Chapter 2
Exponential Rank and Approximate Unitary Equivalence in Simple C^*-Algebras

Let $CU(A)$ be the closure of the commutator subgroup of the unitary group of A and let $u \in CU(A)$. We prove that there exists aself-adjoint element $h \in A$ such that $\|u - \exp(ih)\| < \epsilon$ and $\|h\| \leq 2\pi$. Examples are given that the bound 2π for $\|h\|$ is optimal in general and let C be a unital AH-algebra and A be a unital separable simple C^*-algebra with tracial rank no more than one. Suppose that $\phi, \psi: C \to A$, are two unital monomorphisms. With some restriction on C.

Section (2.1): Exponential Rank and Exponential Length for Z-Stable Simple C*-Algebras

Let A be a unital C^*- algebra and let $U_0(A)$ be the connected component of unitary group of Acontaining the identity. It is well known that every $u \in U_0(A)$. Is a finite product of exponentials, that is $u = \prod_{k=1}^{n} \exp(ih_k)$, where $h_k$is a self-adjoint element in$A$. One of the interesting questions that one can ask about $U_0(A)$ is " are all its element expressible as single exponentials? ".Or moer interesting "are singl exponentials dense in$U_0(A)$?". If $u \in U_0(A)$ one may also ask "how long the length of the path connecting u to the identity" the first questions concern the exponential rank of A and the last questions is related to the exponential length$u$ and A.

Exponential rank and exponential length had been extensively studied see [121], [114], [109], [113], [145], [143], [47], [111], [110], [79], [112] and [132].

Exponential length and rank have played, inevitably, important roles in the study of structure of C^*-algebras, in particular, in the Elliott program, the classification of amenable C^*-algebras by K-theoretic invariant. The renew interest and direct motivation of this study is the recent research project to study the Jiang-Su algebra and its multiplier algebra. It turns out that exponential length again plays an essential role.

Let us briefly summarize some facts about exponential rank and length for unital (simpleand amenable)C^*−algebras in the center of the Elliott program. It was shown by N. C. Phillips([113]) that the exponential rank of a unital purely infinite simple C^*−algebra is $1 + \epsilon$ and itsexponential length isπ. In fact, this holds for any unital C^*−algebras of real rank zero ([79]). Inother words, if$u \in U_0(A)$, where $A$is a unital C^*−algebra of real rank zero, then, for any$\epsilon > 0$, there exists a self-adjoint element $h \in A$with$\|h\| \leq \pi$such that

$$\|u - \exp(ih)\| < \epsilon.$$

Here π is the smallest numbers that one can get. When $A$is not of real rank zero, the situationis very different. For example, if $A$is a unital simple AH-algebra with slow dimension growth,then $\mathrm{cer}(A) = 1 + \epsilon$. But$n$ $\mathrm{cel}(A) = \infty$, whenever Adoes not have real rank zero (see [112]). Recently it was shown ([94]) that $cer(A) \leq 1 + \epsilon$ for any unital simple C^*−algebra $A$with tracial rank at most one (without assuming the amenability).

The classification of unital simple amenable C^*−algebras now includes classes of C^*−algebra far beyond C^*−algebras mentioned above. In fact unital separable simple amenable Z−stable C^*−algebras which are rationally tracial rank at most one and satisfy the UCT can be classified by the Elliott invariant ([73]), we show that, if A is Z−stable, i.e., $A \otimes Z \cong A$, has rational tracial rank at most one, i.e., $A \otimes U$ has tracial rank at most one for some infinite dimensional UHF algebra U, and $u \in U_0(A)$, then, for any $\epsilon > 0$, there exists a self-adjoint

element $h \in A$ such that

$$\|u - \exp(ih)\| < \epsilon. \tag{1}$$

However, in general, there is no control of the norm of h. In fact, $cel(A) = \infty$, i.e., the exponential length of A is infinite.

In the study of classification of simple amenable C^*−algebras, one relies on a fact that exponential length for unitaries in $CU(A)$, the closure of the commutator subgroup of $U_0(A)$ isoften bounded. It seems to suggest that, for exponential length of a unitalC^*−algebra, it is theexponential length of unitaries in $CU(A)$ that needs to be computed. So the question is what isthe norm bound for the above $h$when $u$is in $CU(A)$. We show that, if $A$is a unital separablesimple C^*−algebra with tracial rank at most one, and $u \in CU(A)$,then (1) holds and $h$can be chosen so that$\|h\| \leq 2\pi$. Furthermore, we also prove this holds for any unital separablesimple Z−stable C^*−algebra Asuch that $A \otimes U$hastracial rank at most one. We struggled at first to reduce this bound to π but eventually realized that we were not facing technical difficulty in the proof but that 2π is indeed

We show in general, for a unital simple AH-algebra (or even AI-algebra) A, for any $\sigma > 0$,there are unitaries $u \in U_0(A)$ such that $\|h\| \leq 2\pi - \sigma$ holds for some sufficiently small ϵ. What is more surprising at the first was the answer to the question how long the exponential length of unitaries in $U_0(Z)$ is, where $Z$is the Jiang-Su algebra, the projectionless simple ASH-algebra with $k_0(Z) = Z$ and $k_1(Z) = \{0\}$.It seems that, among experts, one expects the exponential length of $Z$to be infinite since Zdoes not have real rank zero. However, we find that $cel(Z) \leq 3\pi$. In fact, we prove that for any unitary $u \in U_0(z)$, there exists $-\pi < t < \pi$satisfying the following: for any $\epsilon > 0$, there exists a self-adjoint element $h \in Z$with $\|h\| \leq 2\pi$such that

$$\left\|e^{it}u - \exp(ih)\right\| < \epsilon.$$

Definition (2.1.1)[80]: Let A be a unital C^*−algebra. We denote by $U(A)$ the unitary group of$A$.We denote by $U_0(A)$ the connected component of $U(A)$ containing the identity and $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. If $u \in U(A)$,we use the notation $\bar{u}$for its image in $U(A)/CU(A)$.

Let $u \in U_0(A)$.Denote by $cel(u)$ the exponential length of $u$in A. In fact,

$$cel(u) = inf\left\{\sum_{k=1}^{n}\|h_k\| : u = \prod_{k=1}^{n}\exp(ih_k): h_k \in A_{s,a}\right\}.$$

Define

$$\mathrm{cel}(A) = \sup\{\mathrm{cel}(u) : u \in U_0(A)\}.$$

Define

$$\mathrm{cel}_{CU}(A) = \sup\{\mathrm{cel}(u) : u \in CU(A)\}.$$

If $u = \lim_{n\to\infty} u_n$, where $u_n = \prod_{j=1}^k \exp(ih_{n,j})$ for some self-adjoint elements $h_{n,j} \in A$.then we write

$$\mathrm{cer}(u) \leq k + \epsilon.$$

If $u = \prod_{j=1}^k \exp(ih_j)$ for some $h_1, h_2, \ldots, h_k \in A_{s,a}$, we write

$$\mathrm{cer}(u) \leq k.$$

If $\mathrm{cer}(u) \leq k + \epsilon$ but $\mathrm{cer}(u) \nleq k$, we write $\mathrm{cer}(u) = k + \epsilon$. If $\mathrm{cer}(u) \leq k$ but $\mathrm{cer}(u) \nleq (k-1) + \epsilon$, we write $\mathrm{cer}(u) = k$.

By $T(A)$, we mean the tracial state space of A and by $\mathrm{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. Let $\tau \in T(A)$. We also use τ for the trace $\tau \otimes T_r$ on $A \otimes M_n$, where T_r is the standard trace on M_n.

Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the positive homomorphisms defined by $\rho_A([p]) = \tau(p)$ for all projections $p \in M_n(A), n = 1, 2, \ldots$.

Definition(2.1.2)[80]: Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Let $u \in U_0(A)$. Suppose that $\{u(t) : t \in [0,1]\}$ is a continuous path of unitaries which is also piece-wisely smooth such that $u(0) = u$ and $u(1) = 1$. Define de la Harp-Skandalis determinant as follows:

$$Det(u) := Det\big(u(t)\big) := \int_{[0,1]} \tau\left(\frac{du(t)}{dt} w(t)^*\right) dt \quad \text{for all } \tau \in T(A) \qquad (2)$$

Note that, if $u_1(t)$ is another continuous path which is piece-wisely smooth with $u_1(0) = u$ and $u_1(1) = 1$, Then $Det((u(t)) - Det(u_1(t)) \in \rho_A(K_0(A))$. Suppose that $u, v \in U(A)$ and $uv^* \in U0(A)$. Let $\{w(t) : t \in [0,1]\} \subset U(A)$ be a piece-wisely smooth and continuous path such that $w(0) = u$ and $w(1) = v$. Define

$$R_{u,v}(\tau) = Det\big(w(t)\big)(\tau) = \int_{[0,1]} \tau\left(\frac{du(t)}{dt} w(t)^*\right) dt \quad \text{for all } \tau \in T(A)$$

Note that $R_{u,v}$ is well-defined (independent of the choices of the path) up to elements in $\rho_A(K_0(A))$.

Definition (2.1.3)[80]: Denote by $\mathbb{Q}$ the group of rational numbers. Let r be a supernatural number. Denote by M_r the UHF-algebra associated with r. Denote by $\mathbb{Q}_r$ the group $K_0(M_r)$ with order as a subgroup of $\mathbb{Q}$.

Denote by Z the Jiang-Su algebra ([55]) which is a unital separable simple ASH-algebra with $K_0(Z) = \mathbb{Z}$ and $K_1(Z) = \{0\}$. Let p, q be two relatively prime supernatural numbers of infinite type. Denote by

$$Z_{p,q} = \big\{f \in C\big([0,1], M_{pq}\big) : f(0) \in M_p \text{ and } f(1) \in M_q\big\}.$$

Here we identify M_r with $M_r \otimes 1$ as a subalgebra of M_{pq}. One may write Z as a stationary inductive limit of $Z_{p,q}$ (see [124]).

Definition (2.1.4)[80]: Let A be a unital simple C^*-algebra. We write $TR(A) = 0$ if tracial rank of A is zero. We write $TR(A) \leq 1$, if the tracial rank of A is either zero or one (see [91]).

Denote by A_0 the class of unital separable simple C^*-algebras A such that $TR(A \otimes U) = 0$ for some infinite dimensional UHF-algebra U. Note that $Z \in A_0$.

Denote by A_1 the class of unital simple separable C^*-algebras A such that $TR(A \otimes U) \leq 1$, see ([141]), ([86]), ([99]), ([100]), ([85]), ([73]) and ([95]) for some further discussion of these C^*-algebras.

Definition (2.1.5)[80]: Let A be a unital C^*-algebra and $C = C([0,1], A)$. Denote by $\pi_t: C \to A$ the point-evaluation: $\pi_t(f) = f(t)$ for all $f \in C$.

Definition (2.1.6)[80]: Let X be a compact metric space and let $\psi: C(X) \to C$ be a state. Denote by μ_ψ the probability Borel measure induced by ψ.

The following could be easily proved directly.   .

Lemma(2.1.7)[80]: Let $\epsilon > 0$. There exists $\delta > 0$ satisfying the following: Suppose that A is a unital separable simple C^*-algebra with $TR(A) \leq 1$ and suppose that $u \in U(A)$ with $sp(u) = \mathbb{T}$. Then, for any $x \in K_0(A)$ with $\|\rho_A(x)\| < \delta$, there exists a unitary $v \in A$ such that

$$\|[U,V]\| < \epsilon \quad and \quad \text{bott}_1(u,v) = x. \tag{3}$$

The following is also known and we state here for the convenience.

Lemma (2.1.8)[80]: Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Let u and v be two unitaries in A with $[u] = [v]$ in $K_1(A)$. Suppose that there is a unitary $w \in A$ such that

$$\|uw^*v\,w^*\| < 2. \tag{4}$$

Then,

$$R_{u,v}(\tau) - \frac{1}{2\pi i}\tau(\log(uw^*v\,w^*)) \in \rho_A\big(K_0(A)\big). \tag{5}$$

proof: It suffices to show that there is one piece-wisely smooth and continuous path $\{U(t): t \in [0,1]\} \in M_2(A)$ such that $U(0) = \text{diag}(u,1), U(1) = \text{diag}(v,1)$ and

$$\frac{1}{2\pi i}\int_0^1 \tau\,(U(t)'U(t)^*)dt = \frac{1}{2\pi i}\tau(\log(uw^*v\,w^*)).$$

To see this, let $h = \frac{1}{2\pi i}\tau(\log(uw^*vw^*))$. Define $U(t) = \text{diag}(u\exp(i4\pi ht),1)$ for $t \in [0,1/2]$. Define $U_1(t) = U(2t)$ for $t \in [0,1]$. Let $W = \text{diag}(w,w^*)$. Then $W = \prod_{j=1}^k \exp(i2\pi h_j)$ for some self-adjoint elements $h_1, h_2, \ldots, h_m \in M_2(A)$. Define $W(0) = 1$ and

$$W(t) = \left(\prod_{j=1}^{k-1}\exp(i2\pi h_j)\right)\exp(i2\pi mh_k t) \quad for\ all\ t \in (k-1/m, k/m], \tag{6}$$

$k = 1,2,\ldots,m$. Let $Z(t) = W(t)^*\text{diag}(v,1)W(t)$ for $t \in [0,1]$. Then $Z(t)$ is a piece-wisely smooth and continuous path with $Z(0) = \text{diag}(v,1)$ and $Z(1) = $

$W^* \text{diag}(v, 1) W$. It is straightforward to compute that the de la Harpe-Skandalis determinant

$$Det(W(t)) = 0.$$

Define $U(t) = Z(1 - 2t)$ for $t \in (1/2, 1]$ and define $U_2(t) = Z(1 - t)$ for $t \in [0, 1]$. Now $U(t)$ is a continuous and piece-wisely continuous path with $U(0) = \text{diag}(u, 1)$ and $U(1) = \text{diag}(v, 1)$. We then compute that

$$\frac{1}{2\pi} \int_0^1 \tau \left(\frac{dU(t)}{dt} U(t)^* \right) dt = Det(U(t)) \tag{7}$$

$$= Det(U_1(t)) + Det(U_2(t)) \tag{8}$$

$$= Det(U_1(t)) + 0 \tag{9}$$

$$= \frac{1}{2\pi i} \tau(\log(uw^* v w^*)) \tag{10}$$

for all $\tau \in T(A)$.

Lemma (2.1.9)[80]: Let A be a unital separable C^*−algebra of stable rank one. Suppose that $u, v \in U(A)$ with $uv^* \in CU(A)$. Then, for any $\delta > 0$, there exists $a \in A_{s.a}$ with $\|a\| < \delta$ such that

$$\hat{a} - R_{u,v} \in \rho_A(K_0(A)).$$

Proof: This follows from the fact that $R_{u,v} \in \overline{\rho_A(K_0(A))}$.

Lemma (2.1.10)[80]: Let $\epsilon > 0$ and let $\Delta: (0, 1) \to (0, 1)$ be a non-decreasing function. There exists $\delta > 0$ and $\sigma > 0$ satisfying the following: For any unital separable simple C^*−algebra A with $TR(A) \leq 1$ and $u, v \in U(A)$ such that

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \tag{11}$$

and for all arc I_a with length at least $a \geq \sigma$, where $\phi: C(\mathbb{T}) \to A$ is the homomorphism defined by $\phi(f) = f(u)$ for all $f \in C(\mathbb{T})$,

$$\|[u, v]\| < \delta \quad [v] = 0 \quad in \quad k_1(A) \quad and \quad \text{bott}_1(u, v) = 0, \tag{12}$$

there exists a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset U_0(A)$ such that

$$\|[v(t), u]\| < \epsilon \quad for \ all \ t \in [0, 1], \quad v(0) = v \ and \ v(1) = 1. \tag{13}$$

The following is a variation of a special case of 5.1 of [98].

Lemma (2.1.11)[80]: Let $\epsilon > 0$ and let $\Delta: (0, 1) \to (0, 1)$ be a non-decreasing function. There is $\delta > 0, \eta > 0, \sigma > 0$ and there is a finite subset $\mathcal{G} \subset C(T)_{s.a}$ satisfying the following: For any unital separable simple C^*−algebra A with $TR(A) \leq 1$, any pair of unitary s $u, v \in A$ sp$(u) = T$ and $[u] = [v]$ in $K_1(A)$,

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(A)$$

for all intervals I_a with length at least η, where $\phi: C(T) \to A$ is the homomorphism defined by $\phi(f) = f(u)$ for all $f \in C(\mathbb{T})$,

$$|\tau(g(u)) - \tau(g(v))| < \delta \quad \text{for all } \tau \in T(A) \tag{14}$$

and for all $g \in \mathcal{G}$,

$$uv^* \in CU(A). \tag{15}$$

and for any $a \in \text{Aff}(T(A))$ with $a - R_{u,v} \in \rho_A(K_0(A))$ and $\|a\| < \sigma$ and any $y \in K_1(A)$, there is a unitary $w \in A$ such that $[w] = y$,

$$\|u - w^*v\,w^*\| < \epsilon \quad and \tag{16}$$

$$= a(\tau) \text{ for all } \tau \in T(A) \tag{17}$$

proof: Let $\epsilon > 0$ and Δ be given. Choose $\epsilon > \theta > 0$ such that, $\log(u_1), \log(u_2)$ and $\log(u_1 u_2)$ are well defined and

$$\tau\left(\frac{1}{2\pi i}\tau(\log(u^*w^*vw))\log(u_1u_2)\right) = \tau(\log(u_1)) + \tau(\log(u_2)) \tag{18}$$

for all $\tau \in T(A)$ and for any unitaries u_1, u_2 such that

$$\|u_j - 1\| < \theta, \quad j = 1,2.$$

Let $\delta' > 0$ (in place of δ) be required by Lemma(**2.1.7**)for $\theta/2$ (in place of ϵ). Put $\sigma = \delta'/2$. Let $\delta > 0$ and η be required by 4.3 for $\min\{\sigma, \theta, 1\}$(in place of ϵ) and Δ. Suppose that A is a unital separable simple C^*-algebra with $TR(A) \leq 1$ and $u, v \in U(A)$ satisfy the assumption for the above δ ,η and σ. Then, there exists a unitary $z \in U(A)$ such that

$$\|u - z^*vz\| < \min\{\sigma, \theta, 1\} \tag{19}$$

Let $b = \frac{1}{2\pi i}\tau(\log(u^*z^*vz))$ Then $\|b\| < \min\{\sigma, \theta, 1\}$. $\hat{b} - R_{u,v} \in \rho_B(K_0(A))$. let $a \in \text{Aff}(T(A))$ be such that $\|a\| < \sigma$ and $a - R_{u,v} \in \rho_A(K_0(A))$ as given by the lemma. It follows that $a - \hat{b} \in \rho_A(K_0(A))$. Moreover, $\|a - \hat{b}\| < 2\sigma < \delta'$. It follows that there exists a unitary $z_1 \in A$ s uch that

$$[z_1] = -y - [z], \quad \|[u, z_1]\| < \theta/2 \text{ and bott}_1(u, z_1)(\tau) = a(\tau) + \tau(b) \tag{20}$$

for all $\tau \in T(C)$.

Define $w = zz_1^*$ Then

$$[w] = y \text{ and } \|u - w^*vw\| < \theta < \epsilon \tag{21}$$

We compute that

$$\frac{1}{2\pi i}\tau(\log(u^*w^*vw)) = \frac{1}{2\pi i}\tau(\log(u^*z_1z^*vzz_1^*)) \tag{22}$$

$$= \frac{1}{2\pi i}\tau(\log(u^*z_1uu^*z^*vzz_1^*)) \tag{23}$$

$$= \frac{1}{2\pi i}\tau(\log(z_1^* u^*z_1uu^*z^*vz)) \tag{24}$$

$$= \frac{1}{2\pi i}\left(\tau(\log(z_1u^*z_1^*u)) + \tau(\log(u^*z^*vz))\right) \tag{25}$$

$$= \text{bott}_1(u, z_1)(\tau) + \tau(b) \tag{26}$$

$$= a(\tau) \text{ for all } \tau \in T(A), \tag{27}$$

where we use the Exel's formula for bott element in the second last equality.

Lemma (2.1.12)[80]: Let $\epsilon > 0$ and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. There exists $\eta > 0, \delta > 0$ and a finite subset $G \in C(\mathbb{T})_{s.a.}$ satisfying the following: Suppose that A is a $Z-$stable unital separable simple C^*-algebra in A_1 and suppose that $u, v \in U(A)$ are two unitaries such that $sp(u) = \mathbb{T}$,

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{ for all } \tau \in T(A) \tag{28}$$

and for all arcs I_a with length at least $a \geq \eta$, where $\varphi: C(\mathbb{T}) \to A$ is defined by $\phi(f) = f(u)$ for all $f \in C(\mathbb{T})$ and

$$|\tau(g(u)) - \tau(g(v))| < \delta \quad for\ all\ \ a \in \mathcal{G} \ and\ for\ all\tau \in T(A) \quad (29)$$
$$[u] = [v]\ in\ k_1(A) \quad and \quad uv^* \in CU(A).$$

Then there exists a unitary $w \in U(A)$ such that

$$\| w^* u w - v \| < \epsilon. \quad (30)$$

Proof: We first note, by [100], that $TR(A \otimes Mr) \leq 1$ for any supernatural number r. Let $\varphi: C(\mathbb{T}) \to A$ be the monomorphism defined by $\varphi(f) = f(u)$. For any $a \in (0,1)$, denote by

$$\Delta(a) = \inf\{\mu_{r \circ \psi}(O_a);\ \tau \in T(A), I_a \text{ an open arcs of length } a \text{ in } \mathbb{T}\}.$$

Since A is simple, one has that $0 < \Delta(a) \leq 1$ (for all $a \in (0,1)$) and $\Delta(a) \to 0$ as $a \to 0$.

$$\mu_{\tau \circ \varphi}(I_a) \geq \Delta(a) \text{for all} \tau \in T(A) \quad (31)$$

and all arcs with length $a > 0$. Let $\epsilon > 0$.

Let p and q be a pair of relatively prime supernatural numbers of infinite type with $\mathbb{Q}_p + \mathbb{Q}_q = \mathbb{Q}$. Denote by M_p and M_q the UHF-algebras associated to p and q respectively. Let $\iota_r: A \to A \otimes M_r$ be the embedding defined by $\iota_r(a) = a \otimes 1$ for all $a \in A$, where r is a supernatural number. Define $u_r = \iota_r(u)$ and $v_r = \iota_r(v)$. Denote by $\varphi_r : C(\mathbb{T}) \to A \otimes M_r$ the homomorphism given by $\varphi_r(f) = f(u_r)$ for all $f \in C(\mathbb{T})$.

For any supernatural number $r = p, q$, the C^*−algebra $A \otimes M_r$ has tracial rank at most one.

Let $\delta_1 > 0$ (in place of δ) and $d_1 > 0$ (in place of σ) for $\epsilon/6$.

Without loss of generality, we may assume that $\delta_1 < \epsilon/12$ and is small enough and $\mathcal{G}$ is large enough so that $\text{bott}_1(u_1, z_j)$ and $\text{bott}_1(u_1, w_j)$ are well defined and

$$\text{bott}_1(u_1, w_j) = \text{bott}_1(u_1, z_1) + \cdots + \text{bott}_1(u_1, z_j) \quad (32)$$

if u_1 is a unitary and z_j is any unitaries with $\|[u_1, z_j]\| < \delta_1$, where $w_j = z_1 \cdots z_j, j = 1, 2, 3, 4.$

Let $\delta_2 > 0$ (in place δ) for $\delta_1/8$ (in place of ϵ).

Furthermore, one may assume that δ_2 is sufficiently small such that for any unitaries z_1, z_2 in a C^*−algebra with tracial states, $\tau\left(\frac{1}{2\pi i}\log(z_i z_j^*)\right) (i,j = 1,2,3)$ is well defined and

$$\tau\left(\frac{1}{2\pi i}\log(z_1 z_2^*)\right) = \tau\left(\frac{1}{2\pi i}\log(z_1 z_j^*)\right) + \tau\left(\frac{1}{2\pi i}\log(z_3 z_2^*)\right)$$

for any tracial state τ, whenever $\|z_1 - z_3\| < \delta_2$ and $\|z_2 - z_3\| < \delta_2$. We may further assume that $\delta_2 < \min\{\delta 1, \epsilon/6, 1\}$.

Let $\delta > 0, d_2 > 0$ (in place of η) and $\delta_3 > 0$ (in place of σ) required by (3) for δ_2 (in place of ϵ). Let $\eta = \min\{d_1, d_2\}$.

Now assume that u and v are two unitaries which satisfy the assumption of the lemma with above δ and η. Since $uv^* \in CU(A), R_{u,v} \in \rho_A(K_0(A))$. It follows that there is $a \in Aff(T(A))$ with $\|a\| < \delta_3/2$ such that $a - R_{u,v} \in \rho_A(K_0(A))$. Then the image of $a_p -$

R_{u_p,v_p} is in $\rho_A \otimes M_P(K_0(A \otimes M_P)))$, where a_p is the image of $a$under the map induced by ι_P. The same holds for q. Note that

$$\mu_{(\tau \otimes t) \circ \varphi_\tau}(I_a) \geq \Delta(a) \text{for all} \tau \in T(A) \tag{33}$$

where$t$is the unique tracial state on M_r, and for all $a > 0, r = p, q$.By Lemma (2.1.11)there exist unitaries$z_P \in A \otimes M_P$and$z_q \in A \otimes M_q$ such that

$$\|u_p - z_p^* v_p z_p\| < \delta_2 \text{and} \|u_q - z_q^* v_q z_q\| < \delta_2.$$

Moreover,

$$\tau\left(\frac{1}{2\pi i}\log(u_p^* z_p^* v_p z_p)\right) = a_p(\tau) \text{for all} \tau \in T(A_p) \text{and} \tag{34}$$

$$\tau\left(\frac{1}{2\pi i}\log(u_q^* z_q^* v_q z_q)\right) = a_q(\tau) \text{for all} \tau \in T(A_q) \tag{35}$$

We then identify $u_p, u_q$with $u \otimes 1$ and z_p and z_q with the elements in $A \otimes M_P \otimes M_q = A \otimes Q$.

In the following computation, we also identify $T(A)$ with $T(A_p), T(A_q)$,and $T(A_p)$, or $T(A_q)$with $T(A \otimes Q)$ by identify $\tau$with$\tau \otimes t$,where $t$is the unique tracial state on M_P, or M_q, or Q.

In particular,

$$a_p(\tau \otimes t) = \tau(a) \text{for all} \tau \in T(A) \text{and} \tag{36}$$

$$a_q(\tau \otimes t) = \tau(a) \text{for all} \tau \in T(A). \tag{37}$$

We compute that by the Exel formula (see Lemma (2.1.11)),

$$(\tau \otimes t)\left(\text{bott}_1(u \otimes 1, z_p^* z_q)\right) = (\tau \otimes t)\left(\frac{1}{2\pi i}\log\left(z_p^* z_q(u^* \otimes 1)\right) z_q^* z_p(u \otimes 1)\right) \tag{38}$$

$$= (\tau \otimes t)\left(\frac{1}{2\pi i}\log(z_q(u^* \otimes 1) z_q^* z_p(u \otimes 1) z_p^*)\right) \tag{39}$$

$$= (\tau \otimes t)\left(\frac{1}{2\pi i}\log\left(z_q(u^* \otimes 1) z_q^*(u \otimes 1)\right)\right) \tag{40}$$

$$+(\tau \otimes 1)\left(\frac{1}{2\pi i}\log\left((v^* \otimes 1) z_q(u \otimes 1) z_p^*\right)\right) \tag{41}$$

$$= (\tau \otimes t)\left(\frac{1}{2\pi i}\log(u_q^* z_q^* v_q z_q)\right) \tag{42}$$

$$+(\tau \otimes t)\left(\frac{1}{2\pi i}\log(u_p^* z_p^* v_p z_p)\right) \tag{43}$$

$$= \tau(a) - \tau(a) = 0 \tag{44}$$

for all$\tau \in T(A)$. It follows that

$$\tau\left(\text{bott}_1(u \otimes 1, z_p^* z_q)\right) = 0 \tag{45}$$

for all$\tau \in T(A \otimes Q)$

Let y $= \text{bott}_1(u \otimes 1, z_p^* z_q) \in kre\, \rho_{A \otimes Q}$.Since $\mathbb{Q}$, $\mathbb{Q}_P$and$\mathbb{Q}_q$are flat $\mathbb{Z}$ modules,

$$kre\, \rho_{A \otimes Q} = kre\, \rho_A \otimes \mathbb{Q} \tag{46}$$

$$kre \, \rho_{A \otimes M_r} = kre \, \rho_A \otimes \mathbb{Q}_r \quad r = p \quad amd \quad r = q \tag{47}$$

Itfollows that there are $x_1, x_2, \ldots, x_l \in \rho_A(k_0(A))$ and $r_1, r_2, \ldots, r_l \in \mathbb{Q}$ such that

$$y = \sum_{j=1}^{l} x_j \otimes r_j$$

Since $\mathbb{Q} = \mathbb{Q}_p + Q_q$, one has $r_{j,p} \in \mathbb{Q}_p$ and $r_{j,q} \in \mathbb{Q}_q$ such that $r_j = r_{j,p} - r_{j,q}$. So

$$y = \sum_{j=1}^{l} x_j \otimes r_{j,p} - \sum_{j=1}^{l} x_j \otimes r_{j,q}$$

Put $y_p = \sum_{j=1}^{l} x_j \otimes r_{j,p}$ and $y_q = \sum_{j=1}^{l} x_j \otimes r_{j,q}$. Then, by (47), $y_p \in kre \, \rho_{A \otimes M_p}$ and $y_q \in kre \, \rho_{A \otimes M_q}$. It follows that there are unitaries $w_p \in A \otimes M_p$ and $w_q \in A \otimes M_q$ such that

$$\|[u_p, w_p]\| < \delta_1/8, \quad \|[u_q, w_q]\| < \delta_1/8 \tag{48}$$

$$\text{bott}_1(u_p, w_p) = y_p \quad and \quad \text{bott}_1(u_q, w_q) = y_q \tag{49}$$

Put $W_p = z_p w_p \in A \otimes M_p$ and $W_q = z_q w_q \in A \otimes M_q$. Then

$$\|u_p - W_p^* v_p w_p\| < \delta_2 + \delta_1/8 < \epsilon/6 \quad \text{and}$$

$$\|u_p - W_p^* v_p w_p\| < \delta_2 + \delta_1/8 < \epsilon/6 \tag{50}$$

Note, again, that $u_r = u \otimes 1$ and $v_r = v \otimes 1, r = p, q$. With identification of W_r, w_r, z_r with unitaries in $A \otimes Q$, we also have

$$\|[u \otimes 1, W_p^* W_q]\| < \delta_1/4 \tag{51}$$

and

$$\text{bott}_1(u \otimes 1, W_p^* W_q) = \text{bott}_1(u \otimes 1, w_p^* z_p^* z_q w_q) \tag{52}$$

$$= \text{bott}_1(u \otimes 1, w_p^*) + \text{bott}_1(u \otimes 1, z_p^* z_q) + \text{bott}_1(u \otimes 1, w_q) \tag{53}$$

$$= -y_p + (y_p - y_q) + y_q = 0 \tag{54}$$

Let $Z_0 = W_p^* W_q$. Then it follows from the choice of δ_1, (33) that there is a continuouspath of unitaries $\{Z(t) : t \in [0,1]\} \subset A \otimes Q$ such that $Z(0) = Z_0$ and $Z(1) = 1$ and

$$\|[u \otimes 1, Z(t)]\| < \epsilon/6 \quad \text{for all } t \in [0,1] \tag{55}$$

Define $U(t) = w_p Z(t)$. Then $U(0) = w_p$ and $U(1) = w_q$. So, in particular, $U(0) \in A \otimes M_P$ and $U(1) \in A \otimes M_q$. So, $U \in A \otimes Z_{p,q} \subset A \otimes Z$ is a unitary and, by (50) and (55),

$$\|u \otimes 1 - U^*(v \otimes 1)U\| < \epsilon/3. \tag{56}$$

Note that we assume that $A \otimes Z \cong A$. Let $l: A \to A \otimes Z$ be the embedding defined by $l(a) = a \otimes 1$ for all $a \in A$ and $j: A \otimes Z \to A$ such that $j \circ l$ is approximately inner. Let $V \in A$ be a unitary such that

$$\|c - V^* j \circ l(c)V\| < \epsilon/3. \quad for \; all \; c \in \{u, v\}. \tag{57}$$

Then, let $w = Vj(U)V^* \in U(A)$.

$$\|u - w^* uw\| \le \|u - V^* j(u \otimes 1)V\|$$

$$+ \|V^* j(u \otimes 1)V - V^* j(U)^* j(v \otimes 1)j(U)V\| \tag{58}$$

$$+\|V^*j(U)^*j(v \otimes 1)j(U)V - V^*j(U)^*V_vV^*j(U)V\| \qquad (59)$$
$$< \epsilon/3 + \|u \otimes 1 - U^*(v \otimes 1)U\| + \|j \circ l(v) - V^*vV\| \quad (60)$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \qquad (61)$$

Lemma (2.1.13)[80]: Let A be a unital separable simple C^*-algebra in A_1. Then every quasi-trace on A extends to a trace. Moreover, if in addition, A is $Z-$stable, then
$$W(A) = V(A) \sqcup \mathrm{LAff}(T(A)),$$
where $W(A)$ is the Cuntz semi-group of A, $V(A)$ is the equivalence classes of projections in $\cup_{n=1}^{\infty} M_n(A)$ and $\mathrm{LAff}(T(A))$ is the set of all bounded real lower-semi-continuous affine functions on $T(A)$.

proof: Note that $TR(A \otimes Q) \leq 1$. Therefore every quasi-trace on $A \otimes Q$ is a trace. Suppose that s is a quasi-trace on A, then $s \otimes t$ is a trace on A, where t is the unique tracial state of Q.

Therefore $s \otimes t$ on $A \otimes \mathbb{C}1_Q$ is a trace. This implies that s is a trace.

The second part of the statement, A is assumed to be also exact. But that was only used so that every quasi-trace is a trace.

