Chapter 1
Function Model and Estimates in Operator Classes

We obtain new formulas for the dilation and its eigenfunctions , generalizing B. S. Pavlov's formulas for the Schrodinger operator on the semiaxis with a real potential and complex boundary condition. Examples are considered. The use of estimates in operator classes for a difference of functions (e.g., fraction-al powers) of two operators in a Hilbert space is typical for various problems in operator theory and mathematical physics.

Section (1.1): Dissipative Operators

We introduce the concept of boundary spaces and we compute the characteristic function, "attached" to these spaces. This approach, suggested by A. V. Shtraus' investigations, is suitable for the analysis of operators with finite defect indices. We construct self adjoint dilation as the Cayley transform of the standard unitary dilation of a contraction. We derive various forms of dilations (B. S. Pavlov [2, 3], S. N. Naboko [4], A. V. Kuzhel', Yu. L. Kudryashov [5-7]). In the mentioned investigations the dilations have been somehow "guessed." Apparently, the form of the dilation, generalizing B. S. Pavlov's [2] dilation of the Schrodinger operator on the semi axis with a. real-valued potential and dissipative boundary condition, is new. We construct a coordinate-free model for dissipative operators on the basis of the constructed dilations. We derive formulas for the eigen functions of the continuous spectrum, generalizing the case of the above-mentioned Schrödinger operator. On the example of this operator it is shown how to construct the dilation and how to compute the characteristic function and the eigen functions of the dilation.

For $R_{\pm} = \{x \in R: \pm x > 0\}$, $C_{\pm} = \{\Im z \in C: z \in R_{\pm}\}$ D is the unit circle. We shall consider linear operators in a separable Hilbert space H, not necessarily bounded and not necessarily densely defined. If L is such an operator, then $\mathfrak{D}(L)$ is its domain of definition, Ran $L = L\mathfrak{D}(L)$ is the range of the operator, $\sigma(L)$ is the spectrum, i.e., the complement of the set of those λsuch that $(L - \lambda I)^{-1}$ can be continued to a continuous operator in H. We say that the operator L_2 is an extension of L_1 written $L_1 \subset L_2$, if $\mathfrak{D}(L_1) \subset \mathfrak{D}(L_2)$, $L_2|\mathfrak{D}(L_1) = L_1$. By $H^2(D,\varepsilon)$, $H^2(C_{\pm},\varepsilon)$ we shall denote the vector Hardy spaces (with values in the Hilbert space ε) in the circle and the half-plane, respectively. We shall use the (Hilbert) Sobolev spaces $W_2^1(a,b)$, $W_2^2(a,b)$. The symbol clos denotes closure, while Span denotes the closed linear span.

Definition(1.1.1)[1]. An operator L_0 in a Hilbert space H with a dense domain of definition $\mathfrak{D}(L_0)$ is said to be dissipative if

$$\operatorname{Im}(L_0x,x) \geq 0,\ x \in \mathfrak{D}(L_0)\,. \qquad (1)$$

the complication consists in the fact that operator L_0 may be unbounded. We would like to introduce symbol $\Im$ into the inner product and write $\Im L_0 \geq 0$. But operator $\Im L_0 = (L_0 -$

$L_0^*)/2i$ need not exist since, in general, $\mathfrak{D}(L_0) \neq \mathfrak{D}(L_0^*)$.

Example(1.1.2) [1]. **Let $H = L^2(0,1)$,**

$$Ly = -y'', \mathfrak{D}(L) = \{ y \in W_2^2(0{,}1): y'(0) = h_0 y(0), y'(1) = h_1 y(1) \ .$$

We have

$$\mathrm{Im}(Ly, y) = -\mathrm{Im} \int_0^1 y''(t)\bar{y}(t)\,dt = -\mathrm{Im}\, y'^{\bar{y}}|_0^1 = -\mathrm{Im}\, h_1 \cdot |y(1)|^2 + \mathrm{Im}\, h_0 \cdot |y(0)|^2.$$

If $\Im h_0 \geq 0, \Im h_1 \geq 0$ then the operator L is dissipative. Computing the adjoint operator (see, for example, [8]), we find

$$L^* y = -y'',$$
$$\mathfrak{D}(L^*) = \left\{ y \in W_2^2(0{,}1): y'(0) = \bar{h}_0(0), y'(1) = \bar{h}_1(1) \right\}.$$

If $h_0, h_1 \in R$, then the operator L is selfadjoint. If at least one of the numbers h_0, h_1 is not real, $\Im h_0 \geq 0, \Im h_1 \geq 0$ then L is a closed dissipative operator. The spaces $\mathfrak{D}(\mathrm{L})$ and $\mathfrak{D}(\mathrm{L}^*)$ are distinct. Moreover, in $\mathfrak{D}(L) \cap \mathfrak{D}(L^*)$ we have, obviously, $L = L^*$, so that to talk about the imaginary part of the operator is difficult.

We usually, at the analysis of differential operators, first one defines the operator L_0 on smooth functions, while L is defined as its closure. The exact determination of the domain of definition $\mathfrak{D}(\mathrm{L})$ is not always simple.

Special Case.(1.1.3). Let$L = A + iB$, $\mathfrak{D}(\mathrm{A}) \subset \mathfrak{D}(\mathrm{B})$, $A = A^*$, $(Bx, x) \geq 0$, $x \in \mathfrak{D}(\mathrm{A})$; $\mathfrak{D}(\mathrm{L}) = \mathfrak{D}(\mathrm{A})$. Then the operator L is dissipative.

Definition(1.1.4). **[1]**. A dissipative operator is said to be maximal dissipative if it does not have proper dissipative extensions.

LEMMA (1.1.5) [1]. (properties of dissipative operators)

(i) Assume that the operator L_0 is dissipative. Then the operator $\mathcal{X}(L_0) \stackrel{\mathrm{def}}{=} (L_0 - iI)(L_0 + iL)^{-1}$ is a contraction from$(L_0 + iL\ ) \mathfrak{D}(\mathrm{L}_0)$ onto $(L_0 - iL)\, \mathfrak{D}(\mathrm{L}_0)$and $L_0 = \mathcal{X}^{-1}(T_0) = i(I + T_0)(I - T_0)^{-1}$. For each contraction T_0 such that $1 \notin \rho_p(T_0)\sigma_p(\cdot)$ is the point spectrum of the operator), operator $L_0 = \mathcal{X}^{-1}(T_0)$, $\mathfrak{D}(L_0) = (I - T_0)\mathfrak{D}(T_0)$, is dissipative.

(ii) Each dissipative operator L_0 has a maximal dissipative extension L. A maximal dissipative operator is closed.

(iii) A maximal dissipative operator is maximal dissipative if and only if $T = \mathcal{X}\,(L)$ is a contraction such that $\mathfrak{D}(T) = H$ and $1 \notin \rho_p(T)$.

(iv) If L is a maximal dissipative operator, $L = \mathcal{X}^{-1}\,(T)$, then $-L^*$ is also maximal dissipative, $L^* = -\mathcal{X}^{-1}\,(T^*)$.

(v) If L is a maximal dissipative operator, then $\sigma(L) \subset \operatorname{clos} C_+, \|(L - \lambda I)^{-1}\| \leq |\Im\lambda|^{-1}, \lambda \in C_-$.

The transformation $\mathcal{X}$ is called the Cayley transform.

Example (1.1.6) [1]. Let $H = L^2(0, a)$,

$$Ly = iy', \quad \mathfrak{D}(\mathrm{L}) = \{y \in W_2^1(0, a): y(0) = 0\}.$$

We have

$$(Ly, y) = \int_0^a iy\bar{y}' + i|y(a)|^2 = (y, Ly) + i|y(a)|^2. \qquad (2)$$

Thus, the operator L is dissipative. It is easy to verify that the Cayley transform of the operator L is a truncated shift operator $M_\theta = P_\theta S|K_\theta, \theta = \exp(a(z+1)/(z-1))$. Here $K_\theta = H^2 \ominus \theta H^2$, P_θ is a projection onto $K_\theta, Sf = zf$. More exactly, if $\mathcal{F}$ is the Fourier transform in $L^2(\boldsymbol{R}), P_a$ is the projection from $L^2(\boldsymbol{R})$ onto $L^2(0, a)$, while W is a unitary operator, mapping

$L^2(\boldsymbol{T})$ into $L^2(\boldsymbol{R})$, $(Wf)(x) = \pi^{-\frac{1}{2}}(x+i)^{-1} f((x-i)/(x+i))$, then

$$M_\theta = W^{-1}\mathcal{F}P_a^*(L - iI)(L + iI)^{-1}P_a\mathcal{F}^{-1}W$$

The analogous operator in $L^2(0, \infty)$ has a Cayley transform that is unitarily equivalent to the shift operator S.

Semi group of Contractions. We set $W(\lambda) = i(1+\lambda)(1-\lambda)^{-1}$, ω: $D \to C_+$. Let L be a maximal dissipative operator and let T be its Cayley transform. If the operator $(g \circ \omega)(T)$ is defined in the Sz.-Nagy-Foia calculus, then one can set $g(L) \triangleq (g \circ \omega)(T)$. Since $1 \notin \sigma_p(T)$, we can apply to the operator T any function $\varphi \in H^\infty$ such that φ is continuous in clos $D\backslash\{1\}$ ([9, Section II.6], according to which $1 \notin \sigma_p(T) \Longrightarrow 1 \notin \sigma_p(U)$, where U is the minimal unitary dilation). Thus, to a maximal dissipative operator one can apply any function that is bounded and analytic in C_+ and continuous in clos C_+. In particular, we can take $g(z) = e^{itz}$. As a result we obtain the semigroup of contractions

$$\{e^{itL}\}_{t\geq 0} = \{Q_t(T)\}_{t\geq 0}, \quad Q_t = \exp\left(t\frac{z+1}{z-1}\right).$$

It is easy to verify that this semi group is strongly continuous.

LEMMA (1.1.7) [1] . Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous group of contractions, $T(0) = 1$. Then $T(t) = e^{itL}$ for some maximal dissipative operator L.

The operator L is called the generator, while $T = \mathcal{X}(L)$ is the cogenerator of the semigroup $\{T(t)\}$.

We give elementary formulas that connect a dissipative operator L and its Cayley transform T.

We set $\zeta = (\lambda - i)(\lambda + 1)^{-1}$. Then

$$(L - \lambda I)^{-1} = \frac{\xi - 1}{2i}(I - T)(\zeta I - T)^{-1},$$

$$(L^* - \lambda I)^{-1} = \frac{\xi - 1}{2i}(I - T^*)(I - \zeta T^*)^{-1}, \qquad (3)$$

There exist several different ways to define the characteristic function (c.f.). All of them are more or less equivalent, but the specific character of the initial operator may bring a definite advantage in the form in which the c.f. is written. Usually, the c.f. is an analytic operator-valued function, acting from one "defect space" into the other, while these latter measure the deviation of the operator from a unitary or from a self adjoint one.

If L is a maximal dissipative operator, then for the defect subspaces one can take $\mathfrak{D}_T = \operatorname{clos}(I - T^*T)^{\frac{1}{2}}H$, $\mathfrak{D}_{T^*} = \operatorname{clos}(I - TT^*)^{\frac{1}{2}}H$, , where $T = \mathcal{X}(L)$. These spaces can be given also a more explicit description in terms of the operator L**.** An operator A with domain of definition $\mathfrak{D}(A) \subset H$ (not necessarily dense) is said to be Hermitian, if

$$(Ax, x) \in \mathbb{R}, x \in \mathfrak{D}(A).$$

If $\operatorname{clos}\mathfrak{D}(A) = H$ then a Hermitian operator is said to be symmetric. The defect subspaces of a Hermitian operator are defined by the formulas

$$N_{\mp}(A) = H \ominus \operatorname{Ran}(A \pm iI).$$

We mention that the range $\operatorname{Ran}(A \pm iI)$ is closed if the operator A is closed since for a Hermitian operator we have $\|(A - \lambda I)x\| \geq |\Im\lambda|. \|x\|$.

Definition(1.1.8) [1] . Let L be a maximal dissipative operator. By Hermitian domain of L we mean subspace $G_L = \{x \in \mathfrak{D}(L) \cap \mathfrak{D}(L^*): Lx = L^*x\}$. By the Hermitian part of the operator L we mean the restriction $L_H = L|G_L$.

Clearly, the operator L_H is closed since its graph is the intersection of the graphs of the operators Land L^*.

Let T be a contraction. We recall that the operators $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - T^*T)^{1/2}$ are called the defect operators of the contraction T, while the subspaces $\mathfrak{D}_T = \operatorname{clos} D_T H$, $\mathfrak{D}_{T^*} = \operatorname{clos} D_{T^*}H$ are the defect subspaces.

LEMMA(1.1.9) [1] .

Let L be a maximal dissipative operator, $T = \mathcal{X}(L) = (L - iI)(L + iI)^{-1}$. Then $N_-(L_H) = D_T$, $N_+(L_H) = D_{T^*}$. We have the equalities

$$\frac{1}{2}D_T^2 = i(L + iI)^{-1} - i(L^* - iI)^{-1} - 2(L^* - iI)^{-1}(L + iI)^{-1},$$

$$\frac{1}{2}D_{T^*}^2 = i(L + iI)^{-1} - i(L^* - iI)^{-1} - 2(L + iI)^{-1}(L^* - iI)^{-1}, \quad (4)$$

Proof. Formulas (4) are easily verified in a straightforward manner. We prove that $D_T = N_-(L_H)$; the second equality is obtained with the aid of the substitution $L \to -L^*$. We have $D_T = H \ominus \ker D_T$. We show that $(L + iI)G_L = \operatorname{Ker} D_T = \operatorname{Ker} D_{T^2}$. From (4) there follows that

$$1/2\, D_T^2 = i[I - (L^* - iL)^{-1}(L - iI)](L + iI)^{-1}.$$

Consequently, if $x = (L + iI)y$, $y \in G_L$, then $D_T^2 x = 2i[I - (L^* - iI)^{-1}(L - iI)]y = 2i[y - (L^* - iL)^{-1}(L^* - iI)y] = 0$.

Conversely, if $x \in \operatorname{Ker} D_T^2$ and $y = (L + iI)^{-1}x$, then $y = (L^* - iI)^{-1}(L - iI)y$. But then $y \in D(L) \cap D(L^*)$, $L^*y = Ly$. The lemma is proved.

Definition(1.1.10) [1] . Let L be a maximal dissipative operator and let G_L be its Hermitian part. We consider the quotient space $D(L)/G_L$ and the natural projection $\rho: D(L) \to D(L)/G_L$. On the quotient space we define the inner product

$$\langle \rho x, \rho y \rangle = \frac{1}{2}\big((x, Ly) - (Lx, y)\big), \quad x, y \in D(L).$$

The correctness of the definition and the nondegeneracy of the form follows from the definition of the subspace G_L, while its positivity from the dissipativity of the operator L. By $F(L)$ we denote the completion of the quotient space $D(L)/G_L$ with respect to the corresponding norm. In a similar manner we define the space $F_*(L) \stackrel{\text{def}}{=} F(-L^*)$. By ρ_* we denote the projection from $D(L^*)$ onto $D(L^*)/G_L$. We have

$$\|\rho x\|_F^2 = \mathrm{Im}(Lx, x), \|\rho^* x\|_{F_*}^2 = -\mathrm{Im}(L^* x, x).$$

The Hilbert spaces $F(L), F_*(L)$ will be called the boundary spaces of the operator L.

Such spaces and closely related objects (spaces of boundary values) have been repeatedly considered in the literature [11-14]. As a rule, in these investigations one has taken an abstract axiomatic definition of the spaces of boundary values. In [14] it is mentioned that the "canonical" selection of these spaces, given above, is not always convenient. Nevertheless, we restrict ourselves to these.

The above defined spaces $F(L)$, $F_*(L)$ are especially convenient if the operator L has finite defects; in this case the completion is redundant. The term "boundary spaces" is connected with the fact that for differential operators they can be realized as the actual spaces of boundary values. This is illustrated by the following example.

