Chapter 2

Hankel Operators and Extension Trace Formulae

One of the fundamental spectral characteristics, considered in the perturbation theory of unitary and self-adjoint operators ,is the so-called spectral shift function of Krein. We improve the results and obtain sharp conditions under which the Koplienko–Neidhardt trace formulae hold.

Section (2.1): Perturbation Theory of Unitary and Self-Adjoint Operators

A function is associated to a pair of self-adjoint (unitary) operators and one has

$$\text{trace}\big(\varphi(A) - \varphi(B)\big) = \int \xi(\lambda)\varphi'(\lambda)d\lambda. \qquad (1)$$

If the spectrum of the operators A and B is discrete in some interval, then the increment of the function ξ on it coincides with the difference of the number of eigepvalues of the operators A and B. The function ξ is related by simple and natural formulas with the fundamental objects of the spectral theory: the perturbation determinants, the scattering suboperators, and for the Schrodinger operator" also with the limit phase and the amplitude of the corresponding eigenfunctions [45].

However, for the applicability of the formula (1) one has to observe certain conditions. The following two questions raised by Krein [45]are fundamental here: i) for which functions φ is it true that $A - B \in \mathfrak{S}$. implies $\varphi(A) - \varphi(B) \in \mathfrak{S}_1$? ($A$, B are self-adjoint (unitary) operators, while $\mathfrak{S}_1$ is the class of nuclear operators); ii) is formula (1), true for the functions φ, satisfying the previous property?

We introduce a new technique in the considered sphere of questions. The fundamental circumstance consists in the reduction of the mentioned problems to the investigation of the metric properties of Hankel operators, but also the use of tensor products turns out to be of no small importance. The fundamental consequence of this, approach is the determination of new sufficient ($\varphi \in \boldsymbol{B}^1_{\infty 1}$, the Besov class) and necessary ($\varphi \in \boldsymbol{B}^1_{11}$, $\varphi \in L$) conditions for the validity of the formula (1) for all nuclear perturbations and also for the differentiability of the mapping $A \to \varphi(A)$. The comparison of the two last conditions shows that the "tolerance" between them is very small and, apparently, it cannot be expressed in terms of the local (or integral) smoothness of the function φ. In addition, we give criteria for the continuity of double operator integrals .

Every real Borel function f on the R axis generates a mapping $A \to f(A)$ of the set of self-adjoint operators into itself. We investigate the smoothness properties of this mapping. This problem and related ones have been investigated by [45-48].

It is easy to show that if the mapping $A \to f(A)$ is differentiable, then. $f \in C^1(R)$, Also. if $\varphi(A) - \varphi(B) \in \mathfrak{S}_1$ for any bounded self-adjoint operators A, B such that $A - B \in \mathfrak{S}_1$, then $f|I \in L^\infty(I)$ for each bounded interval I. However, it has turned out that the converse statements do not hold [49-51]. Moreover [9], there exist a function f from

$C^1(R)$ and self- adjoint operators A_n, B_n with spectra in[45], such that $\|A_n - B_n\| \to 0$, but $\|f(A_n) - f(B_n)\|.\ (\|A_n - B_n\|)^{-1} \to \infty$. The problem of the characterization of the functions f for which the mapping $A \to f(A)$ is differentiable has been posed also by Widom [52].

The considered questions are closely related with the problem of the description of the functions f for which

$$(s,t) \to \check{f}(s,t)K(s,t), \quad \check{f}(s,t) \stackrel{\text{def}}{=} \frac{f(s) - f(t)}{s - t}$$

is the kernel of a nuclear operator in $L^2(I)$ for any finite interval I and for any function K on $I \times I$ which is the kernel of a nuclear operator in $L^2(I)$. In this case we say that $\check{f} \in \mathfrak{R}$. This question is discussed also in Widom's problem [52].

We obtain a new sufficient condition under which f has the above described property. This condition is given in terms of the Besov class $\boldsymbol{B}^1_{\infty 1}$ and it is suitable . At the same time it is proved that it is not necessary. We also give new necessary conditions, showing that the condition $f \in C^1$ is not sufficient. From here there follow the mentioned results of Yu. B. Farforovskaya for the uniform and the nuclear norms and one shows that such counterexamples can be constructed for any function f which does not satisfy the obtained necessary conditions. For technical reasons it is convenient to consider unitary rather than self-adjoint operators and to deal with functions on the unit circumference T.

We obtain sufficient and necessary conditions on a function f for which $\check{f} \in \mathfrak{R}$ .In particular, we show that the sufficient condition of Birman and Solomyak [47] for the boundedness of double operator integrals is in fact also necessary .See [53].

The machinery of double operator integrals, i.e., integrals of the form

$$\Phi T = \int_\Lambda \int_{\mathrm{M}} \varphi(s,t) dF(s) dE(t),$$

will play a fundamental role. Here F and E are spectral Borel measures on separable metric spaces Λ and M, while T is a bounded operator in a Hilbert space H.

If $\varphi \in L^\infty(\Lambda \times \mathrm{M})$, then the transform Φ continuous on the Hilbert-Schmidt class $\mathfrak{S}_2$ and its norm is equal to$\|\varphi\|_{L^\infty}$. If it is continuous also in $\mathfrak{S}_1$ then one can define by duality the operator ΦT each bounded operator T. If the transform Φ bounded in the symmetric normed ideal S, then we say that φ is a Schur multiplier of the space S. The set of all Schur multipliers of the space $\mathfrak{S}_1$(or of the space $B(\mathcal{H})$ of bounded operators in H) will be denoted by the symbol $R(\Lambda \times \mathrm{M})$. The basic information regarding double operator integrals can be found in [47, 54, 55]. The class $R(\Lambda \times \mathrm{M})$ admits the following description.

THEOREM(2.1.1)[44]. Let $\varphi \in L^\infty(\Lambda \times \mathrm{M})$, and let λ, μ be scalar positive measures on Λ, M, mutually absolutely continuous with F, E. The following statements are equivalent:

i) $\varphi \in \mathfrak{R}(\Lambda \times \mathrm{M})$.

ii) For each function K on $\Lambda \times \mathrm{M}$, which is the kernel of a nuclear operator from $L^2(\Lambda, \lambda)$ into $L^2(\mathrm{M}, \mu)$, the function

$$(s,t) \to \varphi(s,t)K(s,t)$$

is the kernel of a nuclear operator from $L^2(\Lambda,\lambda)$ into $L^2(\mathrm{M},\mu)$.

iii) There exist a measure space $(\mathcal{X},\sigma)$ and measurable functions α on $\Lambda\times\mathcal{X}$, β on $\mathrm{M}\times\mathcal{X}$ such that

$$\varphi(s,t) = \int_{\mathcal{X}} \alpha(s,x)\ \beta(t,x)\, d\sigma(x) \quad ,(s,t) \in \Lambda\times\mathrm{M} \tag{2}$$

$$\int \|\alpha(\cdot,x)\|_{L^\infty(\Lambda)} .\|\beta(\cdot,x)\|_{L^\infty(\mathrm{M})}\, d\sigma(x) \quad < \infty. \tag{3}$$

iv)There exist a measure space $(x,0)$ and a measurable functions α on $\Lambda\times\mathcal{X}$, β on $\mathrm{M}\times\mathcal{X}$ such that conditions holds (2) and

$$\left\|\int_{\mathcal{X}} |\alpha(\cdot,x)|^2 d\sigma(x)\right\|_{L^\infty(\Lambda)} \cdot \left\|\int_{\mathcal{X}} |\beta(\cdot,x)| d\sigma(x)\right\|_{L^\infty(\mathrm{M})} < \infty. \tag{4}$$

The implications iv)⟹i)⟺ii) have been obtained in [47]. In the case of discrete measure spaces, a statement similar to ii) ⟹iii) has been obtained by Bennett [56]. The proof of the most nontrivial implication ii)⟹iii) has been obtained with the collaboration of C. V. Kislyakov and makes use of an idea of E. D. Gluskin, with the aid of which he has obtained a new proof of G. Bennett's theorem [56]. We note that iii) is equivalent to the fact that φ admits the representation (2), for which one has the condition (stronger than (3))

$$\int_{\mathcal{X}} \|\alpha(\cdot,x)\|^2_{L^\infty(\Lambda)} d\sigma(x) \cdot \int_{\mathcal{X}} \|\beta(\cdot,x)\|^2_{L^\infty(\mathrm{M})} d\sigma(x) < \infty. \tag{5}$$

Condition (5) is essentially stronger than (4); nevertheless, Theorem (2.1.1) states that 3) ⟺4). More exactly, there exist representations (2), satisfying (4) but not (5). Nevertheless, in this case there exists another representation for which (3) holds (and even (5)).

Proof. 3) ⟹4). Let ε be a positive function on X such that for $\tilde{\alpha}(\cdot,x) = \varepsilon(x)\alpha(\cdot,x)$, $\tilde{\beta}(\cdot,x) = \varepsilon^{-1}(x)\beta(\cdot,x)$ we have $\|\tilde{\alpha}(\cdot,x)\|_{L^\infty(\Lambda)} = \|\tilde{\beta}(\cdot,x)\|_{L^\infty(\mathrm{M})}$, $x\in\mathcal{X}$. Then

$$\varphi(s,t) = \int_{\mathcal{X}} \tilde{\alpha}(s,x)\tilde{\beta}(t,x)\, d\sigma(x), \qquad s,t \in \Lambda\times\mathrm{M};$$

$$\left(\int_{\mathcal{X}} \|\tilde{\alpha}(\cdot,x)\|^2_{L^\infty(\Lambda)} d\sigma(x)\right)^{1/2} \cdot \left(\int_{\mathcal{X}} \|\tilde{\beta}(\cdot,x)\|^2_{L^\infty(\mathrm{M})} d\sigma(x)\right)^{1/2}$$

$$= \int_{\mathcal{X}} \|\alpha(\cdot,x)\|_{L^\infty(\Lambda)} \cdot \|\beta(\cdot,x)\|_{L^\infty(\mathrm{M})} d\sigma(x) < \infty.$$

Obviously, $\tilde{\alpha}$and $\tilde{\beta}$satisfy inequality (4).

Definition(2.1.2)[44]. Let B_1,B_2 be Banach spaces. We say that the operator $T: B_1 \to B_2$ belongs to the ideal $I(B_1,B_2)$, if there exist a compact Q, a finite Borel measure αon Q, bounded operators $T_1: B_1 \to C(Q), T_2: L^1(\sigma) \to B_2^{**}$ such that the diagram

$$\begin{array}{ccc} B_1 \xrightarrow{T} B_2 & \xrightarrow{j} & B_2^{**} \\ T_1 \downarrow & & \uparrow T_2 \\ C(Q) & \xrightarrow{Id} & L^1(\sigma) \end{array}$$

is commutative (Id is the identity imbedding and j is the natural imbedding of the space B_2 in its second conjugate space).

LEMMA(2.1.3)[44]. Let $T: L^1(\lambda) \to L^\infty(\mu)$ be an integral operator with kernel K,

$$(Tf)(t) = \int_\Lambda K(s,t) f(s) d\lambda(s),$$

and assume that K satisfies the conditions of Theorem (2.1.1). Then $\in I(L^1(\lambda), L^\infty(\mu))$.

Proof. From Theorems 10.3.6 and 19.2.13 of [57] there follows that $I(L^1(\lambda), L^\infty(\mu))$ is the conjugate space to the space of compact operators from $L^\infty(\mu)$ into $L^1(\lambda)$. More exactly, the inclusion $T \in I(L^1(\lambda), L^\infty(\mu))$ follows from the inequality

$$|\text{trace}\, TU| \leq \text{ const } \|U\|$$

for each operator $U: L^\infty(\mu) \to L^1(\lambda)$ of finite rank.

Let $f = \sum_{i=1}^n (f, g_i) h_i$, where $g_i \in (L^\infty(\mu))^*, h_i \in L^1(\lambda)$. By virtue of the principle of local reflexivity (see LemmaE.3.1 from [57]) one can assume that $g_i \in L^1(\mu)$. Further, one can assume that g_i, h_i are finite linear combinations of characteristic functions.

Consequently, there exist finite-dimensional subspaces A^∞, A^1 of the spaces $L^\infty(\mu), L^1(\lambda)$, consisting of step functions isometric to the finite-dimensional spaces L^∞ and L^1 such that the diagram

$$\begin{array}{ccc} L^\infty(\mu) & \xrightarrow{U} & L^1(\lambda) \\ P\downarrow & & \uparrow \text{Id} \\ A^\infty & \xrightarrow{U} & A^1 \end{array}$$

is commutative (P is the natural projection (averaging) from $L^\infty(\mu)$ onto A^∞). One can identify A^∞ with $L^\infty(\tilde{\mu})$, A^1 with $L^1(\tilde{\lambda})$, where $\tilde{\mu}, \tilde{\lambda}$, are measures with a finite number of atoms.

By Grothendieck's theorem [58], U admits the factorization

$$\begin{array}{ccc} L^\infty(\tilde{\mu}) & \xrightarrow{U} & L^1(\tilde{\lambda}) \\ & \xrightarrow{M_\varphi} L_2(\tilde{\mu}) V & \uparrow \end{array}$$

where M_φ is the operator of multiplication by function from $L^2(\tilde{\mu}), V$ is a bounded operator with $\|\varphi\|_{L^2(\tilde{\mu})} \|V\| \leq K_G \|U\|$. where K_G is the Grothendieck constant. We apply now Orot.hendiec'ks theorem [58] to the operator V^* and we obtain that V admits the factorization

$$\begin{array}{ccc} L^2(\tilde{\mu}) & \xrightarrow{U} & L^1(\tilde{\lambda}) \\ W\downarrow & & \uparrow M_\varphi \\ & \longrightarrow & L^2(\tilde{\lambda}) \end{array}$$

where W is a bounded operator $\psi \in L^2(\tilde{\lambda})$, such that $\|\psi\|_{L^2(\tilde{\lambda})} \cdot \|W\| \leq K_G \|V\|$.

From here it follows that U admits the factorization

$$\begin{array}{ccc} L^\infty(\mu) & \xrightarrow{U} & L^1(\lambda) \\ \text{M}_\varphi \downarrow & & \uparrow \text{M}_\psi \\ L^2(\mu) & \xrightarrow{\tilde{W}} & L^2(\lambda) \end{array}$$

where M_φ, M_ψ, are the operators of multiplication by the step functions $\varphi, \psi, \widetilde{W}$ is the operator induced by the operator W and $\|\varphi\|_{L^2(\mu)} \cdot \|W\| \cdot \|\psi\|_{L^2(\lambda)} \le K_G^2 \|U\|$.

