Chapter 4
Simple Nuclear C^*-Algebras and Some Theorem

We are interested in simple C^*-algebras with lower rank.We show that $A \cong B$ if and only if there is an order and unit preserving isomorphism $\gamma = (\gamma_0, \gamma_1, \gamma_2): \left(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\right) \cong \left(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\right)$, where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for each $x \in K_0(A)$ and $\tau \in T(B)$.

Section (4.1) Simple Nuclear C^*-Algebras of Tracial Topological Rank One

In this section we are only interested in simple C^*-algebras with lower rank.We show that $A \cong B$ if and only if there is an order and unit preserving isomorphism $\gamma = (\gamma_0, \gamma_1, \gamma_2): \left(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)\right) \cong \left(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)\right)$, where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for each $x \in K_0(A)$ and $\tau \in T(B)$.

This section is a part of the program to classify separable nuclear C^*-algebras initiated by George A. Elliott (see [38] and [39]). By a classification theorem for a class of nuclear C^*-algebras, one means the following: two C^*-algebras in the class with the same K -theoretical data are isomorphic (as C^*-algebras) and the range of the invariant can be described for the class so that given a set of K-theoretical data in the range there is a C^*-algebra in the class which possesses the given K-theoretical data. By the K-theoretical data, one usually means the Elliott invariant which contains the K-theory and traces, at least for the simple case. In this paper we are only interested in simple C^*-algebras with lower rank. By C^*-algebras of lower rank, one often means that the C^*-algebras have real rank zero, or stable rank one. Many important C^*-algebras which arise naturally are of real rank zero or stable rank one. Notably, all purely infinite simple C^*-algebras have real rank zero and many C^*-algebras arising from dynamical systems are of stable rank one. One of the classical results of this kind states that all irrational rotation C^*-algebras are simple nuclear C^*-algebras with real rank zero and stable rank one (see [34] and [117]).

One may view (simple) C^*-algebras of real rank zero and stable rank one as some kind of generalization of AF-algebras. A more suitable generalization of AF-algebras has been demon-strated to be C^*-algebras with tracial topological rank zero. Simple C^*-algebras with tracial topological rank zero have real rank zero, stable rank one, with weakly unperforated K0 and are quasidiagonal. All simple AH-algebras with slow dimension growth and with real rank zero have tracial topological rank zero. This shows that simple C^*-algebras with zero tracial rank could have rich K-theory. Simple AH-algebras with slow dimension growth and with real rank zero have been classified in [31] (together with [9,21] and [50]). A classification theorem for unital nuclear separable simple C^*-algebras with tracial topological rank zero which satisfy the UCT was given in [93] (see also [32,36] and [24] for earlier

references). Simple C^*-algebras with tracial topological rank zero are also called TAF (tracially AFC^*-algebras.

We studiedly C^*-algebras of tracial rank one. A standard example of a C^*-algebra with stable rank one is of course Mk (C([0, 1])) (which also has tracial rank one). A notion of tracially approximately interval C^*-algebras (TAI C^*-algebras) is introduced in this paper—see Definition 2.2 below. It turns out that simple TAI C^*-algebras are the same as simple C^*-algebras with tracial topological rank no more than one. Roughly speaking, TAI C^*-algebras are those C^*-algebras whose finite subsets can be approximated by C^*-subalgebras which are finite direct sums of finite-dimensional C^*-algebras and matrix algebras over C([0, 1]) in "measure" or rather in trace.

It is proved here that simple TAI C^*-algebras have stable rank one. From a result of G. Gong [51] we observe that all simple AH-algebras with very slow dimension growth are in fact TAI C^*-algebras. It is also shown here that simple TAI C^*-algebras are quasidiagonal, their ordered K0-groups are weakly unperforated and satisfy the Riesz interpolation property, and these C^*-algebras also satisfy the Fundamental Comparison Property of Blackadar.

Elliott, Gong and Li in [36] (also [51]) give a complete classification (up to isomorphism) for simple AH-algebras with bounded dimension growth by their K-theoretical data (an important special case can be found in K. Thomsen's work [131]). G. Gong also has a proof [52] that simple AH-algebras with very slow dimension growth can be rewritten as simple AH-algebras with bounded dimension growth.

These C^*-algebras are nuclear separable simple C^*-algebras of stable rank one. Their work is a significant advance in classifying finite simple C^*-algebras after the remarkable result of [31] which classifies simple AH-algebras of real rank zero (with slow dimension growth). Therefore, it is the time to classify nuclear simple separable finite C^*-algebras with real rank other than zero without assuming that they are inductive limits (AH-algebras are inductive limits of finite direct sums of some standard homogeneous C^*-algebras) of certain special building blocks.

Since then a great deal of progress on the subject has been made. The present paper absorbs both parts of the original preprint and reflects the new developments. But it is significantly shorter than the original preprint. More importantly, the main result of the paper has been greatly im-proved and a technical condition in original preprint has been removed. Let A and B be two unital separable nuclear simple C^*-algebras with TR(A)$\leq$ 1, TR(B)$\leq$ 1 and satisfying the UCT. Then A $\cong$ B if and only if they have the same Elliott invariant.

Consider two C^*-algebras A and B as above. As in [36], we will construct the following approximately commutative diagram:

An important fact is the following classification of monomorphisms from $\bigoplus_{k=1}^{n} M_{r(k)}$ (C([0,1])) to a simple TAI-algebra. For any unital C^*-algebra C, denote by T (C) the tracial

state space of C (it could be an empty set). Let $A = \bigoplus_{k=1}^{n} M_{r(k)}$ $(C([0,1]))$ and B be a unital simple TAI C^*-algebra. Suppose that $\emptyset_i : A \rightarrow B$ $(i = 1,2)$ are two unital monomorphisms which induce the same map at the level of K_0 and satisfy

$$\tau \circ \varphi_1(a) = \tau \circ \varphi_2(a)$$

for all $a \in A$ and all $\tau \in T$ (B). Then there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} u_n^* \varphi_1(a) = \varphi_2(a) \qquad \text{for all } a \in A$$

The uniqueness theorem also has to be adjusted to deal with other complications caused by the fact that our C^*-algebras are no longer assumed to have real rank zero. A careful treatment on exponential length is needed. Our existence theorem also needs to be improved from that in [93]. The existence theorem should also control the exponential length. It turns out that when C^*-algebras are assumed to have only torsion K_1, the proof can be made much shorter. This is done without using de la Harpe and Skandalis determinants as in [36].

Let A be a C^*-algebra.

Definition(4.1.1)[89]: We denote byI the class of all unital C^*-algebras with the form $\bigoplus_{i=1}^{n} B_i$, where each $B_i \cong M_{k(i)}$ for some integer k(i) or $B_i \cong M_{k(i)}$ $(C([0,1]))$. Let A ϵI . We have the following well-known facts.

(i) Every C^*-algebra in I is of stable rank one.

(ii) Two projections p and q in a C^*-algebra A $\in I$ are equivalent if and only if τ (p) = τ (q)for all $\tau \in T$ (A).

(iii) For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: if L : A $\rightarrow$ B is a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where B is a C^*-algebra, then there exists a homomorphism h : A $\rightarrow$ B such that ||h (a)-L (a)||< ε for all a $\in \mathcal{F}$.

Definition(4.1.2)[89]: A unital C^*-algebra A is said to be tracially AI (TAI) if for any finite subset $\mathcal{F} \subset A$ containing a nonzero element b, $\varepsilon > 0$, integer n > 0 and any full element a $\in A_+$, there exist a nonzero projection p $\in$ A and a C^*-subalgebra I $\subset$ A with I $\in I$ and $1_I = p$, such that:

(i) $\|[x , p]\| < \varepsilon \ for \ all \ x \ \in \mathcal{F}$,

(ii)$pxp \ \in \varepsilon \ I \ for \ all \ x \ \in \mathcal{F} \ and \ \|pbp\| \ \geq \ \|b\| \ - \ \varepsilon$,

(iii)$n[1 - p] \leq [p] \ and \ 1 - p \preccurlyeq \ a$.

A non- unital C^*-algebra A is said to be TAI if $\tilde{A}$ is TAI.

In 4.10, we show that, if A is simple, condition (iii) can be replaced by

(iii') $1 - p$ is unitarily equivalent to a projection in eAe for any previously given nonzero projec-tion e $\in$ A.

If A has the Fundamental Comparability (see [5]), condition (iii) can be replaced by

(iii ") τ $(1 - p) < \sigma$ for any prescribed $\sigma > 0$ and for all normalized quasi-traces of A.

From the definition, one sees that the part of A which may not be approximated by C^*-algebras in I has small "measure" or trace. Note in the above, if I is replaced by finite-dimensional C^*-algebras, then it is precisely the definition of TAF C^*-algebras (see [92]). Every AF-algebra is TAI. Every TAF C^*-algebra introduced in [92] is a TAI C^*-algebra. However, in general, TAI C^*-algebras have real rank other than zero. In 4.5 we will show that every simple TAI C^*-algebra has stable rank one, which implies that simple TAI C^*-algebras have real ra-nk one or zero. It is obvious that every direct limit of C^*-algebras in I is a TAI C^*-algebra. These C^*-algebras provide many examples of TAI C^*-algebras that have real rank one. However, TAI C^*-algebras may not be inductive limits of C^*-algebras in I.

Let $A= \lim_{n\to\infty}(A_n, \emptyset_{n,m})$, where $A_n = \bigoplus_{i=1}^{s(n)} P_{n,i} M_{n,i} (C(X_{n,i}))P_{n,i}$, $X_{n,i}$ is a finite-dimensional compact metric space and $P_{n,i} \in M_{n,i} (C(X_{n,i}))$ is a projection for all n and Such a C^*-algebra is called an AH-algebra. Suppose that A is unital. Following [51], A is said to have very slow dimension growth if

$$\lim_{n\to\infty} \min_i \frac{rank\ (P_{n,i})}{(\dim X_{n,i} + 1)^3} = \infty$$

A is said to have no dimension growth if there is an integer m > 0 such that dim $X_{n,i} \leq$ m. Note these C^*-algebras may not be of real rank zero. Since these C^*-algebras could have non-trivial K1-groups (see 10.1), they are not inductive limits of C^*-algebras in I. In [92], example of simple TAF C^*-algebras which are non-nuclear was given. In particular, there are simple TAI C^*-algebras that are not even nuclear.

Lemma (4.1.3)[89]:   Let a be a positive element in a unital C^*-algebra A with sp(a) $\subset$ [0, 1]. Then for any $\varepsilon > 0$, there exists b $\in A_+$ such that sp(b) is a union of finitely many mutually disjoint closed intervals and finitely many points and

$$\|a - b\| < \varepsilon.$$

Proof: Fix $\varepsilon > 0$. Let $I_1, I_2, \ldots, I_K$ be all disjoint closed intervals in sp(a) with length at least $\varepsilon/8$ such that if I $\supset I_j$ is an interval, then I $\not\subset$ sp(a). Let d' = min{dist(I_i , I_j), i $\neq$ j } and d = min(d '/2, $\varepsilon/16$).

   Choose J_i = {$\xi \in$ [0, 1]: dist(ξ , I_j) < d_i}, i = 1, 2, $\ldots$, k with $d_i \leq$ d and the endpoints of J_i are not in sp(a). Since the endpoints of J_i are not in sp(a), there are open inter-Vals $J'_i \subset J_i$ such that $\bar{J}'_i \subset J_i$ and $I_i \subset J'_i$. Set Y=sp(a) \ ($\cup_{i=1}^k J_i$). Then Y = sp(a)\ ($\cup_{i=1}^k J'_i$)= sp (a) \($\cup_{i=1}^k \overline{J'_i}$) .Since Y is compact and Y contains no intervals with length more than $\varepsilon/8$, it is routine to show that there are finitely many disjoint closed intervals $K_1, K_2, \ldots, K_n$ in [0,1]\ ($\cup_{i=1}^k J_i$) with length more than $9\varepsilon /64$ such that Y $\subset \cup_{i=1}^m J_i$.

Note that $\{\overline{J'}_1, \overline{J'}_2, \ldots, \overline{J'}_k, K_1, K_2, \ldots, K_n\}$ are disjoint closed intervals . Fix a point$\xi_j \in K_{j,,}$ j=1,2,……,m. One can define a continuous function $\mathcal{F} : (\cup_{i=1}^k \overline{J'}_i) \cup (\cup_{s=1}^n K_s) \to [0,1]$ which maps each K_1 onto I_i, i=1,2,….,k and maps K_j to single point $\xi_j$such that

$|f(\xi)- \xi|<\varepsilon / 2$ for all $\xi \in [0,1]$

Define b = f (a). We see that b meets the requirements of the lemma.

Theorem(4.1.4)[89]: Let A be a unital simple AH-algebra with very slow dimension growth. Then A is TAI.

Proof. As in [51] (and [35]), to show that A is TAI, it suffices to assume that A= $\lim_{n \to \infty} (A_n, \emptyset_{n,m})$,where$A_n = \oplus_{i=1}^{i(n)} M_{n(i)}(C(X_{n,i}))$,$X_{n,i}$are simplicial complexes

and $\emptyset_{n,m}$are injective . Moreover, we may also assume that A satisfies the condition of very slow dimension growth.

Let $\varepsilon > 0$, $\mathcal{F} \subset A$ be a finite subset and e $\in$ A be a non-zero projection. To verify (1), (ii) and (iii') in 2.2, without loss of generality, we may assume that F $\subset A_1$ and e $\in A_1$. By considering each summand separately, without loss of generality, we may also assume that $A_1 = M_r$ (C(X)) for some finite simplicial complex and integer r$\geq$ 1. Let $\mathcal{F}_1 \subset$ C(X) be a finite subset such that

$\mathcal{F} \subset \{(f_{i,j})_{r \times r} : f_{i,j} \in \mathcal{F}_1\}$.

Let J > r + 1 be an integer. Let $\varepsilon/2r^2 > \eta > 0$ such that $|f (x)- f (x')| < \varepsilon/9r^2$ for all f $\in \mathcal{F}_1$ whenever dist(x, x') < 2η. Let $\delta > 0$ and L be as in Theorem 4.35 corresponding to $\varepsilon/2r^2$, η and $\mathcal{F}_1$ above. Since A is simple, as in [51], each partial map of $\emptyset_{1,m}$ (for sufficiently large m) has the property sdp($\eta/32$, $\varepsilon/2r^2$). To simplify notation, without loss of generality, we may assume that $A_m = M_k$ (C(Y)) and rank($\emptyset_{1,m}(1)) > 2J L^2 2^L (dim X + dim Y + 1)^3$, where Y is a finite simplicial complex. To simplify notation, by considering each summand separately, without loss of generality, we may assume that Y is connected. Since A is simple, by choosing a larger m, we further assume that e $\in M_k$ (C(Y)) is a non-zero projection which has the rank at least rank($\emptyset_{1,m}(1)$)/r.

As   in [51], there are three mutually orthogonal projections $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\mathcal{Q}_2 \in A_m$ and homomorphisms $\psi_i : A_1 \to \mathcal{Q}_i A_m \mathcal{Q}_i$ (i = 0, 1, 2) such that:

(i)$\emptyset_{1,m} = \mathcal{Q}_0 + \mathcal{Q}_1 + \mathcal{Q}_3$;

(ii)$\|\emptyset_{n,m} (f) - (\emptyset_0 (f) \oplus \emptyset_1 (f) \oplus \emptyset_2 (f))\|< \varepsilon/2$ for all f $\in \mathcal{F}$;

(iii)ψ_2 factors through M_r (C([0, 1]);

(iv)ψ_1 has finite-dimensional range;

(v)J $[\mathcal{Q}_0] \leq [\mathcal{Q}_1]$.

Put $\psi = \psi_i \oplus \psi_i$. It follows from Lemma (4.1.3) that there is a unital C^*-subalgebra $B_1 \in I$ of $(\mathcal{Q}_1 + \mathcal{Q}_2)Am(\mathcal{Q}_1 + \mathcal{Q}_2)$ such that

$$\psi (\mathcal{F}) \in \varepsilon B_1 \text{ for f } \in \mathcal{F}.$$

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We also have
$$[Q_0] \leq [e].$$
Thus, A is of TAI.

Lemma (4.1.5)[89]: For any d > 0 there are $f_1, f_2, \ldots, f_m \in C([0, 1])+$ with the following properties. For any n, and any positive element $x \in B = M_n(C([0, 1]))$ with $\|x\| \leq 1$ if there exist $a_{ij} \in B$,

i = 1, 2, . . . , n(j), j = 1, 2, . . . , m with
$$\left\| \sum_{i=1}^{n(j)} a_{i,j}, f_j(x)a_{i,j}^* - 1_A \right\| < 1/2, \quad j = 1, 2, \ldots, m,$$
then, for any subinterval J of [0, 1] with $\mu(J) \geq d$ (μ is the Lebesgue measure), $sp(\pi_t(x)) \cap$ J $\neq \emptyset$ for all t $\in$ [0, 1], where $\pi_t : B \to M_n$ is the point evaluation at t. Moreover, denote by N = max{n(j): j = 1, 2, . . . , m},
$$|sp(\pi_t(x)) \cap J| \geq 1/N |sp(\pi_t(x))|,$$
where |S| means the number of elements in the finite set S (counting multiplicities).

Proof: Divide [0, 1] into m closed subintervals $\{J_j\}$ each of which has the same length < d/4. Let $f_j \in C([0, 1])$ be such that $0 \leq f_j \leq 1$, $f_j(t) = 1$ for t $\in J_j$ and $f_j(t) = 0$ for dist(t, $J_j) \geq \mu(J_j)$. Note that, for any subinterval J with $\mu(J) \geq d$, there exists j such that $J_j \subset J$. For any t $\in$ [0, 1], set
$$I = \{g \in B: g(t) = 0\}.$$
Then I is a (closed) ideal of A. If $sp(\pi_t(x)) \cap J = \emptyset$, there would be j such that $\pi_t(f_j(x)) = 0$. Therefore $f_j \in I$. But this is impossible, since there is an element $z \in B$ with
$$z \left(\sum_{i=1} a_{i,j} f_j(x)a_{i,j}^* \right) = 1_B$$
For the last part of the lemma, fix t $\in$ [0, 1] and an interval J with t $\in J$ and $\mu(J) \geq d$. Let $\pi_t(B) = M_1(t)$. Then $|sp(\pi_t(x))| = l(t)$. Suppose that $J_j \subset J$ so that $f_j(t) = 0$ for all t $\in$/ J. Let q_t be the spectral projection of $\pi_t(x)$ in $M_1(t)$ corresponding to J. Then $qt \geq f_j(\pi_t(x))$. An elementary linear algebra argument shows that rank $q_t \geq (1/N)l(i)$.

Theorem(4.1.6)[89]: Every unital simple C^*-algebra satisfying (i) and (ii) in property (SP), i.e., every hereditary C^*-subalgebra contains a nonzero projection.

Corollary(4.1.7)[89]: Let A be a unital simple C^*-algebra satisfying (i) and (ii) 2. Then, for any

integer N , we may assume that $I = \bigoplus_{i=1}^k M_{mi}(C([0,1])) \bigoplus_{j=1}^l M_{ni} where m_i, n_i \geq N$.

Proof: In the proof of 3.2, we see that if $1/2d \geq N$, since $sp(\pi_t(pap)) \cap J_i = \emptyset$ for each j, then $\pi_t(pap)$ has at least N distinct eigenvalues (see also the proof of 3.1). Therefore, each summand C in the proof 3.2 has rank at least N .

Proposition (4.1.8[89]): Let A be a unital TAI C^*-algebra and e $\in$ A be a full projection. Then eAe satisfies (i) and (ii), and for any full positive element a $\in$ eAe, we can have (iii') $1 - p \preccurlyeq a$.

If A is also simple, eAe is TAI.

Proof: Fix $\varepsilon > 0$, a finite subset $\mathcal{F} \subset$ eAe, an integer n > 0 and nonzero elements a, b $\in$ eAe with a$\geq$ 0 and b $\in \mathcal{F}$. Let $\mathcal{F}_1 = \{e\} \cup \mathcal{F}$. Since A is TAI, there exists q $\in$ A and a C^*-subalgebra C $\in$ I with $1_C = q$ such that:

(i)$\|[x, q]\| < \varepsilon/64$ for all x $\in \mathcal{F}$,

(ii)qxq $\in_{\varepsilon/64}$ C for all x $\in \mathcal{F}$, and $\|qbq\| \geq \|b\| - \varepsilon/64$; and,

(iii)n$[1 - q] \leq [q]$ and $1 - q \preccurlyeq$ a.

Note that, by the second part of (ii), qeq $\neq 0$. We estimate that
$$\|(eqe)^2 - eqe\| < \varepsilon/64 \quad \text{and} \quad \|eqe - qeq\| < \varepsilon/32.$$
Therefore there is a projection p $\in$ eAe such that
$$\|p - eqe\| < \varepsilon/16.$$
Consequently, there is a projection d $\in$ C such that
$$\|d - p\| < \varepsilon/8.$$

Note that
$$\|qp - pq\| < \varepsilon/8 + \|qeqe - eqeq\| < \varepsilon/8 + \varepsilon/32 = 5\varepsilon/32,$$
and B = [dCd $\in$ I. With $\varepsilon/2 < 1/2$, we obtain a unitary u $\in$ A such that
$$\|u - 1\| < \varepsilon/4 \quad \text{and} \quad u^*du = p.$$
Set $C_1 = u^*Bu$. Then $C_1 \in$ I and $C_1 \subset$ eAe. Now $1_{C_1} = p$,

(i) $\|[x, p]\| < \varepsilon/2$ for all x $\in \mathcal{F}$,

(ii) pxp $\in_{\varepsilon/2} C_1$ for all x $\in \mathcal{F}$ and $\|pbp\| \geq \|b\| - \varepsilon/2$.

We also have
$$\|(e - p) - (1 - q)(e - p)(1 - q)\|$$
$$\leq \|(e - p) - (e - p)(1 - q) + q(e - p)(1 - q)\|$$
$$< \|(e - p)q\| + \varepsilon/16 < 5\varepsilon/32 + \varepsilon/64 + \|qeq - qpq\| + \varepsilon/16$$
$$< 5\varepsilon/32 + \varepsilon/64 + \varepsilon/16 + \|qeq - qeqeq\| + \varepsilon/16 < 9\varepsilon/32.$$

We have (with $\varepsilon < 1$)
$$(3\,') \ (e - p) \preccurlyeq (1 - q) \preccurlyeq a.$$

Finally, if we assume that A is simple, there is a nonzero projection $p_1 \leq$ p such that n$[p_1] \leq$ [p]. There is a nonzero projection $p_1 \in \overline{aAa}$. We obtain a nonzero projection $q_1 \leq p_1$ such that n$[q_1] \leq [p_1]$. Applying the first part of the proof to $(e - p)\mathcal{F}(e - p)$, we obtain a projection p'$\leq$ (e − p) and a unital C^*- subalgebra $C_2 \in$ I with $1_{C_2} = $ p' such that:

(i")$\|[(e - p)x(e - p), p']\| < \varepsilon/2$ for all x $\in \mathcal{F}$,

(ii") p' xp'$\in_{\varepsilon/2} C_2$ for all x $\in \mathcal{F}$, and

(iii ") (e − p − p')$\preccurlyeq p_1$.

Now since $n[p_1] \leq [p] \leq [p+p']$ and $(e-p-p') \lesssim (e-p) \lesssim a$, we obtain

(iii) $n[(e-p-p')] \leq [p+p']$ and $(e-p-p') \lesssim a$.

We also have $\|[x,(p+p')]\| < \varepsilon$ and $(p+p')x(p+p') \in_\varepsilon C_1 \oplus C_2$ for all x $\in \mathcal{F}$. Hence eAe is TAI.

