V)\| \le \text{const } \|f\|_{\Lambda_\omega}\omega^*(\|U - V\|).$$
In particular, if $\omega^*(x) \le \text{const}\omega(x)$, then for unitary operators U and V
$$\|f(U) - f(V)\| \le \text{const } \|f\|_{\Lambda_\omega}\omega(\|U - V\|).$$
We have also proved an analog of Theorem (4.1.4) for higher order differences. We denote here by $(\Lambda_\alpha)_+$ the set of functions $f \in \Lambda_\alpha$, for which the Fourier coefficients $\hat{f}(n)$ vanish for $n < 0$.

Recall that an operator T on Hilbert space is called a contraction if $\|T\| \le 1$. The following result is an analog of Theorem (4.1.3) for contractions.

Theorem(4.1.5)[131]. Let n be a positive integer and $0 < \alpha < n$. Then there exists a constant $c > 0$ such that for every $f \in (\Lambda_\alpha)_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds:
$$\left\|\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f\left(T + \frac{k}{n}(T - R)\right)\right\| \le c\|f\|_{\Lambda_\alpha}\|T - R\|^\alpha.$$
Note that an analog of Theorem (4.1.4) also holds for contractions.

Theorem(4.1.6)[131]. Let $0 < \alpha < 1$ and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B$ is bounded. Then $f(A) - f(B)$ is bounded and
$$\|f(A) - f(B)\| \le \text{const}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^\alpha.$$
In this connection we mention the paper [132]where it was proved that for selfadjoint operators A and B with spectra in an interval $[a, b]$ and a function $\varphi \in \Lambda_\alpha(\mathbb{R})$, the following inequality holds:
$$\|\varphi(A) - \varphi(B)\| \le \text{const}\|\varphi\|_{\Lambda_\alpha(\mathbb{R})}\left(\log\left(\frac{b-a}{\|A - B\|} + 1\right) + 1\right)^2 \|A - B\|^\alpha$$
(see also [142] where the above inequality is generalized for general moduli of continuity).

Theorem(4.1.7)[131]. Suppose that n is a positive integer and $0 < \alpha < n$. Let A be a selfadjoint operator and let K be a bounded selfadjoint operator. Then the map,
$$f \to (\Delta_K^n f)(A) \stackrel{\text{def}}{=} \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}f(A + jK), \tag{4}$$
has a unique extension from $L^\infty \cap \Lambda_\alpha(\mathbb{R})$ to a sequentially continuous operator from $\Lambda_\alpha(\mathbb{R})$ (equipped with the weak-star topology) to the space of bounded linear operators on Hilbert space (equipped with the strong operator topology) and
$$\|(\Delta_K^n f)(A)\| \le \text{const}\|f\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^\alpha.$$
We use the same notation $(\Delta_K^n f)(A)$ for the unique extension of the map (4).

We can also prove an analog of Theorem (4.1.4) for selfadjoint operators.

We consider the behavior of functions of selfadjoint operators under perturbations of Schatten-von Neumann class $\boldsymbol{S_p}$. Similar results also hold for unitary operators and for contractions.

Recall that the spaces $\boldsymbol{S_p}$and $\boldsymbol{S_{p,\infty}}$ consist of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S_p}} \stackrel{\text{def}}{=} \left(\sum_{n\geq 0}(s_n(T))^p\right)^{1/p} < \infty \quad \text{and } \|T\|_{\boldsymbol{S_{p,\infty}}} \stackrel{\text{def}}{=} \sup_{n\geq 0}(1+n)^{\frac{1}{p}} s_n(T) < \infty.$$

Theorem(4.1.8)[131]. Let $1 \leq p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B \in \boldsymbol{S_p}$. Then

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{p}{\alpha},\infty} \text{ and } \|f(A) - f(B)\|_{\boldsymbol{S}_{\frac{p}{\alpha},\infty}} \leq \text{const } \|T\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^{\alpha}_{\boldsymbol{S_p}}.$$

Note that in Theorem (4.1.8) in the case $p > 1$ we can replace the condition $A - B \in \boldsymbol{S_p}$ with the condition $A - B \in \boldsymbol{S_{p,\infty}}$.

Using interpolation arguments, we can deduce from Theorem (4.1.8) the following result:

Theorem(4.1.9)[131]. Let $1 < p < \infty$, $0 < \alpha < 1$, and let $f \in \Lambda_\alpha(\mathbb{R})$. Suppose that A and B are selfadjoint operators such that $A - B \in \boldsymbol{S_p}$. Then

$$f(A) - f(B) \in \boldsymbol{S}_{\frac{p}{\alpha}} \quad \text{and} \quad \|f(A) - f(B)\|_{\boldsymbol{S}_{\frac{p}{\alpha}}} \leq \text{const } \|T\|_{\Lambda_\alpha(\mathbb{R})}\|A - B\|^{\alpha}_{\boldsymbol{S_p}}.$$

We state similar results for higher order differences.

Theorem(4.1.10)[131]. Suppose that n is a positive integer, $\alpha$is a positive number such that $n - 1 \leq \alpha < n$, and $n \leq p < \infty$. Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S_p}$. Then the operator $(\Delta^n_K f)(A)$ defined in Theorem (4.1.7) belongs to $\boldsymbol{S}_{\frac{p}{\alpha},\infty}$, and

$$\|(\Delta^n_K f)(A)\|_{\boldsymbol{S}_{\frac{p}{\alpha},\infty}} \leq \text{const } \|T\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^{\alpha}_{\boldsymbol{S_p}}$$

Theorem(4.1.11)[131]. Suppose that n is a positive integer, $\alpha$is a positive number such that $n - 1 \leq \alpha < n$, $f \in \Lambda_\alpha(\mathbb{R})$, and $n < p < \infty$. Let A be a selfadjoint operator and let K be a selfadjoint operator of class $\boldsymbol{S_p}$. Then the operator $(\Delta^n_K f)(A)$ defined in Theorem(4.1.7) belongs to $\boldsymbol{S}_{\frac{p}{\alpha}}$, and

$$\|(\Delta^n_K f)(A)\|_{\boldsymbol{S}_{\frac{p}{\alpha}}} \leq \text{const } \|T\|_{\Lambda_\alpha(\mathbb{R})}\|K\|^{\alpha}_{\boldsymbol{S_p}}.$$

Section (4.2): Normal Operators Under Perturbations

We generalize results of the papers [145], [146], [147], [148], and [149] to the case of normal operators.

A Lipschitz function f on the real line $\mathbb{R}$(i.e., a function satisfying the inequality $|f(x) - f(y)| \leq \text{const}|x - y|, x, y \in \mathbb{R}$) does not have to be operator Lipschitz. In other words, a Lipschitz function f does not necessarily satisfy the inequality

$$\|f(A) - f(B)\| \leq \text{const } \|A - B\|$$

for arbitrary self-adjoint operators A and B on Hilbert space. The existence of such functions was proved in [150]. Later Kato proved in [151] that the function $f(x) = |x|$

is not operator Lipschitz. Note also that earlier McIntosh established in [152] a similar result for commutators (i.e., the function $f(x) = |x|$ is not commutators Lipschitz).

In [146] and [153] necessary conditions were found for a function f to be operator Lipschitz. In particular, it was shown in [146] that if f is operator Lipschitz, and then f belongs locally to the Besov space $\boldsymbol{B}_{11}^{1}(\mathbb{R})$. This also implies that Lipschitz functions do not have to be operator Lipschitz. In [146] and [153] stronger necessary conditions were also obtained. The necessary conditions obtained in [145] and [146] are based on the trace class criterion for Hankel operators; see [145] .

On the other hand, it was shown in [146] and [153] that if f belongs to the Besov class $\boldsymbol{B}_{\infty 1}^{1}(\mathbb{R})$, and then f is operator Lipschitz. We refer to [154] for information on Besov spaces.

It was shown in [147] and [148] that the situation dramatically changes if we consider Hölder classes $\Lambda_{\alpha}(\mathbb{R})$ with $0 < \alpha < 1$. In this case such functions are necessarily operator Hölder of order α, i.e., the condition $|f(x) - f(y)| \le \text{const}|x - y|$, $x, y \in \mathbb{R}$, implies that for self-adjoint operators A and B on Hilbert space,

$$\|f(A) - f(B)\| \le \text{const}\, \|A - B\|^{\alpha}.$$

This result was generalized in [147] and [148] to the case of functions of class $\Lambda_{\alpha}(\mathbb{R})$ for arbitrary moduli of continuity ω. This class consists of functions f on $\mathbb{R}$, for which $|f(x) - f(y)| \le \text{const}\, \omega(|x - y|)$, $x, y \in \mathbb{R}$.

Let us also mention that in [147] and [149] properties of operators $f(A) - f(B)$ were studied for functions f in $\Lambda_{\alpha}(\mathbb{R})$ and self-adjoint operators A and B whose difference $A - B$ belongs to Schatten–von Neumann classes $\boldsymbol{S}_p$.

In [147], [148] and [155]analogs of the above results were obtained for higher order operator differences.

We also mention here that [147], [148], [149], [155], [156], and [157] study problems of perturbation theory for unitary operators, contractions, and dissipative operators.

We study the case of (not necessarily bounded) normal operators.

We prove that if f is a function on $\mathbb{R}^2$ that belongs to the Besov class $\boldsymbol{B}_{\infty 1}^{1}(\mathbb{R}^2)$, then it is an operator Lipschitz function on $\mathbb{R}^2$, i.e.,

$$\|f(N_1) - f(N_2)\| \le \text{const}\, \|N_1 - N_2\|$$

for arbitrary normal operators N_1 and N_2. Note that we say that the operator $N_1 - N_2$ is bounded if the domains $\mathfrak{D}_{N_1}$ and $\mathfrak{D}_{N_2}$ of N_1 and N_2 coincide and $N_1 - N_2$ is bounded on $\mathfrak{D}_{N_1}$. If $N_1 - N_2$ is not a bounded operator, we say that $\|N_1 - N_2\| = \infty$

Note, however, that the proof of the corresponding result for self-adjoint operators obtained in [153] does not work in the case of normal operators. In the case of self-adjoint operators it was shown in [153]that for functions f in the Besov space $\boldsymbol{B}_{\infty 1}^{1}(\mathbb{R})$ and self-adjoint operators A and B with bounded $A - B$, the following formula holds:

$$f(A) - f(B) = \iint\limits_{\mathbb{R}\times\mathbb{R}} \frac{f(x) - f(y)}{x - y}\, dE_A(x)(A - B)\, dE_B(y)\,.$$

The expression on the right is a double operator integral. However, in the case of normal operators a similar formula holds for arbitrary normal operators only for linear functions .We obtain a new formula for $f(N_1) - f(N_2)$ in terms of double operator integrals for suitable functions f on $\mathbb{C}$ and normal operators N_1 and N_2 with bounded $N_1 - N_2$. The validity of this formula depends on the fact that certain divided differences are Schur multipliers.

We prove that as in the case of self-adjoint operators, Hölder functions of order α, $0 < \alpha < 1$, must be operator Hölder of order α. We also consider the case of arbitrary moduli of continuity. Note that in [175] some weaker results were obtained.

We study of properties of $f(N_1) - f(N_2)$, where N_1 and N_2 are normal operators whose difference $N_1 - N_2$ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$ and f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R}^2)$. We obtain analogs for normal operators of the results of [147] and [148] for self-adjoint operators. We also obtain much more general results for normal operators N_1 and N_2 whose difference $N_1 - N_2$ belongs to ideals of operators on Hilbert space.

Finally, We obtain estimates for quasicommutators $f(N_1)R - Rf(N_2)$ in terms of $N_1R - RN_2$ and $N_1^*R - RN_2^*$.

We give a brief introduction to Besov spaces and the spaces $\Lambda_\omega(\mathbb{R}^2)$ of functions of two real variables. We review ideals of operators on Hilbert space.

Note that the results were announced in the note [176]. We identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$.

We collect necessary information on Besov spaces and the spaces $\Lambda_\omega(\mathbb{R}^2)$ of functions of two real variables.

The purpose to give a brief introduction to Besov spaces that play an important role in problems of perturbation theory. We need the Besov spaces on $\mathbb{R}^2$ only.

Let w be an infinitely differentiable function on $\mathbb{R}$ such that

$$w \geq 0, \quad \operatorname{supp} w \subset \left[\frac{1}{2}, 2\right], \quad \text{and} \quad w(x) = 1 - w\left(\frac{x}{2}\right) \quad \text{for } x \in [1,2]. \quad (5)$$

We define the functions W_n on $\mathbb{R}^2$ by

$$\mathcal{F}W_n(x) = w\left(\frac{|x|}{2^n}\right), \quad n \in \mathbb{Z}, \ x = (x_1, x_2), \qquad |x| \stackrel{\text{def}}{=} (x_1^2 + x_2^2)^{1/2},$$

where $\mathcal{F}$ is the Fourier transform defined on $L^1(\mathbb{R}^2)$ by

$$(\mathcal{F}f) = \int_{\mathbb{R}} f(x) e^{-i(x,t)}\, dx, \quad x = (x_1, x_2), t = (t_1, t_2), (x,t) \stackrel{\text{def}}{=} x_1t_1 + x_2t_2.$$

With each tempered distribution $f \in \mathcal{S}'(\mathbb{R}^2)$, we associate a sequence $\{f_n\}_{n\in\mathbb{Z}}$

$$f_n \stackrel{\text{def}}{=} f * W_n. \quad (6)$$

Initially we define the (homogeneous) Besov class $\dot{\boldsymbol{B}}_{pq}^s(\mathbb{R}^2)$, $s > 0, 1 \leq p, q \leq \infty$, as the space of all $f \in \mathcal{S}'(\mathbb{R}^2)$ such that

$$\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}). \quad (7)$$

According to this definition, the space $\dot{\boldsymbol{B}}_{pq}^s(\mathbb{R}^2)$ contains all polynomials. Moreover, the distribution f is defined by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ uniquely up to a polynomial. It is easy to see that the series $\sum_{n\geq 0} f_n$ converges in $\mathcal{S}'(\mathbb{R}^2)$. However, the series $\sum_{n<0} f_n$ can diverge in general. It is easy to prove that the series

$$\sum_{n<0} \frac{\partial^r f_n}{\partial x_1^k \partial x_2^{r-k}} \quad (8)$$

converges uniformly on $\mathbb{R}^2$ for every nonnegative integer $r > s - 2/p$ and $0 \le k \le r$. Note that in the case $q = 1$ the series (8) converges uniformly, whenever $r \ge s - 2/p$ and $0 \le k \le r$.

Now we can define the modified (homogeneous) Besov class $\boldsymbol{B}^s_{pq}(\mathbb{R}^2)$. We say that a distribution f belongs to $\boldsymbol{B}^s_{pq}(\mathbb{R}^2)$ if (7) holds and

$$\frac{\partial^r f}{\partial x_1^k \partial x_2^{r-k}} = \sum_{n\in\mathbb{Z}} \frac{\partial^r f_n}{\partial x_1^k \partial x_2^{r-k}}$$

in the space $\mathcal{S}'(\mathbb{R}^2)$, where r is the minimal nonnegative integer such that $r > s - 2/p (r \ge s - 2/p$ if $q = 1)$ and $0 \le k \le r$. Now the function f is determined uniquely by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ up to a polynomial of degree less than r, and a polynomial φ belongs to $\boldsymbol{B}^s_{pq}(\mathbb{R}^2)$ if and only if $\deg \varphi < r$.

To define a regularized de laVallée Poussin type kernel V_n, we define the C^∞ ∞ function v on $\mathbb{R}$ by

$$v(x) = 1 \quad \text{for} \quad x \in [-1,1] \quad \text{and} \quad v(x) = w(|x|) \quad \text{if} \quad |x| \ge 1,$$

where w is the function defined by (5). Now we can define the de la Vallée Poussin type functions V_n by

$$\mathcal{F}\, V_n(x) = v\left(\frac{|x|}{2^n}\right), \quad n \in \mathbb{Z}, \qquad x = (x_1, x_2).$$

We put $\stackrel{\text{def}}{=} V_0$. Clearly, $V_n(x) = 2^{2n} V(2^n x)$.

Besov classes admit many other descriptions. We give here the definition in terms of finite differences. For $h \in \mathbb{R}^2$, we define the difference operator Δ_h,

$$(\Delta_h f\,)(x) = f(x+h) - f(x), \quad x \in \mathbb{R}^2.$$

It is easy to see that $\boldsymbol{B}^s_{pq}(\mathbb{R}^2) \subset L^1_{\text{loc}}(\mathbb{R}^2)$ for every $s > 0$ and $\boldsymbol{B}^s_{pq}(\mathbb{R}^2) \subset C(\mathbb{R}^2)$ for every $s > 2/p$. Let $s > 0$ and let m be a positive integer such that $m - 1 \le s < m$. The Besov space $\boldsymbol{B}^s_{pq}(\mathbb{R}^2)$ can be defined as the set of functions $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} |h|^{-2-sq} \|\Delta_h^m f\|^q_{L^p} dh < \infty \quad \text{for } q < \infty$$

and

$$\sup_{h\neq 0} \frac{\|\Delta_h^m f\|_{L^p}}{|h|^s} < \infty \quad \text{for } q = \infty. \tag{9}$$

However, with this definition the Besov space can contain polynomials of higher degree than in the case of the first definition given above.

