Chapter 3
Differential Properties of Subalgebras and Fully Operators

It is shown that (D_1^*)-subalgebras are closed under C^∞- calculus. If δ is a closed derivation of A, the algebras $D(\delta^P)$ are (D_p^*)-subalgebras of A. In the case when δ is a generator of a one-parameter semi group of automorphisms of A, it is proved that, in fact, $D(\delta^P)$are (D_1^*)-subalgebras. We also characterize those Banach *-algebras which are isomorphic to (D_1^*)-subalgebras of C*-algebras.It is proved that the following classes of functions from $A(\mathbb{D})$coincide: the class of the sequences of operator Lipschitz functions on the unit circle $\mathbb{T}$; the class of the sequences of operator Lipschitz functions on $\mathbb{D}$; and the class of the sequences of operator Lipschitz functions on all contraction operators. A similar result is obtained for the class of the sequences of operator C_2-Lipschitz functions from $A(\mathbb{D})$.

Section (3 .1): Some Dense Subalgebras of C*-Algebras

It is well-known that C*-algebras are noncommutative analogies of the algebras of continuous functions. We studies some classes of dense C^*-subalgebras of C^*-algebras whose properties are "close" to the properties of the algebras of differentiable functions.

In [100] is investigated dense locally normal Q^*-subalgebras B of C^*-algebras A. These subalgebras retain many properties of the enveloping C^*-algebras: $Sp_A(x) = Sp_B(X)$, $x \in B$, every finite-dimensional semisimple representation of B is automatically continuous and extends to A and, for every injective *-homomorphism φ of B into a Banach *-algebra, $\|x\| \leqq \|\varphi(x)\|$, $x \in B$. All closed (in the topology of A) two-sided ideals of B are obtained by the mapping $I \to I \cap B$ of the set of all closed two-sided ideals I in A, this mapping is one-to-one and it maps the set of all maximal ideals in A onto the set of all maximal ideals in B. From Longo's result [101] it also follows that everywhere defined derivations from B into A are automatically bounded.

Let B be a dense *-subalgebra of a C^*-algebra $(A, \| \|_0)$ and a Banach *-algebra with respect to a norm $\| \|$. We describe in terms of the norm $\| \|$ different classes of locally normal Q^*-subalgebras of A. If, for example, $\| \|$ is such that B is closed under C^∞-functional calculus of selfadjoint elements, then B is a locally normal Q^*-subalgebra of A. Thus if δ is a closed *-derivation of A, if $x = x^* D(\delta^p)$ and a function $f(t)$ has $p+1$ continuous derivatives, it follows from the result of Bratteli, Elliott and Jorgensen [102] that $f(x) \in D(\delta^p)$, so that $D(\delta^p)$ is a locally normal Q^*-subalgebra of A.

Blackadar and Cuntz [103] started the study of smooth *-subalgebras of C^*-algebras. The basic concept in their approach is the one of differential seminorms which generalizes the seminorms associated with the powers of derivations. They showed that any subalgebra which is complete with respect to a differential seminorm of total order k is closed under C^{k+1}-functional calculus.

We consider (D_p)-subalgebras B of Banach algebras $(A, \| \ \|_0)$: dense subalgebras of A for which there exist norms $\{\| \ \|_i\}_{i=0}^p$ and positive constants $\{D_i\}_{i=0}^p$ such that $(B, \| \ \|_p)$ is Banach algebra and
$\|xy\|_i \leqq D_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i)$, $x, y \in B$ and $1 \leqq i \leqq p$.
The differential subalgebras of order p studied by Blackadar and Cuntz are (D_p)-subalgebras and, for $p = 1$, these classes coincide. But for $p \geqq 2$, the growth of $\| \ \|_p$on products and exponentials in (D_p)-subalgebras, which determines the properties of the subalgebra, is much faster than in the differential subalgebras. Because of this, even for $p = 2$, it is not clear whether (D_p)-subalgebras are closed under C^∞-functional calculus.

In Theorem (3.1.6) we show that if A contains an identity 1, then $1 \in B$ and B is a Q-subalgebra of A, i.e., $Sp_B(X) = Sp_A(x)$, for all $x \in B$. For the case when A is a C^*-algebra, $x = x^* \in B$ and $f(t)$ is a function on $Sp_A(x)$, Theorem (3.1.19) gives some sufficient conditions for $f(x)$ to belong to B. Although this condition is much stronger than the condition of Bratteli, Elliott and Jorgensen [102] for the algebras $D(\delta^p)$ and than the condition of Blackadar and Cuntz [103] for differential algebras, nevertheless, as a corollary of this result, we obtain that (D_p)-subalgebras of C^*-algebras are locally normal Q^*-subalgebras (similarly, the Fourier-Wiener algebra is a locally normal Q^*-subalgebra, but it is not closed under Q^∞-functional calculus).

Blackadar and Cuntz [103] studied a special class of flat differential seminorms. They showed that a differential seminorm $T = \{| \ |_i\}_{i=0}^p$on B is flat if and only if there exist a seminormed algebra D and a derivation δ of D such that $B \subseteq D(\delta^p)$ and that $|x|_i = \left\|\delta^i(x)\right\|_D / i!$, $x \in B$ and $0 \leqq i \leqq p$. We show that if δ is a generator of a one-parameter semigroup of automorphism of A, then the flat differential seminorm $T = \left\{\left\|\delta^i(x)\right\|_0 / i!\right\}_{i=0}^p$ of order $p > 1$ on $D(\delta^p)$ is equivalent to the differential seminorm $T' = \left\{\| \ \|_0, \| \ \|_p\right\}$ of order 1, where $\|x\|_p = \sum_{i=0}^p \left\|\delta^i(x)\right\|_0 / i!$. Thus, in this case, the algebras $D(\delta^p)$ are, in fact, (D_1)-subalgebras of A.

(D_1)-Subalgebras of C^*-algebras constitute probably the most interesting subclass of subalgebras of C^*-algebras. Characterizes those Banach *-algebras which are isomorphic to (D_1)-subalgebras of C^*-algebras.

Let

$$a(k, j) = \binom{k}{j} = \begin{cases} k!/j!\,(k-j)!, & \text{if } j \leqq k, \\ 0, & \text{if } j > k. \end{cases}$$

By induction one can prove the following formula:

$$\sum_{k=0}^{m} \binom{k}{j} = \sum_{k=j}^{m} \binom{k}{j} = \binom{m+j}{j+1}, \quad j \leqq m. \tag{1}$$

Set

$$S(k,p)=\sum_{j=0}^{p-1}\binom{k}{j}.$$

Making use of (1), we obtain that

$$\sum_{k=0}^{m}S(k,p)=\sum_{k=0}^{m}\sum_{j=0}^{p-1}\binom{k}{j}=\sum_{j=0}^{p-1}\sum_{k=0}^{m}\binom{k}{j}$$

$$=\sum_{j=0}^{p-1}\binom{m+1}{j+1}=\sum_{j=1}^{p}\binom{m+1}{j}=S(m+1,p+1)-1. \qquad (2)$$

Lemma (3.1.1)[99]. Let B be an algebra and let $\{f_i\}_{i=0}^{p}$ be a set of non-negative functions on B such that for all $i=0,\ldots,p,$ $f_i(x^{n+m})f_i(x^n)f_i(x^m)$, $x\in B$ and $m,n>0$. If there exist positive numbers $\{C_i\}_{i=0}^{p}$ such that
$f_i(x^2)\leqq C_i f_{i-1}(x)f_i(x),\quad x\in B$ and $i=1,\ldots,p,$
then, for all k,

$$f_i\left(x^{2^k}\right)\leqq f_0(x)^{2^k-S(k,i)}\prod_{j=0}^{i-1}[f_{i-1}(x)]^{a(k,j)}\,(C_{i-j})^{a(k,j+1)}.$$

in particular, $f_1\left(x^{2^k}\right)\leqq C_1^k[f_0(x)]^{2^k-1}f_1(x).$

Proof. We have that

$$f_i\left(x^{2^k}\right)\leqq C_i f_{i-1}\left(x^{2^{k-1}}\right)f_i\left(x^{2^{k-1}}\right)\leqq\cdots$$

$$\leqq C_1^k f_{i-1}\left(x^{2^{k-1}}\right)f_{i-1}\left(x^{2^{k-2}}\right)\ldots f_{i-1}(x^2)f_{i-1}(x)f_i(x). \qquad (3)$$

Since $f_0(x^m)\leqq f_0(x)^m$, we obtain that $f_1\left(x^{2^k}\right)\leqq C_1^k[f_0(x)]^{2^k-1}f_1(x)$ and the lemma holds for $i=1$. Suppose that the lemma holds for $i\leqq n$. Let $i=n+1$. By (3),

$$f_{n+1}\left(x^{2^k}\right)\leqq C_{n+1}^k(x)f_{n+1}(x)\prod_{m=0}^{k-1}f_n\left(x^{2^m}\right)$$

$$\leqq C_{n+1}^k f_{n+1}(x)\prod_{m=0}^{k-1}f_0(x)^{2^m-S(m,n)}\prod_{j=0}^{n-1}\left[f_{n-j}(x)\right]^{a(m,j)}\left(C_{n-j}\right)^{a(m,j+1)}.$$

By (2), $\sum_{m=0}^{k-1}(2^m-S(m,n))=2^k-1-S(k,n+1)+1=2^k-S(k,n+1)$. By (1) $\sum_{m=0}^{k-1}a(m,j)=a(k,k,j+1)$. therefore

$$f_{n+1}\left(x^{2^k}\right)\leqq C_{n+1}^k f_{n+1}(x)f_0(x)^{2^k-S(k,n+1)}\prod_{j=0}^{n-1}f_{n-j}(x)^{a(k,j+1)}\,C_{n-j}^{a(k,j+2)}$$

$$=f_0(x)^{2^k-S(k,n+1)}\prod_{j=0}^{n}f_{n+1-j}(x)^{a(k,j)}\,C_{n+1-j}^{a(k,j+1)}.$$

Corollary (3.1.2)[99]. Let $2^m \leqq n < 2^{m+1}$ and set $K(x) = \max_{1\leqq i\leqq p}\{C_i f_i(x), 1\}$. Then for $m > 2p$,
$S(m,p) < m^{p-1}$ and $f_p(x) \leqq K(x)^{2(m+1)^{p+1}}[f_0(x)]^{n-d(n)}$,
where $S(m,p) \leqq d(n) \leqq S(m+1, p+1)$.

Proof. By Lemma (3.1.1), $f_p(x^{2^i}) \leqq K(x)^{b_i}[f_0(x)]^{2^i - S(i,p)}$, where

$$b_i = \sum_{j=0}^{p-1}\left(\binom{i}{j} + \binom{i}{j+1}\right) = 2S(i,p) - 1 + \binom{i}{p} \leqq 2S(i, p+1).$$

Set $f_0(x^0) = 1$. Let $n = \sum_{i=0}^{m} a_i 2^i$, where a_i are either 1 or 0 and $a_m = 1$. Then

$$f_p(x^n) \leqq \prod_{i=0}^{m} f_p\left(x^{a_i 2^i}\right) \leqq \prod_{i=0}^{m} K(x)^{a_i b_i}[f_0(x)]^{a_i\left(2^i - S(i,p)\right)} \leqq K(x)^b [f_0(x)]^{n-d(n)},$$

Where $b = \sum_{i=0}^{m} a_i b_i$ and $d(n) = \sum_{i=0}^{m} a_i S(i,p)$. Since $a_m = 1$, we obtain from (2) that $b \leqq 2S(m+1, p+2)$ and $S(m,p) \leqq d(n) \leqq S(m+1, p+1)$.

For $m > 2p$, we have that $s(m,p) < p\binom{m}{p-1} < m^{p-1}$. Since$1 \leqq K(x)$, then $K(x)^b \leqq K(x)^{2(m+1)^{p+1}}$

Definition (3.1.3)[99]. Let $\{\| \|_i\}_{i=0}^{p}$ be algebraic seminorms on an algebra B, i.e.,$\|xy\|_i \leqq \|x\|_i\|y\|_i$.

(i) We say that B has property $(\boldsymbol{D}_p)$ with respect to $\{\| \|_i\}_{i=0}^{p}$ if there exist numbers $\{\boldsymbol{D}_i\}_{i=0}^{p}$, $\boldsymbol{D}_i \geqq 0$, such that for all $x \in B$,

$$\|xy\|_i \leqq \boldsymbol{D}_i(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i) , \qquad 1 \leqq i \leqq p.$$

(ii) Let $\| \|_0$ and $\| \|_p$ be norms on B and let A be the completion of B with respect to $\| \|_0$. If B is Banach algebra with respect to $\| \|_p$, then we say that B is a $(\boldsymbol{D}_p)$-subalgebra of A. If, in addition, B is a *-algebra and $\|x^*\|_i = \|x\|_i$, $0 \leqq i \leqq p$ then we say that B is a $(\boldsymbol{D}_p^*)$-subalgebra of the Banach *-algebra A.

Since $\|x^{n+m}\| \leqq \|x^n\|\|x^m\|$ for any seminorm $\| \|$ Lemma (3.1.1) and Corollary (3.1.2) hold if B has property $(\boldsymbol{D}_p)$ with $C_i = 2\boldsymbol{D}_i$.

Lemma(3.1.4)[99]. Let B be a *-algebra and let $\|x^*\|_i = \|x\|_i$, $0 \leqq i \leqq p$. Then property $(\boldsymbol{D}_p)$ is equivalent to the following property $(\boldsymbol{D}_p^*)$: there exist numbers $\{\boldsymbol{D}'_i\}_{i=1}^{p}$, $\boldsymbol{D}'_i \geqq 0$, such that for all $x \in B$,
$\|x^*x\|_i \leqq \boldsymbol{D}'_i\|x\|_i\|x\|_{i-1}$, $1 \leqq i \leqq p$

Proof. Let $x = a + ib \in B$. Then $\|a\|_i = \|(x+x^*)/2\|_i \leqq \|x\|_i$ and $\|b\| = \|(x-x^*)/2i\|_i \leqq \|x\|_i$. Property $(\boldsymbol{D}_p)$ clearly implies property $(\boldsymbol{D}_p^*)$. Let B have property $(\boldsymbol{D}_p^*)$. Fix i and set $\| \| = \| \|_i = \| \|_{i-1}$ and $\boldsymbol{D} = \boldsymbol{D}'_i$. We have that
$x^2 = a^2 + i(ab+ba) - b^2 = (1-i)a^2 - (1+i)b^2 + i(a+b)^2$, $x^*x = a^2 + i(ab - ba) + b^2$

Then

$$\|ab - ba\| = \|x^*x - a^2 - b^2\| \leqq D(\|x\||x| + \|a\||a| + \|b\||b|) \leqq 3D\|x\||x|.$$

For $t > 0$,

$$\|2t(ab+ba)\| = \|(a+tb)^2-(a-tb)^2\| \leqq \|(a+tb)^2\|\|(a-tb)^2\|$$
$$\leqq D(\|a+tb\||a+tb| + \|a-tb\||a-tb|)$$
$$\leqq 2D(\|a\||a| + t|a|\|b\| + \|a\||b| + t^2\|b\||b|).$$

Therefore

$$\|ab+ba\| \leqq D(\|a\||b| + |a|\|b\| + t\|b\||b| + \|a\||a|/t)$$

Using the inequality $2(\lambda\mu)^{1/2} \leqq \lambda+\mu$, we obtain that

$$\min(t\|b\||b| + \|a\||a|/t) = 2(\|b\||b|\|a\||a|)^{1/2} \leqq \|a\||b| + |a|\|b\|.$$

Therefore $\|ab+ba\| \leqq 2D(\|a\||b| + |a|\|b\|)$, so that $\|ab+ba\| \leqq 4D\|x\||x|$. Hence we obtain that $\|ab\| = \|(ab+ba)+(ab-ba)\|/2 \leqq 3.5D\|x\||x|$. Since $\|x\| \leqq \|a\|+\|b\|$ and $|x| \leqq |a|+|b|$,

$$\|ab\| \leqq 3.5D(\|a\|+\|b\|)(|a|+|b|).$$

Let now $x = a_1 + ia_2$ and $v = b_1 + ib_2$. For every j and k,

$$\|a_jb_k\| \leqq 3.5D(\|a_j\|+\|b_k\|)(|a_j|+|b_k|) \leqq 3.5D(\|x\|+\|y\|)(|x|+|y|).$$

Then

$$\|xy\| \leqq \sum_{j.k=1}^{2}\|a_jb_k\| \leqq 14D(\|x\|+\|y\|)(|x|+|y|).$$

Set $t = |x|$, $s = |y|$, $u = x/t$ and $v = y/s$. Then $|u| = 1$, $|v| = 1$ and

$$\|xy\| = ts\|uv\| \leqq 28tsD(\|u\|+\|v\|) = 28D(\|x\||y| + |x|\|y\|).$$

Thus B has property $(\boldsymbol{D}_p)$.

Recall that a normed algebra B with identity is a Q-algebra if the group of all invertible elements in B is open in B. If, in addition, B is a *-normed algebra, then B is a Q^*-algebra. Let $Sp_B(x)$ be the spectrum and $r_B(x)$ be the spectral radius of x in B.

Lemma(3.1.5)[99]. ([8,11]). The following conditions are equivalent:

(i) B is a Q-algebra;

(ii) $r_B(x) \leqq \|x\|$ for all $x \in B$;

(iii) $Sp_A(x) = Sp_B(x)$ for all $x \in B$, where A is the completion of B.

Bratteli and Robinson [104] (cf. [105]) proved that if δ is a closed *-derivation of a C^*-algebra with an identity **1**, then **1** belongs automatically to the domain $D(\delta)$ of δ. In [100] this result was extended to the case when δ is a densely defined closed derivation of a Banach algebra A. The following theorem shows that $(\boldsymbol{D}_p)$-subalgebras of Banach algebras are Q-algebras and that **1** automatically belongs to them.

Theorem (3.1.6)[99]. Let B be a $(\boldsymbol{D}_p)$-subalgebra of A and let A contain an identity **1**. Then $\mathbf{1} \in B$ and B is a Q-algebra with respect to $\|\ \|_0$.

Proof. Let y be an element in B such that $\|\mathbf{1}-y\|_0 = \varepsilon < 1$.Set

$$a_n = \mathbf{1} - (\mathbf{1}-y)^n = -\sum_{i=1}^{n}\binom{n}{i}(-y)^i \in B$$

Then $a_{n+1} - a_n = (\mathbf{1} - y)^n y$. Let $K = K(\mathbf{1} - y) = \max_{1 \leqq i \leqq p}\{2\boldsymbol{D}_i, \|\mathbf{1} - y\|_i, 1\}$. From Corollary(3.1.2) we obtain that

$$\|a_{n+1} - a_n\|_p \leqq \|(\mathbf{1} - y)^n\|_p \|y\|_p \leqq K^{2(m+1)^{p+1}} \varepsilon^{n-S(m+1,p+1)} \|y\|_p$$
$$\leqq K^{2(m+1)^{p+1}} \varepsilon^{n-(m+1)^p} \|y\|_p,$$

where $m \leqq lg_2 n \leqq m + 1$. The series $\sum_{n=1}^{\infty} K^{2(lg_2 n+1)^{p+1}} \varepsilon^{n-(lg_2 n+1)^p}$ converges. Therefore its partial sums S_i, converge.

For any $q > 0$,

$$\|a_{n+q} - a_n\|_p \leqq \sum_{j=n}^{n+q-1} \|a_{j+1} - a_j\|_p \leqq \|y\|_p (S_{n+q-1} - S_{n-1}).$$

Since B is a Banach algebra with respect to$\| \|_p$, the sequence $\{a_n\}$ converges to an element x in B. Hence $\{a_n y\}$ converges to xy in B.

On the other hand,

$$a_n y = y - (1 - y)^n y = y - (a_{n+1} - a_n).$$

Since $\|a_{n+1} - a_n\|_p \to 0$, $\{a_n y\}$ converges to y with respect to $\| \|_p$. Thus $xy = y$. Since $\|\mathbf{1} - y\|_0 < 1$, y is invertible in A. Therefore $x = 1 \in B$.

Since $(B, \| \|_p)$ is a Banach algebra, it follows from Corollary (3.1.2) that

$$r_B(x) = \lim_{n\to\infty} (\|x^2\|_p)^{1/n} \leqq \lim_{n\to\infty} K(x)^{2(m+1)^{p+1/n}} \|x\|_0^{(n-d(n))/n},$$

where $K(x) = \max_{1 \leqq i \leqq p}\{2\boldsymbol{D}_i, \|x\|_i, 1\}$ and where $\binom{m}{p-1} \leqq d(n) \leqq (m+1)^p - 1$ and $m \leqq lg_2 n < m + 1$. Hence $r_B(x) \leqq \|x\|_0$ and, by Lemma (3.1.5), B is a Q-algebra with respect to $\| \|_0$.

Example (3.1.7)[99]. Let $(A, \|\cdot\|)$ be Banach algebra. A two-sided ideal I of A is symmetrically normable (see [106]) if I is a Banach algebra with respect to a norm $| |_s$. and

$$|xyz|_s \leqq \|x\| |y|_s \|z\|, \quad \text{for } y \in I \text{ and } x, z \in A.$$

The symmetric Segal algebras of locally compact groups G are symmetrically normable ideals of $L^1(G)$ [107]. Shatten classes of operators give another example of symmetrically normable ideals [108J.

If l is a dense symmetrically normable ideal in A, then it is a $(\boldsymbol{D}_1)$-subalgebra of A with respect to $\| \|$ and $| |_s$. By Theorem (3.1.6), A does not have an identity. The algebras A and I can be canonically embedded in larger Banach algebras $\hat{A} = A + \mathbb{C}\mathbf{1}$ and $\hat{I} = I + \mathbb{C}\mathbf{1}$ with the norms

$\|t\mathbf{1} + x\| = |t| + \|x\|$ and $\|t\mathbf{1} + y\|_1 = |t| + |y|_s$, $t \in \mathbb{C}, x \in A, y \in I$

respectively. The algebra $\hat{I}$ is a $(\boldsymbol{D}_1)$-Subagebra of $\hat{A}$.

Example (3.1.8)[99]. Let $\delta_1, \dots, \delta_p$ be closed derivations of a Banach algebra $(A, \| \|)$some of them may be the same). For every subset $S = \{k_1, \dots, k_m\}$ of $\{p, \dots, 1\}$, $k_1 > k_2 \dots > k_m$, set

$\delta_S(x) = \delta_{k_1}(\ldots(\delta_{k_m}(x))\ldots)$ and $\delta_\emptyset(x) = x$
where $\emptyset$ is the empty set. Let $D(\delta_s)$ be the domain of δ_S and let $D(\delta_s \ldots . \delta_1) \cap_S D(\delta_S)$ where S ranges over all subsets of $\{p, \ldots, 1\}$.

For $x \in D(\delta_S \ldots . \delta_1)$, set $\|x\|_0 = \|x\|$ and, for $1 \leqq i \leqq p$, set

$$\|x\|_1 = \sum_S \|\delta_S(x)\|$$

where S ranges over all subsets of $\{i, \ldots, 1\}$. For example,
$\|x\|_1 = \|x\| + \|\delta_1(x)\|$ and $\|x\|_2 = \|x\| + \|\delta_1(x)\| + \|\delta_2(x)\| + \|\delta_2(\delta_1(x))\|$
For every subset S of $\{p, \ldots, 1\}$, the derivation property implies the identity $\delta_S(xy) = \sum_Q \delta_Q(x)\delta_{S\setminus Q}(y)$, where Q ranges over all subsets of S. Therefore

$$\|xy\|_i \leqq \sum_S \sum_Q \|\delta_Q(x)\| \|\delta_{S\setminus Q}(y)\|$$

where S ranges over all subsets of $\{i, \ldots, 1\}$ and Q ranges over all subsets of S. From this we can deduce that
$\|xy\|_i \leqq \|x\|_i \|y\|_i$ and $\|xy\|_i \leqq \|x\|_i \|y\|_{i-1} + \|x\|_{i-1}\|y\|_i$.
Therefore $D(\delta_p \ldots . \delta_1)$ has property $(\boldsymbol{D}_p)$ with respect to the norms $\{\| \|_i\}_{i=0}^p$. The closedness of all the derivations $\delta_1 \ldots . \delta_p$ implies that $D(\delta_p \ldots . \delta_1)$ is Banach algebra with respect to $\| \|_p$. If $D(\delta_p \ldots . \delta_1)$ is dense in A, then it is a $(\boldsymbol{D}_p)$-subalgebra of A.

Example (3.1.9)[99]. . Let $(A, \| \|)$ be a Banach algebra, let F be a linear closed mapping from a dense subalgebra $D(F)$ of A into a normed space $(H, \| \|_H)$ and let there exist $D \geqq 0$ such that
$\|F(ab)\|_H \leqq D(\|F(a)\|_H \|b\| + \|a\| \|F(b)\|_H)$, $a, b \in D(F)$.
For example, H is Banach A-bimodule and F is a closed derivation from A into H. Set $\|a\|_0 = \|a\|$ and $\|a\|_1 = \|a\| + \|F(a)\|_H$. Then $D(F)$ is a $(\boldsymbol{D}_1)$-subalgebra of A with respect to the norms $\| \|_0$ and $\| \|_1$.

Example (3.1.10)[99].. Let $(\mathcal{A}, | |)$ and $(\mathfrak{B}, | |_1)$ be Banach algebras and let B be a $(\boldsymbol{D}_1)$-subalgebra of A with respect to $| |$ and $| |_1$ By $C(\mathcal{A})$ and $C(\mathfrak{B})$ we denote the Banach algebras of all converging sequences $a = \{a_n\}$, $a_n \in \mathcal{A}$, and $b = \{b_n\}$, $b_n \in \mathfrak{B}$ with the norms $\|a\|_0 = \sup|a_n|$ and $\|b\|_1 = \sup|b_n|_1$ respectively. Then $C(\mathfrak{B})$ is a $(\boldsymbol{D}_1)$-subalgebra of $C(\mathcal{A})$. Let $H = L_1(\mathfrak{B})$ be the linear manifold in $C(\mathfrak{B})$ which consists of all $b = \{b_n\}$ such that

$$\|b\|_H = \sum_{n=1}^{\infty} |b_n|_1 < \infty.$$

Then $L_1(\mathfrak{B})$ is a Banach $C(\mathfrak{B})$-bimodule. Let

$$F(b) = \{b_1, b_2 - b_1, \cdots, b_n - b_{n-1}, \cdots\}$$

be a mapping from $C(\mathfrak{B})$ into $L_1(\mathfrak{B})$. Then $L_1(\mathfrak{B}) \subseteq D(F)$ and $D(F)$ contains every constant sequence $b = \{b_n\} \in \mathfrak{B}$, $b_n = b_m$, for all n and m. Hence $D(F)$ is dense in $C(\mathfrak{B})$, so that F is a closed derivation from $C(\mathfrak{B})$ into $L_1(\mathfrak{B})$. Thus $D(F)$ is a $(\boldsymbol{D}_1)$-subalgebra of

$C(\mathfrak{B})$ with respect to the norms $\|b\|_1$. and $\|b\|_2 = \|b\|_1 + \|F(b)\|_H$. It is easy to check that $D(F)$ is also dense in $C(\mathcal{A})$, so that $D(F)$ is a (D2)-subalgebra of $C(\mathcal{A})$.

We consider a special subclass of $(\boldsymbol{D}_p)$-subalgebras of Banach algebras-differential subalgebras studied by Blackadar and Cuntz [103]. A set of seminorms $T = \{|\cdot|_i\}_{i=0}^{p} = 0$ (not necessarily algebraic on an algebra B is called a differential seminorm of order p if

$$|xy|_i \leqq \sum_{j=0}^{i} K_{j,i-j}|x|_j\,|y|_{i-j}, \quad x, y \in B \quad \text{and } 0 \leqq i \leqq p,$$

where $K_{j,m}$ are nonnegative constants and $K_{0,0} = 1$. Differential seminorms $T = \{|\cdot|_i\}_{i=0}^{p} = 0$ and $T' = \{|\cdot|_i'\}_{i=0}^{m} = 0$ are equivalent if the seminorms $\sum_{i=0}^{p}|\ |_i$ and $\sum_{i=0}^{m}|\cdot|_i'$ are equivalent on B. Every differential seminorm T is equivalent to a differential seminorm $T' = \{|\cdot|_i'\}_{i=0}^{m} = 0$ such $|xy|_i' \leqq \sum_{j=0}^{i}|x|_i'\,|y|_{i-j}'$.

For $0 \leqq i \leqq p$, set $R_i = \min_{j+n\leqq i}(K_{j,n})$.

Lemma (3.1.11)[99]. $\|x\|_i = R_i\sum_{n=0}^{i}|x|_n$, $0 \leqq i \leqq p$ are algebraic seminorms on B and B has property $(\boldsymbol{D}_p)$ with respect to $\{\|\cdot\|_i\}_{i=0}^{p}$ and with constants $\boldsymbol{D}_i = R_i/R_{i-1}$.