Corollary (4.1.9) [89]: If A is a unital simple TAI C^*- algebra, then condition (2) can be strengthened to (ii') $pxp \in_\varepsilon$ B and $\|pxp\| \geq \|x\| - \varepsilon$ for all x $\in \mathcal{F}$.

We omit the proof.

Theorem (4.1.10)[89]: Let A be a unital simple C^*- algebra. Then A is TAI if and only if $M_n(A)$ is TAI for all n (or for some n > 0).

Proof. If $M_n(A)$ is TAI, then by identifying A with a unital hereditary C^*- subalgebra of $M_n(A)$. We know A is TAI. It remains to prove the "only if" part.

We prove this in two steps. The first step is to prove that $M_n(A)$ satisfies (i) and (ii) To do this, we let $\varepsilon > 0$ and $\mathcal{F}$ be a finite subset of the unit ball of $M_n(A)$. Set $\mathcal{G} = \{f_{ij} \in A: (f_{ij})_{n \times n} \in \mathcal{F}\}$. Note that $\mathcal{G} \subset A$. Since A is TAI, there exists a projection p $\in$ A and a unital C^*- subalgebra B $\in I$ such that:

(i)$\|[x,p]\| < \varepsilon/2n^2$,

(ii)$pxp \in_{\varepsilon/2n^2}$ B for all x $\in \mathcal{G}$ and for some $x_1 \in \mathcal{G}$, $\|px_1p\| \geq \|x_1\|1 - \varepsilon/2n^2$.

Put P $=$ diag(p, p, . . . , p) $\in M_n(A)$ and D $= M_n(B)$. Then, it is easy to check that

(i)$\|[f,P]\| < \varepsilon$ and

(ii)Pf P $\in_\varepsilon$ D for all $f \in \mathcal{F}$ and $\|pf_1p\| \geq \|f_1\| - \varepsilon$ (if f_1 is prescribed).

This completes the first step. Now we also know that $M_n(A)$ has (SP). Let a $\in M_n(A)$ be given. Choose any nonzero projection e $\in \overline{aM_n(A)a}$. Since $M_n(A)$ is simple and has (SP), in [92], there is a nonzero projection q$\leq$e and [q]$\leq$[1_A], there exists a nonzero projection $q_1 \leq q$ such that $(n+1)[q_1] \leq [q]$. In the first step, we can also require, for any integer N > 0, that

(iii) N $[1_A - p] \leq [p]$ and $1_A - p \lesssim q_1$.

This implies that

(iii) N $[1_{M_{n(A)}} - P] \leq [P]$ and $(1_{M_{n(A)}} - P) \lesssim q \lesssim e$.

Therefore $M_{n(A)}$ is TAI.

Next we show that every simple TAI C^*- algebra has the property introduced by Popa [116].

Proposition (4.1.11)[89]: Let A be a unital simple TAI C^*- algebra. Then for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there exists a projection p $\in$ A and a finite-dimensional C^*- algebra F $\subset$ A with $1_F = p$ such that

(P1)$\|[x,p]\| < \varepsilon$ and

(P2) $pxp \in_\varepsilon$ F for all x $\in \mathcal{F}$ and $\|pxp\| \geq \|x\| - \varepsilon$ for all x $\in \mathcal{F}$.

Proof. The clear that it suffices to prove the following claim: for any unital C^*- subalgebra B $\in I$, the proposition holds for any finite subset $\mathcal{F} \subset$ B $\subset$ A.

This can be further reduced to the case that $B = C([0, 1]) \otimes M_k$. Moreover, it suffices to prove the claim for the case in which $B = C([0, 1])$. In this reduced case, we only need to consider the case in which $\mathcal{F}$ contains a single element $x \in B$, where x is the identity function on $[0, 1]$. Now, for any $\varepsilon > 0$, let $\xi_1, \xi_2, \ldots, \xi_n$ be $\varepsilon/4$-dense in $[0, 1]$ and $\mathrm{dist}(\xi_i, \xi_j) > \varepsilon/8$ if $i \neq j$. Denote by f_i continuous functions with $0 \leq f_i \leq 1$, which are one on $(\xi_i - \varepsilon/32, \xi_i + \varepsilon/32)$ and zero on $[0, 1] \setminus (\xi_i - \varepsilon/16, \xi_i \varepsilon/16)$. By 3.2, there is a nonzero projection $e_i \in \overline{f_i A f_i}$. Note that $e_i e_j = 0$ if $i \neq j$. Set $p = \sum_{i=1}^{n} e_i$.

$$\|x - [(1 - p)x(1 - p) + \sum_{i=1}^{n} \xi_i e_i]\| < \varepsilon/2 \qquad \text{and} \qquad \|[p, x]\| < \varepsilon \text{ by (P1)}.$$

Let F_1 be the finite-dimensional C^*- subalgebra generated by $e_1, e_2, \ldots, e_n$. Then by (P2), $pxp \in_\varepsilon F_1$ and $\|pxp\| \geq \|x\| - \varepsilon$.

Theorem(4.1.12)[89]: Every unital separable simple TAI C^*- algebra is MF [7].

Proof: Let A be such a C^*- algebra and let $\{x_n\}$ be a dense sequence in the unit ball of A, there are projections pn $\in$ A and finite-dimensional C^*- subalgebras B_n with $1_{B_n} = p_n$ such that

(i) $\|[p_n, x_i]\| < 1/n$, and

(ii) $px_i\, p \in_{1/n} B_n$ and $\|p_n x_i p_n\| \geq \|x_i\| - 1/n$ for $i = 1, 2, \ldots, n$.

Let $id_n : B_n \to B_n$ be the identity map and let $j : B_n \to M_{K_{(n)}}$ be a unital embedding. We note that such j exists provided that $K_{(n)}$ is large enough, there exists a completely positive map $L'_n : p_n A\, p_n \to M_{K_{(n)}}$ such that $L'_n \backslash B_n = j \circ id_n$. Since L'_n is unital.

L'_n is a contraction. We define $L_n : A \to M_{K_{(n)}}$ by $L_n(a) = L'_n(p_n a p_n)$. Let $y_{i,n} \in B_n$ such that $\|p_n x_i p_{n-} y_{i,n}\| < 1/n$, $n = 1, 2, \ldots$. Then

$$\|L_n(x_i) - p_n x_i p_n\| \leq \|L_n(x_i - y_{i,n}) - (y_{i,n} - p_n x_i p_n)\| < 2/n \to 0$$

As $n \to \infty$. Combining this with (i) above, we see that

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0$$

as $n \to \infty$. Define $\Phi : A \to \prod_{n=1}^{\infty} M_{m(n)}$ by sending a to $\{L_n(a)\}$. Then Φ is a completely positive map. Denote by $\pi : \prod_{n=1}^{\infty} M_{m(n)} \to \prod_{n=1}^{\infty} M_{m(n)} / \oplus_{n=1}^{\infty} M_{m(n)}$. Then

$\pi \circ \Phi : A \to \prod_{n=1}^{\infty} M_{m(n)} / \oplus_{n=1}^{\infty} M_{m(n)}$.

is a (nonzero) homomorphism. Since A is simple, $\pi \circ \Phi$ is injective. It follows from [4, 3.22] that A is an MF-algebra.

Corollary (4.1.3)[89]: Every separable unital C^*- algebra satisfying (P1) and (P2) is MF.

Proof: We actually proved this above. Note, simplicity is not needed for injectivity since $\|p_n x p_n\| \to \|x\|$.

Proposition (4.1.14)[89]: Every nuclear separable simple TAI C^*- algebra is quasidiagonal.

Proof. As in [7], a separable nuclear MF C^*- algebra is NF, and it is quasidiagonal. In fact it is strong NF (see [8]).

Corollary. (4.1.15)[89]: Every unital separable simple TAI C^*- algebra has at least one tracial state.

Proof: It is well known that $\prod_{n=1}^{\infty} M_{m(n)} / \bigoplus_{n=1}^{\infty} M_{m(n)}$. has tracial states. Tracial states are defined by weak limits of tracial states on each $M_{m(n)}$. Let τ be such a tracial state. Then, in the proof of 4.2, let $t(a) = \tau \circ \pi \circ \Phi(a)$.

Theorem(4.1.16)[89]: A unital simple TAI C^*- algebra has stable rank one.

Proof: Let A be a unital simple C^*- algebra. Take a nonzero element a $\in$ A. We will show that a is a norm limit of invertible elements in A. So we may assume that a is not invertible and $\|a\|= 1$. Since A is finite, a is not one-sided invertible. For any $\varepsilon > 0$, by [50, 3.2], there is a zero divisor b $\in$ A such that $\|a - b\| < \varepsilon/2$. We further assume that $\|b\| \leq 1$. Therefore, by [122], there is a unitary u $\in$ A such that ub is orthogonal to a nonzero positive element c $\in$ A. Set d = ub. Since A has (SP), there exists a nonzero projection e $\in$ A such that de = ed = 0. Since A is simple and has (SP), we may write $e = e_1 \oplus e_2$ with $e_2 \lesssim e_1$. Note that $d \leq (1 - e) \leq (1 - e_1)$. Moreover, $(1 - e_1)A(1 - e_1)$ is TAI.

Let $\eta > 0$ be a positive number. There is a projection p $\in (1 - e_1)A(1 - e_1)$ and a unital C^*- subalgebra B $\in$ I with $1_B = p$ such that:

(i) $\| [x, p] \| < \eta$,

(ii) $pxp \in_\eta$ B for all x $\in \mathcal{F}$, and

(iii) $[1 - e_1 - p] \leq [e_2]$,

where $\mathcal{F}$ contains d. Thus, with sufficiently small η, we may assume that

$\|d - (d_1 + d_2)\| < \varepsilon/16$,

where $d_1 \in B$ and $d_2 \in (1 - e_1 - p)A(1 - e_1 - p)$.

Since C^*- algebras in I have stable rank one and B $\in$ I, there is an invertible $d'_1 \in$ B such that $\|d_1 - d'_1\| < \varepsilon/8$.

Let v be a partial isometry such that $v*v = (1 - e_1 - p)$ and $vv* \leq e_1$. Set $e'_1 = vv*$ and $d'_2 = \varepsilon/8(e_1 - e'_1) + (\varepsilon/8)v + (\varepsilon/8)v* + d_2$. Note that $(\varepsilon/8)v + (\varepsilon/8)v* + d_2$ has matrix decomposition

$$\begin{pmatrix} 0 & \varepsilon/8 \\ \varepsilon/8 & d_2 \end{pmatrix}$$

Therefore d'_2 is invertible in (1−p)A(1−p). This implies that d'=d'_1+d'_2 is invertible in A.

We also have

$$\|d'_2 - d_2\| < \varepsilon/8$$

whence

$\|d - d'\| < \|d - (d_1 + d_2)\| + \|(d_1 + d_2) - (d'_1 + d'_2)\| < \varepsilon/16 + \varepsilon/8 + \varepsilon/8 < 3\varepsilon /8$

We have

$$\|b - u * d'\| \leq \|u * u(b - u * d')\| = \|ub - d'\| < 3\varepsilon/8.$$

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Finally,
$$\|a - u * d'\| \leq \|a - b\| + \|b - u * d'\| < \varepsilon/2 + 3\varepsilon/8 < \varepsilon.$$
Note that u∗d' is invertible.

Corollary (4.1.17)[89]: Every unital simple TAI C^*-algebra has the cancellation of projections, i.e., if $p \oplus e \sim q \oplus e$ then $p \sim q$.

Theorem (4.1.18)[89]: Every unital simple TAI C^*-algebra has the following Fundamental Compara-

bility [5]: if p, q ∈ A are two projections with $\tau(p) < \tau(q)$ for all tracial states τ on A, then p $\preccurlyeq$ q.

Proof: Denote by T (A) the space of all normalized traces. It is compact. There is d > 0 such that $\tau(q - p) > d$ for all $\tau \in$ T (A). It follows that there exists a nonzero projection e $\leq$ q such that $\tau(e) < d/2$ for all $\tau \in$ T (A). Set q' = q − e. Then $\tau(q' - p) > d/2$ for all $\tau \in$ T (A).

It follows from [8, 6.4] that there exists a nonzero a $\in A_+$ such that q' − p − (d/4) = a + z and there is a sequence $\{u_n\}$ in A
$$z = \sum_n u_n * u_n - \sum_n u_n u_n *.$$
Choose an integer N > 0 such that
$$\|\sum_n u_n * u_n - \sum_{n=1}^{N} u_n * u_n\| < d/128 \text{ and} \|\sum_n u_n u_n * - \sum_{n=1}^{N} u_n * u_n\| < d/128.$$
Let $\mathcal{F} = \{$p, q, q' , e, z, un, u∗n, n = 1, 2, . . . , N $\}$ and let $0 < \varepsilon < 1$. Since A is TAI, there exists a projection P ∈ A and a C∗ -subalgebra B $\in$ I with $1_B = P$ such that:

(i) $\|[x, P]\| < \varepsilon/2N$,

(ii)P xP $\in_{\varepsilon/2N}$ B for all x $\in \mathcal{F}$ and

(iii)$(1 - P) \preccurlyeq$ e.

With sufficiently small ε, using a standard perturbation argument, we obtain projections q" = $q_1 + q_2$, p' = $p_1 + p_2$, where q_1, q_2, p_1, p_2 are projections, $p_1, q_1 \in$ B and $q_2, p_2 \in (1 - P)A(1 - p)$ such that
$$\|q'' - q'\| < d/32 \text{ and } \|p' - p\| < d/32.$$
Furthermore (with sufficiently small ε), we obtain $v_1, v_2, . . . , v_N \in$ B such that
$$\|(q_1 - p_1 - (d/4)P) - (b + \sum_{n=1}^{N} v_n * v_n \sum_{n=1}^{N} v_n v_n *)\| < d/16,$$
where b$\in B_+$and $\|P aP - b\| < \varepsilon/2N$ .Denote by T (B) the space of all normalized traces on B.

Then
$$\tau(q_1 - p_1 - (d/4)P - b) > -d/16$$
for all $\tau \in$ T (B). Therefore
$$\tau(q_1 - p_1) > d/4 - d/16 = 3d/16$$
for all $\tau \in$ T (B). This implies that $p_1 \preccurlyeq q_1$ q1 in B, whence also in A. Since $p_2 \preccurlyeq (1 - P) \preccurlyeq$ e, we conclude that
$$[p] = [p_1 + p_2] \leq [q_1] + [e] \leq [q].$$

Theorem(4.1.19)[89]: Let A be a unital simple TAI C^*-algebra. Then $K_0(A)$ is weakly unperforated and satisfies the Riesz interpolation property.

Theorem (4.1.20)[89]: Let A be a unital simple C^*- algebra. Then A is TAI if and only if the following hold. For any finite subset $\mathcal{F} \subset$ A containing a nonzero element b, $\varepsilon > 0$, integers n > 0 and N >0, and any nonzero projection e $\in$A, there exist a nonzero projection p $\in$A and a C^*- subalgebra

$I = \bigoplus_{i=1}^{k} M_{n_i}(C([0, 1]))$, with $1_I = p$ and $\min\{n_i: 1 \leq i \leq k\} \geq N$, such that:

(i) $\| [x, p] \| < \varepsilon$ for all x $\in$ F,

(ii) pxp $\in_\varepsilon$ I for all x $\in \mathcal{F}$ and $\|pbp\| \geq \|b\| - \varepsilon$, and

(iii) $1 - p$ is unitarily equivalent to a projection in eAe.

Proof: To show that the above is sufficient for A being TAI we note that A has property (SP) Then, a result of Cuntz, there exists a projection q $\in$ eAe such that $(n + 1)[q] \leq [e]$. Then it is clear that the above (iii') implies (iii) (if we use the projection q instead of e).

To see it is also necessary, we use the fact that simple TAI C^*- algebras have stable rank one (so they have cancellation).

Definition(4.1.21)[89]: Let A be a unital simple C^*- algebra. Then A has tracial topological rank no more than one, denote by TR(A) ≤ 1, if the following holds. For any $\varepsilon > 0$, and any finite subset$\mathcal{F} \subset$A containing a nonzero element a $\in A_+$, there is a C^*- subalgebra C in A with $C = \bigoplus_{i=1}^{k} M_{n_i}(C([X_i))$, where each X_i is a finite CW complex with dimension no more than One such that $1_C =$p satisfying the following:

(i)$\|px - xp \| < \varepsilon$ for x $\in \mathcal{F}$,

(ii)pxp $\in_\varepsilon$ C for x $\in \mathcal{F}$ and

(iii)$1 - p$ is equivalent to a projection in $\overline{aAa}$.

In the above definition, if T can be chosen to be a finite-dimensional T* -subalgebra then we write TR(A) = 0 (see [91]). If TR(A)$\leq$ 1 but TR(A) $\neq 0$ (see [91]) then we will write TR(A) = 1.

In the light of [91] in what follows, we will replace unital simple TAI C^*- algebras by unital simple C^*- algebras with tracial topological rank no more than one and write TR(A) ≤ 1.

Definition (4.1.22)[89]: A unital simple C^*- algebra A is said to be tracially approximately divisible if for any $\varepsilon > 0$, any projection e $\in$ A, any integer N > 0 and any finite subset $\mathcal{F} \subset$ A, there exists a projection q $\in$ A and there exists a finite-dimensional C^*- subalgebra B with each simple summand having rank at least N such that:

(i) $\|qx - xq\| < \varepsilon$ for all x $\in \mathcal{F}$,

(ii) $\|y(1 - q)x(1 - q) - (1 - q)x(1 - q)y]\| < \varepsilon$ for all x $\in \mathcal{F}$ and all y $\in$ B with $\| y \| \leq 1$, and

(iii)q is unitarily equivalent to a projection of eAe.

Of course if A is approximately divisible, then A is tracially approximately divisible (see [4]).

Theorem(4.1.23)[89]: Every nonelementary unital simple C^*- algebra with TR(A) ≤ 1 is tracially ap-proximately divisible.

Proof. Let A be a unital simple C^*- algebra with TR(A)$\leq$ 1. Fix $\varepsilon > 0$, $\sigma > 0$, N > 0 and a finite subset $\mathcal{F} \subset$ A. Let b $\in$ A with $\|b\| = 1$ and assume that b $\in$F. There exist a projection p $\in$ A and a C^*- subalgebra C $\in\mathcal{I}$ with $1_C = $ p such that:

(i)$\|px - xp\| < \varepsilon/4$ for all x $\in \mathcal{F}$,

(ii)pxp $\in_{\varepsilon/4}$ C and $\|pbp\| \geq \|b\| - \varepsilon/2$, and

(iii)$\tau (1 - p) < \sigma/2$ for all traces τ on A.

Write C $=\oplus$ C_i , where $C_i = M_{l(i)}(C[0, 1])$, or $C_i = M_{l(i)}$. It will become clear that, without loss of generality, to simplify notation, we may assume that C = C_i (i.e., there is only one summand). If C = M_l , let {eij } be matrix units for M_l . Since A is not elementary, there is a positive element a $\in e_{11}Ae_{11}$ such that sp(a) = [0, 1] (see [1, p. 6.1]). This implies that C $\subset M_l$ (C([0, 1])). So, we may assume that C = M_l (C([0, 1])). Let $\mathcal{G}_1\subset$ C be a finite subset such that dist(pxp, $\mathcal{G}_1$) $< \varepsilon/4$

for all x $\in \mathcal{F}$. Let $\mathcal{G}$ be a finite subset of C containing {e_{ij} } and e_{ij} ge $*_{ij}$ for all g $\in\mathcal{G}_1$.

Let $\eta > 0$. Denote by δ the positive number in [59] corresponding to η (instead of ε). Let {f_1, f_2, . . . , f_m} $\subset$ C([0, 1]) respect to δ (= d). We identify C([0, 1]) with $e_{11}C e_{11}$. Since $e_{11}A e_{11}$ is simple, there are $b_{ij}\in e_{11}Ae_{11}$ such that

$$\left\|\sum_{i=1} b_{ij}f_j b *_{ij}- e_{11}\right\| < 1/16$$

j= 1, 2, . . . , m. Let $\mathcal{G}_2$ be a finite subset containing {f_j , b_{ij} , $b *_{ij}$ }$\cup$ {$a_{ij}\in e_{11}Ae_{11}$: $(a_{ij})_{l\times l}\in\mathcal{G}$ }. By 3.4, TR($e_{11}Ae_{11}$) 1. So for any 0 $<\sigma < \eta/2$ and any finite subset $\mathcal{G}_3\supset\mathcal{G}_2$, there exist aprojection q $\in e_{11}Ae_{11}$ and a C^*-subalgebra $C_1\subset e_{11}Ae_{11}$ with $1_{C_1} = $ q and $C_1\in$I satisfying the following:

    (a) $\|qx - xq\| < \sigma$,

    (b) qxq $\in_\sigma C_1$ for all x $\in\mathcal{G}_3$,

    (c)$\tau (e_{11} - q) < \sigma/2l$ for all traces τ.

With sufficiently small σ and sufficiently large $\mathcal{G}_2$, we may assume that there exists a homomorphism $\varphi : C([0, 1]) \rightarrow C_1$ such that

(b') $\|\emptyset (x) - qxq\| < \eta/2$ for all x $\in\mathcal{G}_2 \cap$ C([0, 1]).

Note that we also have $c_{ij}\subset C_1$ such that

$\left\|\sum_{i=1} c_{ij} \emptyset (f_j)c *_{ij}- q\right\| < 1/8$, j = 1, 2, . . . , m.

We are now it follows that Sp($\emptyset$t) is δ-dense in [0, 1]. Byapplying in [59] there is a homomorphism $\psi : C([0, 1]) \to C_1$ and there is a finite-dimensional C^*-subalgebra F $=\bigoplus_i F_i$, where each F_i is simple and dim $F_i \geq N$, with $1_F= q$ such that

$$\|\psi (f) - \emptyset (f)\|< \eta/2 \text{ for all f}\in G_2 \text{and}$$
$$\|\psi (g), b\|=0$$

for all $g\in C([0,1])$ and $b \in$ F. Set F'= diag(F , F , . . . , F) in F$\otimes M_l$, $\psi' = \psi \otimes id_{M_l}$, $\emptyset'=\emptyset\otimes id_{M_l}$ and P = diag(q, q, . . . , q) $\in M_l (C_1)$. With sufficiently small η and large G_2, we have

$$\|\psi' (g) - \emptyset' (g)\|< \varepsilon/2 \text{ for g} \in G.$$

We also have

$$\|[\psi' (f),c] \| = 0 \text{ for f} \in C \text{ and c} \in F'.$$

These imply that

$$\|[P xP ,c]\|< \varepsilon \quad \text{ for all x} \in F \text{ and c} \in F'.$$

Note that $1_F = P$. Wealso have
$\tau (1 - P) \leq \sigma/2 + l\sigma/2l = \sigma.$
We conclude that A is tracially approximately divisible. When C $=\bigoplus C_i$, it is clear that we can do exactly the same as above for each summand. Let $d_i = 1_{C_i}$. If we find a matrix algebra $F_i\in d_i Ad_i$ with rank greater than N which commutes with C_i , then $\bigoplus F_i$commutes with C.

Lemma(4.1.24)[89]: Let A be a unital nuclear simple C^*-algebra with TR(A)$\leq$ 1. Then for any $\varepsilon > 0$, any $\sigma > 0$, any integer n > 0, and any finite subset F$\subset$ A, there exist mutually orthogonal projections q, $p_1, p_2, \ldots, p_n$ with q$\lesssim p_1$ and $[p_1] = [p_i]$ (i = 1, 2, . . . , n), a C^*-subalgebra C $\in$I with $1_C = p_1$ and completely positive linear contractions $L_1 : A \to$ qAq and $L_2 : A \to$ C such that

$$\|x - (L_1(x) \oplus diag(L_2(x), L_2(x),\ldots,L_2(x)))\| < \varepsilon \text{ and}$$
$$\|L_i (xy) - L_i (x)L_i (y)\|< \varepsilon,$$

where $L_2 (x)$ is repeated n times, for all x, y $\in$F and $\tau(q) < \sigma$ for all $\tau \in$ T (A).