We use the notation $\boldsymbol{B}^s_p(\mathbb{R}^2)$ for $\boldsymbol{B}^s_{pq}(\mathbb{R}^2)$.

For $\alpha > 0$, denote by $\Lambda_\alpha(\mathbb{R}^2)$ the Hölder–Zygmund class that consists of functions $f \in C(\mathbb{R}^2)$ such that

$$|(\Delta_h^m f)(x)| \le \text{const}\, |h|^\alpha, \quad x, h \in \mathbb{R}^2,$$

where m is the smallest integer greater than α. By (9), we have $\Lambda_\alpha(\mathbb{R}^2) = \boldsymbol{B}^\alpha_\infty(\mathbb{R}^2)$.

We refer the reader to [154] and [158] for more detailed information on Besov spaces.

Let ω be a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ for $x > 0$, and

$$\omega(x+y) \le \omega(x) + \omega(y), \quad x, y \in [0, \infty).$$

We denote by $\Lambda_\omega(\mathbb{R}^2)$ the space of functions on $\mathbb{R}^2$such that

$$\|f\|_{\Lambda_\omega(\mathbb{R}^2)} \stackrel{\text{def}}{=} \sup_{x \ne y} \frac{|f(x) - f(y)|}{\omega(|x-y|)} < \infty.$$

Theorem(4.2.1)[144]. There exists a constant $c > 0$ such that for an arbitrary modulus of continuity ω and for an arbitrary function f in $\Lambda_\omega(\mathbb{R}^2)$, the following inequality holds:

$$\|f - f * V_n\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R}^2)}, \quad n \in \mathbb{Z}. \tag{10}$$

Proof. We have

$$|f(x) - (f * V_n)(x)| = 2^{2n} \left| \int_{\mathbb{R}^2} \big(f(x) - f(x-y)\big) V(2^n y) dy \right|$$

$$\le 2^{2n} \|f\|_{\Lambda_\omega(\mathbb{R}^2)} \int_{\mathbb{R}^2} \omega(|y|)\, |V(2^n y)| dy$$

$$= 2^{2n}\|f\|_{\Lambda_\omega(\mathbb{R}^2)} \int_{\{|y| \le 2^{-n}\}} \omega(|y|)\, |V(2^n y)| dy + 2^{2n}\|f\|_{\Lambda_\omega(\mathbb{R}^2)} \int_{\{|y| > 2^{-n}\}} \omega(|y|)\, |V(2^n y)| dy$$

Clearly,

$$2^{2n} \int_{\{|y| \le 2^{-n}\}} \omega(|y|)\, |V(2^n y)| dy \le \omega(2^{-n}) \|V\|_{L^1}.$$

On the other hand, keeping in mind the obvious inequality $2^{-n}\omega\,(|y|) \le 2|y|\omega(2^{-n})$ for $|y| \ge 2^{-n}$, we obtain

$$2^{2n} \int_{\{|y| > 2^{-n}\}} \omega(|y|)\, |V(2^n y)| dy \le 2 \cdot 2^{3n} \omega(2^{-n}) \int_{\{|y| > 2^{-n}\}} |y|\, |V(2^n y)| dy$$

$$= 2\omega(2^n) \int_{\{|y| > 1\}} |y| \cdot |V(y)| dy \le \text{const}\ \omega(2^{-n}).$$

This proves (10).

Corollary (4.2.2)[144]. There exists $c > 0$ such that for every modulus of continuity ω and for every $f \in \Lambda_\omega(\mathbb{R}^2)$, the following inequalities hold:

$$\|f * W_n\|_{L^\infty} \le c\omega(2^{-n}) \|f\|_{\Lambda_\omega(\mathbb{R}^2)}, \quad n \in \mathbb{Z}.$$

We give a brief introduction to quasinormed ideals of operators on Hilbert space. Recall a functional $\|\cdot\|: X \to [0, \infty)$ on a vector space X is called a quasinorm on X if

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$, for every $x \in X$ and $\alpha \in \mathbb{C}$;

(iii) there exists a positive number c such that $\|x + y\| \le c(\|x\| + \|y\|)$ for every x and y in X .

We say that a sequence $\{x_j\}_{j \ge 1}$ of vectors of a quasinormed space X converges to $x \in X$ if $\lim_{j \to \infty} \|x_j - x\| = 0$. It is well known that there exists a translation invariant metric on X which induces an equivalent topology on X . A quasinormed space is called quasi-Banach if it is complete.

Recall that for a bounded linear operator T on Hilbert space, the singular values $s_j(T), j \geq 0$, are defined by

$$s_j(T) = \inf\{ \|T - R\|: \text{rank } R \leq j\} .$$

Clearly, $s_0(T) = \|T\|$ and T is compact if and only if $s_j(T) \to 0$ as $j \to \infty$. We also introduce the sequence $\{\sigma_n(T)\}_{n\geq 0}$ defined by

$$\sigma_n(T) \stackrel{\text{def}}{=} \frac{1}{n+1}\sum_{j=0}^{n} s_j(T). \tag{11}$$

Definition(4.2.3) [144]. Let $\mathcal{H}$ be a Hilbert space and let $\mathfrak{I}$ be a linear manifold in the set $\mathfrak{B}(\mathcal{H})$of bounded linear operators on $\mathcal{H}$ that is equipped with a quasi-norm $\|\cdot\|_{\mathfrak{I}}$ that makes $\mathfrak{I}$ a quasi-Banach space. We say that $\mathfrak{I}$ is a quasinormed ideal if for every A and B in $\mathfrak{B}(\mathcal{H})$ and $T \in \mathfrak{I}$,

$$ATB \in \mathfrak{I} \text{ and } \|ATB\|_{\mathfrak{I}} \leq \|A\| \cdot \|B\| \cdot \|T\|_{\mathfrak{I}} . \tag{12}$$

A quasinormed ideal $\mathfrak{I}$ is called a normed ideal if $\|\cdot\|_{\mathfrak{I}}$ is a norm.

Note that we do not require that $\mathfrak{I} \neq \mathfrak{B}(\mathcal{H})$.

It is easy to see that if T_1 and T_2 are operators in a quasinormed ideal $\mathfrak{I}$ and $s_j(T_1) = s_j(T_2)$ for $j \geq 0$, then $\|T_1\|_{\mathfrak{I}} = \|T_2\|_{\mathfrak{I}}$. Thus there exists a function $\Psi = \Psi_1$ defined on the set of nonincreasing sequences of nonnegative real numbers with values in $[0, \infty]$ such that $T \in \mathfrak{I}$ if and only if $\Psi(s_0(T), s_1(T), s_2 (T),\cdots) < \infty$ and

$$\|T\|_{\mathfrak{I}} = \Psi(s_0(T), s_1(T), s_2 (T),\cdots), \ T \in \mathfrak{I}.$$

If T is an operator from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$, we say that T belongs to $\mathfrak{I}$ if $\Psi(s_0(T), s_1(T), s_2 (T),\cdots) < \infty$.

For a quasinormed ideal $\mathfrak{I}$ and a positive number p, we define the quasinormed ideal $\mathfrak{I}^{\{p\}}$ by

$$\mathfrak{I}^{\{p\}} = \{T: (T^*T)^{p/2} \in \mathfrak{I}\}, \|T\|_{\mathfrak{I}^{\{p\}}} \stackrel{\text{def}}{=} \left\|(T^*T)^{p/2}\right\|_{\mathfrak{I}}^{1/p}.$$

If T is an operator on a Hilbert space $\mathcal{H}$ and d is a positive integer, we denote by $[T]_d$ the operator $\oplus_{j=1}^{d} T_j$ on the orthogonal sum $\oplus_{j=1}^{d}\mathcal{H}$ of d copies of $\mathcal{H}$, where $T_j = T$,$1 \leq j \leq d$. It is easy to see that

$$s_n([T]_d) = s_{[n/d]}(T), \ n \geq 0,$$

where $[x]$ denotes the largest integer that is less than or equal to x.

We denote by $\beta_{\mathfrak{I},d}$ the quasinorm of the transformer $T \to [T]_d$ on $\mathfrak{I}$. Clearly, the sequence $\{\beta_{\mathfrak{I},d}\}_{d\geq 1}$ is nondecreasing and submultiplicative, i.e., $\beta_{\mathfrak{I},d_1,d_2} \leq \beta_{\mathfrak{I},d_1}\beta_{\mathfrak{I},d_2}$. It is well known (see e.g., § 3 of [149]) that the last inequality implies that

$$\lim_{d\to\infty} \frac{\log \beta_{\mathfrak{I},d}}{\log d} = \inf_{d\geq 2} \frac{\log \beta_{\mathfrak{I},d}}{\log d}. \tag{13}$$

Definition(4.2.4) [144]. If $\mathfrak{I}$ is a quasinormed ideal, the number

$$\beta_{\mathfrak{I}} \stackrel{\text{def}}{=} \lim_{d\to\infty} \frac{\log \beta_{\mathfrak{I},d}}{\log d} = \inf_{d\geq 2} \frac{\log \beta_{\mathfrak{I},d}}{\log d}.$$

is called the upper Boyd index of $\mathfrak{I}$.

It is easy to see that $\beta_{\mathfrak{I}} \leq 1$ for an arbitrary normed ideal $\mathfrak{I}$. It is also clear that $\beta_{\mathfrak{I}} < 1$ if and only if $\lim\limits_{d\to\infty} d^{-1}\beta_{\mathfrak{I},d} = 0$.

Note that the upper Boyd index does not change if we replace the initial quasinorm in the quasinormed ideal with an equivalent one that also satisfies (12). It is also easy to see that

$$\beta_{\mathfrak{I}\{p\}} = p^{-1}\beta_{\mathfrak{I}}\,.$$

The proof of the following fact can be found in [149], § 3.

Let $\mathfrak{I}$ be a quasinormed ideal. The following are equivalent:

(i) $\beta_{\mathfrak{I}} < 1$;

(ii) for every nonincreasing sequence $\{s_n\}_{n\ge0}$ of nonnegative numbers,

$$\Psi_{\mathfrak{I}}(\{\sigma_n\}_{n\ge0}) \le \text{ const } (\Psi_{\mathfrak{I}}\{s_n\}_{n\ge0})\ , \tag{14}$$

where $\sigma_n \stackrel{\text{def}}{=} (1+n)^{-1}\sum_{j=0}^{n} s_j$.

For a normed ideal $\mathfrak{I}$ let $C_{\mathfrak{I}}$ be the best possible constant in inequality (14). Then (see [149], § 3)

$$C_{\mathfrak{I}} \le 3\sum_{k=0}^{\infty} 2^{-k}\beta_{\mathfrak{I},2^k}. \tag{15}$$

Let $\boldsymbol{S}_p$, $0 < p < \infty$, be the Schatten–von Neumann class of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S}_p} \stackrel{\text{def}}{=} \left(\sum_{j\ge0}\big(s_j(T)\big)^p\right)^{1/p}.$$

This is a normed ideal for $p \ge 1$. We denote by $\boldsymbol{S}_p$, $0 < p < \infty$, the ideal that consists of operators T on Hilbert space such that

$$\|T\|_{\boldsymbol{S}_{p,\infty}} \stackrel{\text{def}}{=} \left(\sup_{j\ge0}(1+j)\big(s_j(T)\big)^p\right)^{1/p}.$$

The quasinorm $\|\cdot\|_{\boldsymbol{S}_{p,\infty}}$ is not a norm, but it is equivalent to a norm if $p > 1$. It is easy to see that

$$\beta_{\boldsymbol{S}_p} = \beta_{\boldsymbol{S}_{p,\infty}} = \frac{1}{p},\ \ 0 < p < \infty.$$

Thus $\boldsymbol{S}_p$ and $\boldsymbol{S}_{p,\infty}$ with $p > 1$ satisfy the hypotheses of Theorem on ideals with upper Boyd index less than 1.

It follows easily from (15) that for $p > 1$,

$$C_{\boldsymbol{S}_p} \le 3(1-2^{1/p-1})^{-1}.$$

Suppose now that $\mathfrak{I}$ is a quasinormed ideal of operators on Hilbert space. With a nonnegative integer l we associate the ideal $(l)_{\mathfrak{I}}$ that consists of all bounded linear operators on Hilbert space and is equipped with the norm

$$\Psi_{(l)_{\mathfrak{I}}}(s_0, s_1, s_2, \cdots) = \Psi(s_0, s_1, s_2, \cdots, s_l, 0, 0, \dots).$$

It is easy to see that for every bounded operator T,

$$\|T\|_{(l)_{\mathfrak{I}}} = \sup\{\|RT\|_{\mathfrak{I}}: \|R\| \le 1, \operatorname{rank} R \le l+1\} = \sup\{\|TR\|_{\mathfrak{I}}: \|R\| \le 1, \operatorname{rank} R \le l+1\}.$$

It is easy to verify (see [149], § 3) that if $\mathfrak{I}$ is a quasinormed ideal, then for all

$$C_{(l)_{\mathfrak{I}}} \le C_{\mathfrak{I}}. \tag{16}$$

Note that if $\mathfrak{I} = \boldsymbol{S}_p, p \ge 1$, then $\boldsymbol{S}_p^l \stackrel{\text{def}}{=} \boldsymbol{S}_p$ is the normed ideal that consists of all bounded linear operators equipped with the norm

$$\|T\|_{S_p^l} \overset{\text{def}}{=} \left(\sum_{j=0}(s_j(T))^p\right)^{1/p}.$$

It is well known that $\|\cdot\|_{S_p^l}$ is a norm for $p \geq 1$ (see [160]).

It is also well known (see [149], § 3) that

$$\|T_1 T_2\|_{S_r^l} \leq \|T_1\|_{S_p^l}\|T_2\|_{S_q^l}, \tag{17}$$

where $T_1$and T_2 bounded operator on Hilbert space and $1/p + 1/q = 1/r$.

We say that a quasinormed ideal $\mathfrak{I}$ has majorization property (respectively weak majorization property) if the conditions

$$T_1 \in \mathfrak{I},\ T_2 \in \mathfrak{B},\ \text{ and }\ \sigma_l(T_2) \leq \sigma_l(T_1) \quad \text{for all} \quad l \geq 0$$

imply that

$$T_1 \in \mathfrak{I} \ \text{ and } \ \|T_2\|_{\mathfrak{I}} \leq \|T_1\|_{\mathfrak{I}} \qquad (\text{respectively} \quad \|T_2\|_{\mathfrak{I}} \leq C\|T_1\|_{\mathfrak{I}})$$

(see [158]). Note that if a quasinormed ideal $\mathfrak{I}$ has weak majorization property, then we can introduce on it the following new equivalent quasinorm:

$$\|T\|_{\tilde{\mathfrak{I}}} \overset{\text{def}}{=} \sup\{\|R\|_{\mathfrak{I}}: \sigma_l(R) \leq \sigma_l(T) \ \text{ for all } l \geq 0\}$$

such that $(\mathfrak{I}, \|\cdot\|_{\tilde{\mathfrak{I}}})$ has majorization property.

It is well known that every separable normed ideal and every normed ideal that is dual to a separable normed ideal has majorization property, see [158]. Clearly, $\boldsymbol{S}_1 \subset \mathfrak{I}$ for every quasinormed ideal $\mathfrak{I}$ with majorization property. Note also that every quasinormed ideal $\mathfrak{I}$ with $\beta_{\mathfrak{I}} < 1$ has weak majorization property (see, for example, § 3 of [149] and § 3 of [155]).

We need the following fact on interpolation properties of quasinormed ideals that have majorization property (see e.g., [155]):

Theorem on interpolation of quasinormed ideals. Let $\mathfrak{I}$ be a quasinormed ideal with majorization property and let $\mathfrak{A}: \mathfrak{L} \to \mathfrak{L}$ be a linear transformer on a linear subset $\mathfrak{L}$ of $\mathfrak{B}$ such that $\mathfrak{L} \cap \boldsymbol{S}_1$ is dense in $\boldsymbol{S}_1$. Suppose that $\|\mathfrak{A}T\| \leq \|T\|$ and $\|\mathfrak{A}T\|_{\boldsymbol{S}_1} \leq \|T\|_{\boldsymbol{S}_1}$for every $T \in \mathfrak{L}$.

We refer to [158] and [160] for further information on singular values and normed ideals of operators on Hilbert space.

We give a brief introduction in double operator integrals. Double operator integrals appeared in [161] by Daletskii and S.G. Krein. However, the beautiful theory of double operator integrals was developed later by Birman and Solomyak in [162], [163], and [164], see also their survey [165].