We have

$$\text{trace}\, TU = \text{trace}\, TM_\psi \widetilde{W} M_\varphi = \text{trace}\, M_\varphi T M_\psi \widetilde{W}.$$

It is easy to see that

$$\left(M_\varphi T M_\psi f\right)(t) = \int_\Lambda K(s,t)\varphi(t)\psi(s)f(s)d\lambda(s).$$

Since $(s,t) \to \varphi(t)\psi(s)$ is the kernel of a nuclear operator from $L^2(\lambda)$ into $L^2(\mu)$ with norm $\|\varphi\|_{L^2(\mu)} \cdot \|\psi\|_{L^2(\lambda)}$, we have $M_\varphi T M_\psi \in \mathfrak{S}_1$ and $\|M_\varphi T M_\psi\|_{\mathfrak{S}_1} \le c\|\varphi\|_{L^2(\mu)} \cdot \|\psi\|_{L^2(\lambda)}$.
Consequently,

$$\left|\text{trace} M_\varphi T M_\psi \widetilde{W}\right| < c\|\varphi\|_{L^2(\mu)} \cdot \|\psi\|_{L^2(\lambda)} \|\widetilde{W}\| \le cK_G^2\|U\|$$

Now we prove that 2)⟹3). Since the space L^∞ is complemented in $(L^\infty)^{**}$, by virtue of Lemma1 the operator T with the kernel K admits the factorization

$$\begin{array}{ccc} L^\infty(\lambda) & \xrightarrow{T} & L^\infty(\mu) \\ T_1 \downarrow & & \uparrow T_2 \\ L^\infty(\mathrm{X},\sigma) & \xrightarrow{Id} & L^1(\mathrm{X},\sigma) \end{array}$$

where T_1 and T_2 are bounded operators, while σ is a finite measure on $\mathcal{X}$.
It is well known (see [16, Chap. XI, Sec. 2]) that every bounded operator from L^1 into L^∞ is an integral operator with a bounded kernel. Consequently, there exist $\alpha \in L^\infty(\Lambda \times \mathcal{X})$, $\beta \in L^\infty(\mathrm{M} \times \mathcal{X})$ such that

$$(T_1 f)(x) = \int_\Lambda \alpha(s,t)f(s)d\lambda(s), \quad f \in L^1(\Lambda,\lambda),$$

$$(T_2 g)(x) = \int_{\mathcal{X}} \beta(t,x)g(x)d\sigma(s), \quad g \in L^1(\mathcal{X},\sigma).$$

Consequently, K admits representation (2), satisfying inequality (3).
The function K satisfies trivially the condition of statement 3) of Theorem1 if $K \in L^\infty(\lambda)\widehat{\bigoplus} L^\infty(\mu)$, i.e. $K(s,t) = \sum_{n\ge 0}\alpha_n(s)\beta_n(t)$, where $\sum_{n\ge 0}\|\alpha_n\|_{L^\infty(\lambda)}\|\beta_n\|_{L^\infty(\mu)} < \infty$. This means that an integral operator with kernel K is a nuclear operator from $L^1(\lambda)$ into $L^\infty(\mu)$.

Let A and B be self-adjoint operators in H, let E and F be their spectral measures, and let $\varphi \in C^1(\boldsymbol{R})$.
We assume that $B - A$ is a bounded operator. Then, as shown in [47], we have

$$\varphi(B) - \varphi(A) = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \check{\varphi}(s,t)dF(t)(B-A)dE(s), \tag{6}$$

if $\check{\varphi}$ is a Schur multiplier of the space $B(\mathcal{H})$. Also, it is proved in [47] that if $B - A \in \mathfrak{S}_1$ for some separable (or conjugate to a separable) symmetric normed ideal S, and φ is Schur multiplier of the space S, then one has (6) and $\varphi(B) - \varphi(A) \in \mathfrak{S}$, and, in particular, $\varphi(B) - \varphi(A) \in \mathfrak{S}_2$ if $A - B \in \mathfrak{S}_2$ and $\varphi' \in L^\infty$, Thus, there arises the question, when is the function $\check{\varphi}$ on $I \times I$ or on $T \times T$ (I is a finite interval in the case of bounded self-adjoint operators and an infinite interval in the case of unbounded operators) a Schur multiplier of the space $B(\mathcal{H})$. We denote by the

symbol $\mathfrak{R}_1(\mathfrak{R}_T)$ the set of Schur multipliers of the space $B(\mathcal{H})$ relative to any spectral Borel measures on I (on T).
Let A and T be self-adjoint operators, let $T \in \mathfrak{B}(\mathcal{H})$, and let $\check{\varphi}$ be a Schur multiplier of the space $\mathfrak{B}(\mathcal{H})$. Then, in view of (6), we have

$$\big(\varphi(A + rT) - \varphi(A)\big)/r = \int_{\boldsymbol{R}} \int_{\boldsymbol{R}} \check{\varphi}(s,t)dE_r(s)TdE(t), \tag{7}$$

where E and E_r are the spectral measures of the operators A and $A + rT$.

It is established in [47] that if φ admits the representation (2), in which $\alpha \in C(\bar{\boldsymbol{R}}, L^2(\mathcal{X},\sigma))$ (the set of such function will be denoted by the symbol $A(\boldsymbol{R})$), then in this case in the right-hand side in (7) one can take the limit with respect to the norm and one has the equality

$$\frac{d_\varphi(A+rT)}{dr}\bigg|_{r=0} = \int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \check{\varphi}(s,t)dE(s)TdE(t). \tag{8}$$

Moreover, the function $A \to \varphi(A)$, defined on the class of bounded self-adjoint operators, is Fréchet differentiable. Formula (9) for bounded A and for $\varphi \in C^2(\boldsymbol{R})$ has been obtained earlier in [2].

In a similar manner one considers the problem of the differentiability of the function $r \to \varphi(e^{irT}U)$, where U is a unitary operator while T is a bounded self-adjoint operator, and one has similar results for functions φ from $A(\boldsymbol{T})$ (i.e., such φ for which $\check{\varphi}$ admits the representation (2), in which $\alpha \in C\big(\boldsymbol{T}, L^2(\mathcal{X},\sigma)\big), \beta \in L^\infty(\boldsymbol{T}, L^2(\mathcal{X},\sigma))$.

Krein [45]has associated to each pair of unitary operators U, V, for which $V - U \in \mathfrak{S}_1$ a function ξ from $L^1(\boldsymbol{T})$ (the spectral shift function corresponding to the perturbation $U \to V$) such that

$$\operatorname{trace}\big(\varphi(U) - \varphi(V)\big) = \int_{\boldsymbol{T}} \xi(\zeta)\varphi'(\zeta)d\zeta \tag{9}$$

for each function φ whose derivative has an absolutely convergent Fourier series. In a similar manner, the concept of the spectral shift function is introduced also for nuclear perturbations of self-adjoint operators and one has a formula, similar to (9), for functions φ whose derivative is the Fourier transform of a finite measure [1].

It is proved in [48] that formula (9) holds for $\varphi \in \mathfrak{A}(\boldsymbol{T})(\varphi \in \mathfrak{A}(\boldsymbol{R}))$ in the case of self- adjoint operators).

In [56] one has obtained sufficient conditions on φ, for which $\varphi \in \mathfrak{A}(\boldsymbol{T})$. In particular, let $\varphi_{(n)} = \varphi - \sum_{m=-n}^{n} \check{\varphi}(m)z^m$. In this case, if

$$\sum_{n=1}^{\infty} \left\|\varphi_{(n)}\right\|_{L^\infty} < \infty, \tag{10}$$

then $\varphi \in \mathfrak{A}(\boldsymbol{T})$. Also there one finds sufficient conditions for functions on R.
We show that $\check{\varphi} \in \mathfrak{R}_T$ (and $\varphi \in \mathfrak{A}(\boldsymbol{T})$), if $\varphi \in \boldsymbol{B}^1_{\infty 1}$ (see the definition below, of the Besov classes). This condition is weaker than (10) and it is easier to handle.

For $\varphi \in L^\infty(\boldsymbol{T})$, the Hankel operator $H_\varphi: H^2 \to H^2_- \overset{\text{def}}{=} L^2 \ominus H^2$ is defined by the equality $H_\varphi f = \boldsymbol{P}_- \varphi f$ ($\boldsymbol{P}_-$ and $\boldsymbol{P}_+$ are orthoprojections onto H^2_- and H^2_+).

As we have already seen, the discussed problems lead to the investigation of the kernels $\check{\varphi}$. Assume now that $\boldsymbol{T}$ is an integral operator in $L^2(\boldsymbol{T})$ with kernel $\frac{\bar{z}_2}{2\pi i}\check{\varphi}(z_1, z_2)$. It is well known (and easy to verify in a straight forward manner) that $Tf = \boldsymbol{P}_-\varphi f_+ - \boldsymbol{P}_+\varphi f_-$ where $f_+ \stackrel{\text{def}}{=} \boldsymbol{P}_+ f$, $f_- \stackrel{\text{def}}{=} \boldsymbol{P}_- f$. Then $\boldsymbol{P}_-\varphi f_+ = H_\varphi f_+$, while the operator $f_- \to \boldsymbol{P}_+\varphi f_-$ from H_-^2 into H^2 is analogous to the Hankel operator (these operators will be also called Hankel operators). Consequently, T can be represented in the form of an orthogonal sum of two Hankel operators and, in particular, $T \in \mathfrak{S}_1$ if and only if $H_\varphi \in \mathfrak{S}_1$ and $H_{\overline{\varphi}} \in \mathfrak{S}_1$. In a similar manner, an operator in $L^2(\boldsymbol{R})$ with kernel $\check{\varphi}/2\pi i$ is the orthogonal sum of two Hankel operators on the Hardy classes H^2, H_-^2 in the upper and the lower half planes.

We need the following criterion, obtained in [59], for the nuclearity of Hankel operators:

$H_\varphi \in \mathfrak{S}_1$ if and only if $\boldsymbol{P}_-\varphi \in B_1^1$ (see the definition below).

Let $1 \le p \le \infty$, $1 \le q \le \infty$, $s > 0$. The Besov class $\boldsymbol{B}_{pq}^s$ of functions on $\boldsymbol{T}$ is defined by the equality

$$\boldsymbol{B}_{pq}^s = \left\{f : \frac{1}{|t|^s}\|\Delta_t^n f\|_{L^p(\boldsymbol{T})} \in L^p\left([-\pi, \pi], \frac{dt}{|t|}\right)\right\},$$

where $(\Delta_t f)(e^{ix}) = f\left(e^{i(x+t)}\right) - f(e^{ix})$, $\Delta_t^n = \Delta_t \Delta_t^{n-1}$, while n is the least integer greater than s , We also set $\boldsymbol{B}_p^s \stackrel{\text{def}}{=} \boldsymbol{B}_{pq}^s$.

These classes admit several other descriptions [61-63]. We need the following one. Let $W_n, n > 0$, be functions on $\boldsymbol{T}$, whose Fourier coefficients $\widetilde{W}_n(k)$ satisfy the conditions $\widetilde{W}_n(2^n) = 1, \widetilde{W}_n \equiv 0$, outside $(2^{n-1}, 2^n)$, and $\widetilde{W}_n$ is a linear function on $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$. For $n < 0$ we set $W_n = \widetilde{W}_{-n}$, $W_0(z) \stackrel{\text{def}}{=} 1 + z + \bar{z}$. Then

$$f \in \boldsymbol{B}_{pq}^s \Leftrightarrow \left\{2^{|n|s}\|f * W_n\|_{L^p}\right\}_{n\in\boldsymbol{Z}} \in \ell^q(\boldsymbol{Z}).$$

In a similar manner one defines the (homogeneous) Besov classes $\boldsymbol{B}_{pq}^s(\boldsymbol{R})$ of functions on $\boldsymbol{R}$.

we need a statement whose proof is similar to the proof of Lemma2.1, Chap. IV of [64]. Let F be a continuous function on $\boldsymbol{R}$ with a compact support. We consider the polynomials

$$F_n = \sum_{k\in\boldsymbol{Z}} F\left(\frac{k}{n}\right) z^k, \quad n > 0.$$

LEMMA(2.1.4)[44]. If the Fourier transform, FF of the function F is in L^1, then $\|F_n\|_{L^1(\boldsymbol{T})} <$ const, and for each positive number δ we have

$$\lim_n \left(\int_{1/2}^{-\delta} \left|F_n(e^{2\pi it})\right| dt + \int_\delta^{1/2} \left|F_n(e^{2\pi it})\right| dt\right) = 0.$$

Proof. We set $\psi(x) = F(x/n)e^{2\pi ixt}$. Now we make use of the Poisson summation formula for ψ [21, Chap. VII, No.2] and we obtain

$$\sum_{k\in\boldsymbol{Z}} F\left(\frac{k}{n}\right) e^{2\pi ixt} = \sum_{k\in\boldsymbol{Z}} n \cdot (\mathcal{F}F)\big(n(k-t)\big).$$

From here

$$\int_{-1/2}^{1/2}\left|F_n(e^{2\pi it})\right|dt \le \sum_{k\in\mathbf{Z}} n\int_{-1/2}^{1/2}|(\mathcal{F}F)(n(k-t))|\,dt = \|\mathcal{F}F\|_{L^1(\mathbf{R})},$$

$$\left(\int_{-\frac{1}{2}}^{-\delta}+\int_{0}^{\frac{1}{2}}\right)\ \left|F_n(e^{2\pi it})\right|dt \le \left(\int_{-\infty}^{-n\delta}\int_{n\delta}^{\infty}\right)|(\mathcal{F}F)(t)|dt \underset{n\to\infty}{\longrightarrow} 0.$$

We show that $\breve{\varphi}\in\Re_{\boldsymbol{T}}$ if $\in \boldsymbol{B}^1_{\infty 1}$. However, the converse statement is not true. In Theorem (2.1.7) we construct a class of functions not contained in $\boldsymbol{B}^1_{\infty 1}$ but all of its elements φ satisfy the condition $\breve{\varphi}\in\Re_{\boldsymbol{T}}$. This class, as well as the class $\boldsymbol{B}^1_{\infty 1}$ is contained in $A(\boldsymbol{T})$.

We have mentioned the sufficient condition (10), obtained in [47]. We note that (10) implies that $\varphi\in\boldsymbol{B}^1_{\infty 1}$. This follows from the known characterization of the class $\boldsymbol{B}^1_{\infty 1}$: $\varphi\in\boldsymbol{B}^1_{\infty 1}\Leftrightarrow\sum_{n\ge 1} dist_{L^\infty}(\varphi,\rho_n)<\infty$ where ρ_n is the set of polynomials of degree n. The converse statement does not hold.

THEOREM (2.1.5)[44]. If $\varphi\in\boldsymbol{B}^1_{\infty 1}$ then $\breve{\varphi}\in L^\infty\widehat{\otimes}L^\infty$ and $\varphi\in\Re_{\boldsymbol{T}}$.

