Chapter 3

Homotopy and Result of Equivalence Approximate

Let C be a unital separable amenable simple C^*-algebra with tracial rank no more than one which also satisfies the UCT. Suppose that $\phi: C \to A$ is a unital monomorphism and suppose that $v \in A$ is a unitary with $[v] = 0$ in $K_1(A)$ such that v almost commutes with ϕ. It is shown that there is a continuous path of nitaries $\{v(t): t \in [0,1]\}$ in A with $v(0) = v$ and $v(1) = 1$ such that the entire path $v(t)$ almost commutes with ϕ, provided that an induced Bott map vanishes. Other versions of the so-called Basic Homotopy Lemma are also presented.

Section (3.1) Homotopy of Unitaries in Simple C^*-Algebras with Tracial Rank One

Fix a positive number $\epsilon > 0$. Can one find a positive number δ such that, for any pair of unitary matrices u and v ($K_1(M_n) = \{0\}$ for any integer $n \geq 1$) with $\|uv - vu\| < \delta$, there exists a continuous path of unitary matrices $\{v(t): t \in [0,1]\}$ for which $v(0) = v, v(1) = 1$ and $\|uv(t) - v(t)u\| < \epsilon$ for all $t \in [0,1]$? The answer is negative in general. A Bott element associated with the pair of unitary matrices may appear. The hidden topological obstruction can be detected in a limit process. This was first found by Dan Voiculescu [29]. On the other hand, it has been proved that there is such a path of unitary matrices if an additional condition, $\mathrm{bott1}(u, v) = 0$, is provided (see, for example, [57] and also in [70]).

It was recognized by Bratteli, Elliott, Evans and Kishimoto [57] that the presence of such continuous path of unitaries in general simple C^* −algebras played an important role in the study of classification of simple C^* −algebras and perhaps plays important roles in some other areas such as the study of automorphism groups (see, for example, [12,24,21]). They proved what they called the Basic Homotopy Lemma: For any $\epsilon > 0$, there exists $\delta > 0$ satisfying the following:

For any pair of unitaries u and v in A with $\mathrm{sp}(u)$ δ −dense in $\mathbb{T}$ and $[v] = 0$ in $K_1(A)$ for which

$$\|uv - vu\| < \delta \quad \text{and} \quad \mathrm{bott}_1(u, v) = 0,$$

there exists a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset A$ such that

$$v(0) = v, \qquad v(1) = 1_A \quad and \quad \|v(t)u - uv(t)\| < \epsilon$$

for all $t \in [0,1]$, where A is a unital purely infinite simple C^* −algebra or a unital simple C^* −algebra with real rank zero and stable rank one. Define $\phi : C(\mathbb{T}) \to A$ by $\phi(f) = f(u)$ for all $f \in C(\mathbb{T})$. Instead of considering a pair of unitaries, one may consider a unital homomorphism from $C(\mathbb{T})$ into A and a unitary $v \in A$ for which v almost commutes with ϕ.

In the study of asymptotic unitary equivalence of homomorphisms from an AH −algebra to a unital simple C^* −algebra, as well as the study of homotopy theory in simple C^* −algebras, one considers the following problem: Suppose that X is a compact metric space and ϕ is a unital homomorphism from $C(X)$ into a unital simple C^* −algebra A. Suppose that there is a unitary $u \in A$ with $[u] = 0$ in $K_1(A)$ and u almost commutes with ϕ. When can one find a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset A$ with $u(0) = u$ and $u(1) = 1$ such that $u(t)$ almost commutes with ϕ for all $t \in [0,1]$?

Let C be a unital AH −algebra and let A be a unital simple C^* −algebra. Suppose that ϕ, ψ : $C \to A$ are two unital monomorphisms. Let us consider the question when ϕ and ψ are asymptotically unitarily equivalent, i.e., when there is a continuous path of unitaries $\{w(t): t \in [0,\infty)\} \subset A$ such that

$$\lim_{t\to\infty} w(t)^* \phi(c) w(t) = \psi(c) \text{ for all } c \in C.$$

We study the case that A is no longer assumed to have real rank zero, or tracial rank zero. The result of W. Winter in [30] provides the possible classification of simple finite C^* −algebras far beyond the cases of finite tracial rank. However, it requires to understand much more about asymptotic unitary equivalence in those unital separable simple C^* −algebras which have been classified. An immediate problem is to give a classification of monomorphisms (up to asymptotic unitary equivalence) from a unital separable simple AH −algebra into a unital separable simple C^* −algebra with tracial rank one. For that goal, it is paramount to study the Basic Homotopy Lemmas in a simple separable C^* −algebras with tracial rank one. This is the main purpose.

A number of problems occur when one replaces C^* −algebras of tracial rank zero by those of tracial rank one. First, one has to deal with contractive completely positive linear maps from $C(X)$ into a unital C^* −algebra C with the form $C([0,1], M_n)$ which are *not* homomorphisms but almost multiplicative. Such problem is already difficult when $C = M_n$ but it has been proved that these above mentioned maps are close to homomorphisms if the associated K-theoretical data of these maps are consistent with those of homomorphisms. It is problematic when one tries to replace M_n by $C([0,1], M_n)$. In addition to the usual K-theory and trace information, one also has to handle the maps from $U(C)/CU(C)$ to $U(A)/CU(A)$, where $CU(C)$ and $CU(A)$ are the closure of the subgroups of $U(C)$ and $U(A)$ generated by commutators, respectively.

Other problems occur because of lack of projections in C^* −algebras which are not of real rank zero.

The main theorem is stated as follows: Let C be a unital separable simple amenable C^* −algebra with tracial rank one which satisfies the Universal Coefficient Theorem. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset K(C)$ satisfying the following:

Suppose that A is a unital simple C^* −algebra with tracial rank no more than one, suppose that $\phi : C \to A$ is a unital homomorphism and $u \in U(A)$ such that

$$\|[\phi(c), u]\| < \delta \quad \text{for all } c \in \mathcal{G} \text{ and } \mathrm{Bott}(\varphi, u)|P = 0. \tag{1}$$

Then there exists a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset A$ such that

$$u(0) = u, u(1) = 1 \text{ and } \|[\varphi(c), u(t)]\| < \epsilon \text{ for all } c \in \mathcal{F} \tag{2}$$

and for all $t \in [0,1]$.

We also give the following Basic Homotopy Lemma in simple C^* −algebras with tracial rank one.

Let $\epsilon > 0$ and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map.We show that there exist $\delta > 0$ and $\eta > 0$ (which does not depend on Δ) satisfying the following:

Given any pair of unitaries u and v in a unital simple C^* −algebra A with tracial rank no more than one such that $[v] = 0$ in $K_1(A)$,

$$\|[u,v]\| < \delta, \qquad \text{bott}_1(u,v) = 0 \quad \text{and} \quad \mu_{\tau\circ l}(I_a) \geq \Delta(a)$$

for all open arcs I_a with length $a \geq \eta$, there exists a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset A$ such that

$$v(0) = v, \qquad v(1) = 1 \text{ and } \|[u,v(t)]\| < \epsilon \text{ for all } t \in [0,1],$$

where $\iota: C(T) \to A$ is the homomorphism defined by $\iota(f) = f(u)$ for all $f \in C(\mathbb{T})$ and $\mu_{\tau\circ l}$ is the Borel probability measure induced by the state $\tau \circ l$. It should be noted that, unlike the case that A has real rank zero, the length of $\{v(t)\}$ cannot be controlled. In fact, it could be as long as one wishes.

In a subsequent paper [23], we use the main homotopy result Theorem (3.1.34) and the results in [22] to establish a K −theoretical necessary and sufficient condition for homomorphisms from unital simple AH-algebras into a unital separable simple C^* −algebra with tracial rank no more than one to be asymptotically unitarily equivalent which, in turn, combining with a result of W. Winter, provides a classification theorem for a class of unital separable simple amenable C^* −algebras which properly contains all unital separable simple amenable C^* −algebras with tracial rank no more than one which satisfy the UCT as well as some projectionless C^* −algebras such as the Jiang–Su algebra.

Let A be a unital C^* −algebra. Denote by $T(A)$ the tracial state space of A and denote by Aff$(T(A))$ the set of affine continuous functions on $T(A)$.

Let $C = C(X)$ for some compact metric space X and let $L: C \to A$ be a unital positive linear map. Denote by $\mu_{\tau\circ l}$ the Borel probability measure induced by the state $\tau \circ l$, where $\tau \in T(A)$.

Let a and b be two elements in a C^* −algebra A and let $\epsilon > 0$ be a positive number. We write $a \approx_\epsilon b$ if $\|a - b\| < \epsilon$. *Let* $L_1, L_2 : A \to C$ be two maps from A to another C^* −algebra C and let $\mathcal{F} \subset A$ be a subset. We write

$$L_1 \approx_\epsilon L_2 \quad \text{on} \quad \mathcal{F},$$

if $L_1(a) \approx_\epsilon L_2(a)$ for all $a \in \mathcal{F}$.

Suppose that $B \subset A$. We write $a \in_\epsilon B$ if there is an element $b \in B$ such that $\|a - b\| < \epsilon$..

Let $\mathcal{G} \subset A$ be a subset. We say L is $\epsilon - \mathcal{G}$ −multiplicative if, for any $a, b \in \mathcal{G}$,

$$L(ab) \approx_\epsilon L(a)L(b)$$

For all $a, b \in \mathcal{G}$.

Let A be a unital C^* −algebra. Denote by $U(A)$ the unitary group of A. Denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of those unitaries in the path connected component of $U(A)$containing the identity. Let $u \in U_0(A)$. Define

$$\text{cel}_A(u) = \inf\{\text{length}(\{u(t)\}): u(t) \in C\left([0,1], U_0(A)\right),$$
$$u(0) = u \text{ and } u(1) = 1_A\}$$

We use $cel(u)$ if the C^* −algebra A is not in question.

Denote by $CU(A)$ the *closure* of the subgroup generated by the commutators of $U(A)$. For $u \in U(A)$, we will use. u for the image of u in $U(A)/CU(A)$. If . $\bar{u}.\bar{v} \in U(A)/CU(A)$, define

$$\operatorname{dist}(\bar{u},\bar{v}) = \inf\{\|x - y\|: x,y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v}.$$

If $u,v \in U(A)$, then

$$\operatorname{dist}(\bar{u},\bar{v}) = inf\{\|uv^* - x\|: x \in CU(A)\}.$$

Let A and B be two unital C^* −algebras and let $\phi: A \to B$ be a unital homomorphism.

It is easy to check that ϕ maps $CU(A)$ to $CU(B)$. Denote by $\phi^{\ddagger}$ the homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$ induced by ϕ. We also use $\phi^{\ddagger}$ for the homomorphism from $U(M_k(A))/CU(M_k(A))$ into $U(M_k(B))/CU(M_k(B))$ $(k = 1,2,\ldots)$.

Let A and C be two unital C^* −algebras and let $F \subset U(C)$ be a subgroup of $U(C)$. Suppose that $L: F \to U(A)$ is a homomorphism for which $L(F \cap CU(C)) \subset CU(A)$. We will use $L^{\ddagger}: F/CU(C) \to U(A)/CU(A)$ for the induced map.

Let A and B be as in 2.6, let $1 > \epsilon > 0$ and let $\mathcal{G} \subset A$ be a subset. Suppose that L is a $\epsilon-\mathcal{G}$ multiplicative unital completely positive linear map. Suppose that $u,u^* \in \mathcal{G}$. Define $\langle L\rangle(u) = L(u)L(u^*u)^{-1/2}$.

Definition (3.1.1)[84]:

Let A and B be two unital C^* −algebras. Let $h: A \to B$ be a homomorphism and let $v \in U(B)$ such that

$h(g)v = vh(g)$ for all $g \in A$.

Thus we obtain a homomorphism. $\bar{h}: A \otimes C(S^1) \to B$ by $\bar{h}(f \otimes g) = h(f)g(v)$ for $f \in A$ and $g \in C(S^1)$. From the following splitting exact sequence

$$0 \to SA \to A \otimes C(S^1) \leftrightarrows A \to 0 \qquad (3)$$

and the isomorphisms $K_i(A) \to K_{1-i}(SA)$ $(i = 0,1)$ given by Bott periodicity, one obtains two injective homomorphisms

$$\beta^{(0)}: K_0(A) \to K_1\big(A \otimes C(S^1)\big) \qquad (4)$$

$$\beta^{(1)}: K_1(A) \to K_0\big(A \otimes C(S^1)\big) \qquad (5)$$

Note, in this way, one can write $K_i(A \otimes C(S^1)) = K_i(A) \oplus \beta^{(1-i)}(K_{1-i}(A))$. We use $\widehat{\beta^{(i)}}: K_i(A \otimes C(S^1)) \to \beta^{(1-i)}(K_{1-i}(A))$ for the projection to the summand $\beta^{(1-i)}(K_{1-i}(A))$For each integer $k \geq 2$, one also obtains the following injective homomorphisms

$$\beta_k^{(i)}: K_i(A,\mathbb{Z}/k\mathbb{Z}) \to K_{1-i}(A \otimes C(S^1),\mathbb{Z}/k\mathbb{Z}), \ i = 0,1. \qquad (6)$$

Thus we write

$$K_{1-i}(A \otimes C(S^1),\mathbb{Z}/k\mathbb{Z}), = K_{1-i}(A,\mathbb{Z}/k\mathbb{Z}) \otimes \beta_k^{(i)}(K_i(A,\mathbb{Z}/k\mathbb{Z})), i = 0,1. \quad (7)$$

Denote by $\widehat{\beta_k^{(i)}}: K_i(A \otimes C(S^1)) \to \beta_k^{(1-i)}K_{1-i}(A,\mathbb{Z}/k\mathbb{Z})$ similarly to $\widehat{\beta^{(i)}}$, $i = 1.2$.. If $x \in \underline{K}(A)$, we use $\boldsymbol{\beta}(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$if $x \in K_i(A,\mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\boldsymbol{\beta}: \underline{K}(A) \to \underline{K}(A \otimes C(S^1))$ as well as $\boldsymbol{\beta}: \underline{K}(A \otimes C(S^1)) \to \boldsymbol{\beta}(\underline{K}(A))$. Thus one may write $K(A \oplus C(S1)) = K(A) \oplus \boldsymbol{\beta}(K(A))$.