Let $(\mathcal{X}, E_1)$ and $(\mathcal{Y}, E_2)$ be spaces with spectral measures E_1 and E_2 on a Hilbert space $\mathcal{H}$. The idea of Birman and Solomyak is to define first double operator integrals

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y)\,dE_1(x)T\,dE_2(y) \tag{18}$$

for bounded measurable functions Φ and operators T of Hilbert Schmidt class $\boldsymbol{S}_2$. Consider the spectral measure $\mathcal{E}$ whose values are orthogonal projections on the Hilbert space $\boldsymbol{S}_2$, which is defined by

$$\mathcal{E}(\Lambda \times \Delta)T = E_1(\Lambda)TE_2(\Delta), T \in \boldsymbol{S}_2,$$

Λ and Δ being measurable subsets of $\mathcal{X}$ and $\mathcal{Y}$. It was shown in [166] that $\mathcal{E}$ extends to a spectral measure on $\mathcal{X} \times \mathcal{Y}$ and if Φ is a bounded measurable function on $\mathcal{X} \times \mathcal{Y}$, by definition,

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) = \left(\int_{\mathcal{X}\times\mathcal{Y}} \Phi d\mathcal{E}\right)T.$$

Clearly,

$$\left\| \int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) \right\|_{\boldsymbol{S}_2} \le \|\Phi\|_{L^\infty} \|T\|_{\boldsymbol{S}_2}.$$

If

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) \in \boldsymbol{S}_1$$

for every $T \in \boldsymbol{S}_1$, we say that Φ is a Schur multiplier of $\boldsymbol{S}_1$ associated with the spectral measures E_1 and E_2.

In this case the transformer

$$T \to \int_{\mathcal{X}} \int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y), \quad T \in \boldsymbol{S}_2, \tag{19}$$

extends by duality to a bounded linear transformer on the space of bounded linear operators on $\mathcal{H}$ and we say that the function Ψ on $\mathcal{Y} \times \mathcal{X}$ defined by

$$\Psi(y,x) = \Phi(x,y)$$

is a Schur multiplier (with respect to E_2 and E_1) of the space of bounded linear operators. We denote the space of such Schur multipliers by $\mathfrak{M}(E_2, E_1)$. The norm of Ψ in $\mathfrak{M}(E_2, E_1)$ is, by definition, the norm of the transformer (19) on the space of bounded linear operators.

In [164] it was shown that if A and B are a self-adjoint operators (not necessarily bounded) such that $A - B$ is bounded and if f is a continuously differentiable function on $\mathbb{R}$ such that the divided difference $\mathfrak{D}f$,

$$(\mathfrak{D}f)(x,y) = \frac{f(x) - f(y)}{x - y},$$

is a Schur multiplier of $\boldsymbol{S}_1$ with respect to the spectral measures of A and B, then

$$f(A) - f(B) = \iint (\mathfrak{D}f)(x,y)\, dE_A(x)(A - B)dE_B(y) \tag{20}$$

and

$$\|f(A) - f(B)\| \le \text{const}\, \|f\|_{\mathfrak{M}(E_2,E_1)} \|A - B\|,$$

i.e., f is an operator Lipschitz function.

It is easy to see that if a function Φ on $\mathcal{X} \times \mathcal{Y}$ belongs to the projective tensor product $L^\infty(E_1) \hat{\otimes} L^\infty(E_2)$ of $L^\infty(E_1)$ and $L^\infty(E_2)$ (i.e., Φ admits a representation

$$\Phi(x,y) = \sum_{n\ge 0} \varphi_n(x)\psi_n(y)$$

where $\varphi_n \in L^\infty(E_1)$, $\psi_n \in L^\infty(E_2)$, and

$$\sum_{n\ge 0} \|\varphi_n\|_{L^\infty} \|\psi_n\|_{L^\infty} < \infty,$$

then $\Phi \in \mathfrak{M}(E_2, E_1)$. For such functions Φ we have

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) = \sum_{n\ge 0}\left(\int_{\mathcal{X}} \varphi_n dE_1\right)T\left(\int_{\mathcal{Y}} \psi_n\, dE_2\right).$$

More generally, $\Phi \in \mathfrak{M}(E_2, E_1)$ if Φ belongs to the integral projective tensor product $L^\infty(E_1)\,\widehat{\otimes}\,L^\infty(E_2)$ of $L^\infty(E_1)$, and $L^\infty(E_2)$ i.e., Φ admits a representation

$$\Phi(x,y) = \int_\Omega \varphi(x,w)\,\psi(y,w)d\lambda(w), \qquad (21)$$

where (Ω, λ) is a σ-finite measure space, ϕ is a measurable function on $\mathcal{X} \times \Omega$, ψ is a measurable function on $\mathcal{Y} \times \Omega$, and

$$\int_\Omega \|\varphi(\cdot,w)\|_{L^\infty(E_1)}\,\|\psi(\cdot,w)\|_{L^\infty(E_2)}d\lambda(w) < \infty. \qquad (22)$$

If $\Phi \in L^\infty(E_1)\,\widehat{\otimes}_i\,L^\infty(E_2)$, then

$$\int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) = \int_\Omega\left(\int_{\mathcal{X}} \varphi(x,w)dE_1(x)\right)T\left(\int_{\mathcal{Y}} \psi(y,w)dE_2(y)\right)d\lambda(w).$$

Clearly, the function

$$s \to \left(\int_{\mathcal{X}} \varphi(x,w)dE_1(x)\right)T\left(\int_{\mathcal{Y}} \psi(y,w)dE_2(y)\right)$$

is weakly measurable and

$$\int_\Omega \left\|\left(\int_{\mathcal{X}} \varphi(x,s)dE_1(x)\right)T\left(\int_{\mathcal{Y}} \psi(y,w)dE_2(w)\right)\right\| d\lambda(w) < \infty$$

It turns out that all Schur multipliers can be obtained in this way. More precisely, the following result holds (see [146]):

Theorem on Schur multipliers. Let Φ be a measurable function on $\mathcal{X} \times \mathcal{Y}$. The following are equivalent:

(i) $\Phi \in \mathfrak{M}(E_2, E_1)$;

(ii) $\Phi \in L^\infty(E_1)\,\widehat{\otimes}_i\,L^\infty(E_2)$;

(iii) there exist measurable functions ϕ on $\mathcal{X} \times \Omega$ and ψ on $\mathcal{Y} \times \Omega$ such that (21) holds and

$$\left\|\left(\int_\Omega |\varphi(\cdot,w)|^2 d\lambda(w)\right)^{1/2}\right\|_{L^\infty(E)} \left\|\left(\int_\Omega |\psi(\cdot,w)|^2\, d\lambda(w)\right)^{1/2}\right\|_{L^\infty(E)} < \infty. \qquad (23)$$

The implication (iii)⇒(i) was established in [164]. In the case of matrix Schur multipliers (this corresponds to discrete spectral measures of multiplicity 1) the fact that (i) implies (ii) was proved in [167].

Note that the infimum of the left-hand side in (23) over all representations of the form (21) is the so-called Haagerup tensor norm of two L∞ spaces.

It is interesting to observe that if ϕ and ψ satisfy (22), then they also satisfy (23), but the converse is false. However, if Φ admits a representation of the form (21) with ϕ and ψ satisfying (23), then it also admits a (possibly different) representation of the form (21) with ϕ and ψ satisfying (22). We refer the reader to [168] for related problems.

It is also well known that $\mathfrak{M}(E_2, E_1)$ is Banach algebra (see [146]).

We would like to observe that it follows from the Theorem on interpolation of quasinormed ideals that if $\Phi \in \mathfrak{M}(E_2, E_1)$ and $\mathfrak{I}$ is a quasinormed ideal with majorization property, then

$$T \in \mathfrak{I} \quad \Rightarrow \quad \int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) \ \in \mathfrak{I}$$

and

$$\left\| \int_{\mathcal{X}}\int_{\mathcal{Y}} \Phi(x,y)dE_1(x)TdE_2(y) \right\| \le \|\Phi\|_{\mathfrak{M}(E_2,E_1)} \|T\|_{\mathfrak{I}}. \qquad (24)$$

Recall that a function f on $\mathbb{R}^2$ is called operator Lipschitz if

$$\|f(N_1) - f(N_2)\| \le \text{const}\ \|N_1 - N_2\| \qquad (25)$$

for every normal operators N_1 and N_2 on Hilbert space. Clearly, if f is operator Lipschitz, then f is a Lipschitz function. The converse is false, because it is false for self-adjoint operators (see the Introduction).

The first natural try to prove that a function on $\mathbb{R}^2$ is operator Lipschitz is to attempt to generalize formula (20) to the case of normal operators. Suppose that the divided difference

$$(z_1, z_2) \to \frac{f(z_1) - f(z_2)}{z_1 - z_2}, \qquad z_1, z_2 \in \mathbb{C},$$

is a Schur multiplier with respect to arbitrary Borel spectral measures on $\mathbb{C}$. Then as in the case of self-adjoint operators, for arbitrary normal operators N_1 and N_2 with bounded difference $N_1 - N_2$, the following formula holds

$$f(N_1) - f(N_2) = \iint\limits_{\mathbb{C}\times\mathbb{C}} \frac{f(z_1) - f(z_2)}{z_1 - z_2} dE_1(z_1)\,(N_1 - N_2)dE_2(z_2), \qquad (26)$$

where E_j is the spectral measure of N_i, i = 1,2. Moreover, in this case f is operator Lipschitz.

However, it follows from the results of [169] that under the above assumptions f must have complex derivative everywhere. In other words, f must be an entire function. In addition to this f must be Lipschitz. Therefore in this case f is a linear function, but the fact that linear functions are operator Lipschitz is obvious.

Thus to prove that a given function on $\mathbb{R}^2$ is operator Lipschitz, we have to find something different.

We introduce the following notation. Given normal operators N_1 and N_2 on Hilbert space, we put

$A_j \stackrel{\text{def}}{=} \text{Re}N_j$, $B_j \stackrel{\text{def}}{=} \text{Im}N_j$, E_j is the spectral measure of N_j, j = 1,2.

In other words, $N_j = A_j + iB_j$, j = 1,2, where A_j and B_j are self-adjoint operators. Since the operators N_j are normal, A_j commutes with B_j.

With a function f on $\mathbb{R}^2$ that has partial derivatives everywhere, we associate the following divided differences

$$(\mathfrak{D}_x f)(z_1, z_2) \stackrel{\text{def}}{=} \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2}, \qquad z_1, z_2 \in \mathbb{C}.$$

and

$$\left(\mathfrak{D}_y f\right)(z_1, z_2) \stackrel{\text{def}}{=} \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2}, \qquad z_1, z_2 \in \mathbb{C}.$$

We use the notation

$$x_j \stackrel{\text{def}}{=} \operatorname{Re} z_j, \quad y_j \stackrel{\text{def}}{=} \operatorname{Im} z_j, \quad j = 1,2.$$

Note that in the above definition by the values of $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ on the sets

$$\{(z_1, z_2): x_1 = x_2\} \quad \text{and} \quad \{(z_1, z_2): y_1 = y_2\}$$

we mean the corresponding partial derivatives of f.

Theorem (4.2.5)[144]. Let f be a continuous bounded function on $\mathbb{R}^2$ whose Fourier transform $\mathcal{F}f$ has compact support. Then the functions $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ are Schur multipliers with respect to arbitrary Borel spectral measures E_1 and E_2.

Moreover, if

$$\operatorname{supp} \mathcal{F}f \subset \{\zeta \in \mathbb{C} : |\zeta| \le \sigma\}, \qquad \sigma > 0,$$

then

$$\|\mathfrak{D}_x f\|_{\mathfrak{M}(E_1,E_2)} \le \text{const}\, \sigma \|f\|_{L^\infty} \quad \text{and} \quad \left\|\mathfrak{D}_y f\right\|_{\mathfrak{M}(E_1,E_2)} \le \text{const}\, \sigma \|f\|_{L^\infty} \tag{27}$$

Proof . The result follows from Theorem (4.2.9), because

$$\|\Phi\|_{\mathfrak{M}(E_1,E_2)} \le \|\Phi\|_{C_b(\mathbb{C})\widehat{\otimes}_h C_b(\mathbb{C})}$$

for every $\Phi \in C_b(\mathbb{C})\, \widehat{\otimes}_h\, C_b(\mathbb{C})$ and for every Borel spectral measures E_1 and E_2 on $\mathbb{C}$.

Theorem (4.2.6)[144]. Let f be a continuous bounded function on $\mathbb{R}^2$ whose Fourier transform $\mathcal{F}f$ has compact support. Suppose that N_1 and N_2 are normal operators such that the operator $N_1 - N_2$ is bounded. Then

$$f(N_1) - f(N_2) = \iint_{\mathbb{C}^2} \left(\mathfrak{D}_y f\right)(z_1, z_2)\, dE_1(z_1)\, (B_1 - B_2)\, dE_2(z_2)$$

$$+ \iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1, z_2)\, dE_1(z_1)\, (A_1 - A_2)\, dE_2(z_2) \tag{28}$$

We postpone the proof of Theorem (4.2.5). Let us deduce here Theorem (4.2.6) from Theorem (4.2.5).

Proof. Consider first the case when N_1 and N_2 are bounded operators. Put

$$d = \max\{\|N_1\|, \|N_2\|\} \quad \text{and} \quad D \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C}: |\zeta| \le d\}.$$

By Theorem (4.2.5), both $\mathfrak{D}_y f$ and $\mathfrak{D}_x f$ are Schur multipliers. We have

$$\iint_{\mathbb{C}^2} (\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,(B_1-B_2)dE_2(z_2) = \iint_{D\times D} (\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,(B_1-B_2)dE_2(z_2)$$
$$= \iint_{D\times D} (\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,B_1 dE_2(z_2)$$
$$- \iint_{D\times D} (\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,B_2 dE_2(z_2)$$
$$= \iint_{D\times D} y_1(\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,dE_2(z_2)$$
$$- \iint_{D\times D} y_2(\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,dE_2(z_2)$$
$$= \iint_{D\times D} (y_1-y_2)(\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,dE_2(z_2)$$
$$= \iint_{D\times D} (f(x_1,y_1)-f(x_1,y_2))dE_1(z_1)\,dE_2(z_2).$$

Since $\mathfrak{M}(E_1,E_2)$ is a Banach algebra, it is easy to see that the function

$$(z_1,z_2) \to f(x_1,y_1)-f(x_1,y_2) = (y_1-y_2)(\mathfrak{D}_y f)(z_1,z_2)$$

is a Schur multiplier. Similarly,

$$\iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1,z_2)dE_1(z_1)\,(A_1-A_2)dE_2(z_2) = \iint_{D\times D} (f(x_1,y_2)-f(x_2,y_2))dE_1(z_1)\,dE_2(z_2).$$

It follows that

$$\iint_{\mathbb{C}^2} (\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,(B_1-B_2)dE_2(z_2) + \iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1,z_2)dE_1(z_1)\,(A_1-A_2)dE_2(z_2)$$
$$= \iint_{D\times D} (f(x_1,y_1)-f(x_2,y_2))dE_1(z_1)\,dE_2(z_2)$$
$$= \iint_{D\times D} f(x_1,y_1)dE_1(z_1)\,dE_2(z_2) - \iint_{D\times D} f(x_2,y_2)dE_1(z_1)\,dE_2(z_2)$$
$$= f(N_1)-f(N_2).$$

Consider now the case when N_1 and N_2 are unbounded. Put

$$P_k \stackrel{\text{def}}{=} E_1(\{\zeta\in\mathbb{C}: |\zeta|\le k\}) \quad \text{and} \quad Q_k \stackrel{\text{def}}{=} E_2(\{\zeta\in\mathbb{C}: |\zeta|\le k\}), \quad k>0.$$

Then

$$N_{1,k} \stackrel{\text{def}}{=} P_k N_1 \quad \text{and} \quad N_{2,k} \stackrel{\text{def}}{=} Q_k N_2$$

are bounded normal operators. Denote by $E_{j,k}$ the spectral measure of $N_{j,k}$, $j=1,2$. It is easy to see that

$$N_{1,k} = P_k A_1 + iP_k B_1 \quad \text{and} \quad N_{2,k} = A_2 Q_k + iB_2 Q_k, \quad k>0.$$

We have

$$P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,(B_1-B_2)dE_2(z_2)\right)Q_k$$

$$= P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_y f)(z_1,z_2)dE_{1,k}(z_1)\,(P_kB_1-B_2Q_k)dE_{2,k}(z_2)\right)Q_k$$

and

$$P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1,z_2)dE_1(z_1)\,(A_1-A_2)dE_2(z_2)\right)Q_k$$

$$= P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1,z_2)dE_{1,k}(z_1)\,(P_kA_1-A_2Q_k)dE_{2,k}(z_2)\right)Q_k.$$

If we apply identity (28) to the bounded normal operators $N_{1,k}$ and $N_{2,k}$, we obtain

$$P_k\left(f(N_{1,k})-f(N_{2,k})\right)Q_k$$

$$= P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_y f)(z_1,z_2)dE_{1,k}(z_1)\,(P_kB_1-B_2Q_k)dE_{2,k}(z_2)\right)Q_k$$

$$+ P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1,z_2)dE_{1,k}(z_1)\,(P_kA_1-A_2Q_k)dE_{2,k}(z_2)\right)Q_k.$$

Since obviously,

$$P_k\left(f(N_{1,k})-f(N_{2,k})\right)Q_k = P_k\big(f(N_1)-f(N_2)\big)Q_k,$$

we have

$$P_k\big(f(N_1)-f(N_2)\big)Q_k$$

$$= P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_y f)(z_1,z_2)dE_1(z_1)\,(B_1-B_2)dE_2(z_2)\right)Q_k$$

$$+ P_k\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1,z_2)dE_1(z_1)\,(A_1-A_2)dE_2(z_2)\right)Q_k$$

It remains to pass to the limit in the strong operator topology.