Lemma (2.1.14)[80]: Let $A \in A_1$ be a unital separable simple $Z-$stable C^*-algebra. Let $\Gamma : C([0,1])_{s.a.} \to \mathrm{Aff}(T(A))$ be a continuous affine map with $\Gamma(1)(\tau) = 1$ for all $\tau \in T(A)$ for some $a \in A_+$ with $\|a\| \leq 1$. Then there exists a unital monomorphism $\varphi : C([0,1]) \to A$ such that
$$\tau(\varphi(f)) = \Gamma(f)(\tau) \quad for\ all\ \tau \in T(A)$$
$and\ f \in C([0,1]).$

Let p and q be relatively prime supernatural numbers with $\mathbb{Q}_p + \mathbb{Q}_q = \mathbb{Q}$. Let M_r be the UHF-algebra associated with the supernatural number $r, r = p, q$. Let Q be the UHF-algebra such that $(K_0(Q), (K_0(Q)+, [1_\mathbb{Q}]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$. By the assumption $TR(A \otimes M_r) \leq 1$ and $TR(A \otimes Q) \leq 1$, $h_r \in (A \otimes M_r)_{s.a.}$ such that $\mathrm{sp}(h) = [0,1]$ and $(\tau \otimes t) \circ \varphi_r(f) = \Gamma(f)(\tau)$ for all $\tau \in T(A)$ and $f \in C([0,1])_{s.a.}$, where t is the unique tracial state on $M_r, r = p, q$. We use the same notation for φ_r for the unital monomorphisms $C([0,1]) \to A \otimes M_r \to A \otimes Q$ composed by φ_r and the embedding from $A \otimes M_r \to A \otimes Q$. Note that $K_0(C([0,1])) = \mathbb{Z}$ and $K_1(C([0,1])) = \{0\}$. Then $[\varphi_p] = [\varphi_q]\ in\ KK(C([0,1]), A \otimes Q)$ and, φ_p and φ_q induce the same map from $T(A \otimes Q)$ into $T(C([0,1]))$ as well as the same map from $U(C([0,1]))/CU(C([0,1])$ into $CU(A \otimes Q)/CU(A \otimes Q)$. Moreover since $K_1(C([0,1])) = \{0\}$. They induce zero rotation map. φ_p and φ_q are strongly asymptotically unitarily equivalent, i.e., there exists a continuous path of unitaries $\{u(t) : t \in [0,1)\} \subset A \otimes Q$ such that
$$\lim_{t\to\infty} u(t)^* \varphi_p(f)u(t) = \varphi_q(f) \quad for\ all\ f \in C([0,1])$$
Define $\psi : C([0,1]) \to A \otimes Z_{p,q}$ by
$$\psi(f)(t) = u(t)^* \varphi_p(f)u(t)\ for\ all\ t\ \in [0,1)\ and$$

$$\psi(f)(1) = \varphi_q(f) \quad for \ all \ f \in C([0,1]).$$

Note $\psi(f)(0) = \varphi_p(f) \in A \otimes M_p$ and $\psi(f)(1) = \varphi_q(f) \in A \otimes M_q$ for all $f \in C([0,1])$. By embedding $A \otimes Z_{p,q}$ into $A \otimes Z$, we obtain a unital monomorphism $\varphi: C([0,1]) \to A \otimes Z \cong A$. It is easy to check that so defined φ meets the requirements.

Let A be a unital simple C^*−algebra with $T(A) \neq \emptyset$. Let $u \in U(A)$ be a unitary with $sp(u) = T$. For each τ, let μ_τ be the Borel probability measure on T induced by state $\tau \circ f(u)$ (for all $f \in C(T)$) on T. Fix $n \geq 1$, let log: $\{e^{it}: t \in [-\pi + \pi/n, \pi]\} \to [-\pi + \pi/n, \pi]$ be the usual logarithm map. Consider the measure $\nu_{\tau,n}$ on $(-\pi, \pi]$ defined by

$$\nu_{\tau,n}(E) = \mu_\tau(\{e^{it}: t \in E \cap [-\pi + \pi/n, \pi]\})$$

for all Borel sets $E \subset (-\pi, \pi]$. Define

$$\nu_\nu(E) = \lim_{t\to\infty} \mu_{\tau,n}(E)$$

for all Borel sets $E \subset (-\pi, \pi]$. It is easy to check that ν_τ is a measure on $(-\pi, \pi]$. Let $f \in C_0((-\pi,\pi])_{s.a.}$ defined

$$\Gamma(f)(\tau) = \int_{(-\pi,\pi]} f \, d\nu_\tau.$$

Note that

$$\Gamma(f)(\tau) = \int_{(-\pi+\pi/n,\pi]} f \circ \log d\mu_\tau.$$

Let $g_n(t) = 1$ if $t \in \left[-\pi + \frac{\pi}{n}, \pi\right], g_n(t) = 0$ if $t \in \left[-\pi, -\pi + \frac{\pi}{2n}\right]$ and $g(t)$ is linear in $\left(-\pi + \frac{\pi}{2n}, -\pi + \frac{\pi}{n}\right)$. Note that $0 \leq g_n \leq 1$ and $g_n \in C(\mathbb{T})_+$. It is clear that $\Gamma(g_n) \in$ Aff$(T(A))$ and $\Gamma(g_n) \leq \Gamma(g_{n+1})$ and $\Gamma(g_n)(\tau) \to 1$ for each $\tau \in$ Aff$(T(A))$. It follows from the Dini theorem that $\Gamma(g_n)$ converges to 1 uniformly on $T(A)$. On the other hand

$$\left| \int_{(-\pi,\pi)} f(1 - g_n) \, d\nu_\tau \right| \leq \int_{(-\pi,\pi]} |f|^2 \, d\nu_\tau \int_{(-\pi,\pi]} (1 - g_n)^2 \, d\nu_\tau \quad (62)$$

$$\leq \int_{(-\pi,\pi]} |f|^2 \, d\nu_\tau \int_{(-\pi,\pi]} (1 - g_n)^2 \, d\nu_\tau \to 0 \quad (63)$$

uniformly on $T(A)$. This implies that $\Gamma(f)$ is continuous on $T(A)$. If $g \in C([-\pi, \pi])_{s.a.}$, we may write $g(t) = g(0) + (g(t) - g(0))$. Define $\Gamma(g) = g(0) + \Gamma(g - g(0))$. This provides an affine continuous map from $C([-\pi, \pi])_{s.a.}$ to Aff$(T(A))$.

We check that

$$\tau(f(u)) = \int_{(-\pi,\pi]} f \circ \exp(it) \, d\nu_\tau(t) = \Gamma(f \circ \exp(it))(\tau) \text{ for all } f \in C(\mathbb{T})_{s.a.}.$$

In the above, we can replace $-\pi$ by 0 and π by 2π. We will keep this notation in the next proof.

Theorem (2.1.15)[80]: Let $A \in A_1$ be a unital separable simple Z −stable C^* −algebra. Let $u \in U_0(A)$ be a unitary. Then, for any $\epsilon > 0$, there exists a self-adjoint element $h \in A$ such that

$$\|u - \exp(ih)\| < \epsilon \tag{64}$$

In the other words

$$\mathrm{cer}(A) \leq 1 + \epsilon.$$

Proof: Let $u \in U_0(A)$. If $sp(u) \neq \mathbb{T}$, then u is an exponential. So we may assume that $sp(u) = \mathbb{T}$.

Let $\epsilon > 0$. Let $\varphi: C(\mathbb{T}) \to A$ be defined by $\varphi(f) = f(u)$ for all $f \in C(T)$. It is a unital monomorphism. The a non-decreasing function $\Delta_1: (0,1) \to (0,1)$ such that

$$M_\tau(O_a) \geq \Delta_1(a) \text{ for all } \tau \in T(A) \tag{65}$$

for all arcs I_a of T with length $a \in (0,1)$. Define $\Delta = (1/2)\Delta_1$.

Let $\eta > 0, \delta > 0$ and let $\mathcal{G} \subset C(T)$ be a finite subset required by (4) for $\epsilon/2$ (in place of ϵ).

Without loss of generality, we may assume that $\|g\| \leq 1$ for all $g \in \mathcal{G}$. Let $\sigma = \min\{\eta/2, \delta/2\}$.

Let $\Gamma: C([0,2\pi])_{s.a.} \to \mathrm{Aff}(T(A))$ be the map defined in (7) (using $[0,2\pi]$ instead of $[-\pi, \pi]$). Define $\Gamma_1: C([0,2\pi])_{s.a.} \to \mathrm{Aff}(T(A))$ as follows: define

$$\Gamma_1(f)(\tau) = (1-\sigma)\Gamma(f)(\tau) \text{ for all } \tau \in T(A)$$

and for all $f \in C_0((0,2\pi])$ and define

$$\Gamma_1(f) = f(0) + \Gamma_1(f - f(0))(\tau) \quad \text{for all } \tau \in T(A)$$

and for all $f \in C([0,2\pi])_{s.a.}$. It follows that, for any $f \in C(T)_{s.a.}$. With $\|f\| \leq 1$,

$$|\tau(f) - \Gamma_1(f \circ \exp)| = |\Gamma(f \circ \exp) - \Gamma_1(f \circ \exp)| < \sigma \text{ for all } \tau \in T(A), \tag{66}$$

where $\exp: [0,2\pi] \to T$ is defined by $\exp(t) = e^{it}$ for all $t \in [0,2\pi]$. Note since A is simple and $sp(u) = T, \Gamma_1$ is strictly positive. It follows (6) that there is a self-adjoint element $b \in A$ such that $sp(b) = [0,2\pi]$ and $\tau(f(b)) = \Gamma_1(f)(\tau)$ for all $f \in C_0((0,2\pi])$. It follows that

$$d_\tau(b) = \lim_{t \to \infty} \tau(b^{1/n}) \leq (1-\sigma) \text{ for all } \tau \in T(A). \tag{67}$$

Note that since A is also Z −stable, by (5), $W(A) = V(A) \sqcup \mathrm{LAff}(T(A))$. There are mutually orthogonal elements $a_1, c_1, c_2 \in M_K(A)_+$ with $0 \leq a_1, c_1, c_2 \leq 1$ for some integer $K \geq 1$ such that

$$d_\tau(a_1) = 1 - \sigma/2, \quad d_\tau(c_1) = d_\tau(c_2) = \sigma/5 \text{ for all } \tau \in T(A). \tag{68}$$

Put $a_2 = a_1 + c_1 + c_2$. Note that $0 \leq a_2 \leq 1$ and

$$d_\tau(a_2) = 1 - \frac{9\sigma}{10} < 1 \quad \text{for all } \tau \in T(A). \tag{69}$$

By the strict comparison, (67), (69) and the fact that A has stable rank one, we may assume, without loss of generality, that

$$a_2 \in A \quad and \quad b \in \overline{a_1 M_K(A) a_1}.$$

Suppose that
$$\text{Det}(u)(\tau) = s(\tau) \quad \text{for all } \tau \in T(A) \tag{70}$$
for some $s \in \text{Aff}(T(A))$.

The above argument also shows that there are $b_1 \in \overline{a_1 A c_1}$. and $b_2 \in \overline{a_2 A c_2}$. such that
$$\tau(b_1) = \sigma s(\tau)/6 \text{ and } \tau(b_2) = \sigma \tau(b)/6 \quad \text{for all } \tau \in T(A). \tag{71}$$

Let
$$h_1 = \frac{-6b_2}{\sigma} + \frac{6b_1}{\sigma} + b.$$

Note that
$$\tau(h_1) = (6/\sigma)\tau(b_1) = s(\tau) \quad \text{for all } \tau \in T(A). \tag{72}$$

Define $v = \exp(ih_1)$. One checks, by (72), that
$$\text{Det}(v) = \text{Det}(u). \tag{73}$$

Therefore
$$uv^* \in CU(A). \tag{74}$$

Let $\text{sp}(h_1) \subset [-m_1\pi, m_2\pi]$ for some integers $m_1, m_2 \geq 0$. Note that $f(b_j) \in \overline{c_j A c_j}$ iff $\in C\left(\text{sp}(b_j)\right)$, $j = 1, 2$. So, if, in addition, $\|f\| \leq 1$, by (68),
$$|\tau(f(b_j))| < \sigma/5 \quad \text{for all } \tau \in T(A). \tag{75}$$

We have, for any $f \in C([-m_1\pi, m_2\pi])_{s.a.}$ with $\|f\| \leq 1$, by (75),
$$\left|\tau(f(h_1)) - \Gamma_1(f|_{[0,2\pi]})\right| = \left|\tau(f(b)) + \tau(f(b_1)) + \tau(f(b_2)) - \Gamma_1(f|_{[0,2\pi]})\right| \tag{76}$$
$$= \sigma/5 + \sigma/5 + \left|\tau(f(b)) - \Gamma_1(f|_{[0,2\pi]})\right| = 2\sigma/5 \tag{77}$$

For all $\tau \in T(A)$. Therefore, by (66), (76), and (77), we have that
$$|\tau(g(v)) - \tau(g(u))| < \sigma + 2\sigma/5 < \delta \quad \text{for all } g \in \mathcal{G}. \tag{78}$$

It follows from (4) that there exits a unitary $w \in U(A)$ such that
$$\|u - w^*vw\| < \epsilon.$$

Let $h = w^*h_1w$. Then
$$\|u - \exp(ih)\| < \epsilon.$$

Corollary (2.1.16)[80]: Let $\mathcal{Z}$ be the Jiang-Su algebra. Then
$$cer(\mathcal{Z}) = 1 + \epsilon.$$

We will prove much stronger result than the above for $\mathcal{Z}$. The following is known (something similar could be found in [129] and [109]). We state here for the convenience.

Lemma (2.1.17)[80]: Let u be a unitary in $C([0,1], M_n)$. Then, for any $\epsilon > 0$, there exist continuous functions $h_j \in C([0,1])_{s.a.}$ such that
$$\|u - u_1\| < \epsilon,$$
where $u_1 = exp(i\pi H)$, $H = \sum_{j=1}^n h_j p_j$ and $\{p_1, p_2, \ldots, p_n\}$ is a set of mutually orthogonal rankone projections in $C([0,1], M_n)$, and $exp(i\pi h_j(t)) \neq exp(i\pi h_k(t))$ if $j \neq k$ for all $t \in [0,1]$. Moreover, suppose that $u(0) = \sum_{j=1}^n \exp(ia_j) p_j(0)$ for some real number a_j which are distinct, we may assume that $h_j(0) = a_j$.

Furthermore, if $det(u(t)) = 1\ for\ all\ t \in [0,1]$, then we may also assume that $det(u_1(t)) = 1\ for\ all\ t \in [0,1]$.

Proof:

The last part of the statement follows from [109]. By Lemma (2.1.5), if $det(u(t)) = 1\ for\ all\ t \in [0,1]$, then we can choose u_1 such that $\|u - u_1\| < \epsilon$ and $det(u_1(t)) = 1\ for\ all\ t \in [0,1]$ and $u(t)$ has distinct eigenvalues. Therefore $u_1 = \sum_{j=1}^{k} z_j(t)\, p_j(t)$, where $p_j(t) \in C([0,1], M_n)$ is a rank one projection, $\sum_{j=1}^{k} p_j(t) = 1$ and $z_j(t) \in C([0,1])$ with $|z_j(t)| = 1$ for all $t \in [0,1]$ Let $z_j(t) = e^{ia(0)}$ for some real number $a(0)$. But $z_j(t) = e^{ib_j(t)}$ for some real $b_j \in C([0,1]), j = 1, 2, \ldots, n$. Note that $a_j(0) - b_j(0) = 2k\pi$ for some integer k. By replacing b_j by $a_j(t) = h_j(t) + (a_j(0) - h_j(0))$. Then $z_j(t) = e^{ia_j(0)}$ and $z_j(0) = a_j(0), j = 1, 2, \ldots, n$. In particular,

$$u_1(t) = \sum_{j=1}^{n} e^{ia_j(t)}\, p_j(t) \quad for\ all\ t \in [0,1].$$

Lemma (2.1.18)[80]: Let $u \in C([0,1], M_n)$ be a unitary with $det(u)(t) = 1$ for each $t \in [0,1]$. Then, for any $\epsilon > 0$, there exists a self-adjoint element $h \in C([0,1], M_n)$ such that $\|h\| \le 1, \tau(h) = 0$ for each $\tau \in T(C([0,1], M_n))$ and

$$\|u - \exp(i2\pi h)\| < \epsilon.$$

In particular $length(u) \le 2\pi$.

Proof: First, without loss of generality, we may assume that $u(0)$ has distinct eigenvalues. Suppose that

$$u(0) = \sum_{j=1}^{n} \exp(i2\pi b_j)\, p_j(0),$$

where $b_j \in (-1/2, 1/2], j = 1, 2, \ldots, n$.

Then $\sum_{j=1}^{n} b_j = k$ for some integer k. Since $b_j \in (-1/2, 1/2], k \le n$. Keep in mind that b_j are distinct. If $k \ge 1$, to simplify notation, we may assume that $b_j > 0, j = 1, 2, \ldots, k, b_{k+l} < b_k < b_j$ for $j < k$ and $l > 0$. Define $a_j = b_j - 1, j = 1, 2, \ldots, k$ and $a_j = b_j, j > k$. Then

$$\sum_{j=1}^{n} a_j = 0 \quad and\ |a_j| < 1. \tag{79}$$

Note that $max_j a_j < b_k$. Since $b_j > -1/2, min_j a_j = b_k - 1$. Therefore, we also have

$$\max_j a_j - \min_j a_j < 1. \tag{80}$$

If $k < -1$, we may assume that $b_j < 0, j = 1, 2, \ldots, k, b_{k+l} \ge b_k > b_j$ for $j \le k$ and $l > 0$. Define $a_j = b_j + 1, j = 1, 2, \ldots, k$ and $a_j = b_j$ if $j > k$. Then (79) and (80) also hold in this case.

We may assume, without loss of generality, that

$$u(t) = \sum_{j=1}^{n} \exp(i2\pi h_j(t))\, p_j(t), \qquad (81)$$

where $h_j(t) \in C([0,1])_{s.a.}$ and $\{p_1, p_2, \ldots, p_n\}$ is a set of mutually orthogonal rank one projections. Moreover, we may assume that $det(u(t)) = 1$ for all $t \in [0,1]$ and $u(t)$ has distinct eigenvalues at each point $t \in [0,1]$. We may also assume that $h_j(0) = a_j, j = 1, 2, \ldots, n$. We also have that $|h_j(0)| < 1$,

$$\sum_{j=1}^{n} h_j(0) = 0 \quad and \quad \max_j a_j - \min_j a_j < 1. \qquad (84)$$

Since $det(u(t)) = 1$ for all $t \in [0,1]$,

$$\sum_{j=1}^{n} h_j(t) \in \mathbb{Z} \quad for\ all\ t \in [0,1]. \qquad (83)$$

Since $\sum_{j=1}^{n} h_j(t) \in C([0,1])$, it follows that it is a constant. By (81),

$$\sum_{j=1}^{n} h_j(t) = 0 \quad for\ all\ t \in [0,1]. \qquad (84)$$

Since $u(t)$ has distinct eigenvalues, $h_j(t) - h_k(t) \notin \mathbb{Z}$, for any $t \in [0,1]$ when $j \neq k$. We also have $\max_j h_j(t) - \min_j h_j(t)$ is a continuous function. It follows from (82) that

$$0 < \max_j h_j(t) - \min_j h_j(t) < 1 \quad for\ all\ t \in [0,1]. \qquad (85)$$

Now by (84), either $h_j(t) = 0$ for all j, which is not possible, since $u(t)$ has n distinct eigenvalues, or, for some $j, h_j(t) < 0$ and for some other $j', h_{j'} > 0$, it follows from (85) that

$$|h_j| < 1 \quad for\ all\ t \in [0,1]. \qquad (86)$$

Now let $h = \sum_{j=1}^{n} h_j \in C([0,1], M_n)_{s.a.}$. Then

$$\|h\| < 1, \quad \tau(h) = 0 \ for\ all\ \tau \in T(A) \quad and \quad u = \exp(i2\pi h). \qquad (87)$$

We will use the following theorem .

Theorem (2.1.19)[80]: Let $A \in \mathcal{A}_1$ be a unital separable simple $\mathcal{Z}$-stable C^*-algebra. Suppose that $u \in CU(A)$. Then, for any $\epsilon > 0$, there exists a self-adjoint element $h \in A$ with $\|h\| < 1$ such that

$$\|u - \exp(i2\pi h)\| < \epsilon. \qquad (88)$$

In particular, $cel_{CU}(A) \leq 2\pi$.

Proof: We may assume that $sp(u) = \mathbb{T}$. Let $\epsilon > 0$. Let $\varphi : C(\mathbb{T}) \to A$ be defined by $\varphi(f) = f(u)$. It is a unital monomorphism. That there is a non-decreasing function $\Delta : (0,1) \to (0,1)$ such that

$$\mu_\tau(O_a) \geq \Delta(a) \quad for\ all\ \tau \in T(A) \qquad (89)$$

for all open balls O_a of $\mathbb{T}$ with radius $a \in (0,1)$.

Note, by [95], for any supernatural number p of infinite type, $TR(A \otimes M_p) \leq 1$. Consider $u \otimes 1$. Denote by u_p for $u \otimes 1$ in $A \otimes M_p$. For any $\epsilon/2 > \epsilon_0 > 0$, there is a self-adjoint element $h_p \in A \otimes M_p$ with $sp(h_p) = [-2\pi, 2\pi]$ such that

$$\|u_p - \exp(ih_p)\| < \epsilon_0 \quad and \quad \tau(h_p) = 0 \quad for \ all \ \tau \in T(A \otimes M_p). \tag{90}$$

Let $\psi_0 : C(T) \to A \otimes M_p$ be the homomorphism defined by $\psi_0(f) = f(exp(ih_P))$ for all $f \in C(T)$. Let $\eta > 0, \delta > 0$ and let $\mathcal{G}$ be a finite subset as required by (4) for $\epsilon/2$ (in place of ϵ) and Δ. Choose ϵ_0 sufficiently small, so the following holds: For any unitary $v \in A \otimes M_p$, if $\|u_p - v\| < \epsilon_0$, then

$$\left| \tau\left(g(u_p)\right) - \tau(g(v)) \right| < \delta \ for \ all \ \tau \in T(A) \tag{91}$$

and for all $g \in \mathcal{G}$. Note each $\tau \in A \otimes M_p$ may be written as $s \otimes t$, where $s \in T(A)$ is any tracial state and $t \in T(M_p)$ is the unique tracial state.

Let $\Gamma : C([-2\pi, 2\pi])_{s.a.} \to Aff(T(A))$ be defined by

$$\Gamma(f)(\tau) = (\tau \otimes t)\left(f(h_p)\right) \quad for \ all \ f \in C([-2\pi, 2\pi])_{s.a.} \tag{92}$$

and for all $\tau \in T(A)$, where t is the unique tracial state on M_p.

It follows from (6) that there exists a self-adjoint element $h \in A$ with $sp(h) = [-2\pi, 2\pi]$ such that

$$\tau(f(h)) = \Gamma(f)(\tau) = (\tau \otimes t)\left(f(h_p)\right) \quad for \ all \ f \in C([-2\pi, 2\pi]) \tag{93}$$

and for all $\tau \in T(A)$. In particular,

$$\tau(h) = 0 \quad for \ all \ \tau \in T(A). \tag{94}$$

Define $v_1 = exp(ih) \in A$. Note that, by (92),

$$\tau(g(v_1)) = (\tau \otimes t)g\left(\exp(ih_p)\right) \quad for \ all \ \tau \in T(A) \tag{95}$$

and for all $f \in C(T)$. By the choice of ϵ_0, as in (91),

$$|\tau(g(u)) - \tau(g(v_1))| = \left| (\tau \otimes t)\left(g(u_p)\right) - (\tau \otimes t)\left(g\left(\exp(ih_p)\right)\right) \right| < \delta \tag{96}$$

for all $\tau \in T(A)$ and for all $g \in \mathcal{G}$. We also have $[v_1] = [u] = 0$ in $K_1(A)$. Furthermore, $v_1 \in CU(A \otimes Z)$. Thus, by applying (4), there exists a unitary $w \in A$ such that

$$\|u - w^* \exp(ih) w\| < \epsilon/2. \tag{97}$$

Theorem (2.1.20)[80]: Let A be a unital separable simple Z-stable C^*-algebra in $\mathcal{A}_0$ with a unique tracial state. Then, for any unitary $u \in U_0(A)$, there exists a real number $-\pi < a < \pi$ such that, for any $\epsilon > 0$, there exists a self-adjoint element $h \in A$ with $\|h\| \leq 2\pi$ and

$$\|u - \exp(i(h + a))\| < \epsilon.$$

Consequently

$$cel (A) \leq 3\pi.$$

Proof: Let $u \in U_0(A)$ and let $\epsilon > 0$. Since A has a unique tracial state τ, $U_0(A)/CU(A) = \mathbb{R}/\overline{\rho_A(K_0(A))}$. Therefore there is $t \in (-1, 1)$ such that

$$Det(u) = t + \overline{\rho_A(K_0(A))}. \tag{98}$$

Consequently

$$e^{-\pi t}u \in CU(A). \tag{99}$$

It follows from theorm (2.1.19) that there is a self-adjoint element $h \in A$ with $\|h\| \le 2\pi$ such that

$$\|e^{-\pi t}u - \exp(ih)\| < \epsilon. \tag{100}$$

Therefore

$$\|u - e^{i\pi t}\exp(ih)\| < \epsilon. \tag{101}$$

Let $a = \pi t$. Note that $e^{i\pi t}\exp(ih) = exp(i(h + a))$. Put $h_1 = h + a$. We conclude that

$$\|u - \exp(ih_1)\| < \epsilon.$$

Note that $\|h_1\| \le \|h\| + |a| < 3\pi$. There is $h_2 \in A_{s.a.}$ with $\|h_2\| < 2arcsin(\pi/2)$ such that

$$u = \exp(ih_1)\exp(ih_2). \tag{102}$$

If we choose ϵ so that

$$2\arcsin(\epsilon/2) < 3\pi - \|h\| + |a|,$$

then

$$\|h_1\| + \|h_2\| < 3\pi.$$

Corollary (2.1.21)[80]: Let $u \in U_0(Z)$ be a unitary. There exists $t \in (-\pi, \pi)$ such that, for any $\epsilon > 0$, there exists a self-adjoint element $h \in Z$ with $\|h\| \le 2\pi$ and a real number $-\pi < t \le \pi$ satisfying

$$\|e^{it}u - \exp(ih)\| < \epsilon. \tag{103}$$

Let $u \in C([0,1], M_n)$ be defined as follows:

$$u(t) = e^{\pi it(2-1/(n-1))}e_1 + e^{-\pi i\frac{t(2-1/(n-1))}{(n-1)}}\left(\sum_{k=2}^{n} e_k\right) \ and \tag{104}$$

$$h(t) = t(2 - 1/(n-1))e_1 - \frac{t(2-1/(n-1))}{(n-1)}\left(\sum_{k=2}^{n} e_k\right) \ for \ t \in [0,1], \tag{105}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a set of mutually orthogonal rank one projections.
Then

$$u(t) = \exp(i\pi h) \quad and \quad \tau(h) = 0 \quad for \ all \ \tau \in T(C[0,1], M_n).$$

Therefore $det(u(t)) = 1$ for all $t \in [0, 1]$ and $u \in CU(C[0,1], M_n)$. Note also $\|h\| = \pi(2 - 1/(n-1))$. In what follows we will show that $cel(u) \ge (2 - 1/(n-1))\pi$. It should be noted that it ismuch easier to show that if $u(t) = exp(iH)$ for some self-adjoint element in $C([0,1], M_n)$then$\|H\| \ge (2 - 1/(n-1))\pi$. Suppose that $cel(u) = r_1$. Fix $r_1 > \epsilon > 0$ and put $r = r_1 + \epsilon/16$. Then there are self-adjointelements $h_1, h_2, \ldots, h_k \in C([0,1], M_n)$ such that

$$u = \prod_{j=1}^{k} \exp(ih_j) \quad and \quad \sum_{j=1}^{k} \|h_j\| = r. \tag{106}$$

Define $u_s = \prod_{j=1}^{k} exp\,(ih_j(1-s))$. Then us is continuous and piecewise smooth on $[0,1]$. Moreover length$\{u(t)\} \leq r$. Since $h_j(t)(1-s)$ is continuous on $[0,1] \times [0,1]$, one shows that $W(t,s) = u_s(t)$ is continuous on $[0,1] \times [0,1]$. Furthermore

$$\|u_{s_1} - u_{s_2}\| \leq r|s_1 - s_2| \quad for\ all\ s_1, s_2 \in [0,1]. \tag{107}$$

Lemma (2.1.22)[80]: Let u and v be two unitaries in a unital C^*-algebra A. Suppose that there is acontinuous path of unitaries $\{w(t): t \in [0,1]\} \subset A$ with $w(0) = u\ and\ w(1) = v$. Then, if $\lambda \in sp(u)$, there is a continuous path $\{\lambda(t) \in T: t \in [0,1]\}$ such that $\lambda(0) = \lambda, \lambda(t) \in sp(w(t))\ for\ all\ t \in [0,1]$. If furthermore, length$\{w(t): t \in [0,1]\} = r \leq \pi/2$, then one can require that

$$length\ \{\lambda(t): t \in [0,1] \leq r.$$

Proof: The proof of this was originally taken from an argument of Phillips . One obtains a sequence of partitions $\{\mathcal{P}_n\}$ of $[0,1]$ such that $\mathcal{P}_n \subset \mathcal{P}_{n+1}, n = 1, 2, \ldots$, foreach partition $\mathcal{P}_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots t_{k(n)}^{(n)} = 1\}$, there are $\lambda(n,i) \in sp(w(t_i^{(n)}))$ such that

$$|\lambda(n,i) - \lambda(n,i+1)| = \left\| w\left(t_i^{(n)}\right) - w\left(t_{(i+1)}^{(n)}\right) \right\| \quad and \tag{108}$$

$$\sum_{i=1}^{k(n)} |\lambda(n,i) - \lambda(n,i+1)| \leq \sum_{i=1}^{k(n)} \left\| w\left(t_i^{(n)}\right) - w\left(t_{(i+1)}^{(n)}\right) \right\| \leq r, \tag{109}$$

if$\{w(t)\}$ is rectifiable with $length\{w(t): t \in [0,1]\} = r$. Write $\lambda(n,j) = e^{i\theta(n,j)}$ with $\theta(n,j) \in [0,2\pi), j = 1,2,\ldots, k(n)$ and $n = 1,2,\cdots$. Define

$$\theta(t) = \sup\left\{\theta(n,j): t_j^{(n)} \leq t\right\}. \tag{110}$$

By the uniform continuity of $w(t)$, one checks that $\lambda(t) = exp(i\theta(t))$ is continuous on $[0,1], \lambda(t) \in sp(w(t))$ and $length(\{\lambda(t)\} \leq r$.

Suppose that $u(t) \in C([c,d], M_n)$ is a unitary which has the form:

$$u(t) = f(t)q_1 + z(t) \quad for\ all\ t \in [c,d],$$

where$f(t) \in C([0,1], \mathbb{T})$, q_1 is a rank one projection and $z \in (1-q_1)\,C([c,d],M_n)(1-q_1)$ isa unitary. Let $cel(u) = r_1$ and fix $r_1/2 > \epsilon > 0$. Let $r = r_1 + \epsilon/16$. let$\{W(t,s) : s \in [0,1]\}$ be a continuous rectifiable path such that $W(t,s) \in C([c,d] \times [0,1], M_n)$ with$W(t,0) = u(t)$ and $W(t,1) = 1$ with length r. Fix $s_0 \in (0,1]$. Suppose that $length\{W(t,s): s \in [0,s_0]\} = r_0$. Define S_1 the subset of $\mathbb{T}$such that every point of S_1 can be connected to a point in $sp(z)$ by a continuous path of length at most r_0.

Then we have the following:

Lemma (2.1.23)[80]: Let $\{f(t) : t \in [c,d]\} = \{e^{it}: t \in [t_0,t_1]\}$ with $f(c) = e^{it_0}$ and $f(d) = e^{it_1}$ such that $t_1 - t_1 = r_2$. Suppose that

$$dist(\lambda_t(s), S_1) > 0 \quad for\ all\ t \in [t_0,t_1] \quad and\ s \in [0,s_0],$$

where $length\{W(t,s): s \in [0,s_0]\} = r_0 < r_2/2$. Then

$$W(t,s_0) = g_{s_0}(t)q_1 + v_1(t) \quad for\ all\ t \in [c_1,d_1]$$

for some $[c_1, d_1] \subset [c, d]$ with $d_1 > c_1$. Moreover $g_{s_0}(t)$ is a continuous function and
$$\{g_{s_0}(t) : t \in [c_1, d_1]\} \supset \{e^{it} : t \in [t_0 + r_0, t_1 - r_0]\}.$$

Proof: View $Z = W(t, s)|_{[c,d] \times [0, s_0]}$ as a unitary in $C([c, d] \times [0, s_0], M_n)$. Then the assumptionimplies that
$$sp(Z) \subset J \sqcup S_1,$$
where$J = \{\lambda t(s) : t \in [t_0, t_1]$ and $s \in [0, s_0]\}$, Note that $J \cap S_1 = \emptyset$. Then, there is a non-zeroprojection $q_1' \in C([c, d] \times [0, s_0], M_n)$ such that
$$Z = z_1 + z_2, \tag{111}$$
where$z_1 \in q_1' C([c, d] \times [0, s_0], M_n) q_1'$ and $z_2 \in (1 - q_1') C([c, d] \times [0, s_0], M_n)(1 - q_1')$ are unitariessuch that $sp(z_1) \subset J$ and $sp(z_2) \ S_1$. Since q_1'has rank one in $[c, d] \times \{0\}$, we conclude that q_1'has rank one everywhere. Thus
$$Z(t, s) = g_s(t) q_1'(t, s) + z_2(s, t) \quad for \ all \ (t, s) \in [c, d] \times [0, s_0].$$
Note that $g_s(t) \in C([c, d] \times [0, s_0])$. Therefore $\{g_{s_0}(t) : t \in [c, d]\}$is an are containing $g(c) = \lambda_c(s_0)$and $g_{s_0}(d)$. By the assumption, $g_{s_0}(c) \in \{e^{it} : t \in [t_0 - r_0, t_0 + r_0]\}$and $g_{s_0}(d) \in \{e^{it} t \in [t_1 - r_0, t_1 + r_0]\}$. The lemma follows.

Lemma (2.1.24)[80]: Suppose that $length(\{W(t, s) : s \in [0, s_1]\} = C_1 < \pi/4$. If $[c, d] \subset [a, b]$ such that
$$dist(\{f(t) : t \in [c, d]\}, \{sp(v_1(t)) : t \in [c, d]\}) = r_1 = 4 \sin(C_1/2) + \delta$$
for some $0 < \delta < \pi/8$, $\{f(t) : t \in [c, d]\} = \{e^{it} : t \in [t_0, t_1]\}$ with $t_0, -t_1 > 2r_1$, then, for any $\delta > 0$,there exists an interval $[c_1, d_1] \subset [c, d]$ with $c_1 < d_1$, a rank one projection $q_1 \in C([c_1, d_1], M_n)$such that
$$W(t, s_1) = g_{s_1}(t) q_1 + v_1'(t), \tag{112}$$
where$g_{s_1}(t) \in C([c_1, d_1])$ with
$$\{g_1(t) : t \in [c_1, d_1]\} = \{e^{it} : t \in [t_0 + C_1 + \delta, t_1 - C_1 - \delta]\}, \tag{113}$$
where $2\pi > t_1' > t_0' \geq 0$, and where $v_1' \in (1 - q_1) C([c_1, d_1], M_n)(1 - q_1)$ is a unitary with$sp(v_1(t)) \subset S_1$, where S_1 is a subset of $\mathbb{T}$ such that every point of S_1 can be connected by a pointin $sp(v(t))$ $(t \in [c, d])$ by a continuous path with length at most C_1.