On the other hand $.h$ induces homomorphisms $.\bar{h}_{*i,k} : K_i(A \otimes C(S^1), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z})$, $k = 0, 2, \dots$, and $i = 0, 1$. We use $\mathrm{Bott}(h, v)$ for all homomorphisms. $\bar{h}_{*i,k} \circ \beta_k^{(i)}$. We write

$$\mathrm{Bott}(h, v) = 0,$$

if $.\bar{h}_{*i,k} \circ \beta_k^{(i)} = 0$ for all $k \geq 1$ and $i = 0, 1$.

We will use $\mathrm{bott}_1(h, v)$ for the homomorphism $.\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B)$, and $\mathrm{bott}_0(h, u)$ for the homomorphism $\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)$.

Since A is unital, if $\mathrm{bott}_0(h, v) = 0$, then $[v] = 0$ in $K_1(B)$.

For a fixed finite subset $\mathcal{P} \subset K(A)$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, if $v \in B$ is a unitary for which

$$\|h(a)v - vh(a)\| < \delta \text{ for all } a \in \mathcal{G},$$

then $Bott(h, v)|P$ is well defined. In what follows, whenever we write $Bott(h, v)|P$, we mean that δ is sufficiently small and $\mathcal{G}$ is sufficiently large so it is well defined.

Now suppose that $K_i(A)$ is finitely generated ($i = 0, 1$). For example, $A = C(X)$, where X is a finite CW complex. When $K_i(A)$ is finitely generated, $\mathrm{Bott}(h, v)|P_0$ defines $\mathrm{Bott}(h, v)$ for some sufficiently large finite subset P_0. In what follows such P_0 may be denoted by P_a Suppose that $P \subset K(A)$ is a larger finite subset, and $\mathcal{G} \supset \mathcal{G}_0$ and $0 < \delta < \delta_0$.

$Bott(h, v)|P$ defines the same map $Bott(h, v)$ as $\mathrm{Bott}(h, v)|P_0$ defines, if

$$\|h(a)v - vh(a)\| < \delta \text{ for all } a \in \mathcal{G},$$

when $K_i(A)$ is finitely generated. In what follows, in the case that $K_i(A)$is finitely generated, whenever we write $\mathrm{Bott}(h, v)$, we always assume that δ is smaller than δ_0 and $\mathcal{G}$ is larger than $\mathcal{G}_0$ so that $\mathrm{Bott}(h, v)$ is well defined (see [70] for more details).

In the case that $A = C(S^1)$, there is a concrete way to visualize $\mathrm{bott}_1(h, v)$. It is perhaps helpful to describe it here. The map $\mathrm{bott}_1(h, v)$ is determined by $\mathrm{bott}_1(h, v)([z])$where z is the identity map on the unit circle.

Denote $u = h(z)$ and define

$$f(e^{2\pi it}) = \begin{cases} 1 - 2t, & \text{if } 0 \leq t \leq 1/2, \\ -1 + 2t, & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

$$\mathrm{g}(e^{2\pi it}) = \begin{cases} \left(f(e^{2\pi it}) - f(e^{2\pi it})^2\right)^{1/2,}, & \text{if } 0 \leq t \leq 1/2, \\ 0, & \text{if } 1/2 < t \leq 1, \end{cases}$$

and

$$\mathrm{h}(e^{2\pi it}) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1/2, \\ \left(f(e^{2\pi it}) - f(e^{2\pi it})^2\right)^{1/2}, & \text{if } 1/2 < t \leq 1, \end{cases}$$

These are non-negative continuous functions defined on the unit circle. Suppose that $uv = vu$.

Define

$$b(u, v) = \begin{pmatrix} f(v) & \mathrm{g}(v) + h(v)u^* \\ \mathrm{g}(v) + uh(v) & 1 - f(v) \end{pmatrix}. \tag{8}$$

Then $b(u, v)$ is a projection. There is $\delta_0 > 0$ (independent of unitaries u, v and A) such that if $\|[u, v]\| < \delta_0$, the spectrum of the positive element $p(u, v)$ has a gap at $1/2$. The

Bott element of u and v is an element in $K_0(A)$ as defined in [9,8] which may be represented by

$$\text{bott}_1(u,v) = [\chi[1/2,\infty)b(u,v)] - \left[\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right]. \quad (9)$$

Note that $\chi[1/2,\infty)$ is a continuous function on $\text{sp}(b(u,v))$. Suppose that $\text{sp}(b(u,v)) \subset (-\infty,a] \cup [1-a,\infty)$ for some $0 < a < 1/2$. Then $\chi[1/2,\infty)$ can be replaced by any other positive continuous function F for which $F(t) = 0$ if $t \leq a$ and $F(t) = 1$ if $t \geq 1/2$.

Definition (3.1.2)[84]:

Let A and C be two unital C^*-algebras. Let $N : C_+ \setminus \{0\} \to N$ and $K : C_+ \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ be two maps. Define $T = N \times K : C_+ \setminus \{0\} \to N \times \mathbb{R}_+ \setminus \{0\}$ by $T(c) = (N(c), K(c))$ for $c \in C_+ \setminus \{0\}$. Let $L : C \to A$ be a unital positive linear map. We say L is T $-$full if for any $c \in C_+ \setminus \{0\}$, there are $x_1, x_2, \ldots, x_{N(c)} \in A$ with $\|x_i\| \leq K(c)$ such that

$$\sum_{i=1}^{N(c)} x_i^* L(c) x_i = I_A.$$

Let $H \subset C + \setminus \{0\}$. We say that L is $T - H$ $-$full if

$$\sum_{i=1}^{N(c)} x_i^* L(c) x_i = I_A.$$

for all $c \in H$.

Definition (3.1.3)[84]:

Denote by I the class of unital C^*-algebras with the form $\otimes_{i=1}^{m} C(X_i, M_{n(i)})$, where $X_i = [0,1]$ or X_i is one point

Definition (3.1.4)[84]:

Let $k \geq 0$ be an integer. Denote by I_k the class of all C^*-algebras B with the form $= PM_m(C(X))P$, , where X is a finite CW complex with dimension no more than k, P is a projection in $M_m(C(X))$.

Recall that a unital simple C^*-algebra A is said to have tracial rank no more than k (write $TR(A) \leq k$) if the following holds: For any $\epsilon > 0$, any positive element $a \in A_+ \setminus \{0\}$ and any finite subset $\mathcal{F} \subset A$, there exist a non-zero projection $p \in A$ and a C^*-subalgebra $B \in I_k$ with $1_B = p$ such that

(i) $\|xp - px\| < \epsilon$ for all $x \in \mathcal{F}$;

(i)$pxp \in_\epsilon B$ for all $x \in \mathcal{F}$; and

(iii) $1 - p$ is von Neumann equivalent to a projection in $\overline{aAa}$.

If $TR(A) \leq k$ and $TR(A) \neq k-1$, we say A has tracial rank k and write $TR(A) = k$. It has been shown that if $TR(A) = 1$, then, in the above definition, one can replace B by a C^*-algebra in I (see [91]). All unital simple AH-algebra with slow dimension growth and real rank zero have tracial rank zero (see [31] and also [88]) and all unital simple AH-algebras with no dimension growth have tracial rank no more than one (see [51], or, Theorem 2.5 of [89]). Note that all $AH-$algebras satisfy the Universal Coefficient Theorem. There is unital separable simple C^*-algebra A with $TR(A) = 0$ (and $TR(A) = 1$) which is not amenable.

The following is taken from an argument of N.C. Phillips [25].

Lemma (3.1.5)[84]:

Let $H > 0$ be a positive number and $let\ N \geq 2$ be an integer. Then, for any unital C^* −algebra A, any projection e ∈ A and any $u \in U_0(eAe)$ with $\mathrm{cel}_{eAe}(u) < H$,

$$\mathrm{dist}(\overline{u + (1 - e)}, \bar{1}) < H/N, \quad (10)$$

if there are mutually orthogonal and mutually equivalent projections $e_1, e_2, \ldots, e_{2N} \in (1 - e)A(1 - e)$ such that e1 is also equivalent to e.

Proof:

Since $\mathrm{cel}_{eAe}(u) < H$, there are unitaries $u_0, u_1 \ldots, u_N \in eAe$ such that

$$u_0 = u, \quad u_N = 1\ and\ \|u_i - u_{i-1}\| < H/N, \quad i = 1, 2, \ldots, N. \quad (11)$$

We will use the fact that

$$\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix} = \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, $\begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}$ is a commutator. Note that

$$\|(u \oplus u_1^* \oplus u_1 \oplus u_2^* \oplus \ldots \oplus u_N^* \oplus u_N) - (u \oplus u^* \oplus u_1 \oplus u_1^* \oplus \ldots \oplus u_{N-1}^* \oplus u_N)\| < H/N. \quad (12)$$

Since $u_N = 1, u \oplus u^* \oplus u_1 \oplus u_1^* \oplus \ldots \oplus u_{N-1}^* \oplus u_N$ is a commutator

Now we write

$$u \oplus e_1 \oplus \ldots \oplus e_{2N} = (u \oplus u_1^* \oplus u_1 \oplus u_2^* \oplus \ldots \oplus u_N^* \oplus u_N)(e \oplus u_1 \oplus u_1^* \oplus \ldots \oplus u_{N-1}^* \oplus u_N)$$

We obtain $z \in CU\left(\left(e + \sum_{i=1}^{2N} e_i\right)A\left(e + \sum_{i=1}^{2N} e_i\right)\right)$ such that

$$\|u \oplus e_1 \oplus \ldots \oplus e_{2N} - z\| < H/N.$$

It follows that

$$\mathrm{dist}(\overline{u + (1 - e)}, \bar{1}) < H/N.$$

Definition (3.1.6)[84]:

Let $= PM_k(C(X))P$, where X is a compact metric space and $P \in M_k(C(X))$ is a projection. Let $u \in U(C)$. Recall (see [27]) that

$$D_c(u) = \inf\{\|a\|: a \in C_{s.a}.\ \text{such that}\ \det(\exp(ia).u)(x) = 1\ \text{for all}\ x \in X\}.$$

If no self-a djoint element $a \in A_{s.a}$. exists for which $\det(\exp(ia).u)(x) = 1$ for all $x \in X$, define $D_c(u) = \infty$.

Lemma (3.1.7)[84]:

Let $K \geq 1$ be an integer. Let A be a unital simple C^* −algebra with $TR(A) \leq 1$, let $e \in A$ be a projection and let $u \in U_0(eAe)$. Suppose that $w = u + (1 - e)$ and suppose $\eta > 0$. Suppose also that

$$[1 - e] \leq K[e]\ \text{in}\ K_0(A) \quad \text{and} \quad \mathrm{dist}(\bar{w}, \bar{1}) < \eta. \quad (13)$$

Then, if $\eta < 2$,

$$\mathrm{cel}_{eAe}(u) < \left(\frac{k\pi}{2} + 1/16\right)\eta + 8\pi \quad \text{and} \quad \mathrm{dist}(\bar{u}, \bar{e}) < (k + 1/8)\eta.$$

and if $\eta = 2$,

$$\mathrm{cel}_{eAe}(u) < \frac{k\pi}{2}\mathrm{cel(w)} + 1/16 + 8\pi\,.$$

Proof:

We assume that (13) holds. Note that $\eta \leq 2$. Put $L = \mathrm{cel}(w)$.
We first consider the case that $\eta < 2$. There is a projection $e' \in M_2(A)$ such that

$$[(1-e)+e'] = k[e].$$

To simplify notation, by replacing A by $(1_A - e')M_2(A)(1_A - e')$ and w by $w + e'$, without loss of generality, we may now assume that

$$(1-e) = k[e] \text{ and } \mathrm{dist}(\overline{w}, \overline{1}) < \eta. \tag{14}$$

There is $R_1 > 1$ such that $max\{L/R_1, 2/R_1, \eta\pi/R_1\} < min\{\eta/64, 1/16\pi\}$.
For any $\frac{\eta}{32K(K+1)\pi} > \epsilon > 0$ with $\epsilon + \eta < 2$, since $TR(A) \leq 1$, there exist a projection $p \in A$ and a $C^* -$ subalgebra $D \in I$ with $1_D = p$ such that
(i) $\|[p,x]\| < \epsilon$ for $x \in \{u, w, e, (1-e)\}$;
(ii) $pwp, pup, pep, p(1-e)p \in_\epsilon D$;
(iii) there is a projection $q \in D$ and a unitary $z_1 \in qDq$ such that $\|q - pep\| < \epsilon$, $\| z_1 - quq\| < \epsilon, \| z_1 \oplus (p-q) - pwp\| < \epsilon$ and $\| z_1 \oplus (p-q) - c_1\| < \epsilon + \eta$;
(iv) there is a projection $q_0 \in (1-p)A(1-p)$ and a unitary $z_0 \in q_0 A_{q_0}$ such that $\|q_0 - (1-p)e(1-p)\| < \epsilon$, $\|z_0 - (1-p)u(1-p)\| < \epsilon$, $\|z_0 \oplus (1-p-q_0) - (1-p)w(1-p)\| < \epsilon$, $\|z_0 \oplus (1-p-q_0) - c_0\| < \epsilon + \eta$;
(v) $[p-q] = K[q]$ in $K_0(D)$, $[(1-p)-q_0] = K[q_0]$ in $K_0(A)$;
(vi) $2(K+1)R_1[1-p] < [p]$ in $K_0(A)$;
(vii) $\mathrm{cle}_{(1-p)A(1-p)}\left(z_0 \oplus (1-p-q_0)\right) \leq \mathrm{L} + \epsilon$,
where $c_1 \in CU(D)$ and $c_0 \in CU((1-p)A(1-p))$.
Note that $D_D(c_1) = 0$. Since $\epsilon + \eta < 2$, there is $h \in D_{s.a.}$ with $\|h\| \leq 2\arcsin\left(\frac{\epsilon+\eta}{2}\right)$ such that (by (iii) above)

$$\left(z_1 \oplus (p-q)\right)\exp(ih) = c_1. \tag{15}$$

It follows that

$$D_D\left(z_1 \oplus (p-q)\right)\exp(ih) = 0. \tag{16}$$

By (v) above and applying in [27], one obtains that

$$\left|D_{qD_D} z_1\right| \leq k2\arcsin\left(\frac{\epsilon+\eta}{2}\right). \tag{17}$$