Example(1.1.11). We consider the operator L from Example(1 .l.6):

$$Ly = iy', D(L) = \{y \in W_2^1(0, a): y(0) = 0\}$$

From (2) there follows that

$$L^* y = iy', D(L^*) = \{y \in W_2^1(0, a): y(a) = 0\}$$

$$G_L = \{y \in W_2^1(0, a): y(0) = y(a) = 0\}$$

Clearly, $F(L) \cong \mathbb{C}, F_*(L) \cong \mathbb{C}$. From (2) we have

$$\|\rho y\|_F^2 = \frac{1}{2}|y(a)|^2.$$

Identifying $F(L)$ with $\boldsymbol{C}$, we obtain that the projection ρcorresponds (to within a constant) to the calculation of the value at the point a. Similarly, ρ_* corresponds to the calculation of the value at zero (to within a constant).

LEMMA(1.1.12) [1] . Let L be a maximal dissipative operator and let $T = \mathcal{X}(L)$ Then there exist isometric isomorphisms $\rho: F(L) \to D_T, \rho_*: F_*(L) \to D_{T^*}$, defined by the equalities

$$\rho\rho(I - T) = D_T \ , \ \rho_*\rho_*(I - T^*) = D_{T^*} \qquad (5)$$

Proof. Taking into account that $L = i(I + T)(I - T)^{-1}$, we have

$$\begin{aligned}\|\rho(I - T)x\|_F^2 &= \mathrm{Im}(L(I - T)x, (I - T)x) = \mathrm{Im}\, i\big((I + T)x, (I - T)x\big) \\ &= \mathrm{Re}\big((I + T)x, (I - T)x\big) = ([(I - T)^*(I + T) + (I + T)^*(I - T)]x, x)/2 \\ &= ((I - T^*T)x, x) = \|D_T x\|^2.\end{aligned}$$

Consequently, equality (5) defines an isometry ρ. Since $\rho(I - T)H = \rho D(L) = D(L)/G_L$ is dense in $F(L)$, while DrH is dense in D_T ,we obtain what we intended to prove. Passing to the operator T^*, we obtain the assertion regarding ρ^*.

COROLLARY(1.1.13) [1] .

$$\dim D_T = \dim[(L - L^*)(D(L) \cap D(L^*))] + \dim(D(L)/D(L) \cap D(L^*)),$$
$$\dim D_{T^*} = \dim[(L - L^*)(D(L) \cap D(L^*))] + \dim(D(L^*)/D(L) \cap D(L^*)).$$

Proof . We verify the first equality. We have

$$\dim D_T = \dim F(L) = \dim(D(L)/D(L) \cap D(L^*)) + \dim(D(L) \cap D(L^*)/G_L)$$

It remains to make use of the fact that $\dim Ran\ (L - L^*) = \dim(D(L - L^*)/Ker\ (L - L^*))$.

Discussion(1.1.14). The corollary shows intuitively the two-fold reason for the appearance of defects at dissipative operators. It is natural to consider the "extreme" cases, i.e., the operators for which one of the terms vanishes. The first class of operators consists of those for which $\mathfrak{D}(L) = \mathfrak{D}(L^*)$; for them the defect indices are necessarily equal. The "defect" is defined by the imaginary part. A model example is the Schrodinger operator with a complex-valued potential and real boundary condition. The second class consists of those operators for which $G_L = \mathfrak{D}(L) \cap \mathfrak{D}(L^*)$. For them the "defect" is defined by the difference of the domains of definition of the operator and of the conjugate. A model example is the Schrodinger operator with a real-valued potential and complex boundary condition. We also note that in this class one has all the dissipative extensions of the symmetric operators. Indeed, for them G_L is dense in H. If $y \in \mathfrak{D}(L) \cap \mathfrak{D}(L^*)$, then $(Ly, x) = (y, L^*x) = (y, Lx) = (L^*y, x)$ for x from the dense set G_L. Thus,
$Ly = L^*y$, i.e., $\mathfrak{D}(L) \cap \mathfrak{D}(L^*) = G_L$.

A powerful device in the investigation of a dissipative operator is the characteristic function (c.f.). In a coordinate-free approach to a model, the c.f. is defined starting from functional imbeddings [15, 16J. This will be done , while here we wish only to derive some formulas for the c.f.

A completely non unitary contraction T is defined to within a unitary equivalence of the c.f. $\theta_T \in H^\infty(\boldsymbol{D}: E \to E_*)$ [16], where E, E_* are auxiliary Hilbert spaces, isomorphic to $\mathfrak{D}_T$, $\mathfrak{D}_{T^*}$ respectively. In turn, neither the c.f. θ_T is defined uniquely, but to within a multiplication on the right and on the left by an isometry from E onto E' and from E_* onto E'_*, respectively. We fix arbitrary isometric isomorphisms $\Omega: E \to \mathfrak{D}_T$, $\Omega_*: E_* \to \mathfrak{D}_{T^*}$. Then [9]

$$\theta_T(\zeta) = \Omega_*^*(-T + \zeta\mathfrak{D}_{T^*}(I - \zeta T^*)\mathfrak{D}_T)\Omega, \qquad (6)$$
$$\theta_T(\zeta)\Omega^*\mathfrak{D}_T = \Omega_*^*\mathfrak{D}_{T^*}(I - \zeta T^*)^{-1}(\zeta I - T). \qquad (7)$$

The characteristic function of a dissipative operator is the conformal transplant from the circle into the half-plane of the c.f. of its Cayley transform. It turns out that the various forms of the c.f., are obtained from the "abstract" c.f. if one selects E, E_* in a special manner. But first it is necessary to isolate the selfadjoint part of a dissipative operator.

Definition(1.1.15) [1]. Let L be a maximal dissipative operator in H and let $H_0 \subset H$ be a closed subspace. It is said to be reducing if $P_{H_0}L \subset LP_{H_0}$. The operator L is said to be completely nonselfadjoint if there is no nontrivial reducing subspace H_0 such that the restriction $L|H_0$ is a selfadjoint operator in H_0.

From the decomposition of a contraction into the unitary and the completely nonunitary parts, we obtain at once the following statement.

LEMMA (1.1.16) [1]. Let L be a maximal dissipative operator in H. Then there exists a unique decomposition $H = H_0 \oplus H_1$ of the space H into reducing subspaces such that $L_0 = L|H_0$ is selfadjoint and $L_1 = L|H_1$ is a completely nonselfadjoint maximal dissipative operator.

Definition(1.1.17) [1]. Let L be a completely non selfadjoint, maximal dissipative operator and let $T = \mathcal{X}(L)$. By the characteristic function of the operator L we mean the operator-valued function

$$S_L(\lambda) = \theta_T\left(\frac{\lambda - i}{\lambda + i}\right), \qquad S_L \in H^\infty(\mathbb{C}_+, E \to E_*).$$

In view of Lemma (1.1.12) we can set $\Omega = \rho$, $\Omega_* = \rho_*$, $E = F(L), E_* = F_*(L)$.

LEMMA(1.1.18) [1] . Let L be a completely nonselfadjoint, maximal dissipative operator. Then the characteristic function $S_L \in H^\infty(\mathbb{C}_+, F(L) \to F_*(L))$ on the sense set $\mathfrak{D}(L)/G_L$ is defined by the equality

$$S_L(\lambda) = \rho_*(L^* - \lambda I)^{-1}(L - \lambda I)\rho^{-1} \tag{8}$$

Proof. First we note that the operator $\rho_*(L^* - \lambda I)^{-1}(L - \lambda I)\rho^{-1}$ is well defined on $\mathfrak{D}(L)/G_L$. Indeed, if $\rho x = 0$, then $x \in G_L, Lx = L^*x$,

$$(L^* - \lambda I)^{-1}(L - \lambda I)x = (L - \lambda I)^{-1}(L^* - \lambda I)x = x, \quad \rho_* x = 0.$$

Further, making use of (2.3') and setting $\zeta = (\lambda - i)(\lambda + i)^{-1}$ on $\mathfrak{D}(L)/G_L$ we have

$$\mathcal{S}_L(\lambda)\rho = \theta_{T(\zeta)\Omega^*\rho}\rho = \theta_{T(\zeta)\Omega^*\mathfrak{D}_T}(I - T)^{-1} = \Omega_*^*\mathfrak{D}_{T^*}(I - \zeta T^*)^{-1}(\zeta I - T)(I - T)^{-1}$$
$$= \Omega_*^*\rho_*(I - T^*)(I - \zeta T^*)^{-1}(\zeta I - T)(I - T)^{-1}$$

From here, taking into account (3), we obtain equality (8).

Special Case(1.1.19): $\mathfrak{D}(L) = \mathfrak{D}(L^*)$. In this case one can consider the operator $Q(L - L^*)/2i$. It is expressed in terms of the Cayley transform, $T = \mathcal{X}(L)$ in the following manner [9, Section IX.4]:

$$Q = (I - T)^{-1}(I - TT^*)(I - T^*)^{-1},$$
$$Q = (I - T^*)^{-1}(I - T^*T)(I - T)^{-1} . \tag{9}$$

Since $(Qx, x) \geq 0$, $x \in \mathfrak{D}(Q), Q$ admits a positive extension (for example, the Friedrichs extension). We fix such an extension and we denote it also by Q. Then the operator $Q^{1/2}$ is defined. We set $\mathfrak{B}_Q = \text{clos}\, Q^{\frac{1}{2}}\mathfrak{D}(L)$.

LEMMA(1.1.20) [1]. Let $\mathfrak{D}(L) = \mathfrak{D}(L^*)$. Then there exist isometric isomorphisms $\mu: \mathfrak{B}_Q \to \mathfrak{D}_T$, $\mu_*: \mathfrak{B}_Q \to \mathfrak{D}_{T^*}$ defined by the equalities

$$\mu Q^{1/2}(I - T) = \mathfrak{D}_T , \mu_* Q^{1/2}(I - T^*) = \mathfrak{D}_{T^*} , \tag{10}$$

Proof From (9) we obtain that $(I - T^*)Q(I - T) = D_{T^2}$, from where $\left\|Q^{1/2}(I - T)x\right\|^2 = \|D_T x\|^2$. This means that equality (10) defines an isometry μ. In a similar manner we proceed with μ_*

The following obvious statement establishes the connection between, $\mathfrak{B}_Q$ and the boundary spaces.

LEMMA(1.1.21) [1]. Let $\mathfrak{D}(L) = \mathfrak{D}(L^*)$. Then $F(L) = F(L^*)$, $\rho = \rho_*$, $Q^{1/2}|\mathfrak{D}(L) = \mu^*\rho = \mu_*^*\rho_*$. Lemma (1.1.20) shows that for the characteristic function one can set $E = E_* = \mathfrak{B}_Q$ $\Omega = \mu, \Omega_* = \mu_*$.

LEMMA(1.1.22) [1]. Let L be a completely nonselfadjoint, maximal dissipative operator such that $\mathfrak{D}(L) = \mathfrak{D}(L^*)$. Then the c.f. $S_L = H^\infty(\mathbb{C}_+ : \mathfrak{B}_Q \to \mathfrak{B}_Q)$ is defined on the dense set $Q^{1/2}\mathfrak{D}(L)$ by the equality

$$\mathcal{S}_L(\lambda) = I + 2iQ^{1/2}(L^* - \lambda I)^{-1}Q^{1/2}. \qquad (11)$$

Proof. We set $\zeta(\lambda - i)(\lambda + i)^{-1}$. From (10), (2.3'), (3) we have

$$\begin{aligned}\mathcal{S}_L(\lambda)Q^{\frac{1}{2}}(I-T) &= \mathcal{S}_L(\lambda)\mu^* D_T = \theta_T(\zeta)\Omega^*\mathrm{D_T} = \Omega_*^*\mathfrak{D}_{T^*}(I-\zeta T^*)^{-1}(\zeta I - T)\\ &= \mu_*^*\mathfrak{D}_{T^*}(I-\zeta T^*)^{-1}(\zeta I - T) = Q^{\frac{1}{2}}(I-T^*)(I-\zeta T^*)^{-1}(\zeta I - T)\\ &= Q^{\frac{1}{2}}(L^*-\lambda I)^{-1}(L-\lambda I)(I-T) = Q^{\frac{1}{2}}(L^*-\lambda I)^{-1}(L^*-\lambda I + 2iQ)(I-T)\\ &= [I + 2iQ^{\frac{1}{2}}(L^*-\lambda I)^{-1}]Q^{\frac{1}{2}}(I-T).\end{aligned}$$

From here we obtain (11).

The characteristic function in the form (11) has appeared for the first time in a paper by M. S. Lipschitz [18]; see also [9, Sec IX.4].

LEMMA(1.1.23) [1]. Let L be a maximal dissipative operator in H, and let $\mathcal{Z}$ be a selfadjoint operator in a Hilbert space $\mathcal{X} \supset H$. The following statements are equivalent:

(a) $(L - \lambda I)^{-1} = P_H(\mathcal{Z} - \lambda I)^{-1}|H$, $\lambda \in \mathbb{C}_-$;

(b) $(L + iI)^{-n} = P_H(\mathcal{Z} + iI)^{-n}|H$, $n \geq 0$;

(c) $e^{iLt} = P_H e^{i\mathcal{L}t}|H$, $t > 0$;

(d) the operator $U = \mathcal{X}(\mathcal{Z})$ is a unitary dilation of the contraction $T = \mathcal{X}(L)$.

Proof. Since $\sigma(L) \subset clos\ \mathbb{C}_-$ the conditions (a), (b), (d) are, obviously, equivalent. Condition (c) follows from (d) since $e^{iLt} = \theta_t(T), e^{i\mathcal{Z}t} = \theta_t(U)$. The implication (c)⇒(a) follows from the known formula

$$i(L - \lambda I)^{-1} \int_0^\infty e^{-it\lambda}e^{itL}dt\ , \qquad \lambda \in \mathbb{C}_-.$$

A selfadjoint operator $\mathcal{Z}$, satisfying the conditions of the lemma, is called a selfadjoint dilation of the operator L. The dilation is minimal if Span$((\mathcal{Z} - \lambda I)^{-1}H, \lambda \in \mathbb{C}_-) = \mathcal{H}$. Clearly, also the minimality can be formulated by using, instead of the resolvent $(\mathcal{Z} - \lambda I)^{-1}$, the operator from the statements (b)-(d) of Lemma (1.1.21).

Discussion(1.1.24). From the existence of a unitary dilation of a contraction and from condition (d) of Lemma (1.1.21) there follows that a selfadjoint dilation exists. However, one would want to have an expression for the dilation in terms of a dissipative operator, The complication (in comparison with the unitary dilation) consists in the fact that, in general, the restriction $\mathcal{L}|H$ is not equal to the operator L. Frequently it occurs even that

$\mathfrak{D}(\mathcal{Z}) \cap H = \{0\}$. Nevertheless, there exist sufficiently simple and nice formulas for selfadjoint dilation, especially in particular cases. They can be found in the investigations of B. S. Pavlov [2, 3]; see also [4]. The general case has been considered by A. V. Kuzhel' and Yu. L. Kudryashov [5-7]. In all the mentioned investigations one presents at once a certain operator and then one verifies that it is selfadjoint dilation. We obtain these formulas, as well as some of their generalizations, in a natural manner, namely, by the Cayley transformation of a unitary dilation.

Let L be a maximal dissipative operator, let $T = \mathcal{X}(L)$ be its Cayley transform. Then the minimal unitary dilation U of the contraction T acts in the space $\mathcal{H}_T = G_* \oplus H \oplus G$,where G and G_* are U_- and U_-^*invariant, respectively [16]. One can select the following realization:

$$\mathcal{H}_T = H_-^2(\mathbb{D}, E_*) \oplus H \oplus H^2(\mathbb{D}, E); \quad U|G = \mathbb{Z}, \qquad U^*|G_* = \overline{\mathbb{Z}},$$

where E, E_* are isomorphic to $\mathfrak{D}_T, \mathfrak{D}_{T^*}$, respectively. We wish to obtain a selfadjoint dilation $\mathcal{Z}$, in the space

$\mathcal{H}_L = L^2(\mathbb{R}_-, E_*) \oplus H \oplus L^2(\mathbb{R}_+, E)$, $e^{it\mathcal{Z}}|L^2$ - is a shift by t.

For this it is necessary to perform

i) passage from the circle to the half-plane by a linear fractional transformation;

ii) the Fourier transform, i.e., passage to the translation representation of the semigroup;

iii) the Cayley transform.

This can be performed in a different order.