Proof: Let S_1 be the subset of $\mathbb{T}$ such that every point in S_1 can be connected to a point in$sp(v_1)$ with length at most C_1. Since $length(\{W(t, s) : s \in [0, s_1]\} = C_1 < \pi/4$,
$$dist(\lambda_t(s), S_1) > 0 \ for \ all \ t \ \in [t_0, t_1] and \ s \ \in [0, s_1]. \tag{114}$$

Theorem (2.1.25)[80]: Let $u(t) \in C([0, 1], M_n)$ be the unitary. For any $\epsilon > 0$,
$$length \{u(t)\} \geq \pi \left(2 - \frac{1}{n-1}\right) - \epsilon. \tag{115}$$
If $h \in C([0, 1], M_n)_{s.a.}$ such that
$$\|u - \exp(ih)\| < \epsilon, \tag{116}$$
then $\|h\| \geq \pi \left(2 - \frac{1}{n-1}\right) - 2\arcsin(\epsilon/2)$. Moreover,
$$cel_{CU}(C[0,1], M_n) \geq \pi \left(2 - \frac{1}{n-1}\right).$$

Proof: From anterior definition. We write
$$u(t) = f(t)e_1 + v(t).$$
Let $0 < d \leq \epsilon/2$ and $k \geq 1$ be an integer such that $kd = \pi(1 - 1/n - 1)$. Let $0 < a_0 < b_0 < 1$ such that
$$f(a_0) = e^{id + \epsilon/k2} \quad \text{and} \quad f(b_0) = e^{i\pi(2 - 1/n - 1) - d - \epsilon/k2}.$$
Let $0 < s_1 < 1$ such that $length\{W(t,s) : s \in [0, s_0]\} = d$. It follows from lemma(2.1.24) that there exists $a_0 < a_1 < b_1 < b_0$ such that
$$W(t, s_0) = g_1(t)q_1 + z_1(t) \quad for \ all \ t \in [a_1, b_1], \tag{117}$$
where q_1 is a rank one projection, $z_1 \in (1 - q_1)C([a_1, b_1], M_n)(1 - q_1)$,
$$\{g_1(t) : t \in [a_1, b_1]\} = \{e^{it} : t \in [2d + \epsilon/k2, \pi(2 - 1/(n - 1)) - 2d - \epsilon/k2]\}$$
with $g_1(a_1) = e^{i(2d + \epsilon/k2)}$ and $g_1(b_1) = e^{i(\pi(2 - 1/n - 1)) - 2d - \epsilon/k2}$, $sp(z_1) \subset S_1$, where S_1 is the subset of $\mathbb{T}$ such that every point in $S1 S_1$ is connected by a rectifiable continuous path from $\{e^{it} : t \in [-(2 - 1/n - 1)\pi n - 1, 0]\}$. In particular,
$$S_1 \subset \left\{ e^{it} : t \in \left[-\frac{(2 - 1/(n - 1))\pi}{n - 1} - d, +d \right] \right\}. \tag{118}$$
Let $1 > s_1 > s_0$ such that $length\{W(t, s) : s \in [s_0, s_1]\} = d$. By repeating above, one obtains $a_1 < a_2 < b_2 < b_1$ such that
$$W(t, s_1) = g_2(t)q_2 + z_2(t), \quad t \in [a_2, b_2], \tag{119}$$
where q_2 is a rank one projection, $z_2 \in (1 - q_2)C([a_2, b_2], M_n)(1 - q_2)$,
$$\{g_2(t) : t \in [a_2, b_2]\} \tag{120}$$
$$= \left\{ e^{it} : t \in \left[2d + \frac{\epsilon}{k2} + d + \frac{\epsilon}{k4}, \pi\left(2 - \frac{1}{n - 1}\right) - 2d - \frac{\epsilon}{k2} - d - \frac{\epsilon}{k4} \right] \right\} \tag{121}$$
$$= \left\{ e^{it} : t \in \left[3d + \frac{\epsilon}{k2} + d + \frac{\epsilon}{k4}, \pi\left(2 - \frac{1}{n - 1}\right) - 3d - \frac{\epsilon}{k2} - d - \frac{\epsilon}{k4} \right] \right\} \tag{122}$$
with $g_2(a_2) = e^{i(3d + \varrho/k2 + \varrho/k4)}$ and $g_2(b_2) = e^{i(\pi(2 - 1/n - 1)) - 3d - \epsilon/k2 - \epsilon/k4}$, $sp(z_1) \subset S_2$, where S_2 is the subset of $\mathbb{T}$ such that every point in S_2 is connected by a rectifiable continuous path from $\{e^{it} : t \in [-2(1 - 1/n - 1)\pi/(n - 1) - d, d]\}$. In particular,
$$S_2 \subset \left\{ e^{it} : t \in \left[-\frac{(2 - 1/(n - 1))\pi}{n - 1} - 2d, d \right] \right\}. \tag{123}$$
By repeating this argument $k - 1$ times, We obtain $1 > s_{k-1} > s_{k-2}$ such that
$$length\{W(t, s) : s \in [0, s_{k-1}]\} = (k - 1)d = \pi(1 - 1/(n - 1)) - d \quad and$$
$$\pi \in \left\{ e^{it} : t \in \left[(k - 1)d + \sum_{j=1}^{k-1} \frac{\epsilon}{k2^j}, \left(2 - \frac{1}{n - 1}\right)\pi - (k - 1)d + \sum_{j=1}^{k-1} \frac{\epsilon}{k2^j} \right] \right\}$$
$$\subset sp(W(t, s_{k-1})).$$
Thus the minimum length of continuous path from $W(t, s_{k-1})$ to 1 is at least π. Thus
$$length\{u(t)\} + \frac{\epsilon}{16} \geq \pi + (k - 1)d \tag{124}$$
$$= \pi + \pi\left(1 - \frac{1}{n - 1}\right) - d \geq \pi(2 - 1/(n - 1)) - \epsilon/2 \tag{125}$$
for all $\epsilon > 0$. It follows that

$$length\{u(t)\} \geq \pi(2 - 1/(n-1)). \tag{126}$$

Fix an integer $n > 12$ and let $k_0 \geq 0$. Suppose that $0 \leq k \leq k_0$. Let $N = mn + k$ and let $N_0 = mn$. Consider a unitary $u_{00} \in C([0,1], M_{N_0})$:

$$u_{00} = e^{\pi it(2-1/(n-1))}P_1 + e^{-\pi it(2-1/(n-1))}P_2, \tag{127}$$

where $P_1, P_2 \in C([0,1], M_{N_0})$ are constant projections with $rank\ P_1 = m$ and $rank\ P_2 = (n-1)m$. Define

$$u_0 = u_{00} + v_0 \in C([0,1], M_N), \tag{128}$$

where $v_0 \in (1 - (P_1 + P_2))C([0,1], M_N)(1 - (P_1 + P_2))$ is another unitary such that $det(v_0(t)) = 1$ for each $t \in [0,1]$ and $v_0 = \sum_{j=1}^{k} \lambda_j e_j$, where $\{e_1, e_2, \ldots, e_k\}$ is a set of mutually orthogonal rank one constant projections in $(1 - P_1 - P_2)C([0,1], M_N)(1 - P_1 - P_2)$. Note that $rank(1 - (P_1 + P_2)) = k$.

Lemma (2.1.26)[80]: Let $n \geq 1$ be a given integer and let $\epsilon > 0$. There exists $\delta > 0$ satisfying the following: Choose $m_0 > 128(k_0 + 1)n\pi/\epsilon$, for any unitary u with $m \geq m_0$, if $v \in C([0,1], M_N)$ is another unitary such that

$$\|u - v\| < \delta,$$

then

$$\left|\mu_{tr,t,v}(\{e^{it}: s \in [t\theta_0 - \epsilon/2, t\theta_0 + \epsilon/2]\}) - 1/n\right| < \epsilon \qquad and \tag{129}$$

$$\left|\mu_{tr,t,v}(\{e^{is}: s \in [-t\theta_0/(n-1) - \epsilon/2, -t\theta_0/(n-1) + \epsilon/2]\}) - (n-1)/n\right|$$

$$< \epsilon \tag{130}$$

for all $t \in [1/(n-1), 1]$, where $\mu_{tr,t,v}$ is the probability measure given by $tr \circ \pi_t \circ \psi$, where $\psi : C(\mathbb{T}) \to C([0,1], M_N)$ is the homomorphism defined by $\psi(f) = f(v)$ for all $f \in C(\mathbb{T})$, and where tr is the normalized trace on M_N.

Lemma (2.1.27)[80]: Let $n \geq 12$. There exists $\delta > 0$ and integer $m_0 > 2^{15}(k_0 + 1)n^3\pi^2$ satisfying the following: If $h \in C([0,1], M_N)_{s.a.}$ with $\|h\| \leq 2\pi$ such that

$$\|u - \exp(ih)\| < \delta, \tag{131}$$

then

$$\|h\| \geq 2(1 - 1/(n-1))\pi. \tag{132}$$

Lemma (2.1.28)[80]: Let (G, G_+) be a countable unperforated ordered group. Then there exists a unital simple C^*-algebra A which is an inductive limit of interval algebras with $(K_0(A), K_0(A)_+) = (G, G_+)$ satisfying the following:
For any $\epsilon > 0$, there exists a unitary $u \in CU(A)$ and $\delta > 0$ satisfying the following: if $h \in A_{s.a.}$ with $\|h\| \leq 2\pi$ such that

$$\|u - \exp(ih)\| < \delta,$$

then

$$\|h\| \geq 2\pi - \epsilon.$$

Proof:
Fix $1/2 > \epsilon$. Choose $n \geq 12$ such that $\pi/(n-1) < \epsilon/4$. Let $k_0 = n - 1$. Let $m_0' > 2^{15}(k_0 + 1)n^3\pi^2$ (in place of m_0) be an integer required by lemma(2.1.27) for the above mentioned n and k_0. Let $m_0 = 2m_0'$.

Let $C = \lim_{k\to\infty}(C_k, \varphi_k)$ be a unital simple AF-algebra, where each C_k is a unital finitedimensional C^*-algebra, such that $(K_0(C), K_0(C)_+) = (G, G_+)$. We may assume that the map $\varphi_k : C_k \to C_{k+1}$ is unital and injective. We write

$$C_k = M_{r(1,k)} \oplus M_{r(2,k)} \oplus \cdots \oplus M_{r(m(k),k)}.$$

Let $\varphi_{k,j}: M_{r(j,k)} \to C_{k+1}$ be the homomorphism defined by $\varphi_{j,i,k} = (\varphi_k)|_{M_{r(j,k)}}$. Define $\pi_{k,j} : C_k \to M_{r(j,k)}$ by the projection to the summand. Set $\varphi_{j,i,k} = \pi_{k+1,i} \circ \varphi_{k,j} : M_{r(j,k)} \to M_{r(k+1,j)}$. Note that $(\psi_{j,i,k})_{*0}$ is determined by its multiplicity $M(j,i,k)$. Since C is simple, without lossof generality, we may assume that $r(j,1) := r(j) \geq 2n(m_0 + 1), j = 1, 2, \ldots, m(1)$. By passingto a subsequence if necessary, we may assume that $M(i,j,k) \geq (2m_0 + 1)k$. There is a set of $M(j,i,k)$ mutually orthogonal projections $\{e_{j,i,k,s} : s\}$ in $M_{r(i,k+1)}$ such that each $e_{j,i,k}$ has rank $r(j,k)$,

$$\sum_{j=1}^{m(j)} \sum_{s=1}^{M(j,i,k)} e_{j,i,k,s} = 1_{M_{r(i,k+1)}}.$$

Put

$$e(j,i,k) = \sum_{s=1}^{M(j,i,k)} e_{j,i,k,s}.$$

We write

$$r(j) = d(j)n + k(j), \quad k(j) < n,$$

where $d(j) \geq 2m_0$.

Denote $\theta_0 = (2 - 1/(n-1))\pi, j = 1, 2, \ldots, m(1)$. Let $B_{j,k} = C([0,1], M_{r(j,k)})$, $j = 1, 2, \ldots, m(k), k = 1, 2, \ldots$. Let $\{t(0,k), t(1,k), \ldots, t(k,k)\}$ be a partition of $[0,1]$ such that $t(0,k) = 0, t(k,k) = 1$ and $t(i,k) - t(i-1,k) = 1/(k+1), i = 1, 2, \ldots, k, k = 1, 2, \ldots$. Define $\psi_{j,i,k}: B_{j,k} \to e(j,i,k)B_{i,k+1}e(j,i,k)$ as follows:

$$\psi_{j,i,k}(f) = \mathrm{diag}\left(\underbrace{f, f, \ldots, f}_{M(j,i,k)-k}, f(t,(1,k)), f(t(2,k)), \ldots, f(t(k,k))\right) \tag{133}$$

for all $f \in M_{j,k}$. Define $A_k = C([0,1]) \otimes C_k$. Note that

$$A_k = \bigoplus_{j=1}^{m(k)} C([0,1], M_{r(j,k)}).$$

Let $\psi_k: A_k \to A_{k+1}$ be the unital homomorphism given by the partial maps $\psi_{j,i,k}$. Define $A = \lim_{n\to\infty}(A_k, \psi_k)$. It is known such defined A is a unital simple C^*-algebra. Moreover,

$$\left(K_0(A), \left(K_0(A)\right)_+\right) = (G, G_0).$$

Consider the unitaries

$$u_j = e^{i\theta_0}p_{1,j} + e^{-i\theta_0/(n-1)}p_{2,j} + p_{3,j},$$

where $\{p_{1,j}, p_{2,j}, p_{3,j}\} \subset M_{r(1,j)}$ are mutually orthogonal constant projections, $p_{1,j}$ has $rank\ d(j)$, $p_{2,j}$ has $rank\ (n-1)r(j)$ and $p_{3,j}$ has $rank\ k(j) < n, j = 1, 2, \ldots, m(1)$. Define

$$w = u_1 \oplus u_2 \oplus \ldots \oplus u_{m(1)}. \tag{134}$$

Let $u = \psi_{1,\infty}(w)$, where $\psi_{1,\infty}$ is the homomorphism induced by the inductive limit system. Sinceeach $u_j \in CU(C([0,1], M_{r(1,j)}), u \in CU(A)$. We now verify that u satisfies the assumption. Let$\delta_1 > 0$ be as in lemma (2.1.27) for $\epsilon/2$ (in place of ϵ) and $k_0 = n - 1$. Let $\delta = \delta_1/2$. Suppose that there is aself-adjoint element $h \in A_{s.a.}$ with $\|h\| \leq 2\pi$ such that

$$\|u - \exp(ih)\| < \delta. \tag{135}$$

There is, for a sufficiently larger k, a self-adjoint element $h_1 \in A_k$ for some $k \geq 1$ such that

$$\left\|h - \psi_{k,\infty}(h_1)\right\| < \frac{\epsilon}{4} \quad and \quad \left\|\psi_{k,\infty}(h_1) - \exp(ih_1)\right\| < 2\delta = \delta_1. \tag{136}$$

Consider a summand $A_{i,k}$ of A_k. Note that $A_{i,k} = C([0,1], M_{r(i,k)})$. We compute that

$$\psi_{1,k}(w) = e^{i\theta_0}P_{1,i,k} + e^{-i\theta_0/(n-1)}P_{2,i,k} + v_{0,i}, \tag{137}$$

where$v_{0,i} \in P_{3,i,k}E_{i,k}P_{3,i,k}$ is a constant unitary, $P_{1,i,k}, P_{2,i,k}, P_{3,i,k}$ are mutually orthogonalprojections with

$$P_{1,i,k} + P_{2,i,k} + P_{3,i,k} = id_{A_{i,k}},$$

$P_{2,i,k}$has$rank\ n - 1$ times as much as $P_{1,i,k}$ and $P_{1,i,k}$ has rank at least m_0 times that of therank of $P_{3,i,k}$. Denote by K_0 the rank of $P_{3,i,k}$. Then we have

$$rank P_{3,i,k} > 2^{15}n^3(K_0 + 1)\pi^2.$$

It follows from lemma (2.1.27) that

$$\|h_1\| \geq 2(1 - 1/(n-1))\pi = 2\pi - \frac{2\pi}{n-1} \geq 2\pi - \frac{\epsilon}{2}. \tag{138}$$

Note that each ψ is injective. Therefore

$$\left\|\psi_{k,\infty}(h_1)\right\| = \|h_1\| \geq 2\pi - \frac{\epsilon}{2}.$$

By (136),

$$\|h\| \geq 2\pi - \epsilon.$$

Theorem (2.1.29)[80]: Let $(G_0, (G_0)_+)$ be a countable weakly unperforated Riesz group and let G_1 be any countable abelian group. There exists a unital simple AH-algebra $A$with tracial rank onesuch that

$$(K_0(A). K_0(A)_+, K_1(A)) = (G_0, (G_0)_+, G_1).$$

Moreover, for any $\epsilon > 0$, there exists a unitary $u \in CU(A)$ and there exists $\delta > 0$ satisfying thefollowing: If $h \in A_{s.a.}$ such that

$$\|u - \exp(ih)\| < \delta,$$

then

$$\|h\| \geq 2\pi - \epsilon.$$

Corollary (2.1.30)[80]: Let $(G_0, (G_0)_+)$ be a countable weakly unperforated Riesz group and let G_1 be any countable abelian group. There exists a unital simple AH-algebra A with tracial rank one such that

$$(K_0(A). K_0(A)_+, K_1(A)) = (G_0, (G_0)_+, G_1) \quad and \tag{139}$$
$$cel_{CU}(A) > \pi. \tag{140}$$

Proof: Let A be in the conclusion of theorem(2.1.29). Let $\epsilon = \pi/16$. Choose a unitary u in A and δ satisfy the conclusion of theorem(2.1.29) for this ϵ. We may assume that $\delta < 1/64$. We will show that $cel(u) > \pi$. Otherwise, one obtains a self-adjoint element $h \in A$ with $\|h\| \le \pi$ such that

$$\|u - \exp(ih)\| < \delta.$$

This is not possible.

Section (2.2): Approximate Unitary Equivalence in Simple C^*-Algebras of Tracial Rank One

Let T_1 and T_2 be two normal operators in M_n, the algebra of $n \times n$ matrices. Then T_1 and T_2 are unitary equivalent, or there exists a unitary U such that $U^* T_1 U = T_2$ if and only if

$$sp(T_1) = sp(T_2),$$

counting the multiplicities. Let $X = sp(T_1)$. Define $\phi_i : C(X) \to M_n$ by

$$\phi(f) = f(T_i) \, for f \in C(X), \quad i = 1, 2.$$

Let $\tau : M_n \to \mathbb{C}$ be the normalized tracial state on M_n. Then $\tau \circ \phi_i$ $(i = 1, 2)$ gives a Borel probability measure μ_i on $C(X), i = 1, 2$. Then ϕ_1 and ϕ_2 are unitarily equivalent if and only if $\mu_1 = \mu_2$. More generally, one may formulate the following Let X be a compact metric space and let $\phi_1, \phi_2 : C(X) \to M_n$ be two homomorphisms. Then ϕ_1 and ϕ_2 are unitarily equivalent if and only if

$$\tau \circ \phi_1 = \tau \circ \phi_2. \tag{141}$$

For an infinite dimensional situation, one has the following classical result: two bounded normal operators on an infinite dimensional separable Hilbert space are unitary equivalent if and only if they have the same equivalent spectral measures and multiplicity functions. Perhaps a more interesting and useful statement is the following: let T_1 and T_2 be two bounded normal operators in $B(l^2)$. Then there exists a sequence of unitary $U_n \in B(l^2)$ such that

$$\lim_{n \to \infty} \|U_n^* T_1 U_n - T_2\| = 0 \, and$$
$$U_n^* T_1 U - T_2 \, is compact$$

if and only if

$$(i) sp_e(T_1) = sp_e(T_2),$$
$$(ii) \dim \text{null}(T_1 - \lambda I) = \dim \text{null}(T_2 - \lambda I) \, for all \lambda \in \mathbb{C} \setminus sp_e(T_1).$$

Here $sp_e(T_i)$ is the essential spectrum of T_i, i.e., $sp_e(T_i) = sp(\pi(T_i))$, where $\pi : B(l^2) \to B(l^2)/\kappa$ is the quotient map, i = 1, 2. Let X be a compact subset of the plane and let $\phi_1, \phi_2 : C(X) \to B(l^2)/\kappa$ be two unital monomorphisms. In the study of essentially normal operators on the infinite dimensional separable Hilbert space, one asks when ϕ_1

and ϕ_2 are unitarily equivalent. This was answered: ϕ_1 and ϕ_2 are unitarily equivalent if and only if $(\phi_1)_{*1} = (\phi_2)_{*1}, where (\phi_i)_{*1} : K_1(C(X)) \to K_1((B(l^2)/\kappa)) \cong \mathbb{Z} is$ the induced homomorphism (Fredholm index), $i = 1, 2$ (cf. [15]). In fact, one has the following more general BDF-theorem:

Theorem (2.2.1)[71]: If X is a compact metric space, then ϕ_1 and ϕ_2 are unitarily equivalent if and only

$$[\phi_1] = [\phi_2] in KK\left(C(X), \frac{B(l^2)}{K}\right).$$

It is known that the Calkin algebra $B(l^2)/K$ is a unital simple C^*-algebra with real rank zero. It is also purely infinite. We will study approximate unitary equivalence in a unital separable simple stably finite C^*-algebra.

Definition (2.2.2)[71]: Let A and B be two unital C^*-algebras and let $\phi_1, \phi_2 : A \to B$ be two homomorphisms. We say that ϕ_1 and ϕ_2 are approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\} \subset B$ such that

$$\lim_{n \to \infty} ad\, u_n \phi_1(a) = \phi_2(a) for all a \in A. \tag{142}$$

In Definition (2.2.2), suppose that $J = \mathrm{kcr}\phi_1$. Then $\ker\phi_2 = J$ if ϕ_1 and ϕ_2 are approximately unitarily equivalent. Thus one may study the induced monomorphisms from A/I to B instead of homomorphisms from A. To simplify matters, we will only study monomorphisms.

We note that M_n is a unital finite dimensional simple C^*-algebra with a unique tracial state. We now replace A by an infinite dimensional simple C^*-algebra. First we consider AF-algebras, approximately finite dimensional C^*-algebras.

Let A be a unital simple AF-algebra and let X be a compact metric space. Let $\phi_1, \phi_2 : C(X) \to A$ be two unital monomorphisms. When are ϕ_1 and ϕ_2 approximately unitarily equivalent or when are there unitaries $u_n \in A$ such that

$$\lim_{n \to \infty} u_n^* \phi_1(a) u_n = \phi_2(a)$$

for all $a \in C(X)$?

Let C be a unital stably finite C^*-algebra. Denote by $T(C)$ the tracial state space of C. Suppose that $\phi_1, \phi_2 : C(X) \to A$ are two unital monomorphisms. Let $\tau \in T(A)$ be a tracial state. Then $\tau \circ \phi_j$ is a normalized positive linear functional ($j = 1, 2$). It gives a Borel probability measure μ_j. Furthermore, it is strictly positive in the sense that $\mu_j(O) > 0$ for every non-empty open subset $O \subset X$. If ϕ_1 and ϕ_2 are approximately unitarily equivalent, then it is obvious that $\mu_1 = \mu_2$, or equivalently, $\tau \circ \phi_1 = \tau \circ \phi_2$. In fact, one has the following:

Let X be a compact metric space and let A be a unital simple AF-algebra with a unique tracial state τ. Suppose that $\phi_1, \phi_2 : C(X) \to A$ are two unital monomorphisms. Then ϕ_1 and ϕ_2 are approximately unitarily equivalent if and only if

$$(\phi_1)_{*0} = (\phi_2)_{*0} and \tau \circ \phi_1 = \tau \circ \phi_2.$$

Here $(\phi_1)_{*0}$ is an induced homomorphism from $K_0(C(X))$ into $K_0(A)$. Note in this case that X is connected and $K_0(A)$ has no infinitesimal elements, i.e., $\tau(p) = \tau(q)$ implies $[p] = [q]$ in $K_0(A)$ for any pair of projections p and q, as in the case that $A = M_n$, or in the case that A is a UHF-algebra, the condition $(\phi_1)_{*0} = (\phi_2)_{*0}$ is automatically satisfied if the two measures are the same. Note also that $K_1(A) = \{0\}$. In general, ϕ_j also gives another homomorphism:

$$(\phi_j)_{*1} : K_1(C(X)) \to K_1(A), j = 1, 2.$$

Theorem (2.2.3)[71]: Let X be a compact metric space and let A be a unital simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and with a unique tracial state τ. Suppose that $\phi_1, \phi_2 : C(X) \to A$ are two unital monomorphisms. Then ϕ_1 and ϕ_2 are approximately unitarily equivalent if and only if

$$[\phi_1] = [\phi_2] in KL(C(X), A) and \tau \circ \phi_1 = \tau \circ \phi_2.$$

In the case that $K_*(C(X))$ is torsion free, the condition that $[\phi_1] = [\phi_2]$ $in KL(C(X), A)$ can be replaced by $(\phi_1)_{*i} = (\phi_2)_{*i}$, where $(\phi_j)_{*i} : K_i i(C(X)) \to K_i(A)$ $(i = 0, 1$ $and j = 1, 2)$, is the induced homomorphism.

Recall that an AH-algebra is an inductive limit of C^*-algebras with the form $P_n M_{k(n)}(C(X_n))P_n$, where X_n is a (not necessarily connected) finite CW complex and P_n is a projection in $M_{k(n)}(C(X_n))$. More recently, for the situation that $T(A)$ has no restriction, we have the following:

Theorem (2.2.4)[71]: Let C be a unital AH-algebra with property (J) and let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \quad \text{in} \quad KL(C, A), \tag{143}$$
$$\phi = \psi \quad \text{and} \quad \phi^\ddagger = \psi^\ddagger. \tag{144}$$

Let X be a compact metric space, let $x \in X$ and let $a > 0$. Denote by $B_a(x)$ the open ball of X with radius a and center x. Let A be a unital C^*-algebra and $\xi \in X$. Denote by $\pi_\xi : C(X) \to A$ the point-evaluation defined by $\pi_\xi(f) = f(\xi) \cdot 1_A$ for all $f \in C(X)$. Let A and B be two C^*-algebras and let $L_1, L_2 : A \to B$ be two maps. Suppose that $\mathcal{F} \subset A$ is a subset and $\epsilon > 0$. We write

$$L_1 \approx_\epsilon L_2 \quad on \quad \mathcal{F}$$

If $\|L_1(a) - L_2(a)\| < \epsilon$ for all $a \in \mathcal{F}$.
The map L_1 is said to be ϵ-$\mathcal{F}$-multiplicative if

$$\|L_1(ab) - L_1(a)L_1(b)\| < \epsilon \text{ for all } a, b \in \mathcal{F}.$$

Let A be a C^*-algebra. Set $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$.
Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A. Denote by $U_0(A)$ the normal subgroup of $U(A)$ consisting of the path connected component of $U(A)$ containing the identity. Suppose that $u \in U_0(A)$ and $\{u(t) : t \in [0, 1]\}$ is a continuous path with $u(0) = u$ and $u(1) = 1$. Denote by length $(\{u(t)\})$ the length of the path. Put

$$\mathrm{cel}(u) = \inf\{\mathrm{length}(\{u(t)\})\}.$$

Definition (2.2.5)[71]: Let X be a compact metric space and let $P \in M_l(C(X))$ be a projection. Put $C = PM_l(C(X))P$. Let $u \in U(C)$. Define, as in [113],

$$D_C(u) = \inf\{\|a\| : a \in A_{s.a} \text{ such that} \det(e^{ia} \cdot u) = 1\}. \tag{145}$$

Let A be a unital C^*-algebra. Denote by $CU(A)$ the closure of the subgroup generated by the commutators of $U(A)$. For $u \in U(A)$, we will use $\bar{u}$ for the image of u in $U(A)/CU(A)$.

If $\bar{u}, \bar{v} \in U(A)/CU(A)$, define

$$dist(\bar{u}, \bar{v}) = \inf\{x - y : x, y \in U(A) \text{ such that } \bar{x} = \bar{u}, \bar{y} = \bar{v}\}.$$

$If\ u, v \in U(A)$, then

$$dist(\bar{u}, \bar{v}) = \inf\{\|uv^* - x\| : x \in CU(A)\}.$$

Let A and B be two unital C^*-algebras and let $\phi : A \to B$ be a unital homomorphism. It is easy to check that ϕ maps $CU(A)$ to $CU(B)$. Denote by $\phi^{\ddagger}$ the homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$ induced by ϕ. We also use $\phi^{\ddagger}$ for the homomorphism from $U(M_k(A))/CU(M_k(A))$ into $U(M_k(B))/CU(M_k(B))$ $(k = 1, 2, \dots)$.

Definition (2.2.6)[71]: Let A be a C^*-algebra. Following [21], denote

$$\underline{K}(A) = \bigoplus_{i=0,1} K_i(A) \bigoplus \bigoplus_{i=0,1 \ k \geq 2} K_i(A, \mathbb{Z}/k\mathbb{Z}).$$

Let B be a unital C^*-algebra. Furthermore,

$$Hom_\Lambda(\underline{K}(A), \underline{K}(B)) = KL(A, B).$$

Here $KL(A, B) = KK(A, B)/Pext(K_*(A), K_*(B))$ (see [21] for details). Let $k \geq 1$ be an integer. Denote

$$F_k\underline{K}(A) = \bigoplus_{i=0,1} K_i(A) \bigoplus_{n|k} K_i(A, \mathbb{Z}/k\mathbb{Z})$$
$$i = 0,1$$

Suppose that $K_i(A)$ is finitely generated $(i = 0, 2)$. It follows from [21] that there is an integer $k \geq 1$ such that

$$Hom_\Lambda\left(F_k\underline{K}(A), F_k\underline{K}(B)\right) = Hom_\Lambda\left(\underline{K}(A), \underline{K}(B)\right). \tag{146}$$

Let A and B be two unital C^*-algebras and let $L : A \to B$ be a unital contractive completely positive linear map. Let $\mathcal{P} \subset \underline{K}(A)$ be a finite subset. It is well known that, for some small δ and large finite subset $\mathcal{G} \subset A$, if L is also δ-$\mathcal{G}$-multiplicative, then $[L]|_{\mathcal{P}}$ is well defined. In what follows whenever we write $[L]|_{\mathcal{P}}$.

we mean δ is sufficiently small and $\mathcal{G}$ is sufficiently large so that it is well defined (see 2.3 of [70]). If $u \in U(A)$, we will use $\langle L \rangle(u)$ for the unitary $L(u)|L(u)^*L(u)|^{-1}$.

For an integer $m \geq 1$ and a finite subset $\mathcal{U} \subset U(M_m(A))$, let $F \subset U(A)$ be the subgroup generated by $\mathcal{U}$. There exists a finite subset $\mathcal{G}$ and a small $\delta > 0$ such that a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map L induces a homomorphism $L^{\ddagger} : \overline{F} \to U(M_m(B))/CU(M_m(B))$. Moreover, we may assume $\overline{\langle L \rangle(u)} = L^{\ddagger}(\bar{u})$.

If there are $L_1, L_2 : A \to B$ and $\epsilon > 0$ is given, suppose that both L_1 and L_2 are δ-$\mathcal{G}$-multiplicative and $L_1^{\ddagger}$ and $L_2^{\ddagger}$ are well defined on $\overline{F}$. Whenever we write

$$\text{dist}(L_1^{\ddagger}(\bar{u}), L_2^{\ddagger}(\bar{u})) < \epsilon$$

for all $u \in \mathcal{U}$, we also assume that δ is sufficiently small and $\mathcal{G}$ is sufficiently large so that

$$\text{dist}\left(\overline{\langle L_1 \rangle(u)}, \overline{(u)\langle L_2 \rangle}\right) < \epsilon \quad \text{for all } u \in \mathcal{U}.$$

Definition(2.2.7)[71]: Let A and B be two unital C^*-algebras. Let $h : A \to B$ be a homomorphism and $v \in U(B)$ such that

$$h(g)v = vh(g) \text{for all } g \in A.$$

Thus we obtain a homomorphism $\bar{h} : A \otimes C(\mathbb{T}) \to B$ by $\bar{h}(f \otimes g) = h(f)g(v)$ for $f \in A$ and $g \in C(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \to K_1(A \otimes C(\mathbb{T})) and \tag{147}$$

$$\beta^{(1)} : K_1(A) \to K_0(A \otimes C(\mathbb{T})). \tag{148}$$

The second one is the usual Bott map. Note that, in this way, one writes

$$K_i(A \otimes C(\mathbb{T})) = K_i(A) \oplus \beta^{(i-1)}(K_{i-1}(A)).$$

We use $\widehat{\beta^{(i)}} : K_i(A \otimes C(\mathbb{T})) \to \beta^{(i-1)}(K_{i-1}(A))$ for the projection to $\beta^{(i-1)}(K_{i-1}(A))$.

For each integer $k \geq 2$, one also obtains the following injective homomorphisms:

$$\beta_k^{(i)} : K_i\left(A, \frac{\mathbb{Z}}{k\mathbb{Z}}\right) \to K_{i-1}\left(A \otimes C(\mathbb{T}), \frac{\mathbb{Z}}{k\mathbb{Z}}\right), i = 0, 1. \tag{149}$$

Thus we write

$$K_{i-1}\left(A \otimes C(\mathbb{T}), \frac{\mathbb{Z}}{k\mathbb{Z}}\right) = K_{i-1}\left(A, \frac{\mathbb{Z}}{k\mathbb{Z}}\right) \oplus \beta_k^{(i)}\left(K_i\left(A, \frac{\mathbb{Z}}{k\mathbb{Z}}\right)\right), i = 0, 1. \tag{150}$$

Denote $\widehat{\beta_k^{(i)}} : K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to \beta_k^{(i-1)}(K_{i-1}(A, \mathbb{Z}/k\mathbb{Z}))$, similar to that of $\widehat{\beta^{(i)}}., i = 1, 2$. If $x \in \underline{K}(A)$, we use $\beta(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x \in k_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\beta : \underline{K}(A) \to \underline{K}(A \otimes C(\mathbb{T}))$ as well as $\hat{\beta} : \underline{K}(A \otimes C(\mathbb{T})) \to \beta(\underline{K}(A))$. Therefore one may write $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \beta(\underline{K}(A))$ On the other hand h induces homomorphisms

$$\bar{h}_{*i,k} : K_i(A \otimes C(\mathbb{T})),, \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}), \ k = 0, 2, \ldots, \ i = 0, 1.$$

We use Bott (h, v) for all homomorphisms $\bar{h}_{*i,k} \circ \beta_k^{(i)}$. We write

$$\text{Bott}(h, v) = 0$$

if $\bar{h}_{*i,k} \circ \beta_k^{(i)} = 0$ for all $k \geq 1$ and $i = 0, 1$. We will use $\text{bott}_1(h, v)$ for the homomorphism $\bar{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B)$ and $\text{bott}_0(\text{h, u})$ for the homomorphism $\bar{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)$. Since A is unital, if $\text{bott}_0(h, v) = 0$, then $[v] = 0$ in $K_1(B)$. In what follows, we will use z for the standard generator of $C(\mathbb{T})$ and we will often identify $\mathbb{T}$ with the unit circle without further explanation. With this identification z is the identity map from the circle to the circle.

Given a finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$Bott(h, v)|_{\mathcal{P}}$$

is well defined if
$$\|[h(a), v]\| = \|h(a)v - vh(a)\| < \delta_0 \text{ for all } a \in \mathcal{F}$$
(see [70]). There is $\delta_1 > 0$ ([101]) such that $\text{bott}_1(u, v)$ is well defined for any pair of unitaries$u$ and v such that $\|[u, v]\| < \delta_1$. As in 2.2 of [32], if $v_1, v_2, \ldots, v_n$ are unitaries such that
$$\|[u, v_j]\| < \delta_1/n, \qquad j = 1, 2, \ldots, n,$$
then
$$\text{bott}_1(u, , v_1, v_2, \ldots, v_n) = \sum_{j=1}^{n} \text{bott}_1(u, v_j).$$

By considering unitaries $z \in A \otimes C$ ($C = C_n$ for some commutative C^*-algebra with torsion K_0 and $C = SC_n$) from the above, for a given unital C^*-algebra A and a given finite subset $\mathcal{P} \subset \underline{K}(A)$, one obtains a universal constant $\delta > 0$ and a finite subset $\mathcal{F} \subset A$ satisfying the following:
$$Bott(h, v_j)|_{\mathcal{P}} \tag{151}$$
is well defined and$Bott(h, v_1, v_2, \ldots, v_n) = \sum_{j=1}^{n} \text{bott}(h, v_j)$
for any unital homomorphism h and unitaries $v_1, v_2, \ldots, v_n$ for which
$$\|[h(a), v_j]\| < \frac{\delta}{n}, \, j = 1, 2, \ldots, n, \, \text{for all } a \in \mathcal{F}. \tag{152}$$
Furthermore, if $K_i(A)$ is finitely generated, then (146) holds. Therefore, there is a finite subset $Q \subset \underline{K}(A)$ such that
$$Bott(h, v)$$
is well defined if $\text{Bott}(h, v)|_Q$ is well defined (see [70]). See [70] for further information.