Proof. We have

$$\|xy\|_i = R_i\sum_{m=0}^{i}|xy|_m \leqq R_i\sum_{m=0}^{i}\sum_{j=0}^{m}K_{j,m-j}\,|x|_j|y|_{m-j}$$
$$\leqq R_i^2\sum_{m=0}^{i}\sum_{j=0}^{m}|x|_j|y|_{m-j} \leqq \|x\|_i\|y\|_i, \tag{4}$$

so that $\|\ \|_i$ are algebraic seminorms on B. From (4) it follows that

$$\|xy\|_i = R_i^2\sum_{m=0}^{i}\sum_{j=0}^{m}|x|_j|y|_{m-j} \leqq R_i^2\left[\left(\sum_{j=0}^{i}|x|_j\right)\left(\sum_{j=0}^{i-1}|y|_j\right) + \left(\sum_{j=0}^{i-1}|x|_j\right)\left(\sum_{j=0}^{i}|y|_j\right)\right]$$
$$\leqq \boldsymbol{D}_1(\|x\|_i\|y\|_{i-1} + \|x\|_{i-1}\|y\|_i).$$

Definition(3.1.12)[99]. Let $T = \{|\cdot|_i\}_{i=0}^{p}$ be a differential seminorm on B, let $|\ |_0 = \|\ \|_0$ and $\|\ \|_p$ be norms on B and let $(B, \|\ \|_p)$ be Banach algebra. By A we denote the completion of B with respect to $\|\ \|_0$.Then (B,T) is called a differential subalgebra of A of order p. If, in addition, B is a *-algebra and $|x^*|_i = |x|_i$, $0 \leqq i \leqq p$ then (B,T) is a differential *-subalgebra of the Banach *-algebra A.

Lemma (3.1.1) establishes the growth of $\left\|x^{2^k}\right\|_p$, as $k \to \infty$, in the case when B has property $(\boldsymbol{D}_p)$. Lemma (3.1.13) below is similar to Lemma (3.1.1) and estimates the growth of $\left\|x^{2^k}\right\|_p$, as $k \to \infty$, when B has a differential seminorm $T = \{|\cdot|_i\}_{i=0}^{p}$ (cf. [103]). Set $C_i = K_{i,0} + K_{0,i}$ and $N_i = \max C_{j_1}C_{j_2}\cdots C_{j_n}$ $i = 1, \dots, p$, where maximum is taken over all the sets of integers $(j_1, \dots, j_n)$ such that $j_1 + \dots + j_n = i$. Changing slightly, if necessary, the constants C_i, we can assume that $C_i \neq N_j N_{i-j}$ for $1 < i$ and $1 \leqq j < i$. Let $N = \max_{1 \leqq j \leqq i} N_i$ be the total order of T (cf. [103]).

Lemma(3.1.13)[99]. Let $x \in B$ and $|x|_0 = 1$. Then there are constants $K_i(x), i = 1, \dots, p$, continuous with respect to $\| \, \|_p$ such that, for all k,

$\left|x^{2^k}\right|_i \leqq K_i(x) N_i^k$ ana $\left\|x^{2^k}\right\|_p \leqq K(x) N^k$, where $K(x) = R_p \sum_{i=1}^{p} K_i(x)$.

Proof. Let $\{u_k\}_{k=0}^{\infty} = 0$, a, $\{b_i\}_{i=1}^{m}$ and $\{c_i\}_{i=1}^{m}$ be positive numbers such that $a \neq b$ $1 \leqq i \leqq p$, and

$$u_k \leqq a u_{k-1} + \sum_{i=1}^{m} c_i b_i^{k-1}.$$

Let $d = \max_{1 \leqq i \leqq m} (a, b_i)$ and $\lambda = u_0 + \sum_{i=1}^{m} c_i / |b_i - a|$. Then

$$u_k \leqq a^k u_0 + \sum_{i=1}^{m} c_i (b_i^k - a^k)/(b_i - a) \leqq \lambda d^k, \tag{5}$$

Set $L_{j,i-j} = K_{j,i-j} + K_{i-j,j}$. Then for $1 \leqq i \leqq p$,

$$\left|x^{2^k}\right|_i \leqq \sum_{j=0}^{i} K_{j,i-j} \left|x^{2^{k-1}}\right|_j \left|x^{2^{k-1}}\right|_{i-j} \leqq C_i \left|x^{2^{k-1}}\right|_i + \sum_{j=1}^{[i/2]} L_{j,i-j} \left|x^{2^{k-1}}\right|_j \left|x^{2^{k-1}}\right|_{i-j}.$$

For $i = 1$, $\left|x^{2^k}\right|_1 \leqq C_i \left|x^{2^{k-1}}\right|_1 = N_1 \left|x^{2^{k-1}}\right|_1$ By (5), $\left|x^{2^k}\right|_1 \leqq K_1(x) N_1^k$. where $K_1(x) = |x|_1$ is continuous with respect to $\| \, \|_p$ Assume that the lemma holds for $i \leqq m - 1$. Since $N_i N_j \leqq N_{i+j}$, for $i = m$,

$$\left|x^{2^k}\right|_m \leqq C_m \left|x^{2^{k-1}}\right|_m + \sum_{j=1}^{[i/2]} L_{j,m-j} \left|x^{2^{k-1}}\right|_j \left|x^{2^{k-1}}\right|_{m-j}$$

$$\leqq C_m \left|x^{2^{k-1}}\right|_m + \sum_{j=1}^{[i/2]} L_{j,m-j} K_j(x) \left(N_j N_{m-j}\right)^{k-1}$$

Since $N_m = \max_j \left(C_m, N_j N_{m-j}\right)$, it follows from (5) that $\left|x^{2^k}\right|_m \leqq K_m(x) N_m^k$, where $K_m(x)$ is continuous with respect to $\| \, \|_p$.

We also have that

$$\left\|x^{2^k}\right\|_p = R_p \sum_{i=1}^{p} \left|x^{2^k}\right|_i \leqq R_p \sum_{i=1}^{p} K_i(x) N_i^k \leqq K(x) N^k.$$

Let a graded algebra $B = B_0 + \cdots + B_p$ be the direct sum of subspaces B_i, let $B_i B_j \subseteq B_{i+j}$(we assume that $B_n = \{0\}$ if $p < n$) and let P_i, be the projections onto B_i Then B_0 is a subalgebra of B and all B, are B_0-bimodules. Suppose that every B_i, has a space seminorm $s_i(x)$ and that there are constants $K_{i,j}$ such that $K_{0,0} = 1$, and

$s_{i+j}(xy) \leqq K_{i,j} s_i(x) s_j(y)$, $x \in B_i$ and $y \in B_j$.
For $x \in B$, set $s_i(x) = s_i(P_i(x))$, $0 \leqq i \leqq p$. Then S_i are seminorms on B, $x = \sum_{i=0}^{p} P_i(x)$ and

$$s_i(xy) = s_i\big(P_i(xy)\big) = s_i\left(\sum_{j=0}^{i} P_j(x)P_{i-j}(y)\right) \leqq \sum_{j=0}^{i} s_i\big(P_j(x)P_{i-j}(y)\big)$$

$$\leqq \sum_{j=0}^{i} K_{j,i-j} s_j\left(P_j(x)\right) s_{i-j}\left(P_{i-j}(y)\right) = \sum_{j=0}^{i} K_{j,i-j}\, S_j(x) S_{i-j}(y).$$

Therefore $S = \{S_i\}_{i=0}^{p}$ is a differential seminorm on B, If all s_i are norms on B_i, then, by (4), B is a graded normed algebra with respect to the norm $\|x\|_p = R_p \sum_{i=0}^{p} S_i(x)$.

As an example, we can consider a nest algebra $\mathfrak{B}$ = Alg $N = \{V \in B(H): VL_k \subseteq L_k\}$, where N is a nest of subspaces $\{L_k\}_{k=0}^{p+1}$, $\{0\} = L_0 \subset L_1 \subset \ldots \subset L_{p+1} = H$, of a Hilbert space H. Let Q_k be the projections on L_k and let $R_k = Q_k - Q_{k-1}$, $k = 1, \ldots, p+1$. For $i = 0, \ldots, p$, set

$$B_i = \{V \in \mathfrak{B} : VR_k = R_{k-1}R_k, \text{for all } k = 1, \ldots, p+1\}$$

and $s_i(V) = \|V\|$, $V \in B_i$, where $\| \; \|$ the usual norm of the operator V on H. Let P_i, be the projections onto B_i. Then $B = B_0 + \cdots + B_p$ is a graded algebra and $S = \{S_i\}_{i=0}^{p}$, where $s_i(x) = s_i(P_i(V))$, is a differential seminorm on Alg N of order p.

A differential seminorm $T = \{| \; |_i\}_{i=0}^{p}$ on an algebra B is called flat ([1, Def. 4.3]) if there is a homomorphism $\emptyset$ from B into a graded algebra B with a differential seminorm $S = \{S_i\}_{i=0}^{p}$ such that $|x|_i = S_i(\emptyset(x))$.

Let, in particular, all $B_i = B_0$, $1 \leqq i \leqq p$, and δ be a derivation of B_0. Set $\varphi_i(x) = \delta^i(x)/i!$ $x \in D(\delta^p)$. Since, for every i,

$$\varphi_i(xy) = \delta^i(xy)/i! = \sum_{j=0}^{i} \binom{i}{j} \delta^i(x)\delta^{i-j}(y)/i! = \sum_{j=0}^{i} \varphi_i(x)\varphi_{i-j}(y),$$

the mapping $\varphi(x) = \left(x, \varphi_1(x), \ldots, \varphi_p(x)\right)$ is a homomorphism from $D(\delta^p)$ into B. Let $| \; |_0$ be an algebraic seminorm on B_0. Set $|x|_i = |\varphi_i(x)|_0$ for $x \in D(\delta^p)$. Then $T = \{|\cdot|_i\}_{i=0}^{p}$ is a flat differential seminorm on $D(\delta^p)$. If $(B_0, | \; |_0)$ is Banach algebra, δ is a closed derivation and $D(\delta^p)$ is dense in B_0, then $D(\delta^p)$ is a differential subalgebra of $B_0$of order p.

Blackadar and Cuntz ([1,Th. 4.4]) proved that $T = \{|\cdot|_i\}_{i=0}^{p}$ is flat if and only if there exist a seminormed algebra D and a derivation δ of D such that $B \subseteq D(\delta^p)$ and $|x|_i = \left\|\delta^i(x)\right\|_D / i!$ and $0 \leqq i \leqq p$. We shall now show that, under some conditions on δ, T is equivalent to the differential seminorm $T' = \{|\cdot|_0, \|\cdot\|_0\}$ of order 1 where $\|x\|_p = \sum_{i=0}^{p} |x|_i$

Let $F = \cup_{i=1}^{n} [a_i, b_i]$, $n < \infty$, be the union of disjoint segments and $C(\mathcal{F})$ be the C^*-algebra of all continuous functions on F with the norm $\|f\| = \sup_{t \in \mathcal{F}} |f(t)|$. Let $\delta = d/dt$. For any p, $D(\delta^p) = D_p(\mathcal{F})$ is a dense *-subalgebra of $C(\mathcal{F})$ which consists of all functions $f(t)$

such that $f^{(p)}(t) \in C(\mathcal{F})$. Set $|f|_k = \|f^k\|/k!$ Then $T = \{|\cdot|_k\}_{k=0}^{p}$ is a differential seminorm on $C(\mathcal{F})$ of order p. For $0 \leqq m \leqq p$,

$$\|f\|_m = \sum_{k=0}^{m} \|f^k\|/k!.$$

are norms on $D_p(\mathcal{F})$ and $D_p(\mathcal{F})$ is a Banach *-algebra with respect to $\|\ \|_p$.

Theorem (3.1.14)[99]. $T' = \{|\cdot|_0, \|\cdot\|_0\}$ is a differential seminorm of order 1 on $D_p(\mathcal{F})$ equivalent to T, i.e., there exists $D_1(p) > 0$ such that

$$\|fg\|_p \leqq D_1(p)\big(\|f\|_p\|y\|_0 + \|f\|_0\|g\|_p\big), \quad f, g \in D_p(\mathcal{F}).$$

and $(D_p(\mathcal{F}), T)$ is a differential *-subalgebra of $C(\mathcal{F})$ of order 1.

Proof. First assume that $F = [a, b]$ and set $h = b - a$. In [113] it is proved that, for all k and $m, k \leqq m$,

$$\|f^{(k)}\| \leqq C(k, m)\|f\|^{(m-k)/m} M_m(f)^{k/m},$$

where $C(k, m) = 4e^{2k}(m/k)^k$ and $M_m(f) = \max\{\|f\|m!\,h^{-m}, \|f^{(m)}\|\}$. Let $G(h, m) = (m!)\max\{1, m!\,h^{-m}\}$. Then

$$M_m(f) \leqq G(h, m)(\|f\|\|f^m\|/m!) \leqq G(h, m)\|f\|_m.$$

Thus $\|f^{(k)}\| \leqq R(k, m, h)\|f\|^{(m-k)/m}\|f\|_m^{k/m}$ where $R(k, m, h) = C(k, m)G(h, m)^{k/m}$. Hence

$$\|f^{(k)}\|\|g^{(m-k)}\| \leqq R(k, m, h)R(m-k, m, h)(\|f\|\|g\|_m)^{(m-k)/m}(\|f\|_m\|g\|)^{k/m}.$$

Using the inequality $\alpha^x\beta^{1-x} \leqq \alpha x + \beta(1-x), \alpha, \beta \geqq 0, 0 \leqq x \leqq 1$, we obtain that

$$\|f^{(k)}\|\|g^{(m-k)}\| \leqq L(k, m, h)(\|f\|\|g\|_m(m-k)/m + \|f\|_m\|g\|k/m),$$

$$\leqq L(k, m, h)(\|f\|\|g\|_m + \|f\|_m\|g\|), \qquad (6)$$

where $L(k, m, h,) = R(k, m, h)R(m-k, m, h)$. Therefore

$$\|fg\|_p = \sum_{m=0}^{p} \|(fg)^m\|\,/m! \leqq \sum_{m=0}^{p} \frac{1}{m!}\left[\sum_{k=0}^{m} \binom{m}{k} \|f^{(i)}\|\|g^{(m-i)}\|\right]$$

$$\leqq \sum_{m=0}^{p} \frac{1}{m!}\left[\sum_{k=0}^{m} \binom{m}{i} L(i, m, h)(\|f\|_m\|g\| + \|f\|\|g\|_m)\right].$$

Since $\|f\|_m \leqq \|f\|_p$, for $m \leqq p$,

$$\|fg\|_p \leqq D(p, h)\big(\|f\|_p\|g\| + \|f\|\|g\|_p\big), \qquad (7)$$

where $D(p, h) = \sum_{m=0}^{p} \frac{1}{m!}\left[\sum_{k=0}^{m} \binom{m}{i} L(i, m, h)\right]$.

Assume now that $F = \bigcup_{i=1}^{n}[a_i, b_i], n < \infty$, and let $h_i = b_i - a_i$. For $f \in D_p(\mathcal{F})$ and $1 \leqq i \leqq n$, set

$$\|f\|^{[i]} = \sup_{t \in [a_i, b_i]} |f(t)| \quad \text{and} \quad \|f\|_p^{[i]} = \sum_{k=0}^{p} \frac{1}{k!} \|f^{(k)}\|^{[i]}.$$

Then

$$\|f\|_0 = \max_{1\leqq i\leqq n}\left(\|f\|_0^{[i]}\right) \quad \text{and} \quad \max_{1\leqq i\leqq n}\left(\|f\|_p^{[i]}\right) \leqq \|f\|_p \leqq \sum_{i=1}^{n}\frac{1}{k!}\|f\|_p^{[i]}.$$

Set $D_1(p) = \sum_{i=1}^{n} D_1(p, h_i)$. Then by (7), for $f, g \in D_p(\mathcal{F})$,

$$\|fg\|_p \leqq \sum_{i=1}^{n}\|f\|_p^{[i]} \leqq D_1(p)\left(\|f\|_p\|g\|_0 + \|f\|_0\|g\|_p\right).$$

We shall now extend the result of Theorem (3.1.14) to generators δ of strongly continuous one-parameter semigroups of bounded automorphism α_i of Banach algebras A. There exists a dense subalgebra $D(\delta)$ in A such that $\delta(x) = \lim_{t\to 0}(\alpha_i(x) - x)/t, \ x \in D(\delta)$, is a closed derivation on A. For every p,

$$D(\delta^p) = \{x \in D(\delta): \delta^k(x) \in D(\delta), 1 \leqq k \leqq p-1\}$$

is a subalgebra of A. It is Banach algebra with respect to the norm $\|x\|_p = \sum_{k=0}^{p}\|\delta^k(x)\| / k!$ and $T = \{|\ |_i\}_{i=0}^{p}$, $|x|_i = \left\|\delta^{(i)}(x)\right\|/i!$ is a flat differential seminorm on $D(\delta^p)$. Since $D(\delta^p)$ is dense in A, it is a differential subalgebra of A of order p.

Theorem(3.1.15)[99]. There exists a constant $D_1(p) > 0$ such that

$$\|xy\|_p \leqq D_1(p)\left(\|x\|_p\|y\| + \|x\|\|y\|_p\right), \qquad x, y \in D(\delta^p),$$

so that $T' = \left\{\|\ \|, \|\ \|_p\right\}$ is a differential seminorm of order 1 on $D(\delta^p)$ equivalent to T. Thus $D(\delta^p)$ is a differential subalgebra of A of order 1.

Proof. We have (see [108, v. I]) that, for $t \geqq 0$, $\alpha_i(x) \in D(\delta)$ and $d\alpha_i(x)/dt = \delta\left(\alpha_i(x)\right) = \alpha_i(\delta(x))$, $x \in D(\delta)$ Therefore if $x \in D(\delta^p)$, then

$\alpha_i(x) \in D(\delta^k)$ and $d\alpha_i\left(\delta^{k-1}(x)\right)/dt = \delta(\alpha_i(\delta^{k-1}(x))) = \alpha_i\left(\delta^k(x)\right)$.

Hence $\delta^k\left(\alpha_i(x)\right) = \alpha_i\left(\delta^k(x)\right)$.

Let f be a bounded functional on A such that $\|f\| = 1$. For $x \in A$, the function $F(x,t) = f(\alpha_i(x))$ is continuous with respect to t. If $x \in D(\delta^p)$, then $F(x,t)$ has p continuous derivatives with respect to t and

$$F^{(p)}(x,t) = f\left(\delta^p\left(\alpha_i(x)\right)\right) = f\left(\alpha_i\left(\delta^k(x)\right)\right) = F(\delta^p(x), t).$$

Let $C(\mathcal{F})$ be the C^*-algebra of all continuous functions on $F = [0,1]$. If $x \in D(\delta^p)$, $F(x,t)D_p(\mathcal{F})$. By Lemma VIII.1.3 [108, v. I], there is $C < \infty$ such that $\sup_{0\leqq t\leqq 1}\|\alpha_i\| = C$. Therefore

$$\|F\| = \sup_{0\leqq t\leqq 1}|f(\alpha_i(x))| \leqq \sup_{0\leqq t\leqq 1}\|\alpha_i(x)\| \leqq \|x\| \sup_{0\leqq t\leqq 1}\|\alpha_i\| \leqq C\|x\|.$$

and, for $k \leqq p$,

$$\left\|F^{(k)}\right\| = \sup_{0\leqq t\leqq 1}\left|f\left(\alpha_i\left(\delta^k(x)\right)\right)\right| \leqq \sup_{0\leqq t\leqq 1}\left\|\alpha_i\left(\alpha_i\left(\delta^k(x)\right)\right)\right\| \leqq C\|\delta^k(x)\|.$$

Then

$$\|F\|_p = \sum_{k=0}^{p}\left\|F^{(k)}\right\|/k! \leqq \sum_{k=0}^{p} C\|\delta^k(x)\|/k! \leqq C\|x\|_p$$

Let now f and g be bounded functionals on A such that $\|f\| = \|g\| = 1$, let $x, yD(\delta^p)$ and let $F(x,t) = f(\alpha_i(x))$ and $G(y,t) = g(\alpha_i(y))$. By (6), for every $k \leqq m$, there exists a constant $L(k,m,1)$ such that

$$\|F^{(k)}\|\|G^{(m-k)}\| \leqq L(k,m,1)(\|F\|_m\|G\| + \|F\|\|G\|_m).$$

Therefore

$$\sup_{0\leqq t\leqq 1}\left|f\left(\alpha_i(\delta^k(x))\right)\right| \sup_{0\leqq t\leqq 1}\left|g\left(\alpha_i(\delta^{m-k}(y))\right)\right| \leqq CL(k,m,1)(\|x\|_m\|y\| + \|x\|\|y\|_m).$$

Set $t = 0$.Then $|f(\delta^k(x))||g(\delta^{m-k}(y))| \leqq CL(k,m,1)(\|x\|_m\|y\| + \|x\|\|y\|_m)$. Since this inequality holds for all $f, g \in A^*$, such that $\|f\| = \|g\| = 1$,

$$\|\delta^k(x)\|\|\delta^{m-k}(y)\| \leqq CL(k,m,1)(\|x\|_m\|y\| + \|x\|\|y\|_m).$$

From this it follows that

$$\|xy\|_p = \sum_{m=0}^{p}\|\delta^m(xy)\|/m! \leqq \sum_{m=0}^{p}\frac{1}{m!}\left[\sum_{k=0}^{m}\binom{m}{k}\|\delta^k(x)\|\|\delta^{m-k}(y)\|\right]$$

$$\leqq \sum_{m=0}^{p}\frac{1}{m!}\left[\sum_{k=0}^{m}CL(k,m,1)(\|x\|_m\|y\| + \|x\|\|y\|_m)\right].$$

Since $\|x\|_m \leqq \|x\|_p$, for $m \leqq p$, $\|xy\|_p \leqq D_1(p)(\|x\|_p\|y\| + \|x\|\|y\|_p)$, where

$$D_1(p) = C\sum_{m=0}^{p}\frac{1}{m!}\sum_{k=0}^{m}CL(k,m,1).$$

Let $(A, \|\ \|_0)$ be a C^*-algebra and let $x = x^* \in A$. We call $\{a,b\}$,$a \leqq b$, endpoints in $Sp(x)$ if $a, b \in Sp(x)$ and there is $\varepsilon > 0$ such that

$$(a-\varepsilon, a) \cap Sp(x) = (b, b+\varepsilon) \cap Sp(x) \neq \emptyset.$$

If $p_{a,b}(t)$ is a continuous function such that $p_{a,b}(t) = 1, t \in [a,b]$, and $p_{a,b}(t) = 0$, $t \notin (a-\varepsilon, b+\varepsilon)$, then $p_{a,b}(x) \in A$ is a projection and $\|p_{a,b}(x\|_0 = 1$.

Corollary (3.1.16)[99]. Let δ be a closed *-derivation of a commutative C^*-algebra $(A, \|\ \|_0)$ and let $D(\delta^2)$ be a differential subalgebra of order 1 with respect to the norms $\|x\|_0$and $\|x\|_2 = \|x\|_0 + \|\delta(x)\|_0 + \|\delta^2(x)\|_0/2$. Let $y = y^* \in D(\delta^2)$ and let there be endpoints $\{a_i, b_i\}_{i=0}^{\infty}$ in $Sp(y)$ such that $b_i - a_i \to 0$, as $i \to \infty$. Then $\|p_{a_i,b_i}(y)\delta(y)\|_0 \to 0$. In particular, $\delta(y) \neq 1$.

Proof. Let $g_i(t)$ be functions which have two continuous derivatives and such that $g_i(t) = t - a_i$, $t \in [a_i, b_i]$, and $g_i(t) = 0$, if t is outside $(a_i - \varepsilon_i, b_i + \varepsilon_i)$. Then $g_i(y) \in D(\delta^2)$ and $\|g_i(y)\|_0 = a_i - b_i$. Since $g_i'(t) = p_{a_i,b_i}(t)$ and since A is commutative,

$$\delta(g_i(y)) = g_i'(y)\delta(y) = p_{a_i,b_i}(y)\delta(y) \quad \text{and} \quad p_{a_i,b_i}(y) \in D(\delta).$$

For any projection p in $D(\delta)$, $\delta(p) = \delta(p^2) = 2p\delta(p)$, so that $p\delta(p) = 2p\delta(p)$. Therefore $\delta(p) = 0$. Hence $\delta\left(p_{a_i,b_i}(y)\right) = 0$ and

$$\delta^2(g_i(y)) = \delta\left(p_{a_i,b_i}(y)\right) = p_{a_i,b_i}(y)\delta^2(y).$$

Thus

$$\|g_i(y)\|_2 = \|g_i(y)\|_0 + \|\delta(g_i(y))\|_0 + \|\delta^2(g_i(y))\|_0/2$$
$$\leqq (a_i - b_i) + \|\delta(y)\|_0 + \|\delta^2(y)\|_0/2.$$

Set $h_i(y) = (g_i(y))^2$. Then $\|h_i(y)\|_0 = \|g_i(y)\|_0^2 = (a_i - b_i)^2$,
$$\|\delta(h_i(y))\|_0 = \|2g_i(y)\delta(g_i(y))\|_0 \leqq 2(a_i - b_i)\|\delta(y)\|_0$$
and
$$\|\delta^2(h_i(y))\|_0 = 2\left\|g_i(y)\delta^2(g_i(y)) + [\delta(g_i(y))]^2\right\|_0$$
$$= 2\left\|g_i(y)p_{a_i,b_i}(y)\delta^2(y) + [p_{a_i,b_i}(y)\delta(y)]^2\right\|_0.$$

Since $D(\delta^2)$ is a differential subalgebra of A of order 1, there is $D > 0$ such that $\|(g_i(y))^2\|_2 \leqq 2D\|g_i(y)\|_2\|g_i(y)\|_0$. Therefore
$$\left\|g_i(y)p_{a_i,b_i}(y)\delta^2(y) + [p_{a_i,b_i}(y)\delta(y)]^2\right\|_0$$
$$\leqq 2D[(a_i - b_i) + \|\delta(y)\|_0 + \|\delta^2(y)\|_0/2](b_i - a_i) \to 0.$$
as $i \to \infty$. Hence $\left\|p_{a_i,b_i}(y)\delta(y)\right\|_0 \to 0$.

If we put $n = \infty$ and $\min_{1\leqq i<\infty}(h_i) = 0$ in Theorem (3.1.10), we obtain an example which shows that $T' = \{\|\ \|_0, \|\ \|_p\}$ is not necessarily a differential seminorm on $D(\delta_p)$.

Example(3.1.17) [99].. Let
$$\mathcal{F}_1 = \left(\bigcup_{i=1}^{\infty}\left[\frac{2}{2^i}, \frac{3}{2^i}\right]\right)\bigcup\{0\},$$
let $A = C(\mathcal{F}_1)$ and let $\delta = (d/dt)$. The function $y = y(t) = t, t \in \mathcal{F}_1$ belongs to $D(\delta^2)$ and $\delta(y) = 1$. The points $a_i = (2/2^i)$ and $b_i = (3/2^i)$ are endpoints of y and $b_i - a_i \to 0$ as $i \to \infty$, By Corollary (3.1.16), $T' = \{\|\ \|_0, \|\ \|_2\}$ is not a differential seminorm on $D(\delta^2)$.

Powers [109] (cf. [105] and [110]) proved that if $\delta$is a closed *-derivation of a C^*-algebra A, if $x = x^* \in D(\delta)$ and a function $f(t)$ has two continuous derivatives on $Sp_A(x)$, then $f(x) \in D(\delta)$. Bratteli, Elliott and Jorgensen [102] generalized this result and showed that $f(x) \in D(\delta^p)$ if $f(t)$ has $p + 1$ continuous derivatives and $x = x^* \in D(\delta^p)$. Lackadar and Cuntz [103] extended this result to differential subalgebras of C^*-algebras. $(\boldsymbol{D}_1^*)$-subalgebras are differential subalgebras of order1, so that Blackadar's and Cuntz's result holds for them. $(\boldsymbol{D}_p)$-subalgebras, $p \geqq 2$, however, are not, generally speaking, differential subalgebras. Considers some sufficient conditions for $f(x)$ to belong to B if $x^* = x \in B$ and B is a $(\boldsymbol{D}_p^*)$-subalgebra of A. This will allows us to show that $(\boldsymbol{D}_p^*)$-subalgebras of C^*-algebras are locally normal.

Lemma(3.1.18)[99]. If B is a $(\boldsymbol{D}_1^*)$-subalgebra of a unital C^*-algebra A, then $\|\mathbf{1}\|_1 = 1$ and $\boldsymbol{D}_1 > 1/2$.