We would like to extend formula (28) to the case of arbitrary functions f in $\boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$. Since $\boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$ consists of Lipschitz functions, it follows that for $f \in \boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$,

$$|f(\zeta)| \le \mathrm{const}(1+|\zeta|), \quad \zeta \in \mathbb{C}. \tag{29}$$

Hence, for $f \in \boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$,

$$D_{f(N)} \supset D_N.$$

Theorem (4.2.7)[144]. Let N_1 and N_2be normal operators such that $N_1 - N_2$ is bounded. Then (28) holds for every $f \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$

Proof. It suffices to prove that for $u \in D_{N_1} = D_{N_2}$,$(\mathrm{f(N_1) - f(\ N_2)})\mathrm{u} =$
$\left(\iint_{\mathbb{C}^2}(\mathfrak{D}_\mathrm{y}\mathrm{f})(\mathrm{z_1, z_2})\mathrm{dE_1(z_1)\ (B_1 - B_2)dE_2(z_2)}\right)\mathrm{u}$

$$+\left(\iint\limits_{\mathbb{C}^2}(\mathfrak{D}_\mathrm{x}\mathrm{f})(\mathrm{z_1, z_2})\mathrm{dE_1(z_1)\ (A_1 - A_2)dE_2(z_2)}\right)\mathrm{u}$$

Indeed, if N is a normal operator and f satisfies (29), then $f(N)$ is the closure of its restriction to the domain of N

We have

$$\left(f(N_1) - f(\ N_2)\right)u = \Big(\big(f - f(0)\big)(N_1)\Big)u - \Big(\big(f - f(0)\big)(N_2)\Big)u$$

$$\Big(\big(f - f(0)\big)(N_1)\Big)u = \sum_{n\in\mathbb{Z}}\Big(\big(f_n - f_n(0)\big)(N_1)\Big)u \qquad (30)$$

and

$$\Big(\big(f - f(0)\big)(N_2)\Big)u = \sum_{n\in\mathbb{Z}}\Big(\big(f_n - f_n(0)\big)(N_2)\Big)u \qquad (31)$$

where the functions f_n are defined by (6). Moreover, the series on the right-hand sides of (30) and (31) converge absolutely in the norm.

Thus

$$\left(f(N_1) - f(\ N_2)\right)u = \sum_{n\in\mathbb{Z}}\left(f_n(N_1) - f_n(N_2)\right)u$$

It remains to observe that

$$\iint\limits_{\mathbb{C}^2}\left(\mathfrak{D}_y f\right)(z_1, z_2)dE_1(z_1)\,(B_1 - B_2)dE_2(z_2)$$

$$= \sum_{n\in\mathbb{Z}}\iint\limits_{\mathbb{C}^2}\left(\mathfrak{D}_y f_n\right)(z_1, z_2)dE_1(z_1)\,(B_1 - B_2)dE_2(z_2)$$

and

$$\iint\limits_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1, z_2)dE_1(z_1)\,(A_1 - A_2)dE_2(z_2)$$

$$= \sum_{n\in\mathbb{Z}}\iint\limits_{\mathbb{C}^2}(\mathfrak{D}_x f_n)(z_1, z_2)dE_1(z_1)\,(A_1 - A_2)dE_2(z_2),$$

and the series on the right-hand sides converge absolutely in the norm which is an immediate consequence of inequalities (27).

We are going to prove Theorem(4.2.5) that gives sharp estimates for the norms of $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ in the space of Schur multipliers. Consider the function $\mathfrak{D}_x f$,

$$(\mathfrak{D}_x f)(z_1, z_2) = \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_x}, z_1, z_2 \in \mathbb{C}.$$

The first natural thought would be to fix the variable y_2 and represent the function

$$(x_1, x_2) \to \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_x}$$

in terms of the integral projective tensor product $L^\infty \widehat{\otimes}_i L^\infty$ in the same was as it was done in [153] for functions of one variable. However, it turns out that if we do this, we obtain in the integral tensor representation terms that depend on the mixed variables (x_1, y_2), and so this would not help us.

The first proof of Theorem(4.2.5) we have found was based on a modification of the integral tensor representation obtained in [153] and an estimate in terms of the tensor norm (23) rather than the integral projective tensor norm.

We give a different approach based on an expansion of entire functions of exponential type σ in the series in the orthogonal basis $\left\{\frac{\sin \sigma x}{\sigma x - \pi n}\right\}_{n\in\mathbb{Z}}$.

For a topological space $\mathcal{X}$, we denote by $C_b(\mathcal{X})$ the set of bounded continuous (complex) functions on $\mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$are topological spaces, we denote by $C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$ the set of functions Φ on $\mathcal{X} \times \mathcal{Y}$ that admit a representation

$$\Phi(x, y) = \sum_{n\geq 0} \varphi_n(x)\psi_n(y), \qquad (x, y) \in \mathcal{X} \times \mathcal{Y}. \tag{32}$$

such that $\varphi_n \in C_b(\mathcal{X}), \psi_n \in C_b(\mathcal{Y})$ and

$$\left(\sup_{x\in\mathcal{X}} \sum_{n\geq 0} |\varphi_n(x)|^2\right)^{1/2} \left(\sup_{y\in\mathcal{Y}} \sum_{n\geq 0} |\psi_n(y)|^2\right)^{1/2} < \infty. \tag{33}$$

For $\Phi \in C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$, its norm in $C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$ is, by definition, the infimum of the left-hand side of (33) over all representations (32).

For $\sigma > 0$, we denote by $\mathcal{E}_\sigma$ the set of entire functions (of one complex variable) of exponential type at most σ.

It follows from the results of [153]] that

$$f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R}) \implies \left\|\frac{f(x) - f(y)}{x - y}\right\|_{\mathfrak{M}(E_1,E_2)} \leq \text{ const } \sigma\|f\|_{L^\infty(\mathbb{R})} \tag{34}$$

for every Borel spectral measures E_1 and E_2 on $\mathbb{R}$.

It was shown in [155]that inequality (34) holds with constant equal to1.

The following result allows us to obtain an explicit representation of the divided difference $\frac{f(x)-f(y)}{x-y}$ as an element of $C_b(\mathbb{R}) \widehat{\otimes}_h C_b(\mathbb{R})$.

Theorem (4.2.9)[144]. Let $f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R})$. Then

$$\frac{f(x) - f(y)}{x - y} = \sum_{n\in\mathbb{Z}} (-1)^n\sigma \cdot \frac{f(x) - f(\pi n\sigma^{-1})}{\sigma x - \pi n} \cdot \frac{\sin \sigma y}{\sigma y - \pi n} \tag{35}$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x) - f(t)}{x - t} . \frac{\sin(\sigma(y - t))}{y - t} dt, \quad x, y \in \mathbb{R}. \tag{36}$$

Moreover,

$$\sum_{n\in\mathbb{Z}} \frac{|f(x) - f(\pi n\sigma^{-1})|^2}{(\sigma x - \pi n)^2} = \frac{1}{\pi\sigma}\int_{\mathbb{R}} \frac{|f(x) - f(t)|^2}{(x - t)^2} dt, \quad x \in \mathbb{R}. \tag{37}$$

and

$$\sum_{n\in\mathbb{Z}} \frac{\sin^2 \sigma y}{(\sigma x - \pi n)^2} = 1 = \frac{1}{\pi\sigma}\int_{\mathbb{R}} \frac{\sin^2(\sigma(y-t))}{(y-t)^2}dt, \quad y \in \mathbb{R}. \tag{38}$$

Proof. Clearly, it suffices to consider the case $\sigma = 1$. Let us first observe that the identities in (38) are elementary and well known.

We are going to use the well-known fact that the family $\left\{\frac{\sin \sigma x}{\sigma x - \pi n}\right\}_{n\in\mathbb{Z}}$ orthogonal basis in the space $\mathcal{E}_1 \cap L^2(\mathbb{R})$, forms an

$$F(z) = \sum_{n\in\mathbb{Z}} (-1)^n F(\pi n) \frac{\sin z}{z - \pi n}, \tag{39}$$

and

$$\sum_{n\in\mathbb{Z}} |F(\pi n)|^2 = \frac{1}{\pi}\int_{\mathbb{R}} |F(t)|^2 dt. \tag{40}$$

for every $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$, see, e.g., [170], Lect. 20.2,Th. 1. It follows immediately from (6.9) that

$$\sum_{n\in\mathbb{Z}} F(\pi n)\overline{G(\pi n)} = \frac{1}{\pi}\int_{\mathbb{R}} F(t)\overline{G(t)}dt. \text{ for every } F, G \in \mathcal{E}_1 \cap L^2(\mathbb{R}). \tag{41}$$

Given $x \in \mathbb{R}$, we consider the function F defined by $F(\lambda) = \frac{f(x)-f(\lambda)}{x-\lambda}$, $\lambda \in \mathbb{C}$. Clearly, $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$.

It is easy to see that (35) is a consequence of (39) and the equality in(37) is a consequence of (40). It is also easy to see that (36) follows from (41).

It remains to prove that

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{|f(x) - f(t)|^2}{(x-t)^2} dt \le 3\|f\|^2_{L^\infty(\mathbb{R})}$$

for every $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$ and $x \in \mathbb{R}$. Without loss of generality we may assume that $\|f\|_{L^\infty(\mathbb{R})} = 1$. Then $\|f'\|_{L^\infty(\mathbb{R})} \le 1$ by the Bernstein inequality. Hence, $|f(x) - f(t)| \le \min(2, |x-t|)$, and we have

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{|f(x) - f(t)|^2}{(x-t)^2} dt \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{\min(4, (x-t)^2)}{(x-t)^2} dt = \frac{2}{\pi}\int_0^2 dt + \frac{8}{\pi}\int_2^\infty \frac{dt}{t^2} = \frac{8}{\pi} < 3.$$

Theorem (4.2.9)[144]. Let $\sigma > 0$ and let f be a function in $C_b(\mathbb{R}^2)$ such that

$$\operatorname{supp} \mathcal{F}f \subset \{\zeta \in \mathbb{C}: |\zeta| \le \sigma\}.$$

Then $\mathfrak{D}_x f, \mathfrak{D}_y f \in C_b(\mathbb{C}) \widehat{\otimes}_h C_b(\mathbb{C})$,

$$\|\mathfrak{D}_x f\|_{C_b(\mathbb{C})\widehat{\otimes}_h C_b(\mathbb{C})} \le \sigma \|f\|_{L^\infty(\mathbb{C})}$$

and

$$\left\|\mathfrak{D}_y f\right\|_{C_b(\mathbb{C})\widehat{\otimes}_h C_b(\mathbb{C})} \le \sigma \|f\|_{L^\infty(\mathbb{C})}$$

Proof. Clearly, f is the restriction to $\mathbb{R}^2$of an entire function of two complex variables. Moreover, $f(\cdot, a), f(a, \cdot) \in \mathcal{E}_1 \cap L^2(\mathbb{R})$ for every $a \in \mathbb{R}$. It suffices to consider the case $\sigma = 1$. By Theorem(4.2.8), we have

$$(\mathfrak{D}_x f)(z_1, z_2) \stackrel{\text{def}}{=} \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2} = \sum_{n\in\mathbb{Z}} (-1)^n \frac{f(\pi n, y_2) - f(x_2, y_2)}{\pi n - x_2} \cdot \frac{\sin x_1}{x_1 - \pi n}$$

and

$$\left(\mathfrak{D}_y f\right)(z_1,z_2) \stackrel{\text{def}}{=} \frac{f(x_1,y_1)-f(x_1,y_2)}{y_1-y_2} = \sum_{n\in\mathbb{Z}}(-1)^n \frac{f(x_1,y_1)-f(x_1,\pi n)}{y_1-\pi n}\cdot\frac{\sin y_2}{y_2-\pi n}.$$

Note that the functions $\frac{\sin x_1}{x_1-\pi n}$ and $\frac{f(x_1,y_1)-f(x_1,\pi n)}{y_1-\pi n}$ depend on $z_1=(x_1,y_1)$ and do not depend on $z_2=(x_2,y_2)$ while the functions $\frac{f(\pi n,y_2)-f(x_2,y_2)}{\pi n-x_2}$ and $\frac{\sin y_2}{y_2-\pi n}$ depend on $z_2=(x_2,y_2)$ and do not depend on $z_1=(x_1,y_1)$. Moreover, by Theorem(4.2.8) we have

$$\sum_{n\in\mathbb{Z}}\frac{|f(x_1,y_1)-f(x_1,\pi n)|^2}{(y_1-\pi n)^2} \le 3\|f(x_1,\cdot)\|^2_{L^\infty(\mathbb{R})} \le 3\|f\|^2_{L^\infty(\mathbb{C})},$$

$$\sum_{n\in\mathbb{Z}}\frac{|f(\pi n,y_2)-f(x_2,y_2)|^2}{(\pi n-x_2)^2} \le 3\|f(\cdot,y_2)\|^2_{L^\infty(\mathbb{R})} \le 3\|f\|^2_{L^\infty(\mathbb{C})},$$

and

$$\sum_{n\in\mathbb{Z}}\frac{\sin^2 x_1}{(x_1-\pi n)^2} = \sum_{n\in\mathbb{Z}}\frac{\sin^2 y_2}{(y_2-\pi n)^2} = 1.$$

This implies the result.

We show that functions in the Besov space $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ are operator Lipschitz. We also show that if $f\in\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ then,

$$N_1-N_2\in\mathfrak{I} \quad\Longrightarrow\quad f(N_1)-f(N_2)\in\mathfrak{I},$$

whenever $\mathfrak{I}$ is a quasinormed operator ideal with majorization property. In particular, this is true if $\mathfrak{I}=\boldsymbol{S}_1$.

Recall that in the case $\mathfrak{I}=\boldsymbol{S}_1$ one cannot replace the Besov class $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ with the Lipschitz class. Indeed, even in the case of self-adjoint operators a Lipschitz function f on $\mathbb{R}$ does not possess the property

$$A-B\in\boldsymbol{S}_1 \quad\Rightarrow\quad f(A)-f(B)\in\boldsymbol{S}_1.$$

This was shown for the first time in [171]. Later necessary conditions were found in [146] and [153]] that also show that Lipschitzness is not sufficient.

The following lemma is an immediate consequence Theorems (4.2.5) and(4.2.6).

Lemma (4.2.10)[144]. Let f be a function in $C_b(\mathbb{R}^2)$ such that

$$\operatorname{supp}\mathcal{F}f\subset\{\zeta\in C: |\zeta|\le\sigma\},\quad \sigma>0.$$

If N_1 and N_2 are normal operators, then

$$\|f(N_1)-f(N_2)\|\le \text{const}\,\sigma\|f\|_{L^\infty}\|N_1-N_2\|.$$

Theorem(4.2.11)[144]. Let f belong to the Besov space $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ and let N_1 and N_2 be normal operators whose difference is a bounded operator. Then (28) holds and

$$\|f(N_1)-f(N_2)\|\le \text{const}\,\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_1-N_2\|.$$

Proof. It follows from Lemma (4.2.10) that

$$\|f(N_1)-f(N_2)\|\le\sum_{n\in\mathbb{Z}}\|f_n(N_1)-f_n(N_2)\|\le \text{const}\sum_{n\in\mathbb{Z}}2^n\|f\|_{L^\infty}\|N_1-N_2\|$$
$$\le \text{const}\,\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_1-N_2\|$$

In other words, functions in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ must be operator Lipschitz.

We can obtain similar results for operator ideals.

Lemma (4.2.12) [144]. Let $\mathfrak{I}$ be a quasinormed ideal of operators on Hilbert space that has majorization property and let f be a function in $C_b(\mathbb{R}^2)$ such that

$$\text{supp}\,\mathcal{F}f \subset \{\zeta \in C: |\zeta| \le \sigma\}, \quad \sigma > 0.$$

If N_1 and N_2 are normal operators such that $N_1 - N_2 \in \mathfrak{I}$, then

$$f(N_1) - f(N_2) \in \mathfrak{I} \text{ and } \|f(N_1) - f(N_2)\|_{\mathfrak{I}} \le c\sigma\|f\|_{L^\infty}\|N_1 - N_2\|_{\mathfrak{I}}$$

for a numerical constant c.