If $2k\arcsin\left(\frac{\epsilon+\eta}{2}\right) \geq \pi$, then

$$2k\left(\frac{\epsilon+\eta}{2}\right)\frac{\pi}{2} \geq \pi.$$

It follows that

$$k(\epsilon+\eta) \geq 2 \geq \mathrm{dist}(\overline{z_1}, \overline{\mathrm{q}}). \tag{18}$$

Since those unitaries in D with $det(u) = 1$ (for all points) are in $CU(D)$ from (3.17), one computes that, when $2k\arcsin\left(\frac{\epsilon+\eta}{2}\right) < \pi$,

$$\mathrm{dist}(\overline{z_1}, \overline{\mathrm{q}}) < 2\sin\left(k\arcsin\left(\frac{\epsilon+\eta}{2}\right)\right) \leq \mathrm{k}(\epsilon+\eta). \tag{19}$$

By combining both (18) and (19), one obtains that

$$\mathrm{dist}(\overline{z_1}, \overline{\mathrm{q}}) \leq \mathrm{k}(\epsilon+\eta) \leq \mathrm{k}\,\eta + \frac{\eta}{32(k+1)\pi}. \tag{20}$$

By (17), it follows in [27] that

$$\mathrm{cel_q D_q} \le 2k\arcsin\frac{\epsilon+\eta}{2} + 6\pi \le \mathrm{k}(\epsilon+\eta)\frac{\pi}{2} + 6\pi$$
$$\le \left(k\frac{\pi}{2} + \frac{1}{64(\mathrm{k}+1)}\right)\eta + 6\pi \tag{21}$$

By (v) and (vi) above,

$$(K+1)[q] = [p-q] + [q] = [p] > 2(K+1)R_1[1-p].$$

Since $K_0(A)$ is weakly unperforated, one has

$$2R_1[1-p] < [q]. \tag{22}$$

There is a unitary $v \in A$ such that

$$v^*(1-p-q_0)v \le q. \tag{23}$$

Put $v_1 = q_0 \oplus (1-p-q_0)v$. Then

$$v_1^*(z_0 \oplus (1-p-q_0))v_1 = z_0 \oplus v^*(1-p-q_0)v. \tag{24}$$

Note that

$$\|(z_0 \oplus v^*(1-p-q_0)v)v_1^*c_0^*v_1 - q_0 \oplus v^*(1-p-q_0)v\| < \epsilon+\eta. \tag{25}$$

Moreover, by (vii) above

$$\mathrm{cel}(z_0 \oplus v^*(1-p-q_0)v) \le L+\epsilon. \tag{26}$$

It follows from (22) and Lemma (4.1.8) of [89] that

$$\mathrm{cel}_{(q_0+\mathrm{q})\mathrm{A}(q_0+\mathrm{q})}(z_0 \oplus \mathrm{q}) \le 2\pi + (L+\epsilon)/R_1. \tag{27}$$

Therefore, combining (21),

$$\mathrm{cel}_{(q_0+\mathrm{q})\mathrm{A}(q_0+\mathrm{q})}(z_0+z)$$
$$\le 2\pi + \frac{L+\epsilon}{R_1} + \left(k\frac{\pi}{2} + \frac{1}{64(\mathrm{k}+1)}\right)\eta + 6\pi\,. \tag{28}$$

By (26), (22), in $U_0\big((q_0+\mathrm{q})\mathrm{A}(q_0+\mathrm{q})\big)/CU\big((q_0+\mathrm{q})\mathrm{A}(q_0+\mathrm{q})\big)$,

$$\mathrm{dist}(\overline{z_0+\mathrm{q}}, \overline{q_0+\mathrm{q}}) < \frac{(L+\epsilon)}{R_1}. \tag{29}$$

Therefore, by (19) and (29),

$$\mathrm{dist}\big(\overline{z_0 \oplus z_1}, \overline{q_0+\mathrm{q}}\big) < \frac{(L+\epsilon)}{R_1} + k\,\eta + \frac{\eta}{32(\mathrm{k}+1)\pi} < (\mathrm{k}+1/6)\eta. \tag{30}$$

We note that

$$\|e - (q_0+\mathrm{q})\| < 2\epsilon \quad \text{and} \quad \|u - (z_0+z_1)\| < 2\epsilon. \tag{31}$$

It follows that

$$\mathrm{dist}(\bar{u}, \bar{e}) < 4\epsilon + (K+1/16)\,\eta < (K+1/8)\eta\,. \tag{32}$$

$$\mathrm{cel_{eAe}}(u) \le 4\epsilon\pi + 2\pi + (L+\epsilon)/R_1 + \left(k\frac{\pi}{2} + \frac{1}{64(\mathrm{k}+1)}\right)\eta + 6\pi \tag{33}$$
$$< \left(k\frac{\pi}{2} + 1/16\right)\eta + 8\pi. \tag{34}$$

This proves the case that $\eta < 2$.

Now suppose that $\eta = 2$. Define $R = [\mathrm{cel}(w)+1]$. Note that $\frac{\mathrm{cel}(w)}{R} < 1$. There is a projection $e' \in M_{R+1}(A)$ such that

$$[(1-e)+e'] = (K+RK)[e].$$

It follows that

$$\mathrm{dist}(\overline{w \oplus e'}, \overline{1_A + e'}) < \frac{\mathrm{cel}(w)}{R+1}. \quad (35)$$

Put $K_1 = K(R+1)$. To simplify notation, without loss of generality, we may now assume that

$$[1-e] = K_1[e] \text{ and } \mathrm{dist}(\bar{w}, \bar{1}) < \frac{\mathrm{cel}(w)}{R+1}. \quad (36)$$

It follows from the first part of the lemma that

$$\mathrm{cel}_{eAe}(u) < \left(\frac{K_1\pi}{2} + \frac{1}{16}\right)\frac{\mathrm{cel}(w)}{R+1} + 8\pi \quad (37)$$

$$\leq \frac{k\pi\mathrm{cel}(w)}{2} + \frac{1}{16} + 8\pi \quad (38)$$

Theorem (3.1.8)[84]:

Let A be a unital simple C^* −algebra with $TR(A) \leq 1$ and let $e \in A$ be a non-zero projection. Then the map $u \mapsto u + (1-e)$ induces an isomorphism j from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$.

Proof:

It was shown in in[89] that j is a surjective homomorphism. So it remains to show that it is also injective. To do this, fix a unitary $u \in eAe$ so that. $u \in \ker j$. We will show that $u \in CU(eAe)$.

There is an integer $K \geq 1$ such that

$$K[e] \geq [1-e] \text{ in } K_0(A).$$

Let $1 > \epsilon > 0$. Put $v = u + (1-e)$.Since. $u \in \ker j, v \in CU(A)$.In particular $\mathrm{dist}(\bar{v}, \bar{1}) < \epsilon/(K\pi/2 + 1)$.

It follows from Lemma (4.1.7)that

$$\mathrm{dist}(\bar{v}, \bar{1}) < \left(\frac{k\pi}{2} + 1/16\right)(\epsilon/(K\pi/2 + 1)) < \epsilon.$$

It then follows that

$$u \in CU(eAe).$$

Corollary (3.1.9)[84]:

Let A be a unital simple C^* −algebra with $TR(A) \leq 1$. Then the map $j : a \to \mathrm{diag}(a, \overbrace{1, 1, \dots, 1}^{m})$ from A *to* $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ *onto* $U(M_n(A))/CU(M_n(A))$ for any integer $n \geq 1$

Lemma(3.1.10)[84]:

Let X be a path connected finite CW complex, let $C = C(X)$ and *let* $A = C([0,1], M_n)$ for some integer $n \geq 1$. For any unital homomorphism $\phi: C \to A$, any finite subset $\mathcal{F} \subset C$ and any $\epsilon > 0$, there exists a unital homomorphism $\psi : C \to B$ such that

$$\|\phi(c) - \psi(c)\| < \epsilon \text{ for all } c \in \mathcal{F} \tag{39}$$

$$\psi(f)(t) = W(t)^* \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t)) \end{pmatrix} w(t), \tag{40}$$

where $W \in U(A), s_j \in C([0,1], X), j = 1, 2, \ldots, n,$ and $t \in [0,1]$.

Proof:

To simplify the notation, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of C. Since X is also locally path connected, choose $\delta_1 > 0$ such that, for any point $x \in X, B(x, \delta_1)$ is path connected. Put $d = 2\pi/n$. Let $\delta_1 > 0$ (in place of δ) be as required [69] for$\epsilon/2$.

We will also apply in [28], there exists a finite subset $\mathcal{H}$ of positive functions in $C(X)$ and $\delta_3 > 0$ satisfying the following: For any pair of points and $\{y_i\}_{i=1}^n$, if $\{h(x_i)\}_{i=1}^n$and $\{h(y_i)\}_{i=1}^n$ can be paired to within δ_3 one by one, in increasing order, counting multiplicity, for all $h \in \mathcal{H}, then$ $\{x_i\}_{i=1}^n$and $\{y_i\}_{i=1}^n$,i=1 can be paired to within $\delta_3/2$, one by one.

Put $\epsilon_1 = min\{\epsilon/16, \delta_1/16, \delta_2/4, \delta_3/4\}$. There exists $\eta > 0$ such that

$$|f(t) - f(t')| < \epsilon_1/2 \quad \text{for all} \quad f \in \phi(\mathcal{F} \cup \mathcal{H}). \tag{41}$$

provided that $|t - t'| < \eta$. C$\{x_i\}_{i=1}^n$ hoose a partition of the interval:
$0 = t_0 < t_1 < \cdots < t_N = 1.$

Such that $|t_i - t_{i-1}| < \eta,\ i = 1, 2, \ldots, N$. Then

$$\|\phi(f)(t_i) - \phi(f)(t_{i-1})\| < \epsilon_1 \text{ for all } f \in \mathcal{F} \cup \mathcal{H}. \tag{42}$$

$i = 1, 2, \ldots, N$. There are unitaries $U_i \in M_n$ and $\{x_{i,j}\}_{j=1}^n, i = 1, 2, \ldots, N$, such that

$$\phi(f)(t_i) = U_i^* \begin{pmatrix} f(x_{i,1}) & & \\ & \ddots & \\ & & f(x_{i,n}) \end{pmatrix} U_i \tag{43}$$

By the Weyl spectral variation inequality (see [69]), the eigenvalues of $\{h(x_{i,j})\}_{i=1}^n$ and $\{h(x_{i-1,j})\}_{i=1}^n$j=1 can be paired to within δ_3, one by one, counting multiplicity, in decreasing order. It follows in [28] that $\{x_{i,j}\}_{i=1}^n$j=1 and $\{x_{i-1,j}\}_{i=1}^n$can be paired within $\delta_3/2$. We may assume that

$$\text{dist}(x_{i,\sigma_i(j)}, x_{i-1,j}) < \delta_3/2, \tag{44}$$

where $\sigma_i: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation. By the choice of δ_3, there is a continuous path $\{x_{i-1}j\ (t): t \in [t_i - 1, (t_i + t_{i-1})/2]\} \subset B(x_{i-1}, \delta 3/2)$ such that

$$x_{i-1,j}(t_{i-1}) = x_{i-1,j} \quad \text{and} \quad x_{i-1,j}((t_{i-1} + t_i)/2) = x_{i,\sigma_i(j)} \tag{45}$$