Let A be a unital C^*-algebra. Denote by $T(A)$ the tracial state space of A. Suppose that $T(A) \neq \emptyset$. Let B be another unitalC^*-algebra with $T(B) \neq \emptyset$. Suppose that $\phi : A \to B$ is a unital homomorphism. Denote by $\phi : Aff(T(A)) \to Aff(T(B))$ the positive homomorphism defined by $\phi(\hat{a})(\tau) = \tau \circ \phi(a)$ for all $a \in A_{s.a}$.

Let X be a compact metric space and let A be a unital C^*-algebra with $T(A) \neq \emptyset$. Let $L : C(X) \to A$ be a unital positive linear map. For each $\tau \in T(A)$ denote by $\mu_{\tau \circ L}$ the Borel probability measure induced by $\tau \circ L$.

Let $X_1, X_2, \ldots, X_m$ be compact metric spaces. Fix a base point $\xi_i \in X_i, i = 1, 2, \ldots, m$. We write $X_1 \vee X_2 \vee \ldots \vee X_m$ as the space resulted by gluing $X_1, X_2, \ldots, X_m$ together at ξ_i (by identifying all base points at one point ξ_1). Denote by ξ_0 the common point. If $x, y \in X_i$, then $\text{dist}(x, y)$ is defined to be the same as that in X_i. If $x \in X_i, y \in X_j$ with $i \neq j$, and $x \neq \xi_0, y \neq \xi_0$, then we define
$$\text{dist}(x, y) = \text{dist}(x, \xi_0) + \text{dist}(y, \xi_0).$$

Definition (2.2.8)[71]: Let A be a unital simple C^*-algebra. A is said to have tracial rank no more than one ($TR(A) \leq 1$) if the following hold for any $\epsilon > 0$, any $a \in A_+ \setminus \{0\}$ and any finite subset $\mathcal{F} \subset A$, there exists a projection $p \in A$ and a C^*-subalgebra $B =$

$\oplus_{i=1}^k M_{r(i)}(C(X_i))$, where each X_i is a finite CW complex with covering dimension no more than 1, with $1_B = p$ such that:

(i) $\|px - xp\| < \epsilon$ for all $x \in \mathcal{F}$,

(ii) $\text{dist}(pxp, B) < \epsilon$ for all $\mathcal{F}$ and

(iii) $1 - p$ is equivalent to a projection in $\overline{aAa}$.

If in the above definition X_i can always be chosen to be a point, then we say A has tracial rank zero and write $T R(A) = 0$. If $T R(A) \leq 1$ but $T R(A) \neq 0$, then we write $T R(A) = 1$ and say A has tracial rank one. As in [91], if $T R(A) \leq 1$, then A has TAI, i.e., in the above definition, one may replace X_i by $[0, 1]$ or by a point.

Let A be a unital separable simple C^*-algebra with $T R(A) \leq 1$. Then A is tracially approximately divisible. For example, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset A$, any $a \in A_+ \setminus \{0\}$ and any integer $N \geq 1$, there exists a projection $p \in A$ and a finite dimensional C^*-subalgebra $D = \oplus_{i=1}^k M_{r(i)}$ with $r(j) \geq N$ and with $1_D = p$ such that:

(i) $\|[x, y]\| < \epsilon$ for all $x \in \mathcal{F}$ and for all $y \in D$ with $\|y\| \leq 1$;

(ii) $1 - p$ is equivalent to a projection in $\overline{aAa}$

Lemma (2.2.9)[71]: Let X be a connected simplicial complex, let $\mathcal{F} \subset C(X)$ be a finite subset, let $\epsilon > 0, \epsilon_1 > 0$ be positive numbers, and let $N \geq 1$ be an integer. There exists $\eta_1 > 0$ with the following properties.

For any $\sigma_1 > 0$ and any $\sigma > 0$, there exists a positive number $\eta > 0$ and an integer $K > 4/\epsilon$ (which are independent of σ), and there exists a positive number $\delta > 0$, an integer $L > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following.

Suppose that $\phi, \psi : C(X) \to P M_n(C(Y))P$ (where Y is a connected simplicial complex with $\dim Y \leq 3$), where $\text{rank}(P) \geq L$ are two unitalhomomorphisms such that

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1, \ \mu_{\tau \circ \phi}(O_\eta) \geq \sigma \eta \text{ for all } \tau \in T\left(PM_n(C(X))P\right) \qquad (153)$$

and for all open balls O_{η_1} with radius η_1 and open balls O_η with radius η_2, respectively, and

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}. \qquad (154)$$

Then there exist mutually orthogonal projections P_0 and P_1 (with $P_0 + P_1 = P$), a unital homomorphism $\phi_1 : C(X) \to P_1(M_n(C(Y))P_1)$ factoring through $C([0, 1])$, and a unitary $u \in P(M_n(C(Y)))P$ such that

$$\phi(f) - [P_0\phi(f)P_0 + \phi_1(f)] < 1/4K \qquad (155)\text{and}$$

$$\|ad \ u \circ \psi(f) - [P_0(ad \ u \circ \psi(f))P_0 + \phi_1(f)]\| < \frac{1}{4K} \text{ for all } f \in \mathcal{F}, \qquad (156)$$

$$\text{rank}P_0 \geq \frac{rank \ P}{K}, \qquad (157)$$

there are mutually orthogonal projections $q_1, q_2, \ldots, q_m \in P_1(M_n(C(Y)))P_1$ and an ϵ_1-dense subset $\{x_1, x_2, \ldots, x_m\}$ such that

$$\left\|\phi_1(f) - [(P_1 - \sum_{j=1}^m q_j)\phi_1(f)(P_1 - \sum_{j=1}^m q_j) + \sum_{j=1}^m f(x_j)q_j]\right\| < \epsilon (158)$$

for all $f \in \mathcal{F}$ and

$$\text{rank}(q_j) \geq N \cdot (\text{rank } P_0 + 2\dim Y), j = 1, 2, \ldots, m. \tag{159}$$

Proof: This is a reformulation of Proposition 4.47$'$ of [51] and follows from that immediately.

We now will apply in [51]. Let $\epsilon > 0, \epsilon_1 > 0$, N and $\mathcal{F}$ be given. Choose $\eta_0 > 0$ such that

$$|f(x) - f(x')| < \frac{\epsilon}{2} \text{ for all } f \in \mathcal{F}. \tag{160}$$

Choose $\epsilon_2 = \min\{\epsilon_1/3N, \eta_0/3N\}$. Let $\eta_1' > 0$ (in place of η) be as in [51] for $\epsilon/2, \epsilon_2$ (in place of ϵ_1) and $\mathcal{F}$. Let $\sigma_1 > 0$ and $\sigma > 0$. Put $\delta_1 = \sigma_1 \cdot \eta_1'/32$. Let $K > 4/\epsilon$ and $\tilde{\eta}$ be as in [51] for the above $\epsilon/4$, ϵ_2 (in place of ϵ_1) and δ_1 (in place of δ). Let $\delta = \sigma \cdot \tilde{\eta}/32$. Let $L \geq 1$ be an integer and let $\mathcal{G} \subset C(X)$ be a finite subset which corresponds to the finite subset H in [51]. Let $\eta_1 = \eta_1'/32, \eta = \tilde{\eta}/32$ and let $0 < \delta < \tilde{\delta}/4$. Suppose that ϕ and ψ satisfy the assumption of the lemma for the above $\eta_1, \eta, \delta, K, L$ and $\mathcal{G}$.

It follows that ϕ has the properties $\text{sdp}(\eta_1/32, \delta_1)$ and $\text{sdp}(\tilde{\eta}/32, \tilde{\delta})$ (see [51]). One then applies Proposition 4.47$'$ of [51] to obtain

$$\|\phi(f) - [P_0 \phi(f) P_0 + \phi_1(f)]\| < \frac{1}{4K} \tag{161}$$

and

$$\|ad\, u \circ \psi(f) - [P_0(ad\, u \circ \psi(f))P_0 + \phi_1(f)]\| < 1/4K \text{ for all } f \in \mathcal{F}. \tag{162}$$

and mutually orthogonal projections $e_1, e_2, \ldots, e_{m_1}$ in $P_1(M_n(C(Y)))P_1$ and $\epsilon_2/4$-dense subset $\{x_1', x_2', \ldots, x_{m_1}'\}$ of X such that

$$\left\| \phi_1(f) - \left[(P_1 - \sum_{i=1}^{m_1} c_i)\phi_1(f)(P_1 - \sum_{i=1}^{m_1} c_i) + \sum_{i=1}^{m} f(x_i')c_i\right] \right\| < \frac{\epsilon}{2} \tag{163}$$

for all $f \in \mathcal{F}$,

$$\text{rank } P_0 \geq \frac{\text{rank } P}{K} \text{ and } \text{rank} e_i \geq \text{rank } P_0 + 2\dim Y. \tag{164}$$

Since there are at least N many disjoint open balls with radius ϵ_2 in an open ball of radius ϵ_1, by moving points within $N_{\epsilon_2} < \min\{\epsilon_1/2, \eta_0\}$, by (37), one may write

$$\left\| \phi_1(f) - \left[(P_1 - \sum_{i=1}^{m_1} e_i)\phi_1(f)(P_1 - \sum_{i=1}^{m_1} e_i) + \sum_{i=1}^{m} f(x_i)q_i\right] \right\| < \epsilon$$

for all $f \in \mathcal{F}$ \hfill (165)

and

$$\text{rank} q_i \geq N(\text{rank} P_0 + 2\dim Y) \tag{166}$$

where $\sum_{i=1}^{m} q_i = \sum_{i=1}^{m_1} e_i$.

The following is a generalization of [36]. The proof is essentially the same, but we will also apply [45].

Theorem (2.2.10)[71]: Let X be a finite simplicial complex, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\epsilon > 0$. There exists $\eta_1 > 0$ with the following property.

For any $\sigma_1 > 0$ and $\sigma > 0$, there exists $\eta > 0$ and an integer K (which are independent of σ), there exists $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{U} \subset P^{(1)}(C(X))$ and a positive integer L satisfying the following.

Suppose that $\phi, \psi : C(X) \to PM_k(C(Y))P$, where Y is a connected simplicial complex with $\dim Y \leq 3$, are two unital homomorphisms such that
$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1 \text{ and } \mu_{\tau \circ \phi}(O_\eta) \geq \sigma \eta \tag{167}$$
for all open balls O_{η_1} with radius η_1 and open balls O_η with radius η, and
$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \tag{168}$$
and for all $\tau \in T(PM_k(C(Y))P)$,
$$\mathrm{rank}(P) \geq L, \tag{169}$$
$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \tag{170}$$
and
$$\mathrm{dist}\left(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})\right) < \frac{1}{8K\pi} \tag{171}$$
for all $z \in \mathcal{U}$. Then there exists a unitary $u \in PM_k(C(X))P$ such that
$$\|\phi(f) - \mathrm{ad}\, u \circ \psi(f)\| < \text{for all } f \in \tag{172}$$

Proof: It is clear that we may assume that X is connected. Since X is a simplicial simplex, there is $k_0 \geq 1$ such that for any unital separable C^*-algebra A,
$$Hom_\Lambda\left(\underline{K}(C(X)), \underline{K}(A)\right) = Hom_\Lambda\left(F_{k_0}\underline{K}(C(X)), F_{k_0}\underline{K}(A)\right) \text{(see [21])}.$$
Let C_j be a commutative C^*-algebra with $K_0(C_j) = \mathbb{Z}/j\mathbb{Z}$ and $K_1(C_j) = \{0\}$, $j = 1, 2, \ldots, k_0$. Put $D_0 = C(X)$ and $D_j = (C(X) \otimes C_j), j = 1, 2, \ldots, k_0$. There is an integer $m_1 \geq 1$ such that $U(M_{m_1}(D_j))/U_0(M_{m_1}(D_j)) = K_1(D_j), j = 0, 1, 2, \ldots, k_0$. Put $N_1 = (m_1)^2$. Let $r : \mathbb{N} \to \mathbb{N}$ such that $r(n) = 3k_0 n$. Let $b : U(M_\infty(C(X))) \to \mathbb{R}_+$ be defined by $b(u) = (8 + 2N_1)\pi$.

Let $\epsilon > 0$ and $\mathcal{F}$ be given. We may assume, without loss of generality, that $\mathcal{F}$ is in the unit ball of $C(X)$. Let $1 > \delta_1 > 0$ (in place of δ), let $\mathcal{G}_1 \subset C(X)$, let $l \geq 1$ be an integer, let $\mathcal{P}_0 \subset P^{(0)}(C(X))$ and let $\mathcal{U} \subset P^{(1)}(C(X))$ be as required by Theorem 1.1 of [45] for $\epsilon/4$ and $\mathcal{F}$ (and for the above r and b). We may assume that $\mathcal{U} \subset \cup_{j=0}^{k_0} M_{m_1}(D_j))$. We may also assume that there is $l_1 \geq 1$ such that $\mathcal{P}_0 \subset \cup_{j=0}^{k_0} M_{l_1}(D_j)$.

We also assume that, for any unital C^*-algebra A, if u is a unitary and e is a projection for which
$$\|eu - ue\| < \delta',$$
there is a unitary $v \in eAe$ such that
$$\|eue - v\| < 2\delta'$$
for any $0 < \delta' < \delta_1$.

Set $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_1$. Let $\epsilon_1 > 0$ be such that
$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F}_1, \tag{173}$$
if $\mathrm{dist}(x, x') < \epsilon_1$.

Put $N = l + 1$ and $\epsilon_2 = \min\{\delta_1/4, \epsilon/4\}$. Let $\eta_1 > 0$ be required by Lemma (2.2.9) for $\epsilon/2$ (in place of ϵ), ϵ_1, $\mathcal{F}_1$ (in place of $\mathcal{F}$) and N. Fix $\sigma_1 > 0$. Let $\eta > 0$ and $K_1 > 4N_1/\epsilon_2$ (in place of K) be required by Lemma (2.2.9) Fix $\sigma > 0$. Let $\delta > 0$, an integer $L > 0$ and let $\mathcal{G} \subset C(X)$ be a finite subset required by Lemma(2.2.9)for ϵ_2 (in place of ϵ), $\mathcal{F}_1$ (in place of $\mathcal{F}$), σ, σ_1, and N.

We may assume that $\mathcal{G} \supset \mathcal{F}_1$. Let $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset which consists of the image of $\mathcal{P}_0$ and the image of $\mathcal{U}$ in $\underline{K}(C(X))$, and let $K = 2N_1K_1$.

Now suppose that $\phi, \psi : C(X) \to PM_k(C(Y))P$ are unital homomorphisms such that (167), (168), (169), (170) and (171)) hold. It follows from Lemma (2.2.9) that there are mutually orthogonal projections P_0 and P_1 with $P_0 + P_1 = P$, a unital homomorphism $\phi_1 : C(X) \to P_1(M_n(C(Y))P_1)$ factoring through $C([0,1])$, and a unitary $v \in P(M_n(C(Y)))P$ such that

$$\phi(f) - [P_0\phi(f)P_0 + \phi_1(f)] < \frac{1}{4K_1} \tag{174}$$

and

$$\text{ad } v \circ \psi(f) - [P_0(\text{ad } v \circ \psi(f))P_0 + \phi_1(f)] < 1/4K_1 \text{ for all } f \in \mathcal{F}_1, \tag{175}$$

$$\text{rank}P_0 \geq \frac{\text{rank}P}{K_1} \tag{176}$$

there are mutually orthogonal projections $q_1, q_2, \ldots, q_m \in P_1(M_n(C(Y)))P_1$ and an ϵ_1-dense subset $\{x_1, x_2, \ldots, x_m\}$ such that

$$\left\| \phi_1(f) - [(P_1 - \sum_{j=1}^{m_1} q_j)\phi_1(f)(P_1 - \sum_{j=1}^{m_1} q_j) + \sum_{j=1}^{m} f(x_j)q_j] \right\| < \epsilon_2$$

$$\text{for all } f \in \mathcal{F}_1 \tag{177}$$

and

$$\text{rank}(q_j) \geq N(\text{rank}P_0 + 2\dim Y), j = 1, 2, \ldots, m. \tag{178}$$

Note that $1/4K_1 < \delta_1/16(N_1)$. For each C_j, we may assume that

$$C_j = C_0(Z_j \setminus \{\xi_j\}),$$

where Z_j is a path connected CW complex with $K_0(Z_j) = \mathbb{Z} \oplus \mathbb{Z}/j\mathbb{Z}$ and $K_1(Z_j) = \{0\}$ and where $\xi_j \in Z_j$ is a point, $j = 1, 2, \ldots, k_0$.

For each $z \in \mathcal{U}$ and $z \in M_{m_1}(D_j)$, denote $z_1 = (\tilde{\phi} \otimes id_{m_1})(z)$ and $z_2 = (\text{ad } v \circ \psi) \otimes id_{m_1}(z)$, where $\tilde{\phi}, \text{ad } v \circ \psi : D_j \to (C(Y) \otimes \tilde{C_j})$ is the induced homomorphism.

Identify $(M_k(C(Y)\tilde{C_j}))$ with a C^*-subalgebra of $C(Z_j, M_k(C(Y)))$ and denote by P_0' the constant projection which is P_0 at each point of Z_j and by P' the constant projection which is P at each point of Z_j. There are unitaries $z_1', z_2' \in M_{m_1}(P_0'M_k(C(Y) \otimes \tilde{C_j})P_0')$ such that

$$\|z_1' - \bar{P}_0z_1\bar{P}_0\| < \frac{2N_1}{4K_1} < \frac{\delta_1}{8}, \|z_2' - \bar{P}_0z_2\bar{P}_0\| < \frac{2N_1}{4K_1} < \frac{\delta_1}{8} \tag{179}$$

$$\|z_1 - z_1' \oplus \phi_1(z)\| < \frac{3(N_1)^2}{4K_1} < \delta_1/4 \text{ and} \|z_2 - z_2' \oplus \phi_1(z)\| < \frac{3(N_1)^2}{4K_1} < \frac{\delta_1}{4}, \tag{180}$$

where $\bar{P} = \text{diag} \overbrace{(P', P', \ldots, P')}^{m_1}$ and $\bar{P}_0 = \text{diag} \overbrace{(P'_0, P'_0, \ldots, P'_0)}^{m_1}$. By (171)), one computes that

$$\overline{\text{dist}(z'_1 \oplus \phi_1(z), (z'_2 \oplus \phi_1(z)))} \leq \frac{1}{4K\pi} + \frac{6N_1}{4K_1} < \frac{1 + 6N_1^2\pi}{4N_1K_1\pi}, \tag{181}$$

where $\overline{(z'_1 \oplus \phi_1(z)}$ and $\overline{(z'_2 \oplus \phi_1(z)}$ are the images of $(z'_1 \oplus \phi_1(z)$ and $(z'_2 \oplus \phi_1(z)$. It follows that

$$D\left(z'_1(z'_2)^* \oplus (\bar{P} - \&\bar{P}_0)\right) + \frac{1 + 6N_1^2\pi}{4N_1K_1\pi} \tag{182}$$

where D is the determinant defined in Definition (2.2.5)
Since $\text{rank}P_0 \geq \frac{\text{rank }P}{K_1}$, see [113],

$$D_{P_0M_k(C(Y))P_0}(z'_1(z'_2)^* + \frac{1 + 6N_1^2\pi}{4N_1} \tag{183}$$

By the choice of $\mathcal{P}$ and the assumption (170), since $\dim Y \leq 3$,

$$(z'_1(z'_2)^* \oplus \text{diag} \overbrace{(P'_0, P'_0, \ldots, P'_0)}^{3k_0m_1}) \quad \in U_0(M_{3k_0m_1}(P'_0M_k(D_j)P'_0)). \tag{184}$$

By the theorem (2.2.11) of [113],

$$\text{cel}\left(z'_1(z'_2)^* \oplus \text{diag} \overbrace{(P'_0, P'_0, \ldots, P'_0)}^{3k_0m_1}\right) \leq (2N_1\pi + \pi) + 6\pi \leq (2N_1 + 7)\pi \tag{185}$$

for all $z \in \mathcal{U}$. Denote $\phi' = P_0\phi P_0$ and $\psi' = P_0(ad\, u \circ \psi)P_0$. Then both are $\delta_1 - \mathcal{F}_1$-multiplicative. By the assumption (170),

$$[\phi']|_{\mathcal{P}} = [\psi']|_{\mathcal{P}}. \tag{186}$$

Since $\dim Y \leq 3$, for any $p \in \mathcal{P}_0$, it follows that

$$[\phi'](p) \oplus \text{diag} \overbrace{(P_0, P_0, \ldots, P_0)}^{3k_0l_1} \sim [\psi'](p) \oplus \text{diag} \overbrace{(P_0, P_0, \ldots, P_0)}^{3k_0l_1} \tag{187}$$

for all $p \in \mathcal{P}_0$. Note that $3k_0l_1 = r(l_1)$ and $(2N_1 + 7)\pi + \delta_1/4 < b(z)$ for any z. Since (178) holds, $N \geq l$ and $\{x_1, x_2, \ldots, x_m\}$ is 1-dense in X. By (remark) of [45], there exists a unitary

$$u_1 \in ((P_0 + \sum_{j=1}^{m} q_j))PM_k(C(Y))P(P_0 + \sum_{j=1}^{m} q_j)$$

such that

$$\left\| u_1^*(\psi'(f) \oplus \sum_{j=1}^{m} f(x_j)q_j)u_1 - \phi'(f) \oplus \sum_{j=1}^{m} f(x_j)q_j \right\| < \frac{\epsilon}{4} \tag{188}$$

for all $f \in \mathcal{F}$.
Define $u = (u_1 \oplus P - (P_0 \oplus \sum_{j=1}^{m} q_j))v \in PM_k(C(Y))P$. Then, by (265),(177),(174) and (124),

$$\|ad\, u \circ \psi(f) - \phi(f)\| < \quad for\ all\ f \in \mathcal{F}. \tag{189}$$

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Theorem (2.2.11)[71]: Let X be a compact metric space and $L: U(M_\infty(A)) \to R_+$ be a map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exists a positive number $\delta > 0$, a finite subset $\mathcal{G}$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{U} \subset U(M_\infty(A))$, an integer $l \geq 1$ and $\epsilon_1 > 0$ satisfying the following. If $\phi, \psi: C(X) \to \mathcal{B}$ (where $\mathcal{B} = \oplus_{j=1}^m C(X_j, M_{r(j)})$, $X_j = [0,1]$, or X_j is a point) are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps with

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } cel(\phi(v)^*\psi(v)) \leq L(u) \tag{190}$$

for all $v \in \mathcal{U}$, then there exists a unitary $u \in M_{lm+1}(B)$ such that

$$\|u^* \text{diag}(\phi(f), \sigma(f))u - \text{diag}(\psi(f), \sigma(f))\| < \epsilon \tag{191}$$

for all $f \in \mathcal{F}$, where $\sigma(f) = \sum_{i=1}^m f(x_i)e_i$ for any ϵ_1-dense set $\{x_1, x_2, \ldots, x_m\}$ and any set of mutually orthogonal projections $\{e_1, e_2, \ldots, e_m\}$ in $M_{lm}(B)$ such that e_i is equivalent to $\text{id}_{M_l}(B)$.

To prove the above theorem, we note that B has stable rank one, K_0-divisible rank $T(n,k) = [n/k] + 1$, and exponential length divisible rank $E(L,n) = 8\pi + L/n$ (see [46]). Therefore we have the following.

Corollary (2.2.12)[71]: Let X be a simplicial finite CW complex, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\epsilon > 0$. There exists $\eta_1 > 0$ with the following property.

For any $\sigma_1 > 0$ and $\sigma > 0$, there exists $\eta > 0$ and an integer K (which are independent of σ), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{U} \subset U(M_\infty((C(X)))$ and a positive integer L satisfying the following. Suppose that $\phi, \psi: C(X) \to B = \oplus_{j=1}^m C(X_j, M_{r(j)})$ (where $X_j = [0,1]$ or X_j is a point) are two unital homomorphisms such that

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1 \text{ and } \mu_{\tau \circ \phi}(O_\eta) \geq \sigma\eta, \tag{192}$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \tag{193}$$

and for all $\tau \in T(B)$,

$$\min_j\{rank(r(j))\} \geq L, [\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and} \tag{194}$$

$$dist\left(\phi^\ddagger(z), \psi^\ddagger(z)\right) < \frac{1}{8K\pi} \tag{195}$$

for all $z \in \mathcal{U}$. Then there exists a unitary $u \in B$ such that

$$\|\phi(f) - \text{ad } u \circ \psi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \tag{196}$$

$$\lim_{n \to \infty}\|\phi_n(f)\phi_n(g) - \phi_n(fg)\| = 0 \text{ for all } f, g \in C(\mathbb{T} \times \mathbb{T}) \tag{197}$$

And $\{\phi_n\}$ is away from homomorphisms. Therefore $\{\phi_n\}$ are not approximately unitarily equivalent to homomor-phisms. This is because $[\phi_n](b) \neq 0$, where b is the bott element. However, even when X is contractive, as long as $\dim X > 2$, one always has a sequence of contractive completely positive linear maps $\phi_n: C(X) \to M_n$ such that (197) holds and $\{\phi_n\}$ is away from any homomorphisms (see [44]). Therefore the condition on KK-theory (212) as well as the condition on the measure (213) in Lemma (2.2.15) are essential.

The following is a version in [65] and follows from that immediately.

Lemma (2.2.13)[71]: Let X be a compact metric space, $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\eta > 0$ which depends on ϵ and $\mathcal{F}$ for which
$$|f(x) - f(x')| < \epsilon/8 \text{ for all } f \in \mathcal{F},$$
if $\text{dist}(x, x') < \eta$, and for which the following holds.

For any $\eta/2$-dense subset $\{x_1, x_2, \ldots, x_m\}$ and any integer $s \geq 1$ for which $O_i \cap O_j = \emptyset \ (i \neq j)$, where
$$O_i = \{x \in X : \text{dist}(x_i, x) < \eta/2s\},$$
and for any $\sigma > 0$ for which $1/2s > \sigma > 0$, there exist $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi, \psi : C(X) \to A$ (for any unital simple C^*-algebra with tracial rank zero, infinite dimensional or finite dimensional) are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{198}$$
$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \tau \in T(A), \tag{199}$$
$$\mu_{\tau \circ \phi}(O_i) \geq \sigma\eta \text{ and } \mu_{\tau \circ \psi}(O_i) \geq \sigma\eta \tag{200}$$

$i = 1, 2, \ldots, m$.

Then there exists a unitary $u \in A$ such that
$$\text{ad } u \circ \phi \approx \psi \text{ on } \mathcal{F}. \tag{201}$$

Lemma (2.2.14)[71]: Let X be a compact metric space, let $\sigma_1 > 0, 1 > \eta_1 > 0$ and let $\sigma > 0$. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C(X)$, there exist $\eta > 0$(which depends on and $\mathcal{F}$ but not on σ_1, σ, or η_1), $\delta > 0$, and a finite subset $\mathcal{G}$ (*both depend on* $\epsilon, \mathcal{F}, \sigma_1, \sigma$ *and* η_1) satisfying the following.

Suppose that $\phi : C(X) \to M_n$ (for any integer $n \geq 1$) is a δ-G-multiplicative contractive completely positive linear map such that
$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1\eta_1 \text{ and } \mu_{\tau \circ \phi}(O_\eta) \geq \sigma\eta \tag{202}$$
for all open balls with radius η_1 and η, respectively.

Then there exists a unital homomorphism $h : C(X) \to M_n$ such that
$$|\tau \circ h(f) - \tau \circ \phi(f)| < \epsilon \text{ for all } f \in \mathcal{F}, \tag{203}$$
$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1 \text{ and } \mu_{\tau \circ h}(O_\eta) \geq (\sigma/2)\eta, \tag{204}$$
for all $\tau \in T(A)$.

Proof: We apply Lemma (2.2.15) of [65]. Let $\gamma > 0$ and $\mathcal{F}_1 \subset C(X)$ be a finite subset. It follows from Lemma (2.2.15) of [65] that, for a choice of δ and $\mathcal{G}$, there is a projection $p \in M_n$ and a unital homomorphism $h_0 : C(X) \to pM_np$ such that
$$\|\phi(f) - [(1-p)\phi(f)(1-p) + h_0(f)]\| < \gamma \text{ for all } f \in \mathcal{F}_1 \tag{205}$$
and
$$\tau(1-p) < \gamma. \tag{206}$$
Moreover, for any open ball O_η with radius η,
$$\int_{O_\eta} h_0 d\mu_{\tau \circ h_0} > \left(\frac{\sigma}{2}\right)\eta \tag{207}$$

Let $h_1: C(X) \to (1-p)M_n(1-p)$ be a unital homomorphism and define $h = h_1 \oplus h_0$. Therefore

$$\mu_{\tau \circ h}(O_\eta) > \left(\frac{\sigma}{2}\right)\eta \tag{208}$$

for any open ball with radius η. Moreover,

$$|\tau \circ \phi(f) - \tau \circ h(f)| < 2\gamma \; for \; all \; f \in \mathcal{F}_1 \tag{209}$$

We choose $\gamma < \epsilon/2$ and $\mathcal{F}_1 \supset \mathcal{F}$. It is easy to see that, if we choose sufficiently small γ and sufficiently large $\mathcal{F}_1$, we may also have

$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1.$$

Lemma (2.2.15)[71]: Let X be a path connected compact metric space, let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset, and let $\sigma_1 > 0, \sigma > 0$ and $1 > \eta_1 > 0$. Then, there exists $\eta > 0$ (which depends on ϵ and $\mathcal{F}$ but not on σ_1, σ or η_1), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to M_n$ (for any integer $n \geq 1$) is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map such that

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma \cdot \eta \; and \; \mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \cdot \eta_1 \tag{211}$$

for all open balls with radius η and η_1, respectively, and

$$[\phi]|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}} \tag{212}$$

for some point $\xi \in X$. Then there exists a unital homomorphism $h : C(X) \to M_n$ such that

$$\|\phi(f) - h(f)\| < \epsilon \; forall \; f \in \mathcal{F}, \tag{213}$$
$$\mu_{\tau \circ h}(O_{\eta_1}) \geq (\sigma_1/2)\eta_1 \; and \; \mu_{\tau \circ h}(O_n) \geq (\sigma/2)\eta. \tag{214}$$

Proof: Fix $\epsilon > 0$, a finite subset $\mathcal{F} \subset C(X), \sigma_1, \sigma \; and \; 1 > \eta_1 > 0$. Let $\eta_2 > 0$ be a positive number such that

$$|f(x) - f(x')| < \epsilon/16,$$

if $\text{dist}(x, x') < \eta_2$. We may assume that $\eta_2 < \eta_1$. Let $s, \mathcal{G}_1$ (in place of $\mathcal{G}$), δ_1 (in place of δ) and $\mathcal{P} \subset \boldsymbol{P}(C(X))$ be as in Lemma (2.2.13) (for the above $\epsilon/2, \eta_2$ and σ).

Let $\eta > 0, \delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ be as in Lemma (2.215) required for γ (in place of ϵ), $\mathcal{G}_1 \cup \mathcal{F}$ (in place of $\mathcal{F}$), σ (with $\sigma_1 = \sigma$) and η_2 (in place of η_1) above.

Now suppose that $\phi : C(X) \to M_n$ is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map satisfying the assumption with the above $\eta, \delta, \mathcal{G}$ and $\mathcal{P}$. By applying Lemma (2.2.14), one obtains a unital homomorphism $h_1 : C(X) \to M_n$ such that

$$|\tau \circ \phi(g) - \tau \circ h_1(g)| < \delta_1 \; for \; all \; g \in \mathcal{G}_1,$$
$$\mu_{\tau \circ h_1}(O_n) \geq \left(\frac{\sigma}{2}\right)\eta \; and \tag{215}$$
$$\mu_{\tau \circ h_1}(O_{\eta_2}) \geq \left(\frac{\sigma}{2}\right)\eta_2. \tag{216}$$

Since X is a path connected,

$$[h_1] = [\pi_\xi].$$

It follows that

$$[h_1]|_{\mathcal{P}} = [\phi]|_{\mathcal{P}}.$$

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It then follows from Lemma(2.2.13) that there exists a unitary $u \in M_n$ such that
$$\text{ad } u \circ h \approx_\epsilon \phi \text{ on } \mathcal{F}.$$
Put $h = \text{ad } u \circ h_1$. One also has that
$$\mu_{\tau \circ h}(O_n) = \mu_{\tau \circ h_1}(O_n) \geq \sigma \cdot \eta/2.$$
Note that, if one can choose δ_1 sufficiently smaller and G_1 sufficiently larger, one may also require that
$$\mu_{\tau \circ h}(O_{n_1}) \geq (\sigma_1/2)\eta_1.$$

Lemma (2.2.16)[71]: Let X be a compact metric space and let A be a finite dimensional C^*-algebra. Suppose that $\phi : C(X) \to A$ is a unital homomorphism and $u \in A$ is a unitary such that
$$\phi(f)u = u\phi(f) \text{ for all } f \in C(X).$$
Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ such that $u(0) = u, u(1) = u, \phi(f)u(t) = u(t)\phi(f)$ for all $f \in C(X)$ and $\text{Length}(\{u(t)\}) \leq \pi$.

Proof: Define $H : C(X \times \mathbb{T}) \to A$ by $H(f \otimes g) = \phi(f)g(u)$ for $f \in C(X)$ and $g \in C(\mathbb{T})$. Note that $H(C(X))$ is a commutative finite dimensional C^*-algebra. The lemma follows immediately.

Lemma (2.2.17)[71]: Let X be a compact path connected metric space, let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\eta > 0$ such that the following holds.

For any $\sigma > 0$, there exists an integer $s \geq 1, \delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset K(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to M_n$ (for some integer n) is a unital homomorphism and a unitary $u \in M_n$ such that there is a δ-G-multiplicative contractive completely positive linear map $\Phi : C(X \times \mathbb{T}) \to M_n$ such that
$$\|\Phi(f \otimes 1) - \phi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \|u - \Phi(1 \otimes z)\| < \delta, \tag{217}$$
where z is the identity map on the unit circle,
$$\text{Bott}(\phi, u)|_{\mathcal{P}} = \{0\} \tag{218}$$
$$\text{and} \quad \mu_{\tau \circ \Phi}(O_{n/2s}) \geq \sigma\eta \tag{219}$$
for any open ball $O_{n/2s}$ of $X \times \mathbb{T}$ with radius $\eta/2s$.

Then there is a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ such that $u(0) = u, u(1) = 1$ $\|\phi(f), u(t)\| < \epsilon \text{ for all } f \in \mathcal{F}$ and
$$\text{length}(\{u(t)\}) \leq \pi + \epsilon\pi.$$

Proof: Let $\epsilon > 0$ and $\mathcal{F}$ be as in the statement. We may assume that $\epsilon < 1/4$. Let $Y = X \times \mathbb{T}$ and
$$\mathcal{F}_1 = \{f \times g : f \in \mathcal{F} \cup \{1\}, g = 1 \text{ and } g = z\},$$
where z is the identity map of the unit circle.