Proof. By Theorem (3.1.6), $\mathbf{1} \in B$. Therefore $\|\mathbf{1}\|_1 = \|\mathbf{1}^2\|_1 \leqq \|\mathbf{1}\|_1\|\mathbf{1}\|_0$, so that $1/2 \leqq \boldsymbol{D}_1$. Let $\boldsymbol{D}_1 = 1/2$. By Lemma (3.1.1), for $x^* = x \in B$ and for all k,
$$\|\exp(ix)\|_1 = \left\|(\exp(ix/2^k))^{2^k}\right\|_1 \leqq (2\boldsymbol{D}_1)^k\|\exp(ix/2^k)\|_0^{2^k-1}\|\exp(ix/2^k)\|_1$$
$$= \|\exp(ix/2^k)\|_1 \leqq \exp(\|x\|_1/2^k) \to 1,$$
as $k \to \infty$, since $\|\exp(ix/2^k)\|_0 = 1$. Therefore

$$\|\mathbf{1}\|_1 = \|\exp(ix)\exp(-ix)\|_1 \leqq \|\exp(ix)\|_1\|\exp(-ix)\|_1 \leqq 1,$$

so that $\|\mathbf{1}\|_1 = 1$ and $\|\exp(ix)\|_1 = 1$. Since $(B, \|\ \|_1)$ is a Banach *-algebra, it follows from Theorem 38.14 [111] that B is a C^*-algebra. Since B is dense in A, $B = A$. Thus $\boldsymbol{D}_1 > 1/2$.

Theorem (3.1.19)[99]. Let B be a dense ·-subalgebra of a *-algebra $(A, \|\ \|_0)$ with identity, let $x = x^* \in B$ and let $[a, b]$ contain $Sp_A(x)$. Let $f(t)$ be a continuous function on $(-\infty, \infty)$ such that $f(t) = 0$ outside $[a, b]$ and let $\hat{f}(s)$ be its Fourier transform.

(i) Let B be a $(\boldsymbol{D}_1^*)$-subalgebra of A and $M = \max_{1\leqq j\leqq p}\{2\boldsymbol{D}_1, \exp(\|x\|_j)\}$. If $\int_{-\infty}^{\infty}|2s|^{(lg_2|2s|)^{p-1}lg_2M/(p-1)!}|\hat{f}(s)|\,ds < \infty$, for $p \geqq 2$, or $\int_{-\infty}^{\infty}|2s|^{1+lg_2(\boldsymbol{D}_1)}|\hat{f}(s)|\,ds < \infty$, for $p = 1$, then $f(x) \in B$.

(ii) Let B be a $(\boldsymbol{D}_1^*)$-subalgebra of A. If $f(t)$ has $q \geqq 2 + lg_2(\boldsymbol{D}_1)$ continuous derivatives, then $f(x) \in B$.

(iii) (cf [103]). Let B be a differential subalgebra of A with respect to a differential seminorm $T = \{|\ |_i\}_{i=0}^{p}$, where $|\ |_0 = \|\ \|_0$ and let N be the total order of T (see Lemma (3.1.13)). If $\int_{-\infty}^{\infty}|2s|^{lg_2(N)}|\hat{f}(s)|\,ds < \infty$ or if $f(t)$ has $q \geqq 1 + lg_2(N)$ continuous derivatives on $[a, b]$, then $f(x) \in B$.

Proof. By Theorem (3.1.6), $1 \in B$, so that $\exp(isx) \in B$ for real s. Let k be the integer such that $2^{k-1} < |s| \leqq 2^k$, so that $k - 1 < lg_2|s| \leqq k$. Set $y = isx/2^k$. Then $\|\exp(y)\|_0 = 1$ and

$$\|\exp(y)\|_j = \|\exp(isx/2^k)\|_j \leqq \exp(|s|\|x\|_j/2^k) \leqq M.$$

for $1 \leqq j \leqq p$. For every $z \in B$, $\|z^2\|_i \leqq 2\boldsymbol{D}_i\|z\|_i\|z^2\|_{i-1}$ Making use of Lemma (3.1.1)and replacing there C_i by $2\boldsymbol{D}_i$, we obtain

$$\begin{aligned}\|\exp(isx)\|_p &= \|\exp(2^k y)\|_p = \left\|(\exp(y))^{2^k}\right\|_p \\ &\leqq \|\exp(y)\|_0^{2^k - S(k,p)} \prod_{j=0}^{p-1}\|\exp(y)\|_{p-j}^{a(k,p)}\left(2\boldsymbol{D}_{p-j}\right)^{a(k,j+1)} \\ &= \prod_{j=0}^{p-1}\|\exp(y)\|_{p-j}^{a(k,p)}\left(2\boldsymbol{D}_{p-j}\right)^{a(k,j+1)} \leqq M^b \qquad (8)\end{aligned}$$

where $a(k, j) = \binom{k}{j}$ and $b = \sum_{j=0}^{p-1}[a\,(k, j) + a(k, j + 1)]$. For $2p < k$,

$$\begin{aligned}b &\leqq 1 + 2(p - 1)a(k, p - 1) + a(k, p) \leqq 1 + 2(p - 1)k^{p-1}/(p - 1)! + k^p/p! \\ &\leqq 1 + k^p/(p - 1)!\end{aligned}$$

Since $1 \leqq M$,

$$\|\exp(isx)\|_p \leqq M^{1+k^p/(p-1)!} \leqq M^{1+(lg_2|s|+1)^p/(p-1)!} = M|2s|^{\psi(s)}, \qquad (9)$$

where $\psi(s) = (lg_2|2s|)^{p-1}lg_2M/(p - 1)!$.

The rest of the proof of (i) follows the proof of Proposition 3.3.6 [112] with insignificant changes. Since $\hat{f}$ is continuous on $(-\infty, \infty)$ and vanishes at infinity and since $lg_2M \geqq 0$, $\int_{-\infty}^{\infty}|2s|^{\psi(s)}|\hat{f}(s)|ds < \infty$ implies $\int_{-\infty}^{\infty}|\hat{f}(s)|ds < \infty$. Since f is continuous and

$$\hat{\hat{f}}(t) = 1/(2\pi)^{1/2}\int_{-\infty}^{\infty}\exp(its)\hat{f}(s)ds$$

is continuous and since $\hat{\hat{f}} = f$ almost everywhere, $\hat{\hat{f}} = f$. Therefore

$$f\big(x(\lambda)\big) = 1/(2\pi)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp(ix(\lambda)s)\hat{f}(s)ds\,, \quad \text{for } \lambda \in Sp_A(x).$$

It follows from (9) that

$$\int_{-\infty}^{\infty}\|\exp(isx)\|_p\,\big|\hat{f}(s)\big|ds \leqq \int_{-\infty}^{\infty}M|2s|^{\psi(s)}\big|\hat{f}(s)\big|ds < \infty.$$

Therefore $\int_{-\infty}^{\infty}\exp(ixs)\hat{f}(s)ds$ is absolutely convergent in $\|\ \|_p$. Hence

$$y = 1/(2\pi)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp(ixs)\hat{f}(s)ds \in B \quad \text{and} \quad y(\lambda) = f\big(x(\lambda)\big),$$

for $\lambda \in Sp_A(x)$. Thus $y = f(x) \in B$.

For $p = 1$, it follows from (8) that

$$\|\exp(isx)\|_1 \leqq \|\exp(y)\|_1(2\boldsymbol{D}_1)^k \leqq M(2\boldsymbol{D}_1)^{lg_2|2s|} = M|2s|^{\mathbf{1}+lg_2(\boldsymbol{D}_1)}.$$

Repeating the above argument, we obtain that $f(x) \in B$ if $\int_{-\infty}^{\infty}|s|^{lg_2(\boldsymbol{D}_1)}\big|\hat{f}(s)\big|ds < \infty$. Part (i) is proved.

Part (ii) follows from (i) and from the proof of Theorem 3.3.7 [112].

It follows from Lemma (3.1.9) that

$$\|\exp(isx)\|_p = \left\|(\exp(isx))^{2^k}\right\|_p \leqq K(\exp(itx))N^k$$

where $t = s/2^k$, so that $1/2 \leqq |t| \leqq 1$. Since $K(\exp(itx))$ is continuous with respect to $\|\ \|_p$, there exists $M(x) = \sup\limits_{1/2\leqq t\leqq 1} K(\exp(itx)) < \infty$. Hence

$$\|\exp(isx)\|_p \leqq M(x)N^{lg_2(2|s|)} = M(x)|2s|^{lg_2(N)}$$

and this case is similar to the case of $(\boldsymbol{D}_1^*)$-subalgebra of A where $\mathbf{1} + lg_2(\boldsymbol{D}_1)$ is substituted by $lg_2(N)$.

Recall that a family F of functions on a topological space X is said to be normal (see, for example, [106, §15]) if for any disjoint closed subsets S and T in X, there exists a function $f \in F$ such that

$f(x) = 0$ on T and $f(x) = 1$ on S.

Definition (3.1.20)[99]. Let B be a dense subalgebra of Banach algebra A with an identity **1** and let $\mathbf{1} \in B$.

(1) Let A be commutative. The algebra B is said to be normal if the algebra of functions $\{x(s){:}\ x \in B\}$ on the space of all maximal ideals of A is normal.

(2) Let A and B be *-algebras. Then B is said to be locally normal if for every selfadjoint x e B, there is a commutative Banach *-subalgebra $A(x)$ in A such that **1** and x belong to $A(x)$ and such that $B(x) = B \cap A(x)$ is a dense normal subalgebra of $A(x)$.

Theorem (3.1.21)[99]. Let $(A, \|\ \|_0)$ be a C^*-algebra with identity and let B be a $(\boldsymbol{D}_p^*)$-subalgebra of A. Then

(i) B is a locally normal Q^*-subalgebra of A;
(ii) everywhere defined derivations from B into A are bounded;
(iii) the mapping $I \to I \cap B$ is a one-to-one mapping of the set of all closed two-sided ideals in A onto the set of all closed (in the topology of A) two-sided ideals in Band of the set of all maximal ideals of A onto the set of all maximal ideals of B.

Proof. By Theorem (3.1.6), $1 \in B$. It was shown in (9) that, for $x = x^* \in B$, $\|\exp(isx)\|_p \leqq M|2s|^{\psi(s)}$, where $\psi(s) = (lg_2|2s|)^{p-1} lg_2 M/(p-1)!$ and $M = \max_{1\leqq j\leqq p}\{2D_j, \exp(\|x\|_j)\}$. Therefore

$$\int_{-\infty}^{\infty} \ln\|\exp(isx)\|_p \frac{ds}{1+s^2} < \infty.$$

It follows from the Shilov's condition of regularity ([11, §15, 6]) that B is locally normal. Part (ii) follows from (i) and from [101] and part (iii) follows from (i) and from [100].

$(\boldsymbol{D}_1^*)$-Subalgebras of C^*-algebras constitute the simplest and the most interesting subclass of C^*-algebras. We showed that even some differential subalgebras of order $p \geqq 2$ are, in fact, $(\boldsymbol{D}_1^*)$-subalgebras. We characterize those Banach *-algebras which are isomorphic to $(\boldsymbol{D}_1^*)$-subalgebras of C^*-algebras.

Definition(3.1.22)[99]. Let $(B, \| \|)$ be a *-Banach algebra and r_B be the spectral radius on B. We say that B has property $(\boldsymbol{D}^*, r)$ if there exists $D \geqq 0$ such that
$\|xy\| \leqq D(\|x\| r_B(y) + \|y\| r_B(x))$, for $x = x^* \in B$ and $y = y^* \in B$.

Example (3.1.23)[99]. Let B be a C^*-algebra with a norm $\| \|$.
(1) Let

$$B = \left\{b = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \in \mathfrak{B}\right\}, \qquad b^* = \begin{pmatrix} x^* & y^* \\ 0 & x^* \end{pmatrix} \text{ and } \|b\| = |x| + |y|.$$

Then B is a Banach *-algebra and the radical

$$R(B) = \left\{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} : y \in \mathfrak{B}\right\}.$$

We have that $Sp_{\mathfrak{B}}(x) = Sp_B(b)$, so that $r_{\mathfrak{B}}(x) = r_B(b)$. If $b = b^*$, then $x = x^*$, so that $r_B(b) = |x|$. If $b_1^* = b_1 \in B$, and $b_2^* = b_2 \in B$, then

$$\|b_1 b_2\| = |x_1 x_2| + |x_1 y_2 + y_1 x_2| \leqq |x_1| r_B(b_2) + |y_2| r_B(b_1) + |y_1| r_B(b_2)$$
$$\leqq \|b_1\| r_B(b_2) + \|b_2\| r_B(b_1).$$

Hence B has property $(\boldsymbol{D}^*, r)$.
(2) Let δ be a closed *-derivation of B. Then $D(\delta)$ is a $(\boldsymbol{D}_1^*)$-subalgebra of B with respect to the norms $|x|$ and $\|x\| = |x| + |\delta(x)|$.Therefore

$$\|xy\| \leqq \|x\||y| + |x|\|y\|, \qquad x, y \in B,$$

and, by Theorem (3.1.6), $D(\delta)$ is a Q-subalgebra of B. Hence, by Lemma (3.1.5), for $x = x^* \in D(\delta), r_B(x) = r_{\mathfrak{B}}(x) = |x|$. so that

$$\|xy\| \leqq \|x\| r_B(y) + \|y\| r_B(x), \qquad x = x^* \in B \text{ and } y = y^* \in B.$$

Therefore B has property $(\boldsymbol{D}^*, r)$.

Let B be a Banach *-algebra and let $\mathcal{P}(B)$ be the set of all positive functional on B. Then (see [106, §18, 111])

$$I(B) = \{x \in B: (x^*x) = 0 \text{ for all } f \in \mathcal{P}(B)\}$$
is a closed symmetric two-sided ideal of B and the radical $R(B)I \subseteq I(B)$. A Banach *-algebra is called reduced if $I(B) = \{0\}$.

Theorem(3.1.24)[99]. Let $(B, \| \|)$ a Banach *-algebra with identity. The following conditions are equivalent:

(i) B is a reduced algebra with property $(\boldsymbol{D}_1^*, r)$;

(ii) B is a $(\boldsymbol{D}_1^*)$-subalgebra of a C^*-algebra;

(iii) B is a reduced algebra and there exists $D > 0$ such that $\|x^*x\| \leqq 2D\|x\|r_B(x^*x)^{1/2}$,

Proof. (ii) => (i). Let $(A, \| \|_0)$ be a C^*-algebra with identity, let B be a dense *-subalgebra of A and let there exist $D_1 > 0$ such that $\|xy\| \leqq D_1(\|x\|\|y\|_0 + \|x\|_0\|x\|)$, $x, y \in B$. If r_A is the spectral radius on A, then, for selfadjoint $x, \|x\|_0 = r_A(x)$, so that

$\|xy\| \leqq D_1(\|x\|r_A(y) + \|y\|r_A(x))$, for $x = x^* \in B$ and $y = y^* \in B$.

By Theorem (3.1.6), B is a Q^*-subalgebra of A. Therefore it follows from Lemma (3.1.5)that $r_A(x) = r_B(x)$and we obtain that

$\|xy\| \leqq D_1(\|x\|r_B(y) + \|y\|r_B(x))$, for $x = x^* \in B$ and $y = y^* \in B$.

Thus B has property $(\boldsymbol{D}^*, r)$.

Since A is reduced, B is also reduced, so that B is a reduced algebra with property $(\boldsymbol{D}^*, r)$

(ii) $\Longrightarrow$ (iii). Since B is a $(\boldsymbol{D}_1^*)$-subalgebra of A,

$$\|x^*x\| \leqq D_1(\|x^*\|\|x\|_0 + \|x^*\|_0\|x\|) = 2D_1\|x\|\|x\|_0.$$

Since A is a C^*-algebra, $\|x\|_0^2 = \|x^*x\|_0 = r_A(x^*x)$. Since B is a Q^*-subalgebra of A, it follows from Lemma (3.1.5) that $r_B(x^*x) = r_A(x^*x)$. Hence $\|x^*x\| \leqq 2D_1\|x\|r_B(x^*x)^{1/2}$. Since A is reduced, B is also reduced.

(i) $\Longrightarrow$ (ii). Let B be a reduced algebra with property $(\boldsymbol{D}^*, r)$. Then

$$\|x\|_0 = \left(\sup_{f \in \mathcal{P}(B)} f(x^*x)^{1/2}\right), x \in B,$$

is a norm on B such that the completion A of B with respect to $\| \|_0$ is the enveloping C^*-algebra (see [106, §18, 102]).

Let $x = x^* \in B$ and let M be a maximal commutative *-subalgebra of B which contains x. Then $Sp_B(z) = Sp_M(z)$ for all $z \in M$. If $z = z^*$ belongs to the radical of M, $r_B(z) = r_M(z) = 0$ and $\|z^2\| \leqq 2D_1\|z\|r_B(z) = 0$. Hence $z^2 = 0$ and $z \in I(B)$. Since, by assumption, $I(B) = \{0\}$, M is semisimple.

Let T be the space of all maximal ideals of the algebra M and let $C(T)$ be the C^*-algebra of all continuous functions on T with the norm $|f| = \sup_{t \in T}|f(t)|$, $f \in C(T)$. Then M is a dense *-subalgebra of $C(T)$ and

$r_B(z) = r_M(z) = \sup_{t \in T}|z(t)| = |z|$, for all $z \in M$.

Therefore, for $z = z^* \in M$ and $u = u^* \in M$,

$$\|zu\| \leqq 2D\big(\|z\|r_B(u) + \|u\|r_B(z)\big) = 2D(\|z\||u| + |z|\|u\|)$$

and, by Theorem (3.1.21), M is a locally normal Q^*-subalgebra of $C(T)$.

In Theorem 8 [100] it was proved that if F is a dense locally normal Q^*-subalgebra of a C^*-algebra $F$and $\pi$is an injective *-homomorphism of F into a C^*-algebra $\mathcal{A}$, then $\|\pi(y)\|_{\mathcal{A}} = \|y\|_{\mathcal{A}}$ for all $y \in F$. Hence, since M is injectively imbedded in A, we obtain that $\|z\|_0 = |z|$. For all $z \in M$. Therefore $r_B(x) = |x| = \|x\|_0$. From this it follows that for $x = x^* \in B$ and $y = y^* \in B$,

$\|xy\| \leqq D(\|x\|r_B(y) + \|y\|r_B(x)) = D(\|x\|\|y\|_0 + \|x\|_0\|y\|)$. (10)

Now let $z = x + iy \in B$. Then $x = (z + z^*)/2$ and $x = (z - z^*)/2i$ are selfadjoint, $\|x\|_j \leqq \|z\|_j$ and $\|y\|_j \leqq \|z\|_j$,$j = 0,1$. Hence, by (10),

$$\|z^*z\| = \|x^2 + y^2 + i(xy - yx)\| \leqq \|x^2\| + \|y^2\| + \|xy\| + \|yx\|$$
$$\leqq 2D\|x\|\|x\|_0 + 2D\|y\|\|y\|_0 + 2D(\|x\|\|y\|_0 + \|x\|_0\|y\|) \leqq 8D\|z\|\|z\|_0.$$

It follows from Lemma (3.1.4) that B has property $(\boldsymbol{D}_1)$.

(iii)⟹(ii). Let, as above, A be the enveloping C^*-algebra of B, let $x = x^* \in B$ and M be a maximal commutative *-subalgebra of B which contains x. Then, as in (i) ⟹ (ii), we obtain that M is semisimple and that $r_B(z) = r_M(z) = |z|$ for all $z \in M$. Therefore

$$\|z^*z\| \leqq 2D\|z\|r_B(z^*z)^{1/2} = 2D\|z\||z^*z|^{1/2} = 2D\|z\||z|.$$

From Lemma (3.1.4) it follows that M is a $(\boldsymbol{D}_1^*)$-subalgebra of $C(T)$. Hence, by Theorem (3.1.21), M is a locally normal Q^*-subalgebra of $C(T)$. Then, as in (i) ⟹ (ii), $\|z\|_0 = |z|$. for $z \in M$. Therefore $r_B(x) = |x| = \|x\|_0$. Therefore, for every $y \in B$,

$$\|y^*y\| \leqq 2D\|y\|r_B(y^*y)^{1/2} = 2D\|y\|\|y^*y\|_0^{1/2} = 2D\|y\|\|y\|_0.$$

It follows from Lemma (3.1.4) that B is a $(\boldsymbol{D}_1^*)$-subalgebra of A.

Recall that a Banach *-algebra B is called symmetric if $\mathbf{1} + x^*x$ is invertible for all $x \in B$. For a symmetric algebra $B, R(B) = I(B)$ (see [11, §23, 3]). The following lemma shows that the condition $R(B) = I(B)$ is sufficient for algebra with property $(\boldsymbol{D}^*, r)$ to be symmetric.

Lemma(3.1.25)[99]. Let B be a Banach *-algebra with identity and let B have property $(\boldsymbol{D}^*, r)$.

(i) For every $z \in R(B)$, $z^2 = 0$ and the semisimple Banach *-algebra $B/R(B)$ also has property$(\boldsymbol{D}^*, r)$.

(ii) If $R(B) = I(B)$, then the algebra B is symmetric.

Proof. If $z = z^* \in R(B)$, then $\|z^2\| \leqq 2D\|z\|r_B(z) = 0$, so that $z^2 = 0$. If $z = x + iu \in R(B)$,$y = y^* \in R(B)$ and $u = u^* \in R(B)$, then $y^2 = u^2 = 0$ and $z^2 = y^2 + i(yu + uy) - u^2 = i[(y + u)^2 - y^2 - u^2] = 0$

Let $x \to \hat{x}$ be the canonical mapping of B onto the quotient Banach *-algebra $\hat{B} = B/R(B)$. If $z \in R(B)$, then $\mathbf{1} + z$ is invertible. If $x \in B$ is invertible, then $x + z = x(\mathbf{1} + x^{-1}z)$ is also invertible. Therefore $Sp_B(x) = Sp_B(x + z)$ and $r_B(x) = r_B(x + z)$. From this it follows that

$Sp_B(x) = Sp_{\hat{B}}(\hat{x})$ and $r_B(x) = r_{\hat{B}}(\hat{x})$. (11)

Hence, for selfadjoint x and y,

$$\|\hat{x}\hat{y}\| = \inf_{z\in R(B)}\|xy+z\| \leqq \inf_{z\in R(B)}\|(x+z)(y+z)\| \leqq \inf_{z=z^*\in R(B)}\|(x+z)(y+z)\|$$
$$\leqq D\inf_{z=z^*\in R(B)}\left(\|x+z\|r_B(y+z)+\|y+z\|r_B(x+z)\right)$$
$$\leqq D\left(r_{\hat{B}}(\hat{y})\inf_{z=z^*\in R(B)}\|x+z\|+r_{\hat{B}}(\hat{x})\inf_{z=z^*\in R(B)}\|y+z\|\right).$$

For all selfadjoint u and z, $x+z=[(x+z+iu)+(x+z-iu)]/2$, so that
$$\|x+z\| \leqq (\|x+z+iu\|+\|x+z-iu\|)/2=\|x+z+iu\|.$$
Therefore $\inf_{z=z^*\in R(B)}\|x+z\| = \inf_{z\in R(B)}\|x+z\| = \|\hat{x}\|$. Hence
$$\|\hat{x}\hat{y}\| \leqq D(\|\hat{x}\|r_B(\hat{y})+\|\hat{y}\|r_B(\hat{x})), \qquad \hat{x}^*=\hat{x}\in\hat{B} \text{ and } \hat{y}^*=\hat{y}\in\hat{B},$$
and $\hat{B}$ has property $(\boldsymbol{D}^*,r)$. Part (i) is proved.

If $I(B)=R(B)$, the algebra $\hat{B}$ is reduced and has property$(\boldsymbol{D}^*,r)$. Let A be the enveloping C^*-algebra of $\hat{B}$. By (i) and by Theorem (3.1.24), $\hat{B}$ is a $(\boldsymbol{D}_1^*)$-subalgebra of A. It then follows from Theorem (3.1.6) that $\hat{B}$ is a Q^*-subalgebra of A. From this and from (11) we obtain that
$$Sp_B(\mathbf{1}+x^*x)=Sp_{\hat{B}}(\hat{\mathbf{1}}+\hat{x}^*\hat{x})=Sp_A(\hat{\mathbf{1}}+\hat{x}^*\hat{x}) \text{ for all } x\in B.$$
Since $\hat{\mathbf{1}}+\hat{x}^*\hat{x}$ is invertible in A, $\mathbf{1}+x^*x$ is invertible in B. Thus B is a symmetric algebra.

Section (3.2): Operator Lipschitz Functions

Let $\mathbb{D}$ be the closed unit disk. The disc algebra $A(\mathbb{D})$ consists of all continuous complex-valued functions on $\mathbb{D}$ holomorphic in its interior $\mathbb{D}^\circ$. It is a closed subalgebra of the C*-algebra of all continuous complex-valued functions on $\mathbb{D}$ with norm $\|g\|=\sup_{z\in\mathbb{D}}|g(z)|$. The algebra $A(\mathbb{D})$ can be naturally identified with the algebra $A(\mathbb{T})$ of all continuous functions on the unit circle $\mathbb{T}$ with vanishing negative Fourier coefficients and norm $\|f\|=\sup_{z\in\mathbb{T}}|f(z)|$. So we will not distinguish between these two algebras and often identify $f\in A(\mathbb{T})$ with its holomorphic extension $\tilde{f}$ to $\mathbb{D}$.

Denote by $B(H)$ the algebra of all bounded operators on a Hilbert space . An operator $T\in B(H)$ is a contraction if $\|T\|\leq 1$. Von Neumann's inequality states that $\|p(T)\|\leq\|p\|$, for all polynomials p and contractions T. Since the subalgebra of all polynomials is dense in $A(\mathbb{D})$, the operator $g(T)$ can be correctly defined for each $g\in A(\mathbb{D})$ and contraction T, and
$$\|g(T)\|\leq\|g\|. \qquad (12)$$
Hence, considering each $f\in A(\mathbb{T})$ as a function from $A(\mathbb{D})$, we can define the operator $f(T)$.

A continuous function f on a compact $\alpha\in\mathbb{C}$ is called an operator Lipschitz function on α if
$$\|f(T)-f(S)\|\leq C\|T-S\|, \qquad (13)$$
for all normal operators T,S with spectra in a. Thus $f\in A(\mathbb{T})$ is an operator Lipschitz function on $\mathbb{T}$ if (13) holds for all unitary operators T,S. Considering $f\in A(\mathbb{T})$ as a function from $A(\mathbb{D})$, we say that it is operator Lipschitzian on $\mathbb{D}$, if (13) holds for all normal

contractions T, S . Furthermore, we say that it is a fully operator Lipschitz function on $\mathbb{D}$ if (13) holds for all contractions T, S .

The problem whether all these three classes of operator Lipschitz functions in $A(\mathbb{T})$ coincide was posed in [9]. We give the positive solution of this problem:

Theorem(3.2.1)[114]. Let $f \in A(\mathbb{T})$. Then the following conditions are equivalent:

(i) f is an operator Lipschitz function on $\mathbb{T}$;

(ii) f is an operator Lipschitz function on $\mathbb{D}$;

(iii) f is a fully operator Lipschitz function on $\mathbb{D}$.

It is clear that each fully operator Lipschitz function on $\mathbb{D}$ is an operator Lipschitz function on $\mathbb{D}$, and that each operator Lipschitz function on $\mathbb{D}$ is an operator Lipschitz function on $\mathbb{T}$. So we only have to prove implication (i) ⇒(iii).

We prove Theorem (3.2.1). Making use of the interpolation theory, we obtain an analogue of (13) for Schatten ideals C_p of compact operators with norms $\|\cdot\|_p$, $1 \leq p < \infty$: if $f \in A(\mathbb{T})$ is an operator Lipschitz function on $\mathbb{T}$ with constant D, then

$\|f(T) - f(S)\|_p \leq D\|T - S\|_p$, for all contraction T, S with $T - S \in C_p$. (14)

It would be natural to prove (14) for all $f \in A(\mathbb{T})$ that are C_p-Lipschitz functions on $\mathbb{T}$. The classes of C_p-Lipschitz functions are wider than the class of operator Lipschitz functions; in fact no examples of continuously differentiable functions are known which do not belong to them. However, we were only able do this for $p = 2$, using Ando's theorem on common unitary dilations of two commuting contractions. Since C_2-Lipschitz functions are just the usual Lipschitz functions, the result can be written in a quite general form: if $f \in A(\mathbb{T})$ is a Lipschitz function on $\mathbb{T}$ with constant D then If $\|f(T) - f(S)\|_2 \leq D\|T - S\|_2$ for all contractions T, S with $T - S \in C_2$. For all p, a weaker inequality is obtained: if $f \in A(\mathbb{T})$ is a C_p-Lipschitz function on $\mathbb{T}$ with constant D, then, for all contractions T, S with $T - S \in C_{p/2}$,

$$\|f(T) - f(S)\|_p \leq D2^{1/p}(1 + 2^{1/2})\|T - S\|_{p/2}^{1/2}.$$

The proof is based on the study of Lipschitz properties of the multivalued map that takes each contraction to the set of all its power unitary dilations.