The starting point is the following formula for the minimal unitary dilation in $\mathcal{H}_T$ [15, 16]:

$$U = \begin{bmatrix} P_- z & 0 & 0 \\ A & T & 0 \\ c & B & z \end{bmatrix}, \tag{12}$$

Here $V: H \to G, V_*: H \to G_*$ are partial isometries with initial spaces $\mathfrak{D}_T, \mathfrak{D}_{T^*}$ and final spaces $G \ominus UG, G_* \ominus U^* G_*$, respectively.

We select as the free parameters of the dilation the isometries

$$\Omega: E \to \mathfrak{D}_T \quad , \quad \Omega_*: E_* \to \mathfrak{D}_{T^*}$$

Then $V = P_0^* \Omega^*$, $V_* = P_{-1}^* \Omega_*^*$,where $P_0: H^2(\boldsymbol{D}, \varepsilon) \to \varepsilon$, $P_{-1}: H_-^2(\boldsymbol{D}, \varepsilon) \to \varepsilon$ are the Fourier series of the zeroth and $(-1)$the coefficients, respectively. We set

$$(Wf) = \frac{1}{\sqrt{\pi}} = \frac{1}{x+i} = f\left(\frac{x-i}{x+i}\right). \tag{13}$$

The operator W is an isometric isomorphism of $H^2(\boldsymbol{D}, \varepsilon)$ onto $H^2(\boldsymbol{R}, \varepsilon)$ for any space e. Moreover, $WH^2(\boldsymbol{D}, \varepsilon) = H^2(\boldsymbol{C}_+, \varepsilon)$, $WH_-^2(\boldsymbol{D}, \varepsilon) = H^2(\boldsymbol{C}_-, \varepsilon)$. We denote by $\mathcal{F}$ the Fourier transform

$$(\mathcal{F}h) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} h(x) e^{ixz} dx. \tag{14}$$

The operator $\mathcal{F}$ acts unitarily in all of $L^2(\boldsymbol{R}, \varepsilon)$ and maps $L^2(\boldsymbol{R}_\pm, \varepsilon)$ into $H^2(\boldsymbol{C}_\pm, \varepsilon)$. Our aim is the computation of the operator $\mathcal{Z}$:

$$\mathcal{Z} = i(I + \tau^{-1}U\tau)(I - \tau^{-1}U\tau), \tau \overset{\text{def}}{=} \begin{bmatrix} W^{-1}\mathcal{F} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & W^{-1}\mathcal{F} \end{bmatrix} \quad (15)$$

LEMMA (1.1.25) [1]. (on the unitary dilation in the half-plane). The operator

$$U_1 \overset{\text{def}}{=} \begin{bmatrix} W & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & W \end{bmatrix} \cdot U \cdot \begin{bmatrix} W^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & W^{-1} \end{bmatrix}$$

is the space $H^2(\boldsymbol{C}_-, E_*) \oplus H \oplus H^2(\boldsymbol{C}_+, E)$ has the form

$$U_1 = \begin{bmatrix} I - 2i\Gamma_{-i} & 0 & 0 \\ -2i\sqrt{\pi} D_{T^*}\Omega_* h_{-i} & T & 0 \\ 2i(z + iI)^{-1}\Omega^* T^* \Omega_* h_{-i} & \pi^{-\frac{1}{2}}(z + iI)^{-1}\Omega^* D_T & (z + iI)(z + iI)^{-1} \end{bmatrix}. \quad (16)$$

Here h_{-i} denotes the evaluation functional at the point $(-i)$.

Proof. By $U_1(k, \ell)$ we shall denote the corresponding matrix element. From (12) and from the last formulas we have

$$U_1 = \begin{bmatrix} WP\_zW^{-1} & 0 & 0 \\ D_{T^*}\Omega_* P_{-1} W^{-1} & T & 0 \\ -WP_0^*\Omega^* T^* \Omega_* P_{-1} W^{-1} & WP_0^*\Omega^* D_T & WzW^{-1} \end{bmatrix}.$$

It is easy to see that $WzW^{-1} = (z - iI)(z + iI)^{-1}$ (the multiplication operator). We have

$$(W^{-1}f)(z) = f\left(i\frac{1+z}{1-z}\right) \cdot \frac{2\sqrt{\pi}i}{1-z}; \qquad P_{-1}g = zg(z)|_{z=\infty}, \qquad g \in H_-^2(\mathbb{D}, E_*).$$

Thus, $P_{-1}W^{-1} = -2\sqrt{\pi} i h_{-i}$. Further, it is clear that $WP_0^*: e \to \pi^{-1/2}(z + 1)^{-1}e$. Finally,

$$WP_{-1}zW^{-1} = WzW^{-1} - WP_0^* P_{-1} W^{-1} = (z - iI)(z + iI)^{-1} + WP_0^* 2\sqrt{\pi}\, i\, h_{-i},$$

$$WP_0^* 2\sqrt{\pi}\, i\, h_{-i} f = 2i(z + i)^{-1} f(-i).$$

Introducing all these formulas, we obtain (16) .

LEMMA(1.1.26) [1]. (on unitary dilation in a translation representation on a line). Operator

$$U_2 \overset{\text{def}}{=} \begin{bmatrix} \mathcal{F}^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{F}^{-1} \end{bmatrix} U_1 \begin{bmatrix} \mathcal{F} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{F} \end{bmatrix}$$

in the space $L^2(\boldsymbol{R}_-, E_*) \oplus H \oplus L^2(\boldsymbol{R}_+, E)$ has the form

$$U_2 = \begin{bmatrix} I - 2iK_- & 0 & 0 \\ -2i\sqrt{\pi} D_{T^*}\Omega_*(\cdot, e_{R_-}^x) & T & 0 \\ 2e_{R_+}^{-x}\Omega^* T^* \Omega_*(\cdot, e_{R_-}^x) & -2i\sqrt{\pi} i e_{R_+}^{-x}\Omega^* D_T & I - 2iK_+ \end{bmatrix}, \quad (17)$$

where

$$(k_- g_-)(x) = \int_{-\infty}^{x} g_-(t) e^{t-x} dt\,, \quad g_- \in L^2(\boldsymbol{R}_-, E_*),$$

$$(k_+ g_+)(x) = \int_{0}^{x} g_+(t) e^{t-x} dt\,, \quad g_+ \in L^2(\boldsymbol{R}_+, E).$$

The operators $K_\pm$ are operations of convolution on $\boldsymbol{R}$ with the function $e_{R_+}^{-x} \overset{\text{def}}{=} e^{-x}|\boldsymbol{R}_+$ By $(\cdot, f)$ we denote the inner product in the corresponding L^2 space.

Proof i) From (14) we find that $h_{-i}\mathcal{F}|L^2(\boldsymbol{R}_-, E_*) = (2\pi)^{-\frac{1}{2}}(\cdot, e_{R_-}^x)$. This computes, starting from (16), the matrix element $U_2(2,1)$.
ii) We have

$$\mathcal{F}^{-1}(z+i) = (2\pi)^{-1/2}\int_-^{+\infty}(z+i)^{-1}e^{-ixz}dz = \begin{cases} -(2\pi)^{\frac{1}{2}}ie^{-x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

From here we find $U_2(3,2)$. and also $U_2(3,1)$:

$$\mathcal{F}^{-1}2i(z+iI)^{-1}\Omega^*T^*\Omega_*h_{-i}\mathcal{F} = 2e_{R_+}^{-x}\Omega^*T^*\Omega_*\left(\cdot, e_{R_-}^x\right).$$

iii) We have $\mathcal{F}^{-1}(g_1g_2) = (2\pi)^{-\frac{1}{2}}\mathcal{F}^{-1}(g_1) * \mathcal{F}^{-1}(g_2)$ from where

$$\mathcal{F}^{-1}2i(z+iI)^{-1}\mathcal{F} = 2k, \qquad (kg)(x) = \int_{-\infty} g(t)e^{t-x}dt.$$

If $g = g_+ \in L^2(\boldsymbol{R}_+, E)$, then from here we obtain at once a formula for the matrix element $U_2(3,3)$. It remains to evaluate $U_2(1,1) = I - 2(k - i\mathcal{F}^{-1}(z+i)^{-1}h_{-i}, \mathcal{F})$. Let $g_- \in L^2(\boldsymbol{R}_-, E_*)$. We have

$$(kg_-)(x) = \begin{cases} \int_{-\infty}^x g_-(t)e^{t-x}dt, & x < 0, \\ e^{-x}\left(g_-(t), e_{R_-}^t\right), & x > 0. \end{cases}$$

Taking into account that $i\mathcal{F}^{-1}(z+i)^{-1}h_{-i}\mathcal{F} = e_{R_+}^{-x}(\cdot, e_{R_-}^x)$,we obtain the required formula.

It remains to perform the Cayley transform and one obtains the formula for the selfadjoint dilation. We note that this cannot be written in the form of a matrix since the decomposition into a direct sum need not be consistent with the domain of definition.

THEOREM(1.1.27) [1]. Let L be a maximal dissipative operator in the space $H, T = \mathcal{X}(L)$. Then its minimal selfadjoint dilation $\mathcal{Z}$, in the space $\mathcal{X}_L = L^2(\boldsymbol{R}_-, E_*) \oplus H \oplus L^2(\boldsymbol{R}_+, E_*)$ has the form

$$\mathcal{Z}\begin{bmatrix}\mathcal{V}_- \\ f \\ \mathcal{V}_+\end{bmatrix} = \begin{bmatrix} i\mathcal{V}_-' \\ i\left\{2(I-T)^{-1}\left[f - \frac{i}{\sqrt{2}}\mathfrak{D}_{T^*}\Omega_*\mathcal{V}_-(0)\right] - f\right\} \\ i\mathcal{V}_+' \end{bmatrix} \tag{18}$$

and the domain of definition $\mathfrak{D}(\mathcal{Z})$ is given by the conditions:

$$\mathcal{V}_- \in W_2'(\boldsymbol{R}_-, E_*) \ , \ \mathcal{V}_+ \in W_2'(\boldsymbol{R}_+, E_*); \tag{19}$$

$$f - \tfrac{i}{\sqrt{2}}\mathfrak{D}_{T^*}\Omega_*\mathcal{V}_-(0) \in (I-T)H = \mathfrak{D}(L); \tag{20}$$

$$\sqrt{2}i\,\mathfrak{D}_T(I-T)^{-1}\left(f - \tfrac{i}{\sqrt{2}}\mathfrak{D}_{T^*}\Omega_*\mathcal{V}_-(0)\right) = T^*\Omega_*\mathcal{V}_-(0) + \Omega\mathcal{V}_+(0). \tag{21}$$

The "free parameters" of the dilation are the isometries $\Omega: \mathrm{E} \to \mathfrak{D}_T$, $\Omega_*: \mathrm{E}_* \to \mathfrak{D}_{T^*}$.

Proof. We have $\mathcal{Z} = i(I+U_2)(I-U_2)^{-1} = i[2(I-U_2)^{-1} - I]$, where the operator U_2 is given by formula (17). Let $(1 - U_2)[g_-, g, g_+]^T = [v_-, f, v_+]^T$. We obtain the following system of equations:

$$\begin{cases} \int_{-\infty}^{x} g_-(t)e^{t-x}dt = v_-(x) \\ (I-T)g + i\sqrt{2}D_{T^*}\Omega_*(g_-, e_{R_-}^{x}) = f \\ 2\int_0^x g_+(t)e^{t-x}dt + i\sqrt{2}e_{R_+}^{-x}\Omega^* D_T g - 2e_{R_+}^{-x}\Omega^* T^*\Omega_*(g_-, e_{R_-}^{x}) = v_+(x) \end{cases}$$

From the first equation it follows that $v_-'(x) = 2g\_(x) - v\_(x), v\_W_2^1C(R\_, E_*)$. In addition, $v_-(0) = 2(g_-, e_{R_-}^{x})$. From the second equation we find that $\Phi \triangleq f - 2^{-1}/2iD_{T^*}\Omega_* v_0 \in \mathfrak{D}(L), g = (I-T)^{-1}\Phi$. From the third equation we obtain that $v_+'(x) = 2g_+(x) - v_+(x)$, from where $v_+ \in W_2^1(R_+, E)$ and

$$v_+(0) = \sqrt{2}i\Omega^* D_T(I-T)^{-1}\phi - \Omega^* T^*\Omega_* v_-(0).$$

Since$\Omega\Omega^* = P_{\mathfrak{D}_T}, T^*\mathfrak{D}_{T^*} \subset \mathfrak{D}_T$, we have verified that $\mathfrak{D}(\mathcal{Z})$ satisfies conditions (19)-(21). It is also obvious that if for v_+, v_-, f the conditions (19)-(21) are satisfied, then $[v_-, f, v_+]^T \in (I-U_2)\mathcal{H}_L$ it,. and the operator $\mathcal{Z}$is given by the equality (18) .

In the case of finite defects dim $\mathfrak{D}_T < \infty$, dim $\mathfrak{D}_{T^*} < \infty$ it is convenient to have the dilation" attached" , not to the spaces $\mathfrak{D}_T, \mathfrak{D}_{T^*}$, but to the boundary spaces F(L), F",(L) (see Definition 2.4).

THEOREM(1.1.28) [1]. Let L be a maximal dissipative operator in the space H with finite defects. Assume that there are given isometric isomorphisms $\psi: E \to F(L)$, $\psi_*: E_* \to F_*(L)$. Then the minimal selfadjoint dilation $\mathcal{Z}$ in the space $\mathcal{X}_L = L^2(R_-, E_*)\oplus H \oplus L^2(R_+, E)$ has the form

$$\wp \begin{bmatrix} v_- \\ f \\ v_+ \end{bmatrix} = \begin{bmatrix} iv_-' \\ L\left(f - \frac{i}{\sqrt{2}}[\psi_* v_-(0)]\right) + \frac{i}{\sqrt{2}}L^*[\psi_* v_-(0)] \\ iv_+' \end{bmatrix}, \quad (22)$$

where [·] denotes some representative of the quotient class mod G_L. Moreover, the domain of definition $\mathfrak{D}(\wp)$ of the dilation is defined by the condition (19) and also by the following two conditions:

$$f - \tfrac{i}{\sqrt{2}}[\psi_* v_-(0)] \in \mathfrak{D}(L), \quad (23)$$

$$f - \tfrac{i}{\sqrt{2}}[\psi_* v_-(0)] + f + \tfrac{i}{\sqrt{2}}[\psi v_+(0)] \in G_L. \quad (24)$$

Condition (23) can be replaced by the condition (3.12')

$$f + \tfrac{i}{\sqrt{2}}[\psi v_+(0)] \in \mathfrak{D}(L^*), \quad (25)$$

since it is easy to see that the pairs of conditions (23), (24) and (25), (24) are equivalent.

Proof In the case of finite defects we have $F(L) = \mathfrak{D}(L)/G_L$, $F_*(L) = \mathfrak{D}(L^*)/G_L$. In Theorem (1.1.27) we set $\Omega = \rho\psi$, $\Omega_* = \rho_*\psi_*$. From (5) we have

$$\mathfrak{D}_{T^*}\Omega_* v_-(0) = \mathfrak{D}_{T^*}^2(I-T^*)^{-1}\rho_*^{-1}v_-(0) = \mathfrak{D}_{T^*}^2(I-T^*)^{-1}[\psi_*, v_-(0)],$$

and the last expression does not depend on the representative$[\psi_*, v_-(0)] \in \mathfrak{D}(L^*)$. We set $Q \overset{\text{def}}{=} f - \frac{i}{\sqrt{2}}\mathfrak{D}_{T^*}\Omega_* v_-(0)$. We obtain that

$$Q = f - \frac{i}{\sqrt{2}}(I - TT^*)(I - T^*)^{-1}[\psi_*, v_-(0)]$$
$$= f - \frac{i}{\sqrt{2}}[\psi_*, v_-(0)] - \frac{i}{\sqrt{2}}(I - T)T^*(I - T^*)^{-1}[\psi_*, v_-(0)],$$

from where it follows that condition (20) is equivalent to (23). Then

$$(I - T)^{-1}Q = (I - T)^{-1}\left(f - \frac{i}{\sqrt{2}}[\psi_*, v_-(0)]\right) - \frac{i}{\sqrt{2}}T^*(I - T^*)^{-1}[\psi_*, v_-(0)]$$
$$= \frac{1}{2i}(L + iI)\left(f - \frac{i}{\sqrt{2}}[\psi_*, v_-(0)]\right) + \frac{1}{2i}(L^* + iI)^{-1}\frac{i}{\sqrt{2}}[\psi_*, v_-(0)].$$

Introducing this expression into (18), we obtain (22). It remains to rewrite condition (21):

$$\sqrt{2}i\mathfrak{D}_T(I - T)^{-1}Q = T^*\mathfrak{D}_{T^*}(I - T^*)^{-1}[\psi_*, v_-(0)] + \mathfrak{D}_T(I - T)^{-1}[\psi, v_+(0)]$$

is equivalent, taking into account the relation $T^*D_{T^*} = D_T T^*$, to the equality

$$\sqrt{2}i\mathfrak{D}_T(I - T)^{-1}f - \frac{i}{\sqrt{2}}[\psi_*, v_-(0)] + \frac{i}{\sqrt{2}}[\psi, v_+(0)] = 0.$$

The last equality is equivalent to the condition (2.13) since $\ker \mathfrak{D}_T(I - T)^{-1} = \ker \rho = G_L$.