Let $\eta > 0$ be as in Lemma (2.2.15) for $\mathcal{F}_1$ (instead of $\mathcal{F}$) and $\epsilon/4$ (instead of ϵ) for Y. Fix $\sigma_1 = \sigma > 0$ (and $\eta_1 = \eta$). Let $s \geq 1, \delta_0$ (in place of δ), $\mathcal{G}_1$ (in place of $\mathcal{G}$) and $Q \subset K(C(X \times \mathbb{T}))$ (in place of $\mathcal{P}$) be as required by Lemma (2.2.15) for the above $\epsilon/4, F, \eta$

and σ_1 (and for Y). There is $\delta_1 > 0$, a finite subset $\mathcal{G}_1 \subset C(X \times \mathbb{T})$ and a finite subset $Q \subset \beta(\underline{K}(C(X)))$ such that

$$[\Psi]|_{\beta(Q)} = [\pi_\xi]|_{\beta(Q)}$$

for any δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map for which

$$\|\Psi(f \otimes 1) - \phi(f)\| < \delta_1 \text{ for all } f \in \mathcal{G}_1, \|\Phi(1 \otimes z) - v\| < \delta_1 \qquad (220)$$
$$\text{and Bott}(\phi, v)|_{\mathcal{P}} = \{0\} \qquad (221)$$

(for any unitary $v \in M_n$ satisfying the above).

Now suppose that ϕ and u satisfy the assumption for the above $\eta, \delta, \mathcal{G}$ and $\mathcal{P}$. It follows from Lemma 4.3 that there is a unital homomorphism $H: C(X \times \mathbb{T}) \to M_n$ such that

$$\|\Phi(g) - H(g)\| < \frac{\epsilon}{4} \text{ for all } g \in \mathcal{F}_1. \qquad \textbf{(222)}$$

It follows from Lemma (2.2.16) that there exists a continuous path of unitaries $\{u(t) : t \in [1/4, 1]\}$ such that

$$u(1/4) = H(1 \otimes z), u(1) = 1, \qquad (223)$$
$$u(t)H(g \otimes 1) = H(g \otimes 1)u(t) \text{ for all } g \in C(X), t \in [1/4, 1] \text{ and} \qquad (224)$$

$$Length(\{u(t) : t \in [1/4, 1]\}) \leq \pi. \qquad (225)$$

Since

$$\|u - H(1 \otimes z)\| < \epsilon/2,$$

there is a continuous path of unitaries $\{u(t) : t \in [0, 1/4]\}$ such that $u(0) = u$, $u(1/4) = H(1 \otimes z)$ and $Length(\{u(t) : t \in [0, 1/4]\}) \leq \epsilon \cdot \pi$. The lemma then follows.

Lemma (2.2.18)[71]: Let X be a compact metric space without isolated points, $\epsilon > 0$ and $1 \in \mathcal{F} \subset C(X)$ be a finite subset. Let l be a positive integer for which $256\pi M/l < \epsilon$, where $M = \max\{1, \max\{\|f\| : f \in \mathcal{F}\}\}$. Then there exists $\eta > 0$ (which depends on ϵ and $\mathcal{F}$) for any finite $\eta/2$-dense subset $\{x_1, x_2, \ldots, x_N\}$ of X for which $O_i \cap O_j = \emptyset$ ($i = j$), where

$$O_i = \{x \in X : \text{dist}(x, x_i) < \eta/2s\}$$

for some integer $s \geq 1$ and for any $\sigma > 0$ for which $\sigma < 1/2s$, and for any $\delta_0 > 0$ and any finite subset $\mathcal{G}_0 \subset C(X \otimes \mathbb{T})$, there exists a finite subset $\mathcal{G} \subset C(X)$ and there exists $\delta > 0$ satisfying the following.

Suppose that A is a unital separable simple C^*-algebra with tracial rank zero (infinite dimensional or finite dimensional), $h : C(X) \to A$ is a unital homomorphism and $u \in A$ is a unitary such that

$$\|[h(a), u]\| < \delta \text{ for all } a \in \mathcal{G} \text{ and } \mu_{\tau \circ h}(O_i) \geq \sigma\eta \text{ for all } \tau \in T(A). \qquad (226)$$

Then there is a δ_0-$\mathcal{G}_0$-multiplicative contractive completely positive linear map $\phi: C(X) \otimes C(\mathbb{T}) \to A$ and a rectifiable continuous path $\{u_t : t \in [0, 1]\}$ such that

$$u_0 = u, \|[\phi(a \otimes 1), u_t]\| < \epsilon \text{ for all } a \in \mathcal{F}, \qquad (227)$$
$$\|\phi(a \otimes 1) - h(a)\| < \epsilon, \|\phi(a \otimes z) - h(a)u\| < \epsilon \text{ for all } a \in \mathcal{F}, \qquad (228)$$

where $z \in C(\mathbb{T})$ is the standard unitary generator of $C(\mathbb{T})$, and

$$\mu_{\tau \circ \phi}\left(O\left(x_i \times t_j\right)\right) > \frac{\sigma_1}{2l}\eta, i = 1, 2, \ldots, m, j = 1, 2, \ldots, l, \tag{229}$$

for all $\tau \in T(A)$, where $t_1, t_2, \ldots, t_l$ are l points on the unit circle which divide $\mathbb{T}$ into l arcs evenly and where

$$O\left(x_i \times t_j\right) = \{x \times t \in X \times \mathbb{T} : \text{dist}(x, x_i) < \eta/2s \text{ and } \text{dist}(t, t_j) < \pi/4sl\}$$

for all $\tau \in T(A)$

(so that $O\left(x_i \times t_j\right) \cap O\left(x_{i'} \times t_{j'}\right) = \emptyset$ if $(i, j) \neq (i', j')$). Moreover,

$$Length(\{u_t\}) \leq \pi + \pi. \tag{230}$$

Proof: The only diffierence between this lemma and Lemma in [70] is that in the statement of Lemma in [70] h is assumed to be a monomorphism. However, for the case that A is infinite dimensional, it is the condition that

$$\mu_{\tau \circ h}(O_i) \geq \sigma \cdot \eta$$

for all $\tau \in T(A)$ which is actually used. The existence of a monomorphism h implies that A is infinite dimensional.

In the case that M_n, pAp may not have enough projections, a modification is needed for the case where $A = M_n$ for some integer n. Let $\eta > 0$ be such that

$$|f(x) - f(x')| < \epsilon/32 \text{ for all } f \in \mathcal{F},$$

if $\text{dist}(x, x') < \eta$. Suppose that $y_1, y_2, \ldots, y_m \in X$ and $s_1 \geq 1$ such that

$$G_i \cap G_j = \emptyset \text{ if } i \neq j,$$

where $G_i = B_{\eta_1/2s_1}(\mathcal{Y}_i), i = 1, 2, \ldots, m$. Let $\{x_1, x_2, \ldots, x_{2ml}\}$ be another subset of$X$ such that each G_i contains $2l$ many points.

Now let δ_0 and G_0 be given. Then there is $s > s_1$ such that

$$O_i \cap O_j = \emptyset, \quad \text{if } i \neq j,$$

where $O_j = B_{\eta_1/2s}(x_j), j = 1, 2, \ldots, m + 2l$. Let $0 < \sigma < 1/2s$, let $\sigma_1 = 2l\sigma$, and let δ and $\mathcal{G}$ be required by Lemma in [70] for the above $\epsilon, \mathcal{F}, l, \eta, s, \sigma_1, \delta_0$ and $\mathcal{G}_0$.

Now suppose that $h : C(X) \to A$ is a unital homomorphism and $u \in A$ is a unitary such that

$$\|[h(f), u]\| < \delta \text{ for all } f \in \mathcal{G} \text{ and } \mu_{\tau \circ h}(O_i) \geq \sigma\eta.$$

Then

$$\mu_{\tau \circ h}(\mathcal{G}_i) \geq \sigma_1 \eta \geq 2l\sigma\eta.$$

In particular, pAp contains $2l - 1$ mutually orthogonal and mutually equivalent non-zero projections. Thus the proof of Lemma in [70] applies.

Lemma(2.2.19)[71]: Let X be a finite CW complex, $\mathcal{F} \subset C(X)$ be a finite subset and $\epsilon > 0$ be a positive number. Let $\sigma > 0$. There exists $\eta > 0$ (which depends on and $\mathcal{F}$ but not on σ), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to A$, where A is a unital separable simple C^*-algebra with tracial rank zero (infinite or finite dimensional), is a unital homomorphism with

$$\mu_{\tau\circ\phi}\left(O_{\frac{\eta}{2}}\right) \geq \sigma\eta \tag{231}$$

for any open ball with radius $\eta/2$ and a unitary $u \in A$ such that
$$\|[\phi(g), u]\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \text{Bott}(\phi, u)|_{\mathcal{P}} = \{0\}. \tag{232}$$

Then there exists a continuous path of unitaries $\{u_t : t \in [0,1]\}$ such that
$$u_0 = u, \ u_1 = 1, \|[\phi(f), u_t]\| < \epsilon \tag{233}$$

for all $f \in \mathcal{F}$ and $t \in [0,1]$ and
$$\text{length}(\{u_t\}) \leq 2\pi + \epsilon.$$

Let X be a locally path connected compact metric space. Let $\phi, \psi : C(X) \to A$ be two unital homomorphisms, where A is a finite dimensional C^*-subalgebra. In this section, we will show that ϕ and ψ, up to some homotopy, are unitary equivalent if they are close and they induce similar measure. See Lemma(2.2.21) below.

Lemma (2.2.20)[71]: Let X be a connected compact metric space. For any $\eta > 0$ and $\sigma > 0$, there is $\delta = (\sigma\eta/16)$ and there is a finite subset $\mathcal{G} \subset C(X)$ such that if $\phi, \psi : C(X) \to A$ are two unital homomorphisms, where A is a unital C^*-algebra with a tracial state τ such that
$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \ for \ all \ g \in \mathcal{G}, \tag{234}$$
$$\mu_{\tau\circ\phi}\left(O_{\frac{\eta}{8}}\right) \geq \frac{\sigma\eta}{8} \ and \ \mu_{\tau\circ\psi}\left(O_{\frac{\eta}{8}}\right) \geq \frac{\sigma\eta}{8}, \tag{235}$$

then, for any compact subset $F \subset X$,
$$\mu_{\tau\circ\phi}(F) \leq \mu_{\tau\circ\psi}\left(B_\eta(F)\right) \ and \ \mu_{\tau\circ\psi}(F) \leq \mu_{\tau\circ\phi}\left(B_\eta(F)\right), \tag{236}$$

where
$$B_\eta(F) = \{x \in X : \text{dist}(x, F) < \eta\}.$$

Proof: There are finitely many open balls $B_{\eta/8}(x_1), B_{\eta/8}(x_2), \ldots, B_{\eta/8}(x_N)$ with radius $\eta/8$ covers X. It is an easy exercise to show that there is a finite subset $\mathcal{G}$ of $C(X)$ satisfying the following. If (311) holds, then, for any subset S of $\{1, 2, \ldots, N\}$,

$$\mu_{\tau\circ\phi}\left(\bigcup_{i\in S} B_{\frac{\eta}{8}}(x_i)\right) \leq \mu_{\tau\circ\psi}\left(\bigcup_{i\in S} B_{\frac{\eta}{4}}(x_i)\right) + \delta \ and \tag{237}$$

$$\mu_{\tau\circ\psi}\left(\bigcup_{i\in S} B_{\frac{\eta}{8}}(x_i)\right) \leq \mu_{\tau\circ\phi}\left(\bigcup_{i\in S} B_{\frac{\eta}{4}}(x_i)\right) + \delta \tag{238}$$

If $\overline{\bigcup_{i\in S} B_{3\eta/4}(x_i)} = X$, then

$$\mu_{\tau\circ\phi}\left(\bigcup_{i\in S} B_{\frac{\eta}{8}}(x_i)\right) \leq \mu_{\tau\circ\psi}\bigcup_{i\in S} \overline{B_{\frac{3\eta}{4}}(x_i)} \ and \tag{239}$$

$$\mu_{\tau\circ\psi}\left(\bigcup_{i\in S} B_{\frac{\eta}{8}}(x_i)\right) \leq \mu_{\tau\circ\phi}\bigcup_{i\in S} \overline{B_{\frac{3\eta}{4}}(x_i)}. \tag{240}$$

Otherwise, since X is path connected, there is an open ball O of X with radius $\eta/8$ such that

$$O \cap \left(\bigcup_{i \in S} B_{\eta/4}(x_i) \right) = \emptyset \ and \ O \subset \left(\bigcup_{i \in S} B_\eta(x_i) \right)$$

Thus, by (237), (316) and (235),

$$\mu_{\tau \circ \phi} \left(\bigcup_{i \in S} B_{\frac{\eta}{8}}(x_i) \right) \leq \mu_{\tau \circ \psi} \left(\bigcup_{i \in S} B_\eta(x_i) \right) \tag{241}$$

Now for any compact subset $\mathcal{F}$, there is $S \subset \{1, 2, \ldots, N\}$ such that

$$F \subset \bigcup_{i \in S} B_{\frac{\eta}{8}}(x_i) \ and \ F \cap B_{\eta/8}(x_i) \neq \emptyset \ for \ all \ i \in S \tag{242}$$

It follows that

$$\mu_{\tau \circ \phi}(F) \leq \mu_{\tau \circ \phi} \left(\bigcup_{i \in S} B_{\frac{\eta}{8}}(x_i) \right) \tag{243}$$

$$\leq \mu_{\tau \circ \psi} \left(\bigcup_{i \in S} B_{\frac{\eta}{8}}(x_i) \right) \leq \mu_{\tau \circ \psi} \left(B_\eta(F) \right). \tag{244}$$

Exactly the same argument shows that the other inequality of (e 6.97) also holds.

Lemma(2.2.21)[71]: Let X be a locally path connected compact metric space without isolated points, let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. Let $\eta > 0$ be such that $|f(x) - f(x')| < /2$ for all $f \in \mathcal{F}$, provided that $\text{dist}(x, x') < \eta$ and such that any open ball B_η with radius η is path connected.

Let $\sigma > 0$. There is $\delta > 0$ and there exists a finite subset $\mathcal{G} \subset C(X)$ satisfying the following. For any two unital homomorphisms $\phi, \psi : C(X) \to M_n (for \ any \ n \geq 1)$ for which

$$\|\phi(f) - \psi(f)\| < \delta \ for \ all \ f \in \mathcal{G} \tag{245}$$

$$and \quad \mu_{\tau \circ \phi} \left(O_{\frac{\eta}{24}} \right), \mu_{\tau \circ \psi} \left(O_{\frac{\eta}{24}} \right) \geq \sigma \eta \tag{246}$$

for any open balls with radius $\eta/24$, there exist two unital homomorphisms $\Phi_1, \Phi_2 :$ $C([0, 1], M_n)$ such that

$$\pi_0 \circ \Phi_1 = \phi, \pi_0 \circ \Phi_2 = \psi, \tag{247}$$

$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < , \|\pi_t \circ \Phi_2(f) - \psi(f)\| < \epsilon \tag{248}$$

for all $f \in \mathcal{F}$ and $t \in [0, 1]$, and there is a unitary $u \in M_n$ such that

$$ad \ u \circ \pi_1 \circ \Phi_1 = \pi_1 \circ \Phi_2. \tag{249}$$

Proof: X is a union of finitely many connected and locally path connected compact metric spaces. It is clear that the general case can be reduced to the case where X is a connected and locally path connected compact metric space.

We will apply the so-called Marriage Lemma (see [53]). Let δ and $\mathcal{G}$ be in Lemma (2.2.19) corresponding to $\eta/3$ and σ. We may assume that $\mathcal{G} \supset \mathcal{F}$. We may write that

$$\phi(f) = \Sigma_{i=1}^{N_1} f(x_i) p_i \ and \ \psi(f) = \Sigma_{j=1}^{N_2} f(y_j) q_j \tag{250}$$

for all $f \in C(X)$, where $\{p_1, p_2, \ldots, p_{N_1}\}$ and $\{q_1, q_2, \ldots, q_{N_2}\}$ are two sets of mutually orthogonal projections such that $\Sigma_{i=1}^{N_1} p_i = 1 = \Sigma_{j=1}^{N_2} q_j$.

By Lemma (2.2.19),
$$\mu_{\tau\circ\phi}(F) < \mu_{\tau\circ\psi}(B_{\eta/3}(F)) \text{ and } \mu_{\tau\circ\psi}(F) < \mu_{\tau\circ\phi}\left(B_{\eta/3}(F)\right) \tag{251}$$
for any compact subset $F \subset X$.

Suppose that p_i has rank $r(i)$. Choose $r(i)$ many points $\{x_{i,1}, x_{i,2}, \ldots, x_{i,r(i)}\} \subset B_{\eta/3}(x_i)$ and define
$$\phi_1(f) = \sum_{i=1}^{N_1}\left(\sum_{k=1}^{r(i)} f(x_{i,k})e_{i,k}\right) \text{ for all } f \in C(X),$$
where $\{e_{i,1}, e_{i,2}, \ldots, e_{i,r(i)}\}$ is a set of mutually orthogonal rank one projections suchthat$\sum_{k=1}^{r(i)} e_{i,k} = p_i$. It follows that
$$\mu_{\tau\circ\phi_1}(F) \le \mu_{\tau\circ\psi}(B_{\eta/3}(F)) \text{ and} \mu_{\tau\circ\psi}(F) \le \mu_{\tau\circ\phi_1}\left(B_{\eta/3}(F)\right) \tag{252}$$
for any compact subset $F \subset X$. Since $B_\eta(x)$ is path connected for every $x \in X$, there is a unital homomorphism $\Phi_1 : C(X) \to C([0,1], M_n)$ such that
$$\pi_0 \circ \Phi_1 = \phi, \ \pi_1 \circ \Phi_1 = \phi_1 \tag{253}$$
and
$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < \epsilon/2 \text{for all } g \in \mathcal{F} \text{ and } t \in [0,1]. \tag{254}$$
We rewrite
$$\phi_1(f) = \sum_{i=1}^{n} f(x_i')e_i \text{for all } f \in C(X), \tag{255}$$
where each e_i is a rank one projection and x_i' is a point in $X, i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} e_i = 1$. Similarly, there is a unital homomorphism $\Phi_2' : C(X) \to C([0, 1/2], M_n)$ such that
$$\pi_0 \circ \Phi_2' = \psi, \ \pi_{\frac{1}{2}} \circ \Phi_2 = \psi_1 \tag{256}$$
$$\text{and} \quad \|\pi_t \circ \Phi_2'(f) - \psi(f)\| < \frac{\epsilon}{2} \text{for all } f \in \mathcal{F} \text{ and } t \in \left[0, \frac{1}{2}\right], \tag{257}$$
where
$$\psi_1(f) = \sum_{i=1}^{n} f(y_i')e_i \text{ for all f} \in C(X), \tag{258}$$
where each $e_i'$is a rank projection, y_i' is a point in $X, i = 1, 2, \ldots, n$, and$\sum_{i=1}^{n} e_i' = 1$.
Moreover,
$$\mu_{\tau\circ\psi_1}(F) \le \mu_{\tau\circ\psi}(B_{\eta/3}(F)) \text{ and} \mu_{\tau\circ\psi}(F) \le \mu_{\tau\circ\psi_1}\left(B_{\eta/3}(F)\right) \tag{259}$$
for any compact subset $F \subset X$. Combining (251), (252) and (259), one has
$$\mu_{\tau\circ\phi_1}(F) \le \mu_{\tau\circ\phi}\left(B_{\eta/3}(F)\right) < \mu_{\tau\circ\psi}\left(B_{2\eta/3}(F)\right) \tag{260}$$
$$\le \mu_{\tau\circ\psi_1}\left(B_\eta(F)\right) \tag{261}$$
and
$$\mu_{\tau\circ\psi_1}(F) < \mu_{\tau\circ\phi_1}(B_\eta(F)) \tag{262}$$
for any compact subset $F \subset X$.

By the Marriage Lemma (see [53]), there is a permutation $\Delta: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\text{dist}(x_i', y_{\Delta(i)}') < \eta, i = 1, 2, \ldots, n. \tag{263}$$

Define $\psi_2 : C(X) \to M_n$ by

$$\psi_2(f) = \sum_{i=1}^{n} f(x_i'), e_{\Delta(i)}' \quad for\ all\ f \in C(X). \tag{264}$$

Since every open ball of radius η is path connected, one obtains anotherunital Homomorphism

$$\Phi_2'' : C(X) \to C([1/2, 1], M_n):$$

$$\pi_1 \circ \Phi_2'' = \psi_2, \qquad \pi_{\frac{1}{2}} \circ \Phi_2'' = \psi_1 \tag{265}$$

and

$$\|\pi_t \circ \Phi_2'(f) - \psi_1(f)\| < \epsilon/2 \text{ for all } f \in \mathcal{F}. \tag{266}$$

Now define $\Phi_2 : C([0,1], M_n)$ by $\pi_t \circ \Phi_2 = \pi_t \circ \Phi_2'$ for $t \in [0, 1/2]$ and $\pi_t \circ \Phi_2 = \pi_t \circ \Phi_2''$

for $t \in [1/2, 1]$. Then Φ_1 and Φ_2 satisfy (247) and (248).

Moreover, by

(255) and (264), there exists a unitary $u \in M_n$ such that

$$ad\ u \circ \phi_1 = \psi2 = \pi_1 \circ \Phi_2 .$$

Lemma (2.2.22)[71]: Let X be a finite CW complex with torsion $K_1(C(X))$ and torsion free $K_0(C(X))$. Let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset and let $\sigma > 0$. There exist $\eta > 0$ (which depends onϵ and $\mathcal{F}$ but not on σ), a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following.

Suppose that $\phi, \psi : C(X) \to M_n$(for any integer n) are two unitalhomomor- phisms such that

$$\|\phi(f) - \psi(f)\| < \delta \ for\ all\ f \in \mathcal{G}, \tag{267}$$

$$\mu_{\tau \circ \phi}(O_\eta) \geq \sigma\eta \ and \ \mu_{\tau \circ \psi}(O_\eta) \geq \sigma\eta \tag{268}$$

for any open ball O_η of radius η, where τ is the normalized trace on M_n and

$$ad\ u \circ \phi = \psi \tag{269}$$

for some unitary $u \in A$. Then there exists a homomorphism $\Phi : C(X) \to C([0, 1], M_n)$ such that

$$\pi_0 \circ \Phi = \phi, \pi_1 \circ \Phi = \psi \ and$$

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \ for\ all\ f \in \mathcal{F}.$$

Proof: It is easy to see that the general case can be reduced to the case where X is connected.

Let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset and let $\sigma > 0$. Let $\eta_1 0$ (in place of η), $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$, a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ be required by Lemma (2.2.19) for $\epsilon/2, \mathcal{F}$ and $\sigma/2$. Let $\eta = \eta_1/2$.

We may assume that $\mathcal{P} \subset K_1(C(X))$. Since $K_1(C(X))$ is torsion and $K_0(M_n)$ is free, for sufficiently small δ and sufficiently large $\mathcal{G}$, and for any pair of ϕ and u for which $\|[\phi(g), u]\| < \delta$ for all $g \in \mathcal{G}$,

$$bott_1(\phi, u)|_{\mathcal{P}} = 0.$$

We may assume that δ and $\mathcal{G}$have this property. We may further assume that

$\delta < \epsilon/2$ and $\mathcal{F} \subset \mathcal{G}$.

Now we assume that ϕ, ψ and u satisfy the assumption of the lemma for the above η, δ and $\mathcal{G}$. Then

$$\mu_{\tau\circ\phi}\left(\frac{O_{\eta_1}}{2}\right) \geq \frac{\sigma\eta_1}{2} = \left(\frac{\sigma}{2}\right)\eta_1 \ and \tag{270}$$

$$\mu_{\tau\circ\psi}\left(\frac{O_{\eta_1}}{2}\right) \geq \left(\frac{\sigma}{2}\right)\eta_1. \tag{271}$$

By applying Lemma (2.2.19), one obtains a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ such that

$$u(0) = u, u(1) = 1 \tag{272}$$

and

$$\|u(t)^*\phi(f)u(t) - \phi(f)\| < \epsilon/2 \ for all \ f \in \mathcal{F}. \tag{273}$$

Define $\Phi : C(X) \to C([0,1], M_n)$ by

$$\pi_t \circ \Phi = ad \, u(1-t) \circ \phi \ for all \ t \in [0,1].$$

Then,

$$\pi_0 \circ \Phi = \phi \ and \ \pi_1 \circ \Phi = \psi.$$

Moreover, by (350) and (346),

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \ for all \ f \in \mathcal{F} \ and \ t \in [0,1].$$

Lemma (2.2.23)[71]: Let X be a finite CW complex with torsion $K_1(C(X))$ and let k be the largest order of torsion elements in $K_i(C(X))$ ($i = 0, 1$). Let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset and let $\sigma > 0$. There exist $\eta > 0$ (which depends on and $\mathcal{F}$ but not on σ), a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following.

Suppose that $\phi, \psi : C(X) \to M_n$ (for any integer n) are two unital homomor-phisms such that

$$\|\phi(f) - \psi(f)\| < \delta \ \text{for all} \ f \in \mathcal{G}, \tag{274}$$

$$\mu_{\tau\circ\phi}(O_\eta) \geq \sigma\eta \ and \ \mu_{\tau\circ\psi}(O_\eta) \geq \sigma\eta \tag{275}$$

for any open ball O_η of radius η, where τ is the normalized trace on M_n, and

$$ad \, u \circ \phi = \psi \tag{276}$$

for some unitary $u \in A$. Then there exists a homomorphism

$$\Phi : C(X) \to M_{k_0}(C([0,1], M_n))$$

such that

$$\pi_0 \circ \Phi = \phi^{(k_0)}, \pi_1 \circ \Phi = \psi^{(k_0)} \ and$$

$$\left\|\psi^{(k_0)}(f) - \pi_t \circ \Phi(f)\right\| < \ for all \ f \in \mathcal{F},$$

where $k_0 = k!$, and where $\phi^{(k_0)}(f) = \text{diag}\,\overbrace{(\phi(f), \phi(f), \ldots, \phi(f))}^{k_0}$ and $\psi^{(k_0)}(f) = \text{diag}\,\overbrace{(\psi(f), \psi(f), \ldots, \psi(f))}^{k_0}$ for all $f \in C(X)$, respectively.

Proof: By [20], one has

$$H \, om_\Lambda(\underline{K}(C(X)), \underline{K}(M_n)) = H \, om_\Lambda(F_k\underline{K}(C(X)), F_k\underline{K}(M_n)).$$

Let $k_0 = k!$. It follows that

$$\overbrace{\lambda + \lambda + \cdots + \lambda}^{k_0} = 0,$$

for any homomorphism λ from $K_1(C(X), \mathbb{Z}/m\mathbb{Z})$ with $m \leq k_0$. (to $\phi(k_0)$ and $\psi(k_0)$). The point is that

$$\text{Bott}\left(\phi^{(k_0)}, u^{(k_0)}\right)\big|_{\mathcal{P}'} = \{0\}$$

for any finite subset $P' \subset K_1(C(X), \mathbb{Z}/m\mathbb{Z})$ for $0 \leq m \leq k_0$ as long as it is defined, where $u^{(k_0)} = \text{diag}(\overbrace{u, u, \ldots, u}^{k_0})$.

Lemma (2.2.24)[71]: Let $X = \mathbb{T}$ or $X = I \times \mathbb{T}$ (*with the product metric*). Let $\mathcal{F} \subset C(X)$ *be a finite subset and* let $\epsilon > 0$. There exists $\eta_1 > 0$ such that, for any $\sigma_1 > 0$, the following holds. There exists a finite subset $\mathcal{G} \subset C(X)$ and there exists $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists $\delta > 0$ satisfying the following.
Suppose that $\phi, \psi : C(X) \to M_n$ (for some integer n) are two unital homomorphisms such that

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \tag{277}$$

$$\mu_{\tau \circ \phi}(O_{\eta_1}) \geq \sigma_1 \eta_1, \ \mu_{\tau \circ \psi}(O_{\eta_1}) \geq \sigma_1 \eta_1, \tag{278}$$

$$\mu_{\tau \circ \phi}(O_{\eta_2}) \geq \sigma_2 \eta_2, \ \mu_{\tau \circ \psi}(O_{\eta_2}) \geq \sigma_2 \eta_2, \tag{279}$$

for any open ball O_{η_j} of radius η_j, $j = 1, 2$, where τ is the normalized trace on M_n, and

$$\text{ad } u \circ \phi = \psi \tag{280}$$

for some unitary $u \in M_n$. Then there exists a homomorphism $\Phi : C(X) \to C([0,1], M_n)$ such that

$$\pi_0 \circ \Phi = \phi, \pi_1 \circ \Phi = \psi \text{ and}$$
$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

Proof: Let $\delta_{00} > 0$ satisfy the following: for any pair of unitaries u_0, v_0 in a unital C^*-algebra, $\text{bott}_1(u_0, v_0)$ is well defined whenever $\|[u_0, v_0]\| < \delta_{00}$. We will prove the case that $X = I \times \mathbb{T}$. The proof for the case that $X = \mathbb{T}$ follows from the same argument but is simpler. Let $\epsilon > 0$ and $\mathcal{F}$ be given as in the lemma. Let $\mathcal{F}_1 = \mathcal{F} \cup \{z\}$, where

$$z(t, e^{2\pi i s}) = e^{2\pi i s} \text{ for all } t \in [0,1] \text{ and } s \in [0,1].$$

Let $\eta_1 > 0$ (in place of η) be required by Lemma (2.2.19) for $\epsilon/4$ (in place of ϵ) and $\mathcal{F}_1$ (in place of $\mathcal{F}$). Let $\sigma_1 > 0$.
Let $\mathcal{G} \subset C(X)$ be a finite subset, let $\delta_0 > 0$ (in place of δ) and let $\mathcal{P} \subset \underline{K}(C(X))$ be a subset required by Lemma(2.2.19) for $\epsilon/4$ (in place of ϵ) and $\mathcal{F}_1$(in place of $\mathcal{F}$) and $\sigma_1/2$ (as well as for $X = I \times \mathbb{T}$).
Since $K_0(C(X)) = \mathbb{Z}$ and $K_1(C(X)) = \mathbb{Z}$, without loss of generality, we may assume that $\mathcal{P} = \{[z]\}$. We assume that $\delta_0 < \delta_{00}/2$. We may also assume that δ_0 satisfies the following. If u_1, u_2 and v are unitaries with

$$\|u_1 - u_2\| < \delta_0 \text{ and } \|[u_1, v]\| < \delta_0,$$

then

$$\text{bott}_1(u_1, v) = \text{bott}_1(u_2, v) \tag{281}$$

(whenever $[u_1, v] < \delta_{00}/2$). Let $\eta_2' > 0$ such that

$$|f(x) - f(x')| < \min\left\{\frac{\delta_0}{2}, \epsilon\frac{1}{16}\right\} \text{ for all } f \in \mathcal{G} \cup \mathcal{F} \tag{282}$$

provided that $\text{dist}(x,y) < \eta_2$. Choose an integer $K > 1$ such that $2\pi/K < \min\{\eta_1/16, \eta_2'/16\}$ and put $\eta_2 = \pi/4K$. Let $\sigma_2 > 0$. Choose $\delta = \min\{\delta_0/2, \sigma_2\eta_2/2\}$.

Suppose that ϕ and ψ satisfy the assumption of the lemma for the above $\mathcal{G}$, $\eta_1, \eta_2, \sigma_1, \sigma_2$ and δ. Let $w_j = e^{2j\pi\sqrt{-1}/K}$ and $\zeta_j = 1 \times w_j$, $j = 1, 2, \ldots, K$. Then, by the assumption,

$$\mu_{\tau\circ\psi}\left(B_{\eta_2}(\zeta_j)\right) \geq \sigma_2\eta_2 > 2\delta, \tag{283}$$

$j = 1, 2, \ldots, K$. Note that

$$B_{\eta_2}(\zeta_j) \cap B_{\eta_2}(\zeta_j') = \emptyset, \tag{284}$$

If $j \neq j'$, $j, j' = 1, 2, \ldots, K$.

Write

$$\psi(f) = \sum_{l=1}^N f(x_l)e_l \ \text{ for all } f \in C(X), \tag{285}$$

where $\{e_1, e_2, \ldots, e_N\}$ is a set of mutually orthogonal projections and $x_1, x_2, \ldots, x_N$ are distinct points in X. Define

$$p_j = \sum_{x_l \in B_{\eta_2}(\zeta_j)} e_l, j = 1, 2, \ldots, K.$$

By (283),

$$\tau(p_j) \geq \sigma_2\eta_2, \ j = 1, 2\ldots, K. \tag{286}$$

Put

$$\gamma = \frac{1}{2\pi i}\tau\left(\log(u^*\phi(z)u\phi(z)^*)\right), \tag{287}$$

where τ is the normalized trace on M_n. Then

$$|\gamma| < \delta. \tag{288}$$

We first assume that $\gamma = 0$. For convenience, we may assume that $\gamma < 0$. By the Exel formula (see [43]), $\gamma = m/n$ for some integer $|m| < n$.