Proof. We show that $\breve{\varphi}\in L^\infty\widehat{\otimes}L^\infty$. It is easy to see that the arguments given below also show that $\varphi\in\Re_{\boldsymbol{T}}$. We have

$$\breve{\varphi}(z_1,z_2)=\sum_{j,k\ge 0}\breve{\varphi}(j+k+1)z_1^j z_2^k+\sum_{j,k<0}\breve{\varphi}(j+k+1)z_1^j z_2^k.$$

We show, for the sake of definiteness, that the first function in the right-hand side is in $L^\infty\widehat{\otimes}L^\infty$. We represent it in the form of a sum of two functions

$$\sum_{j,k\ge 0}\alpha_{jk}\breve{\varphi}(j+k+1)z_1^j z_2^k+\sum_{j,k\ge 0}\beta_{jk}\breve{\varphi}(j+k+1)z_1^j z_2^k,$$

where $\alpha_{00}=1/2$, $\alpha_{jk}=\frac{2j-k}{j+k}$ for $j+k>0$,$k/2\le j\le 2k$, $\alpha_{jk}=0$ for $j\le k/2$,$k>0$; $\alpha_{jk}=1$ for $j\ge 2k, j>0$; $\beta_{jk}=1-\alpha_{jk}$.

Obviously, $\beta_{jk}=\alpha_{jk}$and, consequently, it is sufficient to consider the function

$$\sum_{j,k\ge 0}\alpha_{jk}\breve{\varphi}(j+k)z_1^j z_2^k.$$

We define the functions q,r on $\boldsymbol{R}$ by the equalities

$$q(x)=\begin{cases}0, & x\le 1/2,\\ (2x-1)/(x+1), & 1/2\le x\le 2,\\ 1, & x\ge 2,\end{cases}$$

$$r(x)=\begin{cases}0, & x\le 3/2,\\ (2x-3)/x, & 3/2\le x\le 3,\\ 1, & x\ge 3.\end{cases}$$

Now we set

$$Q_n(z)=\sum_{j\ge 0}q\left(\frac{j}{n}\right)z^j,\quad R_n(z)=\sum_{j\ge 0}r\left(\frac{j}{n}\right)z^j,\qquad n>0;$$

$$Q_n(z)=R_0(z)=\frac{1}{2}+\sum_{j\ge 1}z^j.$$

It is easy to see that

$$\sum_{j,k\ge 0}\alpha_{jk}\breve{\varphi}(j+k+1)z_1^j z_2^k=\sum_{k\ge 0}\left(((S^*)^k\psi*Q_k)(z_1)\right).z_2^k$$

where $\psi \stackrel{\text{def}}{=} \boldsymbol{P}_+\bar{z}_\varphi$, while $S^*\psi \stackrel{\text{def}}{=} \frac{\psi-\psi(0)}{z}$. Consequently,

$$\left\|\sum_{j,k>0} \alpha_{jk}\check{\varphi}(j+k+1)z_1^j z_2^k\right\|_{L^\infty} \leq \sum_{k\geq 0}\|(S^*)^k\psi * Q_k\|_{L^\infty} \leq \sum_{k\geq 0}\|\psi * R_k\|_{L^\infty}.$$

We show the following statement.

LEMMA(2.1.6)[44]. Let ψ be a function from $\boldsymbol{B}^1_{\infty 1}$ analytic in D. Then $\sum_{k\geq 0}\|\psi * Q_k\|_{L^\infty} < \infty$.

Proof: We define the function u on R by the equality $u(x) = 1 - r(|x|), x \in R$.
Since the function u is continuous and piecewise-smooth, we have $Fu \in L^1((\boldsymbol{R})$. We set

$$U_k(z) = \sum_{j\in \boldsymbol{Z}} u\left(\frac{j}{k}\right) z^j.$$

By virtue of Lemma (2.1.4), we have $\|U_k\|_{L^1(\boldsymbol{T})} \leq$ const. Let $k \geq 2^{n+2}$. Then

$$R_k * \psi = R_k * \left(\sum_{m\geq n} \psi * W_m\right),$$

(the kernels W_m). Consequently,

$$\|R_k * \psi\|_{L^\infty} = \left\|R_k * \left(\sum_{m\geq n} \psi * W_m\right)\right\|_{L^\infty} \leq \left\|\sum_{m\geq n} \psi * W_m\right\|_{L^\infty} (1 + \|U_k\|_{L^1})$$
$$\leq \text{const} \sum_{m\geq n} \|\psi * W_m\|_{L^\infty}.$$

From here

$$\sum_{k\geq 0}\|R_k * \psi\|_{L^\infty} = \sum_{n\geq 1}\sum_{k=2^{n+2}}^{2^{n+3}-1}\|R_k * \psi\|_{L^\infty} \leq \text{const}\sum_{n\geq 1} 2^{n+2}\sum_{k=2^{n+2}}^{2^{n+3}-1}\|\psi * W_m\|_{L^\infty}$$
$$\leq \sum_{n\geq 1} 2^n\|\psi * W_m\|_{L^\infty} < \infty.$$

We have proved that $\check{\varphi} \in C(\boldsymbol{T})\hat{\otimes}C(\boldsymbol{T})$.

COROLLARY(2.1.7)[44]. Let $\varphi \in \boldsymbol{B}^1_{\infty 1}$. Then the Hankel operator $\boldsymbol{P}_- H_\varphi \boldsymbol{P}_+$ is a nuclear operator from L^1 into $C(\boldsymbol{T})$.

Now we show that the condition $\varphi \in \boldsymbol{B}^1_{\infty 1}$ is not necessary in order that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$.

THEOREM(2.1.8)[44]. There exists a function φ on $\boldsymbol{T}$, not belonging to the space $\boldsymbol{B}^1_{\infty 1}$and such that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$ and $\varphi \in \mathfrak{A}(\boldsymbol{T})$.

Proof. Let $\varphi = z\psi$ and $\psi = \sum_{n\geq 1}\psi_n$ where ψ_n is a polynomial, whose Fourier coefficients are concentrated in $[2^{N_n}, 2^{N_n+1}]$, where $\{N_n\}$ is an increasing sequence of positive numbers. We select a sequence of "almost disjoint" polynomials $\{\psi_n\}$, such that $\psi \notin \boldsymbol{B}^1_{\infty 1}$, but $\varphi \in \mathfrak{R}_{\boldsymbol{T}}$. We have

$$\psi(z_1, z_2) = \sum_{n\geq 1}\sum_{j,k\geq 0}\left(\alpha_{jk}\hat{\psi}_n(j+k) + \beta_{jk}\hat{\psi}_n(j+k)\right)z_1^j z_2^k$$

(here one makes use of the notations introduced in the course of the proofs of Theorem(2.1.5) and Lemma (2.1.6). Clearly, the functions $\sum_{n\geq 1}\sum_{j,k\geq 0}\alpha_{jk}\hat{\psi}_n(j+k)z_1^j z_2^k$

and $\sum_{n\geq 1}\sum_{j,k\geq 0}\beta_{jk}\hat{\psi}_n(j+k)z_1^j z_2^k$ belong simultaneously to the class $\mathfrak{R}_{\boldsymbol{T}}$. Therefore, we shall consider only the first function.

As we have proved in Theorem2, we have

$$\sum_{j,k\geq 0}\alpha_{jk}\hat{\psi}(j+k)z_1^j z_2^k = \sum_{n>0}\sum_{k=0}^{2^{N_n+2}/3}\left(\left((S^*)^k\psi_n * Q_n\right)(z_1)\right)z_2^k$$

(if $k > 2^{N_n+2}/3$, then $(S^*)^k\psi_n * Q_n = 0$, since $\hat{\psi}_n \subset [2^{N_n}, 2^{N_n+1}]$).
Assume now that $\{\delta_n\}$ is a sequence of positive numbers such that $\sum_{n\geq 1}2^{N_n}\delta_n < \infty$. We have

$$\sum_{j,k\geq 0}\alpha_{jk}\hat{\psi}(j+k)z_1^j z_2^k = \sum_{n\geq 1}\sum_{k=0}^{2^{N_n+2}/3}\delta_n^{-1/2}\left(\left((S^*)^k\psi_n * Q_n\right)(z_1)\right)\cdot\delta_n^{1/2}z_2^k$$

Clearly,

$$\sup_{|\tau|=1}\sum_{n\geq 1}\sum_{k=0}^{2^{N_n+2}/3}\left|\delta_n^{1/2}\tau^k\right|^2 \leq \text{const}\sum_{n\geq 1}2^{N_n}\delta_n.$$

Consequently, in view of Theorem 1, $\check{\varphi}$ is in $\mathfrak{R}_{\boldsymbol{T}}$ as soon as one has the inequality

$$\sup_{|\zeta|=1}\sum_{n\geq 1}\sum_{k=0}^{2^{N_n+2}/3}\delta_n^{-1}\left|\left((S^*)^k\psi_n * Q_n\right)(\zeta)\right|^2 = \sup_{|\zeta|=1}\sum_{n\geq 1}\delta_n^{-1} < \infty. \tag{11}$$

Obviously,

$$|\psi_n * R_k| = |\psi_n|\,,\ k \leq 2^{N_n}/3 \tag{12}$$

Assume now that $\{\Gamma_n\}_{n\geq 1}$ is a sequence of disjoint arcs in $\boldsymbol{T}$, Δ_n, δ_n are arcs with the same centers, $|\Gamma_n| = 2|\Delta_n| = 4|\delta_n|$. We consider a continuous function ω_n on $\boldsymbol{T}$ such that $0 \leq \omega_n \leq 1$, $\operatorname{supp}\omega_n \subset \Delta_n$, $\omega_n|\delta_n = 1$. Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of complex numbers, $|\lambda_n| \leq 1$, and let $\{\varepsilon_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n\geq 1}2^{N_n}\varepsilon_n < \infty$.

Let h_n be a polynomial such that $\|\lambda_n\omega_n - h_n\| < \varepsilon_n$. Then there exists a sequence of integers $\{m_n\}_{n\geq 1}$ such that the Fourier coefficients of the polynomial $z^{m_n}h_n$ are concentrated in $[2^{N_n}, 2^{N_n+1}]$ for some sequence $\{N_n\}_{n\geq 1}$ such that $N_{n+1} - N_n \geq 2$. We set $\psi_n = z^{m_n}h_n$. The polynomials ψ_n are "almost disjoint" on T. From the definition of the class $\boldsymbol{B}_{\infty 1}^1$ in terms of convolutions with W_n, it is easy to see that under the condition $N_{n+1} - N_n \geq 2$ we have

$$\psi \in \boldsymbol{B}_{\infty 1}^1 \Leftrightarrow \sum_{n\geq 1}2^{N_n}\|\psi_n\|_{L^\infty} < \infty. \tag{13}$$

We show that one can select $\{\lambda_n\}_{n\geq 1}$, $\{\delta_n\}_{n\geq 1}$ so that the inequality (11) holds and the series in the right-hand side of (13) diverges.

In view of (12), for $k \leq 2^{N_n}/3$ we have

$$\left|(\psi_n * R_k)(\zeta)\right| \geq |\lambda_n| - \varepsilon_n\,, \qquad \zeta \in \delta_n,$$
$$\left|(\psi_n * R_k)(\zeta)\right| \leq \varepsilon_n\,,\ \zeta \in \boldsymbol{T}/\Delta_n. \tag{14}$$

From Lemma(2.1.4) it follows that if the sequence $\{N_n\}_{n\geq 1}$ is sufficiently sparse, then for $k \leq 2^{N_n}/3$ we have

$$\int_{\{\zeta:|\zeta|>|\Gamma_n|/4\}} |U_k(\zeta)|\, dm(\zeta) < \varepsilon_n.$$

Consequently, for $k \le 2^{N_n}/3$ we have

$$|(\psi_n * R_k)(\zeta)| = |(\psi_n - \psi_n * U_k)(\zeta)| \le \text{const}\ \varepsilon_n\ , \zeta \in \boldsymbol{T}/\Gamma_n \qquad (15)$$

Clearly, for all k we have

$$\|\psi_n * R_k\|_{L^\infty} \le \text{const}\ (|\lambda_n| + \varepsilon_n). \qquad (16)$$

Consequently, the series in (13) converges if and only if $\sum_{n\ge1} 2^{N_n}|\lambda_n| < \infty$, while, in view of (14), (15), (16), condition (10) is equivalent to the fact that

$$\sup_n \delta_n^{-1}\, 2^{N_n}|\lambda_n|^2 < \infty.$$

We set now $\delta_n = |\lambda_n|^2 \cdot 2^{N_n}$ then $\sum_{n\ge1} 2^{N_n}\,\delta_n < \infty$ if and only if $\sum_{n\ge1}(2^{N_n}|\lambda_n|)^2 < \infty$. It remains to choose the sequence $\{\lambda_n\}_{n\ge0}$ so that $\{2^{N_n}\lambda_n\} \in l^2/l^1$.

The given arguments allow us to conclude that the constructed function belongs to $\mathfrak{A}(\boldsymbol{T})$.

Making use of the nuclearity criterion for Hankel operators, we find necessary conditions on φ in order that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$. They show that the condition $\varphi \in \boldsymbol{B}_1^1$ is not sufficient.

THEOREM(2.1.9)[44]. Let φ be a function on T such that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$. Then $\varphi \in \boldsymbol{B}_1^1$.

Proof. Let $g_1, g_2 \in L^2$. Then the function $g_1(z_1) \times g_2(z_2)\check{\varphi}(z_1, z_2)$ is the kernel of a nuclear operator in L^2. We set $g_1 \equiv 1, g_2 = z$. The operator T with the kernel $1/2\pi i(\varphi(z_1) - \varphi(z_2))/(1 - z_1\bar{z}_2)$ acts in L^2 according to the formula $Tf = \boldsymbol{P}_-\varphi f_+ - \boldsymbol{P}_+\varphi f_-$. Consequently,
$H_\varphi \in \mathfrak{S}_1$, $H_{\overline{\varphi}} \in \mathfrak{S}_1$, from where we obtain $\varphi \in \boldsymbol{B}_1^1$.

We note that condition $\varphi \in C^1$ does not imply $\varphi \in \boldsymbol{B}_1^1$. Indeed, the definition of the class $\boldsymbol{B}_1^1$in terms $W_n$implies that for $\varphi \in \boldsymbol{B}_1^1$, we have $\sum_{n\ge0}\left|\hat{\psi}(2^n)\right| < \infty$, where $\psi = \varphi'$. However, for any sequence $\{\lambda_n\}_{n\ge0} \in l^2$ there exists a function $\psi \in C(\boldsymbol{T})$ such that $\hat{\psi}(2^n) = \lambda_n\ , n \ge 0$ (see [66]).

Now we obtain a stronger necessary condition.

Definition(2.1.10)[44]. We say that the function φ on $\boldsymbol{T}$ belongs to the class L if the Hankel operators $H_\varphi, H_{\overline{\varphi}}$ map H^1 into $\boldsymbol{B}_1^1$.

THEOREM(2.1.11)[44]. Let φ be a function on T such that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$. Then $\varphi \in L$.