$j = 1, 2, \ldots, n$. Put

$$\psi(f)(t) = U_{i-1}^* \begin{pmatrix} f(x_{i,1}(t)) & & \\ & \ddots & \\ & & f(x_{i,n}(t)) \end{pmatrix} U_{i-1} \tag{46}$$

for $t \in [t_{i-1}, (t_{i-1}+t_i)/2]$ and for $f \in C(X)$. In particular,

$$\psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) = U_{i-1}^* \begin{pmatrix} f(x_{i,1}(t)) & & \\ & \ddots & \\ & & f(x_{i,n}(t)) \end{pmatrix} U_{i-1} \tag{47}$$

for $f \in C(X)$. Note that

$$\|\phi(f)(t_{i-1}) - \psi(f)(t)\| < \delta_2/4 \text{ and } \|\psi(f)(t) - \phi(f)(t_i)\| < \delta_2/4 + \epsilon_1/2 < \delta_2/2 \tag{48}$$

for all $f \in \mathcal{F}$ and $t \in [t_{i-1}, \frac{t_{i-1}+t_i}{2}]$. There exists a unitary $W_i \in M_n$ such that

$$w_i^* \psi(f) = \left(\frac{t_{i-1}+t_i}{2}\right) w_i = \phi(f)(t_i) \tag{49}$$

for all $f \in C(X)$. It follows from (48) and (49) that

$$\left\| w_i \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) - \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) w_i \right\| < \delta_2 \tag{50}$$

for all $f \in \mathcal{F}$. By the choice of δ_2 and by applying in [69], we obtain $h_i \in M_n$ such that $W_i = \exp(\sqrt{-1}h_i)$ and

$$\left\| h_i \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) - \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) h_i \right\| < \epsilon/4 \tag{51}$$

and

$$\left\| \exp(\sqrt{-1}th_i)\psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) - \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) \exp(\sqrt{-1}th_i) \right\| < \epsilon/4 \tag{52}$$

for all $f \in \mathcal{F}$. and $t \in [0,1]$. From this we obtain a continuous path of unitaries $\{W_i(t): t \in [\frac{t_{i-1}+t_i}{2}, t_i]\} \subset M_n$ such that

$$W_i\left(\frac{t_{i-1}+t_i}{2}\right) = 1, \quad W_i(t_i) = W_i \tag{53}$$

and

$$\left\| w_i \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) - \psi(f)\left(\frac{t_{i-1}+t_i}{2}\right) w_i \right\| < \epsilon/4 \tag{54}$$

for all $f \in \mathcal{F}$ and $t \in \left[\frac{t_{i-1}+t_i}{2}, t_i\right]$. Define $\psi(f)(t) = w_i^*(t)\psi\left(\frac{t_{i-1}+t_i}{2}\right) w_i(t)$ *for* $t \in [\frac{t_{i-1}+t_i}{2}, t_i], i = 1, 2, \ldots, N$. Note that $\psi : C(X) \to A$. We conclude that

$$\|\phi(f) - \psi(f)\| < \epsilon \text{ for all } \mathcal{F} \tag{55}$$

Define

$$U(t) = U_0 \text{ for } t \in \left[0, \frac{t_1}{2}\right), \; U(t) = U_0 W_1(t) \text{ for } t \in \left[\frac{t_1}{2}, t_2\right), \tag{56}$$

$$U(t) = U(t_i) \quad \text{for} \quad t \in \left[t_i, \frac{t_i + t_{i-1}}{2}\right),$$
$$U(t) = U(t_i)W_{i+1}(t) \quad \text{for } t \in \left[\frac{t_i + t_{i-1}}{2}, t_{i+1}\right], \tag{57}$$
$i = 1, 2, \ldots, N - 1$ and define
$$s_j = x_{0,j}(t) \quad \text{for} \quad t \in \left[0, \frac{t_1}{2}\right), \quad s_j(t) = s_j\left(\frac{t_1}{2}\right) \quad \text{for} \quad t \in \left[\frac{t_1}{2}, t_2\right), \tag{58}$$
$$s_j = x_{i,\sigma_i(j)}(t) \quad \text{for} \quad t \in \left[t_i, \frac{t_i + t_{i+1}}{2}\right),$$
$$s_j(t) = s_j\left(\frac{t_i + t_{i+1}}{2}\right) \quad \text{for } t \in \left[\frac{t_i + t_{i+1}}{2}, t_{i+1}\right], \tag{59}$$
$i = 1, 2, \ldots, N - 1$. Thus $U(t) \in A$ and, by (45), $s_j(t) \in C([0,1], X)$.
One then checks that ψ has the form
$$\psi(f) = U(t)^* \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t)) \end{pmatrix} U(t) \tag{60}$$
for $f \in C(X)$. In fact, for $t \in [0, t_1]$, it is clear that (60) holds. Suppose that (60) holds for $t \in [0, t_i]$. Then, by (49), for $f \in C(X)$,
$$\psi(f)(t_i) = U(t_i)^* \begin{pmatrix} f(x_{i,\sigma_i(1)}) & & \\ & \ddots & \\ & & f(x_{i,\sigma_i(n)}) \end{pmatrix} U(t_i)$$
$$= U_I^* \begin{pmatrix} f(x_{i,1}) & & \\ & \ddots & \\ & & f(x_{i,n}) \end{pmatrix} U_i, \tag{61}$$
Therefore, for $t \in \left[t_i, \frac{t_i+t_{i+1}}{2}\right]$,
$$\psi(f)(t) = U_I^* \begin{pmatrix} f(x_{i,1}(t)) & & \\ & \ddots & \\ & & f(x_{i,n}(t)) \end{pmatrix} U_i \tag{62}$$
$$= U(t_i)^* \begin{pmatrix} f(x_{i,\sigma_i(1)}(t)) & & \\ & \ddots & \\ & & f(x_{i,\sigma_i(n)}(t)) \end{pmatrix} U(t_i) \tag{63}$$
$$= U(t)^* \begin{pmatrix} f(s_1(t)) & & \\ & \ddots & \\ & & f(s_n(t)) \end{pmatrix} U(t) \tag{64}$$
For $t \in \left[\frac{t_i + t_{i+1}}{2}, t_{i+1}\right]$,
$$\psi(f)(t) = W_{i+1}(t)^* \psi\left(\frac{t_i + t_{i+1}}{2}\right) W_{i+1}(t) \tag{65}$$

$$= W_{i+1}(t)^* U(t_i)^* \begin{pmatrix} f(s_1(\frac{t_i + t_{i+1}}{2})) & & \\ & \ddots & \\ & & f(s_n(\frac{t_i + t_{i+1}}{2})) \end{pmatrix} U(t_i) W_{i+1}(t) \quad (66)$$

$$= U(t)^* \begin{pmatrix} f\big(s_1(t)\big) & & \\ & \ddots & \\ & & f\big(s_n(t)\big) \end{pmatrix} U(t) \quad (67)$$

This verifies (60).

Lemma (3.1.11)[84]:

Let X be a finite CW complex and let $A \in$. Suppose that $\phi: C(X) \otimes C(T) \to A$ is a unital homomorphism. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists a continuous path of unitaries $\{u(t): t \in [0,1]\}$ in A such that

$$u(0) = \phi(1 \otimes z), \qquad u(1) = 1 \quad and \quad \|[\phi(f \otimes 1), u(t)]\| < \epsilon \quad (68)$$

for $f \epsilon \mathcal{F}$ and $t \in [0,1]$.

Proof:

It is clear that the general case can be reduced to the case that $A = C([0,1], M_n)$. Let $q_1, q_2, \dots, q_n$ be projections of $C(X)$ corresponding to each path connected component of X. Since $\phi(q_i)A\phi(q_i) \cong C([0,1], M_{n_i})$ for some $1 \le n_i \le n$, $i = 1, 2, \dots$, we may reduce the general case to the case that X is path connected and $A = C([0,1], M_n)$.

Note that we use z for the identity function on the unit circle.

For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, obtains a unital homomorphism $\psi : C(X) \otimes C(T) \to A$ such that

$$\|\phi(g) - \psi(g)\| < \epsilon \ \text{ for all } \ g \in \{f \otimes 1 : f \epsilon \mathcal{F}\} \cup \{1 \otimes z\} \quad (69)$$

$$\psi(f)t = U(t)^* \begin{pmatrix} f\big(s_1(t)\big) & & \\ & \ddots & \\ & & f\big(s_n(t)\big) \end{pmatrix} U(t), \quad (70)$$

for all $f \in C(X \times \mathbb{T})$, where $U(t) \in U(C([0,1], M_n)), s_j : [0,1] \to X \times \mathbb{T}$ *is* a continuous map, $j = 1, 2, \dots, n$, and for all $t \in [0,1]$. There are continuous paths of unitaries $\{u_j(r): r \in [0,1]\} \subset C([0,1])$ such that

$$u_j(0)(t) = (1 \otimes z)\left(s_j(1)\right), \quad u_j(1) = 1, \quad j = 1, 2, \dots, n, \quad (71)$$

Define

$$u(r)t = U(t)^* \begin{pmatrix} u_j(r)(t) & & \\ & \ddots & \\ & & u_n(r)(t) \end{pmatrix} U(t). \quad (72)$$

Then

$$u(r)\psi(f \otimes 1) = \psi(f \otimes 1)u(r) \ \text{ for all } \ r \in [0,1].$$

It follows that

$$\|[\phi(f \otimes 1), u(r)]\| < \epsilon \ \text{ for all } \ r \in [0,1] \ \text{ and for all } \ f \in \mathcal{F}.$$

Definition (3.1.12)[84]:

Let X be a compact metric space. We say that X satisfies property (H) if the following holds:

For any $\epsilon > 0$, any finite subsets $\mathcal{F} \subset C(X)$ and any non-decreasing map $\Delta: (0,1) \to (0,1)$, there exists $\eta > 0$ (which depends on ϵ and F but not Δ), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following:

Suppose that $\phi: C(X) \to C([0,1], M_n)$ is a unital $\delta - G$ −multiplicative contractive completely positive linear map for which

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \tag{73}$$

for any open ball O_a with radius $a \geq \eta$ and for all tracial states τ of $C([0,1], M_n)$, and

$$[\phi]|P = [\Phi]|P, \tag{74}$$

where Φ is a point-evaluation.

Then there exists a unital homomorphism $h: C(X) \to C([0,1], M_n)$ such that
for all $f \in \mathcal{F}$.

It is a restricted version of some relatively weakly semi-projectivity property. It has been shown in [22] that any k −dimensional torus has the property (H). So do those finite CW complexes X with torsion free $K_0(C(X))$ and torsion $K_1(C(X))$, any finite CW complexes with form $Y \times \mathbb{T}$ where Y is contractive and all one-dimensional finite CW complexes.

Corollary(3.1.13)[84]:

Let $C = C(X, M_n)$ where $X = [0,1]$ *or* $X = \mathbb{T}$ and $\Delta: (0,1) \to (0,1)$ be a nondecreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists $\delta > 0, \eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:

Suppose that A is a unital simple C^* −algebra with $TR(A) \leq 1, \phi : C \to A$ *is a* unital monomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c), u]\| < \delta \quad \text{for all} \quad c \in \mathcal{G}, \tag{75}$$

$$\mathrm{bott}_0(\phi, u) = \{0\} \quad and \quad \mathrm{bott}_1(\phi, u) = \{0\} \tag{76}$$

Suppose also that there exists a unital contractive completely positive linear map $L : C \otimes C(T) \to A$ such that (with z the identity function on the unit circle)

$$\|L(c \otimes 1) - \phi(c)\| < \delta, \qquad \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G}$$

and

for all open balls O_a *of* $[0,1] \times \mathbb{T}$ with radius $1 > a \geq \eta$, where $\mu_{\tau\circ L}$ is the Borel probability measure defined by restricting L on the center of $C \otimes C(\mathbb{T})$. Then there exists a continuous path of unitaries $\{u(t): t \in [0,1]\}$ such that

$$u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[\phi(c), u(t)]\| < \epsilon \tag{78}$$

for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Corollary (3.1.14)[84]:

Let $C = C([0,1], M_n)$ *and let* $T = N \times K : (C \otimes C(\mathbb{T}))_+ \setminus \{0\} \to N \times \mathbb{R}_+ \setminus \{0\}$ be a map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, a finite subset $H \subset (C \otimes C(T))_+ \setminus \{0\}$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:

Suppose that A is a unital simple C^* −algebra with $TR(A) \leq 1, \phi: C \to A$ *is a* unital monomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c), u]\| < \delta \quad \text{for all} \quad c \in \mathcal{G}, \tag{79}$$
and
$$\mathrm{bott}_0(\phi, u) = \{0\}. \tag{80}$$

Suppose also that there exists a unital contractive completely positive linear map $L: C \otimes C(T) \to A$ which *is* $T - H -$full such that (with z the identity function on the unit circle)
$$\|L(c \otimes 1) - \phi(c)\| < \delta, \qquad \|L(c \otimes z) - \phi(c)u\| < \delta \text{ for all } c \in \mathcal{G} \tag{81}$$
Then there exists a continuous path of unitaries $\{u(t): t \in [0,1]\}$ in A such that
$$u(0) = u, \quad u(1) = 1 \quad \text{and} \quad \|[\phi(c), u(t)]\| < \epsilon \tag{82}$$
for all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Proof:

Fix $T = N \times K : N \times \mathbb{R}_+ \setminus \{0\}.$ *Let* $\Delta: (0,1) \to (0,1)$ be the non-decreasing map associated with T as in [22]. Let $\mathcal{G} \subset C, \delta > 0$ and > 0 , for ϵ and $\mathcal{F}$ given and the above Δ.

It follows in [22] that there exists a finite subset $H \subset (C \otimes C(T))_+ \setminus \{0\}$ such that for any unital contractive completely positive linear map $L : C \otimes C(T) \to A$ which is $T - H -$full, one has that
$$\mu_{\tau \circ L}(O_a) \geq \Delta(a) \tag{83}$$
For all open balls O_a of $X \times \mathbb{T}$ with radius $a \geq \eta$.

Lemma (3.1.15)[84]:

Let $C = M_n$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F}$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: For any unital C^*-algebra A with $K_1(A) = U(A)/U_0(A)$ and any unital homomorphism $\varphi : C \to A$ and any unitary $u \in A$ if
$$\|[\phi(c), u]\| < \delta \quad \textit{and} \quad \mathit{bott}_0(\phi, u) = \{0\}, \tag{84}$$
then there exists a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset A$ such that
$$u(0) = u, \;\; u(1) = 1 \quad \textit{and} \;\; \|[\phi(c), u]\| < \epsilon \tag{85}$$
for all $c \in \mathcal{F}$ *and* $t \in [0,1]$.

Proof:

First consider the case that $\phi(c)$ commutes with u for all $c \in C$. Then one has a unital homomorphism $\Phi: M_n \otimes C(\mathbb{T}) \to A$ defined by $\Phi(c \otimes g) = \phi(c)g(u)$ for all $c \in C$ and $g \in C(\mathbb{T})$. Let $\{e_{i,j}\}$ be a matrix unit for M_n. Let $u_j = e_j, j \otimes z, j = 1, 2, \ldots, n$. The assumption $bott_0(\varphi, u) = \{0\}$ implies that $\Phi_{*1} = \{0\}$. It follows that $u_j \in U_0(A), j = 1, 2, \ldots, n$. One then obtains a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset A$ such that
$$u(0) = u, \qquad u(1) = 1 \quad \textit{and} \quad \|[\phi(c), u(t)]\| = 0$$
for all $c \in C(\mathbb{T})$ and $t \in [0,1]$.

The general case follows from the fact that $C \otimes C(\mathbb{T})$ is weakly semi-projective.