For each j, there is a projection $q_j \leq p_j$ such that

$$\tau(q_j) = |\gamma| \text{ and } q_je_l = e_lq_j, \ j = 1, 2, \ldots, K, \ l = 1, 2, \ldots, N. \tag{289}$$

There is a unitary $v_1 \in (\sum_{j=1}^K q_j)M_n(\sum_{j=1}^K q_j)$ such that

$$v_1^*q_jv_1 = q_{j+1}, j = 1, 2, \ldots, K-1 \text{ and } v_1^*q_Kv_1 = q_1. \tag{290}$$

Define $v = (1 - \sum_{j=1}^K q_j) + v_1$. Note that, by the choice of δ, we have

$$[uv, \phi(f)] < \delta_0 \text{ for all } f \in \mathcal{G}. \tag{291}$$

Write $x_l = s \times e^{2\pi\sqrt{-1}t_l}, l = 1, 2, \ldots, N$. Define $z' = (1 - \sum_{j=1}^K q_j)\psi(z) +$

$$\sum_{j=1}^K w_jq_j \ .\text{Then}$$

$$\|\psi(z) - z'\| < \delta_0 \text{ and } v^*z'v = \left(1 - \sum_{j=1}^K q_j\right)\psi(z) + \sum_{j=1}^K w_jq_{j+1} + w_Kq_1. \tag{292}$$

It follows that

$$\frac{1}{2\pi i}\tau\left(\log(v^*z'v(z')^*)\right) = \tau(q_j) = -\gamma. \tag{293}$$

By the choice of δ_0, we have that

$$\frac{1}{2\pi i}\tau\left(\log(v^*\psi(z)v\psi(z)^*)\right) = \tau(q_j) = -\gamma. \tag{294}$$

By the choice of δ_0 and the Exel formula, we have

$$\frac{1}{2\pi i}\tau\left(\log(v^*u^*\phi(z)uv\phi(z)^*)\right) \tag{295}$$

$$= \frac{1}{2\pi i}\tau\left(\log(u^*\phi(z)u\phi(z)^*)\right) + \frac{1}{2\pi i}\tau\left(\log(v^*\phi(z)v\phi(z)^*)\right)$$

$$= \frac{1}{2\pi i}\tau\left(\log(u^*\phi(z)u\phi(z)^*)\right) + \frac{1}{2\pi i}\tau\left(\log(v^*\psi(z)v\psi(z)^*)\right) \tag{296}$$

$$= \gamma - \gamma = 0. \tag{297}$$

It follows from the Exel formula, $\mathrm{bott}_1(\phi, uv) = \{0\}$ and $\mathrm{Bott}(\phi, uv)|_{\mathcal{P}} = \{0\}$. It follows from Lemma (2.2.19) that there exists a continuous path of unitaries $\{u(t) : t \in [0, 1/2]\}$ such that

$$u(0) = 1, \ u(1/2) = uv \ and \|[\phi(f), u(t)]\| < \epsilon/4 \tag{298}$$

for all $f \in \mathcal{F}$ and for all $t \in [0, 1/2]$.
Define $\Phi_1 : C(X) \to C([0, 1/2], M_n)$ by

$$\pi_t \circ \Phi_1(f) = u(t)^*\phi(f)u(t) \ for all \ f \in C(X) \ and \ t \in [0, 1/2]. \tag{299}$$

Then

$$\|\pi_t \circ \Phi_1(f) - \phi(f)\| < \frac{\epsilon}{2} for all \ f \in \mathcal{F} \ and \ t \in \left[0, \frac{1}{2}\right]. \tag{300}$$

Let

$$q_j\psi(f) = \sum_{k=1}^{N(j)} f(\xi_{k,j})e'_{k,j} \ for \ all \ f \in C(X),$$

where $\{e'_{k,j}\}$ is a set of mutually orthogonal projections and $\xi_{k,j} \in B_{\eta_2}(\zeta_j), j = 1, 2, \ldots, K$. Note that

$$v^*u^*\phi(f)uv = \psi(f)(1 - \sum_{j=1}^{K} q_j) + \sum_{j=1}^{K}(\sum_{k=1}^{N(j)} f(\xi_{k,j})v_1^*e'_{k,j}v_1) \tag{301}$$

for all $f \in C(X)$. It is easy to find a homomorphism $\Phi_2 : C(X) \to C([1/2, 1], \ M_n)$ such that (with $q_{K+1} = q_1, e_k, K+1 = e'_{k,1}$ and $\xi_{k,K+1} = \xi_{k,1}$)

$$\pi_{1/2} \circ \Phi_2(f) = v^*u^*\phi(f)uv, \tag{302}$$

$$\pi_{\frac{3}{4}} \circ \Phi_2(f) = \psi(f)\left(1 - \sum_{j=1}^{K} q_j\right) + \sum_{j=1}^{K} f\left(\xi_j\right)\left(\sum_{k=1}^{N(j)} v_1^*e'_{k,j}v_1\right) \tag{303}$$

$$= \psi(f)(1 - \sum_{j=1}^{K+1} q_j) + \sum_{j=1}^{K-1} f(\xi_j)q_{j+1} + f(\xi_K)q_1 \tag{304}$$

and

$$\pi_1 \circ \Phi_2(f) = \psi(f)(1 - \sum_{j=1}^{K} q_j) + \sum_{j=1}^{K-1}(\sum_{k=1}^{N(j+1)} f(\xi_{k,j+1})e'_{k,j+1}) + \sum_{k=1}^{N(1)} f(\xi_{k,1})e'_{k,1} \tag{305}$$

$$= \psi(f) \tag{306}$$

for all $f \in C(X)$. Moreover,

$$\|\pi_t \circ \Phi_2(f) - \psi(f)\| < \frac{\epsilon}{16} \text{ for all } f \in \mathcal{F}. \tag{307}$$

Now define $\Phi : C(X) \to C([0,1], M_n)$ by

$$\pi_t \circ \Phi = \pi_t \circ \Phi_1 \text{ for all } t \in [0,1/2] \text{ and } \pi_t \circ \Phi = \pi_t \circ \Phi_2 \text{ for all } t \in \left[\frac{1}{2}, 1\right]. \tag{308}$$

One checks that

$$\|\pi_t \circ \Phi(f) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}. \tag{309}$$

Finally, if $\gamma = 0$, we do not need ν and can apply Lemma (2.2.19) directly.

Lemma (2.2.25)[71]: Let $X = \mathbb{T}$ or $X = I \times \mathbb{T}$ (with the product metric). Let $\mathcal{F} \subset C(X)$ be a finite subset and let $\epsilon > 0$. Then there exists $\eta_1 > 0$, for any $\sigma_1 > 0$, satisfying the following. There exists a finite subset $\mathcal{G} \subset C(X)$ and there exists $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists $\delta > 0$ such that the following holds. Suppose that $\phi, \psi : C(X) \to M_n$ (for some integer n) are two unital homomorphisms given by

$$\phi(f) = \sum_{i=1}^{N_1} f(x_i)p_i \text{ and } \psi(f) = \sum_{j=1}^{N_2} f(y_j)q_j$$

for all $f \in C(X)$, where $\{x_1, x_2, \ldots, x_{N_1}\}, \{y_1, y_2, \ldots, y_{N_2}\} \subset X$ and where $\{p_1, p_2, \ldots, p_{N_1}\}$ and $\{q_1, q_2, \ldots, q_{N_2}\}$ are two sets of mutually orthogonal projections such that

$$\|\phi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{G}, \tag{310}$$
$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j, \ \mu_{\tau \circ \psi}(O_{\eta_j}) \geq \sigma_j \eta_j \tag{311}$$

for any open ball O_{η_j} of radius η_j, $j = 1, 2$, where τ is the normalized trace on M_n. Then there exists a homomorphism $\Phi : C(X) \to C([0,1], M_n)$ such that $\pi_0 \circ \Phi = \phi, \pi_1 \circ \Phi = \psi$ and

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

Moreover, $\pi_t \circ \Phi(C(X)) \subset C_1$ for $t \in [0, 1/4]$, $\pi_0 \circ \Phi(C(X)) \subset C_2$ for $t \in [3/4, 1]$ and

$$\pi_t \circ \Phi(f) = u(t)^* \phi(f)u(t) \text{ for all } t \in \left[\frac{1}{4}, \frac{3}{4}\right] \tag{312}$$

and for all $f \in C(X)$, where C_1 is a finite dimensional commutative C^*-subalgebra containing projections $p_1, p_2, \ldots, p_{N_1}, C_2$ is a finite dimensional commutative C^*-subalgebra containing $q_1, q_2, \ldots, q_{N_2}, u(1/4) = 1$ and $u(t) \in C([1/4, 3/4], M_n)$.

Definition (2.2.26)[71]: Let X be a compact metric space. It is said to satisfy the property (H) if the following holds.

For any finite subset $\mathcal{F} \subset C(X)$ and for any $\epsilon > 0$, there exists $\eta_1 > 0$ such that, for any $\sigma_1 > 0$, the following holds. There exists a finite subset $\mathcal{G} \subset C(X)$ and $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists $\delta > 0$ satisfying the following.

Suppose that $\phi, \psi : C(X) \to M_n$ (for any integer n) are two unital homomorphisms such that

$$\|\phi(f) - \psi(f)\| < \delta \ for \ all \ f \ \in \ \mathcal{G}, \tag{313}$$

$$\mu_{\tau \circ \phi}\left(O_{\eta_j}\right) \geq \sigma_j \eta_j \,, \quad \mu_{\tau \circ \psi}\left(O_{\eta_j}\right) \geq \sigma_j \eta_j \tag{314}$$

for any open ball O_{η_j} of X with radius η_j , $j \ = \ 1, 2$, where τ is the normalized trace on M_n and

$$ad \ u \circ \phi = \ \psi \tag{315}$$

for some unitary $u \ \in \ A$. Then there exists a homomorphism $\Phi : C(X) \to C([0, 1], M_n)$ such that

$$\pi_t \circ \Phi \ = \ \varphi, \pi_1 \circ \Phi \ = \ \psi \ and$$
$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \ for \ all \ f \ \in \ \mathcal{F}.$$

We have proved in Lemma (2.2.22) that if X is a finite CW complex with torsion $K_1(C(X))$ and torsion free $K_0(C(X))$, then X satisfies property (H), and we have proved in Lemma (2.2.24) that if $X \ = \ \mathbb{T}$ or $X \ = \ I \ \times \ \mathbb{T}$, then X has property (H).

Lemma (2.2.27)[71]: Let $X \ = \ \overset{m}{\overbrace{\mathbb{T} \vee \mathbb{T} \vee \mathbb{T} \vee \ldots \vee \mathbb{T}}} \vee Y$, where Y is a finite CW complex with torsion $K_1(C(Y))$ and torsion free $K_0(C(Y))$. Then X has property (H).

Lemma (2.2.28)[71]: Let $X \ = \ \overset{m}{\overbrace{\mathbb{T} \times \mathbb{T} \times \ldots \times \mathbb{T}}}$. Then X has property (H).

Proof: Define $z_i\left(e^{2\pi\sqrt{-1}t_1}, e^{2\pi\sqrt{-1}t_2}, \ldots, e^{2\pi\sqrt{-1}t_m}\right) = e^{2\pi\sqrt{-1}t_i}, i \ = \ 1, 2, \ldots, m$.

Let $\delta_{00} > 0$ be as in the proof of Lemma (2.2.24) Let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset. Let $\mathcal{F}_1 = F \ \cup \ \{z_1, z_2, \ldots, z_m\}$. Let $\eta_1 > 0$ be as in the proof of Lemma (2.2.24) and let $\sigma_1 > 0$. Let $\mathcal{G} \subset C(X), \delta_0 > 0$ and $\mathcal{P} \subset \underline{K}(C(X))$ be as in the proof of Lemma (2.2.24) (for this X).

Since $K_0(C(X)) \ = \ \mathbb{Z}^m$ and $K_1(C(X)) \ = \ \mathbb{Z}^m$, we may assume that $\mathcal{P} \ = \ \{[z_1], [z_2], \ldots, [z_m]\}$. Let $\eta_2 > 0, \sigma_2 > 0, K$ and θ be as in the proof Lemma(2.2.27)

Let $w_j = e^{(2\pi\sqrt{-1}+\theta)/K}$ be as in the proof of Lemma (2.2.27) Choose $\zeta_{j,i} = (\overset{i-1}{\overbrace{1, \ldots, 1}}, w_j, \overset{m-1}{\overbrace{1, \ldots, 1}}), j \ = \ 1, 2, \ldots, K$ and $i \ = \ 1, 2, \ldots, m$. Note that

$$B_{\eta_2}(\zeta_{j,i}) \cap B_{\eta_2}(\zeta_{j',i'}) \ = \ \emptyset \tag{316}$$

if $j \neq j', j, j' = 1, 2, \ldots, K, i, i' \ = \ 1, 2, \ldots, m$. Moreover, $1 \notin B_{\eta_2}(\zeta_{j,i}), j \ = \ 1, 2, \ldots, K$ and $i \ = \ 1, 2, \ldots, m$. Write

$$\psi(f) = \sum_{l=1}^{N} f(x_l)e_l \ for \ all \ f \ \in \ C(X), \tag{317}$$

where $\{e_1, e_2, \ldots, e_N\}$ is a set of mutually orthogonal projections and $x_1, x_2, \ldots, x_l$ are distinct points in X. Define

$$p_{j,i} = \sum_{x_l \in B_{\eta_2}(\zeta_{j,i})} e_l, \quad j \ = \ 1, 2, \ldots, K$$

By (318),
$$\tau\left(p_{j,i}\right) \geq \sigma_2\eta_2, \quad j = 1,2\ldots,K \text{ and } i = 1,2,\ldots,m. \tag{318}$$
Put
$$\gamma_i = \frac{1}{2\pi\sqrt{-1}}\tau(\log(u^*\phi(z_i)u\phi(z_i)^*)), \tag{319}$$
where τ is the normalized trace on M_n. Then
$$|\gamma_i| < \delta. \tag{320}$$
By the Exel formula (see [43]), $\gamma_i = m_i/n_i$ for some integer $|m_i| < n_i$.
For each i and j, there is a projection $q_{j,i} \leq p_{j,i}$ such that

$$\tau(q_{j,i}) \ |\gamma_i| \text{and} q_{j,i}e_l = e_lq_{j,i}, \quad j = 1,2,\ldots,K, \ i = 1,2,\ldots,m \qquad \text{and } l = 1,2,\ldots,N. \tag{321}$$

There is a unitary $v_i \in (\sum_{j=1}^{K} q_{j,i})M_n(\sum_{j=1}^{K} q_{j,i})$ such that
$$v_i^*q_{j,i}v_i = q_{j+1,i}, \quad j = 1,2,\ldots,K-1, \text{ and } v_i^*q_{k,i}v_i = q_{1,i}, \text{if } \gamma < 0, \tag{322}$$
and
$$v_i^*q_{j,i}v_i = q_{j-1,i}, \quad j = 1,2,\ldots,K-1, \tag{323}$$
and $v_i^*q_{1,i}v_i = q_{k,i}$if $\gamma_i > 0$. If $\gamma_i = 0$, define $v_i = 1$. Define $v = (1 - \sum_{i=1}^{m}\sum_{j=1}^{K} q_{j,i}) + \sum_{i=1}^{m} v_i$. Note that, by the choice of δ, we have
$$\|[uv,\phi(f)]\| < \delta_0 \ for \ all \quad f \in \mathcal{G}. \tag{324}$$
Moreover, the same computation as in the proof of Lemma (2.2.24)shows that
$$\frac{1}{2\pi\sqrt{-1}}\tau(\log((uv)^*\phi(z_i)uv\phi(z_i)^*)) = 0, \quad i = 1,2,\ldots,m \tag{325}$$
Then, using the Exel formula, obtains that
$$\text{Bott}(\phi,uv)|_{\mathcal{P}} = \{0\}. \tag{326}$$
It follows from Lemma (2.2.19) that there exists a continuous path of unitaries$\{u(t): t \in [0,1/2]\} \subset M_n$ such that
$$u(0) = uv, \quad u\left(\frac{1}{2}\right) = 1 \ and \ \ \|[\phi(f),uv]\| < \frac{\epsilon}{4} \tag{327}$$
for all $f \in \mathcal{F}$ and $t \in [0,1/2]$. The rest of the proof is exactly the same as that of Lemma (2.2.24).

Theorem (2.2.29)[71]: Let X be a finite CW complex which has property (H). Let $\epsilon > 0$ be a positive number and let $\mathcal{F}$ be a finite subset of $C(X)$. There exists $\eta_1 > 0$ such that, for each $\sigma_1 > 0$, the following holds. There exists $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists$\eta_3 > 0$ such that, for any $\sigma_3 > 0$, there are a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ satisfying the following. Suppose that $\phi, \psi: C(X) \to M_n$ (for some integer n) are two unital homomorphisms such that
$$\|\phi(f) - \psi(f)\| < \delta \ for \ all \ f \in \mathcal{G}, \mu_{\tau\circ\phi}(O_{\eta_j}) \geq \sigma_j\eta_j, \ \mu_{\tau\circ\psi}(O_{\eta_j}) \geq \sigma_j\eta_j \tag{328}$$
for any open ball O_{η_j} of radius η_j, $j = 1,2,3$, where τ is the normalized trace on M_n. Then there exists a homomorphism $\Phi: C(X) \to C([0,1],M_n)$ such that $\pi_0 \circ \Phi = \phi, \pi_1 \circ \Phi = \psi$ and

$$\|\psi(f) - \pi_t \circ \Phi(f)\| < \epsilon \, for \, all \, f \in \mathcal{F}.$$

Proof: It is clear that one can reduce the general case to the case that X is connected.
Let $\eta_1' > 0$ (in place of η_1) be given by Definition (2.2.26) for $\epsilon/2$ and $\mathcal{F}$. Let $\eta_1 = \eta_1'/16$. Let $\sigma_1 > 0$. Let $\mathcal{G}_1$ (in place of $\mathcal{G}$) be a finite subset of $C(X)$ and $\eta_2' > 0$ (in place of η_2) be given by Definition (2.2.26) for η_1' and $\sigma_1/16$ (in place of σ_1). Let $\sigma_2 > 0$.
Choose δ_1 (in place of δ) required by Definition (2.2.26) for the given $\epsilon/2 > 0, \mathcal{F}, \mathcal{G}_1, \eta_1', \eta_2'$ and $\sigma_2/16$. We may assume that $\eta_2' < \eta_1$ and $\mathcal{F} \subset \mathcal{G}$. Denote $\eta_2 = \eta_2'/16$.
We may assume that $\mathcal{G}_1$ is larger than the $\mathcal{G}$ required by Lemma (2.2.20) for $\eta_2'/2$ (in place of η) and $\sigma_2/16$ (in place of σ). Choose $\delta_2 = \min\{\delta_1/2, \sigma_2\eta_2/64\}$. Let $\eta_2 > 0$ be such that

$$|f(x) - f(x')| < \delta_2/4 \, for \, all \, f \in \mathcal{G}_1,$$

provided that $\text{dist}(x, x') < \eta_0$.
Let $0 < \eta_3' \le \min\{\eta_0/2, \eta_2'/2\}$. We may also assume, by choosing a smaller η_0, that any open ball with radius η_3' is path connected. Let $\eta_3 = \eta_3'/24$ and let $\sigma_3 > 0$. Let $\delta_3 > 0$ (in place of δ) and let $\mathcal{G} \subset C(X)$ be a finite subset required by Lemma (2.2.21) for $\delta_2/2$ (in place of ϵ), $\mathcal{G}_1$ (in place of $\mathcal{F}$), η_3' (in place of η) and $\sigma_3/24$. Let $\delta = \min\{\delta_3/2, \delta_2/2\}$.
Now suppose that ϕ and ψ satisfy conditions (328) for the above $\eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3, \mathcal{G}$ and δ. In particular,

$$\mu_{\tau\circ\phi}(O_{\eta_3'/24}) \ge (\sigma_3/24)\eta_3' \quad and \quad \mu_{\tau\circ\psi}(O_{\eta_3'/24}) \ge (\sigma_3/24)\eta_3'$$

for every open ball $O_{\eta_3'/24}$ with radius $\eta_3'/24$. It follows from Lemma (2.2.21) that there are unital homomorphisms $\Phi_i : C(X) \to C([0,1], M_n)$ such that

$$\pi_0 \circ \Phi_1 = \phi, \pi_0 \circ \Phi_2 = \psi, \tag{329}$$

$$\|\pi_t \circ \Phi_1(g) - \phi(g)\| < \delta_2/2 \, and \, \|\pi_t \circ \Phi_2(g) - \psi(g)\| < \frac{\delta_2}{2} \tag{330}$$

for all $g \in \mathcal{G}_1$ and $t \in [0,1]$. Moreover, there is a unitary $u \in M_n$ such that

$$ad \, u \circ \pi_1 \circ \Phi_1 = \pi_1 \circ \Phi_2. \tag{331}$$

Note that

$$\mu_{\tau\circ\phi}(O_{\eta_2'/16}) \ge \sigma_2 \eta_2'/16.$$

It follows from the proof of Lemma (2.2.20) (with possibly larger $\mathcal{G}$ which depends on η_2) that

$$\mu_{\tau\circ\phi}(\overline{O_{\eta_2'/16}}) \le \mu_{\tau\circ\pi_1\circ\Phi_1}(O_{\eta_2}). \tag{332}$$

It follows that

$$\mu_{\tau\circ\pi_1\circ\Phi_1}(O_{\eta_2'}) \ge (\sigma_2/16)\eta_2'. \tag{333}$$

Similarly,

$$\mu_{\tau\circ\pi_1\circ\Phi_2}(O_{\eta_2}) \ge (\sigma_2/16)\eta_2. \tag{334}$$

Moreover,

$$\mu_{\tau\circ\pi_1\circ\Phi_1}(O_{\eta_1'}) \ge (\sigma_1/16)\eta_1' \, and \, \mu_{\tau\circ\pi_1\circ\Phi_2}(O_{\eta_1'}) \ge (\sigma_1/16)\eta_1'. \tag{335}$$

Since X has property (H), there is a unital homomorphism $\Phi_3 : C(X) \to C([0,1], M_n)$ such that

$$\pi_0 \circ \Phi_3 = \pi_1 \circ \Phi_1, \quad \pi_1 \circ \Phi_3 = \pi_1 \circ \Phi_2 \tag{336}$$

and

$$\|\pi_t \circ \Phi_3(f) - \pi_1 \circ \Phi_1(f)\| < \frac{\epsilon}{2} \text{ for all } f \in \mathcal{F}. \tag{337}$$

The theorem follows from the combination of (329), (330), (336) and (337).

Theorem (2.2.30)[71]: Let X be a finite CW complex with dimension 1. Let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following. For any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\phi : C(X) \to C([0,1], M_n)$ (for any integer n), there is a unital homomorphism $h : C(X) \to C([0,1], M_n)$ such that
$$\|\phi(f) - h(f)\| < \epsilon$$
for all $f \in \mathcal{F}$.

Definition (2.2.31)[71]: Let X_0 be the family of finite CW complexes which consists of all those with dimension no more than one and all those which have property (H). Note that X_0 contains all finite CW complexes X with finite $K_1(C(X))$ and torsion free $K_0(C(X))$, $I \times \mathbb{T}$, n-dimensional tori and those with the form $\mathbb{T} \vee \ldots \vee \mathbb{T} \vee Y$ with some finite CW complex Y with torsion $K_1(C(Y))$ and torsion free $K_0(C(Y))$.

Let X be the family of finite CW complexes which contains all those in X_0 and those with torsion $K_1(C(X))$.

Let X be a finite CW complex and let $h : C(X) \to C([0,1], M_n)$ be a unital homomorphism. It is easy to see that there are finitely many mutually orthogo-nal projections $p_1, p_2, \ldots, p_m$ and points $\xi_1, \xi_2, \ldots, \xi_m$ in X with one point in each connected component such that
$$[h] = [\Phi] \text{ in } KK(C(X), C([0,1], M_n)),$$
where $\Phi(f) = \sum_{i=1}^m f(\xi_i) p_i \text{ for all } f \in C(X)$.

Theorem (2.2.32)[71]: Let $X \in X_0$. Let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. There exists $\eta_1 > 0$ such that, for any $\sigma_1 > 0$, there exists $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists $\eta_3 > 0$ such that, for any $\sigma_3 > 0$, there exists a finite subset $\mathcal{G}, \delta > 0$, and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to C([0,1], M_n)$ (for any integer $n \geq 1$) is a unital δ-G-multiplicative contractive completely positive linear map for which
$$\mu_{\tau \circ \phi}(O_{\eta_j}) \geq \sigma_j \eta_j \tag{338}$$
for any open ball O_{η_j} with radius η_j, $j = 1, 2, 3$, and for all tracial states τ of $C([0,1], M_n)$, and
$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \tag{339}$$
where Φ is a point-evaluation.

Then there exists a unital homomorphism $h : C(X) \to C([0,1], M_n)$ such that
$$\|\phi(f) - h(f)\| < \epsilon \tag{340}$$

for all $f \in \mathcal{F}$.

Proof: The cases that need to be considered are those X which have property (H). We may assume that X is connected and $\Phi = \pi_\xi$ for some point $\xi \in X$. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be given.

Let $\eta_1 > 0$ be required by Theorem (2.2.29) for $\epsilon/4$ (in place of ϵ) and $\mathcal{F}$ above. Let $\sigma_1 > 0$. Let $\eta_2 > 0$ be as required by Theorem (2.2.29) for $\epsilon/4$ (in place of ϵ), $\mathcal{F}$, η_1 and σ_1. Let $\sigma_2 > 0$. Let $\eta_3' > 0$ (in place of η_3) be required by Theorem (2.2.29) for $\epsilon/4$ (in place of ϵ), $\mathcal{F}, \eta_1, \eta_2, \sigma_1$ and $\sigma_2/4$ (in place of σ_2). Let $\sigma_2 > 0$.

Let $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset and $\delta_1 > 0$ (in place of δ) be required by Theorem (2.2.29) for $\epsilon/4, \mathcal{F}, \eta_1, \eta_2, \eta_3$ (in place of η_3), and $\frac{\sigma_j}{4}$ ($j = 1, 2, 3$) as above. We may assume that $\mathcal{F} \subset \mathcal{G}_1$. Let $G_2 \subset C(X)$ be a finite subset which is larger than $\mathcal{G}_1$ and which also depends on η_1 and σ_1.

Let $\epsilon_1 = \min\{\epsilon/4, \delta_1/4\}$. Let $\eta_3 > 0$ (in place of η), $\delta_2 > 0$ (in place of δ), $\mathcal{G} \subset C(X)$ be a finite subset and $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset required by Lemma (2.2.15) for 1 (in place of ϵ), G_2 (in place of $\mathcal{F}$), σ_2 (in place of σ_1), $\sigma_3/2$ (in place of σ) and η_2 (in place of η_1). We may assume that $\eta_3 < \min\{\eta_3/2, \eta_2/2\}$.

Suppose that ϕ satisfies the assumption of the theorem for the above $\eta_j, \sigma_j (j = 1, 2, 3), \delta, \mathcal{G}$ and $\mathcal{P}$. Consider $\mu_{\tau \circ \phi}$ for each $t \in [0, 1]$. Note that $\underline{K}(C([0, 1], M_n)) = \underline{K}(M_n)$. It follows that

$$[\pi_t \circ \phi]|_{\mathcal{P}} = [\pi_\xi]|_{\mathcal{P}}. \tag{341}$$

Note that

$$\mu_{\tau \circ \phi}(O_{\eta_3}) \geq \sigma_3 \eta_3 \text{ and } \mu_{\tau \circ \phi}(O_{\eta_2}) \geq \sigma_2 \eta_2 \tag{342}$$

for all open balls O_{η_3} with radius η_3, all open balls O_{η_2} with radius η_2 and for all tracial states τ of $C([0, 1], M_n)$.

By applying Lemma (2.2.15), one obtains, for each $t \in [0, 1]$, a unital homomorphism $h_t : C(X) \to M_n$ such that

$$\|\pi_t \circ \phi(g) - h_t(g)\| < \delta_1/4 \quad \text{for all } g \in G_1, \tag{343}$$

$$\mu_{\tau \circ h_t}(O_{\eta_3}) \geq (\sigma_3/2)\eta_3 \text{ and } \mu_{\tau \circ h_t}(O_{\eta_2}) \geq \left(\frac{\sigma_2}{2}\right)\eta_2, \tag{344}$$

where τ is the unique tracial state on M_n. Note that, by choosing the large $\mathcal{G}_2$ (depends on ϵ_1 and σ_1) and smaller δ_1, we may also assume that

$$\mu_{\tau \circ h_t}(O_{\eta_1}) \geq \left(\frac{\sigma_1}{2}\right)\eta_1. \tag{345}$$

There is a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that

$$\|\pi_{t_i} \circ \phi(g) - \pi_{t_{i-1}} \circ \phi(g)\| < \delta_1/4 \text{ for all } g \in \mathcal{G}_1, \tag{346}$$

$i = 1, 2, \ldots, m$. Therefore

$$\|h_{t_i}(g) - h_{t_{i-1}}(g)\| < \|h_{t_i}(g) - \pi_{t_i} \circ \phi(g)\| \tag{347}$$

$$+ \|\pi_{t_i} \circ \phi(g) - \pi_{t_{i-1}} \circ \phi(g)\| + \|\pi_{t_{i-1}} \circ \phi(g) - h_{t_{i-1}} \circ \phi(g)\| \tag{348}$$

$$< \frac{\delta_1}{4} + \frac{\delta_1}{4} + \frac{\delta_1}{4} < \delta_1 \tag{349}$$

for all $g \in \mathcal{G}_1$. Thus, using (342) and (345), and by applying Theorem (2.2.29), there exists, for each i, a unital homomorphism $\Phi_i : C(X) \to C([t_{i-1}, t_i], M_n)$ such that

$$\pi_{t_{i-1}} \circ \Phi_i = h_{t_{i-1}}, \pi_{t_i} \circ \Phi_i = h_{t_i} \text{ and} \|\pi_t \circ \Phi_i(f) - h_{t_{i-1}}(f)\| < \epsilon/4 \tag{350}$$

for all $f \in \mathcal{F}, i = 1, 2, \ldots, m$.

Define $h : C(X) \to C([0,1], M_n)$ by

$$\pi_t \circ h = \pi_t \circ \Phi_i \text{ if } t \in [t_{i-1}, t_i],$$

$i = 1, 2, \ldots, m$. It follows that

$$\|h(f) - \phi(f)\| < \epsilon \text{ for all } f \in \mathcal{F}.$$

Lemma (2.2.33)[71]: Let $X \in \mathrm{X}$. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. Suppose that $k_0 = k!$, where k is the largest finite order of torsion elements in $K_i(C(X))$, $i = 0, 1$.

There exists $\eta_1 > 0$ such that, for any $\sigma_1 > 0$, there exists $\eta_2 > 0$ such that, for any $\sigma_2 > 0$, there exists $\eta_3 > 0$ such that, for any $\sigma_3 > 0$, the following holds. There is a finite subset $\mathcal{G} \subset C(X)$, there is $\delta > 0$ and there is a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to C([0,1], M_n)$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map for which

$$\mu_{\tau \circ \phi}\left(O_{\eta_j}\right) \geq \sigma_j \eta_j \tag{351}$$

for any open ball O_{η_j} with radius $\eta_j, j = 1, 2, 3$, for all tracial states τ of $C([0,1], M_n)$, and

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \tag{352}$$

where Φ is a point-evaluation.

Then there exists a unital homomorphism $h : C(X) \to M_{k_0}(C([0,1], M_n))$ such that

$$\left\|\phi^{(k_0)}(f) - h(f)\right\| < \epsilon \tag{353}$$

for all $f \in \mathcal{F}$, where $\phi^{(k_0)}(f) = \mathrm{diag}(\overbrace{\phi(f), \phi(f), \ldots, \phi(f)}^{k_0})$ for all $f \in C(X)$.

Corollary (2.2.34) [71]: Let $X \in \mathrm{X_0}$. Let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. There exists $\eta > 0$, a finite subset $\mathcal{G}, \delta > 0$, and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that $\phi : C(X) \to C([0,1], M_n)$ (for any integer $n \geq 1$) is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map for which

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \tag{354}$$

for any open ball O_a with radius $a \geq \eta$ and for all tracial states τ of $C([0,1], M_n)$, and

$$[\phi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}, \tag{355}$$

where Φ is a point-evaluation.

Then there exists a unital homomorphism $h : C(X) \to C([0,1], M_n)$ such that

$$\|\phi(f) - h(f)\| < \epsilon \tag{356}$$

for all $f \in \mathcal{F}$.

Proof: Let $\epsilon > 0, \mathcal{F} \subset C(X)$ be a finite subset and be given as described. Let $\eta_1 > 0$ be as required by Theorem (2.2.32). Let $\sigma_1 = \Delta(\eta_1)/\eta_1$. Let $\eta_2 > 0$ be required by Theorem (2.2.32). for the above $\epsilon, \mathcal{F}, \eta_1$ and σ_2. Let $\sigma_2 = \Delta(\eta_2).\eta_2$. Let $\eta_3 > 0$ be required by the above $\epsilon, \mathcal{F}, \eta_j$ and $\sigma_j, j = 1, 2$. Let $\sigma_3 = \Delta(\eta_3)/\eta_3$. Choose $\eta = \min\{\eta_j : j = 1, 2, 3\}$. We then choose $\delta > 0, \mathcal{G}$ and $\mathcal{P}$ as required by Theorem (2.2.32). for the above $\epsilon, \mathcal{F}, \eta_j$ and $\sigma_j, j = 1, 2, 3$. Suppose that ϕ satisfies the assumption for the above $\eta, \delta, \mathcal{G}$ and $\mathcal{P}$. Then ϕ satisfies the assumption of Theorem 8.3 for the above η_j, σ_j, δ and $\mathcal{P}$. We then apply Theorem (2.2.32).

Note that Lemma (2.2.33). also has its version of Corollary (2.2.34).

We collects a number of elementary facts about simple C^*-algebras with tracial rank one. Let $B = \oplus_{j=1}^{m} C(X_j, M_{r(j)})$, where $X_j = [0, 1]$ or X_j is a point. For $j \leq m$, denote by $t_{j,x}$ the normalized trace at $x \in X_j$ for the j-th summand. For example, if $b \in B$, then

$$t_{j,x}(b) = \tau(\pi_j(b)(x)),$$

where $\pi_j : B \to C([0, 1], M_{r(j)})$ is the projection to the j-th summand, $x \in X_j$ and τ is the normalized trace on $M_{r(j)}$.

Corollary (2.2.35)[71]: Let A be a unital simple separable C^*-algebra with tracial rank one or zero and let $a \in A_+ \backslash \{0\}$ with $\|a\| \leq 1$. Suppose that

$$\tau(a) \geq \sigma \text{ for all } \tau \in T(A) \tag{357}$$

for some $\sigma > 0$. Then, for any $1 > r > 0$, there is a projection $e \in \overline{aAa}$ such that

$$\tau(e) \geq r\sigma \text{ for all } \tau \in T(A). \tag{358}$$

Proof: For any $b \in A_+$ and any $\delta > 0$, there exists $\epsilon > 0$ such that

$$\|f_\epsilon(b)b - b\| < \delta,$$

where f_ϵ is as defined in the proof of Lemma (2.2.37). Then one sees that the corollary follows immediately from the previous lemma.

Proposition (2.2.36)[71]: Let A be a unital separable simple C^*-algebra with tracial rank no more than one and let $p \in A$ be a projection. Then, for any $\sigma > 0$ and integers $m > n \geq 1$, there exists a projection $q \leq p$ such that

$$\frac{n+1}{m}\tau(p) > \tau(q) > \frac{n}{m}\tau(p) \quad \text{for all } \tau \in T(A). \tag{359}$$

Proof. This follows from the fact that A is tracially approximately divisible

Lemma (2.2.37)[71]: Let X be a compact metric space, let $\Delta: (0, 1) \to (0, 1)$ be a non-decreasing map, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset.be a finite subset and let $\{x_1, x_2, \ldots, x_m\}$ Let $\eta > 0$ be such that

$$|f(x) - f(x')| < \epsilon/4 \text{ for all } f \in \mathcal{F},$$

if $\text{dist}(x, x') < 2\eta$ and

$$O_{2\eta}(x_i) \cap O_{2\eta}(x_j) = \emptyset \text{ if } i \neq j$$

(so η does not depend onΔ). Let $1 > r > 0$. Then there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following.

For any unital separable simple C^*-algebra A with tracial rank no more than one and any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to A$ for which

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a) \, for \, all \, \tau \in T\,(A) \tag{360}$$

and for all $1 > a \geq \eta$, there exist mutually orthogonal non-zero projections $p_1, p_2, \ldots, p_m$ in A such that

$$\tau(p_i) \geq r\Delta(\eta) \, for \, all \, \tau \in T\,(A), i = 1, 2, \ldots, m, \tag{361}$$

$$and \quad \left\| L(f) - [PL(f)P + \sum_{i=1}^{m} f(x_i)p_i] \right\| < \epsilon \quad for \, all \, f \in \mathcal{F}, \tag{362}$$

where $P = 1 - \sum_{i=1}^{m} p_i$.

Proof: Suppose that the lemma is false (for the above$\epsilon, \mathcal{F}, \Delta$ and$\{x_1, x_2, \ldots, x_m\}$).
Let $\eta > 0$ be such that

$$|f(x) - f(x')| < \frac{\epsilon}{4} \, for \, all \, f \in \mathcal{F}, \tag{363}$$

if $\mathrm{dist}(x, x') < 2\eta$. We may assume that $O_{2\eta}(x_i) \cap O_{2\eta}(x_j) = \emptyset \, if \, i \neq j$, $i, j = 1, 2, \ldots, m$.

Let g_i be a function in $C(X)$ such that $0 \leq g_i(x) \leq 1$ for all $x \in X$, $g_i(x) = 1$ if $\mathrm{dist}(x, x_i) < \eta$ and $g_i(x) = 0$ if $\mathrm{dist}(x, x_i) \geq 2\eta, i = 1, 2, \ldots, m$. Put $\mathcal{G}_0 = \{g_i : i = 1, 2, \ldots, m\}$.