Proof. From the proof of 5.4, we have the following. For any $\eta > 0$, any integer K > 0, any integer $N > 4Kn^2$ and finite subset $G\subset$ A (containing 1_A), there exists a projection P $\in$ A and a finite-dimensional C^*-subalgebra B with $1_B = P$ such that:

(i)$\|[P ,x]\|< \eta$ for all x $\in$ G;

(ii)every simple summand of B has rank at least N ;

(iii)there is a C^*-subalgebra D $\in$I with $1_D = P$ such that [d, g] = 0 for all d $\in$ D, g $\in$ B and

$$dist(x, D) < \eta \qquad \text{for x} \in G; \text{ and}$$

(iv) 5N [(1 $-$ P)] < [P].

Let $F_1 \subset A$ be a finite subset (containing 1_A) and $\sigma > 0$. Since A is nuclear, with sufficiently large $\mathcal{G}$ and sufficiently small η, by [32, 3.2], there are unital completely positive linear contrac-tions $L'_1 : A \to (1-P)A(1-P)$ and $'L_2 : A \to D$ such that $'L_1(a) = (1-P)a(1-P)$,

$$\|x - L'_1(x) \oplus L'_2(x)\| < \sigma \quad \text{and} \quad \|L'_2(x) - P x P\| < \eta + \sigma$$

for all $x \in F_1$. It follows that, with sufficiently small σ and η,

$$\|L'_i(xy) - L'_i(x)L'_i(y)\| < \varepsilon$$

for all $x, y \in F_i$. Write B $= \oplus_{i=1}^k B_i$, where $B_i \cong M_{l(i)}$ with $l(i) \geq N$, and denote by C the C^*-subalgebra generated by D and B. Note that $C \cong \oplus_{i=1}^k D_0 \otimes B_i$, where $D_0 \cong D$. Let $\pi_i : C \to D_0 \otimes B_i$ the projection. Denote $\emptyset_i = \pi_i \circ L'_2$. By (iii), we see that we may write $\emptyset_i =$ diag(ψ_i, ψ_i, . . . , ψ_i), where $\psi_i : A \to e_i(D_0 \otimes M_{l(i)})e_i$ and e_i is a minimal rank-one projection of $M_{l(i)}$. Write $l(i) = k_i n + r_i$, where $k_i \geq n > r_i \geq 0$ are integers. We may rewrite
$\emptyset_i = $ diag ($\emptyset'_i$, . . . , $\emptyset'_i$) $\oplus \psi'_i$,
where $\emptyset'_i = $ diag(ψ_i, . . . , ψ_i) : $A \to D_0 \otimes M_{k_i}$ is repeated n times and $\psi'_i = $ diag(ψ_i, . . . ,
ψ_i): $A \to D_0 \otimes M_{r_i}$.
Define$L_2 = \oplus_{i=1}^k \emptyset'_i$ and $L_1 = L'_1 \oplus_{i=1}^k \psi'_i$. We estimate that
$$\tau\left((1 - P) + \oplus_{i=1}^k \psi'_i(1_A)\right) < (1/5N)\tau(P) + (1/4nK)\tau(P) < (1/2n)\tau(P)$$
$\leq \min(\sigma, \tau L_2[(1_A)]))$,
provided that $1/K < \sigma$. By 4.7, the lemma follows.

The following corollary follows from Lemma 5.5 immediately.

Corollary(4.1.25)[89]: Let A be a unital separable simple C^*-algebra TR(A)≤ 1. Then for any $\varepsilon > 0$, any $\sigma > 0$, any integer $n > 0$, and any finite subset $F \subset A$, there exists a C^*-subalgebra $C \in \mathcal{I}$ such that
$$\|x - (1-p)x(1-p) \oplus diag(y, y, \ldots, y)\| < \varepsilon$$
where $y \in C$ and diag(y, y, . . . , y) $\in M_n(C)$ and $p = 1_{M_n(C)}$ for all x, $\in F$ and $\tau((1-p)) < \sigma$ for all $\tau \in T(A)$. Moreover, we may require that $\|(1-p)x(1-p)\| \geq (1-\varepsilon)\|x\|$ for all x $\in F$.

Proof. Perhaps the last part of the statement needs an explanation. In the proof of (4.1.24), we know that we may require that$\|y\| \geq (1-\varepsilon/2)\|x\|$ for all x $\in F$. Thus we may replace $(1-p)x(1-p)$ by $(1-p)x(1-p) \oplus y$ and replace $(1-p)$ by $1-p \oplus $ diag(1_C, 0, . . . , 0).

Lemma (4.1.26)[89]: Let B $= \oplus_{i=1}^k B_i$ be a unital C^*-algebra in I (where B_i is a single summand).
For any $\varepsilon > 0$, any finite subset $F \subset B$ and any integer $L > 0$, there exist a finite subset $\mathcal{G} \subset B$ depending on ε and F but not on L, and $\delta = 1/4L$ such that the following holds. If A is a unital separable nuclear simple C^*- algebra with TR(A)≤ 1 and $\emptyset_i : B \to A$ are two homomorphisms satisfying the following:

(i) there are $a_{g,i}$, $b_{g,j} \in$ A, i, j $\leqslant$ L with

$$\left\|\sum_i a *_{g,i} \emptyset_1 (g) a_{g,i} - 1_A\right\| < 1/16 \text{ and } \left\|\sum_j b *_{g,i} \emptyset_2 (g) b_{g,i} - 1_A\right\| < 1/16$$

for all g $\in$ G;

(ii)$(\emptyset_1)_* = (\emptyset_2)_*$ on $K_0(B)$; and,

(iii) if $\| \tau °\emptyset_1(g) - \tau °\emptyset_2 (g) \| < \delta$ for all g $\in \mathcal{G}$, then there exists a unitary u $\in$ A such that

$$\|\emptyset_1(f) - u * \emptyset_2(f)u\| < \varepsilon \quad \text{for all f} \in F.$$

Theorem(4.1.27)[89]: Let A be a unital simple C^*- algebra with TR(A) $\leqslant 1$ and C be a C* -subalgebra of A in I. Then for any finite subset $\mathcal{F} \subset$ C and $\varepsilon > 0$, there exist $\delta > 0$, $\sigma > 0$ and a finite subset $\mathcal{G} \subset$ A satisfying the following: if L_1, L_2 : A $\rightarrow$ B are two unital $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps, where B is a unital simple C^*- algebra with TR(B) $\leqslant 1$, with

$$(L_1|C)* = (L_2|C)* \text{ on } K_0(C) \text{ and}$$
$$|\tau(L_1(g)) - \tau°L_2(g)| < \sigma$$

for all g $\in \mathcal{G}$ and for all $\tau \in$ T (B), then there is a unitary u $\in$ A such that

$$\|L_1(f) - u * L_2(f)u\| < \varepsilon \quad \text{for all f} \in F.$$

Proof. Fix $\varepsilon > 0$ and a finite subset F$\subset$ A. Let $\mathcal{G}_1 \subset$ C be the finite subset required by 5.7 (for a given $\varepsilon > 0$ and a given finite subset $\mathcal{F}$). Suppose that $a_{g,i} \in$ A such that

$$\left\|\sum_{i=1}^{n(\mathcal{G})} a *_{g,i} \ g \ a_{g,i} - 1_A\right\| < 1/64$$

for all g $\in \mathcal{G}_1$. Set L = max{n(g): g $\in$ G}. Then, with sufficiently small $\delta > 0$ and large $\mathcal{G} \supset \mathcal{G}_1 \cup \{a_{g,i} : g, i\}$, we have $b_{g,i},$j $\in$ A such that

$$\left\|\sum_{i=1}^{n(\mathcal{G})} b *_{g,i,j} L_j(g) b_{g,i,j} - 1_B\right\| < 1/32$$

for all g $\in \mathcal{G}_1$ and j = 1, 2. Furthermore, for any $\eta > 0$, with sufficiently small δ, there is a homomorphism $\emptyset_j$: C $\rightarrow$ B (j = 1, 2) such that

$$\|\emptyset_j (g) - L_j (g)\| < \eta \quad \text{and} \quad \left\|\sum_{i=1}^{n(\mathcal{G})} b *_{g,i,j} \emptyset_j(g) b_{g,i,j} - 1_B\right\| < 1/16$$

for g $\in \mathcal{G}_1$. We also require that $\sigma < 1/4L$. Then we see the conclusions of the theorem follow from 5.7 (and its proof) immediately.

Theorem(4.1.28)[89]: Let A be a unital simple C^*- algebra with TR(A) ≤ 1 and B $\in$ I. Let $\emptyset_i$: B $\rightarrow$ A be two monomorphisms such that

$$(\emptyset_1)* = (\emptyset_2)* : K_0 (B) \rightarrow K_0 (A) \quad \text{and} \quad \tau ° \emptyset_1 = \tau ° \emptyset_2$$

for all $\tau \in$ T (A). Then there is a sequence of unitaries un $\in$ A such that

$$\lim_{n \to \infty} u *_n \emptyset_1(x)u_n = \emptyset_2(x) \text{for all x} \in B.$$

Proof: As before, we reduce the general case to the case in which B = C([0, 1]). Let $\varepsilon > 0$ and F⊂ B be a finite subset. Let $\mathcal{G}$⊂ B be the finite subset (it does not depend on L). Since A is simple, there exists an integer L > 0 and $a_{i,g}$, $b_{i,g}$∈ A, i = 1, 2, . . . , L (some of them could be zero) such that

$$\left\|\sum_i a*_{i,g}\,\emptyset_1(g)a_{i,g} - 1\right\| < 1/16 \ and \ \left\|\sum_i b*_{i,g}\,\emptyset_2(g)b_{i,g} - 1\right\| 1/16$$

for all g ∈$\mathcal{G}$. Therefore the theorem follows from 5.7.

We start with the following observations.

Let A be a unital C^*- algebra and p, a ∈ A. Suppose that p is a projection, $\|a\| \leqslant 1$ and

$$\|a*a - p\| < 1/16 \quad and \quad \|aa* - p\| < 1/16.$$

A standard computation shows that

$$\|pap - ap\| < 3/16 \quad and \quad \|pa - pap\| < 3/16.$$

Also $\|pa - a\| < 1/2$. Set b = pap. Then

$$\|b*b - p\| \leqslant \|pa*ap - pa*a\| + \|pa*a - p\| < 1/16 + 1/16 = 1/8.$$

So

$$\|(b*b)^{-1} - p\| < \frac{1/8}{1-1/8} = 1/7 \ and \quad \||b|^{-1} - p\| < 2/7,$$

where the inverse is taken in pAp. Set v = $b|b|^{-1}$. Then v*v = p = vv* and
$\|v - b\| \quad < 2/7$.

We denote v by $\tilde{a}$. Suppose that L : A → B is a $\mathcal{G}$-δ-multiplicative contractive completely posi-tive linear map, u is a normal partial isometry and a projection p ∈ B is given so that

$$\|L(u*u) - p\| < 1/32.$$

Note if v' is another unitary in pApwith $\|v' - b\| < 1/3$, then $[v'] = [v\,]$ in U (pAp)/U_0(pAp).We define $\tilde{L}$ as follows. Let L(u) = a. With small δ and large $\mathcal{G}$ we denote by $\tilde{L}$(u) the normal partial isometry (unitary in a corner) v defined above. This notation will be used later. Note also, if u ∈U_0(A), then, with sufficiently large $\mathcal{G}$ and sufficiently small δ, we may assume that

$$\tilde{L}(u) \in U_0(B).$$

Definition(4.1.29)[89]: Let A be a unital C^*- algebra. Let CU(A) be the closure of the commutator sub-group of U (A). Clearly that the commutator subgroup forms a normal subgroup of U (A). It follows that CU(A) is a normal subgroup of A. It should be noted that U (A)/CU(A) is commu-tative. It is an easy fact that if A = M_r (C(X)), where X is a finite CW complex of dimension 1, then CU(A) ⊂U_0(A). If $K_1(A)$ = U (A)/U_0(A), it is known and easy to verify that every commutator is in U_0(A). Therefore CU(A) ⊂U_0(A). If u ∈ U (A), we will use $\bar{u}$ for the imageof u in U (A)/CU(A), and if F ⊂ U (A) is a subgroup of U (A), then$\bar{F}$is the image of F in U (A)/CU(A).

If $\bar{u}$, $\bar{v}$∈ U (A)/CU(A) define

$\text{dist}(\bar{u}, \bar{v}) = \inf \{\| x - y \| : x, y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v} \}$.

If u, v $\in U(A)$ then dist $(\bar{u}, \bar{v}) = \inf \quad \{\|uv* - x\| : x \in CU(A)\}$Let $g = \prod_{i=1}^{n} a_i b_i a_i^{-1} b_i^{-1}$, where$a_i b_i \in U(A)$. Let $\mathcal{G}$ be a finite subset of A, $\delta > 0$ and $L : A \to B$ be a $\mathcal{G}$-δ-multiplicative contractive completely positive linear map, where B is a unital C^*- algebra. From 6.1, for $\varepsilon > 0$, if $\mathcal{G}$ is sufficiently large and δ is sufficiently small,

$$\left\| L(g) - \prod_{i=1}^{n} a'_i b'_i (a'_i)^{-1} (b'_i)^{-1} \right\| < \varepsilon/2,$$

Where $a'_i b'_i U(B)$. Thus, for any g $\in$ CU(A), with sufficiently large $\mathcal{G}$and sufficiently small δ,

$$\|L(g) - u\| < \varepsilon$$

for some u $\in$ CU(B). Moreover, for any finite subset $\mathcal{U} \subset U(B)$ and subgroup F$\subset$U(B) generated by $\mathcal{U}$, and $\varepsilon > 0$, there exists a finite subset $\mathcal{G}$ and $\delta > 0$ such that, for any $\mathcal{G}$-δ-multiplicative contractive completely positive linear map $L : A \to B$, L induces a homomorphism L‡ $: \bar{F} \to U(B)/CU(B)$ such that dist $(\overline{L˘(u)}, $L‡$(\bar{u})) < \varepsilon$ for all $\mathcal{U}$ Note we may also assume that $\bar{F} \cap U_0(A)/CU(A) \subset U_0(B)/CU(B)$

If $\emptyset$: A $\to$ B is a homomorphism then $\emptyset^{\ddagger}$: U (A)/CU(A) $\to$ U (B)/CU(B) is the induced homomorphism. It is continuous.

Recall that, for a unitary u$\in U_0(A)$ in a unital C^*- algebra A, we write cer(u)$\leqslant$ k, if u = $\prod_{j=1}^{k} \exp(ihj)$ for some self-adjoint elements $h_j \in$ A. We write cer(u)$\leqslant$ k + ε if u is a norm limit of unitaries u_n with cer(u_n)$\leqslant$ k.

Let u $\in U_0(A)$. Denote by cel(u) the infimum of the length of continuous paths of unitaries in A from u to 1_A.

Lemma(4.1.30)[89]: (N.C. Phillips). Let A be a unital C^*- algebra and $2 > d > 0$. Let $u_0, u_1, \ldots, u_n$ be n + 1 unitaries in A such that

$$u_n = 1_A \quad \text{and} \quad \|u_i - u_{i+1}\| \leqslant d, \quad i = 0, 1, \ldots, n - 1.$$

Then there exists a unitary v $\in M_{2n+1}(A)$ with exponential length no more than 2π such that

$$\left\|(u_0 \oplus 1_{M_{2n}}(A)) - v\right\| \leqslant d.$$

Moreover, v can be chosen in CU($M_{2n+1}(A)$).

The following is another version of the above lemma.

Lemma(4.1.31)[89]: Let A be a unital C^*- algebra and u $\in U_0(A)$. Then for each L > 0, if u = v $\oplus$ (1 − p) and v $\in U_0(pAp)$ with cel(v)$\leqslant$ L in pAp and there are N (> 2L) mutually orthogonal and mutually equivalent projections in (1 − p)A(1 − p) each of which is equivalent to p, then cel(u)$\leqslant 2\pi$ + (L/n)π. Furthermore, there is a unitary w $\in$ CU(A) such that cel(uw) < (L/n)π.

(See [111] and also [109]. It should be noted that a unitary in $M_2(A)$ with the form diag(u, u∗) is in CU($M_2(A)$).)

Theorem(4.1.32)[89]: Let A be a unital simple C^*- algebra with $TR(A) \leqslant 1$. Let $u \in U_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries $u_1, u_2 \in$ A such that u1 has exponential length no more than 2π, u_2 is an exponential and

$$\|u - u_1 u_2\| < \varepsilon.$$

Moreover, $cer(A) \leqslant 3 + \varepsilon$.

Proof: Let ε be a positive number. Let $v_0, v_1, \ldots, v_n \in U_0(A)$ such that

$$v_0 = u, \ v_n = 1 \quad \text{and} \quad \|v_i - v_{i+1}\| < \varepsilon/16, \quad i = 0, 1, \ldots, n-1.$$

Let $\delta > 0$. Since $TR(A) \leqslant 1$, there exists a projection $p \in$ A and a unital C^*- subalgebra $B \subset A$ with B $\in$ I and with $1_B = p$ such that:

(i)$\|[v_i, p]\| < \delta$, i = 0, 1, \ldots, n,

(ii)$pv_i \ p \in_\delta$ B, 0, 1, \ldots, n, and

(iii)$2(n+1)[1-p] \leqslant$ p.

There are unitaries $w_i \in (1-p)A(1-p)$ with $w_n = (1-p)$ such that

$$\|w_i - (1-p)v_i (1-p)\| < \varepsilon/16, \quad i = 0, 1, \ldots, n$$

for any given $\varepsilon > 0$, provided δ is sufficiently small. Furthermore, there is a unitary $z \in B$ such that

$$\|z - pup\| < \varepsilon/16.$$

Therefore (with $\delta < \varepsilon/32$)

$$\|u - w_1 \oplus z\| \quad < \varepsilon/8.$$

Write $z_1 = w_1 \oplus p$. Since $2(n+1)[1-p] \leqslant p$, , there is a unitary u_1 with exponential length no more than 2π such that

$$\|z_1 - u_1\| \quad < \varepsilon/4.$$

Now since z $\in$ B and it is well known that B has exponential rank $1 + \varepsilon$, there is an exponential $u_2 \in$ A such that

$$\|u_2 - (1-p) - z\| \quad < \varepsilon/3.$$

Therefore

$$\|u - u_1 u_2\| < \varepsilon.$$

Since $cel(u_1) \leqslant 2\pi$, it follows from [121] that $cer(u_1) \leqslant 2 + \varepsilon$. Therefore $cer(u) \leqslant 3 + \varepsilon$. So $cer(A) \leqslant 3 + \varepsilon$.

Lemma (4.1.33)[89]: Let A be a unital C^*- algebra.

(i)$U_0(A)/CU(A)$ is divisible.

(ii)If u $\in$ U (A) such that $u^k \in U_0(A)$. Then there is v $\in U_0(A)$ such that $v^{-k} = u^{-k}$ in U (A)/CU(A).

(3)Suppose that $K_1(A) = $ U (A)/$U_0(A)$ and G $\subset$ U (A)/CU(A) is finitely generated subgroup. Then one has G = G $\cap$ ($U_0(A)$/CU(A)) $\oplus$ $\kappa(G)$, where κ : U (A)/CU(A) $\rightarrow$ U (A)/$U_0(A)$ is the quotient map.

Proof: Let u $\in U_0(A)$. Then there are $a_1, a_2, \ldots, a_n \in$ Asa such that u $= \prod_{j=1}^n \exp(ia_j)$. For

any integer k > 0, let v $=\prod_{j=1}^{n}$ exp(iaj /k). Then $v^{-k}=\bar{u}$. This proves (1).

To see (2), put u^k $=\prod_{j=1}^{n}$ exp(iaj), where $a_j \in A_{sa}$. Let v $=\prod_{j=1}^{n}$ exp(iaj /k)
Thus $\overline{(uv*)^k} = \bar{1}$. So $v^{-k} = u^{-k}$ .To see (3), we note that (1) implies $0 \to U_0$
(A)/CU(A)$\to$G+U_0(A)/CU(A) $\to$ κ(G) $\to$ 0 splits.

Lemma (4.1.34)[89]: Let A be a unital C^*- algebra and $\mathcal{U} \subset U_0$(A) be a finite subset. Then, for any $\varepsilon > 0$, there is a finite subset $\mathcal{G} \subset$ A and $\delta > 0$ satisfying the following: for any $\mathcal{G}$-δ-multiplicative contractive linear map L : A $\to$ B (for any unital C^*- algebra B), there are unitaries v $\in$ B such that
$$\|L(u) - v\| < \varepsilon/2 \text{ and } cel(v) < cel(u) + \varepsilon/2$$
for all u $\in \mathcal{U}$.

Proof: Suppose that z_0(u) = u, z_j (u) $\in U_0$(A), j = 1, 2, . . . , n(u) such that $\frac{cel(u)}{n(u)}$ $\leqslant$1/4 and

Cel(z_j (u)(z_{j-1}(u)) *)$<\frac{cel(u)}{n(u)}$, j = 1, 2, . . . , n(u),

for all u $\in \mathcal{U}$. Let N = max{n(u): u $\in$ U}. It follows that (for sufficiently large $\mathcal{G}$ and sufficiently small δ) there are unitaries w_j (u) $\in$ U (B) such that
$$\|L z_j (u) - w_j (u)\| < \varepsilon/8N \pi$$
for all j and u$\in \mathcal{U}$. Thus for all u $\in \mathcal{U}$,

$\|L(u) - w_0(u)\| < \varepsilon/2\pi$ and cel $(w_0(u)) < n(u)\left[\frac{cel(u)}{n(u)} + (\varepsilon/8N)2\pi\right] < cel(u) + \varepsilon/2$.

Lemma(4.1.35)[89]: Let A be a unital simple C^*- algebra with TR(A)$\leqslant$ 1 and let u $\in$ CU(A). Then u $\in U_0$(A) and cel(u) 8π.

Proof: We may assume that u is actually in the commutator group. Write u = $v_1 v_2 \bullet \bullet \bullet v_k$, where each v_i is a commutator. We write $v_i = a_i b_i a *_i b *_i$, where a_i and b_i are in U (A). Fix integers N > 0 and K > 0. Since TR(A)$\leqslant$l, by Corollary 3.3, there is a projection p$\in$A and a C^*- subalgebra B $\in \mathcal{I}$ with 1_p = B and B=$\oplus_{i=1}^{l} M_{m_i}$(C(0, 1]))$\oplus_{j=1}^{L} M_{n_i}$, where m_i , $n_i \geqslant$K such that
$$\|a_i - (a'_i \oplus a'_i)\| < \varepsilon/4k, \|b_i - (b'_i \oplus b''_i)\| < \varepsilon/4k, \quad i = 1, 2, . . . , k,$$
$$\|u - (\prod_{i=1}^{k} a'_i b'_i (a'_i)^*(b'_i)^* \oplus a''_i b''_i (a''_i)^*(b''_i)^*)\| < \varepsilon/8,$$
a'_i, b'_iU $((1-p)A(1-p)),a''_i, b''_i \in U_0$(B) and N[1-p]$\leqslant$ [p] . Put w $=\prod_{i=1}^{k} a'_i b'_i (a'_i)^*(b'_i)^*$and z $=\prod_{i=1}^{k} a''_i b''_i (a''_i)^*(b''_i)^*$. Then Det(z) = 1. It follows from [43, 3.4](by choosing K large)we conclude that cel(z)$\leqslant$6π in pAp. It is standard to show that$a'_i b'_i (a'_i)^*(b'_i)^* \oplus (1 - $ p)$\oplus$(1-p) is in $U_0(M_4((1 - p)A(1 - p)))$ and it has exponential length no more than 4(2π) + ε/8k. This implies that (in U $((1 - p)A(1 - p)))$ cel(w $\oplus$ (1 − p)) 8kπ + ε/2. Note the length only depends on k. We can then choose N = N (8kπ + ε) as in 6.4. In this way, cel(w $\oplus$ p)$\leqslant$ 2π + ε/2. It follows that

Cel((w $\oplus$ p) ((1 − p) $\oplus$ z)) $\leqslant$8π + ε/2.