Lemma (3.1.16)[84]:

Let $n < 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exist $\frac{\pi}{2n} > \delta > 0$ and a finite subset $\mathcal{G} \subset D \sim= M_n$ satisfying the following: Suppose that A is a unital C^*-algebra with $T(A) \neq \emptyset, D \subset A$ is a C^*-subalgebra with $1_D = 1_A$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$ such that

$$\|[f,x]\| < \delta \quad for\ all\ f \in \mathcal{F}\ and\ x \in \mathcal{G}, \tag{86}$$

and

$$\|[u,x]\| < \delta \quad for\ all\ \ x \in \mathcal{G}, \tag{87}$$

Then, there exist a unitary $v \in D$ and a continuous path of unitaries $\{w(t): t \in [0,1]\} \subset D$ such that

$$\|u, w(t)\| < n\delta < \epsilon,\ \ \|u, w(t)\| < n\delta < \frac{\epsilon}{2} \tag{88}$$

$$for\ all\ f \in \mathcal{F}\ and\ for\ all\ t \in [0,1], \tag{89}$$

$$w(0) = 1, \qquad w(1) = v\ \ and\ \ \mu_{\tau\circ\phi}(I_a) = \frac{2}{3n^2} \tag{90}$$

for all open arcs I_a of $\mathbb{T}$ with length a 4π/n and for all $\tau \in T(A)$, where $\iota: C(\mathbb{T}) \to A$ is defined by $\iota(f\ ) = f$ (vu) for all $f \in C(T)$.

Moreover,

$$length\ (\{w(t)\}) \leq \pi. \tag{91}$$

If, in addition, $\pi > b_1 > b_2 > \cdots > b_m > 0$ and $1 = d_0 > d_1 > d_2 > \cdots > d_m > 0$ are given so that

$$\mu_{\tau\circ l}\left(I_{b_i}\right) \geq d_i \quad for\ all\ \tau \in T(A),\ \ i = 1,2,\ldots,m, \tag{92}$$

where ι_0: $C(\mathbb{T}) \to A$ is defined by $\iota_0(f\ ) = f\ (u)$ for all $f \in C(\mathbb{T})$, then one also has that

$$\mu_{\tau\circ l}\left(I_{c_i}\right) \geq (1-\epsilon_1)d_i \quad for\ all\ \tau \in T(A), \tag{93}$$

where I_{b_i} and I_{c_i} are any open arcs with length bi and c_i, respectively, and where $c_i = b_i + \epsilon_1, i\ =\ 1, 2, \ldots, m$.

Proof:

Let

$$0 < \delta_0 < \min\left\{\frac{\epsilon_1 d_i}{16n^2} : 1 \leq i \leq m\right\}.$$

Let $\{e_{i,j}\}$ be a matrix unit for D and let $\mathcal{G} =\ \{e_{i,j}\}$. Define

$$v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n} e_{i,j}. \tag{94}$$

Let $f_1 \in C(\mathbb{T})$ with $f_1(t)\ =\ 1$ for $|t\ -\ e^{2\sqrt{-1}\pi/n}| <\ \pi/n$ and $f_1(t) = 0$ if $|t - e^{2\sqrt{-1}\pi/n}| \geq 2\pi/n$ and $1 \geq f_1(t) \geq 0$. Define $f_{j+1}(t) = f_1(e^{2\sqrt{-1}j\pi/n}t), j\ =\ 1, 2, \ldots, n\ -\ 1$. Note that

$$f_i\left(e^{2\sqrt{-1}j\pi/n}\right) = f_{i+j}(t) \quad for\ all\ t \in \mathbb{T} \tag{95}$$

where $i, j\ \in \mathbb{Z}/n\mathbb{Z}$.

Fix a finite subset $\mathcal{F}_0 \subset C(\mathbb{T})_+$ which contains $f_i, i\ =\ 1, 2, \ldots, n$.

Choose δ so small that the following hold:

(i) there exists a unitary $u_i \in e_{i,i}A_{e_{i,i}}$ such that $\left\|e^{2\sqrt{-1}i\pi/n} e_{i,i} u e_{i,i} - u_i\right\| <\ \delta_0^2/16n^2, i\ =\ 1, 2, \ldots, n$.

(ii) ;

(iii) $\left\|e_{i,i} f\ (vu)\ -\ e_{i,i} f\ (e^{2\sqrt{-1}i\pi/n} u)\right\|\ < \delta_0^2/16n^2$ for all $f \in \mathcal{F}_0$; and

(iv) $\|e_{i,j}^* f(u)e_{i,j} - e_{j,j}f(u)e_{j,j}\| < \delta_0^2/16n^2$ for all $f \in \mathcal{F}_0$.

Fix k. For each $\tau \in T(A)$, by (i), (iii) and (iv) above, there is at least one i such that

$$\tau(e_{j,j}f_i(u)) \geq \frac{1}{n^2} - \frac{\delta_0^2}{16n^2}. \tag{96}$$

Choose j so that $k + j = i mod (n)$. Then,

$$\tau(f_k(vu)) \geq \tau(e_{j,j}f_k(vu)) \tag{97}$$

$$\geq \tau\left(e_{j,j}f_k(e^{2\sqrt{-1}i\pi/n}u)\right) - \frac{\delta_0^2}{16n^2} \tag{98}$$

$$= \tau(e_{j,j}f_i(u)) - \frac{\delta_0^2}{16n^2} \geq \frac{1}{n^2} - \frac{2\delta_0^2}{16n^2}. \tag{99}$$

It follows that

$$\mu_{\tau\circ l}\left(B\left(e^{2\sqrt{-1}i\pi/n}, \pi/n\right)\right) \geq \frac{1}{n^2} - \frac{2\delta_0^2}{16n^2} \quad for\ all\ \tau \in T(A) \tag{100}$$

and for $k = 1, 2, \ldots, n$.

It is then easy to compute that

$$\mu_{\tau\circ l}(I_a) \geq \frac{2}{3n^2} \qquad for\ all\ \tau \in T(A) \tag{101}$$

and for any open arc with length $a \geq 2\left(\frac{2\pi}{n}\right) = \frac{4\pi}{n}$.

Note that if $\|[x, e_{i,i}]\| < \delta$, then

$$\left\|\left[x, \sum_{i=1}^{n} \lambda_i e_{i,i}\right]\right\| < n\delta < \frac{\epsilon}{2} \ \ and \quad \left\|\left[u, \sum_{i=1}^{n} \lambda_i e_{i,i}\right]\right\| < n\delta < \epsilon/2$$

for any $\lambda_i \in \mathbb{T}$. Thus, one obtains a continuous path $\{w(t): t \in [0,1]\} \subset D$ with $length(\{w(t)\}) \leq \pi$ and with $w(0) = 1$ and $(1) = v$.

Let $\{x_1, x_2, \ldots, x_K\}$ be an $\epsilon_1/64$-dense set of $\mathbb{T}$. Let $I_{i,j}$ be an open arc with center x_j and $length\ b_i, j = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, m$. For each j and i, there is a positive function $g_{j,i} \in C(\mathbb{T})_+$ with $0 \leq g_{j,i} \leq 1$ and $g_{j,i}(t) = 1$ if $|t - x_j| < d_i$ and $g_{j,i}(t) = 0$ if $|t - x_j| \geq d_i + \epsilon_1/64, j = 1, 2, \ldots, K, i = 1, 2, \ldots, m$. Put $g_{i,j,k}(t) = g_{j,i}(e^{2\sqrt{-1}i\pi/n} \cdot t)$ for all $t \in \mathbb{T}, k = 1, 2, \ldots, n$. Suppose that $\mathcal{F}_0$ contains all $g_{j,i}$ and $g_{i,j,k}$. We have, by (ii), (iii) and (iv) above,

$$\tau\big(g_{j,i}(u), e_{l,l}\big), \tau\big(g_{j,i,k}(u), e_{l,l}\big) \geq \frac{d_i}{n} - \frac{\delta^2}{16n^2} \quad for\ all\ \tau \in T(A), \tag{102}$$

$l = 1, 2, \ldots, n, j = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, m$. Thus

$$\tau\left(e_{k,k} g_{j,i}(vu)\right) \geq \tau\left(e_{k,k} g_{j,i}\left(e^{2\sqrt{-1}i\pi/n} u\right)\right) - n\frac{\delta_0^2}{16n^2} \tag{103}$$

$$\geq \frac{\mathrm{d_i}}{\mathrm{n}} - \frac{\delta_0^2}{8\mathrm{n}^2} \quad \text{for all } \tau \in \mathrm{T(A)}, \tag{104}$$

$k = 1, 2, \ldots, n, j = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, m$. Therefore

$$\tau\left(e_{k,k} g_{j,i}(vu)\right) \geq d_i - \frac{\delta_0^2}{8n^2} \geq (1 - \epsilon_1) d_i \qquad for\ all\ \tau \in T(A), \tag{105}$$

$j = 1, 2, \ldots, K$ *and* $i = 1, 2, \ldots, m$.

It follows that

$$\mu_{\tau \circ l}\big(I_{i,j}\big) \geq (1 - \epsilon_1) d_i \qquad for\ all\ \tau \in T(A), \tag{106}$$

$j = 1, 2, \ldots, K$ and $i = 1, 2, \ldots, m$. Since $\{x_1, x_2, \ldots, x_K\}$ is $\epsilon_1/64$-dense in $\mathbb{T}$, it follows that

$$\mu_{\tau \circ l}\big(I_{c_i}\big) \geq (1 - \epsilon_1) d_i \qquad for\ all\ \tau \in T(A), \mathrm{i} = 1,2, \ldots, \mathrm{m}. \tag{107}$$

Lemma (3.1.17)[84]:

Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exist $\frac{\epsilon}{2n} > \delta > 0$ and a finite subset $G \subset D \cong M_n$ satisfying the following:

Suppose that X is a compact metric space, $\mathcal{F} \subset C(X)$ is a finite subset and $1 > b > 0$. Then there exists a finite subset $\mathcal{F}_1 \subset C(X)$ satisfying the following: Suppose that A is a unital C^*-algebra with $T(A) \neq \emptyset, D \subset A$ is a C^*-subalgebra with $1_D = 1_A, \phi: C(X) \to A$ is a unital homomorphism and suppose that $u \in U(A)$ such that

$$\|[x, u]\| < \delta \quad and \quad \|[x, \phi(f)]\| < \delta \quad for\ all\ x \in \mathcal{G}\ and\ f \in \mathcal{F}_1. \tag{108}$$

Suppose also that, for some $\sigma > 0$,

$$\tau(\phi(f))\sigma \qquad for\ all\ \tau \in T(A) \qquad and \tag{109}$$

for all $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball of X with radius b. Then, there exist a unitary $v \in D$ and a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset D$ such that

$$\|u, v(t)\| < n\delta < \epsilon, \ \ \|\phi(f), v(t)\| < n\delta < \epsilon \tag{110}$$

$$for\ all\ f \in \mathcal{F} \ \ and\ \ t \in [0,,1], \tag{111}$$

$$v(0) = 1, \quad v(1) = v \tag{112}$$

and

$$\tau(\phi(f)g(vu)) \geq \frac{2\sigma}{3n^2} \qquad for\ all\ \tau \in T(A) \tag{113}$$

for any pair of $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $2b$ and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc of $\mathbb{T}$ with length at least

$\frac{8\pi}{n}$.

Moreover,

$$length(v(t)) \le \pi. \tag{114}$$

If, in addition, $1 > b_1 > b_2 > \cdots > b_k > 0, 1 > d_1 \ge d_2 \ge \cdots d_k > 0$ are given and

$$\tau(\phi(f')g'(u)) \ge d_i \quad for\ all\ \tau \in T(A) \tag{115}$$

for any functions $f' \in C(X)$ with $0 \le f' \le 1$ whose support contains an open ball of X with radius $b_i/2$ *and* $g' \in C(\mathbb{T})$ with $0 \le g' \le 1$ whose support contains an arc with length b_i, then one also has that

$$\tau(\phi(f'')g''(vu)) \ge (1-\epsilon_1)d_i \qquad for\ all\ \tau \in T(A), \tag{116}$$

where $f'' \in C(X)$ with $0 \le f'' \le 1$ whose support contains an open ball of radius c_i and $g'' \in C(T)$ with $0 \le g'' \le 1$ whose support contains an arc with $length\ 2c_i$ with $c_i = c_i + 1, i = 1, 2, \ldots, k$.

Proof:

Let $0 < \delta_0 = min\left\{\frac{\epsilon_1 d_i}{16n^2} : i = 1, 2, \ldots, k\right\}$.

Let $\{e_{i,j}\}$ be a matrix unit for D and let $G = \{e_{i,j}\}$. Define

$$v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n}\, e_{j,j}. \tag{117}$$

Let $g_j \in C(\mathbb{T})$ with $g_j(t) = 1$ for $|t - e^{2\sqrt{-1}j\pi/n}| < \pi/n$ and $g_j(t) = 0$ if $|t - e^{2\sqrt{-1}j\pi/n}| \ge 2\pi/n$ and $1 \ge g_j(t) \ge 0, j = 1, 2, \ldots, n$. As in the proof of 5.1, we may also assume that

$$g_i\left(e^{2\sqrt{-1}j\pi/n}t\right) = g_{i+1}(t) \quad for\ all\ t \in \mathbb{T} \tag{118}$$

where $i, j \in \mathbb{Z}/n\mathbb{Z}$.

Let $\{x_1, x_2, \ldots, x_m\}$ be a $b/2$-dense subset of X. Define $f_i \in C(X)$ with $f_i(x) = 1$ for $x \in B(x_i, b)$ and $f_i(x) = 0$ if $x \notin B(x_i, 2b)$ and $0 \le f_i \le 1, i = 1, 2, \ldots, m$.