G_L is dense in H, then , we have $G_L = \mathfrak{D}(L) + \mathfrak{D}(L^*)$. In this case the formulas (22)-(24) can be simplified somewhat. Namely, we consider the operator $\tilde{L}$, defined on $\mathfrak{D}(L) + \mathfrak{D}(L^*)$

$$\tilde{L} \stackrel{\text{def}}{=} (L|\mathfrak{D}(L) \cap \mathfrak{D}(L^*))^* = \begin{cases} L & on \quad \mathfrak{D}(L) \\ L^* & on \quad \mathfrak{D}(L^*) \end{cases}$$

Clearly, $L\left(f - \frac{i}{\sqrt{2}}[\psi_* v_-(0)]\right) + \frac{i}{\sqrt{2}}L^*[\psi_* v_-(0)] = \tilde{L}f$ and the pair of conditions (23), (24) is equivalent to the conditions (23), (3.12'). As a result we obtain the following

COROLLARY(1.1.29) [1]. Let L be a maximal dissipative operator with finite defects such that G_L is dense in H. Then its selfadjoint dilation has the form

$$\wp \begin{bmatrix} v_- \\ f \\ v_+ \end{bmatrix} = \begin{bmatrix} iv'_- \\ \tilde{L}f \\ iv'_+ \end{bmatrix}, \ \tilde{L} = (L/G_L)^*. \tag{26}$$

$$\begin{gathered} \mathfrak{D}(\wp) = \{[v_-, f, v_+]^T : v_- \in W_2'(R_-, E_*), v_+ \in W_2'(R_+, E), \\ f - \tfrac{i}{\sqrt{2}}[\psi_* v_-(0)] \in \mathfrak{D}(L), f + \tfrac{i}{\sqrt{2}}[\psi v_+(0)] \in \mathfrak{D}(L^*)\} \end{gathered} . \tag{27}$$

In the corollary we can reject the condition of the finiteness of the defects if we consider that formula (26) defines the dilation only on the essential selfadjointness set. The obtained dilation is already a direct generalization of B. S. Pavlov's formulas [2]. We mention that if L is a differential operator, then $\tilde{L}$ is given by the same differential expression as L.

In the model case of the Schrodinger operator with a bounded potential $q(x), q\mathfrak{I} \geq 0$, the dilation has been constructed by B. S. Pavlov [3]. In [4], S. N. Naboko generalizes this construction to the case when $L = A + iQ$ and the operator Q is strongly subordinate to A. We mention, however, that the formula for the dilation in [4] can be understood only as the definition of the operator on the essential selfadjointness set. We obtain this formula in the assumption that Q is bounded.

LEMMA(1.1.30) [1]. Under the assumptions of Theorem (1.1.31) we have

(i) $\mathfrak{D}_{T^*}|\mathfrak{D}_{T^*} = (I-T)Q^{\frac{1}{2}}\mu_*^*, \mathfrak{D}_{T^*}\mathfrak{D}_{T^*} \subset \mathfrak{D}(L)$;

(ii) $\mu_*^* = \left(Q^{\frac{1}{2}}\mathfrak{D}_{T^*} - \mu^* T^*\right)|\mathfrak{D}_{T^*}$.

Proof (i) We verify the equality; from it the required inclusion follows. The equality can be verified on a dense set since $Q^{\frac{1}{2}}$ is bounded. Let $x = \mathfrak{D}_{T^*}y$. Then, by formulas (9), (10) we have

$$(I-T)Q^{\frac{1}{2}}\mu_*^* x = (I-T)Q(I-T^*)y = (I-TT^*)y = \mathfrak{D}_{T^*}x.$$

(ii) Again we verify the equality on a dense set for $x = \mathfrak{D}_{T^*}y$. We have

$$\mu_*^* x = Q^{\frac{1}{2}}(I-T^*)y = Q^{\frac{1}{2}}\big((I-TT^*)y - (I-T)T^*y\big) = Q^{\frac{1}{2}}\mathfrak{D}_{T^*}x - \mu^*\mathfrak{D}_T T^* y$$
$$= \left(Q^{\frac{1}{2}}\mathfrak{D}_{T^*} - \mu^* T^*\right)x.$$

THEOREM(1.1.31) [1]. Let $L = A + iQ$, where $A = A^*$, $Q \geq 0$, and Q is bounded. Let $\mathcal{B}_Q = \operatorname{clos} Q^{1/2}H$. Let E, E_* be isomorphic to the space $\mathcal{B}_Q$ and assume that the isometric isomorphisms $\kappa: E \to \mathcal{B}_Q, \kappa *: E_* \to \mathcal{B}_Q$ have been fixed. Then the selfadjoint dilation $\wp$. in the space $L^2(R_-, E_*) \oplus H \oplus L^2(R_+, E)$ has the form

$$\wp \begin{bmatrix} v_- \\ f \\ v_+ \end{bmatrix} = \begin{bmatrix} iv'_- \\ Lf + \sqrt{2}Q^{\frac{1}{2}}x_* v_-(0) \\ iv'_+ \end{bmatrix}, \tag{28}$$

$$\mathfrak{D}(\wp) = \left\{[v_-, f, v_+]^T : v_- \in W_2'(R_-, E_*), v_+ \in W_2'(R_+, E), f \in \mathfrak{D}(L), \sqrt{2}iQ^{\frac{1}{2}}f = xv_+(0) - x_* v_-(0)\right\}. \tag{29}$$

Proof. We apply Theorem (1.1.27), setting $\Omega = \mu\kappa, \Omega_* = \mu_*\kappa_*$.

From Lemma (1.1.30) (i) there follows at once that condition (20) turns into the condition $f \in \mathfrak{D}(L)$, while the expression in the second row of formula (18) goes into $Lf + \sqrt{2}Q^{1/2}x_* v_-(0)$. We apply μ^* to the equality (21). We obtain

$$\sqrt{2}iQ^{1/2}f + Q^{\frac{1}{2}}\mathfrak{D}_{T^*}\mu_* x_* v_-(0) = \mu^* T^* \mu_* x_* v_-(0) + xv_+(0).$$

Taking into account Lemma (1.1.30) (ii), this is equivalent to the condition

$$\sqrt{2}iQ^{1/2}f = xv_+(0) - x_* v_-(0),$$

which is what we intended to prove. -

Definition(1.1.32) [1]. Let L be a maximal dissipative operator in H, let $\mathcal{Z}$ be its minimal selfadjoint dilation in the space $\mathcal{X} = G_* \oplus H \oplus G$, where G_* is the "incoming", while G is the "outgoing" subspace, i.e., $e^{i\mathcal{Z}t}$ for $t > 0$, $e^{i\mathcal{Z}t}G_*$ for $t < 0$. By analogy with the case of a contraction [1], [IS], by functional imbeddings we mean isometries such that

$$(\mathcal{Z} + iI)^{-1}\pi^{\mathbb{R}} = \pi^{\mathbb{R}}(z + iI)^{-1}, \qquad (\mathcal{Z} + iI)^{-1}\pi_*^{\mathbb{R}} = \pi_*^{\mathbb{R}}(z + iI)^{-1},$$
$$\pi^{\mathbb{R}}H^2(\mathbb{C}_+, E) = G, \quad \pi_*^{\mathbb{R}}H^2(\mathbb{C}_-, E_*) = G_*.$$

Under these conditions, $\pi^{\mathbb{R}}, \pi_*^{\mathbb{R}}$ are uniquely determined to within multiplications by unitary constants in E, E_*. Clearly, if $T = \mathcal{X}(L)$ is the Cayley transform, then

$$\pi^{\mathbb{R}} = \pi \circ W^{-1}, \quad \pi_*^{\mathbb{R}} = \pi_* \circ W^{-1}, \tag{30}$$

where π, π_* are functional models of the contraction T.

The operator $S = (\pi_*^{\mathbb{R}})^* \pi^{\mathbb{R}}$, acts from $L^2(\boldsymbol{R}, E)$ into $L^2(\boldsymbol{R}, E_*)$, maps $H^2(C_+, E)$ into $H^2(C_+, E_+)$, and commutes with the multiplication by $(z+i)^{-1}$. Consequently, S is multiplication by a function $S_L(\lambda) \in H^\infty(C_+, E \to E_*)$. It is called the characteristic function of the operator L. Besides, from (30) we find at once that

$$S_L(\lambda) = \theta_T\left(\frac{\lambda - i}{\lambda + \mathrm{i}}\right),$$

where θ_T is the characteristic function of the Cayley transform of the operator L, which conforms to Definition (1.1.15).

In its coordinate-free variant does not carry, basically, any new information: each simple maximal dissipative operator is unitarily equivalent to the model one and this latter is the Cayley transform of the model contraction.

However, of interest is the determination of formulas for the functional imbeddings $\pi^{\mathbb{R}}$, $\pi_*^{\mathbb{R}}$. In particular cases, operators that are conjugate to $\pi^{\mathbb{R}}, \pi_*^{\mathbb{R}}$, have arisen basically in several investigations (see, for example, [4]).

First one has to find the expressions for π, π_*, responding to the unitary dilation (12) in the space $\mathcal{X}_T = H_-^2(D, E_*) \oplus H^2 \oplus (D, E)$, where $T = \mathcal{X}(L)$. The following statement has been kindly communicated to me by V. 1. Vasyunin.

LEMMA (1.1.33) [1]. We have the equalities:

$$\pi = \begin{bmatrix} P_-\theta \\ P_0 z(I - zT^*)^{-1} D_T \Omega \\ P_+ \end{bmatrix}, \quad \pi_* = \begin{bmatrix} P_- \\ P_0 z(I - \bar{z}T)^{-1} D_{T^*} \Omega_* \\ P_+\theta^* \end{bmatrix}; \tag{31}$$

$$\pi^*[h_-, h, h_+]^T = \theta^* h_- + \bar{z}\Omega^* D_T (I - \bar{z}T)^{-1} h + h_+, \tag{32}$$

$$\pi_*^*[h_-, h, h_+]^T = h_- + \Omega_*^* D_{T^*}(I - zT^*)^{-1} h + \theta h_+.$$

Proof: We derive formulas (32); obviously, the equalities (31) follow from them. By the definition of π, π_* we have

$$\pi h_+ = [0,0,h_+]^T, h_+ \in H^2(D, E), \qquad \pi_* h_- = [h_-, 0,0]^T, h_- \in H_-^2(D, E_*).$$

It is known that $\theta = \pi_*^* \pi$, therefore,

$$\pi_*^*[0,0,h_+]^T = \theta h_+, \quad \pi^*[h_-, 0,0]^T = \theta^* h_-.$$

It remains to find $1\pi^*[0, h, 0]^T, h \in H$. Let $e \in E$, $e_* \in E_*$. We have $\pi \bar{z}^n e = U^{*n} \pi e, \pi_* z^n e_* = U^{n+1} \pi_*(\bar{z} e_*)$. From formula (12) it is easy to derive (see also (IS]) that

$$\pi \bar{z}^n e = [*, T^{*(n-1)} D_T \Omega e, 0]^T, \qquad \pi_* \bar{z}^n e_* = [0, T^n D_{T^*} \Omega_* e_*, *]^T.$$

Let $\{e_i\}$ be an orthonormal basis in E. Then

$$\pi^*[0, h, 0]^T = \sum_{i;n=1}^{\infty} (h, T^{*(n-1)} D_T \Omega e_i) \bar{z}^n e_i = \sum_i \left(\sum_{n=1}^{\infty} \bar{z}^n\, \Omega^* D_T T^{n-1} h, e_i \right) e_i$$

$$= \Omega^* D_T \sum_{n=1}^{\infty} \bar{z}^n\, T^{n-1} h = \bar{z} \Omega^* D_T (I - \bar{z}T)^{-1} h.$$

In a similar manner we obtain that $\pi_*^*[0,h,0]^T = \Omega_*^* D_{T^*}(I - zT^*)^{-1}h$.

COROLLARY (1.1.34) [1]. $h \to \bar{z}D_T(I - \bar{z}T)^{-1}h$ is a contraction operator from H into $H_-^2(\mathfrak{D}_T)$, $h \to D_{T^*}(I - zT^*)^{-1}h$ is a contraction operator from H into $H^2(\mathfrak{D}_{T^*})$.

LEMMA (1.1.35) [1]. Let L be a maximal dissipative operator in the space H; let $F(L)$, $F_*(L)$ be its boundary spaces, let $\psi: E \to F(L)$, $\psi_*: E_* \to F_*(L)$ be some isometries. Let π^R, π_*^R be the functional imbeddings, corresponding to the selfadjoint dilation in the space $\mathcal{H}_L = L^2(R_-, E_*) \oplus H \oplus L^2(R_+, E)$ with the parameters ψ, ψ_*. We have the following equalities (in them $\mathcal{F}$ is the Fourier transform, S is the operator of multiplication by the characteristic function):

$$(\pi^R)^*[h_-, h, h_+]^T = S^*\mathcal{F}h_- - \pi^{-\frac{1}{2}}\psi^*\rho(L - \bar{\lambda}I)^{-1}\mathcal{F}h_+, \tag{33}$$

$$(\pi_*^R)^*[h_-, h, h_+]^T = \mathcal{F}h_- - \pi^{-\frac{1}{2}}\psi_*^*\rho_*(L^* - \lambda I)^{-1}h + S\mathcal{F}h_+ \tag{34}$$

$$\pi^R f = \begin{bmatrix} \mathcal{F}^{-1}Sf|R_- \\ \dfrac{i}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{+\infty}[(L^* - \lambda I)^{-1}(L - \lambda I) - I]\,\rho^{-1}\psi f(\lambda)d\lambda \\ \mathcal{F}^{-1}f|R_+ \end{bmatrix}, \tag{35}$$

$$\pi_*^R f = \begin{bmatrix} \mathcal{F}^{-1}f|R_- \\ -\dfrac{i}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{+\infty}[(L - \bar{\lambda}I)^{-1}(L^* - \bar{\lambda}I) - I]\,\rho_*^{-1}\psi_* f(\lambda)d\lambda \\ \mathcal{F}^{-1}S^*f|R_+ \end{bmatrix} \tag{36}$$

Proof We verify formulas (33), (35); the other two are obtained in a similar manner. Thus $\pi^R = \begin{bmatrix} \mathcal{F}^{-1}W & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathcal{F}^{-1}W \end{bmatrix} \pi W^{-1}$ where $W: L^2(D, \varepsilon)$ is isometric isomorphism (12), while π is given by formula (31). Clearly, $\mathcal{F}^{-1}WP_+ = W^{-1}f = \mathcal{F}^{-1}P_+f = (\mathcal{F}^{-1}f)|R_+$ (here by P_+ we denote both projections $L^2(T, \varepsilon) \to H^2(T, \varepsilon)$, $L^2(R, \varepsilon) \to H^2(C_+, \varepsilon)$. From (31) we find that

$$\mathcal{F}^{-1}WP\_\theta W^{-1} = \mathcal{F}^{-1}P\_W\theta W^{-1} = \mathcal{F}^{-1}P\_S = (\mathcal{F}^{-1}S)|R_-.$$

Thus we have verified the first and the third rows of formula (35) and, at the same time, also the first and third terms of (33). Taking into account (32), we have

$$N_2(h) \stackrel{\text{def}}{=} (\pi^R)^*[0, h, 0]^T = W\pi^*[0, h, 0]^T = W\Omega^*\bar{z}D_T(I - \bar{z}T)^{-1}h.$$

Just as in Theorem (1.1.28), we set $\Omega = \rho\psi$, $\Omega_* = \rho_*\psi_*$. Let $\lambda = i(1 + z)(1 - z)^{-1}$; then $z = (\lambda - i)(\lambda + i)^{-1}$. From (12), (5) we find

$$N_2(h) = \frac{1}{\sqrt{\pi}(\bar{\lambda} + i)} \cdot \frac{\bar{\lambda} + i}{\bar{\lambda} - i}\Omega^* D_T(I - \bar{z}T)^{-1}h = \frac{1}{\sqrt{\pi}(\bar{\lambda} + i)}\psi^*\rho(I - T)(I - \bar{z}T)^{-1}h$$

Making use of (3), we obtain that $N_2(h) = -\pi^{-\frac{1}{2}}\psi^*\rho(L - \bar{\lambda}I)^{-1}h$.