Proof. We show that $H_{\overline{\varphi}}$ is a bounded operator from H^1 into $\boldsymbol{B}_1^1$. The assertion for H_φ is proved in the same way.

We shall consider nuclear operators of a special type with kernel $K(z_1, z_2)$ and we shall estimate the nuclear norms of the operators with kernel $z_2\check{\varphi}(z_1, z_2) \times K(z_1, z_2)$. We note that

$$z_2\check{\varphi}(z_1, z_2) = \frac{\varphi(z_1) - \varphi(z_2)}{1 - z_1\bar{z}_2} = \sum_{k\ge1, j\ge0} \check{\varphi}(j + k) z_1^j z_2^k + \sum_{k\ge0, j\le-1} \check{\varphi}(j + k) z_1^j z_2^k.$$

Let T be a nuclear operator in L^2 such that $TL^2 \subset zH_-^2$ and $T|H_-^2 = 0$. Then the kernel K of the operator T has the form

$$K(z_1, z_2) = \sum_{r,s\ge0} t_{rs} z_1^{-r} z_2^{-s}.$$

We consider the operator $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_-$, where $\boldsymbol{T}^\varphi$ is the operator with kernel $z_2\check{\varphi}(z_1,z_2)\times K(z_1,z_2)$. Then $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_- \in \mathfrak{S}_1$.

Now we show that $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_-$ is a Hankel operator. Clearly, if $K(z_1,z_2) = z_1^{-r}z_2^{-s}$ then the operator $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_-$ has the kernel

$\sum_{k\geq 1, j\geq 0}\check{\varphi}(j+k+r+s)z_1^j z_2^k$, Consequently, in the general case the operator $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_-$has the kernel

$$\sum_{k\geq 1, j\geq 0}\left(\sum_{m\geq 0}\hat{g}(m)\check{\varphi}(j+k+m)\right)z_1^j z_2^k,$$

where $g = \sum_{m\geq 0} c_m z^m$, while $c_m = \sum_{r=0}^{m} t_{r,m-r}$.

Thus, $\boldsymbol{P}_+\boldsymbol{T}^\varphi\boldsymbol{P}_-$ is a nuclear Hankel operator, whence

$$\sum_{n>0}\sum_{m\geq 0}\check{\varphi}(n+m)\,\hat{g}(m)z^n \in \boldsymbol{B}_1^1 \qquad (17)$$

It is well known (see. for example. [59]) that when T runs through all the nuclear operators of the indicated form, then g runs through all the functions from H^1. Thus, (17) means that $H_{\overline{\varphi}}\colon H^1 \to \boldsymbol{B}_1^1$.

We consider the space J of functions h analytic in D and such that

$$\lim_{r\to 1}\frac{\|h(rz)\|_{L^\infty}}{1-r} = 0, \qquad \|h\|_J \overset{\text{def}}{=} \sup_{|\zeta|<1}\frac{|h(\zeta)|}{1-|\zeta|}.$$

It is well known [61,63] that $J^* = \boldsymbol{P}_+\boldsymbol{B}_1^1$ relative to the duality

$$(h,g) = \lim_{r\to 1}\int_{\boldsymbol{T}} h(r\zeta)g(r\bar{\zeta})dm(\zeta). \qquad (18)$$

We consider the space N of functions f analytic in D admitting the representation

$$f = \sum_{N\geq 0} g_N h_N, \qquad (19)$$

$$\sum_{N\geq 0}\|g_N\|_{H^1}\|h_N\|_J < \infty. \qquad (20)$$

The norm of the function f is defined as the infimum in (20) over all $\{g_N\}_{N\geq 0}, \{h_N\}_{N\geq 0}$ satisfying condition (19).

Since $J^* = \boldsymbol{P}_+\boldsymbol{B}_1^1$, it is easy to show that $H_{\overline{\varphi}}$, maps H^1 into $\boldsymbol{B}_1^1$ if and only if $\boldsymbol{P}_+\varphi \in N^*$(relative to the duality (18), i.e., $N^* = L$.

Setting in (19) $g_0 = 1$,$g_N = h_N \equiv 0$ for $N \geq 1$, we can see that $J \subset N$ and, therefore $L \subset \boldsymbol{B}_1^1$; thus, Theorem(2.1.11) refines Theorem(2.1.9).

Clearly, each function f from N satisfies the condition

$$\lim_{r\to}\frac{1}{1-r}\int_{\boldsymbol{T}} |f(r\zeta)|dm(\zeta) = 0. \qquad (21)$$

Let D be a space of functions satisfying condition (21). It is well known that $D^* = \boldsymbol{P}_+\boldsymbol{B}_{\infty 1}^1$ [61,63]. If each function from D would admit a representation (19) (20), we would have $L = \boldsymbol{B}_{\infty 1}^1$. Unfortunately, this is not so since this would contradict Theorem 3.

Moreover, the condition $\varphi \in L$ is not sufficient in order that $\check{\varphi} \in \mathfrak{R}_{\boldsymbol{T}}$, since $L \not\subset \{\varphi : \varphi' \in L^\infty\}$. Indeed, if $L \subset \{\varphi : \varphi' \in L^\infty\}$, then the system of functions $\{\boldsymbol{P}_+ K_m\}_{m\geq 1}$ ($K_m$is the Fejér kernel) would have bounded norms in N. Now, if $g \in H^1$, then the maximal function g_M,

$$g_M(\zeta) \stackrel{\text{def}}{=} \sup_{r<1} |g(r\zeta)| , \qquad \zeta \in \boldsymbol{T},$$

is in $L^1(\boldsymbol{T})$ and $\|g_M\|_{L^1} \leq \text{const}\, \|g\|_{H^1}$ (see [67, Chap. VIII]). Consequently, if $f \in N$, then the maximal function f^M,

$$f^M(\zeta) \stackrel{\text{def}}{=} \sup_{r<1} \frac{|f(r\zeta)|}{1-r}, \quad \zeta \in \boldsymbol{T},$$

is in $L^1(\boldsymbol{T})$ and $\|f^M\|_{L^1} \leq \text{const}\, \|f\|_N$. Thus, we would have $\|(\boldsymbol{P}_+ K_m)^M\|_{L^1} \leq$ const, from where $\|(\sum_{k\geq 0} z^k)^M\|_{L^1} < \infty$; however, it is easy to see directly that this is not so.

THEOREM(2.1.12)[44]. Let $\varphi \in \boldsymbol{B}^1_{\infty 1}$. The following statements hold:

(i) If U, V are unitary operators, then

$$\|\varphi(V) - \varphi(U)\| \leq \text{const}\, \|\varphi\|_{\boldsymbol{B}^1_{\infty 1}} \cdot \|V - U\|.$$

(ii) If U is a unitary operator, then the function $A \to \varphi(e^{iA}U)$ on the class of bounded self-adjoint operators is Fréchet differentiable and

$$\lim_{t\to 0} \frac{\varphi(e^{itA}U) - \varphi(U)}{t} = -i \int_{\boldsymbol{T}} \int_{\boldsymbol{T}} \frac{\varphi(z_1) - \varphi(z_1)}{1 - \bar{z}_2 z_1} dE(z_1) A dE(z_2),$$

where E is the spectral measure of the operator U.

(iii) If U, V are unitary operators, $V - U \in \mathfrak{S}_1$, then $\varphi(V) - \varphi(U) \in \mathfrak{S}_1$ and formula (9)holds.

(iv) If S is a symmetric normed ideal, interpolational between $\mathfrak{S}_1$ and $\mathfrak{S}_\infty$ (and, in particular, if the ideal S is separable or conjugate to a separable ideal), U, V are unitary operators, $V - U \in \mathfrak{S}$, then $\varphi(V) - \varphi(U) \in \mathfrak{S}$.

As already mentioned, the assertions of Theorem 6 hold for $\varphi \in \mathfrak{A}(\boldsymbol{T})$
[47, 4]. Now the result follows from Theorem (2.1.5).

From Theorem (2.1.7) there follows that the condition $\varphi \in \boldsymbol{B}^1_{\infty 1}$ is not necessary for the validity of the statements (i).

We show that the condition $\varphi \in C^1$ is not sufficient in order that the statements (i) should hold.

THEOREM(2.1.13)[44]. Let $\in C^1(\boldsymbol{T})/L$. The following statements hold:

(i) There exist unitary operators $U, \{V_n\}_{n\geq 1}$ such that

$$\lim_n \|V_n - U\| = 0 , \qquad \lim_n \|\varphi(V_n) - \varphi(U)\| \cdot (\|V_n - U\|)^{-1} = \infty.$$

(ii) There exist unitary operators W, V such that $W - V \in \mathfrak{S}_1$, but $\varphi(W) - \varphi(V) \notin \mathfrak{S}_1$.

We note that φ satisfies the conditions of the theorem if $\varphi \in C^1(\boldsymbol{T})/\boldsymbol{B}^1_1$.

Proof. Let U be the operator of multiplication by z in L^2, $V = e^{itA}$, where , $A \in \mathfrak{S}_2$, $A = A^*$, $t \in \boldsymbol{R}$. We assume now that $\|\varphi(V) - \varphi(U)\| \leq \text{const}\, \|V - U\|$. Then

$$\frac{1}{t}\big(\varphi(V) - \varphi(U)\big) = \int_{\boldsymbol{T}} \int_{\boldsymbol{T}} \check{\varphi}(z_1, z_2) dE_V(z_2) \frac{V - U}{t} dE_U(z_1), \qquad (22)$$

where E_U, E_V are the spectral measures of the operators U, V and the operators in (22) have bounded norms. It is easy to show that the operators in the right-hand side of the equality (22) converge weakly for $t \to 0$ to

$$-i \int_{\boldsymbol{T}} \int_{\boldsymbol{T}} \frac{\varphi(z_1) - \varphi(z_1)}{1 - \bar{z}_2 z_1} dE_U(z_1) A dE_V(z_2),$$

Consequently, the transformation

$$A \to \int_{\boldsymbol{T}} \int_{\boldsymbol{T}} \frac{\varphi(z_1) - \varphi(z_1)}{1 - \bar{z}_2 z_1} dE_U(z_1) A dE_V(z_2), \qquad A \in \mathfrak{S}_2,$$

is bounded in the operator norm and, therefore, in view of Theorem (2.1.1), the function $\check{\varphi}(z_1, z_2) K(z_1, z_2)$ is the kernel of a nuclear operator if K has this property. At the proof of Theorem (2.1.11) we have established that this implies $\varphi \in L$, Contradiction.

(ii) in the same way as for the proof of statement (i)), we can find a sequence $\{V_n\}_{n \geq 1}$ of unitary operators such that $\|V_n - U\|_{\mathfrak{S}_1}$, while $\lim\limits_{n} \|\varphi(V_n) - \varphi(U)\|_{\mathfrak{S}_1} / \|V_n - U\|_{\mathfrak{S}_1} = \infty$.

Now it is sufficient to consider the appropriate direct sums $W = \sum_{n=1}^{\infty} \oplus U$ and $V = \sum_{n=1}^{\infty} \oplus V_n$ such that $\sum_{n \geq 1} \|V_n - U\|_{\mathfrak{S}_1} < \infty$, but $\sum_{n=1}^{\infty} \|\varphi(V_n) - \varphi(U)\|_{\mathfrak{S}_1} = \infty$. Then it is clear that $W - V \in \mathfrak{S}_1$ while $\varphi(W) - \varphi(V) \notin \mathfrak{S}_1$.

We investigate the properties of the smoothness of functions of self-adjoint operators in the case of bounded operators. At the conclusion, we shall dwell briefly on unbounded operators.

As mentioned, the considered problems are closely related with the question of the characterization of functions φ on $\boldsymbol{R}$ for which $\check{\varphi} \in \mathfrak{R}_I$ roll for each finite interval I.

By the symbol $\mathfrak{B}_{pq}^s$ we denote the class of functions f on $\boldsymbol{R}$ such that for each finite interval I the function $f|I$ can be extended to a function of the class $B_{pq}^s(\boldsymbol{R})$.

THEOREM (2.1.14)[44]. (i) If $\check{\varphi} \in \mathfrak{B}_{\infty q}^s$, then $\check{\varphi} \in \mathfrak{R}_I$ for each finite interval I .

(ii) If $\check{\varphi} \in \mathfrak{R}_I$ for each finite interval I, then $\varphi \in \mathfrak{B}_1^1$.

Proof. Let I be a finite interval and let J_1, J_2 be intervals with the same center, $I \subsetneqq J_1 \subsetneqq J_2$ Multiplying the function φ by a smooth function with support in J_1 identically equal to unity on I, we can assume that assume $\varphi \subset J_1$. Let $\delta = |J_2|$. We set $\psi(e^{2\pi i x/\delta}) = \varphi(x), x \in J_2$. Clearly, $\psi \in B_{\infty 1}^1$ and, by Theorem (2.1.5), $\check{\psi} \in \mathfrak{R}_{\boldsymbol{T}}$.

Let K be the kernel of a nuclear operator from $L^2(I, \mu_1)$ into $L^2(I, \mu_2)$. In this case, if $\widetilde{K}(e^{2\pi i x_1/\delta}, e^{2\pi i x_2/\delta}) = K(x_1, x_2)$, $x_1, x_2 \in J_2$ then $\widetilde{K}$ is the kernel of a nuclear operator from $L^2(\boldsymbol{T}, \tilde{\mu}_1)$ into $L^2(\boldsymbol{T}, \tilde{\mu}_2)$, where $\tilde{\mu}_j$ the image of the measure μ_j under the mapping $x \to e^{2\pi i x/\delta}$, $j = 1,2$. Since $\check{\psi} \in \mathfrak{R}_{\boldsymbol{T}}$, the function $\check{\psi}(z_1, z_2) \widetilde{K}(z_1, z_2)$ is the kernel of a nuclear operator from $L^2(\boldsymbol{T}, \tilde{\mu}_1)$ into $L^2(\boldsymbol{T}, \tilde{\mu}_2)$. From here there follows that the function

$$\frac{\varphi(x_1) - \varphi(x_2)}{e^{2\pi i x_1/\delta} - e^{2\pi i x_2/\delta}} K(x_1, x_2)$$

is the kernel of a nuclear operator from $L^2(I, \mu_1)$ into $L^2(I, \mu_2)$, and, since $\varphi \subset J_1$ it follows that the kernel $\check{\varphi}(x_1, x_2)\, K(x_1, x_2)$ has the same property.

(ii) First we note that if, $\check{\varphi}, \check{\psi} \in \mathfrak{R}_{\boldsymbol{I}}$ then $(\varphi\check{\varphi}) \in \mathfrak{R}_{\boldsymbol{I}}$. Indeed, this follows from the equality $(\varphi\psi)(x_1, x_2) = \varphi(x_1)\check{\psi}(x_1, x_2) + \psi(x_2)\check{\varphi}(x_1, x_2)$. Now we can multiply our function by a smooth function with a compact support, identically equal to unity on I. Afterwards, we can repeat the, scheme of the proof of statement (i).