The fact that $\|u - (w \oplus p)((1 - p) \oplus z)\| < \varepsilon/8$ implies that cel(u)$\leqslant 8\pi + \varepsilon$.

Theorem(4.1.36)[89]: Let A be a unital simple C^*- algebra with TR(A)$\leqslant$1. Let u, v $\in$ U (A) such that [u] = [v] in $K_1(A)$ and

$$u^k, v^k \in U_0(A) \quad \text{and} \quad \text{cel}((u^k)^* v^k) < L.$$

Then

$$\text{cel}(u*v) \leqslant 8\pi + L/k.$$

Moreover, there is y $\in$ U_0(A) with cel(y)$\geqslant$L/ k such that $\overline{u * v} = \overline{y}$ in U (A)/CU(A).

Theorem(4.1.37)[89]: Let A be a unital separable simple C^*- algebra with TR(A) $\leqslant$1 and u $\in U_0$(A). Suppose that $u^k \in$ CU(A) for some integer k > 0, then u $\in$ CU(A). In particular, U_0(A)/CU(A) is torsion free.

Proof: Let $\varepsilon > 0$ and let

$$v = \prod_{j=1}^{r} a_i b_i (a'_i)^{-1}(b'_i)^{-1} \quad \text{be such that } \|u^k - v\| < \varepsilon/64$$

Put l= cel(u^k). Let $\delta > 0$ be such that $2(l+\varepsilon)\delta < \varepsilon/64\pi$. Fix a finite subset$\mathcal{G}\subset$A which contains u, u^k, v, $a_i, b_i, a_i^{-1}, b_i^{-1}$among other elements.

Since TR(A)$\leqslant$1, there is a projection p $\in$ A and a unital C^*- subalgebra F $\in$I with 1_F = p such that:

(i)pxp $\in_{\varepsilon/64}$ F for all x $\in\mathcal{G}$,

(ii)$\|v - v_0 \oplus v_1\| < \varepsilon/32$, $\|u - u_0 \oplus u_1\| < \varepsilon/32$, and $\|u^k - u_0^k \oplus u_1^k\| < \varepsilon/32$,

(iii)cel(u_0^k) $< 1 + \varepsilon/32$ in U $((1 - p)A(1 - p))$ and

(iv)$\tau (1 - p) < \delta$ for all $\tau \in$ T (A).

Here $u_0, v_0 \in$ U $((1 - p)A(1 - p))$ and $u_1, v_1 \in$ U (F). Moreover, we may assume that there are $a'_i, b'_i \in$ U (F) such that

$$\|u_1^k - \prod_{j=1}^{r} a'_i b'_i (a'_i)^{-1}(b'_i)^{-1}\| < \varepsilon/32.$$

Put w =$\prod_{j=1}^{r} a'_i b'_i (a'_i)^{-1}(b'_i)^{-1}$. Since U (F) = U_0(F), we may write

$$w = \prod_{m=1}^{L} \exp(id_m) \text{ for}$$

some $d_m F_{sa}$. Put $w_k = \prod_{m=1}^{L} \exp(id_m/k)$ Then $w_k^k = w$ so

$$\text{cel}((u_1)^k(w_k^*)^k) < \frac{\varepsilon\pi}{32}.$$

Write F =$\bigoplus_{s=1}^{N} F_s$where each $F_s = M_{r(s)}C([0, 1])$ or $F_s = M_{r(s)}$, we may assume that each n(s) > max($16\pi^2/\varepsilon$, K(1)), where K(1) is the number described in [113] (with d = 1).

$$\det(\exp(if/K) \exp(ia/ k)u_1 w_k^*) = 1$$

for some a, f $\in F_{sa}$ with $\|f\| \leqslant 2\pi$ and $\|a\| < \varepsilon\pi/32$ (with K > max($16\pi^2/\varepsilon$, K(1))). Exp(if/K) exp(ia/ k)$u_1 w_k^* \in$ CU(F). We also have

$$\|\exp(if/K) \exp(ia/ k) - 1_F\| \quad < \varepsilon/8\varepsilon/32.$$

Thus

$$\text{Dist}(\overline{u_1 w_k^*}, \overline{1}) < \varepsilon/8\varepsilon/32.$$

Since det(w) = 1, as in the proof of Theorem(4.1.36), we also have

$$\det(\exp(ig/K)w_k^*) = 1$$

for some g $\in F_{sa}$ with $\|g\| \leqslant 2\pi$. Again, $\exp(ig/K)w_k^* \in CU(F)$. But

$$\|(exp(ig/K)\,w_k)\,w_k^* - 1\| \leqslant \|exp(ig/K) - 1\| < \varepsilon/4.$$

So

$$\text{dist}(\overline{w_k}, \overline{1}) < \varepsilon/4.$$

Therefore

$$\text{dist}(\overline{u_1}, \overline{1}) \leqslant \text{dist}(\overline{u}, \overline{w_k}) + \text{dist}(\overline{w_k}, \overline{1}) < \varepsilon/8 + \varepsilon/32 + \varepsilon/4 < \varepsilon/2$$

in U (F)/CU(F). On the other hand, the choice of δ,

$$\text{cel}((u_0 \oplus p)z) < \varepsilon/(8\pi)$$

for some z ∈ CU(A). Thus

$$\text{Inf}\{\|u_0 \oplus u_1 - x\|: x \in CU(A)\} < \varepsilon/8 + \varepsilon/8 + \varepsilon/32 + \varepsilon/4 < 3\varepsilon/4.$$

This implies that

$$\text{Inf}\{\|u - x\| : x \in CU(A)\} < \varepsilon.$$

Therefore u ∈ CU(A). Consequently $U_0(A)/CU(A)$ is torsion free.

Corollary (4.1.38)[89]: Let B_n be a sequence of unital simple C^*- algebra with TR(B_n) $\leqslant 1$. Let $\prod_n^b K_1(B_n)$ be the set of sequences z = $\{z_n\}$, where $z_n \in K_1(B_n)$ and z_n can be represented by unitaries in $M_{k(z)}(B_n)$ for some integer K(z) > 0. Then the kernel of the map

$$K_1(\prod_n B_n) \to \prod_n^b K_1(B_n) \to 0$$

is a divisible and torsion free subgroup of $K_1(\prod_n B_n)$.

Proof: By 6.5, the exponential rank of each B_n is bounded by 4. Therefore that the kernel is divisible follows from the fact that each B_n has stable rank one (and has exponential rank bounded by 4) (see [45]). Suppose that $\{u_n\} \in U(M_K(\prod_n B_n))$ such that $[\{u_n\}]$ is in the kernel and k $[\{u_n\}]$ = 0. By changing notation (with different $\{u_n\}$ and larger K), we may assume that $\{u_n^k\} \in U_0(M_K(\prod_n B_n))$. Also each $u_n \in U_0(B_n)$. This implies that there is L > 0 such that cel(u_n^k) $\leqslant$ L for all n. It follows from 6.10 that

$$\text{cel}(u_n) \leqslant 8\pi + L + L/k + \pi/4 \quad \text{for all n.}$$

This implies (see for example [45]) $\{u_n\} \in U_0(M_L(\prod_n B_n))$. Therefore $[\{u_n\}] = 0$ in $K_1(\prod_n B_n)$.

So the kernel is torsion free.

Definition(4.1.39)[89]: Let Y be a connected finite CW complex with dimension no more than three with torsion $K_1(C(Y))$ and set C' = P M_r (C(X))P , where X = $S^1 \vee S^1 \vee \bullet \bullet \bullet \vee S^1 \vee$ Y and P ∈ M_r (C(X)) is a projection and P has rank R $\geqslant$ 6. We assume that S^1 is repeated s ($\geqslant$ 0) times.Note that the above includes the case that X= Y= [0,1]Then $K_1(C')$=tor $(K_1(C')) \oplus G_1$ where G_1 is s copies of Z. Denote by ξ the point in X where each S^1 and Y meet.

Rename each S^1 by Ω_i, i=1, 2, . . . , s. Denote by z'_i the identity map on ith S^1 (Ω_i). Define z''_i (ζ)=ζ on Ω_i(which is identified with S^1) and z''_i (ζ)= ξ for all ζ / Ω_i. There is an obvious homomorphism$\prod$: $PM_r\big(C(X)\big)P \to D' =\oplus_{i=1}^{K} E_i$, where $E_i \cong M_R(C(S^1))$. Note that if s$\geqslant$ 2, then Π is not surjective. We have that G_1 =K_1(D'). We also use $\Pi_i: PM_r\big(C(X)\big)P \to E_i$which is the composition of Π with the projection from D' to E_i. Let z be the identity map on S^1 .

We may write

$$P\,(t) =\begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix}$$

where P_1 is a projection with rank 3 and I = diag(1, 1, . . . , 1) with 1 repeating rank(P) $-$ 3 times.

Note that, since rank(P)$\geqslant$6, tsr(C(X)) = 2 and csr(C(X))$\leqslant$ 2 + 1 as in [119]). It fol-lows that csrM_3(C')$\leqslant$ 2 in [120] . It then follows [120] that U (C')/U_0(C') =K_1(C'). In particular, CU(C') $\subset U_0$(C').

Denote by u_i = diag(z''_i , 1, . . . , 1), where 1 is repeated r $-$ 4 times. If we write $z_i\in$ U (C), we mean the unitary

$$z_i(t) = \begin{pmatrix} P_1 & 0 \\ 0 & u_i \end{pmatrix}.$$

If we write $z_i\in E_i$, we mean Π_i (z''_i). Note that in this case, z_i has the form diag(1, . . . , 1, z,1, . . . , 1), where z is in the 4th position and there are R $-$ 1 many 1's.

Now let C=$\oplus_{j=1}^{l+l_1} C^{(j)}$, where $C^{(j)}$is either of the form $P_j M_{r(j)}\big(C(X_j)\big)P_j$for j $\leqslant$l, where X_j is ofthe form X described above, $C^{(j)}=M_{r(j)}$, or $C^{(j)} = P_j M_{r(j)}\big(C(Y_j)\big)P_j$,where Y_j is a finite CW complex with dimension no more than 3, rank of P_j is R(j)$\geqslant$6 and $K_1(Y_j$) is finite for $l+1$ $\leqslant$j $\leqslant l+l_1$. Let $D^{(j)}$ be as D' above for each $\leq l$. Let $\Pi^{(j)}$ be as Π above for $C^{(j)}=P_j M_{r(j)}\big(C(X_j)\big)P_j$ Put D=$\oplus_{j=1}^{l} D^{(j)}$and Π =$\oplus_{g=D}^{l} \Pi_j^{(j)}$. Since K_1(C) is finitely generated and U_0(C)/CU(C ,we may write

U (C)/CU(C) = U_0(C)/CU(C) $\oplus K_1$(D) $\oplus$ tor K_1(C) .

Let π_1 : U (C)/CU(C) $\to$ K_1(D), π_0 : U (C)/CU(C) $\to$ U_0(C)/CU(C), π_2 : U (C)/CU(C) $\to$ tor(K_1(C)) be fixed projection maps associated with the above decomposition. To avoid possible confusion, by π_i (U (C)/CU(C)), we mean a subgroup of U (C)/CU(C), i = 0, 1, 2. We also assume that $\pi_1(\overline{z_i}) = \overline{z_i}$ (in U (C)/CU(C)).

It is worth pointing out that one could have X = Y = [0, 1].

The notation established above will be used in the rest of this section.

Lemma (4.1.40)[89]: Let C =$\oplus_{i=1}^{l+l_1} C_i$be as above and$\mathcal{U}\subset$ U (C) be a finite subset and F be the group generated by $\mathcal{U}$. Suppose that G is a subgroup of U (C)/CU(C) which contains $\bar{F}$,

π_2(U (C)/CU(C) andπ_1(U (C)/CU (C)).Suppose that the composition map$\gamma:\bar{F}\to$ U (D)/CU(D) $\to$ U (D)/U_0(D) is injective and γ ($\bar{F}$) is free. Let B be a unital C^*- algebra and Λ : G $\to$ U (B)/CU(B) be a homomorphism such that Λ(G$\cap$ U_0(C)/CU(C)) $\subset U_0$(B)/CU(B).Then there are homomorphisms β : U (D)/CU(D) $\to$ U (B)/CU(B) with $\beta(U_0$(D)/CU(D)) $\subset U_0$(B)/CU(B), and θ : π_2(U (C)/CU(C)) $\to$ U (B)/CU(B) such that

$$\beta \circ \Pi^\ddagger \circ \pi_1(\bar{w}) = \Lambda(\bar{w})\, \theta \circ \pi_2(\bar{w})$$

for all w $\in$ F and such that θ (g)=$\Lambda\backslash_{\pi_2\,(U\,(C)/CU(C))}(g^{-1})$ for g$\in \pi_2$(U (C)/CU(C)). Moreover,$\beta(U_0$(D)/CU(D)) $\subset U_0$(B)/CU(B).

If furthermore A is a simple C^*- algebra with TR(B)$\leq$1 and Λ(U (C)/CU(C)) $\subset U_0$(B)/CU(B), then $\beta \circ \Pi^\ddagger \circ (\pi_1)|_{\bar{F}} = \Lambda|_{\bar{F}}$.

Proof: Let κ_1 : U (D)/CU(D) $\to$ K_1(D) be the quotient map. Let η : π_1(U (C)/CU(C)) $\to$ K_1(D) be defined by $\eta = \kappa_1 \circ \Pi \ddagger |_{\pi_1\,(U\,(C)/CU(C))}$. Note that η is an isomorphism. Since γ is injective and γ ($\bar{F}$) is free, we conclude that $\kappa_1 \circ \Pi^\ddagger \circ \pi_1$ is also injective on $\bar{F}$ From this fact and the fact that U_0(C)/CU(C) is divisible (6.6), we obtain a homomorphism λ : K_1(D) $\to$ U_0(C)/CU(C) such that

$$\lambda\backslash_{\kappa_1 \circ \Pi^\ddagger \pi_1(\bar{F})} = \pi_0 \circ ((\kappa_1 \circ \Pi^{\ddagger\circ}\pi_1)\backslash_{\bar{F}})^{-1}$$

Now define $\beta = \Lambda((\eta^{-1} \circ \kappa_1) \oplus (\lambda \circ \kappa_1))$. Then for any $\bar{w} \in$ $\bar{F}$,
$\beta(\Pi^\ddagger \circ \pi_1(\bar{w}))= \Lambda[\ \eta{-1}(\ \kappa_1 \circ \Pi^\ddagger \pi_1(\bar{w})) \oplus \lambda \circ \kappa_1(\Pi^\ddagger(\pi_1(\bar{w})))]= \Lambda\ \pi_1(\bar{w}) \oplus \pi_0(\bar{w})$.
Now define θ : π_2(U (C)/CU(C)) $\to$ U (B)/CU(B) by θ (x) = $\Lambda(x^{-1})$ for x $\in \pi_2$(U (C)/CU(C)). Then

$$\beta(\ \Pi^\ddagger (\pi_1\ (\bar{w}))) = \Lambda(\bar{w})\theta\pi_2(\bar{w}) \qquad \text{for w} \in \text{F} .$$

To see the last statement, we assume Λ(U (C)/CU(C)) $\subset U_0$(B)/CU(B). Then $\Lambda(\pi_2$(U (C)/CU(C))) is a torsion subgroup of U_0(B)/CU(B). By 6.11, U_0(B)/CU(B) is torsion free. Therefore $\theta = 0$.

Lemma(4.1.41)[89]: Let A be a unital separable simple C^*- algebra with TR(A)$\leq$ 1 and C be as de-scribed in 7.1. Let $\mathcal{U}\subset$ U (A) be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that$(\kappa_1)\backslash_{\bar{F}}$ is injective and $\kappa_1(\bar{F}$) is free, where κ_1: U (A)/CU(A)$\to K_1$ (A) is the quotient map.Suppose that α : K_1(C) $\to$ K_1(A) is an injective homomorphism and L : $\bar{F} \to$U (A)/CU(A)is an injective homomorphism with L($\bar{F} \cap U_0$(A)/CU(A)) $\subset U_0$(C)/CU(C) such that $\pi_1\circ$ L is injective (see 7.1 for π_1) and

$$\alpha \circ \kappa'_1 \circ L(g)=\kappa_1(g) \qquad \text{for all g} \in \bar{F}$$

where κ'_1 : U (C)/CU(C) $\to$ K_1(C) is the quotient map. Then there exists a homomorphismβ : U (C)/CU(C) $\to$ U (A)/CU(A) with $\beta(U_0$(C)/CU(C)) $\subset U_0$(A)/CU(A) such that

$$\beta \circ L(\bar{w}) = \bar{w} \qquad \text{for w} \in \text{F} .$$

Proof: Let G be the preimage of $\alpha \circ \kappa'_1$(U (C)/CU(C)) under κ_1. So we have the following short exact sequence:

$$0 \to U_0(A)/CU(A) \to G \to \alpha \circ \kappa'_1 \; U\,(C)/CU(C) \to 0.$$

Since $U_0(A)/CU(A)$ is divisible, there exists an injective homomorphism

$$\gamma : \alpha \circ \kappa'_1 \; U\,(C)/CU(C) \to G$$

such that $\kappa_1 \circ \gamma(g) = g$ for $g \in \alpha \circ \kappa_1(U\,(C)/CU(C))$. Since $\alpha \circ \kappa'_1 \circ L\,(f) = \kappa_1(f)$ for all $f \in \bar{F}$, we have $\bar{F} \subset G$. Moreover, $\left(\gamma \circ \alpha \circ \kappa'_1 \circ L\,(f)\right)^{-1} f \in U_0(A)/CU(A)$ for all $f \in \bar{F}$.

Define $\psi : L(\bar{F}) \to U_0(A)/CU(A)$ by

$$\psi\,(x) = \gamma \circ \alpha \circ \kappa'_1 \circ L([\,(L)^{-1}(x)]^{-1})\; L^{-1}(x)$$

for $x \in L(\bar{F})$. Since $U_0(A)/CU(A)$ is divisible, there is homomorphism $\tilde{\psi}$: U $(C)/CU(C) \to U_0(A)/CU(A)\tilde{\psi}\,|_{L(\bar{F})} = \psi$. Now define

$$\beta(x) = \gamma \circ \alpha \circ \kappa'_1(x)\tilde{\psi}\,(x).$$

Hence $\beta(L(f\,)) = f$ for $f \in \bar{F}$.

Lemma(4.1.42)[89]: Let B be a unital separable simple C^*- algebra with $TR(B) \leqslant 1$ and C be as in 7.1.

Let F be a group generated by a finite subset $\mathcal{U} \subset$ U (C) such that $(\pi_1)|_{\bar{F}}$ is injective. Let G be a subgroup containing $\bar{F}$, $\pi_0(\bar{F}\,)$, $\pi_1(U\,(C)/CU(C))$ and $\pi_2(U\,(C)/CU(C))$. Suppose that α:U $(C)/CU(C) \to$ U $(B)/CU(B)$ is a homomorphism with $\alpha(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$. Then for any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following: if $\varphi = \varphi_0 \oplus \varphi_1 : C \to B$ is a $\mathcal{G}$-η-multiplicative contractive completely positive linear map such that:

(i)both φ_0 and φ_1 are $\mathcal{G}$-η-multiplicative and φ_0 maps the identity of each summand of C into a projection,

(ii)$\mathcal{G}$ is sufficiently large and η is sufficiently small which depend only on C and F (so that $\varphi^{\ddagger}$ is well defined on a G),

(iii)φ_0 is homotopically trivial (see (vi) in Section 1), $(\varphi_0)_{*0}*0$ is a well-defined homomorphism and $[\emptyset]\backslash_{k_1(\bar{F})} = \alpha_*\backslash_{k_1(\bar{F})}$, where$\alpha_* : K_1(C) \to K_1(B)$induced by α and κ_1:U $(C)/CU(C) \to K_1(C)$ is the quotient map,

    (iv) $\tau\,(\emptyset_0(1_C)) < \delta$ for all $\tau \in$ T (B),

then there is a homomorphism $\Phi : C \to e_0Be_0$ $(e_0 = \emptyset_0(1_C))$ such that:

    (i) Φ is homotopically trivial and $\Phi_{*0} = (\emptyset_0)_{*0}$ and

    (ii) $\alpha(\bar{w})^{-1}(\Phi \oplus \varphi_1)^{\ddagger}(\bar{w}) = \overline{g_w}$,

    where $g_w \in U_0(B)$ and $cel(g_w\,) < \varepsilon$ for all $w \in \mathcal{U}$.

Proof: By Lemma (4.1.40), there are homomorphisms $\beta_1, \beta_2 :$ U $(D)/CU(D) \to$ U $(B)/CU(B)$ with $\beta_i\,(U_0(D)/CU(D)) \subset U_0(B)/CU(B)$ $(i = 1, 2)$ and homomorphisms $\theta_1, \theta_2\pi_2(U\,(C)/CU(C) \to$ U $(B)/CU(B)$ such that

$$\beta_1 \circ \Pi^{\ddagger}\left(\pi_1(\bar{w})\right) = \alpha\,(\bar{w})\theta_1(\pi_2(\bar{w}))\text{and}\beta_2 \circ \Pi^{\ddagger}\left(\pi_1(\bar{w})\right) = \emptyset_1^{\ddagger}(\bar{w}\,*)\theta_2(\pi_2(\bar{w}))$$

for all $\bar{w} \in \bar{F}$. Moreover $\theta_1(g)= \alpha(g^{-1})$ and $\theta_2(g)= \emptyset_1^{\ddagger}$ (g) if g $\in \pi_2(\bar{F})$. Since $\emptyset_0$ is homotopi-cally trivial,

$\theta_1(g)\theta_2(g)\in U_0(B)/CU(B)$ for all g $\in\pi_2(\bar{F})$.

Since $\pi_2(U$ (C)/CU(C) is torsion and $U_0(B)/CU(B)$ is torsion free, we conclude that $\theta_1(g)\theta_2(g)= \bar{1}$ for all g$\in$ $\pi_2(\bar{F})$.

To simplify notation, without loss of generality, we may assume that C=$\bigoplus_{j=1}^{l+l_1} C^{(j)}$ (with $l = 1$) such that $C^{(1)}=$ P M_r (C(X))P as described in Definition(4.1.39)and $C^{(j)}$is also as described in Definition(4.1.39) for $2 = l + 1 \leqslant$ j $\leqslant l_1 + 1$. Let D be as described in 7.1.For each w $\in$ U (C), we may write w = ($w_1, w_2, \ldots, w_{1+l_1}$) according to the direct sum C=$\bigoplus_{j=1}^{l+l_1} C^{(j)}$. Note that $\pi_1(w)=\pi_1(w_1)$. Let π $(\bar{w})= (\bar{z}_1^{k(1,w)},\bar{z}_2^{k(2,w)}, \ldots,\bar{z}_s^{k(s,w)})$, where$k(i,w)$is an integer (here z_i is described in Definition(4.1.39)). Then $\prod_i^{\ddagger}(\pi_1(\overline{w_1})) = \bar{z}_i^{k(i,w)}$ .On the other hand, we may also write $\prod_i^{\ddagger}(\overline{w_1}) = \overline{\bar{z}_i^{k(i,w)}}g_{i,w}$ for some $g_{i,w}\in U_0(C(S_1, M_r))$.Let l = max{cel($g_{i,w}$): w $\in\mathcal{U}$, $1\leqslant i \leqslant s$}. Choose δ so that $(2 + l)\delta < \varepsilon/4\pi$. Let $e_0 = \emptyset_0(1_C$) and $e_1 = \emptyset_1(1_C$). Write $e_0 = E_1\oplus E_2\oplus \cdots \oplus E_{1+l_1}$, where $E_j = \emptyset_0(1_{C^{(j)}}$), j = 1, 2, . . ., $1 + l_1$.