Theorem (4.2.14)[144]. Let $\mathfrak{I}$ be a quasinormed ideal of operators on Hilbert space that has majorizati3on property and let f belong to the Besov space $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$. If N_1 and N_2 are normal operators such that $N_1 - N_2 \in \mathfrak{I}$. Then $f(N_1) - f(N_2) \in \mathfrak{I}$ and

$$\|f(N_1) - f(N_2)\|_{\mathfrak{I}} \le c\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_1 - N_2\|_{\mathfrak{I}}$$

for a numerical constant c.

Proof. In the case where $\mathfrak{I}$ is a normed ideal the result is an immediate consequence of Lemma (4.2.12). In particular, Theorem(4.2.13) is true for $\mathfrak{I} = \boldsymbol{S}_1^l$. To complete the proof in the general case it suffices to use the majorization property.

Corollary(4.2.14)[144]. There exists a positive number c such that if $f \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ and let $N_1$and N_2 are normal operators such that $N_1 - N_2 \in \boldsymbol{S}_1$, then

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_1} \le \|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_1 - N_2\|_{\boldsymbol{S}_1}.$$

Recall that $\alpha \in (0,1)$, the class $\Lambda_\alpha(\mathbb{R}^2)$ of Hölder functions of order α is defined by:

$$\Lambda_\alpha(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{f: \|f\|_{\Lambda_\alpha(\mathbb{R}^2)} = \sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < \infty\right\}.$$

We show that in contrast with the class of Lipschitz functions, a Hölder function of order $\alpha \in (0,1)$ must be operator Hölder of order α.

We also consider the more general case of functions in the space $\Lambda_\omega(\mathbb{R}^2)$, where ω is an arbitrary modulus of continuity.

Theorem(4.2.15)[144]. There exists a positive number c such that for every $\alpha \in (0,1)$ and every $f \in \Lambda_\alpha(\mathbb{R}^2)$,

$$\|f(N_1) - f(N_2)\| \le c(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|N_1 - N_2\|^\alpha. \tag{42}$$

for arbitrary normal operators N_1 and N_2.

Proof. The proof is almost the same as the proof of Theorem 4.1 of [148] (see also Remark following Theorem 4.1 in [148]) for self-adjoint operators. All we need is the following:

$$\|f_n(N_1) - f_n(N_2)\| \le \text{const } 2^n\|f_n\|_{L^\infty}\|N_1 - N_2\|, \; n \in \mathbb{Z}, \tag{43}$$

and

$$\|f_n\|_{L^\infty} \le \text{const } 2^{-n\alpha}\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}, \; n \in \mathbb{Z}, \tag{44}$$

where the functions f_n are defined by (6). We remind that (43) is a consequence of Lemma(4.2.10), while (44) is a special case of Theorem(4.2.1).

The deduction of inequality (42) from (43) and (44) is exactly the same as in the proof of Theorem 4.1 of [148], in which inequality (42) for self-adjoint operators is deduced from the corresponding analogs of inequalities (43) and (44).

Consider now more general classes of functions. Let ω be a modulus of continuity. Recall that the class $\Lambda_\omega(\mathbb{R}^2)$is defined by

$$\Lambda_\omega(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{f: \|f\|_{\Lambda_\omega(\mathbb{R}^2)} = \sup_{z_1 \ne z_2} \frac{|f(z_1) - f(z_2)|}{\omega(|z_1 - z_2|)} < \infty\right\}.$$

As in the case of functions of one variable (see [147], [148]), we define the function ω_*

by

$$\omega_*(x) \overset{\text{def}}{=} x\int_x^\infty \frac{\omega(t)}{t^2}\,dt, \qquad x > 0. \tag{45}$$

Theorem (4.2.16)[144]. There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{R}^2)$,

$$\|f(N_1) - f(N_2)\| \le c\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega_*(\|N_1 - N_2\|) \tag{46}$$

for arbitrary normal operators N_1 and N_2 .

Proof. To prove Theorem(4.2.16), we need inequalities (43) and Theorem(4.2.1). The deduction of inequality (46) from (43) and Theorem(4.2.1) is exactly the same as it was done in the proof of Theorem 7.1 of [148] in the case of self-adjoint operators.

Corollary(4.2.17)[144]. Let ω be a modulus of continuity such that

$$\omega_*(x) \le \text{const}\,\omega(x), \quad x > 0,$$

and let $f \in \Lambda_\omega(\mathbb{R}^2)$. Then

$$\|f(N_1) - f(N_2)\| \le \text{const}\,\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega(\|N_1 - N_2\|)$$

for arbitrary normal operators N_1 and N_2 .

Theorem(4.2.16) allows us to estimate $\|f(N_1) - f(N_2)\|$ for Lipschitz functions f and normal operators N_1 and N_2 whose spectra are contained in a given compact convex subset of $\mathbb{C}$.

For a Lipschitz function f on a subset K of $\mathbb{C}$, the Lipschitz constant is, by definition,

$$\|f\|_{\text{Lip}} \overset{\text{def}}{=} \sup\left\{\frac{|f(\zeta_1) - f(\zeta_2)|}{|\zeta_1 - \zeta_2|} : \zeta_1, \zeta_2 \in K, \zeta_1 \ne \zeta_2\right\}.$$

For a Lipschitz function f on a compact convex subset K of $\mathbb{C}$, we extend it to $\mathbb{C}$ by the formula

$$f(\zeta) \overset{\text{def}}{=} f(\zeta_\#), \tag{47}$$

where $\zeta_\#$ is the closest point to ζ in K . It is easy to see that the Lipschitz constant of this extension does not change.

Theorem(4.2.18) [144]. Let N_1 and N_2 be normal operators whose spectra are contained in a compact convex set K and let f be a Lipschitz function on. Then

$$\|f(N_1) - f(N_2)\| \le \text{const}\,\|f\|_{\text{Lip}}\|N_1 - N_2\|\left(1 + \log\frac{d}{\|N_1 - N_2\|}\right), \tag{48}$$

where d is the diameter of K .

Proof. Without loss of generality, we may assume that $\|f\|_{\text{Lip}} = 1$. Let us extend f to C by formula (47). Define the modulus of continuity ω by

$$\omega(\delta) = \begin{cases} \delta, & \delta \le d, \\ d, & \delta > d. \end{cases}$$

Clearly, $f \in \Lambda_\omega(\mathbb{R})$ and $\|f\|_{\Lambda_\omega(\mathbb{R})} \le \|f\|_{\text{Lip}}$. We have

$$\omega_*(\delta) = \delta\int_\delta^d \frac{dt}{t} + \delta d\int_d^\infty \frac{dt}{t^2} = \delta\log\frac{d}{\delta} + \delta, \quad \delta \le d,$$

where ω_* is defined by (45). Now inequality (48) follows immediately from Theorem(4.2.16).

We obtain sharp estimates for $f(N_1) - f(N_2)$ in the case when $f \in \Lambda_\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, and N_1 and N_2 are normal operators such whose difference belong to Schatten–

von Neumann classes $\boldsymbol{S}_p$. We also obtain more general results in the case when the difference of the normal operators belongs to operator ideals.

Let us first state the result for Schatten–von Neumann classes.

Theorem(4.2.19)[144]. Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2 with $N_1 - N_2 \in \boldsymbol{S}_p$, the operator $f(N_1) - f(N_2)$ belongs to $\boldsymbol{S}_{p/\alpha}$ and the following inequality holds:

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_{p/\alpha}} \le c\|f\|_{\Lambda_\alpha(\mathbb{R})}\|N_1 - N_2\|_{\boldsymbol{S}_p}^{\alpha}.$$

We discuss the case $p = 1$ after the proof of Theorem(4.2.21).

Theorem9.1 is an immediate consequence of a more general result for operator ideals, see Theorem(4.2.25) below.

To proceed to operator ideals, we start with the ideals $\boldsymbol{S}_p^l$. Recall that for $l \ge 0$ and $p \ge 1$, the normed ideal $\boldsymbol{S}_p^l$ consists of all bounded linear operators equipped with the norm

$$\|T\|_{\boldsymbol{S}_p^l} \stackrel{\text{def}}{=} \left(\sum_{j=0}^{l}\big(s_j(T)\big)^p\right)^{1/p}.$$

Theorem(4.2.20)[144]. Let $0 < \alpha < 1$. Then there exists a positive number $c > 0$ such that for every $l \ge 0$,$p \in [1,\infty)$,$f \in \Lambda_\alpha(\mathbb{R}^2)$, and for arbitrary normal operators N_1 and N_2on Hilbert space with bounded $N_1 - N_2$, the following inequality holds:

$$s_j(f(N_1) - f(N_2)) \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}(1 + j)^{-\alpha/p}\|N_1 - N_2\|_{\boldsymbol{S}_p}^{\alpha}$$

for every $j \le l$.

Proof. The proof is almost the same as the proof of Theorem(4.2.5) of [149]. To be able to apply the reasonings given in the proof of Theorem(4.2.5) of [149], we need inequality (44) and the following inequality:

$$\|f_n(N_1) - f_n(N_2)\|_{\boldsymbol{S}_p^l} \le \text{const}\, 2^n\|f_n\|_{L^\infty}\|N_1 - N_2\|_{\boldsymbol{S}_p^l}, \quad n \in \mathbb{Z}, \tag{49}$$

where the functions f_n are defined by (6). Inequality (49) is an immediate consequence of Lemma (4.2.12). All the details can be found in the proof of Theorem(4.2.5) of [149].

Theorem (4.2.21)[144]. Let $0 < \alpha < 1$. Then there exists a positive number $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ and arbitrary normal operators N_1 and N_2 on Hilbert space with $N_1 - N_2 \in \boldsymbol{S}_1$, the operator $f(N_1) - f(N_2)$ belongs to $\boldsymbol{S}_{\frac{1}{\alpha},\infty}$ and the following inequality holds:

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_{\frac{1}{\alpha},\infty}} \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|N_1 - N_2\|_{\boldsymbol{S}_1}^{\alpha}.$$

Proof. As in the case of self-adjoint operators (see Theorem 5.2 of [149]), this is an immediate consequence of Theorem(4.2.20) in the case $p = 1$.

Note that the assumptions of Theorem(4.2.21) do not imply that $f(N_1) - f(N_2) \in \boldsymbol{S}_{1/\alpha}$. This is not true even in the case when N_1 and N_2 are self-adjoint operators. This was proved in [149]. Moreover, in [149] a necessary condition on the function f on $\mathbb{R}$ was found for

$$f(A) - f(B) \in \boldsymbol{S}_{1/\alpha}, \quad \text{whenever} \quad A = A^*, B = B^* \quad \text{and} \quad A - B \in \boldsymbol{S}_1.$$

That necessary condition is based on the $\boldsymbol{S}_p$ criterion for Hankel operators ([145] and [172], Ch. 6) and shows that the condition $f \in \Lambda_\alpha(\mathbb{R})$ is not sufficient.

The following result ensures that the assumption that $N_1 - N_2 \in \boldsymbol{S}_1$ for normal operators N_1 and N_2 implies that $f(A) - f(B) \in \boldsymbol{S}_{1/\alpha}$ under a slightly more restrictive assumption on.

Theorem(4.2.22)[144]. Let $0 < \alpha \leq 1$. Then there exists a positive number $c > 0$ such that for every $f \in \boldsymbol{B}^\alpha_{\infty 1}(\mathbb{R}^2)$ and arbitrary normal operators N_1 and N_2on Hilbert space with $N_1 - N_2 \in \boldsymbol{S}_1$, the operator $f(N_1) - f(N_2)$ belongs to $\boldsymbol{S}_{1/\alpha}$ and the following inequality holds:

$$\|f(N_1) - f(N_2)\|_{\boldsymbol{S}_{1/\alpha}} \leq \|f\|_{\boldsymbol{B}^\alpha_{\infty 1}(\mathbb{R}^2)} \| N_1 - N_2\|^\alpha_{\boldsymbol{S}_1}.$$

Note that in the case $\alpha = 1$ turns into Corollary (4.2.15).

Proof. Again, if we apply Lemma (4.2.10), the proof is practically the same as the proof of Theorem(4.2.7) in [149].

Theorem (4.2.23). Let $0 < \alpha < 1$. Then there exists a positive number $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$ and arbitrary normal operators $N_1$and N_1on Hilbert space with bounded $N_1 - N_2$, the following inequality holds:

$$s_j(|f(N_1) - f(N_2)|^{1/\alpha}) \leq c\|f\|^{1/\alpha}_{\Lambda_\alpha(\mathbb{R}^2)} \sigma_j\,(N_1 - N_2), \quad j \geq 0.$$

Recall that the numbers $\sigma_j\ (N_1 - N_2)$ defined by (11).

Proof. As in the case of self-adjoint operators (see [149]), it suffices to apply Theorem (4.2.20) with l = j and p = 1.

Now we are in a position to obtain a general result in the case $f \in \Lambda_\alpha(\mathbb{R}^2)$ and $N_1 - N_2 \in \mathfrak{I}$ for an arbitrary quasinormed ideal $\mathfrak{I}$ with upper Boyd index less than 1.

Theorem(4.2.24). Let $0 < \alpha < 1$. Then there exists a positive number $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$, for an arbitrary quasinormed ideal $\mathfrak{I}$ with $\beta_{\mathfrak{I}} < 1$, and for arbitrary normal operators N_1 and N_2 on Hilbert space with $N_1 - N_2 \in \mathfrak{I}$, the operator $|f(N_1) - f(N_2)|^{1/\alpha}$ belongs to $\mathfrak{I}$ and the following inequality holds:

$$\left\||f(N_1) - f(N_2)|^{1/\alpha}\right\|_{\mathfrak{I}} \leq cC_{\mathfrak{I}}\|f\|^{1/\alpha}_{\Lambda_\alpha(\mathbb{R}^2)} \| N_1 - N_2\|_{\mathfrak{I}}.$$

Proof. The proof is almost the same as the proof of Theorem 5.5 in [149].

We can reformulate Theorem(4.2.24) in the following way.

Theorem(4.2.25). Under the hypothesis of Theorem(4.2.24), the operator $f(N_1) - f(N_2)$ belongs to $\mathfrak{I}^{\{1/\alpha\}}$ and

$$\|f(N_1) - f(N_2)\|_{\mathfrak{I}^{\{1/\alpha\}}} \leq c^\alpha C^\alpha_{\mathfrak{I}}\|f\|_{\Lambda_\alpha(\mathbb{R}^2)} \| N_1 - N_2\|^\alpha_{\mathfrak{I}}.$$

The following result is a consequence of Theorem(4.2.24).

Theorem(4.2.26). Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$, every $l \in \mathbb{Z}_+$, and arbitrary normal operators N_1 and N_2 with bounded $N_1 - N_2$, the following inequality holds:

$$\sum_{j=0}^{l} \left(s_j(|f(N_1) - f(N_2)|^{1/\alpha})\right)^p \leq c\|f\|^{p/\alpha}_{\Lambda_\alpha(\mathbb{R}^2)} \sum_{j=0}^{l} \left(s_j(\, N_1 - N_2)\right)^p.$$

Proof. As in the case of self-adjoint operators (see [149]), the result immediately follows from Theorem(4.2.24) from (16).

We obtain estimates for quasicommutators $f(N_1)R - Rf(N_2)$, where N_1 and N_2 are normal operators and R is a bounded linear operator. In the special case when $R = I$ we arrive at the problem of estimating $f(N_1) - f(N_1)$ that we have discussed above. On the other hand , in the special case when $N_1 = N_2$ we have the problem of estimating commutators $f(N)R - Rf(N)$.

It turns out, however, that it is impossible to obtain estimates of $\| f(N_1)R - Rf(N_1)\|$ in terms of $\|N_1R - RN_2\|$. This cannot be done even for the function $f(z) = \bar{z}$.

Though the well-known Fuglede-Putnam theorem says that the equality $N_1R = RN_2$ for a bounded operator R and normal operators N_1 and N_2 implies that $N_1^*R = RN_2^*$ the smallness of $N_1R - RN_2$ does not imply the smallness of $N_1^*R - RN_2^*$.

Indeed, it follows from Corollary 4.3 of [169] that for every $\varepsilon > 0$ there exists a bounded normal operator N and operator R of norm 1 such that

$$\|NR - RN\| < \varepsilon \quad \text{but} \quad \|N^*R - RN^*\| \geq 1.$$

The results of [169] also imply that if $f \in C(\mathbb{C})$ and

$$\|f(N)Q - Qf(N)\| \leq \text{ const } \|NQ - QN\|$$

for all bounded operators Q and bounded normal operators N , then f is a linear function, i.e., $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

We obtain estimates for quasicommutators $f(N_1)R - Rf(N_2)$ in terms of the quasicommutators $N_1R - RN_2$ and $N_1^*R - RN_2^*$.

Let us explain what we mean by the boundedness of $N_1R - RN_2$ for not necessarily bounded normal operators N_1 and N_2.