We mention that one can also formulate an analog of Theorem (2.1.11).

One can formulate and prove analogs of Theorems(2.12) and (2.1.13) for the case of bounded self-adjoint operators.

In the case of unbounded operators there arises the question of the characterization of those functions φ for which $\breve{\varphi} \in \mathfrak{R}_{\boldsymbol{R}}$. class L. For these classes one has the analogs of Theorems (2.112), (2.113) in the case of unbounded self-adjoint operators.

Section (2 . 2): Koplienko - Neidhardt Trace Formulae

The spectral shift function for a trace class perturbation of a self-adjoint (unitary) operator plays a very important role in perturbation theory. It was introduced in a special case by Lifshitz [69]and in the general case by Krein [70]. He showed that for a pair of self-adjoint (not necessarily bounded) operators A and B satisfying $B - A \in \boldsymbol{S}_2$ there exists a unique function $\xi \in L^1(\mathbb{R})$ such that

$$\operatorname{trace}\left(\varphi(B) - \varphi(A)\right) = \int_{\mathbb{R}} \varphi'(x)\xi(x)dx \qquad (23)$$

whenever c is a function on $\mathbb{R}$ with Fourier transform of φ'in $L^1(\mathbb{R})$. The function ξ is called the spectral shift function corresponding to the pair (A, B). We use the notation $\boldsymbol{S}_1$for the class of nuclear operators (trace class) on Hilbert space.

A similar result was obtained in [71] for pairs of unitary operators (U, V) with $V - U \in \boldsymbol{S}_1$. For each such pair there exists a function ξ on the unit circle $\mathbb{T}$of class $L^1(\mathbb{T})$ such that

$$\operatorname{trace}\left(\varphi(V) - \varphi(U)\right) = \int_{\mathbb{T}} \varphi'(\xi)(\xi)(\xi)d\boldsymbol{m}(\xi) \qquad (24)$$

whenever φ'has absolutely convergent Fourier series. Such a function ξ is unique modulo a constant and it is called a spectral shift function corresponding to the pair (U, V). We refer to Krein [72], in which the above results were discussed in detail (see [73]).

Spectral shift function plays an important role in perturbation theory. We mention here [74], in which the following important formula was found:

$$\det S(x) = e^{-2\pi i \xi(x)},$$

where S is the scattering matrix corresponding to the pair (A, B). I would also like to mention the monograph [75] and more recent papers on the Lifshitz-Krein spectral function: [76,77,78,79,80,81].

It was shown later in [82] that formulae (23) and (24) hold under less restrictive assumptions on φ.

Note that the right-hand sides of (23) and (24) make sense for an arbitrary Lipschitz function φ. However, it turns out that the condition $\varphi \in \mathrm{Lip}$ (i.e., φ is a Lipschitz function) does not imply that $\varphi(B) - \varphi(A)$ (or $\varphi(V) - \varphi(U)$) belong to $\boldsymbol{S}_1$. This is not even true for bounded A and B and continuously differentiable φ. The first such examples were given in [83].

In [84,85] with the help of the nuclearity criterion for Hankel operators (see recent monograph [86]) necessary conditions (in terms of Besov classes and Carleson measures) were found on φ for the operator $\varphi(B)-\varphi(A)$ (or $\varphi(V)-\varphi(U)$) to belong to $\boldsymbol{S}_1$. Those necessary conditions also imply that the condition $\varphi \in C^1$ is not sufficient for those operators to be in $\boldsymbol{S}_1$ (even for bounded A and B).

It is shown in [84] that if $\varphi$is a function on $\mathbb{T}$of Besov class $\boldsymbol{B}^1_{\infty 1}$, then trace formula (24) holds. Similarly, it was shown in [85] that if φ is a function on $\mathbb{R}$ of Besov class $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$, then trace formula (23) holds. The definition of the above Besov classes will be given. Note that though these sufficient conditions are not necessary, the gap between the necessary conditions and the sufficient conditions obtained in [84,85] is rather narrow. Note also that in [87] a better sufficient condition was found; however, it seems to me that the condition $\varphi \in \boldsymbol{B}^1_{\infty 1}(\mathbb{R})$ is easier to work with.

In Koplienko's paper [88] the author considered the case of perturbations of Hilbert-Schmidt class $\boldsymbol{S_2}$. Let A and B be self-adjoint operators such that $K \overset{\text{def}}{=} B = -A \in \boldsymbol{S_2}$. In this case the operator $\varphi(B)-\varphi(A)$ does not have to be in $\boldsymbol{S_1}$ even for very nice functions φ. The idea of Koplienko was to consider the operator

$$\varphi(B)-\varphi(A)-\frac{d}{ds}(\varphi(A+sK))|_{s=0}$$

and find a trace formula under certain assumptions on φ. It was shown in [88] that there exists a unique function $\eta \in L^1(\mathbb{R})$ such that

$$\text{trace}\left(\varphi(B)-\varphi(A)-\frac{d}{ds}(\varphi(A+sK))|_{s=0}\right)=\int_{\mathbb{R}}\varphi''(x)\eta(x)dx \qquad (25)$$

for rational functions $\varphi$with poles off $\mathbb{R}$. The function is called the generalized spectral shift function corresponding to the pair (A,B).

A similar problem for unitary operators was considered by Neidhardt [N]. Let U and V be unitary operators such that $V-U \in \boldsymbol{S_2}$. Then $V=\exp(iA)U$, where A is a self-adjoint operator in $\boldsymbol{S_2}$. Put $U_s=e^{isA}U$, $s \in \mathbb{R}$. It was shown in [N] that there exists a function $\eta \in L^1(\mathbb{R})$ such that

$$\text{trace}\left(\varphi(V)-\varphi(U)-\frac{d}{ds}(\varphi(U_s))|_{s=0}\right)=\int_{\mathbb{T}}\varphi''\eta d\boldsymbol{m} \qquad (26)$$

whenever φ''has absolutely convergent Fourier series. Such a function η is unique modulo a constant and it is called a generalized spectral shift function corresponding to the pair (U,V).

In [89,90] for applications of Koplienko's trace formula [88]. We obtain better sufficient conditions on functions φ, under which trace formulae (25) and (26) hold. We consider the case of unitary operators and the case of self-adjoint operators . We show that formula (25) holds under the assumption that φ belongs to the Besov class $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ while trace formula (26) holds whenever $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$. Note however, that the case of self-adjoint operators is considerably more complicated. First of all, unlike in the case of functions on $\mathbb{T}$the set of rational functions with poles off $\mathbb{R}$is not dense in $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$.

Second, functions in $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ do not have to be Lipschitz and it is not clear how to interpret each of the operators

$$\varphi(B) - \varphi(A) \quad \text{and} \quad \frac{d}{ds}(\varphi(A + sK))|_{s=0}.$$

However, it is still possible to define their difference and show that the difference belongs to $\boldsymbol{S}_1$.

We outline the theory of double operator integrals developed by Birman and Solomyak in [91,92,93], and we define Besov classes and discuss their properties.

We collect necessary information on double operator integrals and Besov classes.

The technique of double operator integrals developed by Birman and Solomyak [91,92,93] plays an important role in perturbation theory.

Let $(\mathcal{X}, E)$ and $(\mathcal{Y}, F)$ be spaces with spectral measures E and F on a Hilbert space $\mathcal{H}$. Double operator integrals are objects of the form

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \psi(x, y) dE(x) T dF(y), \tag{27}$$

where T is an operator on $\mathcal{H}$. Certainly, one has to specify how to understand the expression in (27). Let us first define double operator integrals for bounded functions ψ and operators T of Hilbert Schmidt class $\boldsymbol{S}_2$. Consider the spectral measure $\mathcal{E}$ whose values are orthogonal projections on the Hilbert space $\boldsymbol{S}_2$, which is defined by

$$\mathcal{E}(\Delta \times \Lambda)T = E(\Delta)TF(\Lambda), \qquad T \in \boldsymbol{S}_2,$$

for $\Delta$and Λ being measurable subsets of $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathcal{E}$ extends to a spectral measure on $\mathcal{X} \times \mathcal{Y}$ and if ψ is a bounded measurable function on $\mathcal{X} \times \mathcal{Y}$, by definition

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \psi(x, y) dE(x) T dF(y) = \left(\int_{\mathcal{X} \times \mathcal{Y}} \psi \, d\mathcal{E} \right) T.$$

Clearly,

$$\left\| \int_{\mathcal{X}} \int_{\mathcal{Y}} \psi(x, y) dE(x) T dF(y) \right\|_{\boldsymbol{S}_2} \leq \|\psi\|_{L^\infty} \|T\|_{\boldsymbol{S}_2}.$$

If

$$\int_{\mathcal{X}} \int_{\mathcal{Y}} \psi(x, y) dE(x) T dF(y) \in \boldsymbol{S}_1$$

for every $T \in \boldsymbol{S}_1$, we say that is a Schur multiplier of $\boldsymbol{S}_1$. In this case by duality the map

$$T \to \int_{\mathcal{X}} \int_{\mathcal{Y}} \psi(x, y) dE(x) T dF(y)$$

extends to a bounded transformer on the space of bounded linear operators on $\mathcal{H}$. Suppose now that A is a self-adjoint operator on a Hilbert space $\mathcal{H}$and $B = A + K$, where K is a self-adjoint operator of class $\boldsymbol{S}_2$, and let φ be a Lipschitz function on $\mathbb{R}$, then $\varphi(B) - \varphi(A) \in \boldsymbol{S}_2$ and

$$\varphi(B) - \varphi(A) = \iint_{\mathbb{R} \times \mathbb{R}} \frac{\varphi(x) - \varphi(y)}{x - y} dE_B(x) K dE_A(y), \tag{28}$$

where $E_A$and E_B are spectral measure of A and B and

$$\|\varphi(B) - \varphi(A)\|_{\boldsymbol{S}_2} \leq \sup_{x \neq y} \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \cdot \|K\|_{\boldsymbol{S}_2}. \tag{29}$$

Here we can define the function $(\varphi(x)-\varphi(y))(x-y)^{-1}$ on the diagonal $\{(x,y): x\in\mathbb{R}\}$ in an arbitrary way.

A similar formula holds for unitary operators U and V with $-U\in \boldsymbol{S}_2$:

$$\varphi(U)-\varphi(V) = \iint_{\mathbb{T}\times\mathbb{T}} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} dE_V(\zeta)(V-U)dE_B(\tau), \qquad (30)$$

where φ is a Lipschitz function on $\mathbb{T}$. Again, the right-hand side of this formula does not depend on the values of the function $(\varphi(\zeta)-\varphi(\tau))(\zeta-\tau)^{-1}$ on the diagonal $\{(\zeta,y): \zeta\in\mathbb{T}\}$. We refer the reader to [91,92,93] for more detailed information on double operator integrals. We also mention recent survey article [94].

It follows from the results of [83, 84 , 85] mentioned in the introduction, that the conditions $\varphi\in C^1$and $\varphi'\in L^\infty$ do not imply that the above double operator integrals determine bounded linear operators on $\boldsymbol{S}_1$. On the other hand, it follows from the results of [84, 85], that for functions φin the Besov class $\boldsymbol{B}^1_{\infty 1}$ on the circle and for functions φ in the Besov class $\boldsymbol{B}^1_{\infty 1}$ $(\mathbb{R})$ on $\mathbb{R}$ the following estimates hold:

$$\left\| \iint_{\mathbb{T}\times\mathbb{T}} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} dE_V(\zeta)(V-U)dE_B(\tau)\right\|_{\boldsymbol{S}_1} \le \text{const}\ \|\varphi\|_{\boldsymbol{B}^1_{\infty 1}} \|V-U\|_{\boldsymbol{S}_1} \qquad (31)$$

and

$$\left\| \iint_{\mathbb{R}\times\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{x-y} dE_B(\zeta)(x)dE_A(y)\right\|_{\boldsymbol{S}_1} \le \text{const}\ \|\varphi\|_{\boldsymbol{B}^1_{\infty 1}(\mathbb{R})} \|K\|_{\boldsymbol{S}_1}$$

In their papers [91,92,93] Birman and Solomyak studied the problem of the differentiability of the map $t\to\varphi(A+sK)$ in the operator norm and obtained sufficient conditions (a similar problems was also studied there in the case of functions of unitary operators). Later their results were improved in [84,85].

We need only differentiability results in the norm of $\boldsymbol{S}_2$. Let φ be a function in $C^1(\mathbb{R})$ such that $\varphi'\in L^\infty$. Suppose that A is a self-adjoint operator (not necessarily bounded) and K is a self-adjoint operator of class $\boldsymbol{S}_2$. Then

$$\frac{d}{ds}(\varphi(A+sK))|_{s=0} = \iint_{\mathbb{R}\times\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{x-y} dE_A(x)dE_A(y) \qquad (32)$$

(the derivative exists in the $\boldsymbol{S}_2$ norm). This follows from formula (28) and Proposition 3.2 of [95].

A similar result holds for functions of unitary operators. Let $\varphi\in C^1(\mathbb{T})$. Suppose that U is a unitary operator, A is a self-adjoint operator of class $\boldsymbol{S}_2$. Then

$$\frac{d}{ds}\varphi(e^{isA}U)\Big|_{s=0} = i\iint_{\mathbb{T}\times\mathbb{T}} \tau\frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} dE_U(\zeta)A dE_U(\tau) \qquad (33)$$

The proof of this formula is much simpler than in the case of possibly unbounded self-adjoint operators.

Let $0<p,q\le\infty$ and $s\in\mathbb{R}$. The Besov class $\boldsymbol{B}^s_{pq}$ of functions (or distributions) on $\mathbb{T}$ can be defined in the following way. Let w be a C^∞ function on $\mathbb{R}$ such that

$$w\ge 0\ ,\text{supp}\ w\subset\left[\frac{1}{2},2\right]\quad,\ \text{and}\ \sum_{n=-\infty}^{\infty} w(2^n x) = 1\ \ for\ x>0. \qquad (34)$$

Consider the trigonometric polynomials W_n , and $W_n^\#$ defined by

$$W_n(z) = \sum_{k\in\mathbb{Z}} w\left(\frac{k}{2^n}\right) z^k, \qquad n \ge 1, \quad W_0(z) = \bar{z} + 1 + z, \text{ and}$$

$$W_n^{\#}(z) = \overline{W_n(z)}, \quad n \ge 0.$$

Then for each distribution φ on $\mathbb{T}$

$$\varphi = \sum_{n\ge0} \varphi * W_n + \sum_{n\ge0} \varphi * W_n^{\#}.$$

The Besov class $\boldsymbol{B}_{pq}^s$ consists of functions (in the case $s > 0$) or distributions φ on $\mathbb{T}$ such that

$$\{\|2^{ns}\varphi * W_n\|_{L^p}\}_{n\ge0} \in \ell^q \quad \text{and} \quad \{\|2^{ns}\varphi * W_n^{\#}\|_{L^p}\}_{n\ge1} \in \ell^q$$

Besov classes admit many other descriptions, in particular, for $s > 0$ the space $\boldsymbol{B}_{pq}^s$ and be described in terms of moduli of continuity (or moduli of smoothness).