It remains to compute the second row of formula (35). This can be done by making use of equality (31), but we find it from the definition of the conjugate operator. We have

$$(\pi^R f, h) = (f, (\pi^R)^*[0, h, 0]^T) = -\pi^{-\frac{1}{2}} \int_{-\infty}^{+\infty} (f(\lambda), \psi^* \rho (L - \bar{\lambda} I)^{-1} h) d\lambda.$$

The subsequent computations make sense for $(\lambda) \in C_-$ and f from a dense set of functions with values in $\psi^*(D(L)/G_L)$; see the remark to this lemma (if the defects are finite, then the second stipulation is not required). Denoting by $[\psi f]$ the representative of the quotient class mod G_L, with the aid of Definition (1.1.10) we have

$$\begin{aligned} \frac{2}{i} (f(\lambda), \psi^* \rho (L - \bar{\lambda} I) h) &= \frac{2}{i} \langle \psi f, \rho (L - \bar{\lambda} I)^{-1} h \rangle_F \\ &= ([\psi f], L(L - \bar{\lambda} I)^{-1} h) - (L[\psi f], (L - \bar{\lambda} I)^{-1} h) \\ &= ([\psi f], h) + \lambda([\psi f], (L - \bar{\lambda} I)^{-1} h) - (L[\psi f], (L - \bar{\lambda} I)^{-1} h) \\ &= -(((L^* - \lambda I)^{-1}(L - \lambda I) - I)[\psi f], h). \end{aligned}$$

Clearly, the last expression does not depend on the representative $[\psi f]$ and we obtain the equality (35) .

We assume that the imaginary part $Q = (L - L^*)/2i \geq 0$ makes sense. Then from Lemma (1.1.21) we obtain

$$\pi^{-\frac{1}{2}} \psi^* \rho (L - \lambda I)^{-1} = \pi^{-\frac{1}{2}} z^* Q^{\frac{1}{2}} (L - \lambda I)^{-1},$$

where $z: E \to \beta_Q$ is some isometry. Then

$$(\pi^R)^*[h_-, h, h_+]^T = S^* \mathcal{F} h_- - \pi^{-\frac{1}{2}} z^* Q^{\frac{1}{2}} (L - \lambda I)^{-1} h + \mathcal{F} h_+$$

Similarly,

$$(\pi_*^R)^*[h_-, h, h_+]^T = \mathcal{F} h_- - \pi^{-\frac{1}{2}} z_*^* Q^{\frac{1}{2}} (L^* - \lambda I)^{-1} h + S \mathcal{F} h_+$$

These operators (with $\kappa = \kappa_* = \text{id}$) and their intertwining properties have been applied by S. N. Naboko in [4]. The fact that the second terms are contracting functions forms the content of Theorem 1 of [4].

Since dilation has an absolutely continuous spectrum [9], one can talk only about generalized eigenfunctions (in the riggings). Formulas (35), (36) give the possibility to interpret the collections

$$\begin{bmatrix} S(\lambda) e^{-i\lambda\xi} d, \qquad \xi \in R_- \\ i\sqrt{2}[(L^* - \lambda I)^{-1}(L - \bar{\lambda} I) - I] \rho^{-1} \psi d \\ e^{-1\lambda\xi} d, \qquad \xi \in R_+ \end{bmatrix}, \qquad d \in E, \lambda \in R,$$

as "incoming" eigenfunctions, while the collections

$$\begin{bmatrix} e^{-1\lambda\xi} d_*, \qquad \xi \in R_- \\ -i\sqrt{2}[(L - \lambda I)^{-1}(L^* - \lambda I) - I] \rho_*^{-1} \psi_* d_* \\ S^*(\lambda) e^{-i\lambda\xi} d_*, \qquad \xi \in R_+ \end{bmatrix}, \qquad d_* \in E_*, \lambda \in R,$$

as "incoming" eigenfunctions. For differential operators, this statement can be given a precise meaning.

Let $\wp$ be minimal selfadjoint dilation in the space $\mathcal{H} = G_* \oplus H \oplus G$ let π^R, π_*^R be its corresponding functional imbeddings. Then $\mathcal{H}_+ = \text{Ran}\, \pi^R = \text{Span}\ (e^{i\wp t} G, t \in R), \mathcal{H}_- =$

Ran π_*^R = Span$(e^{i\wp t}G_*, t \in R)$, are subspaces reducing $\wp$. By the residual part of the dilation we mean operator $\wp|\mathcal{H}\oplus\mathcal{H}_-$ It and by the $*$-residual part we mean the operator $\wp|\mathcal{H}\oplus\mathcal{H}_+$.

Making use of the results of [1] for contractions and passing from the circle to the half-plane, we obtain the following formulas:

$$\pi^R - \pi_*^R S = \tau^R \Delta^R,\ \pi_*^R - \pi^R S^* = \tau_*^R \Delta_*^R, \tag{37}$$

where

$$\Delta^R = (I - S^*S)^{1/2}, \qquad \Delta_*^R = (I - SS^*)^{1/2},$$

while

$$\tau^R : \Delta^R L^2(R, E) \to \mathcal{H}\ , \qquad \tau_*^R : \Delta_*^R L^2(R, E_*) \to \mathcal{H}$$

are isometries such that

$$\tau^R (z + iI)^{-1} = (z + iI)^{-1}\tau^R\ ,\quad \tau_*^R (z + iI)^{-1} = (z + iI)^{-1}\tau_*^R,$$

$$\pi^R(\pi^R)^* + \tau_*^R(\tau_*^R)^* = I\quad ,\quad \pi_*^R(\pi_*^R)^* + \tau^R(\tau^R)^* = I.$$

Thus, the operators $(\tau^{\boldsymbol{R}})^*$, $(\tau_*^{\boldsymbol{R}})^*$ realize the spectral representation of the residual and the *-residual parts of a dilation.

Now we write down the formulas connected with a concrete form of the dilation (22). However, to express $\tau^{\boldsymbol{R}}, \tau_*^{\boldsymbol{R}}$, explicitly is difficult

LEMMA(1.1.36): [1].

$$\tau^{\boldsymbol{R}}\Delta^R f = \begin{bmatrix} 0 \\ i(2\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} [(L - \lambda I)^{-1}(L^* - \lambda I)(L^* - \lambda I)^{-1}(L - \lambda I) - I]\rho^{-1}\psi f(\lambda) d\lambda \\ \mathcal{F}^{-1}(\Delta^R)^2 f|R_+ \end{bmatrix},$$

$$(\Delta^R)^2 = -\psi^*\rho\left[\left(L - \bar{\lambda}I\right)^{-1}(L^* - \bar{\lambda}I)(L^* - \lambda I)^{-1}(L - \lambda I) - I\right]\rho^{-1}\psi;$$

$$\tau_*^R \Delta_*^R f = \begin{bmatrix} \mathcal{F}^{-1}(\Delta_*^R)^2 f|R_- \\ -i(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \left[(L^* - \lambda I)^{-1}(L - \bar{\lambda}I)\left(L^* - \bar{\lambda}I\right) - I\right]\rho_*^{-1}\psi_* f(\lambda) d\lambda \\ 0 \end{bmatrix}$$

$$(\Delta_*^R)^2 = -\psi_*^*\rho_*\left[(L^* - \lambda I)^{-1}(L - \lambda I)\left(L - \bar{\lambda}I\right)^{-1}(L^* - \bar{\lambda}I) - I\right]\rho_*^{-1}\psi_*.$$

The proof follows at once from the formulas (35), (4.5'), (8), (2.4'). The second rows of the formulas have to be understood again as the boundary values of functions in the upper half-plane.

Here we apply the results of the Schrodinger operator on the semi axis

$$\ell y = -y'' + q(x)y;\quad y'(0) = hy(0), \operatorname{Im} h > 0, \tag{38}$$

with a real-valued potential in the Weyl limit point case at infinity. All the results have been obtained by B. S. Pavlov [2] by the method of generalized eigen functions. In subsequent investigations of various authors [10, 19], the characteristic function of this operator has been computed also by other methods, including methods closely related to the one presented below.

We consider the operator $L_0 y = \ell y$ in $L^2(\mathbb{R}_+)$ with domain of definition $\mathfrak{D}(L_0) =$

$\{y \in C_0^\infty(\mathbb{R}_+): y(0) = y'(0) = 0\}$. It is assumed that the (real) potential is continuous on $[0, \infty)$. It is easy to verify the following properties.

LEMMA (1.1.37): [1].

(i) L_0 is symmetric;

(ii) $\mathfrak{D}(L_0^*) = \{y \in C^1(\mathbb{R}_+): y'$ is absolutely continuous, $-y + qy \in L^2(\mathbb{R}_+)\}$, $L_0^* = \ell y$;

(iii) The defect indices $n_\pm(L_0) \triangleq \dim\ker(L_0^* \mp iI)$ are equal among them and are equal either to unity or to two.

If $n_\pm(L_0) = 1$, then we say that the Weyl limit point case takes place (we shall write then $q \in (\ell.p.)$; otherwise, we have the limit circle case (see [20]). In the sequel we shall assume that $q \in (\ell.p.)$. A simple sufficient condition (see [20)) consists in the fact that for some differentiable function $M(x)$, such that $M(x) > 0, M'(x) \leq \text{const}\, M(x)^{3/2}$, we should have

$$q(X) \geq -M(X), \int_0^\infty (M(X))^{-1/2}\, dX = \infty.$$

LEMMA (1.1.38): [1]. Let $q \in (\ell.p.)$. We consider the operator

$$L_h \subset L_0^*, \qquad \mathfrak{D}(L_h) = \{y \in \mathfrak{D}(L_0^*): y'(0) = hy(0)\}.$$

Then

$(i)(L_h y, y) - (y, L_h y) = 2i\, Im\, h|y(0)|^2$, $L_{\bar{h}} = L_h^*$,

(ii) if $H \in \mathbb{R}$, then $L_h = L_h^*$. If $\mathfrak{J}h > 0$, then L_h is a maximal dissipative operator.

Let $q \in (\ell.p.)$. We denote by $\Psi(x, \lambda), \varphi(x, \lambda)$ the solutions of the equation $-y'' + qy = \lambda y$, $\lambda \notin \mathbb{R}$, such that

$$\Psi(0, \lambda) = -1, \qquad \varphi(0, \lambda) = 0\,,$$
$$\Psi'(0, \lambda) = 0, \qquad \varphi'(0, \lambda) = 1.$$

Since $\dim \operatorname{Ker}(L_0^* - \lambda I) = n_\pm(L_0)$, there is defined a unique function $\chi(x, \lambda) \in L^2(\mathbb{R})$ such that $\chi(0, \lambda) = -1$ and $-\chi'' + q\chi = \lambda\chi$. We can write

$$\chi(x, \lambda) = \Psi(x, \lambda) + m_\infty(\lambda)\varphi(x, \lambda).$$

The function $m_\infty(\lambda)$ is called the Weil function. It is known [21] that $m_\infty(\lambda)$ is analytic in $C\backslash\mathbb{R}$ and $\mathfrak{J}\, m_\infty(\lambda)/\mathfrak{J}\lambda < 0$.

We proceed to the analysis of the dissipative operator L_n

Since $L = L_n$ is a dissipative extension of a symmetric operator, we have $G_L = \mathfrak{D}(L) \cap \mathfrak{D}(L^*)$ .We have $F(L) = \mathfrak{D}(L)/G_L, F_*(L) = \mathfrak{D}(L^*)/G_L$. By Lemma (1.1.38) (i) we have

$$\|\rho y\|_F^2 = \operatorname{Im} h|y(0)|^2, \qquad \|\rho_* y_*\|_{F_*}^2 = \operatorname{Im} h|y_*(0)|^2.$$

We set $E = E_* = C^1$ and we define the isometric isomorphisms Ψ, Ψ_*:

$$\Psi: E \to F(L), \Psi(\mathrm{a}) = \rho \mathrm{y} : \mathrm{y} \in \mathfrak{D}(L_h), y(0) = a/\sqrt{\operatorname{Im} h}.$$

$$\Psi_*: E_* \to F_*(L), \Psi_*(\mathrm{a}) = \rho_* \mathrm{y} : \mathrm{y}_* \in \mathfrak{D}(L_{\bar{h}}), y_*(0) = a/\sqrt{\operatorname{Im} h}. \qquad (39)$$

We show that we have the formula [2]

$$S_{L_h}(\lambda) = \frac{m_\infty(\lambda) + h}{m_\infty(\lambda) + \bar{h}} \qquad (40)$$

Proof. From (8), setting $L = L_h$ we have

$$S(\lambda) = S_L(\lambda) = \Psi_*^* \rho_* (L^* - \lambda I)^{-1}(L - \lambda I) y \rho^{-1} \Psi,$$

where $\rho: \mathfrak{D}(L) \to F(L)$, $\rho_*: \mathfrak{D}(L^*) \to F_*(L)$ are natural projections. From (39) we obtain that $\rho^{-1}\Psi a \ni y \in \mathfrak{D}(L_h), y(0) = a/\sqrt{\operatorname{Im} h}$. We have

$$(L - \lambda I)y = -y'' + qy - \lambda y\,,\ (L^* - \lambda I)y = y_1,$$

where $-y'' + qy_1 - \lambda y_1 = -y'' + qy - \lambda y\,,\ y_1 \in \mathfrak{D}(L_{\bar{h}})$. Finally, $\Psi_*^* \rho_* y_1 = \sqrt{\operatorname{Im} h}\ y_1(0)$, i.e., $S(\lambda)a = y_1(0)/y(0)a$. We note that $u = y_1 - y$ is a solution of the homogeneous equation $u'' + qu = \lambda u$ from $L^2(\mathbb{R}_+)$. Consequently, $y_1 - y = B(\lambda)\chi(x, y)$, from where

$$(y_1 - y)'(0) = -m_\infty(\lambda)(y_1 - y)(0)$$

But since $y'(0) = hy(0)\,,\ y_1'(0) = \bar{h} y_1(0)$, we obtain that

$$y_1(0) = \frac{m_\infty(\lambda) + h}{m_\infty(\lambda) + \bar{h}} y(0),$$

and the assertion is proved.