We say that the operator $N_1R - RN_2$ is bounded if $R(\mathfrak{D}_{N_2}) \subset \mathfrak{D}_{N_1}$ and

$$\|N_1Ru - RN_2u\| \leq \text{const } \|u\| \text{ for every } u \in \mathfrak{D}_{N_2}.$$

Then there exists a unique bounded operator K such that $Ku = N_1Ru - RN_2u$ for all $u \in \mathfrak{D}_{N_2}$. In this case we write $K = N_1R - RN_2$. Thus $N_1R - RN_2$ is bounded if and only if

$$|(Ru, N_1^*v) - (N_2^*u, R^*v)| \leq \text{const } \|u\| \cdot \|v\| \tag{50}$$

For every $u \in \mathfrak{D}_{N_2}$and $v \in \mathfrak{D}_{N_1^*} = \mathfrak{D}_{N_1}$. It is easy to see that $N_1R - RN_2$ is bounded if and only if $N_1^*R^* - R^*N_2^*$ is bounded, and $(N_1R - RN_2)^* = -(N_1^*R^* - R^*N_2^*)$. In particular, we write $N_1R = RN_2$ if $R(\mathfrak{D}_{N_2}) \subset \mathfrak{D}_{N_1}$ and $N_1Ru = RN_2u$ for every $u \in \mathfrak{D}_{N_2}$. We say that $\|N_1R - RN_2\| = \infty$ if $N_1R = RN_2$ is not a bounded operator.

Theorem (4.2.27). Let f be a function in $C_b(\mathbb{R}^2)$ whose Fourier transform $\mathcal{F}f$ has compact support. Suppose that R is a bounded linear operator, N_1 and $N_2$are normal operators such that the operators $N_1R - RN_2$ and $N_1^*R^* - R^*N_2^*$ are bounded. Then

$$\begin{aligned} f(N_1) - f(N_2) &= \iint_{\mathbb{C}^2} (\mathfrak{D}_y f)(z_1, z_2)dE_1(z_1)(B_1R - RB_2)dE_2(z_2) \\ &+ \iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1, z_2)dE_1(z_1)(A_1R - RA_2)dE_2(z_2) \end{aligned} \tag{51}$$

Proof. The proof is similar to the proof of Theorem(4.2.6), Consider first the case when N_1 and N_2 are bounded operators. Put

$$d = \max\{\|N_1\|, \|N_2\|\} \quad \text{and} \quad D \overset{\text{def}}{=} \{\zeta \in \mathbb{C}: |\zeta| \le d\}.$$

By Theorem(4.2.5), both $\mathfrak{D}_y f$ and $\mathfrak{D}_x f$ are Schur multipliers. We have $\iint_{\mathbb{C}^2} (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1)(B_1 R - RB_2) dE_2(z_2) = \iint_{D\times D} (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1)(B_1 R - RB_2) dE_2(z_2) = \iint_{D\times D} (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1) B_1 R dE_2(z_2) - \iint_{D\times D} (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1) RB_2 dE_2(z_2) = \iint_{D\times D} y_1 (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1) R dE_2(z_2) - \iint_{D\times D} y_2 (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1) R dE_2(z_2) = \iint_{D\times D} (y_1 - y_2)(\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1) R dE_2(z_2) = \iint_{D\times D} \big(f(x_1, y_1) - f(x_1, y_2)\big) dE_1(z_1) R dE_2(z_2).$

Similarly,

$$\iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1, z_2) dE_1(z_1)(A_1 R - RA_2) dE_2(z_2) = \iint_{D\times D} \big(f(x_1, y_2) - f(x_2, y_2)\big) dE_1(z_1) R dE_2(z_2).$$

It follows that

$$\iint_{\mathbb{C}^2} (\mathfrak{D}_y f)(z_1, z_2) dE_1(z_1)(B_1 R - RB_2) dE_2(z_2) + \iint_{\mathbb{C}^2} (\mathfrak{D}_x f)(z_1, z_2) dE_1(z_1)(A_1 R - RA_2) dE_2(z_2) = \iint_{D\times D} \big(f(x_1, y_1) - f(x_2, y_2)\big) dE_1(z_1) R dE_2(z_2)$$

$$= \iint_{D\times D} f(x_1, y_1) dE_1(z_1) R dE_2(z_2) - \iint_{D\times D} f(x_2, y_2) dE_1(z_1) R dE_2(z_2) = f(N_1)R - Rf(N_2).$$

In the general case we use the same approximation procedure as in the proof of Theorem(4.2.6).

As in the case of differences $f(N_1) - f(N_2)$, we can extend Theorem 10.1 to functions f in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$.

Theorem(4.2.28)[144]. Let N_1 and N_2 be normal operators and let R be a bounded linear operator such that the quasicommutators $N_1 R - RN_2$ and $N_1^* R - RN_2^*$ are bounded. Then (51) holds for every $f \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$.

Theorem (4.2.29)[144]. There exists a positive number c such that for every normal operators N_1 and N_2, every bounded linear operator R and an arbitrary function f in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ the following inequality holds:

$$\|f(N_1)R - Rf(N_2)\| \le c\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)} \max\{\| N_1 R - RN_2\|_{\boldsymbol{S}}, \| N_1^* R - RN_2^*\|\}.$$

Theorem(4.2.30)[144]. Let $0 < \alpha < 1$. Then there exists $c > 0$ such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$, for arbitrary normal operators N_1 and N_2 and a bounded operator R the following inequality holds:

$$\|f(N_1)R - Rf(N_2)\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)} \max\{\| N_1 R - RN_2\|_{\boldsymbol{S}}, \| N_1^* R - RN_2^*\|\}^\alpha \|R\|^{1-\alpha}.$$

Theorem (4.2.31)[144]. There exists $c > 0$ such that for every modulus of continuity ω, for every $f \in \Lambda_\omega(\mathbb{R}^2)$, for arbitrary normal operators N_1 and N_2, and a bounded nonzero operator R the following inequality holds:

$$\|f(N_1)R - Rf(N_2)\| \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\|R\|\omega * \left(\frac{\max\{\| N_1R - RN_2\|_{\boldsymbol{S}}, \| N_1^*R - RN_2^*\|\}}{\|R\|}\right).$$

The next result shows that in the case $N_1R - RN_2 \in \boldsymbol{S}_p$, $1 < p < \infty$, and $f \in \Lambda_\alpha(\mathbb{R}^2)$, $0 < \alpha < 1$, we can estimate $\|f(N_1)R - Rf(N_2)\|_{\boldsymbol{S}_{p/\alpha}}$ in terms of $\| N_1R - RN_2\|_{\boldsymbol{S}_p}$, we do not need $\| N_1^*R - RN_2^*\|_{\boldsymbol{S}_p}$.

Theorem(4.2.32)[144]. Let $0 < \alpha < 1$ and $1 < p < \infty$. Then there exists a positive number c such that for every $f \in \Lambda_\alpha(\mathbb{R}^2)$, for arbitrary normal operators N_1 and N_2 and a bounded operator R with $N_1R - RN_2 \in \boldsymbol{S}_p$ and $N_1^*R - RN_2^* \in \boldsymbol{S}_p$, the operator $f(N_1)R - Rf(N_2)$ belongs to $\boldsymbol{S}_{p/\alpha}$ and the following inequality holds:

$$\|f(N_1)R - Rf(N_2)\|_{\boldsymbol{S}_{p/\alpha}} \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)}\| N_1R - RN_2\|_{\boldsymbol{S}_p}^{\alpha}.$$

Proof. In the same way as in the proof of Theorem 9.1, we can prove that

$$\|f(N_1)R - Rf(N_2)\|_{\boldsymbol{S}_{p/\alpha}} \le c\|f\|_{\Lambda_\alpha(\mathbb{R}^2)} \max\left\{\| N_1R - RN_2\|_{\boldsymbol{S}_p}, \| N_1^*R - RN_2^*\|_{\boldsymbol{S}_p}\right\}^{\alpha}.$$

The result follows from the well-known inequality:

$$\| N_1^*R - RN_2^*\|_{\boldsymbol{S}_p} \le \text{const} \| N_1R - RN_2\|_{\boldsymbol{S}_p}, \quad 1 < p < \infty, \tag{52}$$

see [173] and [174].

Inequality (52) does not hold for $p = 1$, see [175]. Thus to obtain analogs of Theorems (4.2.22) and (4.2.23), we have to estimate the quasicommutators $f(N_1)R - Rf(N_2)$ in terms of both $N_1R - RN_2$ and $N_1^*R - RN_2^*$. Let us state e.g., the analog of Theorem(4.2.23).

Theorem(4.2.33)[144]. Let $0 < \alpha < 1$. Then there exists a positive number c such that for every $f \in \boldsymbol{B}_{\infty 1}^{\alpha}(\mathbb{R}^2)$, for arbitrary normal operators N_1 and N_2 and a bounded operator R with $N_1R - RN_2 \in \boldsymbol{S}_1$ and $N_1^*R - RN_2^* \in \boldsymbol{S}_1$, the operator $f(N_1)R - Rf(N_2)$ belongs to $\boldsymbol{S}_{1/\alpha}$ and the following inequality holds:

$$\|f(N_1)R - Rf(N_2)\|_{\boldsymbol{S}_{1/\alpha}} \le c\|f\|_{\boldsymbol{B}_{\infty 1}^{\alpha}(\mathbb{R}^2)} \max\left\{\| N_1R - RN_2\|_{\boldsymbol{S}_1}, \| N_1^*R - RN_2^*\|_{\boldsymbol{S}_1}\right\}^{\alpha}.$$

The proof is almost the same as the proof of Theorem (4.2.22).

Corollary (4.2.34) [293]: There exists a constant $c > 0$ such that for an arbitrary modulus of continuity ω and for an arbitrary function f in $\Lambda_\omega(\mathbb{R}^2)$, the following inequality holds:

$$\|f - f * V_n\|_{L^\infty} \le c\omega(2^{-n})\|f\|_{\Lambda_\omega(\mathbb{R}^2)}, \quad n \in \mathbb{Z}. \tag{53}$$

Proof. We have

$$|f(x_{r-1}) - (f * V_n)(x_{r-1})| = 2^{2n}\left|\int_{\mathbb{R}^2}\big(f(x_{r-1}) - f(x_{r-1} - y_{r-1})\big)V(2^n y_{r-1})dy_{r-1}\right|$$

$$\le 2^{2n}\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\int_{\mathbb{R}^2}\omega(|y_{r-1}|)\,|V(2^n y_{r-1})|dy_{r-1}$$

$$= 2^{2n}\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\int_{\{|y_{r-1}|\le 2^{-n}\}}\omega(|y_{r-1}|)\,|V(2^n y_{r-1})|dy_{r-1}$$

$$+ 2^{2n}\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\int_{\{|y_{r-1}|> 2^{-n}\}}\omega(|y_{r-1}|)\,|V(2^n y_{r-1})|dy_{r-1}$$

Clearly,

$$2^{2n}\int\limits_{\{|y_{r-1}|\le 2^{-n}\}} \omega(|y_{r-1}|)\,|V(2^n y_{r-1})|dy_{r-1} \le \omega(2^{-n})\|V\|_{L^1}.$$

On the other hand, keeping in mind the obvious inequality $2^{-n}\omega(|y_{r-1}|) \le 2|y_{r-1}|\omega(2^{-n})$ for $|y_{r-1}| \ge 2^{-n}$, we obtain

$$2^{2n}\int\limits_{\{|y_{r-1}|>2^{-n}\}} \omega(|y_{r-1}|)\,|V(2^n y_{r-1})|dy_{r-1}$$

$$\le 2\cdot 2^{3n}\omega(2^{-n})\int\limits_{\{|y_{r-1}|>2^{-n}\}} |y_{r-1}|\,|V(2^n y_{r-1})|dy_{r-1}$$

$$= 2\omega(2^n)\int\limits_{\{|y_{r-1}|>1\}} |y_{r-1}|\cdot|V(y_{r-1})|dy_{r-1} \le \text{const}\,\omega(2^{-n}).$$

This proves (53).

Corollary (4.2.35) [293]: Let f be a continuous bounded function on $\mathbb{R}^2$ whose Fourier transform $\mathcal{F}f$ has compact support. Then the functions $\mathfrak{D}_{x_{r-1}}f$ and $\mathfrak{D}_{y_{r-1}}f$ are Schur multipliers with respect to arbitrary Borel spectral measures E_r and E_{r+1}.

Moreover, if

$$\operatorname{supp}\mathcal{F}f \subset \{\zeta_{r-1} \in \mathbb{C} : |\zeta_{r-1}| \le \sigma\}, \qquad \sigma > 0,$$

then

$$\left\|\mathfrak{D}_{x_{r-1}}f\right\|_{\mathfrak{M}(E_r,E_{r+1})} \le \text{const}\,\sigma\|f\|_{L^\infty} \text{and} \left\|\mathfrak{D}_{y_{r-1}}f\right\|_{\mathfrak{M}(E_r,E_{r+1})} \le \text{const}\,\sigma\|f\|_{L^\infty} \qquad (54).$$

Proof :

We are going to show Theorem 5.1 proved by [144] that gives sharp estimates for the norms of $\mathfrak{D}_{x_{r-1}}f$ and $\mathfrak{D}_{y_{r-1}}f$ in the space of Schur multipliers. Consider the function $\mathfrak{D}_{x_{r-1}}f$,

$$\left(\mathfrak{D}_{x_{r-1}}f\right)(z_r, z_{r+1}) = \frac{f(x_r, y_{r+1}) - f(x_{r+1}, y_{r+1})}{x_r - x_{r+1}}, z_r, z_{r+1} \in \mathbb{C}.$$

The first natural thought would be to fix the variable y_2 and represent the function

$$(x_r, y_{r+1}) \to \frac{f(x_r, y_{r+1}) - f(x_{r+1}, y_{r+1})}{x_r - x_{r+1}}$$

in terms of the integral projective tensor product $L^\infty \widehat{\otimes}_i L^\infty$ in the same was as it was done in [153] for functions of one variable. However, it turns out that if we do this, we obtain in the integral tensor representation terms that depend on the mixed variables (x_r, y_{r+1}), and so this would not help us.

The first proof of Corollary (4.2.36) we have found was based on a modification of the integral tensor representation obtained in [153] and an estimate in terms of the tensor norm (4.6) rather than the integral projective tensor norm.

We give a different approach based on an expansion of entire functions of exponential type σ in the series in the orthogonal basis $\left\{\frac{\sin\sigma x_{r-1}}{\sigma x_{r-1}-\pi n}\right\}_{n\in\mathbb{Z}}$.

For a topological space $\mathcal{X}$, we denote by $C_b(\mathcal{X})$ the set of bounded continuous (complex) functions on $\mathcal{X}$. If $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, we denote by $C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$ the set of functions Φ on $\mathcal{X} \times \mathcal{Y}$ that admit a representation

$$\Phi(x_{r-1}, y_{r-1}) = \sum_{n\geq 0} \varphi_n(x_{r-1})\psi_n(y_{r-1}), \qquad (x_{r-1}, y_{r-1}) \in \mathcal{X} \times \mathcal{Y}. \tag{55}$$

such that $\varphi_n \in C_b(\mathcal{X})$, $\psi_n \in C_b(\mathcal{Y})$ and

$$\left(\sup_{x_{r-1}\in\mathcal{X}} \sum_{n\geq 0} |\varphi_n(x_{r-1})|^2\right)^{1/2} \left(\sup_{y_{r-1}\in\mathcal{Y}} \sum_{n\geq 0} |\psi_n(y_{r-1})|^2\right)^{1/2} < \infty. \tag{56}$$

For$\Phi \in C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$, its norm in $C_b(\mathcal{X}) \widehat{\otimes}_h C_b(\mathcal{Y})$ is, by definition, the infimum of the left-hand side of (56) over all representations (55).

For $\sigma > 0$, we denote by $\mathcal{E}_\sigma$ the set of entire functions (of one complex variable) of exponential type at mostσ.

It follows from the results of [153] that

$$f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R}) \Longrightarrow \left\| \frac{f(x_{r-1}) - f(y_{r-1})}{x_{r-1} - y_{r-1}} \right\|_{\mathfrak{M}(E_r, E_{r+1})} \leq \text{const}\sigma \|f\|_{L^\infty(\mathbb{R})} \tag{57}$$

for every Borel spectral measures E_r and E_{r+1} on $\mathbb{R}$.

It was shown in [150] that inequality (57) holds with constant equal to1.

Corollary (4.2.36) [293]: Let f be a continuous bounded function on $\mathbb{R}^2$ whose Fourier transform $\mathcal{F}f$ has compact support. Suppose that N_r and N_{r+1} are extended normal operators such that the operator $N_r - N_{r+1}$ is bounded. Then

$$f(N_r) - f(N_{r+1}) = \iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1})$$
$$+ \iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left(A_1 - A_2\right) dE_{r+1}(z_{r+1}) \tag{58}$$

We postpone the proof of Corollary (4.2.36)till the next section. Let us deduce here Corollary (4.2.37) from Corollary (4.2.36). (see [144]).

Proof : Consider first the case when N_r and N_{r+1} are bounded operators. Put $d = \max\{\|N_r\|, \|N_{r+1}\|\}$ and $D \stackrel{\text{def}}{=} \{\zeta_{r-1} \in \mathbb{C}: |\zeta_{r-1}| \leq d\}$.