Denote by $\mathrm{Con}(H)$ the set of all contractions on a Hilbert space H . Recall that a unitary operator U on a Hilbert space $\mathfrak{H} \supset H$ is called a (power) unitary dilation of $T \in \mathrm{Con}(H)$ if

$T^n = PU|_H$ for all $n \in \mathbb{N}$, (15)

where P is the projection on H in $\mathfrak{H}$. If U is a unitary dilation of T , it follows from (12) and (15) that

$f(T) = Pp(U)|_H$ for each $f \in A(\mathbb{T})$. (16)

It follows from (12) that, for each $T \in \mathrm{Con}(H)$, the homomorphism $f \to f(T)$ from $A(\mathbb{T})$ into $B(H)$ is norm-continuous. Furthermore, for any $f \in A(\mathbb{T})$, the map $T \to f(T)$ is norm-continuous and continuous in the strong operator topology on $\mathrm{Con}(H)$.

A measure space (X, μ) is called standard if there is a topology on X (called admissible) with respect to which μ is a σ-finite Radon measure, that is, for each measurable set A of finite measure and each $\varepsilon > 0$, there is a compact subset F of A such that $\mu(A \backslash F) < \varepsilon$. A standard space (X,μ) is separable if there is an admissible topology in which X has a countable base.

Let $(\mathcal{T},\mu)$ and $(\mathcal{S},\nu)$ be separable standard measure spaces. Denote the Hilbert spaces $L^2(\mathcal{T},\mu)$ and $L^2(\mathcal{S},\nu)$ by H_1 and H_2 , respectively. Every function $g(t,s) \in L^\infty(\mathcal{T} \times \mathcal{S}, \mu \times \nu)$ defines an operator M_g on the space $C_2(H_1,H_2)$ of Hilbert-Schmidt operators from H_1 into H_2; it can be considered as an analogue of the Hadamard multiplication operator in a space of matrices. Namely, if K is a Hilbert-Schmidt operator with integral kernel $\kappa(t,s) \in L^2(\mathcal{T} \times \mathcal{S}, \mu \times \nu)$,
then $M_g(K)$ is the operator with integral kernel $g(t,s)\kappa(t,s)$.

The operator $M_g$is linear and bounded on $C_2(H_1,H_2)$. If it is also bounded with respect to the usual operator norm:

$\|M_g(K)\| \leq C\|K\|$ for all $K \in C_2(H_1,H_2)$, (17)

and some $C > 0$, then g is called a Schur multiplier on $(\mathcal{T} \times \mathcal{S}, \mu \times \nu)$.

Peller [116] characterized Schur multipliers by several equivalent properties, one of which can be formulated as follows: g is a Schur multiplier if and only if there are a separable Hilbert space H and weakly measurable H-valued functions $\vec{x}$ on $\mathcal{T}$ and $\vec{y}$ on $\mathcal{S}$ such that

$g(t,s) = (\vec{x}(t),\vec{y}(s))$ a.e. on $\times\ \mathcal{S}$, and (18)

$\|\vec{x}(t)\| \leq C^{1/2}$, $\|\vec{y}(s)\| \leq C^{1/2}$ a.e. on $\mathcal{T}$ and $\mathcal{S}$ (19)

for some $C > 0$. Choosing an orthonormal basis $\{\vec{e}_n\}_{n\in\mathbb{N}}$ in H and setting $u_n(t) = (\vec{x}(t),\vec{e}_n)$, $u_n(s) = (\vec{e}_n,\vec{y}(s))$, we present g in the form

$$g(t,s) = \sum_{n=1}^{\infty} u_n(t)v_n(s) \quad \text{a.e. on } \mathcal{T} \times \mathcal{S} \tag{20}$$

with

$$\sum_n |u_n(t)|^2 \leq C \ \text{ a.e. on } \mathcal{T}, \quad and \quad \sum_n |v_n(t)|^2 \leq C \ \text{ a.e. on } \mathcal{S}. \tag{21}$$

This implies that there are measurable subsets $\mathcal{T}_0$ and $\mathcal{S}_0$ of $\mathcal{T}$ and $\mathcal{S}$ with $\mu(\mathcal{T}\setminus\mathcal{T}_0) = \nu(\mathcal{S}\setminus\mathcal{S}_0) = 0$ such that the sum in (20) is defined as a bounded function on $\mathcal{T}_0 \times \mathcal{S}_0$ and (21) holds for all $t \in \mathcal{T}_0$ and $s \in \mathcal{S}_0$. One says in this case that the sum is a bounded function marginally almost everywhere (a.e.). This terminology originated in [117], where a subset $M \subset \mathcal{T} \times \mathcal{S}$ was called marginally null if $M \subset (A \times \mathcal{S}) \cup (\mathcal{T} \times B)$, where $A \subset \mathcal{T}$ and $B \subset \mathcal{S}$ have zero measures. Two subsets of $\mathcal{T} \times \mathcal{S}$ are marginally equal if their symmetric difference is marginally null. Two functions are said to be equal marginally a.e. if the set of points, where the equality fails, is marginally null.

A set E in $\mathcal{T} \times \mathcal{S}$ is called ω-open, if there is a countable family of measurable rectangles $A_n \times B_n$ such that $\cup(A_n \times B_n)$ and E are marginally equal. A complex-valued function φ on $\mathcal{T} \times \mathcal{S}$ (it can be defined marginally a.e.) is ω-continuous [118] if, for each open subset $G \subset \mathbb{C}$, the full preimage $\varphi^{-1}(G)$ is ω-open.

Lemma(3.2.2)[114]. If a Schur multiplier g is ω-continuous then the equality (20) holds marginally almost everywhere.

Proof. Note that if an ω-continuous function h on $\mathcal{T} \times \mathcal{S}$ equals zero a.e., then it is zero marginally a.e. Indeed, the set $F = \{(t,s): h(t,s) \neq 0\} = \bigcup_{i\in\mathbb{N}} h^{-1}(U_i)$, where $\mathbb{C}\setminus\{0\} = \bigcup_{i\in\mathbb{N}} U_i$ and all U_i are open. Each $h^{-1}(U_i)$ marginally equals to a union of measurable rectangles $A_n \times B_n$ and has zero measure, as F has zero measure. Thus all $(A_n) = v(B_n) = 0$. Therefore $h^{-1}(U_i)$ is marginally null, so F is marginally null.

Hence it follows that, if two ω-continuous functions coincide a.e., they coincide marginally a.e., as the difference of ω-continuous functions is ω-continuous by [118, Corollary 3.2]. Thus we only need to show that the sum in (20) is an ω-continuous function.

Since u_n , v_n are measurable, the functions $\widehat{u_n}\,(t,s) = u_n(t), \widehat{v_n} = (t,s) = v_n(s)$ are ω-continuous on $\mathcal{T} \times \mathcal{S}$. Hence, by [118, Corollary 3.2], all $g_N(t,s) = \sum_{n=1}^{N} u_n(t)v_n(s)$ are ω-continuous. As $U_N(t) = (\sum_{n=N+1}^{\infty}|u_n(t)|^2)^{1/2}$ and $V_N(s) = (\sum_{n=N+1}^{\infty}|v_n(s)|^2)^{1/2}$ are measurable, the functions $U_N(t)V_N(s)$ are ω-continuou on $\mathcal{T}_0 \times \mathcal{S}_0$. Since

$$\left|\sum_{n=1}^{\infty} u_n(t)v_n(s) - g_N(t,s)\right| = \left|\sum_{n=N+1}^{\infty} u_n(t)v_n(s)\right| \leq U_N(t)V_N(s) \to 0, \text{ as } N \to \infty$$

on $\mathcal{T}_0 \times \mathcal{S}_0$, it follows from Lemma (3.2.8) [118] that $\sum_{n=1}^{\infty} u_n(t)v_n(s)$ is ω-continuous on $\mathcal{T}_0 \times \mathcal{S}_0$ and, hence, on $\mathcal{T} \times \mathcal{S}$.

Suppose now that $\mathcal{T}$ and $\mathcal{S}$ are separable metrizable compacts and μ, v are regular Borel measures withs upp$(\mu) = \mathcal{T}$, supp$(v) = \mathcal{S}$. Our aim is to prove that if a Schur multiplier g is continuous then the vector functions $\vec{x}(t)$ and $\vec{y}(s)$ in (18) can be chosen with some additional properties. For continuous functions, the condition that g is a Schur multiplier does not depend on the choice of μ, v (see [119, 120]), but we will not need this fact, as the measures will be fixed.

For a subset W of a Hilbert space H , by cls(W) we denote its closed linear span. We will say that W generates H if csl$(W) = H$.

Theorem (3.2.3)[114]. Suppose that a continuous function g on $\mathcal{T} \times \mathcal{S}$ is a Schur multiplier. Then the vector functions $\vec{x}(t), \vec{y}(s)$ and the space H can be chosen in such a way that

(i) each of the sets $\{\vec{x}(t): t \in \mathcal{T}\}$ and $\{\vec{y}(s): s \in \mathcal{S}\}$ generates H;

(ii) $\vec{x}(t)$ and $\vec{y}(s)$ are weakly continuous;

(iii) equality (18) and inequality (19) hold for all $(t,s) \in \mathcal{T} \times \mathcal{S}$.

Proof. As $\mathcal{T}, \mathcal{S}$ have countable bases, $\mathcal{T} \times \mathcal{S}$ has a countable base. Hence each open subset of $\mathcal{T} \times \mathcal{S}$ is a countable union of open rectangles, so all continuous functions on $\mathcal{T} \times \mathcal{S}$ are w-continuous. By Lemma (3.2.2), one can assume that (20) holds marginally a.e. So there are $E_1 \subseteq \mathcal{T}, F_1 \subseteq \mathcal{S}$ such that $\mu(\mathcal{T}\setminus E_1) = v(\mathcal{S}\setminus F_1) = 0$ and (18) holds for all $(t,s) = E_1 \times F_1$. Taking (19) into account and removing, if necessary, from $E_1$and F_1some subsets of null measure, we obtain that there are $E \subseteq \mathcal{T}, F \subseteq \mathcal{S}$ with $\mu(\mathcal{T}\setminus E) = v(\mathcal{S}\setminus F) = 0$ such that (18) and (19) hold for all $(t,s) = E \times F$.

Let P_1 be the projection on H_1 = cls$\{\vec{x}(t): t \in E\}$. Then $g(t,s) = \left(\vec{x}(t), P_1\vec{y}(s)\right)$ for $(t,s) = E \times F$. Let now P_2 be the projection on H_2 = cls$\{P_1\vec{y}(s): s \in F\}$. Then $H_2 \subseteq H_1$ and $g(t,s) = \left(P_2\vec{x}(t), P_1\vec{y}(s)\right)$. Note that cls$(\{P_2\vec{x}(t): t \in E\})$ = $P_2\left(\text{cls}(\{\vec{x}(t): t \in E\})\right) = P_2H_1 = H_2$. Replacing H by H_2 and $\vec{x}(t), \vec{y}(s)$ by $P_2\vec{x}(t), P_2\vec{y}(s)$, we obtain the proof of (i).

Let $\mathcal{F}$ be the set of all $\vec{e} \in H$ for which the function $e(s) = (\vec{e}, \vec{y}(s))$ is uniformly continuous on. Let $\vec{e}_n \in \mathcal{F}$ and $\vec{e}_n = \vec{e}$. As (19) holds for all $\in F$,

$$\left|\left(\vec{e}, \vec{y}(s) - \vec{y}(s')\right)\right| \leq \left|\left(\vec{e} - \vec{e}_n, \vec{y}(s) - \vec{y}(s')\right)\right| + \left|\left(\vec{e}_n, \vec{y}(s) - \vec{y}(s')\right)\right|$$
$$\leq 2\|\vec{e} - \vec{e}_n\|D^{1/2} + \left|\left(\vec{e}_n, \vec{y}(s) - \vec{y}(s')\right)\right|.$$

Hence $\vec{e} \in \mathcal{F}$, so $\mathcal{F}$ is a closed linear subspace of. Moreover, $\mathcal{F}$ contains all $\vec{x}(t)$, $t \in E$. Indeed, the function $\varphi(s) = \left(\vec{x}(t), \vec{y}(s)\right)$ on F coincides with the function $s \to g(t,s)$ which is continuous and, therefore, uniformly continuous on $\mathcal{S}$. Thus $\varphi(s)$ is uniformly continuous on F. By (i), $\mathcal{F} = H$.

Let us redefine, if necessary, $\vec{y}(s)$ on $\mathcal{S}\backslash F$ to obtain a weakly continuous H-valued function on $\mathcal{S}$. As supp$(\nu) = \mathcal{S}$, Closure$(F) = \mathcal{S}$. Let $\vec{e} \in H$. As the function $e(s) = \left(\vec{e}, \vec{y}(s)\right)$ is uniformly continuous on , it extends to $\mathcal{S}$by continuity; the result will be also denoted by $e(s)$. As $\left|\left(\vec{e}, \vec{y}(s)\right)\right| \leq \|\vec{e}\|\|\vec{y}(s)\| \leq D^{1/2}\|\vec{e}\|$, for $s \in F$, we have, by continuity, that I $|e(s)| \leq D^{1/2}\|\vec{e}\|$ for all $s \in \mathcal{S}$.

Clearly, for each $s \in F$, the map $\vec{e} \to e(s)$ is linear on H. Hence, by continuity, it is also linear, for each $s \in \mathcal{S}$, so the map $\vec{e} \to e(s)$ is a bounded linear functional on H. Hence, for each $s \in \mathcal{S}/F$, one can find $\vec{v}_s \in H$ such that $e(s) = (\vec{e}, \vec{v}_s)$ for all $e \in H$. Then $\|\vec{v}_s\| \leq D^{1/2}$. Set $\vec{y}(s) = \vec{v}_s$. . As $e(s) = \left(\vec{e}, \vec{y}(s)\right)$ is continuous on $\mathcal{S}$ for each $e \in H$, $\vec{y}(s)$ is weakly continuous on $\mathcal{S}$ and $\|\vec{y}(s)\| \leq D^{1/2}$, for all $s \in \mathcal{S}$.

In the same way we can redefine $\vec{x}(t)$ on $\mathcal{T}\backslash E$ to obtain a weakly continuous function on $\mathcal{T}$ with$\|\vec{x}(t)\| \leq D^{1/2}$, for all $t \in \mathcal{T}$. The redefined function $\left(\vec{x}(t), \vec{y}(s)\right)$ is separately continuous on $\mathcal{T} \times \mathcal{S}$ on by both arguments and coincides with $g(t,s)$ on $E \times F$. As $g(t,s)$ is continuous, equality (18) holds for all $(t,s) \in \mathcal{T} \times \mathcal{S}$. We have proved (ii) and (iii).

Let us reformulate the result of Theorem (3.2.3) in a "scalar" form.

Corollary(3.2.4)[114]. If g is a continuous Schur multiplier on $\mathcal{T} \times \mathcal{S}$ then there are continuous functions $u_n(t), v_n(s)$ such that the equality (20) holds for all $(t,s) \in \mathcal{T} \times \mathcal{S}$ and the inequalities (21) hold for each $t \in \mathcal{T}$ and each $s \in \mathcal{S}$, respectively.

Now we will prove that the functions u_n , v_n inherit some other properties of the function g.

Corollary(3.2.5)[114]. Let $\mathcal{L}$ and $\mathcal{M}$be, respectively, closed subspaces in the spaces $C(\mathcal{T})$ and $C(\mathcal{S})$ of all continuous functions. If, for each $s \in \mathcal{S}$, the function $t \to g(t,s)$ belongs to $\mathcal{L}$, then all functions u_n belong to $\mathcal{L}$. Similarly, if all functions $s \to g(t,s)$, $\in \mathcal{T}$, belong to $\mathcal{M}$, then all v_n belong to $\mathcal{M}$.

Proof. Let $\vec{x}(t)$ and $\vec{y}(s)$ be as in Theorem (3.2.3). Denote by $\mathcal{F}$ the set of all $\vec{e} \in H$ for which the function $t \to (\vec{x}(t), \vec{e})$ belongs to $\mathcal{L}$. Since $\mathcal{L}$is a closed subspace of $C(\mathcal{T})$ and$\|\vec{x}(t)\| \leq C^{1/2}$

, for all $t \in \mathcal{T}$, $\mathcal{F}$ is a closed subspace of H . It contains all $\vec{y}(s)$,$s \in \mathcal{S}$, as$\left(\vec{x}(t), \vec{y}(s)\right) = g(t,s) \in \mathcal{L}$. Since these vectors generate $H, \mathcal{F} = H$. Thus $e_n \in \mathcal{F}$ and this means that $u_n \in \mathcal{L}$. The second statement has a similar proof.

We prove here Theorem (3.2.1). Let $f \in A(\mathbb{T})$ be continuously differentiable on $\mathbb{T}$. Consider f as a function on $\mathbb{D}$and define the function $\hat{f}$ on $\mathbb{D} \times \mathbb{D}$ by

$$\hat{f}(z,w) = \frac{f(z) - f(w)}{z - w} \qquad \text{for } z \neq w,$$
$$\hat{f}(z,z) = \frac{df(z)}{dz} \qquad \text{for } z \in \mathbb{D}^\circ,$$
$$\hat{f}(t,t) = \frac{df(t)}{dt} \qquad \text{for } t \in \mathbb{T}. \tag{22}$$

We will omit the proof of the following lemma.

Lemma (3.2.6)[114]. Let $f \in A(\mathbb{T})$be a continuously differentiable function on $\mathbb{T}$. Then $f' \in A(\mathbb{T})$ and the function $\hat{f}$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$and continuous on $\mathbb{D} \times \mathbb{D}$. For all $z_1, w_1 \in \mathbb{D}$, the functions $z \to \hat{f}(z, w_1)$ and $w \to \hat{f}(z_1, w)$ belong to $A(\mathbb{D})$.

Let $f \in A(\mathbb{T})$ be an operator Lipschitz function on $\mathbb{T}$ with constant D. Suppose that f is continuously differentiable. Then (see, for example, [121]) $\hat{f}(t,s)$ is a Schur multiplier on $(\mathbb{T} \times \mathbb{T}, \mu \times \mu)$ with constant D, where μ is the Lebesgue measure on $\mathbb{T}$. By Lemma (3.2.6), for $t_1, s_1 \in \mathbb{T}$, the functions $s \to \hat{f}(t_1, s)$ and $t \to \hat{f}(t, s_1)$ belong to $A(\mathbb{T})$. Since $A(\mathbb{T})$ is a closed subspace of the space $C(\mathbb{T})$of all continuous functions on $\mathbb{T}$, Corollaries (3.2.4)and (3.2.5)imply that there are $u_n, v_n \in A(\mathbb{T})$ such that, for all $t, s \in \mathbb{T}$,

$$\hat{f}(t,s) = \sum_{n=1}^{\infty} u_n(t) v_n(s) \;\; \text{with} \;\; \sum_n |u_n(t)|^2 \leq D \, , \sum_n |v_n(s)|^2 \leq D. \tag{23}$$

Consider u_n, v_n as elements of $A(\mathbb{D})$ and, for each $n \in \mathbb{N}$, set $\sigma_N(z,w) = \sum_{n=1}^{N} u_n(z) v_n(w)$ for all $z, w \in \mathbb{D}$. For all $z_1, w_1 \in \mathbb{D}$, the functions $z \to \sigma_N(z, w_1)$ and $w \to \sigma_N(z_1, w)$ belong to $A(\mathbb{D})$. If $g \in A(\mathbb{D})$ then (see [122, Chapter III, (1.7)]), for $0 \leq r < 1$,

$$g(re^{i\tau}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \tau - \theta) g(e^{i\theta}) d\theta \;\; \text{where} \;\; P(r,\tau) = \frac{1 - r^2}{1 - 2r\cos\tau + r^2} \tag{24}$$

Lemma (3.2.7)[114]. σ_N uniformly converge to $\hat{f}$on each compact subset of $\mathbb{D}^\circ \times \mathbb{D}^\circ$.

Proof. By the maximum modulus principle and by (23), for all $(z,w) \in \mathbb{D} \times \mathbb{D}$,

$$|\sigma_N(z,w)| \leq \max_{t \in \mathbb{T}} |\sigma_N(t,w)| \leq \max_{t \in \mathbb{T}} \left(\max_{s \in \mathbb{T}} |\sigma_N(t,s)| \right) = \max_{t,s \in \mathbb{T}} \left| \sum_{n=1}^{N} u_n(t) v_n(s) \right|$$
$$\leq \max_{t \in \mathbb{T}} \left(\sum_{n=1}^{N} |u_n(t)|^2 \right)^{1/2} \max_{s \in \mathbb{T}} \left(\sum_{n=1}^{N} |v_n(s)|^2 \right)^{1/2} \leq D.$$

Let $z_1 = re^{i\tau}$, $w_1 = \rho e^{i\alpha}$. The functions $z \to \sigma_N(z, w_1)$, $w \to \sigma_N(z_1, w)$ belong to $A(\mathbb{D})$, so by (24),

$$\sigma_N(z_1,w_1) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\tau-\theta)\sigma_N(e^{i\theta},w_1)d\theta$$
$$= \frac{1}{4\pi^2}\int_0^{2\pi} P(r,\tau-\theta)\left(\int_0^{2\pi} P(\rho,\alpha-\emptyset)\sigma_N(e^{i\theta},e^{i\emptyset})d\emptyset\right)d\theta.$$

By Lemma (3.2.6), the functions $z \to \hat{f}(z,w_1)$and $w \to \hat{f}(z_1,w)$belong to $A(\mathbb{D})$. Hence, as above

$$\hat{f}(z_1,w_1) = \frac{1}{4\pi^2}\int_0^{2\pi} P(r,\tau-\theta)\left(\int_0^{2\pi} P(\rho,\alpha-\emptyset)\hat{f}(e^{i\theta},e^{i\emptyset})d\emptyset\right)d\theta.$$

Let $\max(r,p) \le R < 1$ .Then $|P(r,\tau-\theta)P(\rho,\alpha-\emptyset)| \le \frac{(1+r)(1+\rho)}{(1-r)(1-\rho)} \le \left(\frac{1+R}{1-R}\right)^2$. Hence

$$\left|\hat{f}(z_1,w_1)-\sigma_N(z_1,w_1)\right| \le \frac{1}{4\pi^2}\left(\frac{1+R}{1-R}\right)^2 \int_0^{2\pi}\int_0^{2\pi}\left|\hat{f}\left(e^{i\theta},e^{i\emptyset}\right)-\sigma_N\left(e^{i\theta},e^{i\emptyset}\right)\right| d\emptyset d\theta.$$

Since $\hat{f}$is continuous on $\mathbb{T}\times\mathbb{T}$, $\sup\left|\hat{f}\left(e^{i\theta},e^{i\emptyset}\right)\right| \le M$ for some M . As $\sup\left|\sigma_N\left(e^{i\theta},e^{i\emptyset}\right)\right| \le D$ and $\sigma_N\left(e^{i\theta},e^{i\emptyset}\right) \to \hat{f}\left(e^{i\theta},e^{i\emptyset}\right)$ for all $\emptyset,\theta$, it follows from the Dominated Convergence theorem that

$$\int_0^{2\pi}\int_0^{2\pi}\left|\hat{f}\left(e^{i\theta},e^{i\emptyset}\right)-\sigma_N\left(e^{i\theta},e^{i\emptyset}\right)\right| d\emptyset d\theta \to 0, \quad \text{as } N\to\infty$$

Therefore $\sigma_N$uniformly converge to $\hat{f}$in $\mathbb{D}_R\times\mathbb{D}_R$ where $\mathbb{D}_R$is the closed disk of radius R.

Let $\{u_n\},\{v_n\}$ be functions in $A(\mathbb{T})$ satisfying (23). For $N\in\mathbb{N}$ and contractions T,S, define the bounded operators $\Gamma_N(T,S)$ on $B(H)$ by the formula: $\Gamma_N(T,S)X = \sum_{n=1}^N u_n(T)Xv_n(S)$.

Lemma(3.2.8)[114]. $\|\Gamma_N(T,S)\|_{B(H)} \le D$.

Proof. Set $\emptyset_N(t) = \sum_{n=1}^N u_n(t)\overline{u_n(t)}$. Let U be a unitary dilation of T and P be the projection on H such that $T = PU|_H$ (see (I.4)). Since $0 \le \emptyset_N \le D\mathbf{1}$, the positive operators $\emptyset_N(U) = \sum_{n=1}^N u_n(U)u_n(U)^*$ satisfy the inequality $0 \le \emptyset_N \le D\mathbf{1}$. By (I.5), $u_n(T) = Pu_n(U)|_H$.
Hence

$$\emptyset_N(T) = \sum_{n=1}^N u_n(T)u_n(T)^* = \sum_{n=1}^N Pu_n(U)u_n(U)^*P = P\left(\sum_{n=1}^N u_n(U)u_n(U)^*\right)P \le DP,$$

so $\|\sum_{n=1}^N u_n(T)u_n(T)^*\| \le D$. We obtain similarly that$\|\sum_{n=1}^N v_n(S)^*v_n(S)\| \le D$
by setting $\psi_N(s) = \sum_{n=1}^N \overline{v_n(s)}\, v_n(s)$.
Consider the Hilbert space $\widetilde{H} = H \oplus\ldots\oplus H \oplus\ldots$. For $X\in B(H)$, the operators

$$A_N = \begin{pmatrix} u_1(T)X & \cdots & u_N(T)X & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \ldots & \cdots & \ldots & \ldots & \cdots \end{pmatrix} \quad \text{and}$$
$$B_N = \begin{pmatrix} v_1(S)^* & \cdots & v_N(S)^* & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \ldots & \cdots & \ldots & \ldots & \cdots \end{pmatrix}$$

on $\widetilde{H}$ are bounded, as

$$\|A_N\|^2 = \|A_N A_N^*\| = \left\| \sum_{n=1}^{N} u_n(T)XX^*u_n(T)^* \right\| = \sup_{x\in H, \|x\|=1} \sum_{n=1}^{N} \|X^*u_n(T)^*x\|^2$$
$$\leq \|X^*\|^2 \sup_{x\in H, \|x\|=1} \sum_{n=1}^{N} \|u_n(T)^*x\|^2 = \|X\|^2 \left\| \sum_{n=1}^{N} u_n(T)u_n(T)^* \right\| \leq D\|X\|^2$$

and $\|B_N\|^2 = \|B_N B_N^*\| = \|\sum_{n=1}^{N} v_n(S)^* v_n(S)\| \leq D$. Hence

$$\|\Gamma_N(T,S)X\| = \left\| \sum_{n=1}^{N} u_n(T)Xv_n(S) \right\| = \|B_N B_N^*\| \leq \|A_N\|\|B_N\| \leq D\|X\|$$

which completes the proof.

For $A \in B(H)$, denote by L_A and R_A the operators of the left and right multiplication by A on $B(H)$; they clearly commute. It is well known that $\|L_A\| = \|R_A\| = \|A\|$. So if A is a strict contraction (this means that $\|A\| < 1$) then one may apply functions in $A(\mathbb{D})$ to L_A and R_A. It is evident that $p(L_A)X = p(A)X$ and $p(R_A)X = p(A)X$, for each polynomial p. Approximating uniformly $f \in A(\mathbb{D})$ by polynomials, we have $f(L_A)X = f(A)X$ and $f(R_A)X = f(A)X$, for each strict contraction A, so

$$f(L_A) = L_{f(A)} \quad \text{and} \quad f(R_A) = R_{f(A)}. \tag{25}$$

Let $f \in A(\mathbb{T})$ be a continuously differentiable, operator Lipschitz function on $\mathbb{T}$ and let T, S be strict contractions. Consider f as an element of $A(\mathbb{D})$. The function $f(z) - f(w)$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ and, by Lemma (3.2.6), the function $\hat{f}(z,w)$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$. Therefore (see [123, §116] and [124, III.11.8, Theorem 8]) they can be applied to any two elements of a commutative Banach unital algebra whose spectra are contained in $\mathbb{D}^\circ$and, hence, to the commuting strict contractions L_T and R_T on the Banach space $B(H)$. Thus $\hat{f}(L_T, L_S)$ and $f(L_T) - f(R_S)$ are bounded operators on $B(H)$. By (22), $f(z) - f(w) = \hat{f}(z,w)(z-w)$. Hence, by (25),

$$L_{f(T)} - R_{f(S)} = f(L_T) - f(R_S) = \hat{f}(L_T, R_S)(L_T - R_S)\,. \tag{26}$$

Now we can prove the result and finish the proof of Theorem(3.2.1).

Theorem (3.2.9)[114]. If $f \in A(\mathbb{T})$ is an operator Lipschitz function on $\mathbb{T}$ then there is $D > 0$ such that

$\|f(T)X - Xf(S)\| \leq D\|TX - XS\|$ for all contraction T, S and $X \in B(H)$. (27)

Proof. (i) First assume that f has continuous derivative on $\mathbb{T}$ and that T, S are strict contractions. It follows from (26) that, for all $X \in B(H)$,

$$f(T)X - Xf(S) = (f(L_T) - f(R_S))X = \hat{f}(L_T, R_S)(TX - XS).$$

By Lemma (3.2.7), the analytic functions $\sigma_N(z,w)$ on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ uniformly converge to $\hat{f}(z,w)$ on $\mathbb{D}_r \times \mathbb{D}_r$ where $\mathbb{D}_r$ is the closed disk of radius $r = \max(\|T\|, \|S\|) < 1$. It follows from continuity of the Holomorphic Functional Calculus (see [123, §13]) that $\hat{f}(L_T, R_S)$ is the norm limit of the operators $\sigma_N(L_T, R_S) = \sum_{n=1}^{N} u_n(L_T)v_n(R_S)$. It follows from (25) that $\sigma_N(L_T, R_S) = \Gamma_N(T,S)$. Hence, by Lemma (3.2.8), $\|\sigma_N(L_T, R_S)\|_{B(H)} \leq D$. Therefore $\|\hat{f}(L_T, R_S)\|_{B(H)} \leq D$ and (27) holds.