We make use of Corollary (1.1.34)We interpret conditions $f - \frac{i}{\sqrt{2}}[\Psi_* \mathcal{V}_-(0)] \in \mathfrak{D}(L)$, $f + \frac{i}{\sqrt{2}}[\Psi \mathcal{V}_+(0)] \in \mathfrak{D}(L^*)$. Let $y \in \mathfrak{D}(L_h)$ $y(0) = \mathcal{V}_+(0)/\sqrt{\operatorname{Im} h}$, $y_* \in \mathfrak{D}(L_{\bar{h}})$, $y_*(0) = \mathcal{V}_-(0)/\sqrt{\operatorname{Im} h}$. We have

$$f - \frac{i}{\sqrt{2}}[\Psi_* \mathcal{V}_-(0)] \in \mathfrak{D}(L_h) \Rightarrow f'(0) - \frac{i}{\sqrt{2}} y_*'(0) = hf(0) - \frac{i}{\sqrt{2}} h y_*(0)$$

$$\Leftrightarrow f'(0) - hf(0) = \frac{i}{\sqrt{2}}(\bar{h} - h) y_*(0) = \sqrt{2\operatorname{Im} h}\ \mathcal{V}_-(0).$$

Similarly,

$$f + \frac{i}{\sqrt{2}}[\Psi \mathcal{V}_+(0)] \in \mathfrak{D}(L_{\bar{h}}) \Rightarrow f'(0) + \frac{i}{\sqrt{2}} y'(0) = \bar{h} f(0) + \frac{i}{\sqrt{2}} \bar{h} y(0)$$

$$\Leftrightarrow f'(0) - \bar{h} f(0) = \frac{i}{\sqrt{2}}(\bar{h} - h) y(0) = \sqrt{2\operatorname{Im} h}\ \mathcal{V}_+(0).$$

As a result we obtain the following form of dilation $\wp$ of operator L_n in space $\mathcal{H} = \mathfrak{D}_- \oplus L^2(\boldsymbol{R}_+) \oplus \mathfrak{D}_+$, $\mathfrak{D}_\pm = L^2(\boldsymbol{R}_\pm)$ [2]:

$$\wp = \begin{bmatrix} \mathcal{V}_- \\ f \\ \mathcal{V}_+ \end{bmatrix} = \begin{bmatrix} i\mathcal{V}_-' \\ -f + qf \\ i\mathcal{V}_+ \end{bmatrix},$$

$$\mathfrak{D}(\wp) = \{[\mathcal{V}_-, f, \mathcal{V}_\pm]^T : \mathcal{V}_\pm \in (\boldsymbol{R}_+), f'(0) - hf(0) = \sqrt{2\operatorname{Im} h}\ \mathcal{V}_-(0), f'(0) - \bar{h} f(0) = \sqrt{2\operatorname{Im} h}\ \mathcal{V}_+(0)\}. \qquad (41)$$

"Incoming" and "Outgoing" Eigen functions of Dilation Calculations (with the same notations) yield

$$[(L^* - \bar{\lambda}I)^{-1}(L - \bar{\lambda}) - I]\rho^{-1}\Psi_a = B(\lambda)\chi(x,\lambda),$$
$$B(\lambda) = y(0) - y_1'(0) = \frac{\bar{h} - h}{m_\infty(\lambda) + \bar{h}} y(0) = \frac{i\sqrt{2\mathrm{Im}\, ha}}{m_\infty(\lambda) + \bar{h}}.$$

We obtain the "outgoing" eigen functions [2]:

$$\mathcal{V}_\lambda^- = \begin{bmatrix} S(\lambda)e^{-i\lambda\xi}, \xi \in \boldsymbol{R}_- \\ \sqrt{2\mathrm{Im}\, h}(m_\infty(\lambda) + \bar{h})^{-1}\chi(x,\lambda), x \in \boldsymbol{R}_+ \\ e^{-i\lambda\xi}, \xi \in \boldsymbol{R}_+ \end{bmatrix}.$$

In a similar manner one finds the "incoming" eigenfunctions:

$$\mathcal{V}_\lambda^+ = \begin{bmatrix} e^{-i\lambda\xi}, \xi \in \boldsymbol{R}_- \\ \sqrt{2\mathrm{Im}\, h}(\overline{m_\infty(\lambda)} + \bar{h})^{-1}\overline{\chi(x,\lambda)}, x \in \boldsymbol{R}_+ \\ S^*(\lambda)e^{-i\lambda\xi}, \xi \in \boldsymbol{R}_+ \end{bmatrix}.$$

The dilation $\wp$ is a differential operator and $v_\lambda^\pm$ are "actual" smooth functions, although not from L^2. Lemma (1.1.35) shows that the integral operators with the kernels $v_\lambda^\pm$ and factor $(2\pi)^{-1/2}$ are $\pi^{\boldsymbol{R}}$, $\pi_*^{\boldsymbol{R}}$, i.e., isometries from $L^2(\boldsymbol{R})$ into $\mathcal{H}_L$. In terms of functional imbeddings, the fact that $v_\lambda^\pm$ are eigenfunctions means the commutativity of $(\wp - \lambda I)^{-1}$ with the multiplication by $(z-\lambda)^{-1}$, the normalization and the inversion formula [2] reduce to isometricity. The terms "outgoing" and "incoming" eigenfunctions mean that

$$\mathrm{Ran}\, \mathcal{J}^{\boldsymbol{R}} = \mathrm{Span}\,(e^{it\wp}\mathfrak{D}_+, t \in R), \mathrm{Ran}\, \mathcal{J}_*^R = \mathrm{Span}\,(e^{it\wp}\mathfrak{D}_-, t \in R)$$

From (6) we have

$$(\Delta^{\boldsymbol{R}})^2 = (\Delta_*^{\boldsymbol{R}})^2 = 1 - |S(\lambda)|^2 = 2i\mathrm{Im}\, h(m_\infty(\lambda) - \overline{m_\infty(\lambda)})\left|m_\infty(\lambda) + \bar{h}\right|^{-2}$$

The spectrum of the residual and *-residual parts is set $\{\lambda \in \boldsymbol{R}: m_\infty(\lambda) \notin \boldsymbol{R}\}$. We apply Lemma (1.1.36). Let $y \in \mathfrak{D}(L_h), y_* \in \mathfrak{D}(L_{\bar{h}})$ .The equalities

$$(L^* - \lambda I)^{-1}(L - \lambda I)y = y - y(0) \cdot 2i\mathrm{Im}\, h(m_\infty(\lambda) + \bar{h})^{-1}\chi(x,\lambda)$$
$$(L - \bar{\lambda}I)^{-1}(L^* - \bar{\lambda}I)y_* = y_* + y_*(0) \cdot 2i\mathrm{Im}\, h(\bar{m}_\infty(\lambda) + \bar{h})^{-1}\overline{\chi(x,\lambda)}$$

from where

$$[(L - \bar{\lambda}I)^{-1}(L^* - \bar{\lambda}I)(L^* - \lambda I)^{-1}(L - \lambda I) - I]y =$$
$$y(0) \cdot 2i\mathrm{Im}\, h\left|m_\infty(\lambda) + \bar{h}\right|^{-2}\left(\overline{\chi(x,\lambda)}(m_\infty(\lambda) + h) - \chi(x,\lambda)(\overline{m_\infty(\lambda)} + h)\right).$$

We have $\chi(x,\lambda) = \varphi(x,\lambda)m_\infty(\lambda) + \Psi(x,\lambda)$, and, moreover, from the definition it is clear that '$\varphi(\bar{\lambda}) = \overline{\varphi(x,\lambda)}, \Psi(x,\bar{\lambda}) = \overline{\Psi(x,\lambda)}$. From here it follows that

$$\lim_{\mathrm{Im}\,\lambda \to 0}\left(\overline{\chi(x,\lambda)}(m_\infty(\lambda) + h) - \chi(x,\lambda)(\overline{m_\infty(\lambda)} + h)\right) = (\Psi - h\varphi)(m_\infty(\lambda) - \overline{m_\infty(\lambda)}).$$

Thus, we obtain an expression for $\tau^{\boldsymbol{R}}$:

$$\tau^{\boldsymbol{R}}\Delta^{\boldsymbol{R}} f = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2\pi i}}\int_{-\infty}^{+\infty}\frac{h\varphi(x,\lambda) - \Psi(x,\lambda)}{\sqrt{2\mathrm{Im}\, h}}(1 - |s(\lambda)|^2)d\lambda \\ \mathcal{F}^{-1}(1 - |s(\lambda)|^2)f(\lambda)|\boldsymbol{R}_+ \end{bmatrix}.$$

In a similar manner one derives the formula

$$\tau_*^{\boldsymbol{R}}\Delta_*^{\boldsymbol{R}} f = \begin{bmatrix} \mathcal{F}^{-1}(1-|s(\lambda)|^2)f(\lambda)|\boldsymbol{R}_- \\ \frac{1}{\sqrt{2\pi i}}\int_{-\infty}^{+\infty}\frac{\bar{h}\varphi(x,\lambda)-\Psi(x,\lambda)}{\sqrt{2\operatorname{Im} h}}(1-|s(\lambda)|^2)d\lambda \\ 0 \end{bmatrix}.$$

If we consider the operators $\tau^{\boldsymbol{R}}\Delta^{\boldsymbol{R}}, \tau_*^{\boldsymbol{R}}\Delta_*^{\boldsymbol{R}}$ as integral operators with respect to the measure $(1-|s(\lambda)|^2)d\lambda$, then we obtain the eigenfunctions of the continuous spectrum, found by B. S. Pavlov [2]. For the residual part we have

$$\mathcal{V}_\lambda^{>} = \begin{bmatrix} 0 \\ \varphi_h(\lambda) \\ e^{-i\lambda\xi} \end{bmatrix}, \qquad \begin{cases} -\varphi_h'' + q\varphi_h - \lambda\varphi_h = 0 \\ (\varphi_h' - h\varphi_h)|_0 = 0 \\ (\varphi_h' - \bar{h}\varphi_h)|_0 = \sqrt{2\operatorname{Im} h} \end{cases}$$

and for the *-residual part

$$\mathcal{V}_\lambda^{<} = \begin{bmatrix} e^{-i\lambda\xi} \\ \varphi_{\bar{h}}(\lambda) \\ 0 \end{bmatrix}, \qquad \begin{cases} -\varphi_{\bar{h}}'' + q\varphi_{\bar{h}} - \lambda\varphi_h = 0 \\ (\varphi_{\bar{h}}' - \bar{h}\varphi_{\bar{h}})|_0 = 0 \\ (\varphi_{\bar{h}}' - \bar{h}\varphi_{\bar{h}})|_0 = \sqrt{2\operatorname{Im} h}. \end{cases}$$

Section (1.2): Difference of Functions, From the Pick Class ,of Accretive Operators

The estimates (in the operator norm) was obtained in [23] by Matsaev and Palant; namely, the inequality

$$\|T^\alpha - (T')\| \le 2^{1-\alpha}\sin\pi\alpha/\pi\alpha(1-\alpha)\,\|T-T'\|^\alpha, 0 < \alpha < 1, \tag{42}$$

was proved for two bounded dissipative ($\operatorname{Im} T, T' \ge 0$) operators T and T' in a Hilbert space H .

On the other hand, for the case of self adjoint positive operators A and B ($\mathfrak{D}(A) = \mathfrak{D}(B)$, $(A-B)$ is bounded) in [24] by Birman and Solomyak the estimate

$$\|A^\alpha - B^\alpha\|_\sigma \le \|(|A-B|)^\alpha\|_\sigma, \quad 0 < \alpha \le 1, \tag{43}$$

was obtained, where σ is an arbitrary symmetrically-normed (s.n.) ideal [3J possessing the domination property. Furthermore, in [24] estimates in the quasi-normed classes of power decrease of the s-numbers were obtained. The results of have been extended by [26] to the case of two maximal dissipative (accretive) operators. Recall that a densely defined operator L in H is called maximal accretive (m.a.o.) [27] if $\operatorname{Re}(Lh,h) \ge 0, \forall h \in \mathfrak{D}(L)$ and the left half-plane $\lambda < 0$ consists entirely of regular points of the operator L. Since an arbitrary maximal dissipative operator [27] is distinguished from a maximal accretive operator only by the multiplier (i), analogous results will be valid for a pair of maximal dissipative operators as well. The choice of the accretive case is motivated only by some notational convenience.

In various problems of perturbation theory a demand arises for obtaining analogous estimates for a wider class of functions $(\varphi(T) - \varphi(T'))$ of operators. However, the extension of the results of [23,24,26]to a case of more general functions has necessitated the application of a technique different from that in [24,26]. Which has an auxiliary character, a class of admissible functions of m.a.o. is introduced, which essentially coincides with the well-known Pick class [28] (so-called operator-monotone functions), and a transformation of functions $\varphi \to \check{\varphi}$ is considered. elementary estimates for the s-numbers of functions of m.a.o. and for a difference of such functions in the operator norm are obtained. Inequalities in s.n. ideals which extend (43) to the case of m.a.o. and the functions of class G are proved , and their right-hand members contain the σ-norm of $\check{\varphi}(|T - T'|)$. The results for quasi-normed classes are briefly . The selected ones are problems, important for applications, on the boundary behavior of the nuclear-valued operator $\boldsymbol{R}$-functions [29,30] , and also the relation of the topics of the theory of Volterra operators [31] is illustrated.

There also should be pointed out [32, 33], where estimates [24] are obtained for a difference of functions of self-adjoint operators, and [34].

We define class as the set of functions $\varphi(\lambda)$, analytic in $\boldsymbol{C}\backslash\boldsymbol{R}_-$ and admitting there the representation

$$\varphi(\lambda) = \int_0^\infty \left(\frac{1}{t+\lambda} - \frac{1}{t}\right) dv(t), \qquad \lambda \notin (-\infty, 0], \tag{44}$$

where dv is some complex Borel measure satisfying at 0 and ∞ the conditions

$$\int_0^1 \frac{|dv(t)|}{t} < \infty, \qquad \int_1^\infty \frac{|dv(t)|}{t^2} < \infty. \tag{45}$$

We recall that the Nevanlinna-Pick class consists of the functions analytic in $\boldsymbol{C}_+$ and having there a nonnegative imaginary part. By the Riesz-Herglotz theorem [35], they admit the integral representation

$$f(\lambda) = \alpha + \beta\lambda - \int_{\boldsymbol{R}} \left[\frac{1}{t+\lambda} + \frac{1}{t^2+1}\right] dv(t),$$

Where = $\bar{\alpha}$, and $\beta \geq 0$,=$dv(t) \geq 0$ is a Borel measure on $\boldsymbol{R}$ such that $\int_0^\infty dv(t)/(t^2+1) < \infty$. The Pick class [28] satisfies an additional condition: the support of the measure $dv(t)$ is concentrated on the positive semi-axis. As is well known [28], $P(0,\infty)$ is exactly the same as the class of functions for which the implication$0 \leq A \leq B \Longrightarrow f(A) \leq f(B), A, B \in B(H)$, holds on the set of positive operators (operator-monotone functions). Since in this case f is analytic in $\boldsymbol{C}\backslash\boldsymbol{R}_-$ and monotonically increasing on the positive semi-axis, the additional condition of the finiteness of the quantity $f(+0)$ is equivalent to the condition $\int_0^\infty \frac{dv(t)}{t(t^2+1)} < \infty$ on the measure dv, which coincides with (45). Setting $\beta = 0$ and $\alpha = \int_0^\infty \frac{dv(t)}{t(t^2+1)}$, we obtain representation (44). Thus, class G coincides, to within a linear addend (for which a separate estimation is more convenient), with the class of functions from $P(0,\infty)$satisfying the additional condition $f(+0) = 0$. And we drop the requirement of the positiveness of the measure dv, which is not essential anyway.

A function $\varphi(T)$, where T is a m.a.o., can be defined on $\mathfrak{D}(T)$ with the aid of formula (44):

$$\varphi(T)h \stackrel{\text{def}}{=} \int_0^\infty ((T+t)^{-1} - I \cdot t^{-1})h \, dv(t), h \in \mathfrak{D}(T), \tag{46}$$

where the integral obviously converges in the norm of the space H due the estimate $\|(T+t)^{-1}\| \le 1/t$,$t>0$. For our goals it suffices to define $\varphi(T)$ on $\mathfrak{D}(T)$ by (46), as in the sequel estimations will be carried out for the s-numbers of the difference $\left(\varphi(T)-\varphi(T')\right)$ of the functions of twom.a.o. Clearly, the information about $\varphi(T)$ only on the dense set $\mathfrak{D}(T) = \mathfrak{D}(T')$ suffices for this. To compare this definition with other ways to define functions of m.a.o. [36] and selfadjoint operators, we present without proof some simple assertions.