Note that

$$\tau(\phi(f_i)) \ge \sigma \quad for\ all\ \tau \in T(A), \qquad i = 1,2,\ldots,m. \tag{119}$$

Fix a finite subset $\mathcal{F}_0 \subset C(\mathbb{T})$ which at least contains $\{g_1, g_2, \ldots, g_n\}$ and a finite subset $\mathcal{F}_1 \subset C(X)$ which at least contains $\mathcal{F}$ and $\{f_1, f_2, \ldots, f_m\}$.

Choose δ so small that the following hold:

(i) there exists a unitary $u_i \in e_{i,i}Ae_{i,i}$ such that $\left\|e^{2\sqrt{-1}i\pi/n}e_{i,i}ue_{i,i} - u_i\right\| < \delta_0^2/16n^4, i = 1, 2, \ldots, n$;

(ii) $\left\|e_{i,j}g(u) - g(u)e_{i,j}\right\| < \delta_0^2/16n^4, \left\|e_{i,j}\phi(f) - \phi(f)e_{i,j}\right\| < \delta_0^2/16n^4$, for $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0, j, k = 1, 2, \ldots, n$ and $s = 1, 2, \ldots, m$;

(iii) $\left\|e_{i,i}g(vu) - e_{i,i}g(e^{2\sqrt{-1}i\pi/n}u)\right\| < \delta_0^2/16n^4$ for all $g \in \mathcal{F}_0$; and

(iv) $\left\|e_{i,j}^*g(u)e_{i,j} - e_{i,j}g(u)e_{i,j}\right\| < \delta_0^2/16n^4, \left\|e_{i,j}^*\phi(f)e_{i,j} - e_{j,j}\phi(f)e_{j,j}\right\| < \delta_0^2/16n^4$ for all $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0, j, k = 1, 2, \ldots, n$ and $s = 1, 2, \ldots, m$.

It follows from (iv) that, for any $k_0 \in \{1, 2, \ldots, m\}$,

$$\tau\big(\phi(f_{k_0})e_{j,j}\big) \geq \frac{\sigma}{n} - \frac{\delta_0^2}{16n^4}. \tag{120}$$

Fix k_0 and k. For each $\tau \in T(A)$, there is at least one i such that

$$\tau\left(\phi(f_{k_0})e_{j,j}g_i(u)\right) \geq \frac{\sigma}{n} - \frac{\delta_0^2}{16n^4}. \tag{121}$$

Choose j so that $k + j = i \ mod\ (n)$. Then,

$$\tau\left(\phi(f_{k_0})g_k(vu)\right) \geq \tau\left(\phi(f_{k_0})e_{j,j}g_k\left(e^{2\sqrt{-1}i\pi/n}u\right)\right) - \frac{\delta_0^2}{16n^4} \tag{122}$$

$$= \tau\left(\phi(f_{k_0})e_{j,j}g_i(u)\right) - \frac{\delta_0^2}{16n^4} \tag{123}$$

$$\geq \frac{\sigma}{n^2} - \frac{\delta_0^2}{16n^4} \qquad for\ all\ \tau \in T(A). \tag{124}$$

It is then easy to compute that

$$\tau\big(\phi(f)g(vu)\big) \geq \frac{2\sigma}{3n^2} \qquad for\ all\ \tau \in T(A) \tag{125}$$

and for any pair of $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $2b$ and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc of length at least $8\pi/n$.

Note that if $\|[\phi(f\ ), e_{i,i}]\| < \delta$, then

$$\left\|\left[\phi(f), \sum_{i=1}^{n} \lambda_i\, e_{i,i}\right]\right\| < n\delta < \epsilon$$

for any $\lambda_i \in \mathbb{T}$ and $f \in \mathcal{F}_1$. We then also require that $\delta < \epsilon/2n$. Thus, one obtains a continuous path $\{v(t) \colon t \in [0,1]\} \subset D$ with $length(\{v(t)\}) \leq \pi$ and with $v(0) = 1$ and $(1)\ =\ v$.

Now we consider the last part of the lemma. Note also that, if $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$ with $0 \leq f, g \leq 1$,

$$\tau\big(\phi(f)g(vu)\big) \geq \sum_{j=1}^{n} \tau\left(\phi(f)e_{j,j}g(vu)\right) - \frac{\delta_0^2}{16n^4} \tag{126}$$

$$\geq \sum_{j=1}^{n} \tau\left(\phi(f)e_{j,j}g^{(j)}(vu)\right) - \frac{\delta_0^2}{16n^4} \quad for\ all\ \tau \in T(A), \tag{127}$$

where $g^{(j)}(t)\ =\ g(e^{2\sqrt{-1}j\pi/n} \cdot t)$ for $t \in \mathbb{T}$. If the support of f contains an open ball with radius $b_i/2$ and that of g contains open arcs with length at least b_i, so does that of $g^{(j)}$. So, if $\mathcal{F}_0$ and $\mathcal{F}_1$ are sufficiently large, by the assumptions of the last part of the lemma, we have

$$\tau\big(\phi(f)g(vu)\big) \geq d_i - \frac{\delta_0^2}{16n^4} \quad for\ all\ \tau \in T(A) \tag{128}$$

for all $\tau\ \in\ T(A)$. As in the proof of (3.1.16), this lemma follows when we choose $\mathcal{F}_0$ and $\mathcal{F}_1$ large enough to begin with.

Lemma (3.1.18)[84]:

Let C be a unital separable simple C^*-algebra with $TR(C) \leq 1$ and let $n \geq 1$ be an integer. For any $\epsilon > 0, \eta > 0$, any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, a projection $p \in A$ and a C^*-subalgebra $D \cong M_n$ with $1_D = p$ such that

$$\|[x,p]\| < \epsilon \qquad for\ all\ x \in \mathcal{F}; \tag{129}$$

$$\|[pxp,y]\| < \epsilon \qquad for\ all\ x \in \mathcal{F}\ and\ y \in D\ with\ \|y\| \leq 1 \tag{130}$$

and

$$\tau(1-p) < \eta \qquad for\ all \tau \in T(C). \tag{131}$$

Proof:

Choose an integer $N \geq 1$ such that $1/N < \eta/2n$ and $N \geq 2n$. It follows from (the proof of) Theorem (3.1.18) of [89] that there is a projection $q \in C$ and there exists a C^*-subalgebra B of C with $1_B = q$ and $B \cong \bigoplus_{i=1}^{L} M_{K_i}$ with $K_i \geq N$ such that

$$\|[x,p]\| < \eta/4 \qquad for\ all\ x \in \mathcal{F}; \tag{132}$$

$$\|[pxp,y]\| < \epsilon/4 \qquad for\ all\ x \in \mathcal{F}\ and\ y \in B\ with\ \|y\| \leq 1 \tag{133}$$

and

$$\tau(1-p) < \eta/2n \qquad for\ all \tau \in T(C). \tag{134}$$

Write $K_i = k_i n + r_i$ with $k_i \geq 1$ and $0 \leq r_i < n$ for some integers k_i and $r_i, i = 1,2,\ldots,L$. Let $p \in B$ be a projection such that the rank of p is k_i in each summand MK_i of B. Take $D_1 = pBp$.

We have

$$\|[x,p]\| < \frac{\epsilon}{2} \qquad for\ all\ x \in \mathcal{F}; \tag{135}$$

$$\|[pxp,y]\| < \epsilon \qquad for\ all\ x \in \mathcal{F}\ and\ y \in D_1\ with\ \|y\| \leq 1 \tag{136}$$

and

$$\tau(1-p) < \frac{\eta}{2n} + \frac{n}{N} < \frac{\eta}{2n} + \frac{\eta}{2} < \eta \qquad for\ all \tau \in T(C). \tag{137}$$

Note that there is a unital C^*-subalgebra $D \subset D_1$ such that $D \cong M_n$.

Lemma (3.1.19)[84]:

Let $n \geq 1$ be an integer with $n \geq 64$. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$. Then, for any $\epsilon > 0$, there exist a unitary $v \in A$ and a continuous path of unitaries $\{w(t): t \in [0,1]\} \subset A$ such that

$$\|[x,w(t)]\| < \epsilon \quad for\ all\ f \in \mathcal{F}\ and\ for\ all\ t \in [0,1], \tag{138}$$

$$w(0) = 1, \quad w(1) = v \tag{139}$$

and

$$\mu_{\tau\circ l}(I_a) \geq \frac{15}{24n^2} \tag{140}$$

for all open arcs I_a of $\mathbb{T}$ with $length\ a \geq 4\pi/n$ and for all $\tau \in T(A)$, where $l: C(\mathbb{T}) \to A$ is defined by $l(f) = f(vu)$. Moreover,

$$length\ (\{w(t)\}) \leq \pi. \tag{141}$$

If, in addition, $\pi > b_1 > b_2 > \cdots > b_m > 0$ and $1 = d_0 > d_1 > d_2 > \cdots > d_m > 0$ are given so that

$$\mu_{\tau\circ l_0}(I_{b_i}) \geq d_i \quad for\ all\ \tau \in T(A), \qquad I = 1,2,\ldots,m, \tag{142}$$

where $l_0: C(\mathbb{T}) \to A$ is defined by $l_0(f) = f(u)$ for all $f \in C(\mathbb{T})$, then one also has that

$$\mu_{\tau\circ l}(I_{c_i}) \geq (1-\epsilon_1)d_i \quad for\ all\ \tau \in T(A), \tag{143}$$

where I_{b_i} and I_{c_i} are any open arcs with length b_i and c_i, respectively, and where $c_i = b_i + 1$, $i = 1, 2, \ldots, m$.

Proof:

Let $\epsilon > 0$, and let $n \geq 64$ be an integer. Put $\epsilon_2 = min\{\epsilon_1/16, 1/64n^2\}$. Let $\mathcal{F} \subset A$ be a finite subset and let $u \in U(A)$. Let $\delta_1 > 0$ (in place of δ) for , ϵ, ϵ_2 (in place of ϵ_1) and let $G = \{e_{i,j}\} \subset D \cong M_n$.

Put $\delta = \delta_1/16$, there is a projection $p \in A$ and a C^*-subalgebra $D \cong M_n$ with $1_D = p$ such that

$$\|[x,p]\| < \delta \quad for\ all\ x \in \mathcal{F}; \tag{144}$$

$$\|[pxp,y]\| < \delta \quad for\ all\ x \in \mathcal{F}\ and\ y \in D\ with\ \|y\| \leq 1; \tag{145}$$

and

$$\tau(1-p) < \epsilon_2 \quad for\ all\ \tau \in T(C). \tag{146}$$

There is a unitary $u_0 \in (1-p)A(1-p)$ and a unitary $u_1 \in pAp$. Put $A_1 = pAp$ and $\mathcal{F}_1 = \{pxp: x \in \mathcal{F}\}$. The $A_1, \mathcal{F}_1\ and\ u_1$.

Lemma (3.1.20)[84]:

Let $n \geq 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1, X$ is a compact metric space, $\phi: C(X) \to A$ is a unital homomorphism, $\mathcal{F} \subset C(X)$ is a finite subset and suppose that $u \in U(A)$. Suppose also that, for some $\sigma > 0$ and $1 > b > 0$,

$$\tau(\phi(f)) \in \sigma \quad for\ all\ \tau \in T(A) \quad and \tag{147}$$

for all $f \in C(\mathbb{T})$ with $0 \leq f \leq 1$ whose supports contain an open ball with radius at least b. Then, there exist a unitary $v \in A$ and a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset A$ such that $v(0) = 1, v(1) = v$,

$$\|[\phi(f), v(t)]\| < \epsilon \quad and \quad \|[u, v(t)]\| < \epsilon \quad for\ all\ f \in \mathcal{F}\ and\ t \in [0,1] \tag{148}$$

$$\tau(\phi(f)g(vu)) \geq \frac{15\sigma}{24n^2} \qquad for\ all\ \tau \in T(A) \tag{149}$$

for any $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball of radius at least $2b$ and any $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc of $\mathbb{T}$ with length $a \geq 8\pi/n$.

Moreover,

$$length(\{v(t)) \leq \pi. \tag{150}$$

If, in addition, $1 > b_1 > b_2 > \cdots > b_k > 0, 1 > d_1 > d_2 > \cdots > d_k > 0$ are given and

$$\tau(\phi(f')g'(u)) \geq d_i \quad for\ all\ \tau \in T(A) \tag{151}$$

for any functions $f' \in C(X)$ with $0 \leq f' \leq 1$ whose support contains an open ball with radius $b_i/2$ and any function $g' \in C(\mathbb{T})$ with $0 \leq g' \leq 1$ whose support contains an arc with $length\ b_i$, then one also has that

$$\tau(\phi(f'')g''(u)) \geq (1-\epsilon_1)d_i \quad for\ all\ \tau \in T(A) \tag{152}$$

where $f'' \in C(X)$ with $0 \leq f'' \leq 1$ whose support contains an open ball with radius c_i and $g'' \in C(\mathbb{T})$ with $0 \leq g''$ whose support contains an arc with length $2c_i$, where $c_i = b_i + 1, i = 1, 2, \ldots, k$.

Define

$$\Delta_{00}(r) = \frac{1}{2(n+1)^2} \quad if\ 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \leq \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{153}$$

for $n \geq 64$ and

$$\Delta_{00}(r) = \frac{1}{2(65)^2} \quad if\ r \geq \frac{8\pi}{64} + \frac{4\pi}{2^{65}(64)}. \tag{154}$$

Let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. Define

$$if\ 0 < \frac{8\pi}{n+1} + \frac{4\pi}{2^{n+2}(n+1)} < r \leq \frac{8\pi}{n} + \frac{4\pi}{2^{n+1}n} \tag{155}$$

for $n \geq 64$ and

$$\mathrm{D}_0(\Delta)(\mathrm{r}) = \mathrm{D}_0(\Delta)(4\pi/64) \ \text{ if r} \geq \frac{8\pi}{64} + \frac{4\pi}{2^{65}(64)}. \tag{156}$$

Lemma (3.1.21)[84]:

Suppose that A is a unital separable simple C^*-algebra with $TR(A) \leq 1$, suppose that $\mathcal{F} \subset A$ is a finite subset and suppose that $u \in U(A)$. For any $\epsilon > 0$ and any $\eta > 0$, there exist a unitary $v \in U_0(A)$ and a continuous path of unitaries $\{w(t): t \in [0,1]\} \subset U_0(A)$ such that

$$w(0) = 1, \quad w(1) = v, \quad \|[f, w(t)]\| < \epsilon \quad for\ all\ f \in \mathcal{F}\ and\ t \in [0,1], \tag{157}$$

and

$$\mu_{\tau \circ l}(I_a) \geq \Delta_{00}(a) \quad for\ all\ \tau \in T(A) \tag{158}$$

for any open arc I_a with length $a \geq \eta$, where $l: C(\mathbb{T}) \to A$ is defined by $l(g) = g(vu)$ for all $g \in C(\mathbb{T})$ and Δ_{00}.