Let now T,S be arbitrary contractions. Applying (27) to $,rS$, for $0 < r < 1$, we get

$$\|f(rT)X - Xf(rS)\| \leq D\|rTX - rXS\| \leq D\|TX - XS\|.$$

Hence

$$\|f(T)X - Xf(S)\| \leq \|f(rT)X - Xf(rS)\| + \|(f(T) - f(rT))X\| + \|X(f(S) - f(rS))\|$$
$$\leq D\|TX - XS\| + \|f(T) - f(rT)\|\|X\| + \|X\|\|f(S) - f(rS)\|.$$

Letting $r \to 1$ and using the norm-continuity of the map $T \to f(T)$, we obtain that (27) holds for all contractions. Thus we proved the theorem for continuously differentiable functions.

(ii) Let now f be any operator Lipschitz function on $\mathbb{T}$ from $A(\mathbb{T})$. Let φ be a non-negative infinitely differentiable function on $\mathbb{T}$ with $\int_{\mathbb{T}} \varphi(t)dt = 1$. The convolution

$$h(t) = \varphi * f(t) = \int_{\mathbb{T}} \varphi(s) f(ts^{-1}) ds$$

is also infinitely differentiable and belongs to $A(\mathbb{T})$, since the negative Fourier coefficients

$$\hat{h}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i\theta})\, e^{-in\theta} d\theta = i\hat{f}(n) \int_{-\pi}^{\pi} e^{-i(n-1)\phi}\, \varphi(e^{i\phi}) d\phi = 0 \ \text{ for } n < 0,$$

as $\hat{f}(n) = 0$. Moreover, it is operator Lipschitzian on $\mathbb{T}$ with the same constant. Indeed,

$$h(U) - h(V) = \int_{\mathbb{T}} \varphi(s)\big(f(Us^{-1}) - f(Vs^{-1})\big) ds$$

for unitary operators U,V . Since f is an operator Lipschitz function on $\mathbb{T}$ and $|s| = 1$, we have from (13) that

$$\|h(U) - h(V)\| \leq \int_{\mathbb{T}} \varphi(s)\|f(Us^{-1}) - f(Vs^{-1})\| ds \leq \int_{\mathbb{T}} \varphi(s) D\|U - V\| ds = D\|U - V\|.$$

Since h is infinitely differentiable, (27) holds for it and all $T,S \in \mathrm{Con}(H)$ and $X \in B(H)$. Thus

$$\|h(U)X - Xh(V)\| = \left\|\int_{\mathbb{T}} \varphi(s)\big(f(Ts^{-1})X - Xf(Ss^{-1})\big) ds\right\| \leq D\|TX - XS\|. \quad (28)$$

Set $F(s) = f(Ts^{-1})X - Xf(Ss^{-1})$. For $t,s \in \mathbb{T}$,

$$\|F(t) - F(s)\| \leq \|f(Tt^{-1})X - Xf(Ts^{-1})\|\|X\| + \|X\|\|f(St^{-1})X - Xf(Ss^{-1})\|.$$

Since the map $T \to f(T)$ is norm-continuous, $\|F(t) - F(s)\| \to 0$, as $s \to t$. Take a sequence $\{\varphi_n\}$ of functions as above with the support of φ_n contained in $\mathbb{T}_n = \{t \in \mathbb{T}: |t - 1| \leq \frac{1}{n}\}$. Then

$$\left\|F(1) - \int_{\mathbb{T}} \varphi_n(s) F(s) ds\right\| = \left\|\int_{\mathbb{T}} \varphi_n(s)(F(1) - F(s)) ds\right\| \leq \sup_{s\in\mathbb{T}_n} \|F(t) - F(s)\| \to 0,$$

and

$$\|f(T)X - Xf(S)\| = \|F(1)\| \leq \left\|F(1) - \int_{\mathbb{T}} \varphi_n(s) F(s) ds\right\| + \left\|\int_{\mathbb{T}} \varphi_n(s) F(s) ds\right\|.$$

By (28) $\left\|\int_{\mathbb{T}} \varphi_n(s)F(s)ds\right\| \leq D\|TX - XS\|$. Letting $n \to \infty$, we conclude that (27) holds.

Proof of Theorem (3.2.1). To complete the proof we only need to prove implication (i) ⇒ (iii). This is done by substituting **1** for X in (27).

Denote by C_p ,$1 \leq p < \infty$, the Schatten ideals of compact operators on H with norm $\|\cdot\|_p$ and by C_∞ the ideal $C(H)$ of all compact operators on H . Recall (see [125]) that

$\|AXB\|_p \le \|A\|\|X\|_p\|B\|$ for all $A, B \in B(H)$ and $X \in C_p$ (29)

If A_n tend to A in the strong operator topology in (H): $A_n \xrightarrow{(\text{sot})} A$, then [125, Theorem III.6.3]
$\|A_nX - AX\|_p \to 0$ for all $X \in C_p$. (30)

If H is finite-dimensional, then all C_p coincide with $B(H)$ but the norms are different.

It is well known that $B(H)$ is isomorphic to the dual space of the ideal C_1 of all nuclear operators on H and that $C_1$is isomorphic to the dual space of C_∞ . Both dualities are given by the bilinear form $\langle A, B\rangle =$ tr(AB). For $T \in B(H)$, set

$L_T^p = L_T|_{C_p}$, $R_T^p = R_T|_{C_p}$ for $1 \le p \le \infty$.

For $A \in B(H)$ and $\in C_1$, $\langle L_T(A), B\rangle =$ tr(TAB) = tr$(ABT) = \langle A, R_T^1(B)\rangle$. Using this, we obtain the following identities for the conjugate operators:

$(L_T^\infty)^* = R_T^1$, $(R_T^\infty)^* = L_T^1$, $(L_T^1)^* = R_T$ and $(R_T^1)^* = L_T$ (31)

We obtain the Lipschitz type inequalities for C_p-norms for the action of operator Lipschitz functions from $A(\mathbb{T})$ on Con(H).

Proposition(3.2.10)[114]. Let $f \in A(\mathbb{T})$ and $1 \le p \le \infty$. Let there exists $D > 0$ such that
$\|f(T)X - Xf(S)\|_p \le D\|TX - XS\|_p$, (32)

for all finite rank contractions T, S and all finite rank operators X. Then (32) holds for any pair of contractions T, S and all $X \in C_p$, and

$\|f(T) - f(S)\|_p \le D\|T - S\|_p$ if $T - S \in C_p$ (33)

Proof. Let X be a finite rank operator and $S, T \in$ Con(H) . Choose finite rank contractions S_n , T_n such that $S_n^* \xrightarrow{(\text{sot})} S^*$, $T_n \xrightarrow{(\text{sot})} T$. By (30), $\|(T - T_n)X\|_p \to 0$ and $\|X(S - S_n)\|_p =$ $\|(S_n^* - S^*)X^*\|_p \to 0$. Furthermore $f(T_n) \xrightarrow{(\text{sot})} f(T)$whence, by (30), $\|(f(T) - f(T_n))X\|_p \to 0$.

The function $\tilde{f}(t) = \overline{f(\bar{t})}$ belongs to $A(\mathbb{T})$ and (see [122, Section III.2]) $\tilde{f}(S^*) = f(S)^*$. Hence $f(S_n)^* - f(S)^* = \tilde{f}(S_n^*) - \tilde{f}(S^*) \xrightarrow{(\text{sot})} 0$, so that $\|X(f(S_n) - f(S))\|_p = \|f(S_n)^* - f(S)^*X^*\|_p \to 0$.

Using these norm limits and taking the limit in the inequality $\|(f(T_n)X - Xf(S_n))\|_p \le D\|T_nX - XS_n\|_p$, we obtain (33) for all $T, S \in$ Con(H) and all finite rank operators X. For arbitrary $X \in C_p$, choose finite rank operators X_n such that $\|X - X_n\|_p \to 0$. Now (33) can be proved by taking the limit in the inequality $\|(f(T)X_n - X_nf(S))\|_p \le D\|TX_n - X_nS\|_p$ and using (12) and (29).

Let now $T - S \in C_p$. Let P_n be an increasing sequence of finite-dimensional projections such $P_n \xrightarrow{(\text{sot})} 1$. . Replace in (32) T, S by P_nT, SP_n and X by P_n. This gives

$$\|f(P_nT)P_n - P_nf(SP_n)\|_p \le D\|P_nTP_n - P_nSP_n\|_p = D\|P_n(T - S)P_n\|_p \le D\|T - S\|_p \quad (34)$$

We have that $P_nT \xrightarrow{(\text{sot})} T$ and $SP_n \xrightarrow{(\text{sot})} S$. Hence $f(P_nT) \xrightarrow{(\text{sot})} f(T)$ and $f(SP_n) \xrightarrow{(\text{sot})} f(S)$. By (12), $\|f(P_nT)\| \le \|f\|$. Therefore the finite rank operators $f(P_nT)P_n - P_nf(SP_n) \xrightarrow{(\text{sot})} f(T) - f(S)$. Taking (34) into account, we obtain from [125, Theorem III.5.1] that $f(T) - f(S) \in C_p$ and $\|f(T) - f(S)\|_p \le D\|T - S\|_p$.

Theorem (3.2.11)[114]. Let $f \in A(\mathbb{T})$ be an operator Lipschitz function on $\mathbb{T}$ with constant D. Then, for all contractions , S , and all $1 \le p \le \infty$,

$$\|f(T)X - Xf(S)\|_p \le D\|TX - XS\|_p \quad \text{for } X \in C_p, \qquad (35)$$

$$\|f(T) - f(S)\|_p \le D\|T - S\|_p \quad \text{if } T - S \in C_p. \qquad (36)$$

Proof. First assume that $\sigma(T) \cap \sigma(S) = 0$. By Rosenblum's theorem (see [126]) the operator $\Delta = L_T - R_S$ on $B(H)$ is invertible and we may consider the operator $F = (L_{f(T)} - R_{f(S)})\Delta^{-1}$. It follows from (27) that $\|F\| \le D$. The operator $\Delta^\infty = \Delta|_{C^\infty}$ is also invertible, so F pre serves C^∞ . Set $F^\infty = F|_{C^\infty}$. Then $\|F^\infty\| \le D$, so $\|(F^\infty)^*\| \le D$, where $(F^\infty)^*$ is the conjugate operator on C_1. As Δ^{-1} commutes with $L_{f(T)} - R_{f(S)}$, we see from (31) that $F^\infty = (\Delta^\infty)^{-1}(L^\infty_{f(T)} - R^\infty_{f(S)})$ and. Hence $((\Delta^\infty)^{-1})^* = ((\Delta^\infty)^*)^{-1} = (L^1_T - R^1_S)^{-1}$. Hence

$$(F^\infty)^* = \left((\Delta^\infty)^{-1}(L^\infty_{f(T)} - R^\infty_{f(S)})\right)^* = (L^1_{f(T)} - R^1_{f(S)})((\Delta^\infty)^*)^{-1} == (L^1_{f(T)} - R^1_{f(S)})(L^1_T - R^1_S)^{-1}$$
$$= F|_{C_1}.$$

Since $\max(\|F\|, \|F|_{C_1}\|) \le D$, it follows from the interpolation theory (see, for example [125], [2, Theorem B]) that F preserves C_p and $\left\|F|_{C_p}\right\| \le D$.

For $X \in C_p$, set $Y = (L_T - R_S)X$. Then $Y \in C_p$ and we obtain (35) for this case

$$\|f(T)X - Xf(S)\|_p = \left\|\left(L_{f(T)} - R_{f(S)}\right)X\right\|_p = \|F(Y)\|_p \le D\|Y\|_p = D\|TX - XS\|_p \qquad (37)$$

Let $\dim(H) < \infty$ and $T, S \in \text{Con}(H)$. Choose contractions S_n such that $\sigma(T) \cap \sigma(S_n) = \emptyset$ and $\|S - S_n\| \to 0$. Hence $\|f(S_n) - f(S)\| \to 0$. We have from (29) and (37) that

$$\begin{aligned}\|f(T)X - Xf(S)\|_p &\le \|f(T)X - Xf(S_n)\|_p + \|X(f(S_n) - f(S))\|_p \\ &\le D\|f(T)X - Xf(S_n)\|_p + \|X\|_p\|f(S_n) - f(S)\| \\ &\le D\|TX - XS_n\|_p + \|X\|_p\|f(S_n) - f(S)\|.\end{aligned}$$

By (29), $\|TX - XS_n\|_p \le \|TX - XS\|_p + \|X\|_p\|S - S_n\|$. Taking the limit, we obtain that (35) holds for all $T, S \in \text{Con}(H)$ if $\dim(H) < \infty$. Thus it holds for arbitrary H if T, S are finite rank contractions. Applying Proposition (3.2.10), we conclude the proof.

The result obtained in Theorem (3.2.11) is not the optimal one. It would be desirable to show that (35) and (36) hold if $f \in A(\mathbb{T})$ is a C_p-Lipschitz function on $\mathbb{T}$ (see (46)). Then we would have proved an analogue of Theorem (3.2.1) for C_p-Lipschitz function on $\mathbb{T}$.

Let H be a separable Hilbert space. Consider it as a subspace of a separable Hilbert space $\mathcal{H}$such that the complement of H in $\mathcal{H}$is infinite-dimensional. A natural approach to the studied problems would be a construction of unitary dilations U, V on $\mathcal{H}$for any two contractions T, S on H in such a way that

$U-V\in C_p$ and $\|U-V\|_p\le C\|T-S\|_p$, if $T-S\in C_p$, (38)
where the constant $C>0$ does not depend on the contractions. However, we will show that such construction is in no means possible. Consider the multivalued map Dil that takes each contraction $T\in \mathrm{Con}(H)$ into the set $\mathrm{Dil}(T)$ of all its power unitary dilations U on $\mathcal{H}: P_H U^n|_H$ for all n. We will establish that it is not Lipschitzian. On the other hand, we will estimate the continuity moduli $\omega$of Dil and use it to obtain Lipschitz type inequalities in C_p
norms.
Denote $C_b=B(H)$. For p,q $[1,\infty]\cup b$ and $t>0$, set
$\delta_p(T_1,T_2)=\inf\{\|U_1-U_2\|_p: U_i\in \mathrm{Dil}(T_i)\}$, for $T_1,T_2\in \mathrm{Con}(H)$with $U_1-U_2\in C_p$
and

$$\omega_{p,q}(t)=\sup\{\delta_p(T_1,T_2): T_i\in \mathrm{Con}(H), T_1-T_2\in C_p\ and\ \|T_1-T_2\|_q=t\}.$$

Proposition(3.2.12)[114]. For all p,q $[1,\infty]\cup b$ and each $t\in(0,1)$,
$\omega_{p,q}(t)\ge\sqrt{2t}$.
Proof. It suffices to show that, f or each $t\in(0,1)$, there are contractions T,S such that $\|T-S\|_q=t$ and that $\|T-S\|_p\le\sqrt{2t}$, for all their unitary dilations , V .

Let $e\in H$ and Q be the projection on $\mathbb{C}e$. Set $T=Q$ and $S=(1-t)Q$. Clearly, $\|T-S\|=t$. As $T-S$ is rank one operator, $\|T-S\|_q=\|T-S\|$ for all q .

Let P be the projection on H in $\mathcal{H}$. Then $PU|_H=T$, $PV|_H=S$, so that $(Ue,e)=1$,$(Ve,e)=1-t$. Hence, $Ue=e$. Then $\|U-V\|\ge\sqrt{2t}$, as

$$\|U-V\|^2\ge\|Ue-Ve\|^2=(Ue,Ue)+(Ve,Ve)-(Ue,Ve)-(Ve,Ue)=2t$$

If $U-V\in C_p$ then $\|U-V\|_p\ge\|U-V\|$. Hence $\|U-V\|_p\ge\sqrt{2t}$.
It follows from Proposition (3.2.12) that (38) does not hold. To estimate the continuity moduli $\omega_{p,q}$ of Dil, we will consider the "canonical" unitary dilation of $T\in \mathrm{Con}(H)$ (see [122, Chapter I,§5]). Set

$$D_T=(1-T^*T)^{1/2}, D_{T^*}=(1-TT^*)^{1/2} \text{ and } \mathcal{H}=\bigoplus_{-\infty}^{\infty} H_n \text{ with all } H_n=H$$

Let P be the projection on $\mathfrak{H}=H_0\oplus H_1$ and U_0^T be the operator on $\mathcal{H}$such that

$$U_0^T=PU_0^TP \text{ and } U_0^T|_{\mathfrak{H}}=\begin{pmatrix} D_T & -T^* \\ T & D_{T^*}\end{pmatrix}.$$

Let V be the unitary shift operator on $\mathcal{H}$ such that $(Vx)_n=x_{n+1}$ for each $x=(x_n)\in\mathcal{H}$. Then the operator $U^T=V(\mathbf{1}_{\mathcal{H}}-P+U_0^T)$ is the unitary dilation of.

If S is another contraction on H then $U^T-U^S=V(U_0^T-U_0^S)$, so $\|U^T-U^S\|=\|(U_0^T-U_0^S)\|$ and $\|U^T-U^S\|_p=\|(U_0^T-U_0^S)\|_p$, if $U_0^T-U_0^S\in C_p$. As

$$U_0^T|_{\mathfrak{H}}-U_0^S|_{\mathfrak{H}}=\begin{pmatrix} D_T & 0 \\ 0 & D_{T^*}-D_{S^*}\end{pmatrix}+\begin{pmatrix} 0 & -T^*+S^* \\ T-S & 0\end{pmatrix},$$

we have
$\|U^T-U^S\|\le\|T-S\|+\max(\|D_T-D_S\|,\|D_{T^*}-D_{S^*}\|)$, (39)
$\|U^T-U^S\|_p\le 2^{1/p}\|T-S\|_p+2^{1/p}\max(\|D_T-D_S\|_p,\|D_{T^*}-D_{S^*}\|_p)$, (40)

Proposition (3.2.13)[114].
(i) $\| U^T - U^S\| \le \|T-S\| + 2^{1/p}\|T-S\|^2$.
(ii) let $1 \le p < \infty$ and $T-S \in C_{p/2}$. Then $T-S \in C_p$, $U^T - U^S \in C_p$ and

$\| U^T - U^S\|_p \le 2^{1/p}\|T-S\|_p + 2^{\frac{1}{2}+\frac{1}{p}}\|T-S\|_{p/2}^{1/2}$, if $\frac{p}{2} \ge 1$,

$\| U^T - U^S\|_p \le 2^{1/p}\|T-S\|_p + 2^{\frac{3}{p}}\|T-S\|_{p/2}^{1/2}$, if $\frac{p}{2} < 1$.

Proof. For $R = R^* \in B(H)$, denote by R^+ and R^- the positive and negative parts of $R: R^{\mp} = \frac{1}{2}(|R| \pm R)$. Then $\|R\| = \max(\|R^+\|, \|R^-\|)$. If $R \in C_p$ then $R^+, R^- \in C_p$ and

$\|R\|_p^p = \||R|\|_p^p = \|R^+\|_p^p = \|R^-\|_p^p$ (41)

For $0 < A \in C_p$, we have $A^{1/2} \in C_{2p}$ and $\|A^{1/2}\|_{2p} = \|A\|_p^{1/2}$. Hence, by (41),

$$\|(R^+)^{1/2}\|_{2p}^{2p} + \|(R^-)^{1/2}\|_{2p}^{2p} = \|R^+\|_p^p + \|R^-\|_p^p = \|R\|_p^p$$

We need now the following result of Birman, Koplienko and Solomyak [129, Theorem 1]. Let A, B be positive operators in $B(H)$. Set $G = B^{1/2} - A^{1/2}$ and $F = B - A$. Then
(1) $\|G^+\| \le \|F^+\|^{1/2}$ and $\|G^-\| \le \|F^-\|^{1/2}$;
(2) If $|F|^{1/2} \in C_p$ then $G \in C_p$ and $\|G^+\|_p \le \|(F^+)^{1/2}\|_p$ and $\|G^-\|_p \le \|(F^-)^{1/2}\|_p$.
Combining this with (41) and (42), yields

$\|G\| = \max(\|G^+\|, \|G^-\|) \le \max(\|F^-\|^{1/2}, \|F^+\|^{1/2}) = \|F\|^{1/2}$, (43)

$\|G\|_p^p = \|G^+\|_p^p + \|G^-\|_p^p \le \|(F^+)^{1/2}\|_p^p + \|(F^-)^{1/2}\|_p^p = \|F\|_{2p}^{2p}$. (44)

Set $B = \mathbf{1} - T^*T$ and $A = \mathbf{1} - S^*S$. We obtain from (43) that

$$\|D_T - D_S\| = \|(\mathbf{1} - T^*T)^{1/2} - (\mathbf{1} - S^*S)^{1/2}\| \le \|(\mathbf{1} - T^*T) - (\mathbf{1} - S^*S)\|^{1/2}$$
$$= \|T^*(T-S) + (T^* - S^*)S\|^{1/2} \le (2\|T-S\|)^{1/2}.$$

Similarly, $\|D_{T^*} - D_{S^*}\| \le (2\|T-S\|)^{1/2}$ and part (i) follows from (39).
Let $T - S \in C_{2p}$. Then

$$F = (\mathbf{1} - T^*T) - (\mathbf{1} - S^*S) = T^*(T-S) + (T^* - S^*)S \in C_{2p},$$

so $|F|^{1/2} \in C_p$. Hence $D_T - D_S \in C_p$ and we obtain from (44) that

$$\|D_T - D_S\|_p^p = \|(\mathbf{1} - T^*T)^{1/2} - (\mathbf{1} - S^*S)^{1/2}\|_p^p \le \|(\mathbf{1} - T^*T) - (\mathbf{1} - S^*S)\|_{p/2}^{p/2}$$
$$= \|T^*(T-S) + (T^* - S^*)S\|_{p/2}^{p/2} \le \left(2\|T-S\|_{p/2}\right)^{p/2}.$$

If $p/2 \ge 1$ then $\|D_T - D_S\|_p^p \le \left(2\|T-S\|_{p/2}\right)^{p/2}$, so $\|D_T - D_S\|_p \le 2^{1/2}\|T-S\|_{p/2}^{1/2}$. Similarly, $\|D_{T^*} - D_{S^*}\| \le 2^{1/2}\|T-S\|_{p/2}^{1/2}$ and the first formula in (ii) follows from (40).

If $p/2 < 1$ then (see [1160, Lemma XI.9.9]) $\|D_T - D_S\|_p^p \le 2\|T-S\|_{p/2}^{p/2}$, so $\|D_T - D_S\|_p \le 2^{2/p}\|T-S\|_{p/2}^{1/2}$. Similarly, $\|D_{T^*} - D_{S^*}\|_p \le 2^{2/p}\|T-S\|_{p/2}^{1/2}$ and the second formula in (ii) follows from (40).

Corollary (3.2.14)[114]. Let $t \in (0,1)$. Then $\omega_{p,\frac{p}{2}}(t) \le k_p t^{1/2}$, where

$k_p = 2^{1/p}(1 + 2^{1/2})$, if $p \ge 2$, and $k_p = 2^{1/p} + 2^{3/p}$, if $p \le 2$. (45)

Proof. It suffices to use Proposition (3.2.13) and to note that $\omega_{p,q}(t) \le \|U^T - U^S\|_p$ for $\|T - S\|_p = t$ and $\|T - S\|_p \le \|T - S\|_{p/2} \le \|T - S\|_{p/2}^{1/2}$.

Recall that f is called a C_p-Lipschitz function on $\mathbb{T}$, $1 \le p \le \infty$, if there is $D > 0$ such that

$f(U) - f(V) \in C_p$ and $\|f(U) - f(V)\|_p \le D\|U - V\|_p$, (46)

for all unitary U, V, with $U - V \in C_p$.

For contractions T_1, T_2, set $U_i = U^{T_i}$. If $f \in A(\mathbb{T})$ is a C_p-Lipschitz function then, by (16),

$$\|f(T_1) - f(T_2)\|_p = \|P(f(U_1) - f(U_2))|_H\|_p \le \|f(U_1) - f(U_2)\|_p \le D\|U_1 - U_2\|_p \le D\delta_p(T_1, T_2).$$

Hence $\max\{\|f(T_1) - f(T_2)\|_p : \|T_1 - T_2\|_q = t\} \le D\omega_{pq}(t)$. Combining this with Corollary (3.2.14)yields

Corollary(3.2.15)[114]. If $f \in A(\mathbb{T})$ is a C_p-Lipschit: function on $\mathbb{T}$ with constant D, then

$$\|f(T) - f(S)\|_p \le Dk_p\|T - S\|_{p/2}^{1/2},$$

for all contractions T, S satisfying $T - S \in C_{p/2}$, where k_p is defined in (45).

Now we will show that the inequalities (35) and (36) hold for $p = 2$.

A function f on $\mathbb{T}$ is a C_2-Lipschitz function with constant D (see, for example, [127]) if and only if f is a Lipschitz function on $\mathbb{T}$with constant D in the usual sense, that is,

$|f(t) - f(s)| \le D|t - s|$ (47)

If f has bounded derivative, then one can take $D = \sup_{t\in\mathbb{T}}|f'(t)|$. Our aim is to prove.

Theorem (3.2.16)[114]. If $f \in A(\mathbb{T})$ satisfies (47) then, for all contractions T, S

$\|f(T)X - Xf(S)\|_2 \le D\|TX - XS\|_2$ for $X \in C_2$, and (48)

$\|f(T) - f(S)\|_2 \le D\|T - S\|_2$ if $T - S \in C_2$. (49)

Let $C(\mathbb{D})$ be the C*-algebra of all continuous functions on $\mathbb{D}$. As it is nuclear, the tensor product $C(\mathbb{D})\otimes C(\mathbb{D})$ is isomorphic to $C(\mathbb{D} \times \mathbb{D})$ with $f\otimes g \cong f(z)g(w)$ for $f, g \in C(\mathbb{D})$.

Denote by $A(\mathbb{D} \times \mathbb{D})$ the closure of the algebraic tensor product $A(\mathbb{D})\odot A(\mathbb{D})$ in $C(\mathbb{D} \times \mathbb{D})$. It is well known that $A(\mathbb{D} \times \mathbb{D})$consists of all functions continuous on $\mathbb{D} \times \mathbb{D}$ and holomorphic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ (the bidisk algebra).

Lemma (3.2.17)[114]. Let π_i, $i = 1,2$, be representations of $A(\mathbb{D})$ on a Hilbert space H such that $\|\pi_i\| \le 1$ and $[\pi_1(g), \pi_2(h)] = 0$ for all $g, h \in A(\mathbb{D})$. Then there exists a representation $\pi$of $A(\mathbb{D} \times \mathbb{D})$ on H such that $\|\pi\| \le 1$ and $\pi(f(z)g(w)) = \pi_1(f)\pi_2(g)$ for $g, h \in A(\mathbb{D})$.

Proof. Let $id \in A(\mathbb{D})$ be the function such that $id(z) \equiv z$. Then $T_i = \pi_i(id)$ are commuting contractions in $B(H)$ and, for each polynomial p, $\pi_i(p) = p(T_i)$. Hence $\pi_i(f) = f(T_i)$ for each $f \in A(\mathbb{D})$. Indeed, if polynomials p_n converge to f then, by (12),

$$\|\pi_i(f) - f(T_i)\| \le \|\pi_i(f - p_n)\| + \|p_n(T_i) - f(T_i)\| \le 2\|f - p_n\| \to 0.$$

By Ando's theorem [122, Theorem I.6.4], there are a Hilbert space $K \subset H$ and commuting unitary operators U_i on K such that $T_i = PU_1|_H$, where P is the projection on , and

$T_1^n T_2^m = PU_1^n U_2^m|_H$ for all m, n (50)

The *-representations ρ_i of $C(\mathbb{D})$ on K defined by

$$\rho_i(g) = g(U_i)$$
commute and, by (16), $P\rho_i(f)|_H = Pf(U_i)|_H = f(T_i) = \pi_i(f)$ for $f \in A(\mathbb{D})$. For each $h \in C(\mathbb{D} \times \mathbb{D})$, there is a unique operator $h(U_1, U_2)$ in the commutative C*-subalgebra of $B(H)$ generated by U_1, U_2 (see [12, Corollary 16.7]), $\rho: h \to h(U_1, U_2)$ is a *-representation of $C(\mathbb{D} \times \mathbb{D})$ on K and $\rho(f \otimes g) = \rho(f\ (z)g(w)) = f(U_1)g(U_2) = \rho_1(f)\rho_2(g)$ for all $f, g \in C(\mathbb{D}$. It remains to define a representation $\pi$of $C(\mathbb{D} \times \mathbb{D})$ by setting $\pi(h) = P_\rho(h)|_H$, for $h \in C(\mathbb{D} \times \mathbb{D})$.