Proposition (1.2.1)[22]. Let $\in G$, $\sup_{|\lambda|=r}|\varphi(\lambda)| \underset{r\to 0}{\longrightarrow} 0$, $\sup_{|\lambda|=R}|\varphi(\lambda)|/R \underset{R\to\infty}{\longrightarrow} 0$.Then:

i) if T is a bounded accretive operator such that $0 \notin \sigma(T)$, the function $\varphi(T)$ defined above coincides with the function of T defined with the aid of the Riesz integral of the resolvent [37] along the contour surrounding the spectrum $\sigma(T)$;

ii) if T is a m.a.o., the equality

$$\varphi(T) = P_H\varphi(-i\mathcal{L})|_{\mathfrak{D}(T)}$$

holds, where $\mathcal{L}$ is a selfadjoint dilation of the operator (iT) and P_H is the orthoprojector onto H in the Hilbert space $\mathcal{H} \subset H$ in which the operator $\mathcal{L}$ is defined [5J; furthermore

$$\varphi(T)(T+i)^{-1} = P_H\varphi(-i\mathcal{L})(-i\mathcal{L}+t)^{-1}|_H, \quad \operatorname{Re} t > 0;$$

iii) if $T = T^* \ge 0$,$\varphi(T)$ coincides with the restriction to $\mathfrak{D}(T)$ of the function $\varphi(T)$ defined by the spectral theorem for selfadjoint operators.

Class G can easily be extended to a class G^1 of analytic functions which might have, besides $\boldsymbol{R}_-$, other singularities, lying in a compact portion of $\{\lambda \in \boldsymbol{C}: \operatorname{Re}\lambda < 0\}$. In this case formula (44) should be modified: it must include, besides the integral along $\boldsymbol{R}_+$in the neighborhoods of 0, and$(+\infty)$, also an integral along a suitable circumference$\{t \in \boldsymbol{C}: |t-\varepsilon-R| = R\}$, $\varepsilon > 0, R > 0$, of some finite measure on that circumference. Such a formula arises in a natural way in the construction of functions with the aid of the Riesz integral of a relolvent. In the process, the new addend results from traversing the suitable circumference enclosing the singularities of φ in the left half-plane. Note that in the sequel this new addend is taken into account by means of rough estimates and introduced without any particular difficulties.

In conclusion, consider the transformation of functions from class G

$$\varphi \in G \to d\nu(t) \to \check{\varphi} \in L_\infty(\boldsymbol{R}_+:(x+1)^{-1}),$$

which will be used in the sequel and which carries $\varphi \in G$ into the nonnegative function on $\boldsymbol{R}_+$, computed by the formula

$$\check{\varphi}(c) \overset{\text{def}}{=} 2\left(\int_{(0,c/2)}\frac{|d\nu(t)|}{t} + c/2\int_{[c/2,\infty)}\frac{|d\nu(t)|}{t^2}\right), c > 0. \qquad (47)$$

The case of an absolutely continuous measure $d\nu(t) \equiv \psi(t)$ the transformation $\psi \to \check{\varphi}$, where ψ is the jump of the function $\varphi(-\lambda)/2\pi$ over the slit $\boldsymbol{R}_+$, is equivalent to the transformation of the modulus of continuity of the function under the Hilbert transformation [38]. This transformation easily generalizes to the case of $\varphi \in G^1$. It can easily be verified that $\check{\varphi}$ is a nonnegative ($\check{\varphi}(+0) = 0$), nondecreasing convex function on $\boldsymbol{R}_+$and a majorant forφ: $|\varphi(c)| \leq \breve{\varphi}(c)$, $c \geq 0$.Furthermore $\breve{\varphi}(x+y) \leq \breve{\varphi}(x) + \breve{\varphi}(y)$; $\breve{\varphi}(\alpha c) \leq \alpha\breve{\varphi}(c)$, $\alpha \geq 1$. A direct computation shows that for $\varphi(\lambda) = \lambda^{\alpha}(\ln(-1/\lambda))\beta$,$0 < \alpha < 1$,$\beta \in R(\varphi \in G^1)$ the asymptotic $\breve{\varphi}(c) \sim 2^{1-\alpha} \times (\sin \pi\alpha/\pi\alpha(1-\alpha)) \cdot c^{\alpha}(\ln 1/c)^{\beta}$, $c \to 0$ holds. An immediate generalization of estimate (42) is the following assertion:

THEOREM (1.2.2)[22]. Let T and T' be m.a.o., $\mathfrak{D}(t) = \mathfrak{D}(T')$, $(T-T') \in B(H)$, $\varphi \in G$. Then

i) $\|\varphi(T) - \varphi(T')\| \leq \breve{\varphi}(\|T-T'\|)$, (48)

ii) for $T \in \sigma_\infty$ we have $\varphi(T) \in \sigma_\infty$, and for the s-numbers the estimate

$$s_{4n-1}(\varphi(T)) \leq 2\breve{\varphi}(s_n(T)), \quad n = 1,2,\ldots$$

holds.

Proof. is carried out in analogy to (42), where the case $\varphi(\lambda) = \lambda^{\alpha}$, $0 < \alpha < 1$ was considered. We have $(\delta > 0)\|\varphi(T) - \varphi(T')\| \leq \left\|\int_{(0,\delta)}[(T+t)^{-1} - (T'+t)^{-1}]dv(t)\right\| + \left\|\int_{[\delta,\infty)}(T+t)^{-1}(T-T')(T'+t)^{-1}dv(t)\right\| \leq \int_{(0,\delta)} 2t^{-1}|dv(t)| + \int_{[\delta,\infty)} t^{-2}\,|dv(t)|\|T-T'\|$, where the Hilbert idently and the estimate of the solvent of the a.o. $\|(T+t)^{-1}\| \leq 1/t$,$t > 0$, have been used. Minimizing with respect to the parameter δ, we arrive at the value $\delta = \|T-T'\|/2$, whence it follows that

$$\|\varphi(T) - \varphi(T')\| \leq \breve{\varphi}(2\delta) = \breve{\varphi}(\|T-T'\|).$$

To obtain the estimates for the s-numbers, denote by T_n an operator such that $T_n \leq n$, $s_{n+1}(T) = \|T - T_n\|$(see [25]), $n = 1,2,\ldots$. From inequality (48) and the elementary fact that rank $\varphi(T) \leq$ rank T, the implication $T \in \sigma_\infty \Longrightarrow \varphi(T) \in \sigma_\infty$ easily follows. Further, like in [23]. From $T \equiv \operatorname{Re} T + i \operatorname{Im} T$, we have $K_{4n-1} \stackrel{\text{def}}{=} [(\operatorname{Re} T)_{2n} + i(\operatorname{Im} T)_{2n-1}]$, the corresponding approximation of T with the preservation of accretivity:$(\operatorname{Re} T)_{2n} \geq 0$. It follows that

$$s_{4n-1}(\varphi(T)) \leq \|\varphi(T) - \varphi(K_{4n-1})\| \leq \breve{\varphi}\|T - K_{4n-1}\| \leq \breve{\varphi}(s_{4n-1}(\operatorname{Re} T) + s_{2n}(\operatorname{Im} T))$$
$$\leq \breve{\varphi}(2s_n(T)),$$

where the well-known estimates for the s-numbers of the sum of operators [25] have been used.

A theorem is proved (by a method different from that considered in [24, 26]) which extends estimates (42). (43) to the case of m.a.o. and the functions of class G . Note that it is not hard to extend its scope to class G'.

THEOREM(1.2.3)[22]. Let T ,T' be m.a.o, $\mathfrak{D}(T) = \mathfrak{D}(T')$,$V(T'-T) \in B(H)$ and σ be a symmetrically-normed ideal with domination property. Then the inequality

$$\|\varphi(T) - \varphi(T')\|_{\sigma} \leq 8\|\breve{\varphi}(|V|)\|_{\sigma} \tag{49}$$

holds for $\breve{\varphi}(|V|) \in \sigma$, $\breve{\varphi}(|V|)$being a function (see [28], [25]) of the positive operator$|V| = (V^*V)^{1/2}$, $\varphi \in G$.

Proof. Estimate (49) has been proved above in the uniform norm. Consider now the case $\sigma = \sigma_1$. Let at first $V \geq 0$, $V \equiv \sum_i V_i$, $V_i \stackrel{\text{def}}{=} s_i(V)(\cdot,\varphi_i)\varphi_i$, $(\varphi_i,\varphi_j) = \delta_{ij}$, the spectral resolution of the operator$V \in \sigma_\infty$. Then

$$\|\varphi(T)-\varphi(T')\|_{\sigma_1} = \left\| \int_{\boldsymbol{R}} [(T+t)^{-1} - (T'+t)] dv(t) \right\|_{\sigma_1}$$
$$\leq \sum_i \left\| \int_{\boldsymbol{R}} [(T_i+t)^{-1} - (T_i+V_i+t)^{-1}] d\, v(t) \right\|_{\sigma_1},$$

where $T_i \stackrel{\text{def}}{=} T + \sum_{k<i} V_k$, $i = 1,2,\dots$, is a m.a.o. due to the condition $V \geq 0$ [37]. Let $\delta_i \geq 0, i = 1,2,\dots$, be a sequel. it will be specified below. From the Hilbert identity and the estimate$\|(T_i+t)^{-1}\| \leq 1/t$, $t > 0$, we have

$$\|\varphi(T)-\varphi(T')\|_{\sigma_1}$$
$$\leq \sum_i \left(\int_{[\delta_i,\infty)} \|V\|_{\sigma_1} t^{-2} |dv(t)| + \int_{(0,\delta_i)} \|(T_i+t)^{-1} + (T_i+V_i+t)^{-1}\|_{\sigma_1} |dv(t)| \right)$$
$$\leq \sum_i \left(\int_{[\delta_i,\infty)} t^{-2}\, |dv(t)| s_i(V) + 2 \int_{(0,\delta_i)} t^{-1} |dv(t)| \right),$$

where the inequality rank $((T_i+t)^{-1} + (T_i+V_i+t)^{-1}) \leq 1, t > 0$, has also been used. To minimize the right-hand side of the above double inequality, we choose the sequence δ_i to be $\delta_i \stackrel{\text{def}}{=} s_i(V)/2$. Hence

$$\|\varphi(T)-\varphi(T')\|_{\sigma_1} \leq \sum_i \breve{\varphi}\left(s_i(V)\right) \equiv \|\breve{\varphi}(|V|)\|_{\sigma_1}.$$

In the case of an operator $V \in \sigma_\infty$ of general form make use of the decomposition of its real part [24] into the nonnegative (Re $V)_+$ and negative (Re $V)_-$ components: $V \equiv (\text{Re}\, V)_+ - (\text{Re}\, V)_- + i\text{Im}\, V$. As before, we obtain

$$\|\varphi(T)-\varphi(T')\|_{\sigma_1}$$
$$\leq \|\varphi(T+i\text{Im}\, V)-\varphi(T)\|_{\sigma_1} + \|\varphi(T+i\text{Im}\, V+(\text{Re}\, V)_+)-\varphi(T+i\text{Im}\, V)\|_{\sigma_1}$$
$$+ \|\varphi(T+i\text{Im}\, V+(\text{Re}\, V)_+)-\varphi(T+V)\|_{\sigma_1}$$
$$\leq \sum_i \left(\breve{\varphi}\big(s_n(\text{Im}\, V)\big) + \breve{\varphi}\big(s_n((\text{Re}\, V)_+)\big) + \breve{\varphi}\big(s_n((\text{Re}\, V)_-)\big)\right)$$
$$= \|\breve{\varphi}(\text{Im}\, V)\|_{\sigma_1} + \|\breve{\varphi}(\text{Re}\, V)\|_{\sigma_1}.$$

Here the facts have been used that $(T+i\text{Im}\, V)$, $(T+i\text{Im}\, V+(\text{Re}\, V)_+)$are m.a.o. and that the proof of the foregoing estimate (for the case $V \geq 0$) carries over verbatim to the case Re $V = 0$. Since $s_{2n-1}(\text{Re}\, V) \leq s_n(V)$, $s_{2n-1}(\text{Im}\, V) \leq s_n(V)$,$n = 1,2,\dots$ [25], it follows from the monotonicity of $\breve{\varphi}$ and the Fan-Tsoi lemma [25] that whence

$$\sum_{k\leq n} \breve{\varphi}\big(s_k(\text{Re}\, V)\big) \leq 2 \sum_{k\leq n} \breve{\varphi}\big(s_k(V)\big), \quad \sum_{k\leq n} \breve{\varphi}\big(s_k(\text{Im}\, V)\big) \leq 2 \sum_{k\leq n} \breve{\varphi}\big(s_k(V)\big),$$

hence

$$\|\varphi(T)-\varphi(T')\|_{\sigma_1} \leq 4 \|\breve{\varphi}(|V|)\|_{\sigma_1}. \tag{50}$$

Turning to the general case of a s.n. ideal σ, we shall follow the idea of Calderon's proof of Mityagin's interpolation theorem [39, 31] (see also [18]). By Horn's lemma [25].

$$L \overset{\text{def}}{=} \sum_{k\le n} s_k(\varphi(T+V)-\varphi(T))$$

$$\le \sum_{k\le n}\{s_k(\varphi(T)-\varphi(T+i\mathrm{Im}\,V)) + s_k(\varphi(T+i\mathrm{Im}V+(\mathrm{Re}\,V)_+)-\varphi(T+i\mathrm{Im}\,V))$$
$$+ s_k(\varphi(T+i\mathrm{Im}V+(\mathrm{Re}\,V)_+)-\varphi(T+V))\}$$
$$\le \sum_{k\le n}\big\{\big[s_k\big(\varphi(T+i\mathrm{Im}\,V)-\varphi(T+(\mathrm{Im}\,V)_n)\big)+s_k\big(\varphi(T+(i\mathrm{Im}\,V)_n)-\varphi(T)\big)\big]$$
$$+\big[s_k\big(\varphi(T+i\mathrm{Im}\,V+(\mathrm{Re}\,V)_+)-\varphi(T+i\mathrm{Im}\,V+((\mathrm{Re}\,V)_+)_n)\big)$$
$$+ s_k(\varphi(T+i\mathrm{Im}\,V_+ +((\mathrm{Re}\,V)_+)_n)-\varphi(T+i\mathrm{Im}\,V))\big]$$
$$+\big[s_k\big(\varphi\big(T+i\mathrm{Im}\,V+(\mathrm{Re}\,V)_+\big)-\varphi(T+V+((\mathrm{Re}\,V)_-)_n)\big)$$
$$+ s_k(\varphi(T+V+((\mathrm{Re}\,V)_+)_n)-\varphi(T+V))\big]\big\},$$

where all the φ's are functions of m.a.o. and $(U)_n, U=U^*$, denotes the part of the spectral resolution of a selfadjoint operator U corresponding to the first n eigenvalue in the order of decreasing modulus, taking multiplicity into account. Estimating some of the s-numbers in this sum roughly by means of the operator norm and others with the use of (48), (50), we obtain

$$L \le n\|\varphi(T+i\mathrm{Im}\,V)-\varphi(T+i(\mathrm{Im}\,V)_n)\|$$
$$+\sum_k \breve{\varphi}\big(s_k((\mathrm{Im}\,V)_n)\big) + n\|\varphi(T+i\mathrm{Im}\,V+(\mathrm{Re}\,V)_+)-\varphi(T+i\mathrm{Im}\,V+((\mathrm{Re}\,V)_+)_n)\|$$
$$+\sum_k \breve{\varphi}\,\big(s_k(((\mathrm{Re}\,V)_-)_n)\big) + n\|\varphi(T+i\mathrm{Im}\,V+(\mathrm{Re}\,V)_+)-\varphi(T+V+((\mathrm{Re}\,V)_-)_n)\|$$
$$+\sum_k \breve{\varphi}\,(s_k((\mathrm{Re}\,V)_-)_n))$$
$$\le n\Big[\breve{\varphi}(s_{n+1}(\mathrm{Im}\,V)) + \breve{\varphi}\big(s_{n+1}((\mathrm{Re}\,V)_+)\big)+\breve{\varphi}\big(s_{n+1}((\mathrm{Re}\,V)_-)\big)$$
$$+\sum_{k=1}^{n}\breve{\varphi}\big(s_k(\mathrm{Im}\,V)+\breve{\varphi}(s_k((\mathrm{Re}\,V)_+))+\breve{\varphi}(s_k((\mathrm{Re}\,V)_-))\big)\Big]$$

Hence, from the monotonicity of $\breve{\varphi}$ and from the Fan-Tsoi lemma [25] it follows that

$$L \le 2\sum_{k=1}^{n}\{\breve{\varphi}\big(s_k(i\mathrm{Im}\,V)\big)+\breve{\varphi}(s_k((\mathrm{Re}\,V)_+))+\breve{\varphi}(s_k((\mathrm{Re}\,V)_-))\} \le 2\times(2+2)\sum_{k=1}^{n}\breve{\varphi}\,(s_k(V)).$$

Since the ideal σ assumed to possess the domination property, we obtain (49). In the cases Re $V=0, V\ge 0$, $V\le 0$ the constant 8 in the estimate can obviously be reduced to 2.