Corollary (3.1.22)[84]: Let C be a unital separable simple amenable C^*-algebra with $TR(C) \leq 1$ which satisfies the UCT. Let $\epsilon > 0$, $\mathcal{F} \subset C$ be a finite subset and let $1 > \eta > 0$.

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi: C \to A$ is a unital homomorphism and $u \in U(A)$ is a unitary with

$$\|\phi(c), u\| < \epsilon \quad for\ all\ c \in \mathcal{F}. \tag{159}$$

Then there exist a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset U(A)$ such that

$$u(0) = u, \quad u(1) = w \quad and \quad \|\phi(f), u(t)\| < 2\epsilon \tag{160}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$. Moreover, for any open arc I_a with $length\ a$,

$$\mu_{\tau \circ l}(I_a) \geq \Delta_{00}(r) \quad for\ all\ a \geq \eta, \tag{161}$$

where $l: C(\mathbb{T}) \to A$ is defined by $l(f) = f(w)$ for all $f \in C(\mathbb{T})$.

Proof:

Let $\epsilon > 0$ and $\mathcal{F} \subset C$ be as described. Put $\mathcal{F}_1 = \phi(\mathcal{F})$. The corollary follows by taking $u(t) = w(t)u$.

Lemma (3.1.23)[84]: Let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map, let $\eta > 0$, let X be a compact metric space and let $\mathcal{F} \subset C(X)$ be a finite subset. Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, suppose that $\phi: C(X) \to A$ is a unital homomorphism and suppose that $u \in U(A)$ such that

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(r) \quad for\ all\ \tau \in T(A) \tag{162}$$

for any open ball with radius $a \leq \eta$. For any $\epsilon > 0$, there exist a unitary $v \in U_0(A)$ and a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset U_0(A)$ such that

$$v(0) = 1, \qquad v(1) = v, \tag{163}$$

$$\|\phi(f), v(t)\| < \epsilon, \qquad \|u, v(t)\| < \epsilon, \qquad for\ all\ f \in \mathcal{F}\ and\ t \in [0,1] \tag{164}$$

and

$$\tau(\phi(f)g(vu)) \geq D_0(\Delta)(a) \quad for\ all\ \tau \in T(A) \tag{165}$$

for any $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius $a \geq 4\eta$ and any $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains an open arc with length $a \geq 4\eta$, where $D_0(\Delta)$.

We will prove Theorem (3.1.25) below. We will apply the results of the previous section to produce the map L which was required by using a continuous path of unitaries.

Lemma (3.1.24)[84]: Let X be a compact metric space, let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map, let $\epsilon > 0$, let $\eta > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exist $\delta > 0$ and a finite subset $G \subset C(X)$ satisfying the following:

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, suppose that $\phi: C(X) \to A$ and suppose that $u \in U(A)$ such that

$$\|\phi(f), u\| < \delta \quad for\ all\ f \in \mathcal{G} \tag{166}$$

and

$$\mu_{\tau\circ\phi}(O_b) \geq \Delta(a) \quad for\ all\ \tau \in T) \tag{167}$$

for any open balls O_b with radius $b \geq \eta/2$. There exist a unitary $v \in U_0(A)$, a unital completely positive linear map $L: C(X \times \mathbb{T}) \to A$ and a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset U_0(A)$ such that

$$v(0) = u,\ \ v(1) = v,\ \ \|\phi(f), v(t)\| < \epsilon,\ \ for\ all\ f \in \mathcal{F}\ and\ t \in [0,1], \tag{168}$$

$$\|L(f \otimes z) - \phi(f)v\| < \epsilon,\ \ \|L(f \otimes 1) - \phi(f)\| < \epsilon \quad for\ all\ f \in \mathcal{F} \tag{169}$$

and

$$\mu_{\tau\circ L}(O_a) \geq (2/3)D_0\Delta\left(\frac{a}{2}\right) \quad for\ all\ \tau \in T) \tag{170}$$

for any open balls O_a of $X \times \mathbb{T}$ with radius $a \geq 5\eta$.

Proof:

Fix $\epsilon > 0, \eta > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\mathcal{F}_1 \subset C(X)$ be a finite subset containing $\mathcal{F}$. Let $0 = min\{\epsilon/2, D_0(\Delta)(\eta)/4\}$. Let $\mathcal{G} \subset C(X)$ be a finite subset containing $\mathcal{F}$, $1_{C(X)}$ and z. There is $\delta_0 > 0$ such that there is a unital completely positive linear map $L': C(X \times \mathbb{T}) \to B$ (for unital C^*-algebra B) satisfying the following:

$$\|L'(f \otimes z) - \phi'(f)u'\| < \epsilon_0 \quad for\ all\ f \in \mathcal{F}_1 \tag{171}$$

for any unital homomorphism $\phi': C(X) \to B$ and any unitary $u' \in B$ whenever

$$\|[\phi'(g), u']\| < \delta_0 \quad for\ all\ g \in \mathcal{G}. \tag{172}$$

Let $0 < \delta < min\{\delta_0/2, \epsilon/2, \epsilon_0/2\}$ and suppose that

$$\|[\phi(g), u]\| < \delta \qquad for\ all\ g \in \mathcal{G}. \tag{173}$$

It follows that there is a continuous path of unitaries $\{z(t): t \in [0,1]\} \subset U_0(A)$ such that

$$z(0) = 1,\ \ z(1) = v_1, \tag{174}$$

$$\|[\phi(f), z(t)]\| < \frac{\delta}{2} \qquad \|[u, z(t)]\| < \frac{\delta}{2} \quad for\ all\ t \in [0,1] \tag{175}$$

and

$$\tau(\phi(f)g(v_1u)) \geq D_0(\Delta)(a) \tag{176}$$

for any $f \in C(X)$ with $0 \leq f \leq 1$ whose support contains an open ball with radius 4η and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ whose support contains open arcs with length $a \geq 4\eta$.

Put $v = v_1u$. Then we obtain a unital completely positive linear map $L: C(X \times \mathbb{T}) \to A$ such that

$$\|L(f \otimes z) - \phi(f)v\| < \epsilon_0\ and\ \|L(f \otimes 1) - \phi(f)\| < \epsilon_0\ for\ all\ f \in \mathcal{F}_1. \tag{177}$$

If $\mathcal{F}_1$ is sufficiently large (depending on η only), we may also assume that

$$\mu_{\tau\circ L}(B_a \times J_a) \geq \left(\frac{2}{3}\right) D_0\Delta\left(\frac{a}{2}\right) \tag{178}$$

for any open ball B_a with radius a and open arcs with $length\ a$, where $a \geq 5\eta$.

Theorem (3.1.25)[84]:

Let X be a finite CW complex so that $X \times \mathbb{T}$ has the property (H). Let $C = PC(X, M_n)P$ for some projection $P \in C(X, M_n)$ and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0, \eta > 0$ and there exists a finite subset $\mathcal{G} \subset C$ satisfying the following:

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi: C \to A$ is a unital homomorphism and $u \in A$ is a unitary and suppose that

$$\|[\phi(c), u]\| < \delta\ \ for\ all\ c \in \mathcal{G}\ \ and\ \ Bott(\phi, u) = \{0\}. \tag{179}$$

Suppose also that

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \tag{180}$$

for all open balls O_a of X with radius $1 > a \geq \eta$, where $\mu_{\tau\circ\phi}$ is the Borel probability measure defined by restricting ϕ on the center of C. Then there exists a continuous path of unitaries $\{u(t): t \in [0,1]\}$ in A such that

$$u(0) = u, \quad u(1) = 1\ \ and\ \ \|[\phi(c), u(t)]\| < \epsilon \tag{181}$$

For all $c \in \mathcal{F}$ and for all $t \in [0,1]$.

Proof:

First it is easy to see that the general case can be reduced to the case that $C = C(X, M_n)$. It is then easy to see that this case can be further reduced to the case that $C = C(X)$.

Corollary (3.1.26)[84]:

Let $k \geq 1$ be an integer, let $\epsilon >, 0$ and let $\Delta: (0,1) \to (0,1)$ be any nondecreasing map. There exist $\delta > 0$ and $\eta > 0$ (η does not depend on Δ) satisfying the following:

For any k mutually commutative unitaries $u_1u_2, \ldots, u_k$ and a unitary $v \in U(A)$ in a unital separable simple C^*-algebra A with tracial rank no more than one for which

$$\|[u_i, v]\| < \delta,\ \ bott_j(u_i, v) = 0,\ \ j = 0,1, \qquad i = 1,2, \ldots, k,$$

and

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \quad for\ all\ \tau \in T(A),$$

for any open ball O_a with radius $a \geq \eta$, where $\phi : C(\mathbb{T}^k) \to A$ is the homomorphism defined by $\phi(f) = f\ (u_1, u_2, \ldots, u_k)$ for all $f \in C(\mathbb{T}^k)$, there exists a continuous path of unitaries $\{v(t): t \in [0,1]\} \subset A$ such that $v(0) = v, v(1) = 1$ and

$$\|[u_i, v(t)]\| < \epsilon \quad for\ all\ t \in [0,1], \qquad i = 1,2,\dots,k.$$

Section (3.2) Result of Equivalence Approximate Unitary with Tracial Rank One

Theorem (3.2.1)[84]:

Let C be a unital separable amenable C^*-algebra satisfying the UCT. Let $b \geq 1$, let $T: \mathbb{N}^2 \to \mathbb{N}, L: U(M_\infty(C)) \to \mathbb{R}_+, E: \mathbb{R}_+ \times \mathbb{N} \to \mathbb{R}_+$ and $T_1 = N \times K: C_+\backslash\{0\} \to \mathbb{N} \times \mathbb{R}_+\backslash\{0\}$ be four maps. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset C$, a finite subset $\mathcal{H} \subset C_+\backslash\{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$, a finite subset $\mathcal{U} \subset U(M_\infty(C))$, an integer $l > 0$ and an integer $k > 0$ satisfying the following:

For any unital C^*-algebra A with stable rank one, K_0-divisible rank T, exponential length divisible $rank\, E$ and $cer(M_m(A))$ b (for all m), if $\phi, \psi: C \to A$ are two unital δ-Gmultiplicative contractive completely positive linear maps with

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \ \ and\ cel\big(\langle\phi\rangle(u)^*\langle\psi\rangle(u)\big) \leq L(u) \tag{182}$$

for all $u \in U$, then for any unital $\delta - \mathcal{G}$-multiplicative contractive completely positive linear map $\theta: C \to M_l(A)$ which is also $T - \mathcal{H}$-full, there exists a unitary $u \in M_{lk+1}(A)$ such that

$$\left\| u^* diag\Big(\phi(a), \overbrace{\theta(a), \theta(a), \dots, \theta(a)}^{k}\Big) u - diag\Big(\psi(a), \overbrace{\theta(a), \theta(a), \dots, \theta(a)}^{k}\Big) \right\| < \epsilon \quad for\ all\ a \in \mathcal{F}. \tag{183}$$

Theorem (3.2.2)[84]: Let C be a unital separable simple amenable C^*-algebra with $TR(C) \leq 1$ satisfying the UCT and let $D = C \otimes C(\mathbb{T})$. Let $T = N \times K: D_+\backslash\{0\} \to \mathbb{N}_+ \times \mathbb{R}_+\backslash\{0\}$.

Then, for any > 0 and any finite subset $\mathcal{F} \subset D$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset D$, a finite subset $\mathcal{H} \subset D_+\backslash\{0\}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$ and a finite subset $U \subset U(D)$ satisfying the following: Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$ and $\phi, \psi: D \to A$ are two unital δ- $\mathcal{G}$-multiplicative contractive completely positive linear maps such that ϕ, ψ are T - $\mathcal{H}$-full,

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \ \ for\ all\ g \in \mathcal{G} \tag{184}$$

for all $\tau \in T(A)$,

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \tag{185}$$

and

$$dist\left(\phi^{\ddagger}(\overline{w}), \psi^{\ddagger}(\overline{w})\right) < \delta \tag{186}$$

for all $w \in \mathcal{U}$. Then there exists a unitary $u \in U(A)$ such that

$$and\ u \circ \psi \approx_{\epsilon} \phi \ \ on\ \mathcal{F}. \tag{187}$$

Corollary (3.2.3)[84]:

Let C be a unital separable amenable simple C^*-algebra with $TR(C) \leq 1$ which satisfies the UCT, let $D = C \otimes C(\mathbb{T})$ and let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\phi, \psi: D \to A$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent, i.e., there exists a sequence of unitaries $\{u_n\} \subset A$ such that

$$\lim_{n\to\infty} ad\, u_n \circ \psi(d) = \phi(d) \quad for\ all\ d \in D,$$

if and only if

$$[\phi] = [\psi] \quad in\ KL(D, A),$$
$$\tau \circ \phi = \tau \circ \psi \quad for\ all\ \tau \in T(A)\ \ and\ \psi^{\ddagger} = \phi^{\ddagger}.$$

Lemma (3.2.4)[84]:

Let C be a unital separable simple C^*-algebra with $TR(C) \le 1$ and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. There exists a map $T = N \times K : D_+ \setminus \{0\} \to \mathbb{N}_+ \times \mathbb{R}_+ \setminus \{0\}$, where $D = C \otimes C(\mathbb{T})$, satisfying the following:

For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset C$ and any finite subset $\mathcal{H} \subset D_+ \setminus \{0\}$, there exist $\delta > 0, \eta > 0$ and a finite subset $\mathcal{G} \subset C$ satisfying the following: for any unital separable unital simple C^*-algebra A, any unital homomorphism $\phi: C \to A$ and any unitary $u \in A$ such that

$$\|[\phi(c), u]\| < \delta \quad for\ all\ c \in \mathcal{G} \tag{188}$$

and

$$\mu_{\tau\circ l}(O_a) \ge \Delta(a) \quad for\ all\ \tau \in T(A) \tag{189}$$

and for all open balls O_a with radius $a \ge \eta$, where $l: C(\mathbb{T}) \to A$ is defined by $l(f) = f(u)$, there is a unital completely positive linear map $L: D \to A$ such that

$$\|L(c \otimes 1) - \phi(c)\| < \epsilon \|L(c \otimes z) - \phi(c)u\| < \epsilon \quad for\ all\ c \in \mathcal{F} \tag{190}$$

and L is $T - \mathcal{H}$-full.