Then, there exists a sequence of unital separable simple C^*-algebras with tracial rank no more than one and a sequence of δ_n-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps $L_n : C(X) \to A_n$ for a sequence of decreasing positivenumbers $\delta_n \to 0$ and a sequence of finite subsets $\{\mathcal{G}_n\}$ with$\bigcup_{n=1}^{\infty} \mathcal{G}_n$ dense in $C(X)$
such that

$$\mu_{\tau \circ L}(O_a) \geq \Delta(a) \, for \, all \, \tau \in T\,(A) \, and \, for \, all \, 1 > a \geq \eta \tag{364}$$

$$\liminf \left\{ \inf \left\{ \max \left\{ \left\| L_n(f) - [P_n L_n(f)P_n + \sum_{i=1}^{m} f(x_i)p_{i,n}] \right\| : f \in F \right\} \right\} \right\}$$

$$\geq \epsilon \, n \tag{365}$$

where infimum is taken among all possible mutually orthogonal non-zero projections$p_{1,n}, p_{2,n}, \ldots, p_{m,n}$ with $\tau(p_{i,n}) \geq r\Delta(\eta)$ for all $\tau \in T\,(A_n)$ and $p_n = 1_{A_n} - \sum_{i=1}^{n} p_{i,n}$in A_n.

Let $B = \prod_{n=1}^{\infty} A_n$, let $Q = B/\bigoplus_{n=1}^{\infty} A_n$ and let $\Pi : B \to Q$ be the quotient map. Define $\Phi : C(X) \to B$ by $\Phi(f) = L_n(f)$and $\phi = \Pi \circ \Phi$. Then$\phi : C(X) \to Q$is a unital homomorphism.
By (488),

$$\tau(L_n(g_i)) \geq \mu_{\tau \circ L_n}(O_\eta) \geq \Delta(\eta)$$

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for all $\tau \in T(A)$. It follows from Corollary 9.4 that there exists a projection $p'_{i,n} \in \overline{L_n(g_i)AL_n(g_i)}$ such that

$$\tau(p'_{i,n}) \geq r\Delta(\eta) \text{ for all } \tau \in T(A_n), i = 1, 2, \ldots, m, \tag{366}$$

for all $n \geq n_0$ for some $n_0 \geq 1$. Define $p_i = \{p'_{i,n}\}$ (with $p'_{i,n} = 1$ for $n = 1, 2, \ldots, n_0$) and $q_i = \Pi(P_i), i = 1, 2, \ldots, m$. Note that

$$q_i \in \overline{\phi(g_i)A\phi(g_i)}, i = 1, 2, \ldots, m. \tag{367}$$

It follows that

$$\left\| \phi(f) - [q\phi(f)q + \sum_{i=1}^{m} f(x_i)q_i] \right\| < \frac{\epsilon}{2 for} \text{ all } f \in \mathcal{F}, \tag{368}$$

where $q = 1 - \sum_{i=1}^{m} q_i$. It follows that, for some sufficiently large $n_1 \geq n_0$,

$$\left\| L_n(f) - [P_n L_n(f)P_n + \sum_{i=1}^{m} f(x_i)p'_{i,n}] \right\| < \epsilon \text{ for all } f \in \mathcal{F}. \tag{369}$$

for all $n \geq n_1$, where $P_n = \sum_{i=1}^{m} p'_{i,n}$. This contradicts (489).

Lemma (2.2.38)[71]: Let X be a connected finite CW complex, let $\xi \in X$ be a point and let $Y = X \setminus \{\xi\}$. Suppose that $K_0(C_0(Y)) = \mathbb{Z}^k \oplus \text{Tor}(K_0(C_0(Y)))$ and $g_1, g_2, \ldots, g_k$ are generators of $\mathbb{Z}^k$. Suppose that $\phi : C(X) \to A$ (for some unital separable simple C^*-algebra with tracial rank one or zero) is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map for which $[\phi](g_i)$ is well defined ($i = 1, 2, \ldots, k$), where δ is a positive number and $\mathcal{G}$ is a finite subset of $C(X)$, and

$$|\tau([\phi](g_i))| < \sigma \text{ for all } \tau \in T(A), i = 1, 2, \ldots, k, \tag{370}$$

for some $1 > \sigma > 0$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F}$, any $1 > r > 0$ and any finite subset $\mathcal{H} \subset A$, there exists a projection $p \in A$ and a unital C^*-subalgebra $B = \oplus_{j=1}^{m} C(X_j, M_{r(j)})$, where $X_j = [0,1]$ or X_j is a single point, with $1_B = p$ and a unital $(\delta + \epsilon)$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to B$ such that

$$\|\phi(f) - [(1-p)\phi(f)(1-p) + L(f)]\| < \epsilon \text{ for all } f \in \mathcal{F} \tag{371}$$

$$\text{and } |t_{j,x}([L](g_i))| < (1+r)\sigma, j = 1, 2, \ldots, k, \text{ and } x \in X_j. \tag{372}$$

(We use $t_{j,x}$ for $\tau_{j,x} \otimes Tr_R$ on $B \otimes M_R$, where Tr_R is the standard trace on M_R.) Moreover,

$$\|pa - ap\| < \epsilon \text{ for all } a \in \mathcal{H}.$$

Proof: The proof is similar. Let $p_j, q_j \in M_R(C(X))$ such that

$$[p_j] - [q_j] = g_j, \quad j = 1, 2, \ldots, k,$$

for some integer $R \geq 1$. There exists a sequence projections $p_n \in A$ such th

$$\lim_{n \to \infty} \|cp_n - p_n c\| = 0 \quad \text{for all} \quad c \in A, \tag{373}$$

and there exists a sequence of C^*-subalgebras $B_n = \oplus_{j=1}^{m(n)} C(X_{j,n}, M_{r(j,n)})$ (where $X_{j,n} = [0,1]$ or X is a single point) with $1_{B_n} = p_n$ such that

$$\lim_{n \to \infty} \text{dist}(p_n cp_n, B_n) = 0 \text{ and } \lim_{n \to \infty} \sup_{\tau \in T(A)} \{\tau(1 - p_n)\} = 0. \tag{374}$$

For sufficiently large n, there exists a contractive completely positive linear map $L'_n : p_n A p_n \to B_n$ such that
$$\lim_{n\to\infty} \|L'_n(a) - p_n a p_n\| = 0 \ for \ all \ a \in A.$$
We have
$$\lim_{n\to\infty} \|\phi(f) - [(1 - p_n)\phi(f)(1 - p_n) + L'_n \circ \phi(f)]\| = 0 \ for \ all \ f \in C(X). \tag{375}$$
Define $L'_{n,R} : M_R(A) \to M_R(A)$ by $L'_n \otimes id_{M_R}$ and $\phi_R : M_R(C(X)) \to M_R(A)$ by $\phi_R = \phi \otimes id_{M_R}$.

Suppose that $(for \ some \ fixed \ 1 > r > 0)$ there exists a subsequence $\{n_k\}, \{j_k\}$ and $\{x_k\} \in [0,1]$ such that
$$t_{j_k, x_k}\left(L'_{n,R} \circ \phi_R(p_i - q_i)\right) \geq (1 + r)\sigma \tag{376}$$
for all k. Define a state $T_k : A \to C$ by $T_k(a) = t_{j_k, x_k}(a), k = 1, 2, \ldots$. Let T be a limit point. Note $T_k(1_A) = 1$. Therefore T is a state on A. Then, by (500),
$$T([\phi](g_i)) \geq (1 + r)\sigma. \tag{377}$$
However, it is easy to check that T is a tracial state. This contradicts (484). So the lemma follows by choosing B to be B_n, p to be p_n and L to be $L'_n \circ L$ for some sufficiently large n.

Lemma (2.2.39)[71]: Let A be a unital separable simple C^*-algebra with tracial rank no more than one. Let $p_1, p_2, \ldots, p_n$ be a finite subset of projections in A, and let $L : C(X) \to A$ be a contractive completely positive linear map with $L(1_{C(X)})$ being a projection. Let $d_1, d_2, \ldots, d_n$ be positive numbers, $\Delta : (0, 1) \to (0, 1)$ be a nondecreasing map and let $\eta > 0$. Suppose that
$$\tau(p_i) \geq a_i \ and \ \mu_{\tau \circ L}(O_\eta) \geq \Delta(a) for all \ a \geq \eta \tag{378}$$
for all $\tau \in T(A)$.

Then, for any $1 > r > 0$, any $1 > \delta > 0$, any finite subset $\mathcal{G} \subset C(X)$ and any finite subset $\mathcal{H} \subset A$, there exists a projection $E \in A$, a C^*-subalgebra $B = \oplus_{j=1}^L C(X_j, M_{r(j)})$ with $1_B = E$ ($X_j = [0, 1]$ or X_j is a point), projections p'_i, p''_1 with $p'_i \in B$, and a contractive completely positive linear map $L_1 : C(X) \to B$ with $L_1(1_{C(X)})$ being a projection satisfying the following:
$$\|Ea - aE\| < \delta \ for all \ a \in \mathcal{H} \cup \{L(f) : g \in \mathcal{G}\}, \tag{379}$$
$$\|p_i - (p'_i \ominus p''_i)\| < \delta, \quad i = 1, 2, \ldots, n, \tag{380}$$
$$L(f) - [EL(f)E + L_1(f)] < \delta \ for all \ f \in G, \tag{381}$$
$$t_{j,x}(p'_i) \geq rd_i, \quad i = 1, 2, \ldots, n, \tag{382}$$
$$and \ \mu_{t_{j,x} \circ L_1}(O_a) \geq r\Delta(O_a) for \ all \ a \geq \eta \tag{383}$$
for all $x \in X_j$ and $j = 1, 2, \ldots, L$. Moreover,
$$\tau(1 - E) < \ for \ all \ \tau \in T(A).$$
If $L' : C(X) \to A$ is another δ-$\mathcal{G}$-multiplicative contractive completely positive linear map such that
$$|\tau \circ L'(g)\tau \circ L(g)| < \delta \ for \ all \ g \in \mathcal{G}, \tag{384}$$

we may further require that
$$\|EL'(f) - L(f)E\| < \delta, |L'(f) - [EL'(f)E + L_1(f)]\| < \delta \, for \, all \, f \, \in \, \mathcal{G}, \quad (385)$$
$$\left|t_{j,x} \circ L_1(f) - t_{j,x} \circ L'_1(f)\right| < \quad and \, \mu_{t_{j,x} \circ L'_1}(O_a) \geq r\Delta(a) \quad (386)$$
for all $x \in X_j$, $j = 1, 2, \ldots, L$, for $a \geq \eta$ and for all $f \in \mathcal{G}$.

Proof: There exists a sequence of projections $E_n \in A$ and a sequence of C^*-subalgebra
$B = \oplus_{j=1}^{L_n} C(X_{j,n}, M_{r(j,n)})$ such that
$$\lim_{n\to\infty} \|E_n a - aE_n\| = 0 \, for \, all \, a \, \in \, A. \quad (387)$$
One then obtains a sequence of projections $p'_{i,n} \in B_n, p''_{i,n} \in (1 - E_n)A(1 - E_n)$ and a
sequence of contractive completely positive linear maps $\Phi_n : A \to B_n$ (see[69]) such that
$$\lim_{n\to\infty} \left\|p_i - (p'_{i,n} + p''_{i,n})\right\| = 0 \text{ and } \lim_{n\to\infty} \|a - [E_n aE_n + \Phi_n(a)]\| = 0 \quad (388)$$
for all $a \in A$. Moreover
$$\lim_{n\to\infty} \sup_{\tau \in T(A)} \{\tau(1 - e_n)\} = 0. \quad (389)$$
Suppose that there exists a subsequence $\{n_k\}$ such that
$$t_{j_{n,k},x_k}(p'_{i,n}) < rd_i, i = 1, 2, \ldots, n. \quad (390)$$
Define $T_k(a) = t_{j_{n,k},x_k}(\Phi_{n_k}(a))$ for $a \in A$. Let T be a limit point. Then $T(1_A) = 1$. So
T is a state. It is easy to see that it is also a tracial state. Then
$$T(p_i) \leq rd_i, \quad i = 1, 2, \ldots, n, \quad (391)$$
a contradiction.
Suppose that there exists a subsequence $\{n_k\}$ such that
$$\mu_{t_{j_{n_k},x_k} \circ \Phi_{n_k} \circ L}(O_{a_k}) < r\Delta(a_k) \quad (392)$$
for some $1 > a_k \geq \eta$ and for all k. Again, use the above notation T for a limit
of$\{t_{j_{n_k},x_k} \circ \Phi_{n_k}\}$. Then $T$is a tracial state so that
$$\mu_{T \circ L}(O_a) \leq r\Delta(a) \quad (393)$$
for some $a \geq \eta$, another contradiction.
The first part of the lemma follows by choosing L_1 to be $\Phi_n \circ L, p'_i$to be $p'_{1,n}$
and p''_i to be $p''_{1,n}$ for some sufficiently large n.
The last part follows from a similar argument.

Lemma (2.2.40)[71]: Let A be a unital separable simple C^*-algebra with tracial rank no
more than one. Suppose that $p, q \in A$ are two projections such that
$$\tau(p) \geq D \, and \, \tau(q) \geq D \, for \, all \, \tau \, \in \, T(A).$$
Then, for any $1 > r > 1$, there are projections $p_1 \leq p$ and $q_1 \leq q$ such that
$$[p_1] = [q_1] in \, K_0(A) and \, \tau(p_1) = \tau(q_1) \geq r \cdot D \quad (394)$$
for all $\tau \in T(A)$.

Proof: Fix $1 > r_1 > r > 0$.
A similar argument as in Lemma (2.2.39) leads to the following. There are mutually
orthogonal projections p'_0, p'_1 and mutually orthogonal projectionsq'_0, q'_1 such that
$$\|p_0 + p'_1 - p\| < 1/2, \quad \|q_0 + q'_1 - q\| < \frac{1}{2} \quad (395)$$

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and $p_1', q_1' \in B = \oplus_{j=1}^{L} C(X_j, M_{r(j)})$, where $X_j = [0, 1]$ or where X_j is a single point

$$t_{j,x}(p_1') > r_1 D \text{ and } t_{j,x}(q_1') > r_1 D \tag{396}$$

for $x \in X_j$ and $j = 1, 2, \ldots, L$. Moreover, as [90], $r(j) \geq \frac{2}{(r_1 - r)D}$. There is a projection

$$p_{1,j} \in C(X_j, M_{r(j)}) \text{ such that } p_{1,j} \leq \pi_j(p_1') \text{ and}$$
$$r_1 D \geq t_{j,x}(p_{1,j}) > rD \tag{397}$$

for $x \in X_j$ and $j = 1, 2, \ldots, L$, where $\pi_j : B \to C(X_j, M_{r(j)})$ is a projection.
Hence

$$t_{j,x}(p_{1,j}) \leq t_{j,x}(q_1')$$

for all $x \in X_j$, $j = 1, 2, \ldots, L$. There exists a partial isometry $v_j \in X_j, M_{r(j)}$ such that

$$v_j^* v_j = p_{1,j} \text{ and } v_j v_j^* \leq \pi_j(q_1'),$$

$j = 1, 2, \ldots, L$.
Define $p_1'' = \sum_{j=1}^{L} p_{1,j}$ and $v = \sum_{j=1}^{L} v_j$. Then

$$p_1'' \leq p_1, v^* v = p_1'' \text{ and } vv^* \leq q_1'.$$

Moreover,

$$\tau(p_1'') \geq rD \text{ for all } \tau \in T(A).$$

By (519), there exists a projection $p_1 \leq p$ and a projection $q_1 \leq q$ such that

$$[p_1] = [p_1''] = [vv^*] = [q_1]. \tag{398}$$

Note that

$$\tau(p_1) = \tau(q_1) \geq r \cdot D \text{ for all } \tau \in T(A).$$

Lemma (2.2.41)[71]: Let B be a unital separable amenable C^*-algebra and let A be a unital simple C^*-algebra with $TR(A) \leq 1$. For any $\epsilon > 0$, any finite subset $\mathcal{F} \subset B$, any $\sigma > 0$, any integer $k \geq 1$, any integer $K \geq 1$ and any finite subset $\mathcal{F}_1 \subset A$. Suppose that $\phi, \psi : B \to A$ are two unital positive linear maps. Then there is a projection $p \in A$, a C^*-subalgebra $C_0 = \oplus_{i=1}^{n_1}(C([0,1], M_{d(i)}) \oplus \oplus_{j=1}^{n_2} M_{r(j)}$ with $d(i), r(j) \geq K$ and a C^*-subalgebra C of A with $C = M_k(C_0)$ and with $1_C = p$ and unital positive linear maps $\phi_0, \psi_0 : B \to C_0$ such that

$$\|\phi(f), p\| < \epsilon, \ \|\psi(f), p\| < \epsilon \text{ for all } f \in \mathcal{F}, \tag{399}$$
$$\|x, p\| < \epsilon \text{ for all } x \in \mathcal{F}_1, \tag{400}$$
$$\left\|\phi(f) - ((1-p)\varphi(f)(1-p) \oplus \phi_0^{(k)}(f))\right\| < \epsilon, \tag{401}$$
$$\left\|\psi(f) - \left((1-p)\psi(f)(1-p) \oplus \psi_0^{(k)}(f)\right)\right\| < \epsilon \text{ for all } f \in \mathcal{F} \tag{402}$$

and

$$\tau(1-p) < \sigma \text{ for all } \tau \in T(A), \tag{403}$$

where

$$\phi_0^{(k)}(f) = \text{diag}(\overbrace{\phi_0(f), \phi_0(f), \cdots \phi_0(f)}^{k}) \tag{404}$$

and

$$\psi_0^{(k)}(f) = \text{diag}(\overbrace{\psi_0(f), \psi_0(f), \ldots, \psi_0(f)}^{k}) \text{ for all } f \in B. \tag{405}$$

Lemma (2.2.42)[71]: Let X be a connected finite CW complex and let $Y = X\backslash\{\xi\}$, where $\xi \in X$ is a point. Let $K_0\big(C(Y)\big) = G = \mathbb{Z}^k \oplus Tor(G)$ and $K_0\big(C(X)\big) = \mathbb{Z} \oplus G$. Fix $\kappa \in Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$. Put

$$K = \max\{|\kappa(g_i)| : g_i = (\overbrace{0,\dots,0}^{i-1}, 1, 0,\dots,0) \in \mathbb{Z}^k\}.$$

Then, for any $\delta > 0$ any finite subset $\mathcal{G} \subset C(X)$ and any finite subset $\mathcal{P} \subset \underline{K}(C_0(Y))$, there exists an integer $N(K) \geq 1$ (which depends on $K, \delta, \mathcal{G}$ and $\mathcal{P}$, but not κ) and a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to M_{N(k)}$ such that

$$\big[L|_{C_0(Y)}\big]|_\mathcal{P} = \kappa|_\mathcal{P} . \tag{406}$$

$$(Note\ that\ the\ lemma\ includes\ the\ case\ that\ K = 0.)$$

Proof: Choose $\delta_0 > 0$ and a finite subset $\mathcal{G}_0 \subset C_0(Y)$ such that, for any pair of δ_0-$\mathcal{G}_0$-multiplicative contractive completely positive linear maps from $C_0(Y)$ to any C^*-algebra, $[L_i]|_\mathcal{P}$ is well defined and

$$[L_1]|_\mathcal{P} = [L_2]|_\mathcal{P} , \tag{407}$$

provided that

$$L_1 \approx_{\delta_0} L_2 \quad on\ \mathcal{G}_0.$$

It follows that there exists an asymptotic morphism$\{\phi_t : t \in [1,\infty)\} : C_0(Y) \to \mathcal{K}$ such that

$$[\{\phi_t\}] = \kappa. \tag{408}$$

Note that, for each $t \in [1,\infty)$, ϕ_t is a contractive completely positive linear map and

$$\lim_{t\to\infty}\|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0$$

for all $a, b \in C_0(Y)$. Define $\delta_1 = \min\{\delta_0/2, \delta/2\}$ and $\mathcal{G}_1 = \mathcal{G}_0 \cup \mathcal{G}$. It follows that, for sufficiently large t,

$$[\phi_t]|_\mathcal{P} = \kappa|_\mathcal{P} \tag{409}$$

and ϕ_t is δ_1-$\mathcal{G}_1$ multiplicative. Choose a projection $E \in \kappa$ such that

$$\|E\phi_t(a) - \phi_t(a)E\| < \delta_1/4 \text{ for all } a \in \mathcal{G}_2, \tag{410}$$

where $\mathcal{G}_2 = \mathcal{G}_1 \cup \{ab : a, b \in \mathcal{G}_1\}$. Define $L : C(X) \to E\mathcal{K}E$ by $L(f) = f(\xi)E + E\phi_t(f - f(\xi))E$ for $f \in C(X)$. It is easy to see that L is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and

$$\big[L|_{C_0(Y)}\big]|_\mathcal{P} = \kappa|_\mathcal{P} . \tag{411}$$

Define the rank of E to be $N(\kappa)$. Note that $E\mathcal{K}E \cong M_{N(\kappa)}$. Note that since $K_i(C_0(Y))$ is finitely generated, by [21],

$$Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K})) = Hom_\Lambda(F_m\underline{K}(C_0(Y)), F_m\underline{K}(\mathcal{K}))$$

for some integer $m \geq 1$. Thus, when K is given, there are only finitely many di erent κ so that $|\kappa(g_i)| \leq K$ for $i = 1, 2,\dots, k$. Thus such $N(K)$ exists by takingthe maximum of those $N(\kappa)$.

Lemma (2.2.43)[71]: Let X be a connected finite CW complex and let $Y = X\backslash\{\xi\}$, where$\xi \in X$ is a point. Let $K_0\big(C(Y)\big) = G = \mathbb{Z}^k \oplus Tor(G)$ and $K_0(C(X)) = \mathbb{Z} \oplus G$.

Forany $\delta > 0$, any finite subset $\mathcal{G} \subset C(X)$ and any finite subset $\mathcal{P} \subset \underline{K}(C_0(Y))$, there exists an integer $N(\delta, \mathcal{G}, \mathcal{P}) \geq 1$ satisfying the following.

Let $\kappa \in Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$ and let

$$K = \max\{|\kappa(g_i)| : g_i = (\overbrace{0,\ldots,0}^{i-1}, 1, 0, \ldots, 0) \in \mathbb{Z}^k\}$$

There exists an integer $N(K) \geq 1$ and a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to M_{N(k)}$ such that

$$[L]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \ and \ \frac{N(K)}{\max\{K,1\}} \leq N(\delta, \mathcal{G}, \mathcal{P}). \tag{412}$$

Proof: Fix $\delta, \mathcal{P}$ and $\mathcal{G}$. Let $N(0)$ and $N(1)$ be as in Lemma (2.2.42) corresponding to the case that $K = 0$ and $K = 1$. Define

$$N(\delta, \mathcal{G}, \mathcal{P}) = kN(1) + N(0).$$

Fix $\kappa \in Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$. Suppose that $\kappa(g_i) = m_i$, $i = 1, 2, \ldots, k$. For each i ($i = 0, 1, 2, \ldots, k$) there is $\kappa_i \in Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(\mathcal{K}))$ such that

$$\kappa_0(g_i) = 0, \qquad i = 1, 2, \ldots, k, \tag{413}$$
$$\kappa_0(g_i) = 0, \ if \ m_i = 0, \qquad j = 1, 2, \ldots, k, \tag{414}$$
$$\kappa_0(g_i) = \ sign(m_i) \cdot 1 \ (in \ \mathbb{Z}) \ and \ \kappa_0(g_j) = 0 \ if \ j \neq i, \tag{415}$$
$$if \ m_i \neq 0, \quad i = 1, 2, \ldots, k, \tag{416}$$

and

$$\kappa_0 + \sum_{i=1}^{k} m_i \kappa_i = \kappa. \tag{417}$$

By Lemma (2.2.42), there exists a unitalδ-$\mathcal{G}$-multiplicative contractive completely positive linear map$L_i : C(X) \to M_{N(1)}$ such that

$$[L_i|_{C_{0(Y)}}]|_{\mathcal{P}} = \kappa_i|_{\mathcal{P}}, \qquad i = 0, 1, 2, \ldots, k. \tag{418}$$

Put $N = N(0) + \sum_{i=1}^{k} |m_i|N(1)$. Define L $: C(X) \to M_N$by

$$L(f) = L_0(f) \oplus \bigoplus_{i=1}^{k} \overline{L}_i(f), \tag{419}$$

for all $f \in C(X)$, where

$$\overline{L}_i(f) = \text{diag}\left(\overbrace{L_i(f), L_i(f), \ldots, L_i(f)}^{|m_i|}\right), i = 1, 2, \ldots, k. \tag{420}$$

One estimates that

$$\frac{N}{max\{K,1\}} = \frac{N(0) + \sum_{i=1}^{k} |m_i|N(1)}{max\{K,1\}} \leq N(0) + kN(1) = N(\delta, \mathcal{G}, \mathcal{P}). \tag{421}$$

Lemma(2.2.44)[71]: Let X be a connected finite CW complex with $K_0(C(X)) = \mathbb{Z} \oplus G$where$G = \mathbb{Z}^k \oplus T \ or \ (G) = K_0(C_0(X))$ and $Y = X \setminus \{\xi\}$ for some point $\xi \in X$. For any $\sigma > 0$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following.

For any unital separable C^*-algebra A with $T(A) \neq \emptyset$ and any unital δ-$\mathcal{G}$-

multiplicative contractive completely positive linear map $L : C(X) \to A$, one has
$$|\tau \circ [L](g_i)| < \sigma \text{ for all } \tau \in T(A), \tag{422}$$
where $g_i = (\overbrace{0,\dots,0}^{i-1}, 1, 0, \dots, 0) \in \mathbb{Z}^k$ and τ is the state on $K_0(C(X))$ induced by the tracial state τ.

Proof: Suppose that the lemma is false.

Then there exists a sequence of unital separable C^*-algebras A_n and a sequence of δ_n-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps $L_n : C(X) \to A_n$, where $\delta_n \downarrow 0$ and $\mathcal{G}_n$ is a sequence of finite subsets such that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $C(X)$ and that there exists $\tau_n \in T(A_n)$ such that
$$|\tau_n \circ [L_n](g_i)| \geq \frac{\sigma}{2} \tag{423}$$
for some $i \in \{1, 2, \dots, k\}$.

Let $B = \prod_{n=1}^{\infty} A_n$. Define $t_n(\{a_n\}) = \tau_n(a_n)$. Then t_n is a tracial state of B. Let T be a limit point of $\{t_n\}$. One obtains a subsequence $\{n_k\}$ such that
$$T(\{a_n\}) = \lim_{k \to \infty} \tau_{n_k}(a_{n_k}) \tag{424}$$
for any $\{a_n\} \in B$. Note that for any $a \in \bigoplus_{n=1}^{\infty} A_n \subset B, T(a) = 0$. It follows that defines a tracial state $\bar{T}$ on $B/\bigoplus_{n=1}^{\infty} A_n$. Let $\Pi : B \to B/\bigoplus_{n=1}^{\infty} A_n$ be the quotient map. Define $L : C(X) \to B$ by $L(f) = \{L_n(f)\}$. Put $\phi = \Pi \circ L$. Then ϕ is a unital homomorphism. Therefore
$$T \circ \phi_{*0}(g_i) = 0. \tag{425}$$
It follows that there is a subsequence $\{n_k'\} \subset \{n_k\}$ such that
$$\lim \tau_{n_k'} \circ \left[L_{n_k'}\right](g_i) = 0. \tag{426}$$
However, this contradicts (347).

Lemma (2.2.45)[71]: Let $C(X)$ be a connected finite CW complex and $\mathcal{P} \subset \underline{K}(C(X))$. There exists $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following. For any unital C^*-algebra A and any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to A$, there exists $\kappa \in Hom_\Lambda(\underline{K}(C(X)), \underline{K}(A))$ such that
$$[L]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} . \tag{427}$$
This is known (see [70]).

Lemma (2.2.46)[71]: Let $X \in \mathbf{X}$ be a finite simplicial complex. Let $\epsilon > 0$, let $\epsilon_1 > 0$, let $\eta_0 > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset, let $N \geq 1$ and $K \geq 1$ be positive integers and let $\Delta : (0, 1) \to (0, 1)$ be a non-decreasing map. There exist $\eta > 0, \delta > 0$, a finite subset $\mathcal{G}$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ satisfying the following.

Suppose that A is a unital separable simple C^*-algebra with tracial rank no more than one and $\phi, \psi : C(X) \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that
$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \tag{428}$$
$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G} \tag{429}$$

for all $\tau \in T(A)$ and .

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \tag{430}$$

Then, for any $\epsilon_0 > 0$, there are four mutually orthogonal projections P_0, P_1, P_2 and P_3 with $P_0 + P_1 + P_2 + P_3 = 1_A$, there is a unital C^*-subalgebra $B_1 \subset (P_1 + P_2 + P_3)A(P_1 + P_2 + P_3)$ with $1_B = P_1 + P_2 + P_3$, where B_1 has the form $B_1 = \bigoplus_{j=1}^{s} C(X_j, M_{r(j)})$ with $P_1, P_2, P_3 \in B_1$, where $X_j = [0,1]$ or X_j is a point.

There are also unital homomorphisms $\phi_1, \psi_1 : C(X) \to B$, where $B = P_3 B_1 P_3$, there exists a finite dimensional C^*-subalgebra $C_0 \subset P_1 B P_1$ with $1_{C_0} = P_1$ and there exists a unital ϵ-$\mathcal{F}$-multiplicative contractive completely positive linear map $\phi_2 : C(X) \to C_0$ and mutually orthogonal projections $p_1, p_2, \ldots, p_m \in P_2 B_1 P_2$ and a unitary $u \in A$ such that

$$\|\phi(f) - [P_0\phi(f)P_0 + \phi_2(f) + \sum_{i=1}^{m} f(x_i)p_i + \phi_1(f)]\|\epsilon/2 \tag{431}$$

and

$$\|ad\,u \circ \psi(f) - [P_0(ad\,u \circ \psi(f))P_0 + \phi_2(f) + \sum_{i=1}^{m} f(x_i)p_i + \psi_1(f)]\| < \epsilon/2 \tag{432}$$

for all $f \in \mathcal{F}$, where $\{x_1, x_2, \ldots, x_m\}$ is ϵ_1-dense in X and $P_2 = \sum_{i=1}^{m} p_i$,

$$N\tau(P_0 + P_1) < \tau(p_i)\,Kt_{j,x}(P_1 + P_2) \le t_{j,x}(P_3), \tag{433}$$

$$\mu_{T \circ \phi_1}(O_a) \ge \frac{\Delta(a)}{4}, \mu_{T \circ \psi_1}(O_a) \ge \frac{\Delta(a)}{4}\,for\,all\,a \ge \eta_0, \tag{434}$$

$$|T \circ \psi_1(f) - T \circ \phi_1(f)| < \quad for\,all\,f \in \mathcal{F}, \tag{435}$$

for all $\tau \in T(A), i = 1, 2, \ldots, m$, for all $x \in X_j, j = 1, 2, \ldots, s$, and for all $T \in T(B)$.

Moreover, for any finite subset $\mathcal{H} \subset A$, one may require that

$$\|aP_0 - P_0a\| < \epsilon_0 \ and \ (1 - P_0)a(1 - P_0) \in_\epsilon B_1 \ for\ all\ a \in \mathcal{H}. \tag{436}$$

Corollary (2.2.47)[71]: Let $X \in X$. Let $\epsilon > 0$, let $\eta_0 > 0$, let $\mathcal{F} \subset C(X)$ and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. Then there exists $\eta > 0, \delta > 0$, and a finite subset $\mathcal{G} \subset C(X)$ satisfying the following.

Suppose that A is a unital separable simple C^*-algebra with $TR(A) \le 1$ and $\phi: C(X) \to A$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map such that

$$\mu_{T \circ \phi}(O_a) \ge \Delta(a) \ for\ all\ a \ge \eta. \tag{437}$$

Then, for any $\epsilon_0 > 0$, for any integer $K \ge 1$, there are mutually orthogonal projections $P_0 P_1$ and P_2 with $P_0 + P_1 + P_2 = 1_A$, there exists a unital C^*-subalgebra $B = \bigoplus_{j=1}^{s} C(X_j, M_{r(j)})$ with $P_1 = 1_B$, where $X_j = [0,1]$ or X_j is a point, a finite dimensional C^*-subalgebra D, a unital completely positive linear map $\phi_1 : C(A) \to D$ and there exists a unital homomorphism $\phi_1 : C(X) \to B$ such that $\|\phi(f) - (P_0\phi(f)P_0 + \phi_2(f) + \phi_1(f))\| < \epsilon$ for all $f \in \mathcal{F}$ \tag{438}

and

$$K\tau(P_0 + P_2) < \tau(P_1)\,for\,all\,\tau \in T(A). \tag{439}$$

Moreover, for any finite subset $\mathcal{H} \subset A$, one may require that

$$\|aP_0 - P_0a\| < \epsilon_0 \ for\ all\ a \in \mathcal{H} \cup \phi(F). \tag{440}$$

Proof: Choose $\psi = \phi$ and then apply Lemma (2.2.46).

Theorem (2.2.48)[71]: Let X be a finite simplicial complex in X. Let $\phi > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. There exists $\eta > 0, \delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and a finite subset $\mathcal{U} \subset \mathcal{U}(M_\infty(C(X)))$ satisfying the following,

Suppose that A is a unital separable simple C^*-algebra with tracial rank no more than one and $\phi, \psi : C(X) \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that

$$\mu_{T \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta, \tag{441}$$

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G}, \tag{442}$$

for all $\tau \in T(A)$, and

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } dist\left(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})\right) < \delta \tag{443}$$

for all $z \in \mathcal{U}$. Then there exists a unitary $u \in A$ such that

$$ad\, u \circ \psi \approx \phi \text{ on } \mathcal{F}. \tag{444}$$

Proof: Let $\eta_1 > 0$ be as in Corollary (2.2.12) for $\epsilon/4$ and $\mathcal{F}$. Let $\sigma_1 = \Delta(\eta_1)/4\eta_1$. Let $\eta_0 > 0$ (in place of η) and $K_1 \geq 1$ (in place of K) be as in Corollary (2.2.12) for $\epsilon/4$ and $\mathcal{F}$ above. Let $\sigma_0 = \Delta(\eta_0)/4\eta_0$ (in place of σ). Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$), $\mathcal{P}_1 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$), $\mathcal{U}_1 \subset U(M_\infty(C(X)))$ (in place of $\mathcal{U}$) and $L_1 \geq 1$ (in place of L) be finite subsets required by Corollary (2.2.12)

Let $L = 8\pi + 1$. Let $\delta_2 > 0$ (in place of δ), $\mathcal{G}_2 \subset C(X)$ (in place of G), $\mathcal{P}_2 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$), $\mathcal{U}_2 \subset \mathcal{U}(M_\infty(C(X)))$ (in place of $\mathcal{U}$), $l \geq 1$ and $\epsilon_1 > 0$ be as required by Theorem (2.2.11) for $\epsilon/4$ and $\mathcal{F}$. Let $\epsilon_2 = min\{\delta_1/2, \delta_2/2\}$ and $\mathcal{F}_2 = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2$. Let $\epsilon_3 > 0$ be a number smaller than ϵ_2. Let $N = l$ and $K > 16/min\{\sigma\eta, \sigma_1\eta_1, \delta_1\}$. Let $\eta_2 > 0$, let $\delta_3 > 0$ (in place of δ), let $\mathcal{G}_3 \subset C(X)$ (in place of $\mathcal{G}$), let $\mathcal{P}_3 \subset \underline{K}(C(X))$ be required by Lemma (2.2.46) for ϵ_3 (in place of ϵ) ϵ_1, $min\{\eta_1, \eta_0\}$ (in place of η_0) and $\mathcal{F}_2$ (in place of $\mathcal{F}$).