To define (homogeneous) Besov classes $\boldsymbol{B}_{pq}^s(\mathbb{R})$ on the real line, we consider the same function w as in (34) and define the functions W_n and $W_n^{\#}$ on $\mathbb{R}$ by

$$\mathcal{F}W_n(x) = w\left(\frac{x}{2^n}\right), \quad \mathcal{F}W_n^{\#}(x) = \mathcal{F}W_n(-x), \qquad n \in \mathbb{Z}$$

where $\mathcal{F}$is the Fourier transform. The Besov class $\boldsymbol{B}_{pq}^s(\mathbb{R})$consists of distributions φ on $\mathbb{R}$ such that

$$\{\|2^{ns}\varphi * W_n\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z}) \quad \text{and} \quad \{\|2^{ns}\varphi * W_n^{\#}\|_{L^p}\}_{n\in\mathbb{Z}} \in \ell^q(\mathbb{Z})$$

According to this definition, the space $\boldsymbol{B}_{pq}^s(\mathbb{R})$ contains all polynomials. However, it is not necessary to include all polynomials.

We need only Besov spaces $\boldsymbol{B}_{\infty1}^1$and $\boldsymbol{B}_{\infty1}^2$. In the case of functions on the real line it is convenient to restrict the degree of polynomials in $\boldsymbol{B}_{\infty1}^1(\mathbb{R})$ by 1 and in $\boldsymbol{B}_{\infty1}^2(\mathbb{R})$ by 2. It is also convenient to consider the following seminorms on$\boldsymbol{B}_{\infty1}^1(\mathbb{R})$ and in $\boldsymbol{B}_{\infty1}^2(\mathbb{R})$:

$$\|\varphi\|_{\boldsymbol{B}_{\infty1}^1(\mathbb{R})} = \sup_{x\in\mathbb{R}}|\varphi'(x)| + \sum_{n\in\mathbb{Z}} 2^n\, \|\varphi * W_n\|_{L^\infty} + \sum_{n\in\mathbb{Z}} 2^n\, \|\varphi * W_n^{\#}\|_{L^\infty}$$

and

$$\|\varphi\|_{\boldsymbol{B}_{\infty1}^2(\mathbb{R})} = \sup_{x\in\mathbb{R}}|\varphi''(x)| + \sum_{n\in\mathbb{Z}} 2^{2n}\, \|\varphi * W_n\|_{L^\infty} + \sum_{n\in\mathbb{Z}} 2^{2n}\, \|\varphi * W_n^{\#}\|_{L^\infty}.$$

The classes $\boldsymbol{B}_{\infty1}^1(\mathbb{R})$ and $\boldsymbol{B}_{\infty1}^2(\mathbb{R})$ can be described as classes of function on $\mathbb{R}$ in the following way:

$$\varphi \in \boldsymbol{B}_{\infty1}^1(\mathbb{R}) \Leftrightarrow \sup_{t\in\mathbb{R}}|\varphi'(x)| + \int_{\mathbb{R}} \frac{\|\Delta_t^2\varphi\|_{L^\infty}}{|t|^2} dt < \infty$$

and

$$\varphi \in \boldsymbol{B}_{\infty1}^2(\mathbb{R}) \Leftrightarrow \sup_{t\in\mathbb{R}}|\varphi''(x)| + \int_{\mathbb{R}} \frac{\|\Delta_t^3\varphi\|_{L^\infty}}{|t|^2} dt < \infty,$$

where Δ_t is the difference operator defined by $(\Delta_t\varphi)(x) = \varphi(x+t) - \varphi(x)$.

The Besov class $\boldsymbol{B}_{\infty1}^2(\mathbb{R})$ also appears in a natural way in perturbation theory in [96], where the following problem is studied: in which case

$$\varphi(T_f) - T_{\varphi\circ f} \in \boldsymbol{S}_1?$$

(T_g is a Toeplitz operator with symbol g.)

We refer to [97] for more detailed information on Besov classes.

Let U and V be unitary operators such that $V - U \in \boldsymbol{S}_2$. Denote by E_U and E_V the spectral measures of U and V. Let A be a self-adjoint operator such that $\sigma(A) \subset [-n, n]$ and $V = \exp(iA)U$. It is easy to see that $A \in \boldsymbol{S}_2$.
Put

$$U_s = e^{isA}\, U. \tag{35}$$

Consider the class $\mathrm{Lip}\widehat{\otimes}L^\infty$ that consists of functions u on $\mathbb{T} \times \mathbb{T}$that admit a representation

$$u(\zeta, \tau) = \sum_{n \geq 0} f_n(\zeta) g_n(\tau), \quad \zeta, \tau \in \mathbb{T}, \tag{36}$$

where $f_n \in \mathrm{Lip}$,$g_n \in L^\infty$ and

$$\sum_{n \geq 0} \|f_n\|_{\mathrm{Lip}} \cdot \|g_n\|_\infty < \infty. \tag{37}$$

By definition, $\|u\|_{\mathrm{Lip}\widehat{\otimes}L^\infty}$ is the infimum of the left-hand side of (37) over all functions f_n and g_n satisfying (36). We consider here the following seminorm on the space Lip of Lipschitz functions:

$$\|f_n\|_{\mathrm{Lip}} = \sup_{\zeta \neq \tau} \frac{|f(\zeta) - f(\tau)|}{|\zeta - \tau|}.$$

For a differentiable function φon $\mathbb{T}$ we define the function$\breve{\varphi}$ $\mathbb{T} \times \mathbb{T}$ by

$$\breve{\varphi}(\zeta, \tau) = \begin{cases} \dfrac{\varphi(\zeta) - \varphi(\tau)}{\zeta - \tau} & , \zeta \neq \tau, \\ \varphi'(\zeta), & \zeta = \tau. \end{cases}$$

Theorem (2.2.1)[68]. If $\varphi \in \boldsymbol{B}^1_{\infty 1}$, then

$$\varphi(V) - \varphi(U) - \frac{d}{ds}(\varphi(U_s))|_{s=0} \in \boldsymbol{S}_1 \tag{38}$$

and

$$\left\|\varphi(V) - \varphi(U) - \frac{d}{ds}(\varphi(U_s))|_{s=0}\right\|_{\boldsymbol{S}_1} \leq const\ \|\varphi\|_{\boldsymbol{B}^2_{\infty 1}(\mathbb{R})} \|V - U\|_{\boldsymbol{S}_2}.$$

To prove Theorem **(2.2.1).**, we need the following fact.

Theorem (2.2.2)[68]. If $\varphi \in \boldsymbol{B}^2_{\infty 1}$, then $\breve{\varphi} \in \mathrm{Lip}\widehat{\otimes}L^\infty$ and

$$\|\breve{\varphi}\|_{\mathrm{Lip}\widehat{\otimes}L^\infty} \leq \mathrm{const}\|\varphi\|_{\boldsymbol{B}^2_{\infty 1}}$$

Proof. We have

$$\breve{\varphi}(\zeta, \tau) = \sum_{j,k \geq 0} \breve{\varphi}(j + k + 1)\zeta^j \tau^k + \sum_{j,k < 0} \breve{\varphi}(j + k + 1)\zeta^j \tau^k \tag{39}$$

where $\{\breve{\varphi}(j)\}_{j \in \mathbb{Z}}$ is the sequence of Fourier coefficients of φ.

Let us show that the first term on the right-hand side of (39) belongs to $\mathrm{Lip}\widehat{\otimes}L^\infty$.
A similar result for the second term in (39) can be proved in the same way. We use the construction given in the proof of Theorem 2 of Section 3 of [84]. We have

$$\sum_{j,k \geq 0} \hat{\varphi}(j + k + 1)\zeta^j \tau^k$$

$$= \sum_{j,k \geq 0} \alpha_{jk} \hat{\varphi}(j + k + 1)\zeta^j \tau^k + \sum_{j,k \geq 0} \beta_{jk} \hat{\varphi}(j + k + 1)\zeta^j \tau^k, \tag{40}$$

Where

$$\alpha_{jk} = \begin{cases} \frac{1}{2}, & j = k = 0, \\ \frac{2j-k}{j+k}, \quad j+k>0, \frac{k}{2} \le j \le 2k, & and \quad \beta_{jk} = 1 - \alpha_{jk} \\ 0, & j \ge 2k \end{cases}$$

Let us prove that the function

$$(\zeta, \tau) \to \sum_{j,k \ge 0} \beta_{jk} \hat{\varphi}(j+k+1) \zeta^j \tau^k$$

on the right-hand side of (40) belongs to Lip$\hat{\otimes} L^\infty$.

We define the functions q and r on $\mathbb{R}$ by

$$q(x) = \begin{cases} 0, & x \le \frac{1}{2}, \\ \frac{2x-1}{x+1}, & \frac{1}{2} \le x \le 2, \\ 1, & x \ge 2 \end{cases} \quad \text{and} \quad q(x) = \begin{cases} 0, & x \le \frac{3}{2}, \\ \frac{2x-3}{x+1}, & \frac{3}{2} \le x \le 3, \\ 1, & x \ge 3. \end{cases} \tag{40}$$

Put

$$Q_n(z) = \sum_{j \ge 0} q\left(\frac{j}{n}\right) z^j, \qquad R_n(z) = \sum_{j \ge 0} r\left(\frac{j}{n}\right) z^j \quad \text{for } n > 0$$

and

$$Q_0(z) = R_0(z) = \frac{1}{2} + \sum_{j \ge 1} z^j.$$

It is easy to see that

$$\sum_{j,k \ge 0} \beta_{jk} \hat{\varphi}(j+k+1) \zeta^j \tau^k = \sum_{n \ge 0} \zeta^n \big(((S^*)^n \psi * Q_n)(\tau) \big), \tag{41}$$

where $\psi = \mathbb{P}_+ \bar{z} \varphi$ and $S^* \psi = \frac{\psi - \psi(0)}{z}$. We have

$$\sum_{n \ge 0} \|z^n\|_{\mathrm{Lip}} \|(S^*)^n \psi * Q_n\|_\infty \le \text{const} \sum_{n \ge 0} n \|(S^*)^n \psi * Q_n\|_\infty = \text{const} \sum_{n \ge 0} n \|\psi * R_n\|_\infty$$

Let us show that for $\varphi \in \boldsymbol{B}^2_{\infty 1}$,

$$\sum_{n \ge 0} n \|\psi * R_n\|_\infty < \infty.$$

Consider the function $r^\flat$ on $\mathbb{R}$ defined by $r^\flat(x) = 1 - r(|x|), x \in \mathbb{R}$. Put

$$R_n^\flat(z) = \sum_{j \in \mathbb{Z}} r^\flat\left(\frac{j}{n}\right) z^j, \qquad n > 0,$$

then $\|R_n^\flat\|_{L^1} \le$const (see [84, Proof of Lemma 3]). Suppose that $n \ge 2^m$. Then

$$R_n * \psi = R_n * \sum_{k \ge m} \psi * W_k$$

Hence,

$$\|R_n * \psi\|_\infty \le \left\| R_* \left(\sum_{k \ge m} \psi * W_k \right) \right\|_\infty \le (1 + \|R_n^\flat\|_1) \sum_{k \ge m} \|\psi * W_k\|_\infty.$$

It follows that

$$\sum_{n\ge2} n\|\psi * R_n\|_\infty = \sum_{m\ge1}\sum_{n=2^m}^{2^{m+1}-1} n\|\psi * R_n\|_\infty \le \text{const}\sum_{m\ge1} 2^{2m}\sum_{n\ge2}\|\psi * W_k\|_\infty$$
$$\le \text{const}\sum_{n\ge2} 2^{2m}\|\psi * W_m\|_\infty < \infty,$$
since $\varphi \in \boldsymbol{B}^1_{\infty 1}$.

Let us now show that the function
$$(\zeta,\tau) \to \sum_{j,k\ge0} \alpha_{jk}\hat{\varphi}(j+k+1)\zeta^j\tau^k$$
belongs to the space $\text{Lip}\,\widehat{\otimes}\,L^\infty$.

It follows from (41) that
$$\sum_{j,k\ge0} \alpha_{jk}\hat{\varphi}(j+k+1)\zeta^j\tau^k = \sum_{n\ge0}\big(((S^*)^n\psi * Q_n)(\zeta)\big)\tau^n.$$
It suffices to show that
$$\sum_{n\ge0}\|(S^*)^n\psi * Q_n\|_{\text{Lip}} < \infty.$$
By the Bernstein inequality, we have
$$\sum_{n\ge0}\|(S^*)^n\psi * Q_n\|_{\text{Lip}} = \sum_{n\ge0}\|((S^*)^n\psi * Q_n)'\|_\infty \le \sum_{n\ge0}\sum_{k\ge0}\big\|\big(((S^*)^n(\psi * W_k)) * Q_n\big)'\big\|_\infty$$
$$\le \sum_{n\ge0}\sum_{k\ge0} 2^{k+1}\|((S^*)^n(\psi * W_k)) * Q_n\|_\infty \le \sum_{n\ge0}\sum_{k\ge0} 2^{k+1}\|\psi * W_k * R_n\|_\infty$$
$$\le \sum_{0\le n\le 2^{k+2}/3} (1+\|R_n^\flat\|_1)\sum_{n\ge0} 2^{k+1}\|\psi * W_k\|_\infty \le \text{const}\sum_{n\ge0} 2^{2k}\|\psi * W_k\|_\infty$$
$$\le \text{const}\|\psi\|_{\boldsymbol{B}^2_{\infty 1}},$$
since, clearly, $\psi * W_k R_n = 0$ if $n > 2^{k+2}/2$.

Proof of Theorem (2.2.1)[68]. Without loss of generality we may assume that $\hat{\varphi}(0) = \hat{\varphi}(1) = 0$. Since $\varphi \in C^1(\mathbb{T})$, we have by (33),
$$\frac{d}{ds}(\varphi(U_s))|_{s=0} = i\iint_{\mathbb{T}\times\mathbb{T}} \tau\check{\varphi}(\zeta,\tau)dE_U(\zeta)AdE_U(\tau)$$
On the other hand, by (30),
$$\varphi(V)-\varphi(U) = \iint_{\mathbb{T}\times\mathbb{T}} \check{\varphi}(\zeta,\tau)dE_V(\zeta)(V-U)dE_U(\tau)$$
$$= -\iint_{\mathbb{T}\times\mathbb{T}} \tau\check{\varphi}(\zeta,\tau)dE_V(\zeta)(I-VU^*)dE_U(\tau)$$
$$= -\iint_{\mathbb{T}\times\mathbb{T}} \tau\check{\varphi}(\zeta,\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau)$$
Thus

$$\varphi(V)-\varphi(U)-\frac{d}{ds}\big(\varphi(U_s)\big)\Big|_{s=0}$$
$$= -\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau) - i\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)AdE_U(\tau)$$
$$= -\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau)$$
$$+\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(I-e^{iA})dE_U(\tau)$$
$$+\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(e^{iA}-I-iA)dE_U(\tau).$$

It is easy to see that $e^{iA}-I-iA\in\boldsymbol{S}_1$, and so by (31),

$$\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(e^{iA}-I-iA)dE_U(\tau)\in\boldsymbol{S}_1$$

and

$$\left\|\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(e^{iA}-I-iA)dE_U(\tau)\right\|_{\boldsymbol{S}_1}\le \text{const}\|\varphi\|_{\boldsymbol{B}^1_{\infty 1}}\left\|(e^{iA}-I-iA)\right\|_{\boldsymbol{S}_1}$$

Clearly, $\|\varphi\|_{\boldsymbol{B}^1_{\infty 1}}\le \text{const}\|\varphi\|_{\boldsymbol{B}^2_{\infty 1}}$.