By Theorem 5.1, both $\mathfrak{D}_{y_{r-1}} f$ and $\mathfrak{D}_{x_{r-1}} f$ are Schur multipliers. We have

$$\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A+\epsilon)_1 dE_{r+1}(z_{r+1})$$

$$- \iint_{D\times D} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A+\epsilon)_2 dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} y_{r-1} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)\, dE_{r+1}(z_{r+1})$$

$$- \iint_{D\times D} y_{r+1} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)\, dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} (y_r - y_{r+1}) \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)\, dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} \left(f(x_r, y_{r+1}) - f(x_r, y_{r+1})\right) dE_r(z_r)\, dE_{r+1}(z_{r+1}).$$

Since $\mathfrak{M}(E_r, E_{r+1})$ is a Banach algebra, it is easy to see that the function

$$(z_r, z_{r+1}) \to f(x_r, y_r) - f(x_r, y_{r+1}) = (y_r - y_{r-1})\left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1})$$

is a Schur multiplier. Similarly,

$$\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A_1 - A_2) dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} \left(f(x_r, y_{r+1}) - f(x_{r+1}, y_{r+1})\right) dE_r(z_r)\, dE_{r+1}(z_{r+1}).$$

It follows that

$$\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1})$$

$$+ \iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A_1 - A_2) dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} \left(f(x_r, y_r) - f(x_{r+1}, y_{r+1})\right) dE_r(z_r)\, dE_{r+1}(z_{r+1})$$

$$= \iint_{D\times D} f(x_r, y_r) dE_r(z_r)\, dE_{r+1}(z_{r+1}) - \iint_{D\times D} f(x_{r+1}, y_{r+1}) dE_r(z_r)\, dE_{r+1}(z_{r+1})$$

$$= f(N_r) - f(N_{r+1}).$$

Consider now the case when N_r and $N_{r+1}$are unbounded. Put

$P_k \stackrel{\text{def}}{=} E_r(\{\zeta_{r-1} \in \mathbb{C}: |\zeta_{r-1}| \le k\})andQ_k \stackrel{\text{def}}{=} E_{r+1}(\{\zeta_{r-1} \in \mathbb{C}: |\zeta_{r-1}| \le k\})$, $k > 0$.

Then

$N_{r,k} \stackrel{\text{def}}{=} P_k N_r$ and $N_{r+1,k} \stackrel{\text{def}}{=} Q_k N_{r+1}$

are bounded extended normal operators. Denote by $(E_{r-1})_{j,k}$ the spectral measure

of $(N_{r-1})_{j,k}$, $j = \mu, \mu + 1$. It is easy to see that
$N_{r,k} = P_k A_1 + iP_k(A+\epsilon)_1$ and $N_{r+1,k} = A_2 Q_k + i(A+\epsilon)_2 Q_k$, $k > 0$.
We have
$P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1}) \right) Q_k =$
$P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_{r,k}(z_r) \left(P_k(A+\epsilon)_1 - (A+\epsilon)_2 Q_k\right) dE_{r+1,k}(z_{r+1}) \right) Q_k$ and

$$P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A_1 - A_2) dE_{r+1}(z_{r+1}) \right) Q_k$$

$$= P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_{r,k}(z_r) (P_k A_1 - A_2 Q_k) dE_{r+1,k}(z_{r+1}) \right) Q_k.$$

If we apply identity (58) to the bounded extended normal operators $N_{r,k}$ and $N_{r+1,k}$, we obtain

$$P_k \left(f(N_{r,k}) - f(N_{r+1,k}) \right) Q_k$$

$$= P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_{r,k}(z_r) \left(P_k(A+\epsilon)_1 - (A+\epsilon)_2 Q_k\right) dE_{r+1,k}(z_{r+1}) \right) Q_k$$

$$+ P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_{r,k}(z_r) (P_k A_1 - A_2 Q_k) dE_{r+1,k}(z_{r+1}) \right) Q_k.$$

Since obviously,

$$P_k \left(f(N_{r,k}) - f(N_{r+1,k}) \right) Q_k = P_k\left(f(N_r) - f(N_{r+1})\right) Q_k,$$

we have

$$P_k\left(f(N_r) - f(N_{r+1})\right) Q_k$$

$$= P_k \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) \left((A+\epsilon)_1 - (A+\epsilon)_2\right) dE_{r+1}(z_{r+1}) \right) Q_k$$

$$+ \left(\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r) (A_1 - A_2) dE_{r+1}(z_{r+1}) \right) Q_k$$

It remains to pass to the limit in the strong operator topology.

We would like to extend formula (58) to the case of arbitrary functions f in $\boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$. Since $\boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$ consists of Lipschitz functions, it follows that for $f \in \boldsymbol{B}_{\infty 1}^1(\mathbb{R}^2)$,

$$|f(\zeta_{r-1})| \le \text{const}(1 + |\zeta_{r-1}|), \quad \zeta_{r-1} \in \mathbb{C}. \qquad (59)$$

Hence, for $f \in \boldsymbol{B}_{\infty 1}^{1}(\mathbb{R}^2)$,
$D_{f(N_{r-1})} \supset D_{N_{r-1}}$.
Corollary (4.2.37) [293]: Let N_r and N_{r+1} be extended normal operators such that $N_r - N_{r+1}$ is bounded. Then (58) holds for every $f \in \boldsymbol{B}_{\infty 1}^{1}(\mathbb{R}^2)$
Proof. It suffices to prove that for $u \in D_{N_r} = D_{N_{r+1}}$,

$$\big(f(N_r) - f(N_{r+1})\big)u = \left(\iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{y_{r-1}} f\big)(z_r, z_{r+1}) dE_r(z_r)\, \big((A+\epsilon)_1 - (A+\epsilon)_2\big) dE_{r+1}(z_{r+1})\right)u$$
$$+ \left(\iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{x_{r-1}} f\big)(z_r, z_{r+1}) dE_r(z_r)\, (A_1 - A_2) dE_{r+1}(z_{r+1})\right)u$$

Indeed, if N_{r-1} is an extended normal operator and f satisfies (59), then $f(\mathsf{N}_{r-1})$ is the closure of its restriction to the domain of N_{r-1} .
We have

$$\big(f(N_r) - f(N_{r+1})\big)u = \Big(\big(f - f(0)\big)(N_r)\Big)u - \Big(\big(f - f(0)\big)(N_{r+1})\Big)u$$
$$\Big(\big(f - f(0)\big)(N_r)\Big)u = \sum_{n\in\mathbb{Z}} \Big(\big(f_n - f_n(0)\big)(N_r)\Big)u \qquad (60)$$

and

$$\Big(\big(f - f(0)\big)(N_{r+1})\Big)u = \sum_{n\in\mathbb{Z}} \Big(\big(f_n - f_n(0)\big)(N_{r+1})\Big)u \qquad (61)$$

where the functions f_n are defined by (2.2). Moreover, the series on the right-hand sides of (60) and (61) converge absolutely in the norm.
Thus

$$\big(f(N_r) - f(N_{r+1})\big)u = \sum_{n\in\mathbb{Z}} \big(f_n(N_r) - f_n(N_{r+1})\big)u$$

It remains to observe that

$$\iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{y_{r-1}} f\big)(z_r, z_{r+1}) dE_r(z_r)\, \big((A+\epsilon)_1 - (A+\epsilon)_2\big) dE_{r+1}(z_{r+1})$$
$$= \sum_{n\in\mathbb{Z}} \iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{y_{r-1}} f_n\big)(z_r, z_{r+1}) dE_r(z_r)\, \big((A+\epsilon)_1 - (A+\epsilon)_2\big) dE_{r+1}(z_{r+1})$$

and

$$\iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{x_{r-1}} f\big)(z_r, z_{r+1}) dE_r(z_r)\, (A_1 - A_2) dE_{r+1}(z_{r+1})$$
$$= \sum_{n\in\mathbb{Z}} \iint\limits_{\mathbb{C}^2} \big(\mathfrak{D}_{x_{r-1}} f_n\big)(z_r, z_{r+1}) dE_r(z_r)\, (A_1 - A_2) dE_{r+1}(z_{r+1}),$$

and the series on the right-hand sides converge absolutely in the norm which is an immediate consequence of inequalities (54).

Corollary (4.2.38) [293]: Let $f \in \mathcal{E}_\sigma \cap L^\infty(\mathbb{R})$. Then

$$\frac{f(x_{r-1}) - f(y_{r-1})}{x_{r-1} - y_{r-1}} = \sum_{n\in\mathbb{Z}} (-1)^n \sigma \cdot \frac{f(x_{r-1}) - f(\pi n\sigma^{-1})}{\sigma x_{r-1} - \pi n} \cdot \frac{\sin \sigma y_{r-1}}{\sigma y_{r-1} - \pi n} \tag{62}$$

$$= \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x_{r-1}) - f(t_{r-1})}{x_{r-1} - t_{r-1}} . \frac{\sin(\sigma(y_{r-1} - t_{r-1}))}{y_{r-1} - t_{r-1}}\, dt_{r-1}, \quad x_{r-1}, y_{r-1} \in \mathbb{R}. \tag{63}$$

Moreover,

$$\sum_{n\in\mathbb{Z}} \frac{|f(x_{r-1}) - f(\pi n\sigma^{-1})|^2}{(\sigma x_{r-1} - \pi n)^2} = \frac{1}{\pi\sigma}\int_{\mathbb{R}} \frac{|f(x_{r-1}) - f(t_{r-1})|^2}{(x_{r-1} - t_{r-1})^2} dt_{r-1}, \quad x_{r-1} \in \mathbb{R}. \tag{64}$$

and

$$\sum_{n\in\mathbb{Z}} \frac{\sin^2 \sigma y_{r-1}}{(\sigma x_{r-1} - \pi n)^2} = 1 = \frac{1}{\pi\sigma}\int_{\mathbb{R}} \frac{\sin^2(\sigma(y_{r-1} - t_{r-1}))}{(y_{r-1} - t_{r-1})^2} dt_{r-1}, \quad y_{r-1} \in \mathbb{R}. \tag{65}$$

Proof. Clearly, it suffices to consider the case $\sigma = 1$. Let us first observe that the identities in (65) are elementary and well known.

We are going to use the well-known fact that the family $\left\{\frac{\sin \sigma x_{r-1}}{\sigma x_{r-1} - \pi n}\right\}_{n\in\mathbb{Z}}$ orthogonal basis in the space $\mathcal{E}_1 \cap L^2(\mathbb{R})$, forms an

$$F(z_{r-1}) = \sum_{n\in\mathbb{Z}} (-1)^n F(\pi n) \frac{\sin z_{r-1}}{z_{r-1} - \pi n}, \tag{66}$$

and

$$\sum_{n\in\mathbb{Z}} |F(\pi n)|^2 = \frac{1}{\pi}\int_{\mathbb{R}} |F(t_{r-1})|^2 d(t_{r-1}). \tag{67}$$

For every $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$, see, e.g., [167], Lect. 20.2,Th. 1. It follows immediately from (67) that

$$\sum_{n\in\mathbb{Z}} F(\pi n)\overline{G(\pi n)} = \frac{1}{\pi}\int_{\mathbb{R}} F(t_{r-1})\overline{G(t_{r-1})} dt_{r-1}. \text{ for every } F, G \in \mathcal{E}_1 \cap L^2(\mathbb{R}). \tag{67}$$

Given $x_{r-1} \in \mathbb{R}$, we consider the function F defined by $F(\lambda) = \frac{f(x_{r-1}) - f(\lambda)}{x_{r-1} - \lambda}$, $\lambda \in \mathbb{C}$. Clearly, $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$.

It is easy to see that (62) is a consequence of (66) and the equality in (64) is a consequence of (67). It is also easy to see that (63) follows from (6.10).

It remains to prove that

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{|f(x_{r-1}) - f(t_{r-1})|^2}{(x_{r-1} - t_{r-1})^2} dt_{r-1} \le 3\|f\|^2_{L^\infty(\mathbb{R})}$$

for every $F \in \mathcal{E}_1 \cap L^2(\mathbb{R})$ and $x_{r-1} \in \mathbb{R}$. Without loss of generality we may assume that $\|f\|_{L^\infty(\mathbb{R})} = 1$. Then $\|f'\|_{L^\infty(\mathbb{R})} \le 1$ by the Bernstein inequality. Hence, $|f(x_{r-1}) - f(t_{r-1})| \le \min(2, |x_{r-1} - t_{r-1}|)$, and we have

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{|f(x_{r-1}) - f(t_{r-1})|^2}{(x_{r-1} - t_{r-1})^2} dt_{r-1} \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{\min(4, (x_{r-1} - t_{r-1})^2)}{(x_{r-1} - t_{r-1})^2} dt_{r-1}$$

$$= \frac{2}{\pi}\int_0^2 dt_{r-1} + \frac{8}{\pi}\int_2^\infty \frac{dt_{r-1}}{(t_{r-1})^2} = \frac{8}{\pi} < 3.$$

Corollary (4.2.39) [293]: Let $\sigma > 0$ and let f be a function in $C_b(\mathbb{R}^2)$ such that

$$\operatorname{supp}\mathcal{F}f \subset \{\zeta_{r-1} \in \mathbb{C}: |\zeta_{r-1}| \leq \sigma\}.$$

Then $\mathfrak{D}_{x_{r-1}}f, \mathfrak{D}_{y_{r-1}}f \in C_b(\mathbb{C}) \widehat{\otimes}_h C_b(\mathbb{C})$,

$$\left\|\mathfrak{D}_{x_{r-1}}f\right\|_{C_b(\mathbb{C})\widehat{\otimes}_h C_b(\mathbb{C})} \leq \sigma\|f\|_{L^\infty(\mathbb{C})}$$

and

$$\left\|\mathfrak{D}_{y_{r-1}}f\right\|_{C_b(\mathbb{C})\widehat{\otimes}_h C_b(\mathbb{C})} \leq \sigma\|f\|_{L^\infty(\mathbb{C})}$$

Proof. Clearly, f is the restriction to $\mathbb{R}^2$ of an entire function of two complex variables. Moreover, $f(\cdot,a), f(a,\cdot) \in \mathcal{E}_1 \cap L^2(\mathbb{R})$ for every $a \in \mathbb{R}$. It suffices to consider the case $\sigma = 1$. By Theorem 6.1, we have

$$\left(\mathfrak{D}_{x_{r-1}}f\right)(z_r, z_{r+1}) \stackrel{\text{def}}{=} \frac{f(x_r, y_{r+1}) - f(x_{r+1}, y_{r+1})}{x_r - x_{r+1}}$$
$$= \sum_{n\in\mathbb{Z}}(-1)^n \frac{f(\pi n, y_{r+1}) - f(x_{r+1}, y_{r+1})}{\pi n - x_{r+1}} \cdot \frac{\sin x_r}{x_r - \pi n}$$

and

$$\left(\mathfrak{D}_{y_{r-1}}f\right)(z_r, z_{r+1}) \stackrel{\text{def}}{=} \frac{f(x_r, y_{r+1}) - f(x_r, y_{r+1})}{y_r - y_{r+1}}$$
$$= \sum_{n\in\mathbb{Z}}(-1)^n \frac{f(x_r, y_r) - f(x_r, \pi n)}{y_r - \pi n} \cdot \frac{\sin y_{r+1}}{y_{r+1} - \pi n}.$$

Note that the functions $\frac{\sin x_r}{x_r - \pi n}$ and $\frac{f(x_r,y_r)-f(x_r,\pi n)}{y_r-\pi n}$ depend on $z_r = (x_r, y_r)$ and do not depend on $z_{r+1} = (x_{r+1}, y_{r+1})$ while the functions $\frac{f(\pi n,y_{r+1})-f(x_{r+1},y_{r+1})}{\pi n - x_{r+1}}$ and $\frac{\sin y_{r+1}}{y_{r+1}-\pi n}$ depend on $z_{r+1} = (x_{r+1}, y_{r+1})$ and do not depend on $z_r = (x_r, y_r)$. Moreover, by Corollary (4.2.39)we have

$$\sum_{n\in\mathbb{Z}} \frac{|f(x_r, y_r) - f(x_r, \pi n)|^2}{(y_r - \pi n)^2} \leq 3\|f(x_r,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq 3\|f\|^2_{L^\infty(\mathbb{C})},$$
$$\sum_{n\in\mathbb{Z}} \frac{|f(\pi n, y_{r+1}) - f(x_{r+1}, y_{r+1})|^2}{(\pi n - x_{r+1})^2} \leq 3\|f(\cdot, y_{r+1})\|^2_{L^\infty(\mathbb{R})} \leq 3\|f\|^2_{L^\infty(\mathbb{C})},$$

and

$$\sum_{n\in\mathbb{Z}} \frac{\sin^2 x_r}{(x_r - \pi n)^2} = \sum_{n\in\mathbb{Z}} \frac{\sin^2 y_{r+1}}{(y_{r+1} - \pi n)^2} = 1.$$

This implies the result.