Proof of Theorem (3.2.16). First suppose that f is continuously differentiable. By (22), $f(z) - f(w) = \hat{f}(z, w)(z - w)$. By Lemma (3.2.6), $\hat{f}(z, w)$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ and continuous on D x D. Hence it belongs to $A(\mathbb{D} \times \mathbb{D})$.

Let $T, S \in \text{Con}(H)$. Define contractive representations $\pi_i$of $A(\mathbb{D})$ on the Hilbert space C_2 by setting $\pi_1(g)(X) = g(T)X, \pi_2(g)(X) = Xg(S)$, for $X \in C_2$. They commute and, by (12) and (29), $\|\pi_1(g)(X)\|_2 \leq \|g(T)\|\|X\|_2 \leq \|g\|\|X\|_2$, so $\|\pi_1\| \leq 1$. Similarly, $\|\pi_2\| \leq 1$. Let πbe the representation of $A(\mathbb{D} \times \mathbb{D})$ constructed in Lemma (3.2.17). Then $\pi(f(z) - f(w))X = (\pi_1(f) - (\pi_2 f))X = f(T)X - Xf(S)$ and $\pi(z - w)X = TX - XS$. As $\pi\big(f(z) - f(w)\big) = \pi(\hat{f}(z, w))\pi(z - w)$, we have

$$\|f(T)X - Xf(S)\| \leq \|\pi(\hat{f})\|\|TX - XS\| \leq \|\hat{f}\|\|TX - XS\|,$$

where, by (22), $\|\hat{f}\| = \sup_{z \in \mathbb{D} \times \mathbb{D}} |\hat{f}(z, w)| = \sup_{z \in \mathbb{D}} |f'(z)|$. By Lemma (3.2.6), $\hat{f} \in A(\mathbb{T})$ so $\sup_{z \in \mathbb{D}} |f'(z)| = \sup_{t \in \mathbb{T}} |f'(t)|$. Thus (48) holds for continuously differentiable functions f with $D = \sup_{t \in \mathbb{T}} |f'(t)|$.

Let now $f \in A(\mathbb{T})$ be an arbitrary function that satisfies (47). Choose the sequence $\{\varphi_n\}$ of infinitely differentiable functions on $\mathbb{T}$ as in the proof of part (2) in Theorem (3.2.9) and set $h_n = \varphi_n * f$. Then h_n are infinitely differentiable, belong to $A(\mathbb{T})$ and satisfy (47) with the same constant D. As $D = \sup_{t \in \mathbb{T}} |h'_n(t)| \leq D$, it follows that

$$\|h_n(T)X - Xh_n(S)\|_2 \leq D_n\|TX - XS\|_2 \leq D\|TX - XS\|_2 \quad \text{for } X \in C_2.$$

Repeating the end of the proof of Theorem (3.2.9) and replacing the operator norm $\|\cdot\|$ by the norm $\|\cdot\|_2$, we obtain that (48) holds for f . Now it suffices to use Proposition (3.2.10) to obtain that (49) also holds for f .

The above proof extends to a much more general situation. Let $\mathfrak{A}$ be a semi finite von Neumann algebra and τbe a normal faithful trace on $\mathfrak{A}$. By $L_2(\mathfrak{A})$ we denote the non-commutative L_2-space of $\mathfrak{A}$. It coincides with C_2 when $\mathfrak{A} = B(H)$ and with $L_2(X, \mu)$ when $\mathfrak{A} = L_\infty(X, \mu)$ for some measure space (X, μ). Then the following extension of Theorem (3.2.16) is immediate:

Theorem (3.2.18)[114]. Let $f \in A(\mathbb{T})$ have a bounded derivation. Then (49) holds for all contractions $T, S \in \mathfrak{A}$ with $T - S \in L_2(\mathfrak{A})$ with $D = \sup_{t \in \mathbb{T}} |f'(t)|$.

Denote by $B(H_{r-1})$ the algebra of all bounded operators on a Hilbert space . An operator $T_{r-1} \in B(H_{r-1})$ is a sequence of contractions if $\|T_{r-1}\| \leq 1$. Von Neumann's inequality states that

$\|(1+\epsilon)(\mathrm{T}_{r-1})\| \leq \|1+\epsilon\|$, for all polynomials $(1+\epsilon)$ and sequence of contractions T_{r-1}. Since the sub algebra of all polynomials is dense in $A(\mathbb{D})$, the sequence of operators $\mathrm{g}_{r-1}(\mathrm{T}_{r-1})$ can be correctly defined for each $\mathrm{g}_{r-1} \in A(\mathbb{D})$ and contractions T_{r-1}, and

$$\|\mathrm{g}_{r-1}(\mathrm{T}_{r-1})\| \leq \|\mathrm{g}_{r-1}\|. \quad (51)$$

We shall list now a few definitions and facts that will be used later. Denote by $\mathrm{Con}(H_{r-1})$ the set of all contractions on a Hilbert space H_{r-1}. Recall that a unitary operator U_{r-1} on a Hilbert space $\mathfrak{H} \supset H_{r-1}$ is called a (power) unitary dilation of $\mathrm{T}_{r-1} \in \mathrm{Con}(H_{r-1})$ if

$$(\mathrm{T}_{r-1})^n = P_{r-1}(U_{r-1})^n|_{H_{r-1}} \text{for all } n \in \mathbb{N}, \quad (52)$$

where P_{r-1} is the projection on H_{r-1} in $\mathfrak{H}$. If U_{r-1} is a unitary dilation of T_{r-1}, it follows from (51) and (52) that

$$f_{r-1}(\mathrm{T}_{r-1}) = P_{r-1}f_{r-1}(U_{r-1})|_{H_{r-1}} \text{ for each } f_{r-1} \in A(\mathbb{T}). \quad (53)$$

It follows from (51) that, for each $\mathrm{T}_{r-1} \in \mathrm{Con}(H_{r-1})$, the homomorphism $f_{r-1} \to f_{r-1}(\mathrm{T}_{r-1})$ from $A(\mathbb{T})$ into $B(H_{r-1})$ is norm-continuous. Furthermore, for any $f_{r-1} \in A(\mathbb{T})$, the map $\mathrm{T}_{r-1} \to f_{r-1}(\mathrm{T}_{r-1})$ is norm-continuous and continuous in the strong operator topology on $\mathrm{Con}(H_{r-1})$.
The operator $M_{\mathrm{g}_{r-1}}$ is linear and bounded on $(C^{r-1})_2(H_r, H_{r+1})$. If it is also bounded with respect to the usual operator norm:

$$\left\|M_{\mathrm{g}_{r-1}}(K)\right\| \leq C^{r-1}\|K\| \quad \text{for all } K \in (C^{r-1})_2(H_r, H_{r+1}), \quad (54)$$

and some $C^{r-1} > 0$, then g_{r-1} is called a Schur multiplier on $(\mathcal{T}_r \times \mathcal{S}_r, \mu \times v)$.
Peller [116] characterized Schur multipliers by several equivalent properties, one of which can be formulated as follows: g_{r-1} is a Schur multiplier if and only if there are a separable Hilbert space H_{r-1} and weakly measurable H_{r-1}-valued functions $\vec{x}$ on $\mathcal{T}_r$ and $\vec{y}$ on $\mathcal{S}_r$ such that

$$\mathrm{g}_{r-1}(1-\epsilon, s_{r-1}) = (\vec{x}(1-\epsilon), \vec{y}(s_{r-1})) \text{ a.e. on } \mathcal{T}_r \times \mathcal{S}_r, \text{ and} \quad (55)$$

$$\|\vec{x}(1-\epsilon)\| \leq (C^{r-1})^{1/2}, \quad \|\vec{y}(s_{r-1})\| \leq (C^{r-1})^{1/2} \text{ a.e. on } \mathcal{T}_r \text{ and } \mathcal{S}_r \quad (56)$$

for some $C^{r-1} > 0$. Choosing an orthonormal basis $\{\vec{e}_n\}_{n\in\mathbb{N}}$ in H_{r-1} and setting $u_n(1-\epsilon) = (\vec{x}(1-\epsilon), \vec{e}_n)$, $u_n(s_{r-1}) = (\vec{e}_n, \vec{y}(s_{r-1}))$, we present g_{r-1} in the form

$$\mathrm{g}_{r-1}(1-\epsilon, s_{r-1}) = \sum_{n=1}^{\infty} u_n((1-\epsilon))v_n(s_{r-1}) \text{ a.e. on } \mathcal{T}_r \times \mathcal{S}_r \quad (57)$$

with

$$\sum_n |u_n(1-\epsilon)|^2 \leq C^{r-1} \text{ a.e. on } \mathcal{T}_r, \quad and \sum_n |v_n(s_{r-1})|^2 \leq C^{r-1} \text{ a.e. on } \mathcal{S}_r. \quad (58)$$

This implies that there are measurable subsets $\mathcal{T}_{r-1}$ and $\mathcal{S}_{r-1}$ of $\mathcal{T}_r$ and $\mathcal{S}_r$ with $\mu(\mathcal{T}_r \backslash \mathcal{T}_{r-1}) = \nu(\mathcal{S}_r \backslash \mathcal{S}_{r-1}) = 0$ such that the sum in (57) is defined as a bounded function on $\mathcal{T}_{r-1} \times \mathcal{S}_{r-1}$ and (58) holds for all $(1-\epsilon) \in \mathcal{T}_{r-1}$ and $s_{r-1} \in \mathcal{S}_{r-1}$. One says in this case that the sum is a bounded function marginally almost everywhere (a.e.). This terminology originated in [117], where a subset $M \subset \mathcal{T}_r \times \mathcal{S}_r$ was called marginally null if $M \subset (A_i \times \mathcal{S}_{r-1}) \cup (\mathcal{T}_r \times (A_i + \epsilon))$, where $A_i \subset \mathcal{T}_r$ and $(A_i + \epsilon) \subset \mathcal{S}_r$ have zero measures. Two subsets of $\mathcal{T}_r \times \mathcal{S}_r$ are marginally equal if their symmetric difference is marginally null. Two functions are said to be equal marginally a.e. if the set of points, where the equality fails, is marginally null. (see[17])

Corollary (3.2.19) [293]:. If a Schur multiplier g_{r-1} is ω-continuous then the equality (57) holds marginally almost everywhere.

Proof. Note that if an ω-continuous function h_{r-1} on $\mathcal{T}_r \times \mathcal{S}_r$ equals zero a.e., then it is zero marginally a.e. Indeed, the set $F_{r-1} = \{(1-\epsilon, s_{r-1}): h_{r-1}(1-\epsilon, s_{r-1}) \neq 0\} = \bigcup_{i\in\mathbb{N}}(h_{r-1})^{-1}((U_{r-1})_i)$, where $\mathbb{C}\backslash\{0\} = \bigcup_{i\in\mathbb{N}}(U_{r-1})_i$ and all $(U_{r-1})_i$ are open. Each $(h_{r-1})^{-1}((U_{r-1})_i)$ marginally equals to a union of measurable rectangles $(A_i)_n \times (A_i + \epsilon)_n$ and has zero measure, as F_{r-1} has zero measure. Thus all $(\mu(A_i)_n) = v((A_i + \epsilon)_n) = 0$. Therefore $(h_{r-1})^{-1}((U_{r-1})_i)$ is marginally null, so F_{r-1} is marginally null.

Hence it follows that, if two ω-continuous functions coincide a.e., they coincide marginally a.e., as the difference of ω-continuous functions is ω-continuous by [118, Corollary 3.2]. Thus we only need to show that the sum in (57) is an ω-continuous function.

Since u_n , v_n are measurable, the functions $\widehat{u_n}(1-\epsilon, s_{r-1}) = u_n(1-\epsilon), \widehat{v_n}(1-\epsilon, s_{r-1}) = v_n(s_{r-1})$ are ω-continuouson $\mathcal{T}_r \times \mathcal{S}_r$. Hence, by [118, Corollary 3.2], all $(g_{r-1})_N(1-\epsilon, s_{r-1}) = \sum_{n=1}^{N} u_n(1-\epsilon)v_n(s_{r-1})$ are ω-continuous. As $(U_{r-1})_N(1-\epsilon) = (\sum_{n=N+1}^{\infty}|u_n(1-\epsilon)|^2)^{1/2}$ and $(V_{r-1})_N(s_{r-1}) = (\sum_{n=N+1}^{\infty}|v_n(s_{r-1})|^2)^{1/2}$ are measurable, the functions $(U_{r-1})_N(1-\epsilon)(V_{r-1})_N(s_{r-1})$ are ω-continuous on $\mathcal{T}_{r-1} \times \mathcal{S}_{r-1}$. Since

$$\left|\sum_{n=1}^{\infty} u_n(1-\epsilon)v_n(s_{r-1}) - g_{r-1})_N((1-\epsilon), s_{r-1})\right| = \left|\sum_{n=N+1}^{\infty} u_n(1-\epsilon)v_n(s_{r-1})\right|$$
$$\leq (U_{r-1})_N(1-\epsilon)(V_{r-1})_N(s_{r-1}) \to 0, \text{ as } N \to \infty$$

on $\mathcal{T}_{r-1} \times \mathcal{S}_{r-1}$, it follows from Lemma 3.3 [118] that $\sum_{n=1}^{\infty} u_n(1-\epsilon)v_n(s_{r-1})$ is ω-continuous on $\mathcal{T}_{r-1} \times \mathcal{S}_{r-1}$ and, hence, on $\mathcal{T}_r \times \mathcal{S}_r$.(See[114]).

Suppose now that $\mathcal{T}_r$ and $\mathcal{S}_r$ are separable metrizable compacts and μ, v are regular Borel measures with $\text{supp}(\mu) = \mathcal{T}_r$, $\text{supp}(v) = \mathcal{S}_r$. Our aim is to prove that if a Schur multiplier g_{r-1} is continuous then the vector functions $\vec{x}(1-\epsilon)$ and $\vec{y}(s_{r-1})$ in (55) can be chosen with some additional properties. For continuous functions, the condition that g_{r-1} is a Schur multiplier does not depend on the choice of μ, v (see [119, 120]), but we will not need this fact, as the measures will be fixed.

For a subset W_{r-1} of a Hilbert space H_{r-1} , by $\text{cls}(W_{r-1})$ we denote its closed linear span. We will say that W_{r-1} generates H_{r-1} if $\text{csl}(W_{r-1}) = H_{r-1}$.For the following see[114]

Corollary (3.2.20) [293]: Suppose that a continuous function g_{r-1} on $\mathcal{T}_r \times \mathcal{S}_r$ is a Schur multiplier. Then the vector functions $\vec{x}(1-\epsilon), \vec{y}(s_{r-1})$ and the space H_{r-1} can be chosen in such a way that

(i) each of the sets $\{\vec{x}(1-\epsilon): 1-\epsilon \in \mathcal{T}_r\}$ and $\{\vec{y}(s_{r-1}): s_{r-1} \in \mathcal{S}_r\}$ generates H_{r-1};

(ii) $\vec{x}(1-\epsilon)$and$\vec{y}(s_{r-1})$ are weakly continuous;

(iii) equality (55) and inequality (56) hold for all $(1-\epsilon, s_{r-1}) \in \mathcal{T}_r \times \mathcal{S}_r$.

Proof. As $\mathcal{T}_r$, $\mathcal{S}_r$ have countable bases, $\mathcal{T}_r \times \mathcal{S}_r$ has a countable base. Hence each open subset of $\mathcal{T}_r \times \mathcal{S}_r$ is a countable union of open rectangles, so all continuous functions on $\mathcal{T}_r \times \mathcal{S}_r$ are ω-continuous. By Corollary (3.2.19), one can assume that (57) holds marginally a.e. So there are $E_r \subseteq \mathcal{T}_r, F_r \subseteq \mathcal{S}_r$ such that $\mu(\mathcal{T}_r \backslash E_r) = \nu(\mathcal{S}_r \backslash F_r) = 0$ and (55) holds for all $(1-\epsilon, s_{r-1}) \in E_r \times F_r$. Taking (56) into account and removing, if necessary, from E_r and F_r some subsets of null measure, we obtain that there are $E_{r-1} \subseteq \mathcal{T}_r, F_{r-1} \subseteq \mathcal{S}_r$ with $\mu(\mathcal{T}_r \backslash E_{r-1}) = \nu(\mathcal{S}_r \backslash F_{r-1}) = 0$ such that (55) and (56) hold for all $(1-\epsilon, s_{r-1}) \in E_{r-1} \times F_{r-1}$.
Let P_r be the projection on $H_r = \text{cls}\{\vec{x}(1-\epsilon): 1-\epsilon \in E_{r-1}\}$. Then $g_{r-1}(1-\epsilon, s_{r-1}) = \left(\vec{x}(1-\epsilon), P_r\vec{y}(s_{r-1})\right)$ for $(1-\epsilon, s_{r-1}) \in E_{r-1} \times F_{r-1}$. Let now P_{r+1} be the projection on $H_{r+1} = \text{cls}\{P_r\vec{y}(s_{r-1}): s_{r-1} \in F_{r-1}\}$. Then $H_{r+1} \subseteq H_r$ and $g_{r-1}(1-\epsilon, s_{r-1}) = \left(P_{r+1}\vec{x}(1-\epsilon), P_r\vec{y}(s_{r-1})\right)$.Note that $\text{cls}\left(\{P_{r+1}\vec{x}(1-\epsilon): (1-\epsilon) \in E_{r-1}\}\right) = P_{r+1}\left(\text{cls}\left(\{\vec{x}(1-\epsilon): 1-\epsilon \in E_{r-1}\}\right)\right) = P_{r+1}H_r = H_{r+1}$. Replacing H_{r-1} by H_{r+1} and $\vec{x}(1-\epsilon), \vec{y}(s_{r-1})$ by $P_{r+1}\vec{x}(1-\epsilon), P_{r+1}\vec{y}(s_{r-1})$, we obtain the proof of (i).

Let $\mathcal{F}_{r-1}$ be the set of all $\vec{e} \in H_{r-1}$ for which the function $e(s_{r-1}) = (\vec{e}, \vec{y}(s_{r-1}))$ is uniformly continuous on F_{r-1}. Let $\vec{e}_n \in \mathcal{F}_{r-1}$ and $\vec{e}_n = \vec{e}$. As (56) holds for all $s_{r-1} \in F_{r-1}$,

$$\begin{aligned}\left|\left(\vec{e}, \vec{y}(s_{r-1}) - \vec{y}((s_{r-1})')\right)\right| &\leq \left|\left(\vec{e} - \vec{e}_n, \vec{y}(s_{r-1}) - \vec{y}((s_{r-1})')\right)\right| + \left|\left(\vec{e}_n, \vec{y}(s_{r-1}) - \vec{y}((s_{r-1})')\right)\right| \\ &\leq 2\|\vec{e} - \vec{e}_n\|(D^{r-1})^{1/2} + \left|\left(\vec{e}_n, \vec{y}(s_{r-1}) - \vec{y}((s_{r-1})')\right)\right|.\end{aligned}$$

Hence $\vec{e} \in \mathcal{F}_{r-1}$, so $\mathcal{F}_{r-1}$ is a closed linear subspace of H_{r-1}. Moreover, $\mathcal{F}_{r-1}$ contains all $\vec{x}(1-\epsilon)$, $(1-\epsilon) \in E_{r-1}$. Indeed, the function $\varphi(s_{r-1}) = \left(\vec{x}(1-\epsilon), \vec{y}(s_{r-1})\right)$ on F_{r-1} coincides with the function $s_{r-1} \to g_{r-1}(1-\epsilon, s_{r-1})$ which is continuous and, therefore, uniformly continuous on $\mathcal{S}_r$. Thus $\varphi(s_{r-1})$ is uniformly continuous on F_{r-1} . By (i) , $\mathcal{F}_{r-1} = H_{r-1}$.

Let us redefine, if necessary, $\vec{y}(s_{r-1})$ on $\mathcal{S}_r \backslash F_{r-1}$ to obtain a weakly continuous H_{r-1}-valued function on $\mathcal{S}_r$. As $\text{supp}(\nu) = \mathcal{S}_r$, Closure $(F_{r-1}) = \mathcal{S}_r$. Let $\vec{e} \in H_{r-1}$. As the function $e(s_{r-1}) = \left(\vec{e}, \vec{y}(s_{r-1})\right)$ is uniformly continuous on F_{r-1} , it extends to $\mathcal{S}_r$ by continuity; the result will be also denoted by $e(s_{r-1})$. As $\left|\left(\vec{e}, \vec{y}(s_{r-1})\right)\right| \leq \|\vec{e}\|\|\vec{y}(s_{r-1})\| \leq (D^{r-1})^{1/2}\|\vec{e}\|$, for $s_{r-1} \in F_{r-1}$, we have, by continuity, that $|e(s_{r-1})| \leq (D^{r-1})^{1/2}\|\vec{e}\|$ for all $s_{r-1} \in \mathcal{S}_r$.

Clearly, for each $s_{r-1} \in F_{r-1}$, the map $\vec{e} \to e(s_{r-1})$ is linear on H_{r-1} . Hence, by continuity, it is also linear, for each $s_{r-1} \in \mathcal{S}_r$, so the map $\vec{e} \to e(s_{r-1})$ is a bounded linear functional on H_{r-1} . Hence, for each $s_{r-1} \in \mathcal{S}_{r-1}/F_{r-1}$, one can find $\vec{v}_{s_{r-1}} \in H_{r-1}$ such that $e(s_{r-1}) = (\vec{e}, \vec{v}_{s_{r-1}})$ for all $e \in H_{r-1}$. Then $\left\|\vec{v}_{s_{r-1}}\right\| \leq (D^{r-1})^{1/2}$. Set $\vec{y}(s_{r-1}) = \vec{v}_{s_{r-1}}$. . As $e(s_{r-1}) = \left(\vec{e}, \vec{y}(s_{r-1})\right)$ is continuous on $\mathcal{S}_{r-1}$ for each $e \in H_{r-1}$, $\vec{y}(s_{r-1})$ is weakly continuous on $\mathcal{S}_r$ and $\|\vec{y}(s_{r-1})\| \leq (D^{r-1})^{1/2}$, for all $s_{r-1} \in \mathcal{S}_r$.

In the same way we can redefine $\vec{x}(1-\epsilon)$ on $\mathcal{T}_r \backslash E_{r-1}$ to obtain a weakly continuous function on $\mathcal{T}_r$ with $\|\vec{x}(1-\epsilon)\| \leq (D^{r-1})^{1/2}$, for all $(1-\epsilon) \in \mathcal{T}_r$. The redefined function $\left(\vec{x}(1-\epsilon), \vec{y}(s_{r-1})\right)$ is separately continuous on $\mathcal{T}_r \times \mathcal{S}_r$ by both arguments and coincides with $g_{r-1}(1-\epsilon, s_{r-1})$ on $E_{r-1} \times F_{r-1}$ .As $g_{r-1}(1-\epsilon, s_{r-1})$ is continuous, equality (55) holds for all $(1-\epsilon, s_{r-1}) \in \mathcal{T}_r \times \mathcal{S}_r$. We have proved (ii) and (iii).

Let us reformulate the result of Corollary (3.2.20) in a "scalar" form. (see[114]).

Corollary (3.2.21) [293]: Let $\mathcal{L}$ and $\mathcal{M}$ be, respectively, closed subspaces in the spaces $C^{r-1}(\mathcal{T}_r)$ and $C^{r-1}(\mathcal{S}_r)$ of all continuous functions. If, for each $s_{r-1} \in \mathcal{S}_r$, the function $(1-\epsilon) \to g_{r-1}(1-\epsilon, s_{r-1})$ belongs to $\mathcal{L}$, then all functions u_n belong to $\mathcal{L}$. Similarly, if all functions $s_{r-1} \to g_{r-1}(1-\epsilon, s_{r-1})$, $(1-\epsilon) \in \mathcal{T}_r$, belong to $\mathcal{M}$, then all v_n belong to $\mathcal{M}$.

Proof. Let $\vec{x}(1-\epsilon)$ and $\vec{y}(s_{r-1})$ be as in Corollary (3.2.20) Denote by $\mathcal{F}_{r-1}$ the set of all $\vec{e} \in H_{r-1}$ for which the function$(1-\epsilon) \to (\vec{x}(1-\epsilon), \vec{e})$ belongs to $\mathcal{L}$. Since $\mathcal{L}$ is a closed subspace of $C^{r-1}(\mathcal{T}_r)$ and $\|\vec{x}(1-\epsilon)\| \le (C^{r-1})^{1/2}$, for all$(1-\epsilon) \in \mathcal{T}_r$, $\mathcal{F}_{r-1}$ is a closed subspace of H_{r-1}. It contains all $\vec{y}(s_{r-1})$, $s_{r-1} \in \mathcal{S}_r$, as $\left(\vec{x}(1-\epsilon), \vec{y}(s_{r-1})\right) = g_{r-1}(1-\epsilon, s_{r-1}) \in \mathcal{L}$. Since these vectors generate $H_{r-1}, \mathcal{F}_{r-1} = H_{r-1}$. Thus $e_n \in \mathcal{F}_{r-1}$ and this means that $u_n \in \mathcal{L}$.

Corollary (3.2.22) [293]: σ_N uniformly converge to $(\widehat{f_{r-1}})$on each compact subset of $\mathbb{D}^\circ \times \mathbb{D}^\circ$.

Proof. By the maximum modulus principle and by (60), for all $(z_{r-1}, w_{r-1}) \in \mathbb{D} \times \mathbb{D}$,

$$|\sigma_N(z_{r-1}, w_{r-1})| \le \max_{(1-\epsilon)\in\mathbb{T}} |\sigma_N((1-\epsilon), w_{r-1})| \le \max_{(1-\epsilon)\in\mathbb{T}} \left(\max_{s_{r-1}\in\mathbb{T}} |\sigma_N(1-\epsilon, s_{r-1})| \right)$$

$$= \max_{(1-\epsilon), s_{r-1}\in\mathbb{T}} \left| \sum_{n=1}^{N} u_n(1-\epsilon) v_n(s_{r-1}) \right|$$

$$\le \max_{(1-\epsilon)\in\mathbb{T}} \left(\sum_{n=1}^{N} |u_n(1-\epsilon)|^2 \right)^{1/2} \max_{s_{r-1}\in\mathbb{T}} \left(\sum_{n=1}^{N} |v_n(s_{r-1})|^2 \right)^{1/2} \le D^{r-1}.$$

Let $z_r = (1-\epsilon)e^{i\tau}$, $w_r = \rho e^{i\alpha}$. The functions $z_{r-1} \to \sigma_N(z_{r-1}, w_r)$, $w_{r-1} \to \sigma_N(z_r, w_{r-1})$ belong to $A(\mathbb{D})$, so by (61),

$$\sigma_N(z_r, w_r) = \frac{1}{2\pi} \int_0^{2\pi} P_{r-1}(1-\epsilon, \tau-\theta) \sigma_N(e^{i\theta}, w_r) d\theta$$

$$= \frac{1}{4\pi^2} \int_0^{2\pi} P_{r-1}(1-\epsilon, \tau-\theta) \left(\int_0^{2\pi} P_{r-1}(1+\epsilon, \alpha-\emptyset) \sigma_N\,(e^{i\theta}, e^{i\emptyset}) d\emptyset \right) d\theta.$$

the functions $z_{r-1} \to (\widehat{f_{r-1}})(z_{r-1}, w_r)$and $w_{r-1} \to (\widehat{f_{r-1}})(z_r, w_{r-1})$belong to $A(\mathbb{D})$. Hence, as above

$$(\widehat{f_{r-1}})(z_r, w_r) = \frac{1}{4\pi^2} \int_0^{2\pi} P_{r-1}(1-\epsilon, \tau-\theta) \left(\int_0^{2\pi} P_{r-1}(1+\epsilon, \alpha-\emptyset) (\widehat{f_{r-1}})\,(e^{i\theta}, e^{i\emptyset}) d\emptyset \right) d\theta.$$

Let $\max(1-\epsilon, 1+\epsilon) \le R < 1$ .Then$|P_{r-1}(1-\epsilon, \tau-\theta) P_{r-1}(1+\epsilon, \alpha-\emptyset)| \le \frac{(\epsilon^2-4)}{\epsilon^2} \le \left(\frac{1+R}{1-R}\right)^2$. Hence

$$\left|(\widehat{f_{r-1}})(z_r, w_r) - \sigma_N(z_r, w_r)\right| \le \frac{1}{4\pi^2} \left(\frac{1+R}{1-R}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \left|(\widehat{f_{r-1}})(e^{i\theta}, e^{i\emptyset}) - \sigma_N(e^{i\theta}, e^{i\emptyset})\right| d\emptyset d\theta.$$

Since $(\widehat{f_{r-1}})$ is continuous on $\mathbb{T} \times \mathbb{T}$, $\sup\left|(\widehat{f_{r-1}})(e^{i\theta}, e^{i\emptyset})\right| \le M$ for some M. As $\sup\left|\sigma_N(e^{i\theta}, e^{i\emptyset})\right| \le D^{r-1}$ and $\sigma_N(e^{i\theta}, e^{i\emptyset}) \to (\widehat{f_{r-1}})(e^{i\theta}, e^{i\emptyset})$ for all $\theta, \emptyset$, it follows from the Dominated Convergence theorem that

$$\int_0^{2\pi}\int_0^{2\pi}\left|\widehat{(f_{r-1})}\left(e^{i\theta},e^{i\emptyset}\right)-\sigma_N\left(e^{i\theta},e^{i\emptyset}\right)\right|d\emptyset d\theta\to 0,\quad \text{as } N\to\infty$$

Therefore σ_N uniformly converge to $\widehat{(f_{r-1})}$ in $\mathbb{D}_R\times\mathbb{D}_R$ where $\mathbb{D}_R$is the closed disk of radius R.