Recall that P has rank R. Since $\emptyset_0$ is homotopically trivial ,we may also write $E_1 = e_{01}\oplus \cdots \oplus e_{0R}$, where $\{e_{0i} : 1\leqslant i\leqslant R\}$ is a set of mutually orthogonal and mutually equivalent projections. Since e_0Be_0 is simple and has the property (SP), e_{01} can be written as a sum of s mutually orthogonal projections. Thus $E_1 = p_1\oplus p_2\oplus \cdots \oplus p_s$, where each p_i can be written as a direct sum of R mutually orthogonal and mutually equivalent projections$\{q_{i,1},\ldots\ldots,q_{i,R}\}$. For each q_{i1}, we write$q_{i1} = q_{i1,1} \oplus q_{i1,2}$,where both $q_{i1,1}$ and $q_{i1,2}$ are notzero. Let q =$\sum_{i=1}^s q_{i1,1}$ We may view $E_1BE_1 =\bigoplus_{i=1}^s M_R(q_{i1}Bq_{i1}) = M_R(qBq)$

Let z_i be as in 7.1. Put $x'_i\in$ U $(q_{i1,1}Bq_{i1,1})$ such that $\overline{x'}_i = \beta_1 (\bar{z}_i)$, and $y'_i\in$U $(q_{i1,2}Bq_{i1,2})$ such that$\overline{y'}_i=\beta_2(\bar{z}_i)$,i = 1, 2, . . . , s. This is possible because of 6.7. Put $x_i = x'_i\oplus q_{i1,2}, y_i = y'_i\oplus q_{i1,1}$, i = 1, 2, . . . , s. Note that $x_iy_i = y_ix_i$. Define $\emptyset_1 :$ D $\to$ $M_R(qBq)=\bigoplus_{i=1}^s M_R(q_{i1}Bq_{i1})$ by$\emptyset_1(f) = \sum_{i=1}^s f_i(x_iy_i)$, where $f = (f_1,f_2, ,f_s),f_i \in C(S^1, M_R)$

Define h(g)= $\emptyset_1$ (Π (g)) $\oplus\emptyset_1$ (g) for g $\in$C. We compute that

$$\overline{h(w)}=\prod_{i=1}^s \overline{x_i^{k(i,w)}g_{i,w}(x_iy_i)y_i^{k(i,w)}\emptyset_1^{\ddagger}(\bar{w})}$$

$$=\beta_1(\Pi^{\ddagger}(\pi_1(\bar{w})))\emptyset_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^S g_{i,w}}\right)\beta_2(\Pi^{\ddagger}(\pi_1(\bar{w})))\emptyset_1^{\ddagger}(\bar{w})$$

$$=\alpha (\bar{w})\theta_1(\pi_2(\bar{w})) \theta_2(\pi_2(\bar{w}))\emptyset_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^S g_{i,w}}\right) = \alpha (\bar{w})\emptyset_1^{\ddagger}\left(\overline{\bigoplus_{i=1}^S g_{i,w}}\right)$$

for all w $\in \mathcal{U}$. Put $g'_w = \emptyset_1^\ddagger (\oplus_{i=1}^s g_{i,w}) \oplus (1 - \varphi_0(1_C))$. Since $\tau (\varphi_0(1_C)) < \delta$, by the choice of δ, we conclude from Lemma (4.1.33) that there exists $w' \in CU(B)$ such that cel (w' g'_w)< $\varepsilon/2$ for all w $\in \mathcal{U}$.

Note that $\emptyset_1 \circ \Pi$ factors through D and $(\emptyset_1)_{*1} = 0$. In particular, $\emptyset_1 \circ \Pi$ is homotopicallytrivial. Since $\emptyset_0$is homotopically trivial, it is easy to see that there is a point-evaluation map$\emptyset_1: \oplus_{i=2}^{l+l_1} C^{(j)} \to (\oplus_{i=2}^{l+l_1} E_j) B(\oplus_{i=2}^{l+l_1} E_j)$. Now define $\Phi = \Phi_1 \circ \Pi \oplus \Phi_2$. We see that we can make (by a right choice of Φ_2) $\Phi|_{*0}$. It is clear that Φ is homotopically trivial. Let $g''_w = \Phi_2(w) \oplus (1 - (e_0 - E_1))$. Since $\Phi_2(\sum_{i=2}^{l+l_1} C^{(j)})$ is finite-dimensional, $cel(\Phi_2(w)) \leqslant 2\pi$ (in $U_0((e_0 - E_1) B(e_0 - E_1))$ for all w $\in \mathcal{U}$).By the choice of δ, we conclude that there is w" $\in$ CU(B) such that cel(w" g''_w) < $\varepsilon/2$ (see 6.4). Put $g_w = $w g_w w g_w . We have, for all w$\in \mathcal{U}$, $\alpha(\overline{w})^{-1}(\Phi \oplus \emptyset_1)^\ddagger(\overline{w}) = \overline{g_w}$with$g_w \in U_0$ (B) and cel(g_w) < ε.

Lemma (4.1.43)[89]: Let B be a unital separable simple C^*- algebra with TR(B) $\leqslant 1$ and C be as described in 7.1. Let $\mathcal{U} \subset$ U (B) be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1 (\overline{F})$ is free, where $\kappa_1 :$U (B)/CU(B) $\to K_1$ (B) is the quotient map. Let$\emptyset \to B$ be a homomorphism such that $\emptyset_{*1}$is injective. Suppose that j, L : $\overline{F} \to$U (C)/CU(C) are twoinjective homomorphisms with j($\overline{F} \cap U_0$ (B)/CU(B)) , L($\overline{F} \cap U_0$ (B)/CU(B))$\subset$ $(U_0$ (C)/CU(C))such that $\kappa_1 \circ \emptyset^\ddagger \circ L = \kappa_1 \circ \emptyset^\ddagger \circ j = \kappa_1\backslash_{\overline{F}}$ and all three are injective.

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\emptyset = \emptyset_0 \oplus \emptyset_1 : C \to B$, where $\emptyset_0$ and $\emptyset_1$ are homomorphisms satisfying the following:

    (i)$\tau (\emptyset_0(1_C)) < \delta$ for all $\tau \in$ T (B) and

    (ii)$\emptyset_0$ is homotopically trivial,

then there is a homomorphism $\psi : C \to e_0 B e_0$ ($e_0 = \emptyset_0(1_C)$) such that:

    (i)$[\psi] = [\emptyset_0]$ in KL(C, B) and

    (ii)$(\emptyset^\ddagger \circ j (\overline{w}))^{-1}(\psi \oplus \emptyset_1)^\ddagger(L(\overline{w})) = g_w$, where $g_w \in U_0$(B) and cel(g_w) < ε for all w $\in \mathcal{U}$.

Proof: The first part of the proof is essentially the same as that of Lemma (4.1.40). Let $\kappa'_1 :$ U (C)/CU(C) $\to K_1(C)$ be the quotient map and let G be the preimage of $\emptyset_{*1} \circ \kappa'_1($U (C)/CU(C)) un-der κ_1. Since U_0(B)/CU(B) is divisible, there exists an injective homomorphism $\gamma : \emptyset_{*1} \circ \kappa'_1$ (U (C)/CU(C))$\to G$ such that$\kappa_1 \circ \gamma$ (g)= g for g $\in \emptyset_{*1} \circ \kappa'_1($U (C)/CU(C)). Since$\emptyset_{*1} \circ \kappa_1 \circ L(f) = \kappa_1 (f) = \kappa_1 (\emptyset^\ddagger \circ j(f))$ for all $f \in \overline{F},$ we have$\overline{F} \subset$ G Moreover,

$$[\gamma \circ \emptyset_{*1} \circ \kappa'_1 \circ L(f)]^{-1}\emptyset^\ddagger \circ j (f) \in U_0(B)/CU(B)$$

for all $f \in \mathcal{F}$. Define $\psi : L(\overline{F}) \to U_0$(B)/CU(B) by

$$\psi (x) = [\gamma \circ \emptyset_{*1} \circ \kappa'_1 (x)]^{-1}[\emptyset^\ddagger \circ j \circ L^{-1}(x)]$$

for all x $\in L(\overline{F})$. Since U_0(B)/CU(B) is divisible, there is a homomorphism

$\bar{\psi} : U(C)/CU(C) \to U_0(B)/CU(B)$ such that $\bar{\psi}|_{L(\bar{F})} = \bar{\psi}$.

Define $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ by $\alpha(x) = \gamma \circ \emptyset_{*1} \circ \kappa'_1(x)\bar{\psi}(x)$ for all $X \in U(C)/CU(B)$. Note that

$$\alpha(L(f)) = \emptyset^{\ddagger} \circ j(f) \qquad \text{for all } f \qquad \in \bar{F}.$$

Definition (4.1.44)[89]: Let A and B be C^*- algebras. Two homomorphisms $\emptyset, \psi : A \to B$ are said to be stably unitarily equivalent if for any monomorphism $h : A \to B$, $\varepsilon > 0$ and finite subset $F \subset A$, there exists an integer $n > 0$ and a unitary $U \in M_{n+1}(\tilde{B})$ (or in $M_{n+1}(B)$, if B is unital) such

that

$$\left\| U^* \, diag\, (\emptyset(a), h(a), h(a), \ldots, h(a))U - \, diag\, (\psi(a), h(a), h(a), \ldots, h(a)) \right\| < \varepsilon$$

for all $a \in F$, where h(a) is repeated n times on both diagonals.

Let A and B be C^*- algebras and $\emptyset, \psi : A \to B$ be (linear) maps. Let $F \subset A$ and $\varepsilon > 0$. We write

$$\varphi \sim_\varepsilon \psi \qquad \text{on F,}$$

if there exists a unitary $u \in B$ such that

$$\| ad(u) \circ \emptyset(a) - \psi(a) \| \qquad < \varepsilon \qquad \text{for all } a \in F.$$

We write

$$\varphi \approx_\varepsilon \psi \quad \text{on F, if } \| \emptyset(a) - \psi(a) \| \quad < \varepsilon \text{ for all } a \in F.$$

Definition(4.1.45)[89]: Let A be a C^*- algebra.

(i) Denote by **P**(A) the set of all projections and unitaries in $M_\infty(A \widetilde{\otimes} C_n)$, n 1, 2, . . . , where C_n is an abelian C^*- algebra so that
$K_i(A \otimes C_n) = K_*(A; \mathbf{Z}/n\mathbf{Z})$.
(see [126]). As in [25], we use the notation

$$\underline{K}(A) = \bigoplus_{i = 0,1, n \in Z_+} K_i(A; \mathbf{Z}/n\mathbf{Z}).$$

By $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respect to the direct sum decomposition and the so-called Bockstein operations (see [25]). Denote by $Hom_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ those $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ with the property that $\alpha(K_0(A)_+ \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$. It follows from [25] that if A satisfies the Universal Coefficient Theorem, then $Hom_\Lambda(\underline{K}(A), \underline{K}(B)) \cong KL(A, B)$. Moreover, one has the following short exact sequence,

$$0 \to \text{Pext}(K_*(A), K_*(B)) \to KK(A, B) \to KL(A, B) \to 0.$$

A separable C^*- algebra A is said to satisfy Approximate Universal Coefficient Theorem (AUCT) if

$$KL(A, B) = Hom_\Lambda(\underline{K}(A), \underline{K}(B))$$

for any σ -unital C^*- algebra B (see [76]). A separable C^*- algebra A which satisfies the UCT must satisfy the AUCT. If A satisfies the AUCT, for convenience, we will use $KL(A,B)^{++}$ for$Hom_\Lambda(\underline{K}(A),\underline{K}(B))^{++}$.

(ii) Let L : A → B, be a contractive completely positive linear map. We also use L for the extension from A ⊗**K** → B ⊗**K** as well as maps from$A \widetilde{\otimes} C_n \to B \widetilde{\otimes} C_n$ for all n. Given a pro-jection p ∈**P**(A), if L : A → B is an F-δ-multiplicative contractive completely positive linear map with sufficiently large F and sufficiently small δ, $\|L(p) - q'\| < 1/4$ for some projection q' . Define [L](p) = [q'] in $\underline{K}$(B). It is easy to see this is well defined (see [67]). Suppose that q is also in **P**(A) with [q] = k[p] for some integer k. By adding sufficiently many elements (partial isometries) in F, we can assume that [L](q) = k[L](p). Let $\mathcal{P} \subset \mathbf{P}(A)$ be a finite subset. We say $[L]|_{\mathcal{P}}$ is well defined if [L](p) is well de-fined for every p ∈$\mathcal{P}$ and if [p'] = [p] and p' ∈$\mathcal{P}$, then [L](p') = [L](p). This always occurs if F is sufficiently large and δ is sufficiently small. In what follows we write $[L]|_{\mathcal{P}}$ when [L] is well defined on $\mathcal{P}$.

(iii)Let A = $\oplus_{i=1}^n A_i$, where each A_i is a unital C^*- algebra. Suppose that L : A → B is a $\mathcal{G}$-ε-multiplicative contractive completely positive linear map. For any η > 0, if $\mathcal{G}$ is large enough and ε is small enough, we may assume that

$$\|L(1_{A_i}) - p_i\| < \eta, \qquad \|p_j L(1_{A_i})\| < \eta \text{ and } \qquad \|L(1_{A_i}) p_j\| < \eta$$

for some projection $p_i \in$ B and i ≠ j. Let b = $p_1 L(1_{A_1}) p_1$. Then, with sufficiently small η, we may assume that b is invertible in $p_1 B p_1$. Define $L_1(a) = b^{-1/2} p_1 L(a) p_1 b^{-1/2}$. Then $L_1(1_{A_1}) = p_1$. Consider $(1 - p_1)L(1 - p_1)$. It is clear that, for any δ > 0, by induction and choosing a sufficiently large $\mathcal{G}$ and sufficiently small η and ε,

$$\|L - \Psi\| < \delta,$$

where Ψ (a)=$\oplus_{i=1}^n L_i (1_{A_i} a 1_{A_i})$ for a ∈A. So, to save notation in what follows, we may as-sume that L =$\oplus_{i=1}^n L_i$, where each $L_i : A_i \to B$ is a completely positive contraction which maps 1_{A_i} to a projection in B and $\{L_1(1_{A_1}), L_2(1_{A_2}), \ldots, L_n(1_{A_n})\}$ are mutually orthogonal.

Throughout the rest of this section, A denotes the class of separable nuclear C^*- algebras satisfying the Approximate Universal Coefficient Theorem.

Lemma(4.1.46)[89]: Let B be a unital C^*- algebra and let A be a unital C^*- algebra in A which is a unital C^*- subalgebra of B. Let α : A → B and β : A → B be two homomorphisms. Then α and β are stably approximately unitarily equivalent if [α] = [β] in KK(A, B) and if A is simple or B is simple.

The following is a modification in[90]. A proof was given in the earlier version of this section. Since then a more general version of the following appeared in [76]. We will omit the original proof and view the following as a special case.

Theorem (4.1.47)[89]:   (Cf. [35, Theorem 4.8].) Let B be a C^*- algebra with stable rank one and cel(M_M(B))⩽ k for some k ⩾π and for all m, and let A be a unital simple C^*- algebra in A which is a C^*- subalgebra of B. Let α : A → B and β : A → B be two homomorphisms. Then and β are stably approximately unitarily equivalent if [α] = [β] in KL(A, B).

The following uniqueness theorem is a modification of [35, Theorem 5.3].

Theorem (4.1.48)[89]:   (See [90].) Let A be a unital simple C^*- algebra in **A** and **L** : U $(M_\infty(A)) \to \boldsymbol{R}_+$ be a map. For any ε > 0 and any finite subset F⊂ A there exist a positive number δ > 0, a finite subset $\mathcal{G}$⊂ A, a finite subset $\mathcal{P}$⊂**P**(A) and an integer n > 0 satisfying the following: for any unital simple C^*- algebra B with TR(B)⩽ 1, if Ø, ψ, σ : A → C are three $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with

$[Ø]||_{\mathcal{P}} = [\psi\,]||_{\mathcal{P}}$,

cel $(\widetilde{Ø}(u)^*\widetilde{\psi}\,(u)) \leqslant \mathbf{L}(u)$

for all u ∈ U (A) ∩ $\mathcal{P}$ and σ is unital, then there is a unitary u ∈M_{n+1}(B) such that

$\|u * diag(\,Ø\,(a), \sigma\,(a),\ldots,\sigma\,(a))\,u \; - \; diag\,(\psi\,(a),\sigma\,(a),\ldots,\sigma\,(a))\| < ε$

for all a ∈F, where σ (a) is repeated n times.

Proof. Suppose that the theorem is false. Then there are $ε_0$> 0 and a finite subset F⊂ A such that there are a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \downarrow 0$, an increasing sequence of finite subsets $\{\mathcal{G}_n\}$ whose union is dense in the unit ball of A, a sequence of finite subsets $\{\mathcal{P}_n\}$ of **P**(A) with $\overline{U_{n=1}^\infty \mathcal{P}_n} = $ **P**(A) and with $U_n = $ U (A) ∩ $\mathcal{P}_n$, a sequence $\{k(n)\}$ of integers (k(n)↗ ∞)

and sequences $\{Ø_n\}$, $\{\psi_n\}$ and $\{\sigma_n\}$ of $\mathcal{G}_n$-δ_n-multiplicative positive linear maps from A to B_n with $[Ø_n]||_{\mathcal{P}_n} = [\psi\,]||_{\mathcal{P}_n}$ and

cel $(\widetilde{Ø_n}\,(u)^*\widetilde{\psi_n}\,(u)) \leqslant \mathbf{L}(u)$

for all u ∈U_n satisfying the following:

inf{sup{$\|u^*\, diag\,(Ø_n(a), \sigma_n(a),\ldots,\sigma_n(a))u - diag\,(\psi_n(a),\sigma_n(a),\ldots,\sigma_n(a))\|$:   a∈F}}⩾ $ε_0$

where σ_n (a) is repeated k(n) times and the infimum is taken over all unitaries in $M_{k(n)+1}(B_n)$.

Set $D_0 = \bigoplus_{n=1}^\infty B_n$ and D = $\prod_{n=1}^\infty B_n$. Define Φ, Ψ, Σ : A→ D by Φ (a) ={$Ø_n(a)$} , **Ψ** (a) = {$\psi_n(a)$} and Σ (a) = {$\sigma_n(a)$} for a ∈ A. Let π : D → D/D_0 be the quotient map and set $\overline{Φ} = π \circ Φ, \overline{Ψ} = π \circ Ψ$ and $\overline{Σ} = π \circ Σ$. Note that$\overline{Φ}, \overline{Ψ}$, and $\overline{Σ}$ are monomorphisms. For any u ∈U_K, ,

cel $(\widetilde{Ø_n}\,(u)^*\widetilde{\psi_n}\,(u)) \leqslant \mathbf{L}(u)$

for all sufficiently large n (> k). This implies that there is an equi-continuous path $\{v_n(t)\}$ (t $\in [0, 1]$) such that

$$v_n(0) = \widetilde{\emptyset_n}(u) \quad \text{and} \quad v_n(1) = \widetilde{\psi_n}(u)$$

(see, for example, [45]). Therefore, we conclude that
$[\bar{\Phi}]|_{K_1} (A) = [\bar{\Psi}]|_{K_1} (A)$.

Given an element p $\in \mathcal{P}_k \setminus U_k$ (for some k), we claim that

$$[\bar{\Phi}(p)] = [\bar{\Psi}(p)] .$$

We have (see [45])

$$K_0(\textstyle\prod B_n) = \prod^b K_0(B_n) \quad \text{and} \quad K_0(D/D_0) = K_0(B_n)/K_0(B_n),$$

Where $\prod^b K_0(B_n)$ is the sequences of elements $\{[p_n] - [q_n]\}$, where p_n and q_n can be repr

by projections in $M_L(B_n)$ for some integer L. Since each $TR(B_n) \leqslant 1$, B_n has stable sented rank one and $K_0(B_n)$ is weakly unperforated. As in [45] each B_n has K_i–divisible rank T withT (n, k) = 1., $cer(M_k(B_n)) \leqslant 4$ for all k and n, and the kernel of the map $K_0(\prod_n B_n)$ to$\prod_n^b K_1(B_n)$is divisible and torsion free.

We also have

$$K_i\left(\prod_n B_n, \mathbf{Z}/m\mathbf{Z}\right) \subset \prod_n K_i (B_n, \mathbf{Z}/m\mathbf{Z}), \quad m = 2, 3, \ldots .$$

(In fact, by 6.10, each B_n has exponential length divisible rank E with E(L, k) = $8/\pi + L/k$ + 1 so that [45] can be applied directly.

Since $[\emptyset_n(p)] = [\psi_n(p)]$ in $K_0(B_n)$ or in $K_i (B_n, \mathbf{Z}/m\mathbf{Z})$ (i = 0, 1, m = 2, 3, . . .) for large n, $[\bar{\Phi}(p)] = [\bar{\Psi}(p)]$.

Then$\bar{\Phi}_* = \bar{\Psi}_*$. Therefore$[\bar{\Phi}] = [\bar{\Psi}]$ in $KL(A, \prod_n B_n/\oplus_n B_n)$.

By applying 8.4, we obtain an integer N anda unitary u $\in M_{N+1}(D/D_0)$ such that
$$\|u^* \, diag \, (\bar{\Phi}(a), \bar{\Sigma}(a), \ldots, \bar{\Sigma}(a))u - diag \, (\bar{\Psi}(a), \bar{\Sigma}(a), \ldots, \bar{\Sigma}(a))\| < \varepsilon_0/3$$

for all a $\in \mathcal{F}$, where $\bar{\Sigma}$ (a) is repeated N times. It is easy to (see [67] for example) there is a unitary U $\in M_{N+1}(D)$ such that $\pi(U) = u$ and for each a $\in \mathcal{F}$ there exists $c_a \in M_{N+1}(D_0)$ such that

$$\left\|U^* \, diag \, (\Phi(a), \Sigma(a), \ldots, \Sigma(a))U - diag \, (\Psi (a), \Sigma(a), \ldots, \Sigma(a)) + c_a\right\| < \varepsilon_0/3$$

where Σ (a) is repeated N times. Write U = $\{u_n\}$, where $u_n \in M_{N+1}(B_n)$ are unitaries. Since $c_a \in M_{N+1}(D_0)$ and $\mathcal{F}$ is finite, there is $N_0 > 0$ such that for n $\geqslant N_0$.

$$\|u_n^* \, diag \, (\emptyset_n(a), \sigma_n(a), \ldots, \sigma_n(a))u_n - diag \, (\psi_n(a), \sigma_n(a), \ldots, \sigma_n(a))\| < \varepsilon_0/2$$

for all a $\in \mathcal{F}$, where σ_n is repeated N times. This contradicts the assumption that the theorem is false.