Corollary (4.2.40) [293]: Let f belong to the Besov space $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ and let N_r and N_{r+1} be extended normal operators whose difference is a bounded operator. Then (58) holds and

$$\|f(N_r) - f(N_{r+1})\| \leq \text{const}\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_r - N_{r+1}\|.$$

Proof. It follows that

$$\|f(N_r) - f(N_{r+1})\| \leq \sum_{n\in\mathbb{Z}}\|f_n(N_r) - f_n(N_{r+1})\| \leq \text{const}\sum_{n\in\mathbb{Z}} 2^n\|f\|_{L^\infty}\|N_r - N_{r+1}\|$$
$$\leq \text{const}\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_r - N_{r+1}\|$$

(see the definition of $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ in § 2).

In other words, functions in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$ must be operator Lipschitz.

Corollary (4.2.41) [293]: Let $\mathfrak{I}$ be a quasinormed ideal of operators on Hilbert space that has majorization property and let f belong to the Besov space$\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)$. If N_r and N_{r+1} are extended normal operators such that$N_r - N_{r+1} \in \mathfrak{I}$. Then $f(N_r) - f(N_{r+1}) \in \mathfrak{I}$ and

$$\|f(N_r) - f(N_{r+1})\|_{\mathfrak{I}} \le c\|f\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R}^2)}\|N_r - N_{r+1}\|_{\mathfrak{I}}$$

for a numerical constant c.

Proof. In the case where $\mathfrak{I}$ is a normed ideal the result is an immediate consequence In particular, Corollary (4.2.42)is true for$\mathfrak{I} = \boldsymbol{S}_1^l$. To complete the proof in the general case it suffices to use the majorization property.

Corollary (4.2.42) [293]: There exists a positive number c such that for every $\epsilon > 0$ and every $\in \Lambda_{1-\epsilon}(\mathbb{R}^2)$,

$$\|f(N_r) - f(N_{r+1})\| \le c(\epsilon)^{-1}\|f\|_{\Lambda_{1-\epsilon}(\mathbb{R}^2)}\|N_r - N_{r+1}\|^{(1-\epsilon)}. \qquad (68)$$

for arbitrary extended normal operators N_r and N_{r+1}.

Proof. The proof is almost the same as the proof of Theorem 4.1 of [148] (see also Remark following Theorem 4.1 in [148]) for self-adjoint operators. All we need is the following:

$$\|f_n(N_r) - f_n(N_{r+1})\| \le \text{const } 2^n\|f_n\|_{L^\infty}\|N_r - N_{r+1}\|, \quad n \in \mathbb{Z}, \qquad (69)$$

and

$$\|f_n\|_{L^\infty} \le \text{const } 2^{-n(1-\epsilon)}\|f\|_{\Lambda_{1-\epsilon}(\mathbb{R}^2)}, \quad n \in \mathbb{Z}, \qquad (70)$$

where the functions f_n are defined by (2.2). We remind that (69) is a consequence while (70) is a special case of Corollary (4.2.35).

The deduction of inequality (68) from (69) and (70) is exactly the same as in the proof of Theorem 4.1 of [148], in which inequality (68) for self-adjoint operators is deduced from the corresponding analogs of inequalities (69) and (70).

Consider now more general classes of functions. Let ω be a modulus of continuity. Recall that the class $\Lambda_\omega(\mathbb{R}^2)$is defined by

$$\Lambda_\omega(\mathbb{R}^2) \stackrel{\text{def}}{=} \left\{f : \|f\|_{\Lambda_\omega(\mathbb{R}^2)} = \sup_{z_r \neq z_{r+1}} \frac{|f(z_r) - f(z_{r+1})|}{\omega(|z_r - z_{r+1}|)} < \infty\right\}.$$

As in the case of functions of one variable (see [147], [148]), we define the function ω_* by

$$\omega_*(x_{r-1}) \stackrel{\text{def}}{=} x_{r-1}\int_{x_{r-1}}^{\infty} \frac{\omega(t_{r-1})}{(t_{r-1})^2}dt_{r-1}, \qquad x_{r-1} > 0. \qquad (71)$$

Corollary (4.2.43) [293]: There exists a positive number c such that for every modulus of continuity ω and every $f \in \Lambda_\omega(\mathbb{R}^2)$,

$$\|f(N_r) - f(N_{r+1})\| \le c\|f\|_{\Lambda_\omega(\mathbb{R}^2)}\omega_*(\|N_r - N_{r+1}\|) \qquad (72)$$

for arbitrary extended normal operators N_r and N_{r+1} .

Proof. To prove Corollary (4.2.44), we need inequalities (69) and Corollary (4.2.35). The deduction of inequality (72) from (69) and Corollary (4.2.35)is exactly the same as it was done in the proof of Theorem 7.1 of [148] in the case of self-adjoint operators.

For a Lipschitz function f on a compact convex subset $K$of $\mathbb{C}$, we extend it to $\mathbb{C}$ by the formula

$$f(\zeta_{r-1}) \stackrel{\text{def}}{=} f((\zeta_{r-1})_\#), \qquad (73)$$

Corollary (4.2.44) [293]: LetN_r and N_{r+1} be extended normal operators whose spectra are contained in a compact convex set K and let f be a Lipschitz function on. Then

$$\|f(N_r)-f(N_{r+1})\| \le \text{const}\,\|f\|_{\text{Lip}}\|N_r-N_{r+1}\|\left(1+\log\frac{d}{\|N_r-N_{r+1}\|}\right), \quad (74)$$

where d is the diameter of K .

Proof. Without loss of generality, we may assume that $\|f\|_{\text{Lip}}=1$. Let us extend f to C by formula (73). Define the modulus of continuity ω by

$$\omega(\delta) = \begin{cases} \delta, & \delta \le d, \\ d, & \delta > d. \end{cases}$$

Clearly, $f \in \Lambda_\omega(\mathbb{R})$ and $\|f\|_{\Lambda_\omega(\mathbb{R})} \le \|f\|_{\text{Lip}}$. We have

$$\omega_*(\delta) = \delta\int_\delta^d \frac{dt_{r-1}}{t_{r-1}} + \delta d\int_d^\infty \frac{dt_{r-1}}{(t_{r-1})^2} = \delta\log\frac{d}{\delta}+\delta, \quad \delta \le d,$$

where ω_* is defined by (71). Now inequality (74) follows immediately from Theorem 8.2.

Corollary (4.2.45) [293]: Let $\epsilon \ge 0$. Then there exists a positive number $c>0$ such that for every $l \ge 0, \epsilon \ge 0, f \in \Lambda_{1-\epsilon}(\mathbb{R}^2)$, and for arbitrary extended normal operators N_r and N_{r+1} on Hilbert space with bounded N_r-N_{r+1}, the following inequality holds:

$$s_j(f(N_r)-f(N_{r+1})) \le c\|f\|_{\Lambda_{1-\epsilon}(\mathbb{R}^2)}(1+j)^{\epsilon-1/1+\epsilon}\|N_r-N_{r+1}\|_{S_{1+\epsilon}}^{1-\epsilon}$$

for every $j \le l$.

Proof. The proof is almost the same as the proof of Theorem 5.1 of [149]. To be able to apply the reasonings given in the proof of Theorem 5.1 of [149], we need inequality (70) and the following inequality:

$$\|f_n(N_r)-f_n(N_{r+1})\|_{S_{1+\epsilon}^l} \le \text{const}\,2^n\|f_n\|_{L^\infty}\|N_r-N_{r+1}\|_{S_{1+\epsilon}^l}, \quad n \in \mathbb{Z}, \quad (75)$$

where the functions f_n are defined by (2.2). Inequality (75) is an immediate consequence . All the details can be found in the proof of Theorem 5.1 of [149].

Corollary (4.2.46) [293]: Let $\epsilon > 0$. Then there exists a positive number $c > 0$ such that for every $f \in \Lambda_{1-\epsilon}(\mathbb{R}^2)$ and arbitrary extended normal operators N_r and N_{r+1} on Hilbert space with $N_r-N_{r+1} \in \boldsymbol{S}_1$, the operator $f(N_r)-f(N_{r+1})$ belongs to $\boldsymbol{S}_{\frac{1}{1-\epsilon},\infty}$ and the following inequality holds:

$$\|f(N_r)-f(N_{r+1})\|_{S_{\frac{1}{1-\epsilon},\infty}} \le c\|f\|_{\Lambda_{1-\epsilon}(\mathbb{R}^2)}\|N_r-N_{r+1}\|_{S_1}^{1-\epsilon}.$$

Proof. As in the case of self-adjoint operators (see Theorem 5.2 of [149]), this is an immediate consequence of Corollary (4.2.46) in the case $\epsilon = 0$.

Note that the assumptions of Corollary (4.2.47) do not imply that $f(N_r)-f(N_{r+1}) \in \boldsymbol{S}_{1/1-\epsilon}$. This is not true even in the case when N_r and N_{r+1} are self-adjoint operators. This was proved in [149]. Moreover, in [149] a necessary condition on the function f on $\mathbb{R}$ was found for

$$f(A)-f(A+\epsilon) \in \boldsymbol{S}_{1/1-\epsilon}, \quad \text{whenever} \quad A=A^*, A+\epsilon=(A+\epsilon)^* \quad \text{and} \quad \epsilon \in \boldsymbol{S}_1.$$

That necessary condition is based on the $\boldsymbol{S}_{1+\epsilon}$ criterion for Hankel operators ([145] and [172], Ch. 6) and shows that the condition $f \in \Lambda_{1-\epsilon}(\mathbb{R})$ is not sufficient.

The following result ensures that the assumption that $N_r-N_{r+1} \in \boldsymbol{S}_1$ for extended normal operators N_r and N_{r+1} implies that $f(A)-f(A+\epsilon) \in \boldsymbol{S}_{1/\epsilon-1}$ under a slightly more restrictive assumption on.

Corollary (4.2.47) [293]: Let $\epsilon > 0$.Then there exists a positive number $c > 0$ such that for every $f \in \Lambda_{1-\epsilon}(\mathbb{R}^2)$ and arbitrary extended normal operators N_r and N_{r+1} on Hilbert space with bounded$N_r - N_{r+1}$, the following inequality holds:

$$s_j(|f(N_r) - f(N_{r+1})|^{1/1-\epsilon}) \leq c\|f\|^{1/1-\epsilon}_{\Lambda_{1-\epsilon}(\mathbb{R}^2)}\sigma_j(N_r - N_{r+1}), \quad j \geq 0.$$

Recall that the numbers $\sigma_j\ (N_r - N_{r+1})$ defined by (3.1).

Proof. As in the case of self-adjoint operators (see [149]), it suffices to apply Corollary (4.2.46)with $l = j$ and$\epsilon = 0$.

Now we are in a position to obtain a general result in the case $f \in \Lambda_{1-\epsilon}(\mathbb{R}^2)$ and $N_r - N_{r+1} \in \mathfrak{I}$ for an arbitrary quasinormed ideal $\mathfrak{I}$ with upper Boyd index less than 1. Recall that the number $C_{\mathfrak{I}}$ is defined in § 3.

Corollary (4.2.48) [293]: Let f be a function in $C_b(\mathbb{R}^2)$ whose Fourier transform $\mathcal{F}f$ has compact support. Suppose that R is a bounded linear operator, $N_r$and $N_{r+1}$are extended normal operators such that the operators $N_rR - RN_{r+1}$ and $N_r^*R^* - R^*N_{r+1}^*$are bounded. Then

$$f(N_r) - f(N_{r+1}) = \iint_{\mathbb{C}^2} (\mathfrak{D}_{x_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)((A+\epsilon)_1R - R(A+\epsilon)_2)dE_{r+1}(z_{r+1}) + \iint_{\mathbb{C}^2} (\mathfrak{D}_{x_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)(A_1R - RA_2)dE_{r+1}(z_{r+1}) \quad (76)$$

Proof. The proof is similar to the proof of Corollary (4.2.37), Consider first the case when N_r and N_{r+1} are bounded operators. Put

$$d = \max\{\|N_r\|, \|N_{r+1}\|\} \text{ and } D \stackrel{\text{def}}{=} \{\zeta_{r-1} \in \mathbb{C}: |\zeta_{r-1}| \leq d\}.$$

By Corollary (4.2.36), both $\mathfrak{D}_{y_{r-1}}f$ and $\mathfrak{D}_{x_{r-1}}f$ are Schur multipliers. We have

$$\begin{aligned}
&\iint_{\mathbb{C}^2} (\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)((A+\epsilon)_1R - R(A+\epsilon)_2)dE_{r+1}(z_{r+1}) \\
&= \iint_{D\times D} (\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)((A+\epsilon)_1R - R(A+\epsilon)_2)dE_{r+1}(z_{r+1}) \\
&= \iint_{D\times D} (\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)(A+\epsilon)_1RdE_{r+1}(z_{r+1}) \\
&- \iint_{D\times D} (\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)R(A+\epsilon)_2dE_{r+1}(z_{r+1}) \\
&= \iint_{D\times D} y_r(\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)RdE_{r+1}(z_{r+1}) \\
&- \iint_{D\times D} y_{r+1}(\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)RdE_{r+1}(z_{r+1}) \\
&= \iint_{D\times D} (y_r - y_{r+1})(\mathfrak{D}_{y_{r-1}}f)(z_r, z_{r+1})dE_r(z_r)RdE_{r+1}(z_{r+1}) \\
&= \iint_{D\times D} (f(x_r, y_r) - f(x_r, y_{r+1}))dE_r(z_r)RdE_{r+1}(z_{r+1}).
\end{aligned}$$

Similarly,

$$\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)(A_1 R - R A_2) dE_{r+1}(z_{r+1})$$
$$= \iint_{D\times D} \left(f(x_r, y_{r+1}) - f(x_{r+1}, y_{r+1})\right) dE_r(z_r) R dE_{r+1}(z_{r+1}).$$

It follows that

$$\iint_{\mathbb{C}^2} \left(\mathfrak{D}_{y_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)\left((A+\epsilon)_1 R - R(A+\epsilon)_2\right) dE_{r+1}(z_{r+1})$$
$$+ \iint_{\mathbb{C}^2} \left(\mathfrak{D}_{x_{r-1}} f\right)(z_r, z_{r+1}) dE_r(z_r)(A_1 R - R A_2) dE_{r+1}(z_{r+1})$$
$$= \iint_{D\times D} \left(f(x_r, y_r) - f(x_{r+1}, y_{r+1})\right) dE_r(z_r) R dE_{r+1}(z_{r+1})$$
$$= \iint_{D\times D} f(x_r, y_r) dE_r(z_r) R dE_{r+1}(z_{r+1}) - \iint_{D\times D} f(x_{r+1}, y_{r+1}) dE_r(z_r) R dE_{r+1}(z_{r+1})$$
$$= f(N_r)R - Rf(N_{r+1}).$$

Corollary (4.2.49) [293]: Let $\epsilon > 0$. Then there exists a positive number c such that for every $f \in \Lambda_{1-\epsilon}(\mathbb{R}^2)$, for arbitrary extended normal operators $N_r$and $N_{r+1}$and a bounded operator R with $N_r R - R N_{r+1} \in \boldsymbol{S}_{1+\epsilon}$ and$N_r^* R - R N_{r+1}^* \in \boldsymbol{S}_{1+\epsilon}$, the operator $f(N_r)R - Rf(N_{r+1})$ belongs to $\boldsymbol{S}_{1+\epsilon/1-\epsilon}$ and the following inequality holds:

$$\|f(N_r)R - Rf(N_{r+1})\|_{\boldsymbol{S}_{1+\epsilon/1-\epsilon}} \le c\|f\|_{\Lambda_{1+\epsilon}(\mathbb{R}^2)} \|N_r R - R N_{r+1}\|_{\boldsymbol{S}_{1+\epsilon}}^{1-\epsilon}.$$

Proof. we prove that

$$\|f(N_r)R - Rf(N_{r+1})\|_{\boldsymbol{S}_{1+\epsilon/1-\epsilon}} \le c\|f\|_{\Lambda_{1-\epsilon}(\mathbb{R}^2)} \max\left\{\|N_r R - R N_{r+1}\|_{\boldsymbol{S}_{1+\epsilon}}, \|N_r^* R - R N_{r+1}^*\|_{\boldsymbol{S}_{1+\epsilon}}\right\}^{1-\epsilon}.$$

The result follows from the well-known inequality:

$$\|N_r^* R - R N_{r+1}^*\|_{\boldsymbol{S}_{1+\epsilon}} \le \text{const}\|N_r R - R N_{r+1}\|_{\boldsymbol{S}_{1+\epsilon}}, \quad \epsilon > 0, \qquad (77)$$

see [173] and [174].

Note that inequality (77) does not hold for$\epsilon = 0$, see [175]. Thus to obtain analogs of Corollary (4.2.47)we have to estimate the quasicommutators $f(N_r)R - Rf(N_{r+1})$ in terms of both $N_r R - R N_{r+1}$ and$N_r^* R - R N_{r+1}^*$.