Let $\{u_n\},\{v_n\}$ be functions in $A(\mathbb{T})$ satisfying (60). For $N\in\mathbb{N}$ and contractions T_{r-1},S_{r-1}, define the bounded operators $\Gamma_N(T_{r-1},S_{r-1})$ on $B(H_{r-1})$ by the formula: $\Gamma_N(T_{r-1},S_{r-1})X=\sum_{n=1}^N u_n(T_{r-1})Xv_n(S_{r-1})$.

Corollary (3.2.23) [293]: $\|\Gamma_N(T_{r-1},S_{r-1})\|_{B(H_{r-1})}\le D^{r-1}$.

Proof. Set $\emptyset_N(1-\epsilon)=\sum_{n=1}^N u_n(1-\epsilon)\overline{u_n(1-\epsilon)}$. Let U_{r-1} be a unitary dilation of T_{r-1} and P_{r-1} be the projection on H_{r-1} such that $T_{r-1}=P_{r-1}U_{r-1}|_{H_{r-1}}$ (see (52)). Since $0\le\emptyset_N(1-\epsilon)\le D^{r-1}\mathbf{1}$, the positive operators $\emptyset_N(U_{r-1})=\sum_{n=1}^N u_n(U_{r-1})u_n(U_{r-1})^*$ satisfy the inequality $0\le\emptyset_N(U_{r-1})\le D^{r-1}\mathbf{1}$. By (53), $u_n(T_{r-1})=P_{r-1}u_n(U_{r-1})|_{H_{r-1}}$.

Hence

$$\begin{aligned}\emptyset_N(T_{r-1})&=\sum_{n=1}^N u_n(T_{r-1})u_n(T_{r-1})^*=\sum_{n=1}^N P_{r-1}u_n(U_{r-1})u_n(U_{r-1})^*P_{r-1}\\&=P_{r-1}\left(\sum_{n=1}^N u_n(U_{r-1})u_n(U_{r-1})^*\right)P_{r-1}\le D^{r-1}P_{r-1},\end{aligned}$$

so $\|\sum_{n=1}^N u_n(T_{r-1})u_n(T_{r-1})^*\|\le D^{r-1}$. We obtain similarly that$\|\sum_{n=1}^N v_n(S_{r-1})^*v_n(S_{r-1})\|\le D^{r-1}$,for all N , by setting

$\psi_N(s_{r-1})=\sum_{n=1}^N\overline{v_n(s_{r-1})}v_n(s_{r-1})$.

Consider the Hilbert space $\widetilde{(H_{r-1})}=H_{r-1}\oplus\ldots\oplus H_{r-1}\oplus\ldots$. For $X\in B(H_{r-1})$, the operators

$$(A_i)_N=\begin{pmatrix}u_1(T_{r-1})X\cdots u_N(T_{r-1})X & 0\cdots\\ 0\quad\cdots\quad 0 & 0\cdots\\ \cdots\quad\cdots\quad\cdots & \cdots\cdots\end{pmatrix}\text{ and}$$

$$(A_i+\epsilon)_N=\begin{pmatrix}v_1(S_{r-1})^*\cdots v_N(S_{r-1})^* & 0\cdots\\ 0\quad\cdots\quad 0 & 0\cdots\\ \cdots\quad\cdots\quad\cdots & \cdots\cdots\end{pmatrix}$$

on $\widetilde{(H_{r-1})}$ are bounded, as

$$\begin{aligned}\|(A_i)_N\|^2&=\|(A_i)_N(A_i)_N^*\|=\left\|\sum_{n=1}^N u_n(T_{r-1})XX^*u_n(T_{r-1})^*\right\|=\sup_{x\in H_{r-1},\|x\|=1}\sum_{n=1}^N\|X^*u_n(T_{r-1})^*x\|^2\\&\le\|X^*\|^2\sup_{x\in H_{r-1},\|x\|=1}\sum_{n=1}^N\|u_n(T_{r-1})^*x\|^2=\|X\|^2\left\|\sum_{n=1}^N u_n(T_{r-1})u_n(T_{r-1})^*\right\|\\&\le D^{r-1}\|X\|^2\end{aligned}$$

and $\|(A_i+\epsilon)_N\|^2=\|(A_i+\epsilon)_N(A_i+\epsilon)_N^*\|=\|\sum_{n=1}^N v_n(S_{r-1})^*v_n(S_{r-1})\|\le D^{r-1}$. Hence

$$\begin{aligned}\|\Gamma_N(T_{r-1},S_{r-1})X\|&=\left\|\sum_{n=1}^N u_n(T_{r-1})Xv_n(S_{r-1})\right\|=\|(A_i)_N(A_i+\epsilon)_N^*\|\le\|(A_i)_N\|\|(A_i+\epsilon)_N\|\\&\le D^{r-1}\|X\|\end{aligned}$$

which completes the proof.

For $A_i \in B(H_{r-1})$, denote by L_{A_i} and R_{A_i} the operators of the left and right multiplication by A_i on $B(H_{r-1})$; they clearly commute. It is well known that $\|L_{A_i}\| = \|R_{A_i}\| = \|A_i\|$. So if A_i is a strict contraction (this means that $\|A_i\| < 1$) then one may apply functions in $A(\mathbb{D})$ to $L_{A_i}$and R_{A_i}. It is evident that $p(L_{A_i})X = p(A_i)X$ and $p(R_{A_i})X = Xp(A_i)$, for each polynomial p. Approximating uniformly $f_{r-1} \in A(\mathbb{D})$ by polynomials, we have $f_{r-1}(L_{A_i})X = f_{r-1}(A_i)\, X$ and $f_{r-1}(R_{A_i})X = X\, f_{r-1}(A_i)$, for

each strict contraction A_i, so

$$f_{r-1}(L_{A_i}) = L_{f_{r-1}(A_i)} \text{ and } f_{r-1}(R_{A_i}) = R_{f_{r-1}(A_i)}. \quad (62)$$

Let $f_{r-1} \in A(\mathbb{T})$ be a continuously differentiable, sequence of operator Lipschitz functions on $\mathbb{T}$ and let T_{r-1}, S_{r-1} be strict contractions. Consider f_{r-1} as an element of $A(\mathbb{D})$. The function $f_{r-1}(z_{r-1}) - f_{r-1}(w_{r-1})$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ and, by Lemma 3.1, the function $(\widehat{f_{r-1}})(z_{r-1}, w_{r-1})$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$. Therefore (see [123, §13] and [124, III.11.8, Theorem 8]) they can be applied to any two elements of a commutative Banach unital algebra whose spectra are contained in $\mathbb{D}^\circ$and, hence, to the commuting strict contractions $L_{T_{r-1}}$ and $R_{S_{r-1}}$ on the Banach space $B(H_{r-1})$. Thus $(\widehat{f_{r-1}})(L_{T_{r-1}}, L_{S_{r-1}})$ and $f_{r-1}(L_{T_{r-1}}) - f_{r-1}(R_{S_{r-1}})$ are bounded operators on $B(H_{r-1})$. By (59) ,$f_{r-1}(z_{r-1}) - f_{r-1}(w_{r-1}) = (\widehat{f_{r-1}})(z_{r-1}, w_{r-1})(z_{r-1} - w_{r-1})$. Hence,

$$L_{f_{r-1}(T_{r-1})} - R_{f_{r-1}(S_{r-1})} = f_{r-1}(L_{T_{r-1}}) - f_{r-1}(R_{S_{r-1}}) = (\widehat{f_{r-1}})(L_{T_{r-1}}, R_{S_{r-1}})(L_{T_{r-1}} - R_{S_{r-1}}. \quad (63)$$

Now we can show the main result of the section and finish the proof of Theorem 1.1. (see[114]).

Corollary (3.2.24) [293]: If $f_{r-1} \in A(\mathbb{T})$ is a sequence of operator Lipschitz functions on $\mathbb{T}$ then there is $D^{r-1} > 0$ such that

$$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\| \le D^{r-1}\|T_{r-1}X - XS_{r-1}\| \text{ for all contractions } T_{r-1}, S_{r-1} \text{ and } X \in B(H_{r-1}). \quad (64)$$

Proof. (i) First assume that f_{r-1} has continuous derivative on $\mathbb{T}$ and that T_{r-1}, S_{r-1} are strict contractions. It follows from (63) that, for all $X \in B(H_{r-1})$,

$$f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1}) = (f_{r-1}(L_{T_{r-1}}) - f_{r-1}(R_{S_{r-1}}))X = (\widehat{f_{r-1}})(L_{T_{r-1}}, R_{S_{r-1}})(T_{r-1}X - XS_{r-1}).$$

By Corollary (3.2.22), the analytic functions $\sigma_N(z_{r-1}, w_{r-1})$ on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ uniformly converge to $(\widehat{f_{r-1}})(z_{r-1}, w_{r-1})$ on $\mathbb{D}_{1-\epsilon} \times \mathbb{D}_{1-\epsilon}$ where $\mathbb{D}_{1-\epsilon}$ is the closed disk of radius $1-\epsilon = \max(\|T_{r-1}\|, \|S_{r-1}\|) < 1$. It follows from continuity of the Holomorphic Functional Calculus (see [123, §13]) that $(\widehat{f_{r-1}})(L_{T_{r-1}}, R_{S_{r-1}})$ is the norm limit of the operators $\sigma_N(L_{T_{r-1}}, R_{S_{r-1}}) = \sum_{n=1}^{N} u_n(L_{T_{r-1}}) v_n(R_{S_{r-1}})$. It follows from (62) that $\sigma_N(L_{T_{r-1}}, R_{S_{r-1}}) = \Gamma_N(T_{r-1}, S_{r-1})$. Hence, by Corollary (3.2.23), $\|\sigma_N(L_{T_{r-1}}, R_{S_{r-1}})\|_{B(H_{r-1})} \le D^{r-1}$. Therefore $\|(\widehat{f_{r-1}})(L_{T_{r-1}}, R_{S_{r-1}})\|_{B(H_{r-1})} \le D^{r-1}$ and (64) holds.

Let now T_{r-1}, S_{r-1} be arbitrary contractions. Applying (64) to $(1-\epsilon)T_{r-1}, (1-\epsilon)S_{r-1}$, for $\epsilon > 0$, we get

$$\|f_{r-1}((1-\epsilon)T_{r-1})X - Xf_{r-1}((1-\epsilon)S_{r-1})\| \leq D^{r-1}\|(1-\epsilon)T_{r-1}X - (1-\epsilon)XS_{r-1}\| \leq D^{r-1}\|T_{r-1}X - XS_{r-1}\|.$$

Hence

$$\begin{aligned}
&\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\| \\
&\leq \|f_{r-1}((1-\epsilon)T_{r-1})X - Xf_{r-1}((1-\epsilon)S_{r-1})\| \\
&+ \|(f_{r-1}(T_{r-1}) - f_{r-1}((1-\epsilon)T_{r-1}))X\| + \|X(f_{r-1}(S_{r-1}) - f_{r-1}((1-\epsilon)S_{r-1}))\| \\
&\leq D^{r-1}\|T_{r-1}X - XS_{r-1}\| + \|f_{r-1}(T_{r-1}) - f_{r-1}((1-\epsilon)T_{r-1})\|\|X\| \\
&+ \|X\|\|f_{r-1}(S_{r-1}) - f_{r-1}((1-\epsilon)S_{r-1})\|.
\end{aligned}$$

Letting $\epsilon \to 0$ and using the norm-continuity of the map $T_{r-1} \to f_{r-1}(T_{r-1})$, we obtain that (64) holds for all contractions. Thus we proved the theorem for continuously differentiable functions.

(ii) Let now f_{r-1} be any sequence of operator Lipschitz functions on $\mathbb{T}$ from $A(\mathbb{T})$. Let φ be a non-negative infinitely differentiable function on $\mathbb{T}$ with $\int_{\mathbb{T}} \varphi(1-\epsilon)d(1-\epsilon) = 1$. The convolution

$$h_{r-1}(1-\epsilon) = \varphi * f_{r-1}(1-\epsilon) = \int_{\mathbb{T}} \varphi(s_{r-1}) f_{r-1}((1-\epsilon)(s_{r-1})^{-1}) ds_{r-1}$$

is also infinitely differentiable and belongs to $A(\mathbb{T})$, since the negative Fourier coefficients

$$(\widehat{h_{r-1}})(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_{r-1}(e^{i\theta})\, e^{-in\theta} d\theta = i(\widehat{f_{r-1}})(n) \int_{-\pi}^{\pi} e^{-i(n-1)\phi}\, \varphi(e^{i\phi}) d\phi = 0 \text{ for } n < 0,$$

as $(\widehat{f_{r-1}})(n) = 0$. Moreover, it is operator Lipschitzian on $\mathbb{T}$ with the same constant. Indeed,

$$h_{r-1}(U_{r-1}) - h_{r-1}(V_{r-1}) = \int_{\mathbb{T}} \varphi(s_{r-1})\big(f_{r-1}(U_{r-1}(s_{r-1})^{-1}) - f_{r-1}(V_{r-1}(s_{r-1})^{-1})\big) ds_{r-1}$$

for unitary operators U_{r-1}, V_{r-1} . Since f_{r-1} is an operator Lipschitz function on $\mathbb{T}$ and $|s_{r-1}| = 1$, we have from (2) that

$$\begin{aligned}
\|h_{r-1}(U_{r-1}) - h_{r-1}(V_{r-1})\| &\leq \int_{\mathbb{T}} \varphi(s_{r-1})\|f_{r-1}(U_{r-1}(s_{r-1})^{-1}) - f_{r-1}(V_{r-1}(s_{r-1})^{-1})\| ds_{r-1} \\
&\leq \int_{\mathbb{T}} \varphi(s_{r-1}) D^{r-1}\|U_{r-1} - V_{r-1}\| ds_{r-1} = D^{r-1}\|U_{r-1} - V_{r-1}\|.
\end{aligned}$$

Since h_{r-1} is infinitely differentiable, (64) holds for it and all $T_{r-1}, S_{r-1} \in \mathrm{Con}(H_{r-1})$ and $X \in B(H_{r-1})$. Thus

$$\begin{aligned}
&\|h_{r-1}(U_{r-1})X - Xh_{r-1}(V_{r-1})\| \\
&= \left\|\int_{\mathbb{T}} \varphi(s_{r-1})\big(f_{r-1}(T_{r-1}(s_{r-1})^{-1})X - Xf_{r-1}(S_{r-1}(s_{r-1})^{-1})\big) ds_{r-1}\right\| \\
&\leq D^{r-1}\|T_{r-1}X - XS_{r-1}\|. \qquad (65)
\end{aligned}$$

Set $F_{r-1}(s_{r-1}) = f_{r-1}(T_{r-1}(s_{r-1})^{-1})X - Xf_{r-1}(S_{r-1}(s_{r-1})^{-1})$. For $(1-\epsilon), s_{r-1} \in \mathbb{T}$,

$$\begin{aligned}
&\|F_{r-1}(1-\epsilon) - F_{r-1}(s_{r-1})\| \\
&\leq \|f_{r-1}(T_{r-1}(1-\epsilon)^{-1}) - f_{r-1}(T_{r-1}(s_{r-1})^{-1})\|\|X\| \\
&+ \|X\|\|f_{r-1}(S_{r-1}(1-\epsilon)^{-1}) - f_{r-1}(S_{r-1}(s_{r-1})^{-1})\|.
\end{aligned}$$

Since the map $T_{r-1} \to f_{r-1}(T_{r-1})$ is norm-continuous, $\|F_{r-1}(1-\epsilon) - F_{r-1}(s_{r-1})\| \to 0$, as $s_{r-1} \to (1-\epsilon)$. Take a sequence $\{\varphi_n\}$ of functions as above with the support of φ_n contained in $\mathbb{T}_n = \{(1-\epsilon) \in \mathbb{T}: |-\epsilon| \le \frac{1}{n}\}$. Then

$$\left\|F_{r-1}(1) - \int_{\mathbb{T}} \varphi_n(s_{r-1}) F_{r-1}(s_{r-1}) ds_{r-1}\right\| = \left\|\int_{\mathbb{T}} \varphi_n(s_{r-1})(F_{r-1}(1) - F_{r-1}(s_{r-1})) ds_{r-1}\right\|$$
$$\le \sup_{s_{r-1} \in \mathbb{T}_n} \|F_{r-1}(1) - F_{r-1}(s_{r-1})\| \to 0,$$

and

$$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\| = \|F_{r-1}(1)\|$$
$$\le \left\|F_{r-1}(1) - \int_{\mathbb{T}} \varphi_n(s_{r-1}) F_{r-1}(s_{r-1}) ds_{r-1}\right\| + \left\|\int_{\mathbb{T}} \varphi_n(s_{r-1}) F_{r-1}(s_{r-1}) ds_{r-1}\right\|.$$

By (65) $\left\|\int_{\mathbb{T}} \varphi_n(s_{r-1}) F_{r-1}(s_{r-1}) ds_{r-1}\right\| \le D^{r-1}\|T_{r-1}X - XS_{r-1}\|$. Letting $n \to \infty$, we conclude that (64) holds.

Corollary (3.2.25) [293]: Let $f_{r-1} \in A(\mathbb{T})$ and $\epsilon > 0$ Let there exists $D^{r-1} > 0$ such that

$$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\|_{1+\epsilon} \le D^{r-1}\|T_{r-1}X - XS_{r-1}\|_{1+\epsilon}, \qquad (68)$$

for all finite rank contractions T_{r-1}, S_{r-1} and all finite rank operators X. Then (68) holds for any pair of contractions T_{r-1}, S_{r-1} and all $X \in (C^{r-1})_{1+\epsilon}$, and

$$\|f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1})\|_{1+\epsilon} \le D^{r-1}\|T_{r-1} - S_{r-1}\|_{1+\epsilon} \text{ if } (T_{r-1} - S_{r-1}) \in (C^{r-1})_{1+\epsilon}$$
(69)

Proof. Let X be a finite rank operator and $S_{r-1}, T_{r-1} \in \mathrm{Con}(H_{r-1})$. Choose finite rank contractions $(S_{r-1})_n$, $(T_{r-1})_n$ such that $(S_{r-1})_n^* \xrightarrow{(\mathrm{sot})} (S_{r-1})^*$, $(T_{r-1})_n \xrightarrow{(\mathrm{sot})} T_{r-1}$. By (67) $\|(T_{r-1} - (T_{r-1})_n)X\|_{1+\epsilon} \to 0$ and $\|X(S_{r-1} - (S_{r-1})_n)\|_{1+\epsilon} = \|((S_{r-1})_n^* - (S_{r-1})^*)X^*\|_{1+\epsilon} \to 0$. Furthermore, $f_{r-1}((T_{r-1})_n) \xrightarrow{(\mathrm{sot})} f_{r-1}(T_{r-1})$ whence, by (67), $\|(f_{r-1}(T_{r-1}) - f_{r-1}((T_{r-1})_n))X\|_{1+\epsilon} \to 0$.

The function $(\widetilde{f_{r-1}})(1-\epsilon) = \overline{f_{r-1}(\overline{(1-\epsilon)})}$ belongs to $A(\mathbb{T})$ and (see [122, Section III.2]) $(\widetilde{f_{r-1}})((S_{r-1})^*) = f_{r-1}(S_{r-1})^*$. Hence $f_{r-1}((S_{r-1})_n)^* - f_{r-1}(S_{r-1})^* = (\widetilde{f_{r-1}})((S_{r-1})_n^*) - (\widetilde{f_{r-1}})((S_{r-1})^*) \xrightarrow{(\mathrm{sot})} 0$, so that $\|X(f_{r-1}((S_{r-1})_n) - f_{r-1}(S_{r-1}))\|_{1+\epsilon} = \|f_{r-1}((S_{r-1})_n)^* - f_{r-1}(S_{r-1})^*X^*\|_{1+\epsilon} \to 0$.

Using these norm limits and taking the limit in the inequality $\|(f_{r-1}((T_{r-1})_n)X - Xf_{r-1}((S_{r-1})_n))\|_{1+\epsilon} \le D^{r-1}\|(T_{r-1})_nX - X(S_{r-1})_n\|_{1+\epsilon}$, we obtain (69) for all $T_{r-1}, S_{r-1} \in \mathrm{Con}(H_{r-1})$ and all finite rank operators X. For arbitrary $X \in (C^{r-1})_{1+\epsilon}$, choose finite rank operators X_n such that $\|X - X_n\|_{1+\epsilon} \to 0$. Now (69) can be proved by taking the limit in the inequality $\|(f_{r-1}(T_{r-1})X_n - X_nf_{r-1}(S_{r-1}))\|_{1+\epsilon} \le D^{r-1}\|T_{r-1}X_n - X_nS_{r-1}\|_{1+\epsilon}$ and using (51) and (66).

Let now $(T_{r-1} - S_{r-1}) \in (C^{r-1})_{1+\epsilon}$. Let $(P_{r-1})_n$ be an increasing sequence of finite-dimensional projections such $(P_{r-1})_n \xrightarrow{(\text{sot})} 1$. Replace in (68) T_{r-1}, S_{r-1} by $(P_{r-1})_n T_{r-1}, S_{r-1}(P_{r-1})_n$ and X by $(P_{r-1})_n$. This gives

$$\begin{aligned}&\|f_{r-1}((P_{r-1})_n T_{r-1})(P_{r-1})_n - (P_{r-1})_n f_{r-1}(S_{r-1}(P_{r-1})_n)\|_{1+\epsilon}\\ &\le D^{r-1}\|(P_{r-1})_n T_{r-1}(P_{r-1})_n - (P_{r-1})_n S_{r-1}(P_{r-1})_n\|_{1+\epsilon}\\ &= D^{r-1}\|(P_{r-1})_n (T_{r-1} - S_{r-1})(P_{r-1})_n\|_{1+\epsilon} \le D^{r-1}\|T_{r-1} - S_{r-1}\|_{1+\epsilon} \quad (70)\end{aligned}$$

We have that $(P_{r-1})_n T_{r-1} \xrightarrow{(\text{sot})} T_{r-1}$ and $S_{r-1}(P_{r-1})_n \xrightarrow{(\text{sot})} S_{r-1}$. Hence $f_{r-1}((P_{r-1})_n T_{r-1}) \xrightarrow{(\text{sot})} f_{r-1}(T_{r-1})$ and $f_{r-1}(S_{r-1}(P_{r-1})_n) \xrightarrow{(\text{sot})} f_{r-1}(S_{r-1})$. By (59), $\|f_{r-1}((P_{r-1})_n T_{r-1})\| \le \|f_{r-1}\|$. Therefore the finite rank operators $f_{r-1}((P_{r-1})_n T_{r-1})(P_{r-1})_n - (P_{r-1})_n f_{r-1}(S_{r-1}(P_{r-1})_n) \xrightarrow{(\text{sot})} f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1})$. Taking (70) into account, we obtain from [125, Theorem III.5.1] that $f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1}) \in (C^{r-1})_{1+\epsilon}$ and $\|f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1})\|_{1+\epsilon} \le D^{r-1}\|T_{r-1} - S_{r-1}\|_{1+\epsilon}$.

Corollary (3.2.26) [293]: Let $f_{r-1} \in A(\mathbb{T})$ be a sequence of operator Lipschitz functions on $\mathbb{T}$ with constant D^{r-1}. Then, for all contractions T_{r-1}, S_{r-1}, and all $\epsilon > 0$,

$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\|_{1+\epsilon} \le D^{r-1}\|T_{r-1}X - XS_{r-1}\|_{1+\epsilon}$ for $X \in (C^{r-1})_{1+\epsilon}$,
(71)

$\|f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1})\|_{1+\epsilon} \le D^{r-1}\|T_{r-1} - S_{r-1}\|_{1+\epsilon}$ if $T_{r-1} - S_{r-1} \in (C^{r-1})_{1+\epsilon}$.
(72)

Proof. First assume that $\sigma(T_{r-1}) \cap \sigma(S_{r-1}) = \emptyset$. By Rosenblum's theorem (see [126]) the operator $\Delta = L_{T_{r-1}} - R_{S_{r-1}}$ on $B(H_{r-1})$ is invertible and we may consider the operator $F_{r-1} = (L_{f_{r-1}(T_{r-1})} - R_{f_{r-1}(S_{r-1})})\Delta^{-1}$. It follows from (64) that $\|F_{r-1}\| \le D^{r-1}$. The operator $\Delta^\infty = \Delta|_{(C^{r-1})^\infty}$ is also invertible, so F_{r-1} pre serves$(C^{r-1})^\infty$. Set $(F_{r-1})^\infty = F_{r-1}|_{(C^{r-1})^\infty}$.
Then $\|(F_{r-1})^\infty\| \le D^{r-1}$, so $\|((F_{r-1})^\infty)^*\| \le D^{r-1}$, where $((F_{r-1})^\infty)^*$ is the conjugate operator on $(C^{r-1})_1$. As Δ^{-1} commutes with $L_{f_{r-1}(T_{r-1})} - R_{f_{r-1}(S_{r-1})}$, we see from (68) that $(F_{r-1})^\infty = (\Delta^\infty)^{-1}(L^\infty_{f_{r-1}(T_{r-1})} - R^\infty_{f_{r-1}(S_{r-1})})$ and $((\Delta^\infty)^{-1})^* = ((\Delta^\infty)^*)^{-1} = (L^1_{T_{r-1}} - R^1_{S_{r-1}})^{-1}$. Hence

$$\begin{aligned}({F_{r-1}}^\infty)^* &= \left((\Delta^\infty)^{-1}(L^\infty_{f_{r-1}(T_{r-1})} - R^\infty_{f_{r-1}(S_{r-1})})\right)^* = (L^1_{f_{r-1}(T_{r-1})} - R^1_{f_{r-1}(S_{r-1})})((\Delta^\infty)^*)^{-1}\\ &= (L^1_{f_{r-1}(T_{r-1})} - R^1_{f_{r-1}(S_{r-1})})(L^1_{T_{r-1}} - R^1_{S_{r-1}})^{-1} = F_{r-1}|_{(C^{r-1})_1}.\end{aligned}$$

Since $\max\left(\|F_{r-1}\|, \left\|F_{r-1}|_{(C^{r-1})_1}\right\|\right) \le D^{r-1}$, it follows from the interpolation theory (see, for example [125], [128, Theorem B]) that F_{r-1} preserves $(C^{r-1})_{1+\epsilon}$ and $\left\|F_{r-1}|_{C_{1+\epsilon}}\right\| \le D^{r-1}$.
For $\in (C^{r-1})_{1+\epsilon}$, set $Y = (L_{T_{r-1}} - R_{S_{r-1}})X$. Then $Y \in (C^{r-1})_{1+\epsilon}$ and we obtain (71) for this case

$$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\|_{1+\epsilon} = \left\|(L_{f_{r-1}(T_{r-1})} - R_{f_{r-1}(S_{r-1})})X\right\|_{1+\epsilon} = \|F_{r-1}(Y)\|_{1+\epsilon} \le D^{r-1}\|Y\|_{1+\epsilon} = D^{r-1}\|T_{r-1}X - XS_{r-1}\|_{1+\epsilon} \quad (73)$$

Let $\dim(H_{r-1}) < \infty$ and $T_{r-1}, S_{r-1} \in \mathrm{Con}(H_{r-1})$. Choose contractions $(S_{r-1})_n$ such that $\sigma(T_{r-1}) \cap \sigma((S_{r-1})_n) = \emptyset$ and $\|S_{r-1} - (S_{r-1})_n\| \to 0$. Hence

$\|f_{r-1}((S_{r-1})_n)-f_{r-1}(S_{r-1})\| \to 0$. We have from (66) and (73) that

$$\begin{aligned}\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\|_{1+\epsilon} &\le \|f_{r-1}(T_{r-1})X - Xf_{r-1}((S_{r-1})_n)\|_{1+\epsilon} + \|X(f_{r-1}((S_{r-1})_n) - f_{r-1}(S_{r-1}))\|_{1+\epsilon}\\ &\le D^{r-1}\|f_{r-1}(T_{r-1})X - Xf_{r-1}((S_{r-1})_n)\|_{1+\epsilon}\\ &+ \|X\|_{1+\epsilon}\|f_{r-1}((S_{r-1})_n) - f_{r-1}(S_{r-1})\|\\ &\le D^{r-1}\|T_{r-1}X - X(S_{r-1})_n\|_{1+\epsilon} + \|X\|_{1+\epsilon}\|f_{r-1}((S_{r-1})_n) - f_{r-1}(S_{r-1})\|.\end{aligned}$$

By (66), $\|T_{r-1}X - X(S_{r-1})_n\|_{1+\epsilon} \le \|T_{r-1}X - XS_{r-1}\|_{1+\epsilon} + \|X\|_{1+\epsilon}\|S_{r-1} - (S_{r-1})_n\|$. Taking the limit, we obtain that (71) holds for all $T_{r-1}, S_{r-1} \in \mathrm{Con}(H_{r-1})$ if $\dim(H_{r-1}) < \infty$. Thus it holds for arbitrary H_{r-1} if T_{r-1}, S_{r-1} are finite rank contractions. Applying Corollary (3.2.25), we conclude the proof.