Theorem(4.1.49)[89]: Let A be a separable unital nuclear simple C^*-algebra with TR(A)$\leqslant$ 1 satisfying the AUCT and let **L** : U (A) $\to \mathbf{R}_+$. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset$

A, there exist $\delta_1 > 0$, an integer $n > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$, a finite subset $S \subset A$ satisfying the following:

(i) there exist mutually orthogonal projections $q, p_1, \ldots, p_n$ with $q \leqslant p_1$ and $p_1, \ldots, p_n$ mu-tually unitarily equivalent, and there exists a C^*-subalgebra $C \in I$ with $1_C = p_1$ and unital $S\text{-}\delta_1/2$-multiplicative contractive completely positive linear maps $\emptyset_0 : A \to qAq$ and $\emptyset_1 : A \to C$ such tha

$$\|x - (\emptyset_0(x) \oplus \emptyset_1(x), \emptyset_1(x), \ldots, \emptyset_1(x)\| < \delta_1/2$$

for all $x \in S$, where $\emptyset_1(x)$ is repeated n times; moreover, there exist a finite subset $\mathcal{G}_0 \subset A$, a finite subset $\mathcal{P}_0$ of projections in $M_\infty(C)$, a finite subset $\mathcal{H} \subset A_{sa}$, $\delta_0 > 0$ and $\sigma > 0$ (which depend on the choices of C);

for any unital simple C^*-algebra B with $TR(B) \leqslant 1$ and any two $S \cup \mathcal{G}_0$-δ-multiplicative completely positive linear contractions $L_1, L_2 : A \to B$ for which the following hold (with $\delta = \min\{\delta_1, \delta_0\}$):

(ii) $[L_1]|_{\mathcal{P} \cup \mathcal{P}_0} = [L_2]|_{\mathcal{P} \cup \mathcal{P}_0}$;

(iii) $|\tau \circ L_1(g) - \tau \circ L_2(g)| < \sigma$ for all $g \in \mathcal{H}$ and $\tau \in T(A)$;

(iv) $e = L_1 \circ \emptyset_0(1_A) = L_2 \circ \emptyset_0(1_A)$ is a projection;

(v) $\mathrm{cel}(\widetilde{L_1}(\emptyset_0(u))^* \widetilde{L_2}((\emptyset_0(u)))) \leqslant L(u)$ (in U (eBe)) for all $u \in U(A) \cap \mathcal{P}$,

there exists a unitary $U \in B$ such that

$\mathrm{ad}(U) \circ L_1 \approx_\varepsilon L_2$ on $\mathcal{F}$.

Note that (i) holds as long as $TR(A) \leqslant 1$ and does not depend on L, ε and $\mathcal{F}$.

Theorem(4.1.50)[89]: Let A be a unital separable simple C^*- algebra with $TR(A) \leqslant 1$ and with torsion $K_1(A)$. For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there exist $\delta > 0$, $\sigma > 0$, a finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital simple C^*-algebra B with $TR(B) \leqslant 1$, any two $\mathcal{G}$-δ-multiplicative completely positive linear contractions $L_1, L_2 : A \to B$ with

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}$$

and

$$\sup_{\tau \in T(B)} \{|\tau \circ L_1(g) - \tau \circ L_2(g)|\} < \sigma$$

for all $g \in \mathcal{G}$, there exists a unitary $U \in B$ such that

$$\mathrm{ad}(U) \circ L_1 \approx_\varepsilon L_2 \text{ on } \mathcal{F}.$$

Theorem. (4.1.51)[89]: Let A be a unital nuclear simple C^*- algebra with $TR(A) \leqslant 1$ and with torsion $K_1(A)$ which satisfies the AUCT. Then an automorphism $\alpha : A \to A$ is approximately inner if and only if $[\alpha] = [id_A]$ in KL(A, A) and $\tau \circ \alpha(x) = \tau(x)$ for all $x \in A$ and $\tau \in T(A)$.

Proof: If α is approximately inner, then it is clear that

$$\tau \circ \alpha(x) = \tau(x)$$

for all $x \in A$ and $\tau \in T(A)$. The "only if" part follows from [35]. It is also clear that the "if part".

Section (4.2) Existance and Classfication Theorms

Definition(4.2.1)[89]: Let A and B be two unital stably finite C^*- algebras and let $\alpha : K_0(A)$ $\to K_0(B)$ be a positive homomorphism and $\Lambda : T(B) \to T(A)$ be a continuous affine map. We say Λ is compatible with α if $\Lambda(\tau)(x) = \tau(\alpha(x))$ for all $x \in K_0(A)$, where we view τ as a state on $K_0(A)$. Let S be a compact convex set. Denote by $Aff(S)$ the set of all (real) continuous affine functions on S. Let $\Lambda : S \to T$ be a continuous affine map from S to another compact convex set T . We denote by $\Lambda : Aff(T) \to Aff(S)$ the unital positive linear continuous map defined by $\Lambda(f)(s) = f(\Lambda(s))$ for $f \in Aff(T)$. A positive linear map $\xi :$ $Aff\ T(A) \to Aff\ T(B)$ is said to be compatible with α if $\xi(\hat{p})(\tau) = \tau(\alpha(p))$ for all $\tau \in T$ (B) and any projection $p \in M_\infty(A)$. Let A be a unital C^*- algebra (with at least one normalized trace). Define $Q : A_{sa} \to Aff\ T(A)$ by $Q(a)(\tau) = \tau(a)$ for $a \in A$. Then Q is a unital positive linear map.

A C^*- algebra A is said to be KK-attainable for a class of stably finite C^*- algebras C, if for any C^*- algebra $B \in C$, any $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B))^{++}$ (see 8.2) and any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ with $[1_A] \subset \mathcal{P}$, there exists a sequence of completely positive linear contractions $L_n :$ A $\to B \otimes K$such that

$$\|L_n(a)L_n(b) - L_n(ab)\| \to 0 \ \ \text{and} [L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} \ \ \text{for all a, b} \in A.$$

For the rest of the section, when we say a C^*- algebra A is KK-attainable, we mean that A is KK-attainable for unital separable simple C^*- algebras with tracial rank no more than 1. As in [77], if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a C^*- subalgebra A_1 of A which is KK-attainable such that $F \subset_\varepsilon A_1$, then A is KK-attainable.

A unital nuclear separable simple C^*- algebra A with TR(A) $\leqslant 1$ is said to be pre-classifiable if it satisfies the Universal Coefficient Theorem and is KK-attainable, and, in addition to the above, for any unital separable nuclear simple C^*- algebra with TR(A)$\leqslant 1$ and any continuous affine map $\Lambda : T(B) \to T(A)$ compatible with α,

$$\underset{\tau \in T(B)}{sup}\{|\Lambda(\tau)(a) - \tau \circ L_n(a)|\} \to 0 \ \ \ \text{for all a} \in A$$

Or, equivalently, for any contractive positive linear map $\xi :Aff\ T(A) \to Aff\ T(B)$compatible with α,

$$\underset{\tau \in T(B)}{sup}\{|\xi(Q(a))(\tau) - \tau \circ L_n(a)|\} \to 0 \ \ \ \ \text{for all a} \in A_{sa}$$

If h : A $\to$ B is a unital homomorphism, then h induces a unital positive affine map h : $Aff\ T$ (A) $\to Aff\ T$ (B). The map $h_\#$ is contractive. Suppose that Y is a compact metric space and $P \in M_1(C(Y))$ is a non-zero projection with constant rank. It is known and easy to see that

$$Aff\ T\ (P\ M_1(\ C(Y\ ))\ P) = Aff(T\ (M_1(\ C(Y)) = C_R(Y).$$

Theorem(4.2.2)[89]: Let A be a simple unital C^*- algebra with at least one tracial state. Then for any affine function $f \in Aff$ (T (A)) with $\|fE\| \leqslant 1$ and any $\varepsilon > 0$, there exists an

element a $\in A_{sa}$ with $\|a\| < \|f\| + \varepsilon$ such that $\tau(a) = f(\tau)$ for all $\tau \in$ T (A). Furthermore, if $f >$ 0, we can choose a 0.

Proof: We prove this using the results in [19]. We may identify T (A) with the real part of the unit sphere of $(A^q)*$ (see [19] for the notation). By [8, 2.8], it suffices to consider those $f \in Aff$ (T (A)) with $f(\tau) > 0$ for all $\tau \in$ T (A). There is an element b $\in (A^q)***$ such that $b(\tau) = f(\tau)$ for all $\tau \in$ T (A). Since f is (weak-$*$) continuous, b $\in A^q$. Since $b(\tau) > 0$ for all $\tau \in$ T (A), by [19] there is c $\in A_+$ and z $\in A_{sa}$ with $\tau(z) = 0$ for all $\tau \in$ T (A) (i.e., z $\in A_0$ using the notation in [19]) such that b = c + z. Now the theorem follows from [19].

Lemma(4.2.3)[89]: Let A be a separable unital C^*- algebra. Let $\varepsilon > 0$ and $\mathcal{F} \subset$ A be a finite subset. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset$ A satisfying the following: for any unital separable C^*- algebra C with at least one tracial state and any unital contractive positive linear maps L : A $\rightarrow$ C which is $\mathcal{G}$-δ-multiplicative, then, for any t $\in$ T (C) there is a trace $\tau \in$ T (A) such that
$$|\tau(a) - t(L(a))| < \varepsilon \quad \text{for all a} \in \mathcal{F}.$$

Proof. Otherwise, there would be an $\varepsilon 0 > 0$ and a finite subset $\mathcal{F} \subset$ A, a sequence of unital separable C^*- algebra C_n, a sequence of unital contractive positive linear map L_n : A $\rightarrow C_n$ such that
$$\lim_{n \to \infty} \|L_n(a)L_n(b) - L_n(ab)\| = 0 \quad \text{for all a, b} \in A,$$
and a sequence $t_n \in$ T (C_n) such that
$$\inf\{\max\{| t(a) - t_n(L_n(a))|: a \in \mathcal{F}\} : t \in \text{T (A)}\} \geqslant \varepsilon_0$$
for all n. Let s_n be a state of A which extends $t_n \circ L_n$. Let τ be a weak limit of $\{s_n\}$. So there is a subsequence $\{n_k\}$ such that $\tau(a) = \lim_{m \to \infty} s_{n_k}(a)$ for all a $\in$ A. It is a routine to check that $\tau \in$ T (A). Therefore, there exists K > 0, such that
$$\left|\tau(a) - t_{n_k}(L_{n_k}(a))\right| < \varepsilon_0/2$$
for all k $\geqslant$ K. We obtain a contradiction.

Lemma(4.2.4)[89]: Let A = C(X), where X is a path connected finite CW-complex. Let B be a unital separable nuclear non-elementary simple C^*- algebra with TR(B)$\leqslant$ 1 and Λ : T (B) $\rightarrow$ T (A) be a continuous affine map. Then, for any $\sigma > 0$ and any finite subset $\mathcal{H} \subset Aff$ T (A), there exists a unital monomorphism h : A $\rightarrow$ B such that the image of h is in a C^*- subalgebra $B_0 \in I$ and
$$\|h(f) - \Lambda(f)\| < \varepsilon$$
for all f $\in \mathcal{H}$, where h , Λ :Aff T (A) $\rightarrow Aff$ T (B) are the maps induced by h and Λ, respec-tively.

Moreover, if there is positive homomorphism $\alpha : K_0(A) \to K_0(B)$ with $\alpha([1_A]) = [1_B]$ and Λ is compatible with α, then the above is also true for A = P M_1 (C(X))P , where P $\in M_1$ (C(X)) is a projection inM_1 (C(X)). Furthermore, if X is contractible, we can also require that $h_{*0} = \alpha$.

Corollary(4.2.5)[89]:   Let A $\in$I, B be a unital separable nuclear simple C^*- algebra with TR(B) $\leqslant$1, $\gamma : K_0(A) \to K_0(B)$ be a positive homomorphism and $\Lambda : T (B) \to T (A)$ be an affine contin-uous map which is compatible with γ . Then, for any $\sigma > 0$ and any finite subset $\mathcal{G}\subset$ A, there exists a unital monomorphism $\emptyset : A \to B$ such that

$$\sup_{\tau\in T(B)}\{|\tau \circ \emptyset(g) - \Lambda(\tau)(g)|\}< \sigma$$

for all g $\in\mathcal{G}$ and $\emptyset_* = \gamma$.

Proof: Note that 9.5 holds for A = M_n. It is then clear that, by considering each summand of A, the corollary follows from 9.5.

Proposition (4.2.6)[89]:   Every KK-attainable, unital separable nuclear simple C^*-algebra A with TR(A) $\leqslant$1 which satisfies the AUCT is pre-classifiable.

Proof. Let A be a KK-attainable separable nuclear simple C^*-algebra with TR(A)$\leqslant$ 1 satisfying the AUCT and B be a unital nuclear separable simple C^*-algebra with TR(B) $\leqslant$1. Let $\alpha \in Hom_\Lambda(\underline{K}(A),\underline{K}(B))^{++}$, $\mathcal{P}\subset\mathbf{P}(A)$ be a finite subset containing $[1_A]$, and $\Lambda : T (B) \to$ T (A) be a continuous map which is comparable to $\alpha|_{K_0}(A)$. Suppose that e $\in$B is a projection such that $\alpha(1_A) = e$. To save notation, without loss of generality, we may assume that B = e(B $\otimes K$)e. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers with $lim_{n\to\infty}\delta_n = 0$. For each n, since A is a unital simple C^*-algebra with TR(A) $\leqslant$1, there are nonzero projections $p_n\in$ A and a C^*-subalgebra $C_n\in I$ with $1_{C_n} = p_n$, and a sequence of unital completely positive linear contractions $\Phi_n : A \to C_n$ such that:

(i)$\|[x, p_n]\|<\delta_n$,

(ii)$\|p_n x p_n - \Phi_n(x) \|<\delta_n$

(iii)$\|x - (p_n x p_n \oplus \Phi_n(x))\|<\delta_n$ for all x $\in$ A with $\|x\| \leqslant 1$ and

(iv)$\tau (1 - p_n) < 1/2n$ for all $\tau \in$ T (A).

Denote by $\Psi_n(x) = (1 - p_n)x(1 - p_n) + \Phi_n(x)$ (for x $\in$ A). Note that $\|\Psi_n(ab) - \Psi_n(a)\Psi_n(b)\| \to 0$ and $\|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)\| \to 0$ for all a, b $\in$A as n$\to \infty$.

Since A is KK-attainable, for each n, there exists a sequence of completely positive linear contractions $L_n : A \to B \otimes K$ such that

$$[\Psi_n]|_\mathcal{P} = [id]|_\mathcal{P}, \quad [L_n]|_\mathcal{P} = \alpha|_\mathcal{P}, \quad [L_n \circ \Psi_n]|_\mathcal{P} = \alpha|_\mathcal{P},$$
$$\|L_n \circ \Psi_n(ab) - L_n \circ \Psi_n(a)L_n \circ \Psi_n (b)\| \to 0 \text{ and}$$
$$\|L_n \circ \Phi_n(ab) - L_n \circ \Phi_n(a)L_n \circ \Phi_n(b)\| \to 0$$

as n$\to \infty$for all a, b $\in$ A. Suppose that$C_n=\bigoplus_{i=1}^{t(n)} D_{n,i}$, where $D_{n,i} \cong M_{(n,i)}$ or $D_{n,i} \cong M_{(n,i)}$ (C(0, 1])). Let $d_{n,i}= id_{D_{n,i}}$. We may also assume that, for each n, $L_n(d_{n,i})$ is a projection and $[L_n] ([d_{n,i}]) = \alpha ([d_{n,i}])$ for all n, i.

Let $\gamma_n : T (A) \to T (C_n)$ be defined by $\gamma_n(\tau) = (1/\tau (p_n))\tau |_{C_n}$. Let $\mathcal{G}_n$ be a finite subset (containing generators) of C_n and let $\{d_n\}$ be a decreasing sequence of positive numbers with

$lim_{n\to\infty} d_n = 0$. For large n, we obtain a homomorphism $h_n : C_n \to q_n B q_n$, where $[d_{n,i}] = \alpha([d_{n,i}])$, such that

$$|\tau \circ h_n(g) - \gamma_n \circ \Lambda(\tau)(g)| < d_n \quad \text{for all } \tau \in T(B) \text{ and}$$

for all $g \in \mathcal{G}_n$. Put $\emptyset_n(x) = L_n \circ ((1 - p_n)x(1 - p_n)) + h_n \circ \Phi_n(x)$ for $x \in A$. It is easy to see, by choosing a large n, $\emptyset_n : A \to B$ meets the requirements of Definition (4.2.1)

Lemma(4.2.7)[89]: Let A be a unital C^*-algebra, B be a unital separable simple C^*-algebra with TR(B) $\leqslant 1$ and F $\in$ I be a C^*-subalgebra of B. Let G be a subgroup generated by a finite subset of **P**(A). Suppose that there is an $\mathcal{F}$-δ -multiplicative contractive completely positive linear map $\psi : A \to F \subset B$ such that $[\psi]|_G$ is well defined. Then for any $\varepsilon > 0$, there exists a finite-dimensional C^*-subalgebra C $\subset$ B and an $\mathcal{F}$-δ–multiplicative contractive completely positive linear map L : A $\to$ C $\subset$ B such that

$$[L]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})} = [\psi]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}, \quad \text{and} \quad \tau(1_C) < \varepsilon$$

for all tracial states τ in T(B) and for all k$\geqslant$ 1 so that G$\cap K_0(A, \mathbf{Z}/k\mathbf{Z}) = \neq \{0\}$, where **L** and ψ are viewed as maps to B. Furthermore, if $[\psi]_{G \cap K_0(A)}$ is positive, so is $[L]|_{G \cap K_0(A)}$.

Proof: Let $0 < \varepsilon < 1$. Without loss of gen-erality, we may assume that F = C([0, 1]) $\otimes M_n$. Let $q_1 \in$ F be a minimal projection. Suppose that

$$G \cap K_0(A, \mathbf{Z}/k\mathbf{Z}) = \{0\} \quad \text{for k > K.}$$

with m = 2lK! + 1 and $1/l < \varepsilon/n$, we may write $q_1 = q + \sum_{i=1}^m p_i$, where $[q] \leqslant [p_1]$, q, $p_1, \ldots, p_m$ are mutually orthogonal projections, $[p_1] = [p_i]$, i = 1, 2, . . ., m and $\tau(p_1)$ $< 1/2l < \varepsilon/2n$. Set $e_1 = q + p_1$ and $q_0 = \sum_{j=2}^{2l+1} p_j$. Then $[e_1] + K![q_0] = [q_1]$ in K_0 (B) and $\tau(e_1) < \varepsilon/n$ for all tracial states τ on B.From this we obtain a C^*-subalgebra C of B such that C $\cong M_n$ and its minimal projection is equivalent to e_1.In particular, $\tau(1_C) < \varepsilon$. Let $\emptyset$: F $\to M_n \to$ C be a unital homomorphism, where the map F $\to M_n$ is a point-evaluation. Let L = $\emptyset \circ \psi$, $j_1 : $ F $\to$ B and $j_2 : $ C $\to$ B be embeddings. By the choice of q_1, $[e_1]$ and $[q_1]$ have the same image in $K_0(B)/kK_0(B)$ for k = 1, 2, . . . , K. Therefore $(j_1)_* = (j_2 \circ \emptyset)_*$ on $K_0(F, \mathbf{Z}/k\mathbf{Z})$ for all k$\leqslant$ K. Since $K_1(F) = K_1(C) = 0$, by the six-term exact sequence in 8.2 (see [32, 1.6]), both [L] and [ψ] map $K_0(A, \mathbf{Z}/k\mathbf{Z})$ to $K_0(B)/kK_0(B)$ and factor through $K_0(F, \mathbf{Z}/k\mathbf{Z})$. Therefore

$$[L]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})} = [\psi]|_{G \cap K_0(A, \mathbf{Z}/k\mathbf{Z})}, \quad k = 1, 2, \ldots, K.$$

The general case in which F is a direct sum of M_1 (C([0, 1])) follows immediately.

Lemma (4.2.8)[89]: Let C $= \bigoplus_{j=1}^n C_j$, where each $C_j = P_j M_{s(j)}(C(X_j))P_j$, P_j is a projection in $M_{s(j)}(C(X_j))$ and X_j is a path connected compact metric space with finitely generated $K_1 (C_j)$ $K_0(C(X_j)) = \mathbf{Z} \oplus tor(K_0(C_j))$, $K_1(C(X_j))$ and $K_0(C_j)) \subset \{(z, x): z \in \mathbf{N}, \text{ or } (z, x) = (0, 0)\}$. Then C is KK-attainable.

Proposition(4.2.9)[89]: Let A be a separable unital simple C^*-algebra with TR(A) $\leqslant 1$. If A is locally AH, then A is pre-classifiable.

Proof: It follows from [76] that A satisfies the AUCT. We may assume that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, where each A_n is a finite direct sums of $P_{n,i} M_{r(n,i)} C(X_{n,i}) P_{n,i}$ and $X_{n,i}$ is a path connected finite CW complex. One may assume that $1_{A_n} = 1_A$. Put $j_n : A_n \to A$ the embedding. Consider $j_n \times \alpha$. If A_n has only one summand, then $K_0(A_n) = \mathbb{Z} \oplus \ker \rho_{A_n}$. Since $\alpha \in KL(A, B)^{++}$, $(j_n \times \alpha) \in KL(A_n, B)^{++}$. By considering each summand separately, we may assume An has only one summand. Since A is simple it suffices to show that, $A = C(X)$ is KK-attainable for every path connected finite CW complex X.

Let $\alpha \in KK(A, B)^{++}$. Suppose that $\alpha(1_A) = [p] (\neq 0)$, where p $\in M_l$ (B) is a projection. Fix a unital nuclear simple C^*-algebra B with TR(B) $\leqslant 1$. By [134], there is a unital simple C^*-algebra C which is direct limit of C^*-algebras such that

$$(K_0(C), K_0(C)_+, [1_C], K_1(C)) = (K_0(B), K_0(B)_+, [1_A], K_1(B)).$$

By [125], there exists $\beta \in KK(C, B)$ which gives the above isomorphism.

Let $\alpha \in KL(A, B)^{++}$ and $\gamma = \alpha \times \beta^{-1} \in KL(A, C)^{++}$. Since K_i (C(X)) is finitely generated, KL(A, C) = KK(A, C). In particular, $\gamma (K_0(A)_+ \setminus \{0\}) \subset K_0(C)_+ \setminus \{0\}$. By [58], there is a homomorphism h : A $\to$ pM_l (C)p such that [h] = γ . But by 9.9, since each C^*-algebra described in Lemma (4.2.8) is KK-attainable, C is KK-attainable (see 9.1). Let $\varepsilon > 0$ and fix finite subsets $\mathcal{F} \subset$ A and $\mathcal{P} \subset \mathbf{P}(A)$. Let $\mathcal{G} = h(\mathcal{F}) \subset$ C and $\mathcal{Q} = [h](\mathcal{P}) \subset \mathbf{P}(C)$. Let $\Lambda : C \to B$ be a $\mathcal{G}$-ε-multiplicative contractive completely positive linear map such that $[\Lambda]|_{\mathcal{Q}} = \beta|_{\mathcal{Q}}$.

Define L = $\Lambda \circ h$. Then L : A $\to$ B is a $\mathcal{F}$-ε–multiplicative contractive completely positive linear map such that $[L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$.

So A is *KK*-attainable.

