Introduction

A C^*-algebra is called primitive if it admits a faithful and irreducible $*$–representation. We show that if A_1 and A_2 are separable, unital, residually finite dimensional C^*-algebras satisfying $(\dim(A_1)-1)(\dim(A_2)-1) \geq 2$, then the unital C^*-algebra full free product, $A = A_1 * A_2$, is primitive. It follows that A is antiliminal, it has an uncountable family of pairwise in equivalent irreducible faithful $*$–representations and the set of pure states is w^*-dense in the state space. We prove the following: Suppose that $\phi, \psi: A \to B$ are unital $*$-monomorphisms. There exists a sequence of unitaries $\{u_n\} \subset B$ such that $\lim_{n\to\infty} u_n^* \phi(a) u_n = \psi(a)\ for\ all\ a \in A$ if and only if $[\phi] = [\psi]\ in\ KL(A,B), \phi_\ddagger = \psi_\ddagger\ and\ \phi^\ddagger = \psi^\ddagger$, where $\phi_\ddagger, \psi_\ddagger: \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ and $\phi^\ddagger, \psi^\ddagger: U(A)/CU(A) \to U(B)/CU(B)$ are the induced maps (where $T(A)$ and $T(B)$ are the tracial state spaces of A and B, and $CU(A)$ and $CU(B)$ are the closures of the commutate subgroups of the unitary groups of A and B, respectively). Also show that there is a unital homomorphism $\phi: A \to B$ so that $([\phi], \phi_\ddagger, \phi^\ddagger) = (\kappa, \lambda, \gamma)$, at least in the case that $K_1(A)$ K1(A) is a free group.

Let A be aunital separable simple Z-stable C^*-algebra which has rational tracial rank almost one and let $u \in U_0(A)$, where $U_0(A)$ is the connected component of the unitary group of A containing the identity. We show that, for any $\epsilon > 0$,there exists aself-adjoint element $h \in A$ such that $\|u - \exp(ih)\| < \epsilon$. But there is no control of $\|h\|$ in general. For the Jiang–Su algebra $\mathcal{Z}$, we show that, if $u \in U_0(\mathcal{Z})$ and $\epsilon > 0$,there exists areal number $-\pi < t \leq \pi$ and aself-adjoint element $h \in \mathcal{Z}$ with $\|h\| \leq \pi$ such that $\|e^{it}u - \exp(ih)\| < \epsilon$. Also we show ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]\ in\ KL(C,A)$,$\tau \circ \phi = \tau \circ \psi$ for all tracial states of A and $\phi^\ddagger = \psi^\ddagger$, where $\phi^\ddagger$ and $\psi^\ddagger$ are homomorphisms from $U(C)/CU(C) \to U(A)/CU(A)$ induced by ϕ and ψ, respectively, and where $CU(C)$ and $CU(A)$ are closures of the subgroup generated by commutators of the unitary groups of C and B. A more practical but approximate version of the above is also presented.

Let $\epsilon > 0$ be a positive number. Is there a number $\delta > 0$ satisfying the following. Given any pair of unitaries u and v in a unital simple C^*-algebra A with $[v] = 0$ in $K_1(A)$ for which $\|uv - vu\| < \delta$, there is a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset A$ such that $v(0) = v, v(1) = 1$ and $\|uv(t) - v(t)u\| < \epsilon$ for all $t \in [0,1]$. An answer is given to this question when A is assumed to be a unital simple C^*-algebra with tracial rank no more than one. Also we study the case that A is no longer assumed to have real rank zero, or tracial rank zero.

We give a classification theorem for unital separable nuclear C^*-algebras with tracial rank no more than one. Let A and B be two unital separable simple nuclear C^*-algebras with $TR(A), TR(B) \leq 1$ which satisfy the universal coefficient theorem.