The result obtained in Corollary (3.2.26) is not the optimal one. It would be desirable to show that (71) and (72) hold if $f_{r-1} \in A(\mathbb{T})$ is a $(C^{r-1})_{1+\epsilon}$-Lipschitz function on $\mathbb{T}$ (see (82)). Then we would have proved an analogue of Theorem 1.1 for $(C^{r-1})_{1+\epsilon}$-Lipschitz function on $\mathbb{T}$.

Corollary (3.2.27) [293]: For all $(1+\epsilon, \frac{1+\epsilon}{\epsilon}) \in [1,\infty] \cup b$ and each $\epsilon > 0$, $\omega_{1+\epsilon,\frac{1+\epsilon}{\epsilon}}(1-\epsilon) \ge \sqrt{2(1-\epsilon)}$.

Proof. It suffices to show that , for each $\epsilon \ge 0$, there are contractions T_{r-1}, S_{r-1} such that $\|T_{r-1} - S_{r-1}\|_{\frac{1+\epsilon}{\epsilon}} = 1-\epsilon$ and that $\|U_{r-1} - V_{r-1}\|_{1+\epsilon} \le \sqrt{2(1-\epsilon)}$, for all their unitary dilations U_{r-1}, V_{r-1} .

Let $e \in H_{r-1}$ and Q be the projection on $\mathbb{C}e$. Set $T_{r-1} = Q$ and $S_{r-1} = (\epsilon)Q$. Clearly, $\|T_{r-1} - S_{r-1}\| = 1-\epsilon$. As $T_{r-1} - S_{r-1}$ is rank one operator, $\|T_{r-1} - S_{r-1}\|_{\frac{1+\epsilon}{\epsilon}} = \|T_{r-1} - S_{r-1}\|$ for all $\frac{1+\epsilon}{\epsilon}$.

Let P_{r-1} be the projection on H_{r-1} in $\mathcal{H}$. Then $P_{r-1}U_{r-1}|_{H_{r-1}} = T_{r-1}$,$P_{r-1}V_{r-1}|_{H_{r-1}} = S_{r-1}$, so that $(U_{r-1}e, e) = 1$,$(V_{r-1}e, e) = 1-\epsilon$. Hence ,$U_{r-1}e = e$. Then $\|U_{r-1} - V_{r-1}\| \ge \sqrt{2(1-\epsilon)}$, as

$$\begin{aligned}\|U_{r-1} - V_{r-1}\|^2 &\ge \|U_{r-1}e - V_{r-1}e\|^2\\ &= (U_{r-1}e, U_{r-1}e) + (V_{r-1}e, V_{r-1}e) - (U_{r-1}e, V_{r-1}e) - (V_{r-1}e, U_{r-1}e) = 2(1-\epsilon)\end{aligned}$$

If $U_{r-1} - V_{r-1} \in (C^{r-1})_{1+\epsilon}$then $\|U_{r-1} - V_{r-1}\|_{1+\epsilon} \ge \|U_{r-1} - V_{r-1}\|$.Hence $\|U_{r-1} - V_{r-1}\|_{1+\epsilon} \ge \sqrt{2(1-\epsilon)}$.

It follows from Corollary (3.2.27) that (73) does not hold. To estimate the continuity moduli $\omega_{(1+\epsilon),\frac{1+\epsilon}{\epsilon}}$ of Dil, we will consider the "canonical" unitary dilation of $T_{r-1} \in \mathrm{Con}(H_{r-1})$ (see [122, Chapter I,§5]). Set

$(D^{r-1})_{T_{r-1}} = (1-(T_{r-1})^*T_{r-1})^{1/2}$, $(D^{r-1})_{(T_{r-1})^*} = (1-T_{r-1}(T_{r-1})^*)^{1/2}$ and $\mathcal{H} = \bigoplus_{-\infty}^{\infty} (H_{r-1})_n$ with all $(H_{r-1})_n = H_{r-1}$

Let P_{r-1} be the projection on $\mathfrak{H} = H_{r-1} \oplus H_r$ and $(U_{r-1})_0^{T_{r-1}}$ be the operator on $\mathcal{H}$such that

$(U_{r-1})_0^{T_{r-1}} = P_{r-1}(U_{r-1})_0^{T_{r-1}}P_{r-1}$ and $(U_{r-1})_0^{T_{r-1}}\big|_{\mathfrak{H}} = \begin{pmatrix}(D^{r-1})_{T_{r-1}} & -(T_{r-1})^* \\ T_{r-1} & (D^{r-1})_{(T_{r-1})^*}\end{pmatrix}$.

Let V_{r-1} be the unitary shift operator on $\mathcal{H}$ such that $(V_{r-1}x)_n = x_{n+1}$ for each $x = (x_n) \in \mathcal{H}$. Then the operator $(U_{r-1})^{T_{r-1}} = V_{r-1}\big(\mathbf{1}_{\mathcal{H}} - P_{r-1} + (U_{r-1})_0^{T_{r-1}}\big)$ is the unitary dilation of T_{r-1}.

If S_{r-1} is another contraction on (H_{r-1}) then $(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}} = V_{r-1}((U_{r-1})_0^{T_{r-1}} - (U_{r-1})_0^{S_{r-1}}$, so $\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\| = \big\|((U_{r-1})_0^{T_{r-1}} - (U_{r-1})_0^{S_{r-1}})\big\|$ and $\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\|_{1+\epsilon} = \big\|((U_{r-1})_0^{T_{r-1}} - (U_{r-1})_0^{S_{r-1}})\big\|_{1+\epsilon}$, if $(U_{r-1})_0^{T_{r-1}} - (U_{r-1})_0^{S_{r-1}} \in (C^{r-1})_{1+\epsilon}$. As

$$\begin{aligned}(U_{r-1})_0^{T_{r-1}}\big|_{\mathfrak{H}} - (U_{r-1})_0^{S_{r-1}}\big|_{\mathfrak{H}} &= \begin{pmatrix}(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}} & 0 \\ 0 & (D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\end{pmatrix} \\ &+ \begin{pmatrix}0 & -(T_{r-1})^* + (S_{r-1})^* \\ T_{r-1} - S_{r-1} & 0\end{pmatrix},\end{aligned}$$

we have

$$\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\| \le \|T_{r-1} - S_{r-1}\| + \max\big(\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|, \|(D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\|\big), \quad (75)$$

$$\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\|_{1+\epsilon} \le 2^{1/1+\epsilon}\|T_{r-1} - S_{r-1}\|_{1+\epsilon} + 2^{1/1+\epsilon}\max\Big(\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon}, \|(D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\|_{1+\epsilon}\Big), \quad (76)$$

Corollary (3.2.28) [293]:

(i) $\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\| \le \|T_{r-1} - S_{r-1}\| + 2^{1/2}\|T_{r-1} - S_{r-1}\|^{1/2}$.

(ii) Let $\epsilon > 0$ and $T_{r-1} - S_{r-1} \in (C^{r-1})_{1+\epsilon/2}$. Then $T_{r-1} - S_{r-1} \in (C^{r-1})_{1+\epsilon}$, $(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}} \in (C^{r-1})_{1+\epsilon}$ and

$\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\|_{1+\epsilon} \le 2^{1/1+\epsilon}\|T_{r-1} - S_{r-1}\|_{1+\epsilon} + 2^{\frac{1}{2}+\frac{1}{1+\epsilon}}\|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}^{1/2}$, if $\frac{1+\epsilon}{2} \ge 1$,

$\|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\|_{1+\epsilon} \le 2^{1/1+\epsilon}\|T_{r-1} - S_{r-1}\|_{1+\epsilon} + (2)^{\frac{3}{1+\epsilon}}\|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}^{1/2}$, if $\frac{1+\epsilon}{2} < 1$.

Proof. For $R = R^* \in B(H_{r-1})$, denote by R^+ and R^- the positive and negative parts of $R: R^{\mp} = \frac{1}{2}(|R| \pm R)$. Then $\|R\| = \max(\|R^+\|, \|R^-\|)$. If $R \in (C^{r-1})_{1+\epsilon}$ then $R^+, R^- \in (C^{r-1})_{1+\epsilon}$ and

$$\|R\|_{1+\epsilon}^{1+\epsilon} = \||R|\|_{1+\epsilon}^{1+\epsilon} = \|R^+\|_{1+\epsilon}^{1+\epsilon} = \|R^-\|_{1+\epsilon}^{1+\epsilon} \quad (77)$$

For $0 < A_i \in (C^{r-1})_{1+\epsilon}$, we have $(A_i)^{1/2} \in (C^{r-1})_{2(1+\epsilon)}$ and $\|(A_i)^{1/2}\|_{2(1+\epsilon)} = \|A_i\|_{1+\epsilon}^{1/2}$. Hence, by (77),

$$\|(R^+)^{1/2}\|_{2(1+\epsilon)}^{2(1+\epsilon)} + \|(R^-)^{1/2}\|_{2(1+\epsilon)}^{2(1+\epsilon)} = \|R^+\|_{1+\epsilon}^{1+\epsilon} + \|R^-\|_{1+\epsilon}^{1+\epsilon} = \|R\|_{1+\epsilon}^{1+\epsilon} \quad (78)$$

We need now the following result of Birman, Koplienko and Solomyak [129, Theorem 1]. Let $A_i, (A_i + \epsilon)$ be positive operators in $B(H_{r-1})$. Set $G_{r-1} = (A_i + \epsilon)^{1/2} - (A_i)^{1/2}$and $F_{r-1} = \epsilon$. Then

(i)$\|{G_{r-1}}^+\| \leq \|{F_{r-1}}^+\|^{1/2}$and $\|{G_{r-1}}^-\| \leq \|{F_{r-1}}^-\|^{1/2}$;

(ii) If $|F_{r-1}|^{1/2} \in (C^{r-1})_{1+\epsilon}$ then $G_{r-1} \in (C^{r-1})_{1+\epsilon}$ and $\|{G_{r-1}}^+\|_{1+\epsilon} \leq \|({F_{r-1}}^+)^{1/2}\|_{1+\epsilon}$and $\|{G_{r-1}}^-\|_{1+\epsilon} \leq \|({F_{r-1}}^-)^{1/2}\|_{1+\epsilon}$.

Combining this with (77) and (78), yields

$$\|G_{r-1}\| = \max(\|{G_{r-1}}^+\|, \|{G_{r-1}}^-\|) \leq \max\left(\|{F_{r-1}}^-\|^{1/2}, \|{F_{r-1}}^+\|^{1/2}\right) = \|F_{r-1}\|^{1/2}, \qquad (79)$$

$$\|G_{r-1}\|_{1+\epsilon}^{1+\epsilon} = \|{G_{r-1}}^+\|_{1+\epsilon}^{1+\epsilon} + \|{G_{r-1}}^-\|_{1+\epsilon}^{1+\epsilon} \leq \|({F_{r-1}}^+)^{1/2}\|_{1+\epsilon}^{1+\epsilon} + \|({F_{r-1}}^-)^{1/2}\|_{1+\epsilon}^{1+\epsilon} = \|F_{r-1}\|_{1+\epsilon/2}^{1+\epsilon/2}.$$
(80)

Set $A_i + \epsilon = \mathbf{1} - (T_{r-1})^* T_{r-1}$and $A_i = \mathbf{1} - (S_{r-1})^* S_{r-1}$. We obtain from (79) that

$$\begin{aligned}\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\| &= \|(\mathbf{1} - (T_{r-1})^* T_{r-1})^{1/2} - (\mathbf{1} - (S_{r-1})^* S_{r-1})^{1/2}\| \\ &\leq \|(\mathbf{1} - (T_{r-1})^* T_{r-1}) - (\mathbf{1} - (S_{r-1})^* S_{r-1})\|^{1/2} \\ &= \|(T_{r-1})^*(T_{r-1} - S_{r-1}) + ((T_{r-1})^* - (S_{r-1})^*)S_{r-1}\|^{1/2} \leq (2\|T_{r-1} - S_{r-1}\|)^{1/2}.\end{aligned}$$

Similarly, $\|(D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\| \leq (2\|T_{r-1} - S_{r-1}\|)^{1/2}$ and part (i) follows from (75).

Let $T_{r-1} - S_{r-1} \in (C^{r-1})_{1+\epsilon/2}$. Then

$$\begin{aligned}F_{r-1} &= (\mathbf{1} - (T_{r-1})^* T_{r-1}) - (\mathbf{1} - (S_{r-1})^* S_{r-1}) \\ &= (T_{r-1})^*(T_{r-1} - S_{r-1}) + ((T_{r-1})^* - (S_{r-1})^*)S_{r-1} \in (C^{r-1})_{1+\epsilon/2},\end{aligned}$$

So $|F_{r-1}|^{1/2} \in C_{1+\epsilon}$. Hence $(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}} \in (C^{r-1})_{1+\epsilon}$ and we obtain from (80) that

$$\begin{aligned}\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon}^{1+\epsilon} &= \|(\mathbf{1} - (T_{r-1})^* T_{r-1})^{1/2} - (\mathbf{1} - (S_{r-1})^* S_{r-1})^{1/2}\|_{1+\epsilon}^{1+\epsilon} \\ &\leq \|(\mathbf{1} - (T_{r-1})^* T_{r-1}) - (\mathbf{1} - (S_{r-1})^* S_{r-1})\|_{1+\epsilon/2}^{1+\epsilon/2} \\ &= \|(T_{r-1})^*(T_{r-1} - S_{r-1}) + ((T_{r-1})^* - (S_{r-1})^*)S_{r-1}\|_{1+\epsilon/2}^{1+\epsilon/2} \\ &\leq \left(2\|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}\right)^{1+\epsilon/2}.\end{aligned}$$

If $(1 + \epsilon)/2 \geq 1$ then $\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon}^{1+\epsilon} \leq \left(2\|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}\right)^{1+\epsilon/2}$,

so $\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon} \leq 2^{1/2}\|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}^{1/2}$. Similarly,

$\|(D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\| \leq 2^{1/2}\|T_{r-1} - S_{r-1}\|_{(1+\epsilon)/2}^{1/2}$

and the first formula in (ii) follows from (75).

If $(1 + \epsilon)/2 < 1$ then (see [130, Lemma XI.9.9]) $\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon}^{1+\epsilon} \leq 2\|T_{r-1} - S_{r-1}\|_{(1+\epsilon)/2}^{(1+\epsilon)/2}$, so $\|(D^{r-1})_{T_{r-1}} - (D^{r-1})_{S_{r-1}}\|_{1+\epsilon} \leq 2^{2/1+\epsilon}\|T_{r-1} - S_{r-1}\|_{(1+\epsilon)/2}^{1/2}$.

Similarly, $\|(D^{r-1})_{(T_{r-1})^*} - (D^{r-1})_{(S_{r-1})^*}\|_{1+\epsilon} \leq 2^{2/(1+\epsilon)}\|T_{r-1} - S_{r-1}\|_{(1+\epsilon)/2}^{1/2}$ and the second formula in (ii) follows from (76). (see[114]).

Corollary (3.2.29) [293]: Let $\epsilon > 0$. Then $\omega_{1+\epsilon,\frac{(1+\epsilon)}{2}}(1-\epsilon) \leq k_{1+\epsilon}(1-\epsilon)^{1/2}$, where

$k_{1+\epsilon} = 2^{1/(1+\epsilon)}(1 + 2^{1/2})$, if $\epsilon > 0$, and $k_{1+\epsilon} = 2^{1/(1+\epsilon)} + 2^{3/(1+\epsilon)}$, if $\epsilon > 0$. (81)

Proof. It suffices to use Corollary (3.2.28) and to note that $\omega_{1+\epsilon,\frac{1+\epsilon}{\epsilon}}(1-\epsilon) \le \|(U_{r-1})^{T_{r-1}} - (U_{r-1})^{S_{r-1}}\|_{\frac{1+\epsilon}{\epsilon}}$ for $\|T_{r-1} - S_{r-1}\|_{1+\epsilon} = 1-\epsilon$ and $\|T_{r-1} - S_{r-1}\|_{1+\epsilon} \le \|T_{r-1} - S_{r-1}\|_{1+\epsilon/2} \le \|T_{r-1} - S_{r-1}\|_{1+\epsilon/2}^{1/2}$.

Recall that f_{r-1} is called a $(C^{r-1})_{1+\epsilon}$-Lipschitz function on $\mathbb{T}$, $\epsilon > 0$, if there is $D^{r-1} > 0$ such that

$$f_{r-1}(U_{r-1}) - f_{r-1}(V_{r-1}) \in (C^{r-1})_{1+\epsilon}$$

and $\|f_{r-1}(U_{r-1}) - f_{r-1}(V_{r-1})\|_{1+\epsilon} \le D^{r-1}\|U_{r-1} - V_{r-1}\|_{1+\epsilon}$ (82)

for all unitary U_{r-1}, V_{r-1}, with $U_{r-1} - V_{r-1} \in (C^{r-1})_{1+\epsilon}$.

For contractions T_r, T_{r+1}, set $(U_{r-1})_i = (U_{r-1})^{(T_{r-1})_i}$. If $f_{r-1} \in A(\mathbb{T})$ is a $(C^{r-1})_{1+\epsilon}$-Lipschitz function then, by (53),

$$\begin{aligned}\|f_{r-1}(T_r) - f_{r-1}(T_{r+1})\|_{1+\epsilon} &= \left\|P_{r-1}(f_{r-1}(U_r) - f_{r-1}(U_{r+1}))|_{H_{r-1}}\right\|_{1+\epsilon} \\ &\le \|f_{r-1}(U_r) - f_{r-1}(U_{r+1})\|_{1+\epsilon} \le D^{r-1}\|U_r - U_{r+1}\|_{1+\epsilon} \le D^{r-1}\delta_{1+\epsilon}(T_r, T_{r+1}).\end{aligned}$$

Hence $\max\left\{\|f_{r-1}(T_r) - f_{r-1}(T_{r+1})\|_{1+\epsilon} : \|T_r - T_{r+1}\|_{\frac{1+\epsilon}{\epsilon}} = 1-\epsilon\right\} \le D^{r-1}\omega_{(1+\epsilon),(\frac{1+\epsilon}{\epsilon})}(1-\epsilon)$.

Corollary (3.2.30) [293]: If $f_{r-1} \in A(\mathbb{T})$ satisfies (83) then, for all contractions T_{r-1}, S_{r-1},

$\|f_{r-1}(T_{r-1})X - Xf_{r-1}(S_{r-1})\|_2 \le D^{r-1}\|T_{r-1}X - XS_{r-1}\|_2$ for $X \in (C^{r-1})_2$, and (84)

$\|f_{r-1}(T_{r-1}) - f_{r-1}(S_{r-1})\|_2 \le D^{r-1}\|T_{r-1} - S_{r-1}\|_2$ if $T_{r-1} - S_{r-1} \in (C^{r-1})_2$. (85)

Let $C^{r-1}(\mathbb{D})$ be the $(C^{r-1})^*$-algebra of all continuous functions on $\mathbb{D}$. As it is nuclear, the tensor product $C^{r-1}(\mathbb{D}) \otimes C^{r-1}(\mathbb{D})$ is isomorphic to $C^{r-1}(\mathbb{D} \times \mathbb{D})$ with $f_{r-1} \otimes g_{r-1} \cong f_{r-1}(z_{r-1})g_{r-1}(w_{r-1})$ for $f_{r-1}, g_{r-1} \in C^{r-1}(\mathbb{D})$.

Denote by $A(\mathbb{D} \times \mathbb{D})$ the closure of the algebraic tensor product $A(\mathbb{D}) \odot A(\mathbb{D})$ in $C^{r-1}(\mathbb{D} \times \mathbb{D})$. It is well known that $A(\mathbb{D} \times \mathbb{D})$ consists of all functions continuous on $\mathbb{D} \times \mathbb{D}$ and holomorphic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ (the bidisk algebra).

Proof : First suppose that f_{r-1} is continuously differentiable. By (59), $f_{r-1}(z_{r-1}) - f_{r-1}(w_{r-1}) = (\widehat{f_{r-1}})(z_{r-1}, w_{r-1})(z_{r-1} - w_{r-1})$. $(\widehat{f_{r-1}})(z_{r-1}, w_{r-1})$ is analytic on $\mathbb{D}^\circ \times \mathbb{D}^\circ$ and continuous on $\mathbb{D} \times \mathbb{D}$. Hence it belongs to $A(\mathbb{D} \times \mathbb{D})$.

Let $T_{r-1}, S_{r-1} \in \mathrm{Con}(H_{r-1})$. Define contractive representations π_i of $A(\mathbb{D})$ on the Hilbert space $(C^{r-1})_2$ by setting $\pi_1(g_{r-1})(X) = g_{r-1}(T_{r-1})X$, $\pi_2(g_{r-1})(X) = Xg_{r-1}(S_{r-1})$, for $X \in (C^{r-1})_2$. They commute and, by (51) and (65), $\|\pi_1(g_{r-1})(X)\|_2 \le \|g_{r-1}(T_{r-1})\|\|X\|_2 \le \|g_{r-1}\|\|X\|_2$, so $\|\pi_1\| \le 1$. Similarly, $\|\pi_2\| \le 1$. Let π be the representation of $A(\mathbb{D} \times \mathbb{D})$ constructed in Corollary (3.2.31) .Then $\pi(f_{r-1}(z_{r-1}) - f_{r-1}(w_{r-1}))X = (\pi_1(f_{r-1}) - (\pi_2 f_{r-1}))X = f_{r-1}(T_{r-1})X - X f_{r-1}(S_{r-1})$ and $\pi(z_{r-1} - w_{r-1})X = T_{r-1}X - XS_{r-1}$. As $\pi(f_{r-1}(z_{r-1}) - f_{r-1}(w_{r-1})) = \pi((\widehat{f_{r-1}})(z_{r-1}, w_{r-1}))\pi(z_{r-1} - w_{r-1})$, we have

$$\begin{aligned}\|f_{r-1}(T_{r-1})X - X f_{r-1}(S_{r-1})\| &\le \left\|\pi((\widehat{f_{r-1}}))\right\|\|T_{r-1}X - XS_{r-1}\| \\ &\le \left\|(\widehat{f_{r-1}})\right\|\|T_{r-1}X - XS_{r-1}\|,\end{aligned}$$

where, by (59),$\|(\widehat{f_{r-1}})\| = \sup_{z_{r-1}\in\mathbb{D}\times\mathbb{D}} |(\widehat{f_{r-1}})(z_{r-1}, w_{r-1})| = \sup_{z_{r-1}\in\mathbb{D}} |(f_{r-1})'(z_{r-1})|$ then $(\widehat{f_{r-1}}) \in A(\mathbb{T})$ so $\sup_{z_{r-1}\in\mathbb{D}} |(f_{r-1})'(z_{r-1})| = \sup_{(1-\epsilon)\in\mathbb{T}} |(f_{r-1})'(1-\epsilon)|$. Thus (85) holds for continuously differentiable functions f_{r-1} with $D^{r-1} = \sup_{(1-\epsilon)\in\mathbb{T}} |(f_{r-1})'(1-\epsilon)|$.

Let now $f_{r-1} \in A(\mathbb{T})$ be an arbitrary function that satisfies (83). Choose the sequence $\{\varphi_n\}$ of infinitely differentiable functions on $\mathbb{T}$ as in the proof of part (ii) in Corollary (3.2.24) and set $(h_{r-1})_n = \varphi_n * f_{r-1}$. Then $(h_{r-1})_n$ are infinitely differentiable, belong to $A(\mathbb{T})$ and satisfy (83) with the same constant D^{r-1}. As $(D^{r-1})_n = \sup_{(1-\epsilon)\in\mathbb{T}} |(h_{r-1})_n'(1-\epsilon)| \le D^{r-1}$, it follows that

$$\|(h_{r-1})_n(T_{r-1})X - X(h_{r-1})_n(S_{r-1})\|_2 \le (D^{r-1})_n \| T_{r-1}X - XS_{r-1}\|_2$$
$$\le D^{r-1}\| T_{r-1}X - XS_{r-1}\|_2 \text{ for } X \in (C^{r-1})_2.$$

Repeating the end of the proof of Corollary (3.2.24)and replacing the operator norm $\|\cdot\|$ by the norm$\|\cdot\|_2$, we obtain that (84) holds for f_{r-1} . Now it suffices to use Corollary (3.2.25)to obtain that (85) also holds for f_{r-1} .

Corollary (3.2.31) [293]: Let π_i , $i = 1,2$, be representations of $A(\mathbb{D})$ on a Hilbert space H_{r-1} such that $\|\pi_i\| \le 1$ and $[\pi_1(g_{r-1}), \pi_2(h_{r-1})] = 0$ for all $g_{r-1}, h_{r-1} \in A(\mathbb{D})$. Then there exists a representation π of $A(\mathbb{D}\times\mathbb{D})$ on H_{r-1} such that $\|\pi\| \le 1$ and $\pi(f_{r-1}(z_{r-1})g_{r-1}(w_{r-1})) = \pi_1(f_{r-1})\pi_2(g_{r-1})$ for$f_{r-1}, g_{r-1} \in A(\mathbb{D})$.

Proof. Let $id \in A(\mathbb{D})$ be the function such that $id(z_{r-1}) \equiv z_{r-1}$. Then $(T_{r-1})_i = \pi_i(id)$ are commuting contractions in $B(H_{r-1})$and, for each polynomial p_{r-1}, $\pi_i(p_{r-1}) = p_{r-1}(T_{r-1})_i)$. Hence $\pi_i(f_{r-1}) = f_{r-1}((T_{r-1})_i)$ for each $f_{r-1} \in A(\mathbb{D})$. Indeed, if polynomials $(p_{r-1})_n$ converge to f_{r-1} then, by (51),

$$\|\pi_i(f_{r-1}) - f_{r-1}((T_{r-1})_i)\| \le \|\pi_i(f_{r-1} - (p_{r-1})_n)\| + \|(p_{r-1})_n((T_{r-1})_i) - f_{r-1}((T_{r-1})_i)\|$$
$$\le 2\| f_{r-1} - (p_{r-1})_n\| \to 0.$$

By Ando's theorem [122, Theorem I.6.4], there are a Hilbert space $K \supset H_{r-1}$ and commuting unitary operators $(U_{r-1})_i$ on K such that $(T_{r-1})_i = P_{r-1}(U_{r-1})_i|_{H_{r-1}}$, where P_{r-1} is the projection on H_{r-1} , and

$$T_r^n T_{r+1}^m = P_{r-1}U_r^n U_{r+1}^m|_{H_{r-1}} \text{for all } m, n \qquad (86)$$

The *-representations ρ_i of $C^{r-1}(\mathbb{D})$ on K defined by

$$\rho_i(g_{r-1}) = g_{r-1}((U_{r-1})_i)$$

commute and, by (53) , $P_{r-1}\rho_i(f_{r-1})|_{H_{r-1}} = P_{r-1} f_{r-1}((U_{r-1})_i)|_{H_{r-1}} = f_{r-1}((T_{r-1})_i) = \pi_i(f_{r-1})$ for $f_{r-1} \in A(\mathbb{D})$. For each $h_{r-1} \in C^{r-1}(\mathbb{D}\times\mathbb{D})$, there is a unique operator $h_{r-1}(U_r, U_{r+1})$ in the commutative (C^{r-1})*-sub algebra of $B(H_{r-1})$ generated by U_r, U_{r+1} (see [124, Corollary 16.7]), $\rho: h_{r-1} \to h_{r-1}(U_r, U_{r+1})$ is a *-representation of $C^{r-1}(\mathbb{D}\times\mathbb{D})$ on K and $\rho(f_{r-1}\otimes g_{r-1}) = \rho(f_{r-1}(z_{r-1})g_{r-1}(w_{r-1})) = f_{r-1}(U_r)g_{r-1}(U_{r+1}) = \rho_1(f_{r-1})\rho_2(g_{r-1})$ for all $f_{r-1}, g_{r-1} \in C^{r-1}(\mathbb{D}$. It remains to define a representation π of $A(\mathbb{D}\times\mathbb{D})$ by setting $\pi(h_{r-1}) = P_{r-1}\rho(h_{r-1})|_{H_{r-1}}$, for $h_{r-1} \in A(\mathbb{D}\times\mathbb{D})$.