Lemma(4.2.10)[89]: Let A be a unital separable C^*-algebra, $\{\mathcal{F}_k\}$ be an increasing sequence of finite subsets of the unit ball of A such that $\bigcup k \mathcal{F}_k$ is dense in the unit ball of A, and let $\Phi_n : A \to A$ be a sequence of unital contractive completely positive linear maps such that $lim_{n\to\infty} \|\Phi_n(a)\| = \|a\|$ for all a $\in$ A and

$$\sum_{k=n}^{\infty} \|\Phi_k(ab) - \Phi_k(a)\Phi_k(b)\| < \sum_{k=n}^{\infty} \delta_n ,$$

for all a, b $\in \mathcal{G}_n$, and for any n finite subset $\mathcal{P} \subset \mathbf{P}(A)$, $[\Phi_n]|_{\mathcal{P}} = [id]|_{\mathcal{P}}$ for all sufficiently large n, where $\mathcal{G}_1 = \mathcal{F}_1$, $\mathcal{G}_{n+1} \supset \bigcup_{k=1}^{n} \Phi_k(\mathcal{F}_n) \cup \mathcal{F}_n \cup \Phi_n(\mathcal{G}_n)$, n = 1, 2, . . . , and where $\sum_{n=1}^{\infty} \delta_n < \infty$

. Let B $lim_{n \to \infty}$ (A, Φ_n) be the generalized inductive limit in the sense of [7]). Then $\{\Phi_n\}$ induces an isomorphism

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Proof: The proof is standard. We sketch here. Write K_i (A) $= \bigcup_{n=1}^{\infty} G_n^{(i)}$, where each $G_n^{(i)}$ is a finitely generated subgroup of $K_0(A)$. Let $\Phi_{n,n+m} = \Phi_{n+m-1} \circ \ldots \circ \Phi_n$ and $\Psi_n: A \to B$ be the map induced by the inductive system which maps the nth A to B. For each $G_n^{(i)}$, we may assume that $[\Psi_n]|_{G_n^{(i)}}$ is well defined for all $m \geqslant n$. The assumption that $[\Phi_n]|_{\mathcal{P}} = [id_A]|_{\mathcal{P}}$

for all sufficiently large n implies that $[\Psi_m]|_{G_n^{(i)}} = [\Psi_m]|_{G_n^{(i)}}$ for all m, $m \geqslant$ n. This gives a homomorphism $\beta_i : K_i$ (A) $\to K_i$ (B) (i = 0, 1).

Suppose that p_1, p_2, $v \in M_l$ (B) such that $v^*v = p_1$ and $vv^* = p_2$. There is a sequence $\{\Psi_n(a_k)\}$, where $a_k \in M_l$ (A), such that it converges to v. Since $v^*v = p_1$, we have $\Psi_{n_k}(a_k^*a_k) \to p_1$ and $\Psi_{n_k}(a_k a_k^*) \to p_2$. Therefore we may assume that

$\|\Psi_{n_k}(a_k^*a_k - (a_k^*a_k))^2\| < 1/2^{k+1}$ and $\|\Psi_{n_k}(a_k^*a_k) - p_1\| < 1/2^{k+1}$.

Since $\|\Psi_m(x)\| = \limsup \|\Phi_{m,n}(x)\|$ for all x $\in$ A and $m \geqslant 1$, by passing to a subsequence and possibly replacing ak by $\Phi_{m_k,n_k}(a_k)$, Ψ_{n_k} by Ψ_{m_k}, if necessary, we may assume that

$\|a_k^*a_k - (a_k^*a_k)^2\| < 1/2^2$, k = 1, 2, ...

It is standard that there is a partial isometry v_k and a projection $q_k \in$ A such that

$v_k^* v_k = q_k$ and $\|v_k - a_k\| < 1/2^{k-1}$

for all large k. Let $q_k = v_k v_k^*$. Note also, for any $\varepsilon > 0$, we have

$\|\Psi_{n_k}(q_k) - p_1\| < \varepsilon$ and $\|\Psi_{n_k}(q_k) - p_2\| < \varepsilon$

for all large k. Hence $[\Psi_{n_k}](q_k) = [p_1]$ and $[\Psi_{n_k}](q_k) = [p_2]$ This, in particular, implies that $[p_1]$ is in the image of β_0. It follows that β_0 is surjective. Note also that $[q_k] = [q_k]$ in $K_0(A)$. It follows that β_0 is also injective. It is also easy to check from the definition that β_0 preserves the order.

In the above, if we let $w*w = p_1$ and $ww* \leqslant p_2$, then exactly the same argument shows that there are partial isometries $v_k \in$ A such that $v_k v_k^* = q_k$, $v_k v_k^* \leqslant q_k$ and $\Psi_{n_k}(v_k) \to$ v, $\Psi_{n_k}(q_k) \to p_1$ and $\Psi_{n_k}(q_k) \to vv^* \leqslant p_2$. These imply that β_0 is an order isomorphism.

A similar argument shows that β_1 is an isomorphism and $K_1(A) = K_1(B)$.

Theorem(4.2.11)[89]: Let A be a unital separable nuclear simple C^*-algebra with TR(A)$\leqslant 1$ satisfy-ing the AUCT. Then there exists a unital separable nuclear simple C^*-algebra B with TR(B) = 0 satisfying AUCT and the following:

(i) $(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (K_0(B), K_0(B)_+, [1_B], K_1(B))$,

(ii) there exists a sequence of contractive completely positive linear maps $\Phi_n : A \to B$ such that:

    (i) $\lim_{n\to\infty} \|\Phi_n$ (ab) $- \Phi_n$ (a)Φ_n (b) $\| = 0$ for a, b $\in$ A,

    (ii) For each finite subset $\mathcal{P} \subset \mathbf{P}(A)$ there exists an integer N > 0 such that

$[\Phi_n]|_{\mathcal{P}} = [\alpha]|_{\mathcal{P}}$

for all n⩾N , where α ∈ KL(A, B) which gives an identification in (i) above.

Theorem(4.2.12)[89]: (Cf. [134].) Suppose that G is a countable, partially ordered abelian group which is simple, weakly unperforated with the Riesz interpolation property, that G/tor(G) is non-cyclic,

u $\in G_+$, H is a countable abelian group, Δ is a metrizable Choquet simplex and λ :Δ → S(G, u) is a continuous affine map with $\lambda(\partial_e \Delta) = \partial_e$S(G, u). Then there is a simple AH-algebra A = $lim_{n\to\infty}(A_n, h_n)$ with TR(A)⩽1 and with $A_n = C_1 \oplus C_2 \oplus \cdots \oplus C_{m(n)}$, where C_1 is of the form as described in 7.1 (a single summand) and C_j is of the form C([0, 1]) $\otimes M_{m(j)}$ (for

j> 1), such that:

$$(1)\, h_n = h_n^{(0)} \oplus h_n^{(1)} \oplus h_n^{(2)}, \text{ where } h_n^{(0)},\ h_n^{(1)} \text{ factor through a } C^*\text{-algebra in I,}$$

$$\text{and } h_n \text{ is injective, in particular, } h_n^{(0)} \text{ is homotopically trivial,}$$

(i)$\tau \circ h_{n+1,\infty} \circ h_n^{(0)}(1_{A_n}) \to 0$ uniformly on T (A),

(ii) $\tau \circ h_{n+1,\infty} \circ h_n^{(2)}(1_{A_n}) \to 0$ uniformly on T (A),

(iii) $(h_n)_{*1}$ is injective and

(iv) $(K_0(A), K_0(A)_+, [1_A], K_1 (A), T (A), r_A) = (G, G_+, u, H, \Delta, \lambda)$.

Proof: The proof of this is a combination of Villadsen's proof of the main theorem in [134]. Let A = $lim_{n\to\infty}(A_n, h_n)$ be as in [75]. This algebra A satisfies (iii), (iv) and $(K_0(A), K_0(A)_+, [1_A], K_1 (A)) = (G, G_+, u, H)$. Moreover each An can be chosen so it has the form as required. Here one needs one modification and one explanation. We use $C_j = M_{m(j)}$ for j > 1. But we can map C([0, 1], $M_{m(j)}$) into $M_{m(j)}$ by a one point-evaluation and then map $M_{m(j)}$ into C([0, 1], $M_{m(j)}$) (as constant functions). So we can assume that A_n has the required form. Note also that the new A_n has the same K-theory as the old one. If $K_1(A) = F = lim_{n\to\infty}(F_n, \gamma_n)$, $K_1(A_n) = F_n$ and the map $\Phi_{n,n+1}$ has the property $(\Phi_{n,n+1})_{*1} = \gamma_n$. However, since F is a countable abelian group, one can always assume that Fn is finitely generated and γ_n is injective (by choosing F_n as subgroups and γ_n as embeddings) so that (iv) holds.

We will revise the map hn to meet the other requirements. Villadsen's proof in [134] is to replace h_n by $\emptyset_n$ without changing its K-theory in such a way that one gets Δ as tracial space and λ as pairing. We will follow his proof with a minor modification. Each h_n may be written as $h_n \oplus h_n$, where hn is a point-evaluation, as in [75]. that holds when X_q^i is a compact connected CW complex with dimension at least one but no more than three. Following Villadsen's proof, as in [134] and its proof, one can replace h_n to achieve exactly what [134]achieved. It should be noted that Villadsen's proof of the main theorem in [134] works when X_q^j has lower dimension (but at least one), since the required maps $i_q^j : [0, 1] \to X_q^j$ and

$k_q^j : X_q^j \to [0, 1]$ still exist. The new map obtained from Villadsen's proof has the form $\widetilde{\psi}_n = h_n \oplus \widetilde{h}_n$, where $\widetilde{h}_n$ is homotopically trivial. Furthermore, it can be chosen so that it factors through a C^*-algebra in I. The construction of Villadsen then gives a simple AH-algebra B with TR(B)$\leqslant$1 and satisfies (5). Moreover, one has $\tau^\circ \widetilde{\psi}_{n+1}{}^\circ h_n(1_{A_n}) \to 0$ uniformly on T (B). The construction does not change (3) and (4). It is also easy to get (1) and (2). For example, consider $h_{n+1}{}^\circ h_n \oplus h_{n+1}{}^\circ \widetilde{h}_n \oplus \widetilde{h}_{n+1}{}^\circ \widetilde{\psi}_n$. Note that $h_{n+1}{}^\circ \widetilde{h}_n$ and $\widetilde{h}_{n+1}{}^\circ \widetilde{\psi}_n$ are homotopically trivial and $\tau^\circ \widetilde{\psi}_{n,\infty}{}^\circ h_{n+1}{}^\circ \widetilde{h}_n(1_{A_n}) \to 0$ uniformly on T (B).

Definition(4.2.13)[89]: Let C be a unital C^*-algebra. We denote by $S_u(K_0(C))$ the set of states on $K_0(C)$, i.e., the set of order and unit preserving homomorphisms from $K_0(C)$ to (the additive group) $\mathbb{R}$. There is an affine map $\lambda : $ T (C) $\to S_u(K_0(C))$ such that $\lambda(t)([p]) = t (p)$ for all projections p $\in M_\infty(C)$ and t $\in$ T (C). Suppose that C is stably finite. It was proved in [123] (for the simple case) that each state in $S_u(K_0(C))$ is induced by a quasitrace t $\in \mathcal{Q}$T (C). If C is exact, or if it is both simple and of tracial rank at most one, then all quasitraces on C are traces.

Let A and B be two unital C^*-algebras. We say
$\gamma : (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B))$
is an order isomorphism if there is an order isomorphism
$\gamma_0 : (K_0(A), K_0(A)_+) = (K_0(B), K_0(B)_+)$
which maps $[1_A]$ to $[1_B]$, there is an isomorphism $\gamma_1 : K_1(A) \to K_1(B)$ and an affine homeomorphism $\gamma_2 : $ T (A) $\to$ T (B) such that $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in$ T (B) and x $\in K_0(A)$, where we view τ as a state on $K_0(A)$.

Theorem(4.1.14)[89]: Let A and B be two unital separable nuclear simple C^*-algebras with TR(A)$\leqslant$1 and TR(B) $\leqslant$1 satisfying AUCT such that
$(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A))$
in the sense of 10.2. Then there is a sequence of contractive completely positive linear maps $\{\Psi_n\}$ from A to B such that:

    (i) $lim_{n\to\infty}\Psi_n(ab) - \Psi_n(a)\Psi_n(b) = 0$ for all a, b $\in$ A,

    (ii)   for any finite subset set $\mathcal{P} \subset \mathbf{P}$(A),

$[\Psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$,

for all sufficiently large n, where $\alpha \in KL(A,B)^{++}$ gives the above identification on K-theory and

    (iii) $\displaystyle \lim_{n \to \infty} \sup_{\tau \in \text{T (B)}} \{|\tau^\circ \Psi_n(a) - \xi(\mathcal{Q}(a))(\tau)|\}$

$f$or all a $\in A_{sa}$, where $\xi : \mathrm{Aff}$ T (A) $\to \mathrm{Aff}$ T (B) is the affine isometry given above.

Proof. It follows from Theorem 9.12 that there is a unital separable simple nuclear C^*- -algebra C with $TR(C) = 0$ satisfying the AUCT such that

$(K_0(A), K_0(A), [1_A], K_1(A)) = (K_0(C), K_0(C)_+, [1_C], K_1(C))$

and a sequence of contractive completely positive linear maps $L_n : A \to C$ satisfying condition

(ii) of theorem(4.1.63) In particular, $[L_n]|_{\mathcal{P}} = \beta|_{\mathcal{P}}$, for any finite subset $\mathcal{P}$ and all sufficiently large n, where $\beta \in KL(A, C)^{++}$ gives the above identification on K-theory. It follows from [93] that there is a unital separable simple AH-algebra C_1 such that $C_1 \cong C$. To simplify notation, we may assume that $C_1 = C$.

It follows from 9.10 that there exists a sequence of contractive completely positive linear maps $\Phi_n : C \to B$ such that:

(i) $lim_{n\to\infty}|\Phi_n(ab) - \Phi_n(a)\Phi_n(b)| = 0$ for all a, b $\in$ C,

(ii) for any finite subset $Q \subset \mathbf{P}(C)$,

$[\Phi_n]|_Q = (\beta^{-1} \times \alpha)|_Q$, for all sufficiently large n.

Thus by choosing a subsequence {k(n)} and defining $\Psi_n = \Phi_{k(n)} \circ L_n : A \to B$ we see that Ψ_n satisfies (i) and (ii). (In fact one can show that A is KK-attainable.) We then apply the proof of 9.7, to obtain a (new) sequence $\{\Phi_n\}$ which also satisfies (iii).

Using the argument of [93], Zhuang Niu gives a different proof of the above theorem.

Theorem(4.2.15)[89]: Let A and B be two unital separable nuclear simple C^*-algebras with $TR(A) \leqslant 1$ and $TR(B) \leqslant 1$ satisfying the AUCT. Suppose that $\lambda(\partial_e(T (A))) = \partial_e(S_u(K_0(A)))$ and $\lambda(\partial_e(T (B))) = \partial_e(S_u(K_0(B)))$. Then A is isomorphic to B if and only if there exists an order isomorphism

$$\gamma = (\gamma_0 , \gamma_1, \gamma_2) :(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A)) \to$$
$(K_0(B), K_0(B)_+, [1_B], K_1(B), T(B)),$

where $\gamma_2^{-1}(\tau)(x) = \tau(\gamma_0(x))$ for all $\tau \in T (B)$ and x $\in K_0 (A)$(see 10.2).

Theorem(4.2.16)[89]: Let A and B be two unital simple AH-algebras with very slow dimension growth and with torsion $K_1(A)$. Then A is isomorphic to B if and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T (A)) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T (B))$$

Let C be a stably finite non-unital C^*-algebra with an approximate identity consisting of projections $\{e_n\}$. Let $T (C)$ denotes the set of traces τ on C such that $sup_n \tau (e_n) = 1$. We refer to these traces as tracial states on C, and to $T (C)$ the tracial state space of C. Note that each tracial state extends to a tracial state on $\tilde{C}$. Therefore $T (\tilde{C})$ is the set of convex combinations of $\tau \in T (C)$ and the tracial state which vanishes on C. We also denote by $S_u (K_0(C))$ the set of those order preserving homomorphisms from $K_0(C)$ to $\mathbb{R}$ such that $sup_n s ([e_n]) = 1$. Then each element in $S_u (K_0(\tilde{C}))$ is the convex combination of s $\in S_u (K_0(C))$ and the state which vanishes on $j_*(K_0(C))$, where j : C $\to \tilde{C}$ is the embedding.

Lemma(4.2.17)[89]: Let A be a unital separable simple C^*-algebra with $TR(A) \leqslant 1$. Then there is a C^*-algebra $C = \lim_{n\to\infty}(C_n, \varphi_n)$, where $C_n \in I$, satisfying the following:

(i) each C_n is a C^*-subalgebra of A and $\{\varphi_{n,\infty}(1_{C_n})\}$ forms an approximate identity for C;

(ii)there is a sequence of contractive completely positive linear maps $L_n : A \to C$ such that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0, \quad a, b \in A;$$

(iii) there is an affine continuous (face-preserving) isomorphism r :T (A) $\to$ T (C) such that

$$r(\tau)(\varphi_{n,\infty}(b)) = \lim_{k\to\infty} \tau\left(\varphi_{n,k}(b)\right) \text{ for all } b \in C_n \text{ and } \tau \in T(A);$$

(iv) there is an affine continuous (face-preserving) isomorphism r : $S_u(K_0(C)) \to S_u(K_0(A))$ such that

$$r(s)([p]) = \lim_{n\to\infty} \tau_s(L_n(p)) \text{ for all } s \in S_u(K_0(C)) \text{ and projection} p \in A,$$

where τ_s is the trace which induces s.

Proof: Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \bullet \bullet \bullet \subset \mathcal{F}_n \subset \bullet \bullet \bullet$be a sequence of finite subsets of A such that $\cup n \ \mathcal{F}_n$is

dense in A. Since TR(A)$\leqslant$1, there is a C^*-subalgebra $C_1 \subset A$ with $C_1 \in I$ and $1_{C_1} = p_1$ such that:

(i) $\|ap_1 - p_1 a\| < 1/2$ for all a $\in \mathcal{F}_1$,

(ii) dist($p_1 a p_1, C_1$) $< 1/2$ for all a $\in \mathcal{F}_1$,

(iii) $\tau(1 - p_1) < 1/4$ for all $\tau \in$ T (A).

Let $1 > \eta_1 > 0$, there is a projection $e_{(1,1)} \leqslant p_1$ such that $e_{(1,1)}$ is equivalent to $1 - p_1$. Since $\tau(p_1 - e_{(1,1)}) > 1/2 > \tau(1 - p_1)$ for all $\tau \in$ T (A), by 4.7 again, we obtain mutually orthogonal projections $e_{(1,1)}, e_{(1,2)}$ such that $e_{(1,i)} \leqslant p_1$, $[e_{(1,1)}] = [e_{(1,2)}] \geqslant [1 - p_1]$. There are $x_{(1,1)}, x_{(1,2)} \in A$ such that $x^*_{(1,i)} x_{(1,i)} \geqslant 1 - p_1$ and $x_{(1,i)} x^*_{(1,i)} = e_{(1,i)}$. Let $\mathcal{G}_1$ be a finite set of generators of C_1 and $\mathcal{G}_2 = \mathcal{F}_2 \cup \mathcal{G}_1 \cup \{x_{(1,i)}, x^*_{(1,i)}, e_{(1,i)} : 1 \leqslant i \leqslant 2\}$. There is a C^*-subalgebra $C_2 \subset A$ with$C_2 \in I$ and $1_{C_2} = p_2$ such that:

(i) $\|a p_2 - p_2 a\| < \eta_1/4$ for all a $\in \mathcal{G}_2$,

(ii) dist($p_2 a p_2, C_2$) $< \eta_1/4$ for all a $\in \mathcal{G}_2$, and

(iii) $\tau(1 - p_2) < 1/8$ for all $\tau \in$ T (A).

With sufficiently smallη_1, there is a homomorphism $\varphi_1 : C_1 \to C_2$ such that $\|\emptyset_1(b) - p_2 b p_2\| < 1/4$ for all b $\in \mathcal{F}_1 \cup \mathcal{G}_1$.

Put $q_2 = \emptyset_1(1_{C_1})$. With sufficiently small η_1, since $x_{(1,i)} \in \mathcal{G}_2$, we may also assume that $2[p_2 - q_2] \leqslant [q_2]$ in $K_0(C_2)$. Note that $q_2 \leqslant p_2$.

We continue in this fashion. Suppose that $C_n \subset$ A is a unital C^*-subalgebra which is in I has been constructed. If $\tau(1 - p_n) < 1/2^{n+1}$ for all $\tau \in$ T (A), there are partial isometries x(n,i) $\in$ A

such that $p_n = 1_{C_1}$, $x^*_{(1,i)} x_{(1,i)} \geq 1 - p_n$, $x_{(1,i)} x^*_{(1,i)} = e_{(n,i)} \leq p_n$, $e_{(n,i)} e_{(n,j)} = 0$ if $i \neq j$ and

$[e_{(n,i)}] = [e_{(n,1)}] \geq [1 - p_n]$, $1 \leq i \leq 2^n$. Let $\mathcal{G}_n$ be a finite set which contains a set of generators of C_n, $\emptyset_{i,n}(\mathcal{G}_i)$ and $\emptyset_{i,n}(p_i)$, i = 1, 2, ..., n − 1, where $\emptyset_{i,n} = \emptyset_{n-1} \circ \emptyset_{n-2} \circ \cdots \circ \emptyset_i$. (Note that $C_i \subset$ A.) Let $\mathcal{G}_{n+1} = \mathcal{F}_{n+1} \cup \mathcal{G}_n \cup \{e_{(n,i)}, x_{(1,i)}, x^*_{(1,i)}: 1 \leq i \leq 2^n\}$. Let $1 > \eta_{n+1} > 0$ be a positive number to be determined (but it depends only on C_n and $\mathcal{F}_n \cup \mathcal{G}_n$). Since A has tracial topological rank one, there exist a C^*-subalgebra $C_{n+1} \subset$ A with $C_{n+1} \in I$ and a projection p_{n+1} with $1_{C_{n+1}} = p_{n+1}$ such that:

(i) $\|a\, p_{n+1} - p_{n+1} a\| < \eta_{n+1}/2^{n+1}$ for all $a \in \mathcal{G}_{n+1}$,
(ii) dist($p_{n+1} a\, p_{n+1}$, C_{n+1}) $< \eta_1/2^{n+1}$ for all $a \in \mathcal{G}_{n+1}$, and
(iii) $\tau(1 - p_{n+1}) < 1/2^{n+2}$ for all $\tau \in$ T (A).

We choose η_{n+1} so small that there exist a homomorphisms $\emptyset_n : C_n \to C_{n+1}$ such that
$\|\emptyset_n(b) - p_{n+1} b\, p_{n+1}\| < 1/2^{n+1}$ for all b $\in \mathcal{F}_n \cup \mathcal{G}_n$.   (e1)
Put $q_{n+1} = \emptyset_n(p_n)$. It is useful to note that $q_{n+1} \leq p_{n+1}$. Since $x_{(n,i)} \in \mathcal{G}_{n+1}$, we may further assume that

(4) $2^n[p_{n+1} - q_{n+1}] \leq [q_{n+1}]$ in $K_0(C_{n+1})$.