COROLLARY(1.2.4) [22]. For $\varphi(\lambda)=\lambda^\alpha, 0<\alpha<1 \Longrightarrow \breve{\varphi}(c)=c^\alpha 2^{1-\alpha}\times \sin\pi\alpha/\pi\alpha(1-\alpha)$; therefore from (49) we have

$$\|\varphi(T)-\varphi(T')\|_\sigma \le 2^{1-\alpha}\frac{\sin\pi\alpha}{\pi\alpha(1-\alpha)}\|(|T'-T|)^\alpha\|_\sigma \tag{51}$$

where the constant on the right-hand side does not exceed 8π and does not increase without bound as $a\to 0$, which is somewhat better than in the analogous estimate in [26].

The mapping $\check{\varphi}(|V|) \to \varphi(T+V) - \varphi(T)$ is not linear, yet the method employed above is in fact a "nonlinear" interpolation of estimates in σ_1 and $B(H)$ and follows the interpolatory proof of B. S. Mityagin's theorem.

We consider below analogues of estimates in quasi-normed classes, including in classes of power decrease of -numbers of operators, which extend the well-known estimates in [24] for power functions of nonnegative operators to the case of accretive operators and the functions of class G. The proof of the theorem, which we omit, can be carried out with the use of the selfadjointness technique from [24] and Sz.-Nagy's theorem on the possession of selfadjoint dilation by an arbitrary maximal dissipative (accretive) operator.

We introduce the notations: $T \in \sigma_\infty$; $\rho_{\beta,n}(T) \overset{\text{def}}{=} \sum_{k=1}^{n} {s_k}^{\beta}(T)$;$r_{\beta,n}(T) \overset{\text{def}}{=} \left(\frac{\rho_{\beta,n}(T)}{n}\right)^{1/\beta}$ On the functionφ (see [25]) we impose this additional restriction in terms of its representing measure :

$$\sum_{n=1}^{\infty}\left(h\int_{[h,\infty)}\frac{|dv(t)|}{(t+c_n h)^2}\right)^{\beta} \leq A(\beta)\check{\varphi}^{\beta}(2h), \qquad (52)$$

where $c_n \overset{\text{def}}{=} 3^{\sqrt{n}/2}$, $h > 0$ and $A(\beta), 0 < \beta < 1$, is some constant.

The following estimate is a generalization of the corresponding inequality for power functions of positive operators [24].

THEOREM(1.2.5)[22]. Let T , T' be m.a.o., $(T) = \mathfrak{D}(T')$,$V \equiv T' - T \in \sigma_\infty$ a function $\varphi \in G$, inequality (52)hold, and$0 < \beta < 1$. Then

$$r_{\beta,n}(\varphi(T) - \varphi(T')) \leq C\big(\beta, A(\beta)\big)\check{\varphi}(r_{\beta,n}(V)) \qquad (53)$$

where $C\big(\beta, A(\beta)\big)$ is some constant depending only on β and $A(\beta)$.

Note that for $\beta = 1$ estimate (53) holds also with the constant 8, which easily follows from Theorem (1.2.3) (now without additional restrictions) and the convexity of $\check{\varphi}$.

COROLLARY (1.2.6) [22]. Let condition (52) on a function $\varphi \in G$ be met, and let T , T' be m.a.o., $V \equiv T' - T \in \sigma_\infty$. Then the implication

$$s_n(V) \leq c/n^{\gamma}, \gamma > 0,\ n = 1,2,\ldots \Longrightarrow s_n(\varphi(T) - \varphi(T')) \leq c(\beta)\check{\varphi}(c/n^{\gamma}) \qquad (54)$$

holds for an arbitrary $\beta < 1$, $\beta < 1/\gamma$ where the constant $c(\beta)$ depends only on $C\big(\beta, A(\beta)\big)$ and β. The case $\gamma < 1$ easily follows from Theorem (1.2.3) for power functions.

In the case $\varphi(\lambda) = \lambda^{\alpha} (0 < \alpha < 1)$ condition (52) is met, whence the validity of the implication (see [26])

$$\overline{\lim_{n\to\infty}}\, s_n(V)\cdot n^{\gamma} = \omega_+ < \infty,\ \gamma > 0 \Longrightarrow \overline{\lim_{n\to\infty}}\, s_n((T')^{\alpha} - T^{\alpha})n^{\alpha\gamma} \leq \omega_+^{\alpha} c(\alpha,\gamma)$$

follows. Note that estimate (53) begins to "worsen" only as $\varphi(\lambda)$ “approaches" the linear function λ in logarithmic scale. Finally, let us observe that the estimates in quasi-normed classes of type (54) (and even simpler estimates of "weak type") can be used for obtaining inequalities (49) in symmetrically-normed ideals, i.e., in classes$\sigma_p, p \geq 1$ (and also in the case$0 < p < 1$), with the use of "interpolatory" technique.

Applications (1.2.7).

(i) As an application, let us first indicate the relation of estimates (51) for $\sigma = \sigma_p$,$1 < p < \infty$, to V. T. Matsaev's inequalities for the σ_p-norms of the real and imaginary parts of a Volterra operator. The simple proofs of those Inequalities are well known [31], so this relation has an illustrative character. The discussion below reproduces almost verbatim P. Stein's proof of the boundedness of a Hilbert transform in class L_p (see also [31] about the application of another proof of the boundedness of a Hilbert transform). Consider this particular case: T is an accretive Volterra operator (Re $T \geq 0$),Im $T \in \sigma_p$,$1 < p < 2$. Then $T^p \equiv T \times T^{(p-1)}$ is also a Volterra operator by the theorem on the mapping of spectra [37]. Using L. A. Sakhnovich's theorem (the case of $p = 2$) and estimates (51), it is easy to show that in this case $T^d \in \sigma_1$ (estimates (51) and the membership Im $T \in \sigma_p$ imply Im $(T^{p/2}) \in \sigma_p$) . By Lidskii's theorem,tr $T^p = 0$ (below, this fact will play the same role as Cauchy's integral theorem does in the proof the boundedness of a Hilbert transform). Further, as in P. Stein's proof [41],

$$
\begin{aligned}
|\cos(p\pi/2)| \cdot \|\mathrm{Im}\, T\|_{\sigma_p}^p &\leq |\mathrm{tr}\ (i\mathrm{Im}\, T)^p| = \left|\mathrm{tr}\left(T^p - ((i\mathrm{Im}\, T)^p)\right)\right| \leq \|T^p - (i\mathrm{Im}\, T)^p\|_{\sigma_1} \\
&\leq \|T(T^{p-1} - (i\mathrm{Im}\, T)^{p-})\|_{\sigma_1} + \|(T - i\mathrm{Im}\, T)(i\mathrm{Im}\, T)^{p-1}\|_{\sigma_1} \\
&\leq \left(\|\mathrm{Re}\, T\|_{\sigma_p} + \|\mathrm{Im}\, T\|_{\sigma_p}\right) \times C(p-1) \times \|(\mathrm{Re}\, T)^{p-1}\|_{\sigma_p} + \|\mathrm{Re}\, T\|_{\sigma_p} \\
&\quad \times \|(|\mathrm{Im}\, T|)^{p-1}\|_{\sigma_p} \\
&= \left(\|\mathrm{Re}\, T\|_{\sigma_p} + \|\mathrm{Im}\, T\|_{\sigma_p}\right) \times C(p-1) \times \|\mathrm{Re}\, T\|_{\sigma_p}^{(p-1)} + \|\mathrm{Re}\, T\|_{\sigma_p}\|\mathrm{Im}\, T\|_{\sigma_p}^{(p-1)},
\end{aligned}
$$

Where $q = p/(p-1)$ and $C(\alpha)$ is the constant from the right-hand side of inequality (51) Applying Young's inequality, we easily obtain $\|\mathrm{Im}\, T\|_{\sigma_p} \leq C_p\|\mathrm{Re}\, T\|_{\sigma_p}$. The dual equality

$$\|\mathrm{Re}\, T\|_{\sigma_p} \leq \tilde{C}_p\|\mathrm{Im}\, T\|_{\sigma_p}$$

is obtained in the analogous way.

(ii) The estimate for the functions of Pick class can also be used for analyzing the boundary behavior of operator-valued functions (o.f.) $T(\lambda)$ analytic in $\boldsymbol{C}_+$ and having a non- negative imaginary part: Im $T(\lambda) \geq 0, T(\lambda) \in \sigma_1$, Im $\lambda > 0$. It has been proved in [7J that such functions have almost everywhere on the real axis nontangential boundary valuest in the norm of σ_p for any$p > 1$. However, it can be shown that in the nuclear norm boundary values mayor may not exist a.e. It is not hard to verify [29] that this problem reduces to the investigation of functions of the "special" form

$$T(\lambda) = V^{1/2}(A-\lambda)^{-1}V^{1/2}, A = A^*, V \geq 0\, , V \in \sigma_1, \mathrm{Im}\,\lambda > 0 \qquad (55)$$

which arise in a natural way in problems of perturbation theory for a pair$\{A, A + V\}$. Clearly, the investigation of the boundary behavior of such o.f. has important applications in perturbation theory. Let $N_\lambda(T)(\lambda > 0, \in \sigma_\infty)$ be the count of the s-numbers of an operator T exceeding or equal to $\lambda, \lambda > 0$. The behavior of the boundary values of $T(k) = s-\lim T(z)$,as$z \to k$ nontangentially, $k \in \boldsymbol{R}$,Im $z > 0$, is described by the following theorem:

Theorem (1.2.8) [22]. Let $T(k)$ have the form (55) with $V \in \sigma_1$. Then for the boundary values of $T(k)$ on the real axis the estimate

$$\sup_{\lambda>0} \int_{\boldsymbol{R}} \lambda N_\lambda(T(k))dk \le c_0 \|V\|_{\sigma_1}. \tag{56}$$

holds, where c_0 is an absolute constant.

Proof. of the theorem is the extension to the case of operators of one of the proofs [43] of Kolmogorov's theorem on the "weak type" of Hilbert transform in$L_1(\boldsymbol{R})$. At that, the only nontrivial fact related to operators (besides, of course, the spectral theorem for self- adjoint operators) which we will use is the estimate of a difference of functions, of the Pick class, of accretive operators in the norm of σ_1 (Theorem (1.2.3)).

Since for $s > 0$ we have $\mathrm{Re}\left(-\mathrm{is}T(\lambda)\right) \ge 0$,$\mathrm{Im}\,\lambda > 0$, by $-\mathrm{is}T(\lambda)$ taking as the argument for the function $\varphi(\lambda) = \ln(1+\lambda)$ of the Pick class (whose representative measure [24] $dv = 0$,$0 < t \le 1$, and $dv = -dt$, $t > 1$) we obtain the analytic o.f. $\ln(\mathrm{I} - \mathrm{is}T(\lambda))$ in the region $\mathrm{Im}\,\lambda > 0$. Since $T(\lambda)$ has nontangential boundary values $T(k)$ almost everywhere on $\boldsymbol{R}$ in the norm of σ_2 (and even σ_p $\forall p > 1$ [29]), the function $\mathrm{Re}\ln(\mathrm{I} - \mathrm{is}T(\lambda))$, nonnegative and harmonic in $\boldsymbol{C}_+$ has nontangential boundary values $\mathrm{Re}\ln(\mathrm{I} - \mathrm{is}T(k))$,$k \in \boldsymbol{R}$ a.e. on $\boldsymbol{R}$ in the nuclear norm. Note that for the function$\ln(\mathrm{I} - \mathrm{is}T(\lambda))$ itself this assertion, generally speaking. is not valid. Considering the scalar nonnegative harmonic function $\mathrm{Re}\ln(\mathrm{I} - \mathrm{is}T(\lambda))$, it is easy to show, as in [43], that ($\lambda \equiv x + iy$)

$$0 \le \mathrm{tr}\, y\, \mathrm{Re}\ln(\mathrm{I} - \mathrm{is}T(\lambda)) \ge \frac{1}{\pi}\int_{\boldsymbol{R}} \frac{y^2}{(x-k)^2 + y^2} \mathrm{tr}\,\mathrm{Re}\ln\left(\mathrm{I} - \mathrm{is}T(k)\right)dk.$$

Passing to the limit as $y \to +\infty$ and computing $\lim\limits_{y\to+\infty} \ln(\mathrm{I} - \mathrm{is}T(\lambda)) = sV$, we obtain

$$\int_{\boldsymbol{R}} \mathrm{tr}\,\mathrm{Re}\,\ln\left(\mathrm{I} - \mathrm{is}T(k)\right)dk \le \pi s\, \mathrm{tr}\, V.$$

Hence and from (50) we have

$$\begin{aligned}\int_{\boldsymbol{R}} \left\|\mathrm{Re}\,\ln\left(\mathrm{I} - \mathrm{is}T(k)\right)\right\|_{\sigma_1} dk &\le \pi s\, \mathrm{tr}\, V + \int_{\boldsymbol{R}} \left\|\ln\left(\mathrm{I} - \mathrm{is}T(k)\right) - \ln(\mathrm{I} - \mathrm{is}(\mathrm{Re}\, T(k)))\right\|_{\sigma_1} dk \\ &\le \pi s\, \mathrm{tr}\, V + 2\int_{\boldsymbol{R}} \left\|\breve{\varphi}\left(s\mathrm{Im}T(k)\right)\right\|_{\sigma_1} dk \le \pi s\, \mathrm{tr}\, V + 2s\int_{\boldsymbol{R}} \|\mathrm{Im}\, T(k)\|_{\sigma_1} dk \\ &\equiv \pi s\, \mathrm{tr}\, V + 2s(2\pi\, \mathrm{tr}\, V) = \pi s 5 \times \mathrm{tr}\, V.\end{aligned}$$

Here the estimate $\breve{\varphi}(c) \le c$ has been used and the computati.on of the integraltr $\int_{\boldsymbol{R}} \mathrm{Im}\, T(k)dk = 2\pi\, \mathrm{tr}\, V$, has been carried out, which can easily be done, e.g., with the aid of the spectral theorem (see [7J). Computing the trace of the normal operator, we have

$$\int_{\boldsymbol{R}} dk \sum_n \ln\left(s \cdot s_n\left(\mathrm{Re}\, T(k)\right)\right) \le 5\pi s \mathrm{tr}\, V,$$

hence, after setting $s = e/\lambda$, $\lambda > 0$, we obtain

$$\int_{\boldsymbol{R}} dk\ N_\lambda\left(\operatorname{Re} T(k)\right) \le 5e\pi \operatorname{tr} V/\lambda, \quad \lambda > 0.$$

Since

$$2\pi \operatorname{tr} V = \int_{\boldsymbol{R}} \operatorname{tr} \operatorname{Im} T(k) dk \ \ge \int_{\boldsymbol{R}} \lambda N_\lambda\left(\operatorname{Im} T(k)\right) dk, \quad \lambda > 0,$$

with the use of the inequality $N_{2\lambda}(T_1 + T_2) \le N_\lambda(T_1) + N_\lambda(T_2)$ [31], we easily obtain the required estimate

$$\int_{\boldsymbol{R}} dk\ N_\lambda\left(T(k)\right) \le \pi[10e + 2] \operatorname{tr} V.$$

Proceeding from estimate (54), one can obtain various information about the boundary limits of an o.f. (see [29]). Let us present without proof just one consequence [29].

THEOREM(1.2.9) [22]. Let an operator-function $T(\lambda)$ have the form (55). If $V \in \sigma_1 \times \sigma_\omega$, i.e., is the product of a nuclear operator and an operator from the Matsaev class σ_ω , then the o.f. $T(\lambda)$ has nontangential boundary values on $\boldsymbol{R}$ in the nuclear norm. On the other hand, for any symmetrically-normed ideal $\sigma \ne \sigma_1$ there exists o.f. $T(\lambda)$ of the form (55) with $V \in \sigma_1 \times \sigma_\omega$ (i.e., V is the product of an operator from class σ_ω and an operator from the s.n. ideal σ) whose boundary values do not belong to class σ_1 almost everywhere on $\boldsymbol{R}$.