On the other hand, let $\{f_n\}_{n\ge 0}$ and $\{g_n\}_{n\ge 0}$ be sequences of functions such that

$$\breve{\varphi}(\zeta,\tau) = \sum_{n\ge 0} f_n(\zeta)g_n(\tau),\qquad \zeta,\tau\in\mathbb{T}$$

and (37) holds. We have

$$\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau)-\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(I-e^{iA})dE_U(\tau)$$
$$=\iint_{\mathbb{T}\times\mathbb{T}}\sum_{n\ge 0} f_n(\zeta)\tau g_n(\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau)$$
$$-\iint_{\mathbb{T}\times\mathbb{T}}\sum_{n\ge 0} f_n(\zeta)\tau g_n(\tau)dE_U(\zeta)(I-e^{iA})dE_U(\tau)$$
$$=\sum_{n\ge 0} f_n(V)(I-e^{iA})g_n(U)U-\sum_{n\ge 0} f_n(U)(I-e^{iA})g_n(U)U$$
$$=\sum_{n\ge 0}(f_n(V)-f_n(U))(I-e^{iA})g_n(U)U.$$

Thus

$$\left\|\iint_{\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_V(\zeta)(I-e^{iA})dE_U(\tau)-\iint_{\mathbb{T}\times\mathbb{T}}\tau\breve{\varphi}(\zeta,\tau)dE_U(\zeta)(I-e^{iA})dE_U(\tau)\right\|_{\boldsymbol{S}_1}$$
$$\le\sum_{n\ge 0}\|(f_n(V)-f_n(U))\|_{\boldsymbol{S}_2}\left\|(I-e^{iA})\right\|_{\boldsymbol{S}_2}\|g_n(U)\|$$
$$\le \text{const}\left\|(I-e^{iA})\right\|_{\boldsymbol{S}_2}\sum_{n\ge 0}\|f_n\|_{\text{Lip}}.\|g_n\|_\infty\le \text{const}\left\|(I-e^{iA})\right\|_{\boldsymbol{S}_2}\|\varphi\|_{\boldsymbol{B}^1_{\infty 1}}$$

This completes the proof.

Let now η be a generalized spectral shift function for the pair (V,U).

Theorem (2.2.3)[68]. Let U and V be unitary operators such that $V - U \in \boldsymbol{S}_2$ and let U_s be defined by (43). Then for any $\varphi \in \boldsymbol{B}^2_{\infty 1}$,

$$\operatorname{trac}\left(\varphi(V) - \varphi(U) - \frac{d}{ds}(\varphi(U_s))\Big|_{s=0}\right) = \int_{\mathbb{T}} \varphi'' \eta \, dm. \qquad (42)$$

Proof. The fact that the operator in (38) belongs to $\boldsymbol{S}_1$ is an immediate consequence of Theorem (2.2.1).

It is easy to see from the definition of the space $\boldsymbol{B}^2_{\infty 1}$ given that the trigonometric polynomials are dense in $\boldsymbol{B}^2_{\infty 1}$. Let φ_n be trigonometric polynomials such that

$$\lim_{n\to\infty} \|\varphi - \varphi_n\|_{\boldsymbol{B}^2_{\infty 1}} = 0.$$

Since $\boldsymbol{B}^2_{\infty 1}$is continuously imbedded in the space C^2 of functions with two continuous derivatives, it follows that $\varphi_n \to \varphi$ in C^2. Since $\eta \in L^1$, it follows that

$$\lim_{n\to\infty} \int_{\mathbb{T}} \varphi_n'' \eta \, d\boldsymbol{m} = \int_{\mathbb{T}} \varphi'' \eta \, d\boldsymbol{m}.$$

On the other hand, it follows from Theorems (2.2.1)and (2.2.2)that

$$\left\| \left(\varphi_n(V) - \varphi_n(U) - \frac{d}{ds}(\varphi_n(U_n))\Big|_{s=0}\right) - \left(\varphi_n(V) - \varphi_n(U) - \frac{d}{ds}(\varphi_n(U_n))\Big|_{s=0}\right) \right\|_{\boldsymbol{S}_1} \to 0$$

as $n \to \infty$. The result follows now from the fact that trace formula (42) is valid for all trigonometric polynomials φ (see [98]).

We extend Koplienko's trace formula for self-adjoint operators to a considerably bigger class of functions.

Let A be a self-adjoint operator (not necessarily bounded) on Hilbert space and let K be a self-adjoint operator of class $\boldsymbol{S}_2$. Put $B = A + K$. As we have already mentioned in the introduction, Koplienko introduced in [88] the generalized spectral shift function $\eta \in L^1$ that corresponds to the pair (A, B) and showed that for rational functions $\varphi$with poles off the real line the following trace formula holds.

$$\operatorname{trace}\left(\varphi(B) - \varphi(A) - \frac{d}{ds}(\varphi(A_n))\Big|_{s=0}\right) = \int_{\mathbb{R}} \varphi''(x)\eta(x)dx, \qquad (43)$$

where $A_s = A + sK$.

We are going to extend this formula to the Besov class $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$. Note however, that the situation with self-adjoint operators is subtler than with unitary operators. First of all, the rational functions are not dense in $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ and this makes it more difficult to extend formula (43) from rational functions to $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$. Secondly, functions in $\boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ do not have to belong to the space Lip of Lipschitz functions on $\mathbb{R}$, which we equip with the seminorm:

$$\|f\|_{\mathrm{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Thus for $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$, none of the operators

$$\varphi(B) - \varphi(A) \quad \text{and} \quad \frac{d}{ds}(\varphi(A_s))|_{s=0}$$

has to be in $\boldsymbol{S}_2$. In fact, it is not clear how one can interpret each of those operators. However, it turns out that their difference still makes sense for functions $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ and formula (43) holds for such functions φ.

To do it, we first prove formula (43) in the case $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R}) \cap \text{Lip}$ and estimate the $\boldsymbol{S}_1$ norm of the left-hand side of (43) in terms of $\|\varphi\|_{\boldsymbol{B}^2_{\infty 1}}$. Then we define the operator on the left-hand side of (43) for functions $f \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$ and prove formula (43) for such functions.

For a differentiable function φon $\mathbb{R}$ we define the function $\check{\varphi}$ on $\mathbb{R} \times \mathbb{R}$ by

$$\check{\varphi}(x, y) = \begin{cases} \dfrac{\varphi(x) - \varphi(y)}{\zeta - \tau}, & x \neq y, \\ \varphi'(x), & x = y. \end{cases}$$

We consider the space Lip $\hat{\otimes} i$ L^∞ of functions u on $\mathbb{R} \times \mathbb{R}$ that admit a representation

$$u(x, y) = \int_\Omega f_\omega(x) g_\omega(y) d\mu(\omega), \tag{44}$$

where (Ω, μ) is a measure space and the functions $(\omega, x) \to f_\omega(x)$ and $(\omega, y) \to g_\omega(y)$ are measurable functions on $\Omega \times \mathbb{R}$ such that $f_\omega \in$ Lip, $g_\omega \in L^\infty$ for almost all $\omega \in \Omega$, and

$$\int_\Omega \|f_\omega\|_{\text{Lip}} \|g_\omega\|_{L^\infty} d\mu(\omega) < \infty\,. \tag{45}$$

By definition, the norm of u in Lip $\hat{\otimes} i$ L^∞ is the infimum of the left-hand side of (45) over all representations of form (44).

Theorem (2.2.4)[68]. Let $M > 0$. Suppose that φ is a bounded function on $\mathbb{R}$ such that supp $\mathcal{F}\varphi \subset [M/2, 2M]$. Then

$$\varphi(B) - \varphi(A) - \frac{d}{ds}(\varphi(A_s))\Big|_{s=0} \in \boldsymbol{S}_1 \tag{46}$$

and

$$\left\| \varphi(B) - \varphi(A) - \left(\frac{d}{ds}\varphi(A_s)\right)\Big|_{s=0} \right\|_{\boldsymbol{S}_1} \leq \text{const} \cdot M^2 \|K\|^2_{\boldsymbol{S}_2} \|\varphi\|_{L^\infty}. \tag{4.5}$$

To prove Theorem (2.2.4), we need the following fact.

Lemma(2.2.5)[68]. Let φ be a function on $\mathbb{R}$ such that supp $\mathcal{F}\varphi \subset [M/2, 2M]$. Then $\check{\varphi} \in$ Lip $\hat{\otimes} i$ L^∞ and

$$\|\check{\varphi}\|_{\text{Lip}\,\hat{\otimes} i\, L^\infty} \leq \text{const} \cdot M^2 \|\varphi\|_{L^\infty}.$$

Proof. Let q and r be the functions on $\mathbb{R}$ defined by (40). Consider the distributions Q_t and R_t , $t > 0$, on $\mathbb{R}$ such that

$$(\mathcal{F}Q_t)(x) = q(x/t) \quad \text{and} \quad (\mathcal{F}R_t)(x) = r(x/t).$$

It was shown in [85] (formula (6)) that

$$\check{\varphi}(x, y) = \int_0^\infty \big((S_t^*\varphi) * Q_t\big)(x) e^{ity} dt + \int_0^\infty \big((S_t^*\varphi) * Q_t\big)(y) e^{itx} dt, \tag{48}$$

where $S_t^*\varphi$ is the function such that

$$(\mathcal{F}(S_t^*\varphi))(x) = \begin{cases} e^{-itx}(\mathcal{F}\varphi)(x), & x > t, \\ 0, & x \leq t. \end{cases}$$

Clearly,

$$\|\check{\varphi}\|_{\text{Lip}\,\hat{\otimes} i\, L^\infty} \leq \int_0^\infty \|(S_t^*\varphi) * Q_t\|_{\text{Lip}} dt + \int_0^\infty \|(S_t^*\varphi) * Q_t\|_{L^\infty} t dt.$$

By the Bernstein inequality,

$$\|(S_t^*\varphi) * Q_t\|_{\text{Lip}} = \left\| \big((S_t^*\varphi) * Q_t\big)' \right\|_{L^\infty} \leq 2M \|(S_t^*\varphi) * Q_t\|_{L^\infty}$$

and so

$$\int_0^\infty \|(S_t^*\varphi)*Q_t\|_{\mathrm{Lip}}dt \le 2M\int_0^\infty \|(S_t^*\varphi)*Q_t\|_{L^\infty}dt = 2M\int_0^\infty \|\varphi*R_t\|_{L^\infty}dt$$
$$= 2M\int_0^{4M/3} \|\varphi*R_t\|_{L^\infty}dt.$$

since, obviously, $(S_t^*\varphi)*R_t = 0$ for $t \ge 4M/3$.

On the other hand,

$$\int_0^\infty \|(S_t^*\varphi)*Q_t\|_{L^\infty}tdt = \int_0^\infty \|\varphi*R_t\|_{L^\infty}tdt = \int_0^{4M/3} \|\varphi*R_t\|_{L^\infty}tdt.$$

It remains to observe that if $R_t^\flat$ is the function on $\mathbb{R}$ such that

$$\left(\mathcal{F}R_t^\flat\right)(x) = 1-(\mathcal{F}R_t)(|x|),$$

then $R_t^\flat \in L^1$, $\left\|R_t^\flat\right\|_{L^1}$ does not depend on t and

$$\|\varphi*R_t\|_{L^\infty} \le \left(1+\left\|R_t^\flat\right\|_{L^1}\right)\|\varphi\|_{L^\infty}.$$

Proof of Theorem (2.2.4). By (28) and (32), we have

$$\varphi(B)-\varphi(A)-\frac{d}{ds}(\varphi(A_n))\Big|_{s=0} = \iint_{\mathbb{R}\times\mathbb{R}} \check{\varphi}(x,y)dE_B(x)KdE_A(y)$$
$$= -\iint_{\mathbb{R}\times\mathbb{R}} \check{\varphi}(x,y)dE_A(x)KdE_A(y)$$

By Lemma (2.2.5), $\check{\varphi}$ admits a representation

$$\check{\varphi}(x,y) = \int_\Omega f_\omega(x)g_\omega(y)d\mu(\omega)$$

such that

$$\int_\Omega \|f_\omega\|_{\mathrm{Lip}}\|g_\omega\|_{L^\infty}d\mu(\omega) \le \mathrm{const}\cdot M^2\|\varphi\|_{L^\infty}.$$

We have

$$\iint_{\mathbb{R}\times\mathbb{R}} \check{\varphi}(x,y)dE_B(x)KdE_A(y) = \int_\Omega \left(\iint_{\mathbb{R}\times\mathbb{R}} f_\omega(x)g_\omega(y)dE_B(x)KdE_A(y)\right)d\mu(\omega)$$
$$= \int_\Omega f_\omega(B)Kg_\omega(A)d\mu(\omega).$$

Similarly,

$$\iint_{\mathbb{R}\times\mathbb{R}} \check{\varphi}(x,y)dE_A(x)KdE_A(y) = \int_\Omega f_\omega(A)Kg_\omega(A)d\mu(\omega).$$

Thus

$$\varphi(B)-\varphi(A)-\frac{d}{ds}(\varphi(A+sK))\Big|_{s=0} = \int_\Omega \big(f_\omega(B)-f_\omega(A)\big)Kg_\omega(A)d\mu(\omega).$$

Using (29), we obtain

$$\left\|\varphi(B)-\varphi(A)-\frac{d}{ds}(\varphi(A_s))\Big|_{s=0}\right\|_{\boldsymbol{S}_1} \le \int_\Omega \|f_\omega(B)-f_\omega(A)\|_{\boldsymbol{S}_2}\|K\|_{\boldsymbol{S}_2}\|g_\omega(A)\|d\mu(\omega)$$
$$\le \|K\|_{\boldsymbol{S}_2}\int_\Omega \|f_\omega\|_{\mathrm{Lip}}\|B-A\|_{\boldsymbol{S}_2}\|g_\omega\|_{L^\infty}d\mu(\omega) = \|K\|_{\boldsymbol{S}_2}^2\int_\Omega \|f_\omega\|_{\mathrm{Lip}}\|g_\omega\|_{L^\infty}d\mu(\omega)$$
$$\le \mathrm{const}\cdot M^2\|K\|_{\boldsymbol{S}_2}^2\|\varphi\|_{L^\infty}.$$

Theorem (2.2.6)[68]. Suppose that $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R}) \cap \mathrm{Lip}$. Then (46) and (43) hold, and

$$\left\| \varphi(B) - \varphi(A) - \frac{d}{ds}\left(\varphi(A_s)\right)\Big|_{s=0} \right\|_{\boldsymbol{S}_1} \le \mathrm{const} \cdot M^2 \|K\|^2_{\boldsymbol{S}_2} \|\varphi\|_{\boldsymbol{B}^2_{\infty 1}}.$$

We need the following lemma.