Proof:

We identify D with $C(\mathbb{T}, C)$. Let $f \in D_+ \setminus \{0\}$. There is positive number $b \ge 1$, $g \in D_+$ with $0 \le g \le b \cdot 1$ and $f_1 \in D_+ \setminus \{0\}$ with $0 \le f_1 \le 1$ such that

$$gfgf_1 = f_1. \tag{191}$$

There is a point $t_0 \in \mathbb{T}$ such that $f_1(t_1) \neq 0$. There is $r > 0$ such that

$$\tau(f_1(t)) \ge \tau(f_1(t_0))/2$$

for all $\tau \in T(C)$ and for all t with $dist(t, t_0) < r$.

Define $\Delta_0(f) = inf\{\tau(f_1(t_0))/4 : \tau \in T(C)\} \cdot (r)$. There is an integer $n \ge 1$ such that

$$n. \Delta_0(f) > 1. \tag{192}$$

Define $T(f) = (n, b)$. Put

$$\eta = inf\{\Delta_0(f) : f \in \mathcal{H}\}/2 \quad and \quad \epsilon_1 = \min\{\epsilon, \eta\}.$$

We claim that there exists an $\epsilon_1 - \mathcal{F} \cup \mathcal{H}$-multiplicative contractive completely positive linear map $L : D \to A$ such that

$$\|L(c \otimes 1) - \phi(c)\| < \epsilon \quad for\ all\ c \in \mathcal{F} \quad \|L(1 \otimes z) - u\| < \epsilon \tag{193}$$

and

$$\left| \tau \circ L(f_1) - \int_{\mathbb{T}} \tau\big(\phi(f_1(s))\big)\, d\mu_{\tau\circ l}(s) \right| < \eta \ \ for\ all\ \tau \in T(A) \tag{194}$$

and for all $f \in \mathcal{H}$. Otherwise, there exists a sequence of unitaries $\{u_n\} \subset U(A)$ for which $\mu_{\tau\circ l_n}(O_a) \ge \Delta(a)$ for all $\tau \in T(A)$ and for any open balls O_a with radius $a \to a_n$ with $a_n \to 0$, and for which

$$\lim_{n\to\infty} \|[\phi(c), u_n]\| = 0 \tag{195}$$

for all $c \in C$ and suppose for any sequence of contractive completely positive linear maps $L_n: D \to A$ with

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \quad for\ all\ a, b \in D, \tag{196}$$

$$\lim_{n\to\infty} \|L_n(c \otimes f) - \phi(c)f(u_n)\| = 0, \tag{197}$$

for all $c \in C, f \in C(\mathbb{T})$ and

$$\liminf_n \left\{ \max \left\{ \left| \tau \circ L_n(f_1) - \int_{\mathbb{T}} \tau\big(\phi(f_1(s))\big)\, d\mu_{\tau \circ l_n}(s) \right| : f \in \mathcal{H} \right\} \right\} \geq \eta \tag{198}$$

for some $\tau \in T(A)$, where $l_n: C(\mathbb{T}) \to D$ is defined by $l_n(f) = f(u_n)$ for $f \in C(T)$ (or no contractive completely positive linear maps L_n exists so that (196), (197) and (197)).

Put $A_n = A$, $n = 1, 2, \ldots$, and $Q(A) = \prod_n A_n / \bigoplus_n A_n$. Let $\pi: \prod_n A_n \to Q(A)$ be the quotient map. Define a linear map $L': D \to \prod_n A_n$ by $L(c \otimes 1) = \{\phi(c)\}$ and $L'(1 \otimes z) = \{u_n\}$. Then $\pi \circ L': D \to Q(A)$ is a unital homomorphism. It follows from a theorem of Effros and Choi [69] that there exists a contractive completely positive linear map $L: D \to \prod_n A_n$ such that $\pi \circ L = \pi \circ L'$. Write L$= \{L_n\}$, where $L_n: D \to A_n$ is a contractive completely positive linear map. Note that

$$\lim_{n\to\infty} \|L_n(a)L_n(b) - L_n(ab)\| = 0 \quad for\ all\ ab \in D.$$

Fix $\tau \in T(A)$, define $t_n: \prod_n A_n \to \mathbb{C}$ by $t_n(\{d_n\}) = \tau(d_n)$. Let t be a limit point of $\{t_n\}$. Then t gives a state on $\prod_n A_n$. Note that if $\{d_n\} \in \bigoplus_n A_n$, then $t_m(\{d_n\}) \to 0$. It follows that t gives a state $\bar{t}$ on $Q(A)$. Note that (by (267))

$$\bar{t}(\pi \circ L(c \otimes 1)) = \tau(\phi(c))$$

for all $c \in C$. It follows that

$$\begin{aligned} \bar{t}(\pi \circ L(f)) &= \int_{\mathbb{T}} \bar{t}(\pi \circ L(f(s) \otimes 1))\, d\mu_{\bar{t} \circ \pi \circ L|_{t \otimes C(\mathbb{T})}} \\ &= \int_{\mathbb{T}} \tau\left(\phi\big(f(s)\big)\right) d\mu_{\bar{t} \circ \pi \circ L|_{t \otimes C(\mathbb{T})}} \end{aligned} \tag{199}$$

for all $f \in C(\mathbb{T}, C)$. Therefore, for a subsequence $\{n(k)\}$,

$$\left| \tau \circ L_n(f_1) - \int_{\mathbb{T}} \tau\left(\phi\big(f(s)\big)\right) d\mu_{\bar{t} \circ \pi \circ L|_{t \otimes C(\mathbb{T})}} \right| < \frac{\eta}{2} \tag{200}$$

for all $f \in \mathcal{H}$. This contradicts with (268). Moreover, from this, it is easy to compute that

$$\mu_{\bar{t} \circ \pi \circ L|_{t \otimes C(\mathbb{T})}}(O_a) \geq \Delta(a)$$

for all open balls O_a of t with radius $1 > a$. This proves the claim.

Note that

$$\int_{\mathbb{T}} \tau \circ \phi\big(f_1(s)\big) d\mu_{\tau \circ l} \geq \left(\tau\big(\phi(f_1(t_0)/2)\big)\right) . \Delta(r)$$

for all $\tau \in T(A)$. It follows that

$$\tau\big(L(f_1)\big) \geq \inf\big\{t\big(f_1(t_0)\big)/2 : t \in T(C)\big\} - \frac{\eta}{2} \geq \left(\frac{4}{3}\right)\Delta_0(f) \tag{201}$$

for all $f \in \mathcal{H}$.

In [22], there exists a projection $e \in \overline{L(f_1)AL(f_1)}$ such that

$$\tau(e) \geq \Delta_0(f) \quad \text{for all } \tau \in T(A). \tag{202}$$

It follows from (262) that there exists a partial isometry $w \in M_n(A)$ such that

$$w^* diag\left(\overbrace{e, e, \dots, e}^{n}\right) w \geq 1_A.$$

Thus there $x_1, x_2, \dots, x_n \in A$ with $\|x_i\| \leq 1$ such that

$$\sum_{i=1}^{n} x_i^* e x_i \geq 1. \tag{203}$$

Hence

$$\sum_{i=1}^{n} x_i^* g f g x_i \geq 1. \tag{204}$$

It then follows that there are $y_1, y_2, \dots, y_n \in A$ with $\|y_i\| \leq b$ such that

$$\sum_{i=1}^{n} y_i^* f y_i = 1. \tag{205}$$

Therefore L is T - $\mathcal{H}$ -full.

Lemma (3.2.5)[84]:

Let C be a unital separable amenable simple C^*-algebra with $TR(C) \leq 1$ satisfying the UCT. For $1/2 > \sigma > 0$, any finite subset $\mathcal{G}_0$ and any projections $p_1, p_2, \dots, p_m \in C$. There is $\delta_0 > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset of projections $P_0 \subset C$ satisfying the following: Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi: C \to A$ is a unital homomorphism and $u \in U_0(A)$ is a unitary such that

$$\|\phi(c), u\| < \delta < \delta_0 \ \ for\ all\ c \in \mathcal{G} \cup \mathcal{G}_0 \ \ and\ \ bott_0(\phi, u)|_{\mathcal{P}_0} = \{0\}. \tag{206}$$

where $\mathcal{P}_0$ is the image of $\mathcal{P}_0$ in $K_0(C)$. Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A with $u(0) = u$ and $u(1) = w$ such that

$$\|\phi(c), u\| < 3\delta \quad for\ all\ c \in \mathcal{G} \cup \mathcal{G}_0 \tag{207}$$

and

$$w_j \oplus \big(1 - \phi(p_j)\big) \in CU(A), \tag{208}$$

where $w_j \in U_0(\phi(p_j)A\phi(p_j))$ and

$$\big\|w_j - \phi(p_j) w \phi(p_j)\big\| < \sigma, \tag{209}$$

$j = 1, 2, \dots, m$.

Moreover,

$$cel\Big(w_j \oplus \big(1 - \phi(p_j)\big)\Big) \leq 8\pi + \frac{1}{4}, \qquad j = 1,2, \dots, m. \tag{210}$$

Lemma (3.2.6)[84]:

Let C be a unital separable simple amenable C^*-algebra with $TR(C) \leq 1$ satisfying the UCT. Let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0, \eta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset \underline{K}(C)$ satisfying the following:

For any unital simpleC^*-algebra A with $TR(A) \leq 1$, any unital homomorphism $\phi: C \to A$ and any unitary $u \in U(A)$ with

$$\|\phi(f), u\| < \delta, \quad Bott(\phi, u)|_{\mathcal{P}} = \{0\} \tag{211}$$

and

$$\mu_{\tau \circ l}(O_a) \geq \Delta(a) \quad for\ all\ a \geq \eta, \tag{212}$$

where $l: C(\mathbb{T}) \to A$ is defined by $l(f) = f(u)$ for all $f \in C(\mathbb{T})$, there exists a continuous path of unitaries $\{u(t): t \in [0,1]\} \subset A$ such that

$$u(0) = u, \;\; u(1) = 1 \;\; and \;\; \|\phi(f), u(t)\| < \epsilon \tag{213}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$.

Theorem (3.2.7)[84]:

Let C be a unital separable amenable simple C^*-algebra with $TR(C) \leq 1$ which satisfies the UCT. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset C$ and a finite subset $\mathcal{P} \subset K(C)$ satisfying the following:

Suppose that A is a unital simple $C^* -$algebra with $TR(C) \leq 1$, suppose that $\phi: C \to A\ is\ a$ unital homomorphism and $u \in U(A)$ such that

$$\|[\phi(c), u]\| < \delta\ for\ all\ c \in \mathcal{G}\ and\ \mathrm{Bott}(\phi, u)|_{\mathcal{P}} = 0. \tag{214}$$

Then there exists a continuous and piece-wise smooth path of unitaries $\{u(t): t \in [0,1]\}$ such that

$$u(0) = u, \;\; u(1) = 1 \;\; and \;\; \|[\phi(c), u(t)]\| < \epsilon\ for\ all\ c \in \mathcal{F} \quad and \;\; for\ all \;\; t \in [0,1] \tag{215}$$

Proof:

Fix $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C$. Let $\delta_1 > 0$ (in place of δ), $\eta > 0, \mathcal{G}_1 \subset C$ (in place of $\mathcal{G}$ be a finite subset and $\mathcal{P} \subset \underline{K}(C)$ be finite subset ,for $\epsilon, \mathcal{F}$ and $\Delta = \Delta_{00}$.

We may assume that $\delta_1 < \epsilon$.

Let $\delta = /2$. Suppose that ϕ and u satisfy the conditions in the theorem for the above $\delta, \mathcal{G}$ and $\mathcal{P}$. It follows that there is a continuous path of unitaries $\{v(t): t \in [\delta_1 0,1]\} \subset U(A)$ such that

$$v(0) = u, \;\; v(1) = u_1 \;\; and \;\; \|[\phi(c), v(t)]\| < \delta_1 \tag{216}$$

for all $c \in \mathcal{G}_1$ and for all $t \in [0,1]$, and

$$\mu_{\tau \circ \iota}(O_a) \geq \Delta(a) \;\; \text{for all} \;\; \tau \in T(A) \tag{217}$$

and for all open balls of radius $a \geq \eta$.

There is a continuous path of unitaries $\{w(t): t \in [0,1]\} \subset A$ such that

$$w(0) = u_1, \;\; v(1) = 1 \;\; and \;\; \|[\phi(c), w(t)]\| < \epsilon \tag{218}$$

for all $c \in \mathcal{F}$ and $t \in [0,1]$. Put

$$u(t) = v(2t) \text{ for all } t \in [0,1/2)\ and\ u(t) = w(2t - 1/2) \text{ for all } t \in [1/2,1].$$