Set $C = \lim_{n \to \infty}(C_n, \emptyset_n)$. Since each C_n is nuclear C^*-subalgebra of A, there is a contractive completely positive linear map $L_n : A \to C_n$ (see, for example, [69]) such that

$$\lim_{n \to \infty} \|L_n(a) - p_n a p_n\| = 0 \tag{1}$$

for all a $\in$ A. Note that, by (1),
$\lim_{n \to \infty} \|L_n(ab) - L_n(a) L_n(b)\| = 0$ for all a,b $\in$ A

Define $L_n = \emptyset_{n,\infty} \circ L_n$ .It is clear that L_n satisfies (ii). Put $\emptyset_{n,n+1} = \emptyset_n$ and for k> n+1, $\emptyset_{n,k} = \emptyset_{k-1} \circ \cdots \circ \emptyset_n$. Define r: T (A) $\to$ T (C) as follows. For each b $\in C_n$, define

r (τ) $(\varphi_{n,\infty}(b)) = \lim_{k \to \infty} \tau\big(\emptyset_{n,k}(b)\big)\, for\ \tau \in T(A)$

Note that $\emptyset_{n,k}(b) \in C_k \subset$ A. We will show that the right-hand side above converges. Since we may replace b by $\emptyset_{n,k}(b)$ (replacing n by a larger integer if necessary), without loss of generality, we may also assume that b $\in \mathcal{F}_n$. From (1), one obtains that

$$\left\|\emptyset_{n,k+j+1}(b) - p_{k+j+2}\emptyset_{n,k+j}(b) p_{k+j+2}\right\| < 1/2^{k+j+2} \tag{2}$$

On the other hand, for any integer k ≥ 0,
$$\left|\tau\big(p_{k+j+2}\emptyset_{n,k+j}(b) p_{k+j+2}\big) - \tau(\emptyset_{n,k+j}(b))\right|$$
$$\leq \left|\tau\big((1 - p_{k+j+2})\emptyset_{n,k+j}(b)\big)(1 - p_{k+j+2})\right| + \left|\tau\big(p_{k+j+2}\emptyset_{n,k+j}(b)\big)(1 - p_{k+j+2})\right|$$

$$+\left|\tau\left((1-p_{k+j+2})\emptyset_{n,k+j}(b)p_{k+j+2}\right)\right|$$
$$< 3\|b\|\tau(1-p_{k+j+2}) \leqslant 3\|b\|/2^{k+j+2} \tag{3}$$

It follows from (2) and (3) that

$$\left|\tau\left(\emptyset_{n,k+j+1}(b)\right) - \tau(\emptyset_{n,k+j}(b))\right| < 1/2^{k+j+2} + 3\|b\|/2^{k+j+2}.$$

Therefore, for any m $\geqslant$ 1,

$$\left|\tau\left(\emptyset_{n,k}(b)\right) - \tau(\emptyset_{n,k+m}(b))\right| < \sum_{j=0}^{m} 1/2^{k+j+2} + 3\|b\| \sum_{j=0}^{m} 1/2^{k+j+2} \to 0$$

as k $\to \infty$. We conclude that $\lim_{k\to\infty} \tau\left(\emptyset_{n,k}(b)\right)$ converges. To see r is well defined, we let c $\in C_m$ so that $\emptyset_{m,\infty}(c) = \emptyset_{n,\infty}(b)$. Then, for any $\varepsilon > 0$, there exists N > max$\{n, m\}$ such that

$$\left\|\emptyset_{n,k}(b) - \emptyset_{m,k}(c)\right\| < \varepsilon \quad (\text{in} C_k)$$

for all $k \geqslant$ N . It follows that $(C_k \subset$ A)

$$\left|\tau(\emptyset_{n,k}(b)) - \tau(\emptyset_{m,k}(c))\right| < \varepsilon$$

for all $\tau \qquad \in$T (A) and $k \geqslant$N . It follows that r is well defined on $\bigcup_{n=1}^{\infty} \emptyset_{n,\infty}(C_n)$. Since $\left|\tau(\emptyset_{n,k}(b))\right| \leqslant \left\|(\emptyset_{n,k}(b)\right\|, r \quad (\tau)$ is bounded linear functional on $\bigcup_{n=1}^{\infty} \emptyset_{n,\infty}(C_n)$. It defines (uniquely) a bounded linear functional on C. One then easily sees that r(τ) is a state. Moreover, one checks that it is a tracial state. Thus r is well defined. It is then easy to see that r is an affine continuous map. Define r $^{-1} : T$ (C) $\to T$ (A) by

$$r^{-1}(t)(a) = \lim_{n\to\infty} t(L_n(a)) = \lim_{n\to\infty} t(\emptyset_{n,\infty}(L_n(a))) \text{ for all } t\in \text{T(C) and } a \in A$$

To justify the definition, we first need to show that $\lim_{n\to\infty} t(\emptyset_{n,\infty}(L_n(a)))$ exists. Let $a \in \mathcal{F}_n$ and k > 0. Define

$$b_{n,k,1}(a) = (p_{n+k+2} - q_{n+k+2})L_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2}),$$
$$b_{n,k,2}(a) = (p_{n+k+2} - q_{n+k+2})L_{n+k+2}(a)q_{n+k+2} \qquad and$$
$$b_{n,k,3}(a) = q_{n+k+2}L_{n+k+2}(a)(p_{n+k+2} - q_{n+k+2})$$

and define

$$b_{n,k,1}(a) = p_{n+k+2}(1 - q_{n+k+1})L_{n+k+2}(a)(1 - p_{n+k+1})p_{n+k+2},$$
$$b_{n,k,2}(a) = p_{n+k+2}(1 - q_{n+k+2})L_{n+k+2}(a)p_{n+k+1}p_{n+k+2} \quad and$$
$$b_{n,k,3}(a) = p_{n+k+2}p_{n+k+1}L_{n+k+1}(a)(1 - p_{n+k+1})p_{n+k+2}$$

Note that $b_{n,k,i}(a) \in C_{n+k+2}$, i = 1, 2, 3. By (2) and (1) above, in A,

$$\left\|[\emptyset_{n+1}\circ L_{n+k+1}(a) - L_{n+k+2}(a)] - [b_{n,k,1}(a)b_{n,k,2}(a)b_{n,k,3}(a)]\right\| < 5/2^{n+k+2} +$$
$$5\eta_{n+k+2}/2^{n+k+2} \tag{4}$$

We also estimate:

$$\left\|\left(b_{n,k,1}(a)b_{n,k,2}(a)b_{n,k,3}(a)\right) - (b_{n,k,1}(a)b_{n,k,2}(a)b_{n,k,3}(a))\right\| < 3/2^{n+k+2}$$

It follows that (in C_{n+k+2})
$$\left\|\left(\emptyset_{n+1}\circ L_{n+k+1}(a) - L_{n+k+2}(a)\right) - \left(b_{n,k,1}(a)b_{n,k,2}(a)b_{n,k,3}(a)\right)\right\|$$
$$< 8/2^{n+k+2} + 5\eta_{n+k+2}/2^{n+k+2} \qquad (5)$$

By (5),

$$\left\|\left(L_{n+k+1}(a) - L_{n+k+2}(a)\right) - \emptyset_{n+k+2,\infty}\left(b_{n,k,1}(a)b_{n,k,2}(a)b_{n,k,3}(a)\right)\right\|$$
$$< 1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2} \quad (6)$$

By (iv), in $K_0(C)$,
$$2^{n+k+1}[\emptyset_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})] \leqslant [\emptyset_{n+k+2,\infty}(q_{n+k+2})]$$
It follows that, for any t $\in$ T (C),
$$t\left(\emptyset_{n+k+2,\infty}(p_{n+k+2} - q_{n+k+2})\right) < 1/2^{n+k+1}.$$
From this, we estimate that
$$t\left(\emptyset_{n+k+2,\infty}(b_{n,k,i}(a))\right) \leqslant \|a\|2^{n+k+1}, i = 1, 2, 3$$
for all t $\in$ T (C). Combing this with (e6), we have
$$\left|t\left((L_{n+k+1}(a)\right) - t\left((L_{n+k+2}(a))\right| < 1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2} + 3\|a\|/2^{n+k+1}.$$
Hence

$$\left|t\left((L_{n+1}(a)\right) - t\left((L_{n+m}(a))\right| < \sum_{k=0}^{m}(1/2^{n+k-1} + 5\eta_{n+k+2}/2^{n+k+2} + 3\|a\|/2^{n+k+1}) \to 0$$

as n $\to \infty$. This proves that $\lim_{n\to\infty}(\emptyset_{n,\infty}(L_n(a))$ exists. Then one shows that $r^{-1}(t)$ is well defined. By (ii), which we have shown, $r^{-1}(t)$ is a trace on A. It is then clear that r^{-1} is an affine continuous map. It should be noted that even if $a \in C_m$ (for $m < n), L_n(a) \in C_n$. Now let $\tau \in T$ (A) and $a \in$ A. To show that $(r^{-1}\circ r)(\tau)(a) = \tau (a)$, we note that
$$(r^{-1}\circ r)(\tau)(a) = \lim_{n\to\infty} r(\tau)(\emptyset_{n,\infty}(L_n(a)) = \lim_{n\to\infty}(\lim_{k\to\infty} \tau(\emptyset_{n,k}(L_n(a)))$$
for all a $\in$ A and $\tau \in$ T (A). Let $\varepsilon > 0$. Without loss of generality, we may assume that $a \in \mathcal{F}_n$ for some integer $n > 0$. Moreover, with sufficiently large n, we may assume that $1/2^n < \varepsilon/8$ and
$$\|L_n(a) - p_n a p_n\| < \varepsilon/4.$$
One estimates, by (e1) (with k > n),

$$\left\|\emptyset_{n,k}(L_n(a)) - p_k p_{k-1}\ldots p_{n+1}p_n a p_n p_{n+1}\ldots p_{k-1}p_k\right\| < \sum_{j=1}^{k-n} 1/2^{n+j} + \varepsilon/4 < \varepsilon/2$$

By (iii) and as in (3), one has
$$|\tau(p_k p_{k-1}\ldots p_{n+1}p_n a p_n p_{n+1}\ldots p_{k-1}p_k) - \tau(a)|$$
$$< 3\|a\|\sum_{j=1}^{k-n} 1/2^{n+2} < 3\|a\|\varepsilon/8 \quad \text{for all } \tau \in T \text{ (A)}$$
It follows that

$$\left| \tau \left(\emptyset_{n,k}(L_n(a)) \right) - \tau(a) \right| < 3\|a\|\varepsilon/8 \, + \, \varepsilon/2 \quad \text{for all } \tau \in T \, (A)$$

if k > n. Therefore

(5) τ (a) $\quad = \lim\limits_{n\to\infty} (\lim\limits_{k\to\infty} \tau \, (\emptyset_{n,k}(L_n(a))))$ for all a $\in$ A and $\tau \in$ T (A).

This also proves $r^{-1} \circ r \, (\tau \,)(a) = \tau \, (a)$ for all a $\in$ A and $\tau \in$ T (A). Therefore $r^{-1} \circ r =$ $id_{T \, (A)}$.

Suppose that t $\in T$ (C) and $b \in C_n$. Then

$$r \circ r^{-1}(\tau)(\emptyset_{n,\infty}(b)) = \lim\limits_{k\to\infty} r^{-1} \, (\tau) \, (\emptyset_{n,k}(b)) = \lim\limits_{k\to\infty} (\lim\limits_{m\to\infty} \tau \, (\emptyset_{m,\infty} \left(L_m \left(\emptyset_{n,k}(b) \right) \right))).$$

Fix $\varepsilon > 0$. Choose k > n such that $1/2^k < \varepsilon/32$. We may assume that $\|b\| \leqslant 1$. For any m > k, put $r_j \, = \, \emptyset_{j,m}(p_j), j \, = \, k, \ldots, m \, - \, 1$. Since $\emptyset_j \, (p_j) \leqslant p_{j+1}, r_j \leqslant r_{j+1}$. By choosing a larger k, applying (1) and (2) above, we may assume that there is $c_1 \in$ A such that (we view $\emptyset_{n,k}(b) \in C_k \subset$ A)

$$r_j c_1 \, = \, c_1 r_j \, , k \, + \, 1 \leqslant \, j \, \leqslant \, m \, - \, 1, and \, \|c_1 \, - \, \emptyset_{n,k}(b)\| < \, \varepsilon/8.$$

We also have

$$\left\| \emptyset_{n,m}(b) - p_m r_{m-1} \ldots r_k \emptyset_{n,k}(b) r_k \ldots r_{m-1} p_m \right\| < \sum_{j=1}^{m-k} 2^{k+j} < \varepsilon/8$$

Put $c_2 \, = \, p_m r_{m-1} \ldots r_k c_1$. It then follows that

$$c_3 \, = \, L_m(c_1) - c_2 \leqslant 2(p_m - r_k)$$

Since each C_j has stable rank one, by (4), there are $y_i \in C_m$ such that $y_i^* y_i = p_m - r_k$ and $y_i y_i^* (1 \leqslant i \leqslant 2^m)$ are mutually orthogonal. Let $z_i \, = \, \emptyset_{m,\infty}(y_i), i = 1, \, 2, \, \ldots, \, 2^m$. Then $z_i^* z_i \, = \, \emptyset_{m,\infty}(p_m - r_k)$ and $z_i z_i^* (1 \leqslant i \leqslant 2^m)$ are mutually orthogonal. It follows that

$$t(\emptyset_{m,\infty} \, (c_3) \leqslant 2(1/2^m) < \, \varepsilon/8$$

for all t $\in T$ (C). On the other hand, from the above estimates,

$$\left\| \left[\emptyset_{m,\infty} \left(L_m \left(\emptyset_{n,k}(b) \right) \right) - \emptyset_{n,\infty}(b) \right] - \emptyset_{m,\infty}(c_3) \right\|$$
$$\leqslant \left\| \left[L_m \left(\emptyset_{n,k}(b) \right) - \emptyset_{n,m}(b) \right] - c_3 \right\|$$
$$\leqslant \left\| L_m \left(\emptyset_{n,k}(b) \right) - L_m(c_1) \right\| + \left\| \emptyset_{n,m}(b) - c_2 \right\| + \left\| (L_m(c_1) - c_2) - c_3 \right\|$$
$$\leqslant \varepsilon/8 \, + \, (\varepsilon/8 \, + \, \varepsilon/8) \, + \, 0 \, = \, 3\varepsilon/8.$$

It follows that

$$\left| t(\emptyset_{m,\infty} \left(L_m \left(\emptyset_{n,k}(b) \right) \right)) - t(\emptyset_{n,m}(b)) \right| < 3\varepsilon/8 \, + t(\emptyset_{n,m}(c_3)) < 3\varepsilon/8 \, + \, \varepsilon/8 \, < \varepsilon$$

for all t $\quad \in T$ (C) if m > k. Thus

(6) $t\left(\emptyset_{n,\infty}(b)\right) = \lim_{k\to\infty}(\lim_{m\to\infty} t(\emptyset_{m,\infty}(L_m(\emptyset_{n,k}(b)))))$ for all $b \in C_n$ and $\tau \in T(C)$

It follows that $r \circ r^{-1} = \mathrm{id}_{T(C)}$. Thus we have shown that r is an affine continuous surjective map with an affine continuous inverse r^{-1}. To see r is face-preserving, let $\tau \in T(A)$, $t_1, t_2 \in T(C)$ and $0 \leqslant a \leqslant 1$ for which

$$r(\tau) = at_1 + (1-a)t_2.$$

Let $\tau_1, \tau_2 \in T(A)$ such that $r(\tau_i) = t_i$, $i = 1, 2$. Then, since $r-1$ is the inverse of r, we see that

$$\tau = a\tau_1 + (1-a)\tau_2.$$

Fix a projection $p \in A$ and $s \in S_u(K_0(C))$. One obtains a sequence of projections $e_n \in C_n$ such that

$$\lim_{n\to\infty} \|p_n p p_n - e_n\| = 0$$

or equivalently

$$\lim_{n\to\infty} \|L_n(p) - e_n\| = 0$$

We have shown that, for each $t \in T(C)$, $\lim_{n\to\infty} t(\emptyset_{n,\infty}(L_n(p)))$ exists. So $\lim_{n\to\infty} t(\emptyset_{n,\infty}(L_n(e_n)))$ exists. If $p \in M_K(A)$ for some integer $K > 0$, by replacing C by $M_K(C)$ $p_n \mathrm{diag}(p_n, \ldots, p_n)$, we also obtain a projection $e_n \in C_n$ such that

$$\lim_{n\to\infty} t(\emptyset_{n,\infty}(L_n(p))) = \lim_{n\to\infty} t\left(\emptyset_{n,\infty}(L_n(e_n))\right) \qquad (7)$$

Since C is an inductive limit of C^*-algebras in I, there exists $\sigma_s \in T(C)$ such that $s([e]) = \sigma_s(e)$ for any projection $e \in M_K(C)$ and for any integer $K \geqslant 1$ (recall that we use σ_s for $\sigma_s \otimes Tr$). Suppose that $\sigma_s, \tau_s \in T(C)$ such that $\sigma_s(e) = \tau_s(e)$ for all projections $e \in M_K(C)$ (for all integer $K \geqslant 1$). For any projection $p \in M_K(C)$, let e_n be a projection in C_n for which (7) holds. Then

$$\lim_{n\to\infty} \sigma_s(\emptyset_{n,\infty}\circ L_n(p)) = \lim_{n\to\infty} \sigma_s(\emptyset_{n,\infty}(e_n)) = \lim_{n\to\infty} \tau_s(\emptyset_{n,\infty}(e_n)) = \lim_{n\to\infty} \tau_s(\emptyset_{n,\infty}\circ L_n(p))$$

It follows that the map

$$r(s)([p]) = r^{-1}(\sigma_s)(p) = \lim_{n\to\infty} \sigma_s(\emptyset_{n,\infty}(L_n(p))) = \lim_{n\to\infty} s([\emptyset_{n,\infty}(e_n)]) \qquad (8)$$

is independent of the choices of τ_s and is well defined from $S_u(K_0(C))$ to $S_u(K_0(A))$. (Here we extend L_n and $\emptyset_{n,\infty}$ to $M_K(A)$ and $M_K(C)$ in the obvious way.) It is clear that r is affine. Let $t \in S_u(K_0(A))$. Since A is a simple C^*-algebra with $\mathrm{TR}(A) \leqslant 1$, there exists $\tau_t \in T(A)$ such that τ_t induces t. Suppose that $\sigma_t \in T(A)$ such that $\tau_t(p) = \sigma_t(p)$ for all projections $p \in M_K(A)$ (for all integer $K \geqslant 1$). Let $e \in M_K(C_n)$ be a projection. Then (note that $C_K \subset A$)

$$\lim_{k\to\infty} \tau_t(\emptyset_{n,k}(e)) = \lim_{k\to\infty} \sigma_t(\emptyset_{n,k}(e))$$

$$r(t)([\emptyset_{n,\infty}(e)]) = r(t)(\emptyset_{n,\infty}(e)) = \lim_{k\to\infty} \tau_t(\emptyset_{n,k}(e)) = \lim_{k\to\infty} t([\emptyset_{n,k}(e)])$$

is independent of the choice of τ_t and it is well-defined affine map (where we view C_n as a

C^*-subalgebra of A).

Now let $p \in$ A be a projection and $t \in S_u (K_0(A))$. By 10.2, t is induced by a trace $\tau_t \in$ T (A). One has, by (5) and (e8),

$$r\big(r\,(t)\big)([p])$$
$$= \lim_{n \to \infty} r\,(t)\big([\emptyset_{n,\infty}(L_n(p))]\big) = \lim_{n \to \infty} \Big(\lim_{k \to \infty} \tau_t(\emptyset_{n,k}(L_n(p))) \Big) = \tau_t(p) = t([p])$$

It follows that $r \circ r = \mathrm{id}_{S_u\,(K_0\,(A))}$. On the other hand, let $e \in M_K\,(C_n)$ be a projection and $s \in S_u\,(K_0(C))$. Let $\sigma_s \in$ T (C) which induces s. Then, by (6),

$$r\,(r(s))\big([\emptyset_{n,\infty}(e)]\big)$$
$$= \lim_{k \to \infty} r(s)\big([\emptyset_{n,k}(e)]\big)$$
$$= \lim_{k \to \infty} \Big(\lim_{m \to \infty} \sigma_s(\emptyset_{m,\infty}(L_m(\emptyset_{n,k}(e)))) \Big) = \sigma_s\big(\emptyset_{n,\infty}(e)\big) = s([\emptyset_{n,\infty}(e)])$$

Thus $r \qquad \circ r = \mathrm{id}_{S_u\,(K_0\,(A))}$.

Lemma(4.2.18)[89]: Let A be a unital separable simple nuclear C^*-algebra with TR(A)$\leqslant$1. Then the map λ : T (A) $\to S_u\,(K_0\,(A))$ maps ∂_e(T (A)) onto $\partial_e(S_u\,(K_0\,(A)))$. Moreover, if A is infinite-dimensional, $K_0(A)/tor(K_0(A)) \ncong \mathbb{z}$. In particular, there is a unital simple AH-algebra B with no dimension growth described in 10.1 such that

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A), T\,(A)\big) = (K_0(B), K_0(B)_+, [1_B], K_1(B), T\,(B))$$

Proof. We will apply Lemma (4.2.17). Let C be the inductive limit of C^*-algebras in I as describedin Lemma (4.2.17). the map from T ($\tilde{C}$) to $S_u\,(K_0\,(\tilde{C}))$ maps extremal points onto extremal points. Let $t_0 \in$ T ($\tilde{C}$) be the trace such that t_0(C)= 0 for all c $\in$ C   and   let $s_0 \in S_u\,(K_0\,(\tilde{C})$ such that s_0(x) =0 for all $x \in j_*(K_0\,(C))$, where j : C $\to \tilde{C}$ is the embedding. Note that T($\tilde{C}$)is the set of convex combinations of $\tau \in$ T (C) and t_0 and $S_u\,(K_0\,(\tilde{C}))$ is the set of convex combinations of s $\in S_u\,(K_0\,(C))$ and s_0. Suppose that $\tau \in \partial_e$(T (A)). Then, by 10.8, $r\,(\tau) \in \partial_e$(T (C)) $\subset \partial_e$(T ($\tilde{C}$)). It follows that $r\quad(\tau)$gives an extremal state s_τ in $S_u\,(K_0\,(\tilde{C}))$. It follows that $s_\tau \in \partial_e(S_u\,(K_0(C)))$. Note that $\lambda(\tau\ ) = r\quad(s_\tau\ )$. This shows that $\lambda(\partial_e$(T (A))) $\subset \partial_e(S_u\,(K_0(A)))$. To see that $\lambda(\partial_e$(T (A))) $= \partial_e(S_u\,(K_0(A)))$, let s $\in \partial_e(S_u\,(K_0(C)))$. Set

$$F = \{\tau \in T\,(A): \lambda(\tau\ ) = s\}.$$

It is clear that F is a closed and convex subset of T (A). Furthermore it is a face. By the Krein–Milman theorem, it contains an extremal point t. Since F is a face, $t \in \partial_e(T\,(A))$.

To see $K_0(A)/tor(K_0(A)) \ncong \mathbb{Z}$ when A is infinite-dimensional, we note that A has (SP). Since A is simple, we obtain, for any integer n > 0, n + 1 mutually orthogonalnonzero projections $p_1, p_2, \ldots, p_n$ and q in A for which $1 = q + \sum_{i=1}^n p_i$, $[p_1] = [p_i]$ $(i = 1, 2, \ldots, n)$ and $[q] \leqslant [p_1]$. This implies that $K_0(A)/tor(K_0(A)) \ncong \mathbb{Z}$.

Theorem(4.2.19)[89]: Let A and B be two unital separable nuclear simple C^*-algebras with TR(A) $\leqslant$ 1 and TR(B) $\leqslant$1 which satisfy the AUCT. Then A $\cong$ B if and only if

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A), T\,(A)\big) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T\,(B))$$

List of Symbols

Symbol	page
$\oplus$: orthogonal sum	1
Aut : Automorphism	1
dim : dimension	2
min : minimum	6
max : maximum	6
Lim : limit	8
$\otimes$: tensor product	19
det : determinanl	21
Hom : homomorphism	22
log : logarithm	25
exp : exponent	27
diag : diagonal	27
TR : trace	28
dist : distance	33
Ker : kernel	38
inf : infimum	51
sup : supremum	52

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