Lemma (2.2.7)[68]. Let $\{f_n\}_{n\ge 0}$ and f be functions in $\mathrm{Lip}(\mathbb{R})$ such that

$$\lim_{n\to\infty} f_n(x) = f(x), \qquad x \in \mathbb{R}, \qquad \text{and} \qquad \sup_n \|f_n\|_{\mathrm{Lip}} < \infty.$$

Then

$$\lim_{n\to\infty}\left(f_n(B) - f_n(A)\right) = f(B) - f(A)$$

in $\boldsymbol{S}_2$

Let us first prove Theorem (2.2.6).

Proof of Theorem (2.2.6) Since $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$, φ is continuously differentiable, and so both operators

$$\varphi(B) - \varphi(A) \qquad \text{and} \qquad \frac{d}{ds}\left(\varphi(A + sK)\right)\Big|_{s=0}$$

belong to $\boldsymbol{S}_2$.

Clearly, if $\varphi$is a linear function, then the operator in (46) is zero. Suppose first that $\mathcal{F}\varphi'' \in L^1$. Then

$$\varphi = \sum_{n\in\mathbb{Z}} (\varphi_n + \varphi_n^{\#}),$$

where

$$\varphi_n = \varphi * \mathcal{F}^{-1}\chi_{[2^n, 2^{n+1}]} \qquad \text{and} \qquad \varphi_n^{\#} = \varphi * \mathcal{F}^{-1}\chi_{[-2^{n+1}, -2^n]}.$$

Clearly,

$$2^{2n}\|\varphi_n\|_{L^\infty} \le \mathrm{const}\|\mathcal{F}\varphi_n''\|_{L^1} \qquad \text{and} \qquad 2^{2n}\|\varphi_n^{\#}\|_{L^\infty} \le \mathrm{const}\|\mathcal{F}\varphi_n^{\#}\|_{L^1} \qquad (50)$$

By Theorem (2.2.4),

$$\sum_{n\in\mathbb{Z}} \left\| \varphi_n(B) - \varphi_n(A) - \frac{d}{ds}\left(\varphi_n(A_s)\right)\Big|_{s=0} \right\| \le \mathrm{const} \sum_{n\in\mathbb{Z}} 2^{2n} \|\varphi_n\|_{L^\infty}$$

and the same estimate also holds for the functions $\varphi_n^{\#}$in place of φ_n. It follows now from (50) that

$$\left\| \varphi_n(B) - \varphi_n(A) - \frac{d}{ds}\left(\varphi_n(A_s)\right)\Big|_{s=0} \right\| \le \mathrm{const} \sum_{n\in\mathbb{Z}} 2^{2n}(\|\varphi_n\|_{L^\infty} + \|\varphi_n^{\#}\|_{L^\infty}) \le \mathrm{const}\|\mathcal{F}\varphi''\|_{L^1}$$

Since the rational functions are dense in the space $\{\varphi: \mathcal{F}\varphi'' \in L^1\}$ and trace formula (43) holds for rational functions with poles outside $\mathbb{R}$ (Koplienko's theorem [88]), it is easy to see that it also holds for arbitrary functions φ with $\mathcal{F}\varphi'' \in L^1$.

Suppose now that $\varphi \in \boldsymbol{B}^2_{\infty 1}(\mathbb{R})$. Since

$$\sum_{n\in\mathbb{Z}} 2^{2n}(\|\varphi * W_n\|_{L^\infty} + \|\varphi * W_n^{\#}\|_{L^\infty}) < \infty$$

and inequality (47) holds, it suffices to show that formula (43) holds for the functions $\varphi * W_n$ and $\varphi * W_n^{\#}$.

The following argument is similar to the argument given in the proof of Theorem 4 of [85] to establish the Lifshitz-Krein trace formula for functions in $\boldsymbol{B}^1_{\infty 1}(\mathbb{R})$. Put $\psi = \varphi * V_n$.

Then supp $\psi \subset [2^{n-1}, 2^{n+1}]$. Consider a smooth nonnegative function h on $\mathbb{R}$ such that supp $h \subset [-1,1]$ and $\int_{-1}^{1} h(x)dx = 1$. For $\varepsilon > 0$ put $h_\varepsilon(x) = \varepsilon^{-1} h(x/\varepsilon)$.
Let ψ_ε be the function defined by $\mathcal{F}\psi_\varepsilon = \mathcal{F}\psi * h_\varepsilon$. Clearly
$\mathcal{F}\psi_\varepsilon \in L^1$, $\lim_{\varepsilon\to 0} \|\psi_\varepsilon\|_{L^\infty} = \|\psi\|_{L^\infty}$ and $\lim_{\varepsilon\to 0} \psi_\varepsilon(x) = \psi(x)$ for $x \in \mathbb{R}$
Then formula (43) holds for ψ_ε. Clearly,

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} \psi_\varepsilon''(x)\eta(x)dx = \int_{\mathbb{R}} \psi''(x)\eta(x)dx.$$

Thus to prove that (43) holds for , it suffices to show that

$$\lim_{\varepsilon\to 0}\left(\psi_\varepsilon(B) - \psi_\varepsilon(A) - \frac{d}{ds}\left(\psi_\varepsilon(A_s)\right)\Big|_{s=0}\right) = \operatorname{trace}\left(\psi(B) - \psi(A) - \frac{d}{ds}\left(\psi(A_s)\right)\Big|_{s=0}\right).$$

By (49), we have

$$\psi_\varepsilon(B) - \psi_\varepsilon(A) - \frac{d}{ds}\left(\psi_\varepsilon(A_s)\right)\Big|_{s=0}$$
$$= \iint_{\mathbb{R}\times\mathbb{R}} \check{\psi}_\varepsilon(x,y)dE_B(x)KdE_A(y) - \iint_{\mathbb{R}\times\mathbb{R}} \check{\psi}_\varepsilon(x,y)dE_A(x)KdE_A(y).$$

By (48), this is equal to

$$\int_0^\infty \iint_{\mathbb{R}\times\mathbb{R}} \big((S_t^*\psi_\varepsilon) * Q_t\big)(x)e^{ity}dE_B(x)KdE_A(y)dt$$
$$+ \int_0^\infty \iint_{\mathbb{R}\times\mathbb{R}} \big((S_t^*\psi_\varepsilon) * Q_t\big)(y)e^{itx}dE_B(x)KdE_A(y)dt$$
$$- \int_0^\infty \iint_{\mathbb{R}\times\mathbb{R}} \big((S_t^*\psi_\varepsilon) * Q_t\big)(x)e^{ity}dE_A(x)KdE_A(y)dt$$
$$- \int_0^\infty \iint_{\mathbb{R}\times\mathbb{R}} \big((S_t^*\psi_\varepsilon) * Q_t\big)(y)e^{itx}dE_A(x)KdE_A(y)dt.$$

It is easy to see that

$$\int_0^\infty \iint_{\mathbb{R}\times\mathbb{R}} \big((S_t^*\psi_\varepsilon) * Q_t\big)(x)e^{ity}dE_B(x)KdE_A(y)dt = \int_0^\infty \big((S_t^*\psi_\varepsilon) * Q_t\big)(B)K\exp(itA)dt$$

and similar equalities hold for the other three integrals. Thus

$$\psi_\varepsilon(B) - \psi_\varepsilon(A) - \frac{d}{ds}\left(\psi_\varepsilon(A_s)\right)\Big|_{s=0}$$
$$= \int_0^\infty \Big(\big((S_t^*\psi_\varepsilon) * Q_t\big)(B) - \big((S_t^*\psi_\varepsilon) * Q_t\big)(A)\Big) K\exp(itA)dt$$
$$+ \int_0^\infty (\exp(itB) - \exp(itA))\, K\, \Big(\big((S_t^*\psi_\varepsilon) * Q_t\big)(A)\Big)\, dt.$$

We have

$$\lim_{\varepsilon\to 0}\big((S_t^*\psi_\varepsilon) * Q_t\big)(A) = \big((S_t^*\psi_\varepsilon) * Q_t\big)(A)$$

in the strong operator topology (see [84, Proof of Theorem 4]). By Lemma (2.2.7),

$$\lim_{\varepsilon\to 0}\Big(\big((S_t^*\psi_\varepsilon) * Q_t\big)(B) - \big((S_t^*\psi_\varepsilon) * Q_t\big)(A)\Big) = \big((S_t^*\psi_\varepsilon) * Q_t\big)(A) - \big((S_t^*\psi_\varepsilon) * Q_t\big)(B)$$

in $\boldsymbol{S}_2$.
It follows that

$$\lim_{\varepsilon\to 0}\operatorname{trace}\left(\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(B)-\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right)K\exp(itA)\right)$$
$$=\operatorname{trace}\left(\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(B)-\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right)K\exp(itA)\right)$$

and

$$\lim_{\varepsilon\to 0}\operatorname{trace}(\exp(itB)-\exp(itA))\,K\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right)$$
$$=\operatorname{trace}(\exp(itB)-\exp(itA))\,K\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right).$$

Thus

$$\lim_{\varepsilon\to 0}\operatorname{trace}\left(\psi_\varepsilon(B)-\psi_\varepsilon(A)-\frac{d}{ds}\left(\psi_\varepsilon(A_s)\right)\Big|_{s=0}\right)$$
$$=\int_0^\infty \operatorname{trace}\left(\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(B)-\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right)K\exp(itA)\right)dt$$
$$+\int_0^\infty \operatorname{trace}\left((\exp(itB)-\exp(itA))\,K\left(\left((S_t^*\psi_\varepsilon)*Q_t\right)(A)\right)\right)dt=\operatorname{trace}\left(\psi(B)\right.$$
$$\left.-\psi(A)-\frac{d}{ds}\left(\psi(A_s)\right)\Big|_{s=0}\right),$$

which proves (2.2.6).

Proof of Lemma (2.2.7). We have

$$f_n(B)-f_n(A)=\iint_{\mathbb{R}\times\mathbb{R}\setminus\Delta}\check{f}_n(x,y)\,dE_B(x)K\,dE_A(y)=\iint_{\mathbb{R}\times\mathbb{R}\setminus\Delta}\check{f}_n(x,y)\,d\mathcal{E}K(x,y),$$

where $\mathcal{E}$ is the spectral measure on the space $\boldsymbol{S}_2$ defined by $\mathcal{E}(\delta\times\sigma)T=E_B(\delta)TE_A(\sigma)$,$\delta,\sigma\subset\mathbb{R}$,$T\in\boldsymbol{S}_2$ and $\Delta\subset\mathbb{R}\times\mathbb{R}$ is the diagonal: $\Delta=\{(x,x): x\in\mathbb{R}\}$. Then

$$\|(f_n(B)-f_n(A))-(f(B)-f(A))\|_{\boldsymbol{S}_2}^2=\iint_{\mathbb{R}\times\mathbb{R}\setminus\Delta}\left|\check{f}_n(x,y)-\check{f}(x,y)\right|^2 d(\mathcal{E}K,K)(x,y)\to 0$$

as $n\to\infty$

Now we are going to extend formula (43) to the whole class $\boldsymbol{B}_{\infty 1}^2(\mathbb{R})$. Consider first the case when $\varphi$is a polynomial of degree at most 2. Clearly, for linear functions $\varphi$the operator on the left-hand side of (43) is the zero operator and the right-hand side of (43) is equal to 0. Suppose now that$\varphi(t)=t^2$. If we perform formal manipulations, we obtain

$$(A+K)(A+K)-A^2-\frac{d}{ds}(A+sK)(A+sK)|_{s=0}$$
$$=KA+AK+K^2-\frac{d}{ds}(A^2+sKA+sAK+s^2K^2)|_{s=0}=K^2.$$

We can put now by definition

$$(A+K)^2-A^2-\frac{d}{ds}(A+sK)^2|_{s=0}=K^2.$$

The following result establishes formula (43) for the function $\varphi(t)=t^2$.

Theorem (2.2.8)[68].

$$\operatorname{trace}K^2=2\int_{\mathbb{R}}\eta(x)\,dx. \tag{51}$$

Proof. To establish (51), we first assume that A is a bounded operator. Consider a sequence $\{g_n\}_{n\geq 1}$ such that

$g_n(x) = x^2$ for $x \in [-n, n]$, $\mathcal{F}g_n'' \in L^1$, and $\sup_{n\geq 1}\|\mathcal{F}g_n''\|_{L^1} < \infty$.

Then for $n \geq \|A\| + \|K\|$ we have

$$trace\ K^2 = trace\ \left(g_n(B) - g_n(A) - \frac{d}{ds}(g_n(A_s))\Big|_{s=0}\right)$$
$$= \int_{\mathbb{R}} g_n''(x)\eta(x)dx \to 2\int_{\mathbb{R}}\eta(x)dx \quad , \qquad as\ n \to \infty.$$

If A is an unbounded operator, consider the bounded self-adjoint operator A_n defined by

$$A_n = AE_A\,([-n, n]).$$

Let $\eta_n$be the generalized spectral shift function that correspond to the pair $(A_n, A_n + K)$. Then

$$\text{trace } K^2 = 2\int_{\mathbb{R}}\eta_n(x)dx$$

and (51) follows form the fact that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\eta_n(x)dx = \int_{\mathbb{R}}\eta(x)dx,$$

which can be found in [88].

Finally, we obtain the following result.

Theorem(2.2.9)[68]. The map

$$\varphi \to \varphi(B) - \varphi(A) - \frac{d}{ds}(\varphi(A_s))\Big|_{s=0}$$

extends from $\boldsymbol{B}_{\infty 1}^2(\mathbb{R}) \cap \text{Lip}$ to a bounded linear operator from $\boldsymbol{B}_{\infty 1}^2(\mathbb{R})$ to $\boldsymbol{S}_1$ and trace formula (4.l) holds for functions φin $\boldsymbol{B}_{\infty 1}^2(\mathbb{R})$.

Proof. Since the linear combinations of quadratic polynomials and functions whose Fourier transforms have compact support in $\mathbb{R}\{0\}$ are dense in $\boldsymbol{B}_{\infty 1}^2(\mathbb{R})$, the result follows immediately from Theorems (2.2.6) and (2.2.8).