Let $\eta = min\{\eta_1, \eta_0, \eta_2\}$ and let $\delta_4 = min\{\Delta(\eta)/4, \delta_3, 1/32K_1\pi\}$.

Let δ be a positive number which is smaller than δ_4 and let $\mathcal{G}$ be a finite subset containing $\mathcal{G}_3$. Let $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset which contains $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and the image of $\mathcal{U}$ in $\underline{K}(C(X))$.

Suppose that A is a unital separable simple C^*-algebra with tracial rank one or zero and suppose $\phi, \psi : C(X) \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps which satisfy the assumption of the theorem for the above $\delta, \mathcal{G}, \mathcal{P}$ and $\mathcal{U}$.

It follows from Lemma (2.2.46) that there are four mutually orthogonal projections $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$ with $\mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 = 1_A$, there is a unital C^*-subalgebra $B_1 \subset (\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)A(\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3)$ with $1_B = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$ and $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3 \in B_1$, where B_1 has the form $B_1 = \bigoplus_{j=1}^s C(X_j, M_{r(j)})$ and where $X_j = [0,1]$ or X_j is a point, there are unital homomorphisms $\phi_1, \psi_1 : C(X) \to \mathcal{P}_3 B_1 \mathcal{P}_3$, and there exists a finite dimensional C^*-subalgebra $C_0 \subset \mathcal{P}_1 B_1 \mathcal{P}_1$ with $1_{C_0} = \mathcal{P}_1$. There also exists a unital ϵ_3-$\mathcal{F}_2$-multiplicative

contractive completely positive linear map $\phi_2 : C(X) \to C_0$ and mutually orthogonal projections $p_1, p_2, \ldots, p_m \in B_1$ and a unitary $v \in A$ such that

$$\|\phi(f) - [P_0\phi(f)P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i)p_i + \phi_1(f)]\| < \frac{\epsilon_3}{2} \qquad (445)$$

and

$$\left\| ad\, v \circ \psi(f) - [P_0(ad\, v \circ \psi(f))P_0 + \phi_2(f) + \sum_{i=1}^m f(x_i)p_i + \psi_1(f)] \right\|$$

$$< \frac{\epsilon_3}{2} \qquad (446)$$

for all $f \in \mathcal{F}_2$, where $\{x_1, x_2, \ldots, x_m\}$ is ϵ_1-dense in X and $P_2 = \sum_{i=1}^m p_i$,

$$N\tau(P_0 + P_1) < \tau(p_i), Kt_{j,x}(P_1 + P_2) \le t_{j,x}(P_3), \qquad (447)$$

$$\mu_{T \circ \phi_1}(O_a) \ge \Delta(a)/4, \mu_{T \circ \psi_1}(O_a) \ge \Delta(a)/4 \text{ for all } a \ge min\{\eta_0, \eta_1\} \qquad (448)$$

$$\text{And} |T \circ \phi_1(f) - T \circ \psi_1(f)| < \epsilon_3 \text{ for all } f \in \mathcal{F}_2, \qquad (449)$$

for all $\tau \in T(A)$, $x \in Xj$, $j = 1, 2, \ldots, m$ and for all $T \in T(B)$. Moreover, for any finite subset $H \subset A$, one may require that

$$\|aP_0 - P_0a\| < \epsilon_3 \text{ and } (1 - P_0)a(1 - P_0) \in_{\epsilon_3} B_1 \text{ for all } a \in \mathcal{H}. \qquad (450)$$

We may also assume that $r(j) \ge L_1$ for $j = 1, 2, \ldots, s$. Put $\phi_0(f) = P_0\phi(f)P_0, \psi_0(f) = P_0(ad\, u \circ \psi(f))P_0, \phi_3(f) = \phi_2(f) + \sum_{i=1}^m f(x_i)p_i + \phi_1(f)$ and $\psi_3(f) = \phi_2(f) + \sum_{i=1}^m f(x_i)p_i + \psi_1(f)$ for $f \in C(X)$.

Since

$$dist\left(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})\right) < \delta \text{ for all } z \in \mathcal{U}, \qquad (451)$$

with a sufficiently large $\mathcal{H}$ (and sufficiently small ϵ_3), in [90], we may assume that

$$dist(\phi_0^\ddagger(\bar{z}), \psi_0^\ddagger(\bar{z})) < 2\delta \text{ for all } z \in \mathcal{U}. \qquad (452)$$

Furthermore, we may also assume that

$$dist(\phi_3^\ddagger(\bar{z}), \psi_3^\ddagger(\bar{z})) < 2\delta \text{ for all } z \in \mathcal{U}. \qquad (453)$$

Denote by D the determinant function on B_1. We compute that

$$D(\phi_1(z)\psi_1(z)^*) < 4\delta \text{ for all } z \in \mathcal{U}. \qquad (454)$$

It follows that

$$dist(\phi_1^\ddagger(\bar{z}), \psi_1^\ddagger(\bar{z})) < 1/8K_1\pi \text{ for all } z \in \mathcal{U}. \qquad (456)$$

We may also assume (with sufficiently large $\mathcal{U}$ and sufficiently small ϵ_3) that

$$[\phi_1]|_{\mathcal{P}} = [\psi_1]|_{\mathcal{P}} \qquad (457)$$

and

$$[\phi_0]|_{\mathcal{P}} = [\psi_0]|_{\mathcal{P}}. \qquad (458)$$

By (660), (661) and (667) and by applying Corollary (2.2.12), we obtain a unitary $w_1 \in B$ such that

$$ad\, w_1 \circ \psi_1 \approx_{\frac{\epsilon}{4}} \phi_1 \text{ on } \mathcal{F}. \qquad (459)$$

By applying Theorem (2.2.11), we also have a unitary $w_2 \in (P_0 + P_2)A(P_0 + P_2)$ such that

$$\left\| w_2^* \left(\psi_0(f) \oplus \sum_{i=1}^{m} f(x_i)p_i \right) w_2 - \left(\phi_0(f) \oplus \sum_{i=1}^{m} f(x_i)p_i \right) \right\| < \frac{\epsilon}{4} \text{ for all } f \in \mathcal{F}.$$

(460)

The theorem then follows from the combination of (446), (447), (459) and (460).

Definition (2.2.49): Let $C = P M_k(C(X))P$ for some finite CW complex X and for some projection $P \in M_k(C(X))$. Suppose that the rank of P is m. Let t be a state on C. Then there is a Borel probability measure μ_t such that

$$t(f) = \int_X L_x(f(x))d\mu_t \quad \text{for all } f \in C,$$

(461)

where L_x is a state on M_m. If $t \in T(C)$, then $L_x(f(x)) = tr(f(x))$, where tr is the normalized trace on M_m. There is an integer $n \geq 1$ and a rank one trivial projection $e \in M_n(C)$ such that $eM_n(C)e \cong C(X)$. It follows that there is a unitary $u \in M_n(C)$ and a projection $Q \in M_{kn}(C)$ such that $u^*Cu = QM_k(eM_n(C)e)Q$. Suppose that A is a unital C^*-algebra, s is a state on A and suppose that $\phi : C \to A$ is a contractive completely positive linear map. Then

$$s \circ \phi(f) = \int_X L_x(f(x))d\mu_{\tau \circ \phi} \quad \text{for all } f \in C,$$

where L_x is a state on M_m.

Let $\tau \in T(C)$ and let $\phi^{(n)} : M_n(C) \to M_n(A)$ be the homomorphism induced by $\tilde{\phi}$. Denote by $\phi : C(X) \to \phi^{(n)}(e)M_n(A)\phi^{(n)}(e)$ the restriction of $\phi^{(n)}$ on $eM_n(C)e$. It follows that the probability measure $\mu_{\tau \circ \tilde{\phi}}$ induced by $\tau \circ \tilde{\phi}$ is equal to $\mu_{\tau \circ \phi}$.

Corollary (2.2.50)[71]: Let X be a finite simplicial complex in X. Let $\epsilon > 0$, let $\mathcal{F} \subset C = P M_n(C(X))P$, where $P \in M_n(C(X))$ is a projection, be a finite subset and let $\Delta: (0,1) \to (0,1)$ be a non-decreasing map. There exists $\eta > 0, \delta > 0$, a finite subset $\mathcal{G}$, a finite subset $\mathcal{P} \subset \underline{K}(C)$ and a finite subset $\mathcal{U} \subset \mathcal{U}(M_\infty(C))$ satisfying the following. Suppose that A is a unital separable simple C^*-algebra with tracial rank no more than one and $\phi, \psi : C \to A$ are two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \text{ for all } a \geq \eta,$$

(462)

$$|\tau \circ \phi(g) - \tau \circ \psi(g)| < \delta \text{ for all } g \in \mathcal{G},$$

(463)

for all $\tau \in T(A)$, and

$$[\phi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}} \text{ and } dist\left(\phi^\ddagger(\bar{z}), \psi^\ddagger(\bar{z})\right) < \delta$$

(464)

for all $z \in \mathcal{U}$. Then there exists a unitary $u \in A$ such that

$$ad \, u \circ \psi \approx \phi \text{ on } \mathcal{F}.$$

(465)

Proof. It is standard (using Definition 10.9) that the general case can be reduced to the case that $C = M_k(C(X))$. It is then clear that this corollary follows from Theorem (2.2.48)

It should be noted in the case that $X = I \times \mathbb{T}$ or X is an n-dimensional torus, in the above Theorem (2.2.48) and Corollary 10.10, one may only consider $\mathcal{U} \subset U(C)$. Moreover, in

the case that X is a finite simplicial complex with torsion $K_1(C(X))$, the map $\phi^{\ddagger}$ and $\psi^{\ddagger}$ can be removed entirely (see [36]).

Let X be a compact metric space and let A be a unital simple C^*-algebra with $T(A) \neq \emptyset$. Suppose that $\phi: C(X) \to A$ is a unital monomorphism. Then $\mu_{\tau \circ \phi}$ is a strictly positive probability Borel measure. Fix $a \in (0,1)$. Let $\{x_1, x_2, \ldots, x_m\} \subset X$ be an $a/4$-dense subset. Define

$$d(a,i) = (1/2)\inf\{\mu_{\tau \circ \phi}(B_{a/4}(x_i)): \tau \in T(A)\}, i = 1, 2, \ldots, m.$$

Fix a non-zero positive function $g \in C(X)$ with $g \leq 1$ whose support is contained in $B_{a/4}(x_i)$. Then, since A is simple, $\inf\{\tau(\phi(g)): \tau \in T(A)\} > 0$. It follows that $d(a,i) > 0$. Put

$$\Delta(a) = min\{d(a,i): i = 1, 2, \ldots, m\}.$$

For any $x \in X$, there exists i such that $B_a(x) \supset B_{a/4}(x_i)$. Thus

$$\mu_{\tau \circ \phi}(B_a(x)) \geq \Delta(a) for all \tau \in T(A). \tag{466}$$

Note that Δ gives a non-decreasing map from $(0,1) \to (0,1)$.

This proves the following.

Proposition (2.2.51)[71]: Let X be a compact metric space and let A be a unital simple C^*-algebra with $T(A) \neq \emptyset$. Suppose that $\phi: C(X) \to A$ is a unitalmonomo-rphism. Then there is a non-decreasing map $\Delta: (0,1) \to (0,1)$ such that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) for all \tau \in T(A) \tag{467}$$

for all open balls O_a of X with radius $a \in (0,1)$.

Definition (2.2.52)[71]: Let C be a C^*-algebra. Let $T = N \times K: C_+ + \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ be a map. Suppose that A is a unitalC^*-algebra and $\phi: C \to A$ is a homomorphism. Let $\mathcal{H} \subset C_+ \setminus \{0\}$ be a finite subset. We say that ϕ is T-$\mathcal{H}$-full if there are $x_{a,i} \in A, i = 1, 2, \ldots, N(a)$ with $x_{a,i} \leq K(a), i = 1, 2, \ldots, N(a)$, such that

$$\sum_{i=1}^{N(a)} x_{a,i}^* \phi(a) x_{a,i} = 1_A$$

for all $a \in \mathcal{H}$. The homomorphism ϕ is said to be T-full if

$$\sum_{i=1}^{N(a)} x_{a,i}^* \phi(a) x_{a,i} = 1_A$$

for all $a \in A_+ \setminus \{0\}$. If ϕ is T-full, then ϕ is injective.

Proposition (2.2.53)[71]: Let X be a finite CW complex, let $P \in M_k(C(X))$ be a projection and let $C_1 = P M_k(C(X))P$. Suppose that $T = N \times N: C_+ \setminus \{0\} \to \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$ is a map. Then there exists a non-decreasing map $\Delta: (0,1) \to (0,1)$ associated with T satisfying the following.

For any $\eta > 0$, there is a finite subset $\mathcal{H} \subset (C_1 \otimes C(\mathbb{T}))_+ \setminus \{0\}$ such that, for anyunitalC^*-algebra B with $T(B) \neq \emptyset$and any unital contractive completely positive linear map $\phi: C \to B$ which is T-$\mathcal{H}$-full, one has that

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \, for \, all \, a \, \geq \, \eta \, for \, all \, a \, \in \, (\eta, 1). \tag{468}$$

Proof: To simplify notation, using Definition 10.9, without loss of generality, we may assume that $C = C(X)$. Fix $1 > a > 0$. Let $\{x_1, x_2, \ldots, x_n\}$ be an $a/4$-dense subset of X. Let f_i be a positive function in $C(X)$ with $0 \leq f_i \leq 1$ whose support is in $B_{a/4}(x_i)$ and contains $B_{a/6}(x_i), i = 1, 2, \ldots, m$. Define $\Delta': (0, 1) \to (0, 1)$ by

$$\Delta'(a) = \frac{1}{\max\{N(f_i)K(f_i)^2 : 1 \leq i \leq m\}} \tag{469}$$

Define

$$\Delta(a) = min\{\Delta'(b) : b \geq a\}.$$

It is clear that Δ is non-decreasing.

Now let B be a unital C^*-algebra with $T(B) \neq \emptyset$ and let $\phi: C \to B$ be a unital contractive completely positive linear map which is T -$\mathcal{H}$-full. For each i, there are $x_{i,j}, j = 1, 2, \ldots, N(f_i)$, with $\|x_{i,j}\| \leq N(f_i)$ such that

$$\sum_{i=1}^{N(f_i)} x_{i,j}^* \phi(f_i) x_{i,j} = 1_B, i = 1, 2, \ldots, m. \tag{470}$$

Fix a $\tau \in T(B)$. There exists j such that

$$\tau(x_{i,j}^* \phi(f_i) x_{i,j}) \geq \frac{1}{N(f_i)}. \tag{471}$$

It follows that

$$\|x_{i,j} x_{i,j}^*\| \tau(\phi(f_i)) \geq \tau\left(\phi(f_i)^{\frac{1}{2}} x_{i,j} x_{i,j}^* \phi(f_i)^{\frac{1}{2}}\right) \tag{472}$$

$$= \tau(x_{i,j}^* \phi(f_i) x_{i,j}) \geq \frac{1}{N(f_i)}. \tag{473}$$

It also follows that

$$\tau(\phi(f_i)) \geq \frac{1}{N(f_i)K(f_i)^2} \quad . \tag{474}$$

This holds for all $\tau \in T(B), i = 1, 2, \ldots, m$. Now for any open ball O_a with radius a, suppose that y is the center. Then $y \in B_{a/4}(x_i)$ for some $1 \leq i \leq m$. Thus

$$O_a \supset B_{a/4}(x_i).$$

It follows that

$$\mu_{\tau \circ \phi}(O_a) \geq \tau(f_i) \geq \frac{1}{N(f_i)K(f_i)^2} \geq \Delta(a) \tag{475}$$

for all $\tau \in T(B)$. It is then clear that, when $\eta > 0$ is given, such a finite subset $\mathcal{H}$ exists.

Definition (2.2.54)[71]: An AH-algebra C is said to have property (J) if C is isomorphic to an inductive limit $\lim_{n \to \infty}(C_n, \phi_j)$, where $\oplus_{j=1}^{R(i)} P_{n,j} M_{r(n,j)}(C(X_{n,j})) P_{n,j}$ where $X_{n,j}$ is a one dimensional finite CW complex or a simplicial complex in X and where $P_{n,j} \in M_{r(n,j)}(C(X_{n,j}))$ is a projection and each ϕ_j is injective.

Section (2.3): Unitaries in a Simple C^*-Algebra of Tracial Rank One

Let M_n be the C^*-algebra of $n \times n$ matrices and let $u \in M_n$ be a unitary. Then u can be diagonalized, i.e., $u = \sum_{k=1}^{n} e^{i(\theta_k)} p_k$, where $\theta_k \in \mathbb{R}$ and $\{p_1, p_2, \ldots, p_n\}$ are mutually orthogonal projections. As a consequence, $u = \exp(ih)$, where $h = \sum_{k=1}^{n} \theta_k p_k$ is a selfadjoint matrix. Now let A be a unital C^*-algebra and let $U(A)$ be the unitary group of A. Denote by $U_0(A)$ the connected component of $U(A)$ containing the identity. Suppose that $u \in U_0(A)$. Even in the case that A has real rank zero, $sp(u)$ can have infinitely many points and it is impossible to write u as an exponential, in general. However, it was shown ([79]) that u can be approximated by unitaries in A with finite spectrum if and only if A has real rank zero. This is an important and useful feature for C^*-algebras of real rank zero. In this case, u is a norm limit of exponentials.

Tracial rank for C^*-algebras was introduced in the connection with the program of classification of separable amenable C^*-algebras, or otherwise known as the Elliott program. Unital separable simple amenable C^*-algebras with tracial rank no more than one which satisfy the universal coefficient theorem have been classified by the Elliott invariant ([36] and [89]). A unital separable simple C^*-algebra A with $TR(A) = 1$ has real rank one. Therefore a unitary $u \in U_0(A)$ may not be approximated by unitaries with finite spectrum. We will show that, in a unital infinite dimensional simple C^*-algebra A with tracial rank no more than one, if u can be approximated by unitaries in A with finite spectrum then u must be in $CU(A)$, the closure of the subgroup generated by commutators of the unitary group. A related problem is whether every unitary $u \in U_0(A)$ can be approximated by unitaries which are exponentials. Our first result is to show that, there are selfadjoint elements $h_n \in A_{s.a}$ such that

$$u = \lim_{n \to \infty} \exp(ih_n)$$

(converge in norm). It should be mentioned that exponential rank has been studied quite extendedly (see [112], [113], [108], [111], etc.). In fact, it was shown by N. C. Phillips that a unital simple C^*-algebra A which is an inductive limit of finite direct sums of C^*-algebras with the form $C(X_{i,n}) \otimes M_{i,n}$ with the dimension of $X_{i,n}$ is bounded has exponential rank $1 + \epsilon$, i.e., every unitary $u \in U_0(A)$ can be approximated by unitaries which are exponentials (see [113]). These simple C^*-algebras have tracial rank one or zero. Theorem (2.3.11) was proved without assuming A is an AH-algebra, in fact, it was proved in the absence of amenability.

Let $T(A)$ be the tracial state space of A. Denote by $\text{Aff}(T(A))$ the space of all real affine continuous functions on $T(A)$. Denote by $\rho A : K0(A) \to \text{Aff}(T(A))$ the positive homomorphism induced by $\rho_A([p])(\tau) = \tau(p)$ for all projections in $M_k(A)$ (with $k = 1, 2, \ldots$) and for all $\tau \in T(A)$. It was introduced by de la Harpe and Scandalis a determinant like map Δ which maps $U_0(A)$ into $\text{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. By a result of $K.$ Thomsen ([133]) the de la Harpe and Scandalis determinant induces an isomorphism between $\text{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ and $U_0(A)/CU(A)$. We found out that if u can be approximated by unitaries in A with finite spectrum then u must be in $CU(A)$. But can every unitary in $CU(A)$ be approximated by unitaries with finite spectrum? To answer this question, we consider even simpler question: when can a self-adjoint element in a unital separable simple C^*-algebra with $TR(A) = 1$ be approximated by self-adjoint elements

with finite spectrum? Immediately, a necessary condition for a self-adjoint element $a \in A$ to be approximated by self-adjoint elements with finite spectrum is that $\widehat{h^n} \in \overline{\rho_A(K_0(A))}$ (for all $n \in \mathbb{N}$). Given a unitary $u \in U_0(A)$, there is an affine continuous map from $\text{Aff}(T(C(\mathbb{T})))$ into $\text{Aff}(T(A))$ induced by u. Let $\Gamma(u): \text{Aff}(T(C(\mathbb{T}))) \to \text{Aff}(T(A))/\rho A(K0(A))$ be the map given by u. Then it is clear that (u) = 0 is a necessary condition for u being approximated by unitaries with finite spectrum. Note that $\Gamma(u) = 0$ if and only if $u^n + \widehat{(u^n)^*}, i(u^n - \widehat{(u^n)^*}) \in \rho_A(K_0(A))$ for all positive integers n. By applying a uniqueness theorem together with classification results in simple C^*-algebras, we show that the condition is also sufficient. From this, we show that a unitary $u \in CU(A)$ can be approximated by unitaries with finite spectrum if and only if $\Gamma(u) = 0$. We also show that $\Delta(u) = 0$ is not sufficient for $\Gamma(u) = 0$. Therefore, there are unitaries in $CU(A)$ which can not be approximated by unitaries with finite spectrum. Perhaps more interesting fact is that $\Gamma(u) = 0$ does not imply that $\Delta(u) = 0$ for $u \in U_0(A)$.

Denote by I the class of C^*-algebras which are finite direct sums of C*-subalgebras with the form $M_k(C([0,1])$ or $M_k, k = 1, 2, \ldots$.

Definition (2.3.1)[94]. Recall that a unital simple C^*-algebra A is said to have tracial rank no more than one (or $TR(A) \leq 1$), if for any $\epsilon > 0$, any $a \in A_+\backslash\{0\}$ and any finite subset $\mathcal{F} \subset A$, there exists a projection $p \in A$ and a C^*-subalgebra B with $1_B = p$ such that

(i) $\|px - xp\| < \epsilon$ for all $x \in \mathcal{F}$;
(ii) $\text{dist}(pxp, B) < \epsilon$ for all $x \in \mathcal{F}$ and
(iii) $1 - p$ is Murry-von Nuemann equivalent to a projection in $\overline{aAa}$.

Recall that, in the above definition, if B can always be chosen to have finite dimension, then A has tracial rank zero ($TR(A) = 0$). If $TR(A) \leq 1$ but $TR(A) \neq 0$, we write $TR(A) = 1$.

Every unital simple AH-algebra with very slow dimension growth has tracial rank no more than one (see [89]). There are C^*-algebras with tracial rank no more than one which are not amenable.

Definition (2.3.2)[94]. Suppose that $u \in U(A)$. We will use $\bar{u}$ for the image of u in $U(A)/CU(A)$. If $x, y \in U(A)/CU(A)$, define
$$\text{dist}(x, y) = \inf\{\|u - v\| : \bar{u} = x \text{ and } \bar{v} = y\}.$$

Let C be another unital C^*-algebra and let $\varphi : C \to A$ be a unital homomorphism. Denote by $\varphi^{\ddagger} : U(C)/CU(C) \to U(A)/CU(A)$ the homomorphism induced by φ.

Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$, then A is quasi-diagonal, stable rank one, weakly unperforated $K_0(A)$ and, if $p, q \in A$ are two projections, then p is equivalent to a projection $p' \leq q$ whenever $\tau(p) < \tau(q)$ for all tracial states τ in $T(A)$.

For unitary group of A, we have the following:
(i) $CU(A) \subset U_0(A)$ [89];
(ii) $U_0(A)/CU(A)$ is torsion free and divisible [89];

Theorem (2.3.3)[94]. [84] Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$ and let $e \in A$ be a non-zero projection. Then the map $u \mapsto u + (1 - e)$ induces an isomorphism j from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$.

Corollary (2.3.4)[94]. Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Then the map $j : a \to \mathrm{diag}(a, \overset{m}{\overbrace{1, 1, .., 1}})$ from A to $M_n(A)$ induces an isomorphism from $U(A)/CU(A)$ onto $U(M_n(A))/CU(M_n(A))$ for any integer $n \geq 1$.

Definition (2.3.5)[94]. Let $u \in U_0(A)$. There is a piece-wise smooth and continuous path $\{u(t) : t \in [0, 1]\} \subset A$ such that $u(0) = u$ and $u(1) = 1$. Define

$$R(\{u(t)\})(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt.$$

$R(\{u(t)\})(\tau)$ is real for every τ.

Definition (2.3.6)[94]. Let A be a unital C^*-algebra with $T(A) \neq \emptyset$. As in [133], define a homomorphism $\Delta \colon U_0(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ by

$$\Delta(u) = \Delta\left(\frac{1}{2\pi} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt\right),$$

where $\Delta \colon \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is the quotient map and where $\{u(t) : t \in [0, 1]\}$ is a piece-wise smooth and continuous path of unitaries in A with $u(0) = u$ and $u(1) = 1_A$. This is well-defined and is independent of the choices of the paths.

The following is a combination of a result of K. Thomsen ([133]). We state here for the convenience.

Theorem (2.3.7)[94]. Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u \in U_0(A)$. Then the following are equivalent:

(i) $u \in CU(A)$;

(ii) $\Delta(u) = 0$;

(iii) for some piecewise continuous path of unitaries $\{u(t): t \in [0, 1]\} \subset A$ with $u(0) = u$ and $u(1) = 1_A$,

$$R(\{u(t)\}) \in \overline{\rho_A(K_0(A))},$$

(iv) for any piecewise continuous path of unitaries $\{u(t): t \in [0, 1]\} \subset A$ with $u(0) = u$ and $u(1) = 1_A$,

$$R(\{u(t)\}) \in \rho_A(K_0(A)).$$

(v) there are $h_1, h_2, \ldots, h_m \in A_{s.a.}$ such that

$$u = \prod_{j=1}^m \exp(ih_j) \text{ and } \sum_{j=1}^m \widehat{h_j} \in \overline{\rho_A(K_0(A))}.$$

(vi) $\sum_{j=1}^m \widehat{h_j} \in \overline{\rho_A(K_0(A))}$ for any $h_1, h_2, \ldots, h_m \in A_{s.a.}$ for which

$$u = \prod_{j=1}^m \exp(ih_j)$$

Proof. Equivalence of (ii), (iii), (iv), (v) and (vi) follows from the definition of the determinant and follows from the Bott periodicy. The equivalence of (i) and (ii) follows on [133].

The following is a consequence.

Theorem (2.3.8)[94]. Let A be a unital simple separable C^*-algebra with $TR(A) \leq 1$. Then $\ker \Delta = CU(A)$. The de la Harpe and Skandalis determinant gives an isomorphism:

$$\bar{\Delta} \colon U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Moreover, one has the following short exact (splitting) sequence

$$0 \to \text{Aff}(T(A))/\overline{\rho_A\big(K_0(A)\big)} \xrightarrow{\overline{\Delta}^{-1}} U(A)/CU(A) \to K_1(A) \to 0.$$

(Note that $U_0(A)/CU(A)$ is divisible in this case, by [89].)

Theorem (2.3.9)[94]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let : $C(\mathbb{T})_{s.a} \to \text{Aff}\big(T(A)\big)$ be a (positive) affine continuous map.

For any $\epsilon > 0$, there exists $\delta > 0$ and there exists a finite subset $\mathcal{F} \subset C(\mathbb{T})_{s,a}$ satisfying the following: If $v \in U_0(A)$ with

$$\big|\tau\big(f(u)\big) - \gamma(f)(\tau)\big| < \delta, \qquad \text{for all } f \in \mathcal{F} \text{ and } \tau \in T(A), \text{and} \qquad (476)$$
$$\text{dist}(\bar{u}, \bar{v}) < \delta \text{ in } U_0(A)/CU(A). \qquad (477)$$

Then there exists a unitary $W \in U(A)$ such that

$$\|u - W^*vW\| < \epsilon. \qquad (478)$$

Proof. The lemma follows immediately on [64]. See [71], [64]. Note that, of [64], we can replace the given map h_1 (in this case a given unitary) by a given map γ.

Corollary (2.3.10)[94]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u \in U_0(A)$ be a unitary. For any $\epsilon > 0$, there exists $\delta > 0$ and there exists an integer $N \geq 1$ satisfying the following: If $v \in U_0(A)$ with

$$\big|\tau(u^k) - \tau(v^k)\big| < \delta, k = 1, 2, \dots, N \text{ for all } \tau \in T(A) \text{and} \qquad (479)$$
$$\text{dist}(\bar{u}, \bar{v}) < \delta \text{ in } U_0(A)/CU(A). \qquad (480)$$

Then there exists a unitary $W \in U(A)$ such that

$$\|u - W^*vW\| < \epsilon. \qquad (481)$$

Proof. Note that (479),

$$|\tau(u^k) - \tau(v^k)| < \delta \quad k = \pm 1, \pm 2, \dots, \pm N. \qquad (482)$$

For any subset $\mathcal{G} \subset C(S^1)$ and any $\eta > 0$, there exists $N \geq 1$ and $\delta > 0$ such that

$$\big|\tau\big(g(u)\big) - \tau\big(g(v)\big)\big| < \eta \quad \text{for all} \quad \tau \in T(A)$$

if (482) holds.

Then the lemma follows from (2.3.9) (or [64]).

Theorem (2.3.11)[94]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u \in U_0(A)$, then, for any $\epsilon > 0$, there exists a selfadjoint element $a \in A_{s.a}$ such that

$$\|u - \exp(ia)\| < \epsilon. \qquad (483)$$

Proof. Since $u \in U_0(A)$, we may write

$$u = \prod_{j=1}^{k} \exp(ih_j). \qquad (484)$$

Let $M = \max\{\|h_j\| : j = 1, 2, \dots, k\} + 1$. Let $\delta > 0$ and N be given in Corollary (2.3.4) for u. We may assume that $\delta < 1$ and $N \geq 3$. We may also assume that $\delta < \epsilon$. Since $TR(A) \leq 1$, there exists a projection $p \in A$ and a C^*-subalgebra $B \in A$ with $1_B = p$ such that $B \cong \bigoplus_{i=1}^{m} C\big(X_i, M_{r(i)}\big)$, where $X_i = [0, 1]$ or a point, and

$$\|pu - up\| < \frac{\delta}{16NMk}, \qquad (485)$$

$$\left\| (1 - p)u(1 - p) - (1 - p)\prod_{j=1}^{k} \exp(i((1 - p)h_j(1 - p))) \right\| < \frac{\delta}{16NMk}, \qquad (486)$$

$$pup \in_{\frac{\delta}{16NMk}} B \text{ and } \tau(1 - p) < \frac{\delta}{2NMk} \text{ for all } \tau \in T(A). \qquad (487)$$

There exist unitary $u_1 \in B$ such that

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$$\|pup - u_1\| < \frac{\delta}{8NMk} \tag{488}$$

Put $u_2 = (1 - p) \prod_{j=1}^{k} \exp(i(1 - p)h_j(1 - p))$. Since $u_1 \in B$, it is well known that there exists a selfadjoint element $b \in B_{s.a}$ such that

$$\|u_1 - p \exp(ib)\| < \frac{\delta}{16NMk}. \tag{489}$$

Let $v_0 = (1 - p) + p \exp(ib)$ and $u_0 = p \exp(ib) + u_2$. Then, by (485), (486), (488) and (489),

$$\|u_0 - u\| < \|u - pup - (1 - p)u(1 - p)\| \tag{490}$$
$$+ \|(pup - p \exp(ib)) + ((1 - p)u(1 - p) - u_2)\| \tag{491}$$
$$< \frac{3\delta}{16NMk} + \frac{\delta}{8NMk} + \frac{\delta}{16NMk} = \frac{3\delta}{8NMk}. \tag{492}$$

and

$$u_0 v_0^* = \prod_{j=1}^{k} \exp(i(1 - p)h_j(1 - p)). \tag{493}$$

Note that

$$\left| \tau \left(\sum_{j=1}^{k} (1 - p)h_j(1 - p) \right) \right| \leq \sum_{j=1}^{k} \left| \tau \left((1 - p)h_j(1 - p) \right) \right| \tag{494}$$

$$= k\tau (1 - p)\max\{\|h_j\| : j = 1, 2, \ldots, k\} < \delta/16N \tag{495}$$

for all $\tau \in T(A)$. It follows that

$$\mathrm{dist}(\bar{u}, \bar{v}_0) < \delta/16N \text{ in } U_0(A)/CU(A). \tag{496}$$

It follows from that

$$\mathrm{dist}(\bar{u}, \bar{v}_0) < \delta/8N. \tag{497}$$

On the other hand, for each $s = 1, 2, \ldots, N$, by (493), (492) and (487)

$$|\tau(us) - \tau(v_0^s)| \leq |\tau(u^s) - \tau(u_0^s)| + |\tau(u_0^s) - \tau(v_0^s)| \tag{498}$$

$$\leq \|u^s - u_0^s\| + \left| \tau \left((1 - p) - (1 - p) \prod_{j=1}^{k} \exp\left(i(1 - p)sh_j(1 - p)\right) \right) \right| \tag{499}$$

$$\leq N\|u - u_0\| + 2\tau (1 - p) \tag{500}$$

$$< \frac{3\delta}{8Mk} + \frac{\delta}{MNk} < \delta \tag{501}$$

for all $\tau \in T(A)$. From the above inequality and (497) and applying Corollary (2.3.4), one obtains a unitary $W \in U(A)$ such that

$$\|u - W^* v_0 W\| < \epsilon. \tag{502}$$

Put $a = W^*((1 - p) + b)W$. Then

$$\|u - \exp(ia)\| < \epsilon. \tag{503}$$

Note that Theorem (2.3.11) does not assume that A is amenable, in particular, it may not be a simple AH-algebra. The proof used a kind of uniqueness theorem for unitaries in a unital simple C^*-algebra A with $TR(A) \leq 1$. This bring us to the following theorem which is an immediate consequence of Corollary (2.3.4).

Theorem (2.3.12)[94]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Let u and v be two unitaries in $U_0(A)$. Then they are approximately unitarily equivalent if and only if

$$\Delta(u) = \Delta(v) \quad \text{and} \tag{504}$$

$$\tau(u^k) = \tau(v^k) \text{ for all } \tau \in T(A), \tag{505}$$

$k = 1, 2, \ldots$.

Since $\Delta: U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ is an isomorphism, one may ask if (505) implies that $\Delta(u) = \Delta(v)$? In other words, would $\tau(f(u)) = \tau(f(v))$ for all $f \in C(S^1)$ imply that $\Delta(u) = \Delta(v)$? This becomes a question only in the case that $\overline{\rho_A(K_0(A))} \neq \mathrm{Aff}(T(A))$. Thus we would like to recall the following:

Theorem (2.3.13)[94].

Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Then the following are equivalent:

(i) $TR(A) = 0$,

(ii) $\rho_A(K_0(A)) = \mathrm{Aff}(T(A))$ and

(iii) $CU(A) = U_0(A)$.

However, when $TR(A) = 1$, at least, one has the following:

Proposition (2.3.14)[94]. Let A be a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$. If $a \in \overline{\rho_A(K_0(A))}$, then

$$ra \in \overline{\rho_A(K_0(A))} \tag{506}$$

for all $r \in \mathbb{R}$. In fact, $\overline{\rho_A(K_0(A))}$ is a closed $\mathbb{R}$-linear subspace of $\mathrm{Aff}(T(A))$.

Proof. Note that $\overline{\rho_A(K_0(A))}$ is an additive subgroup of $\mathrm{Aff}(T(A))$. It suffices to prove the following: Given any projection $p \in A$, any real number $0 < r_1 < 1$ and $\epsilon > 0$, there exists a projection $q \in A$ such that

$$|r_1\tau(p) - \tau(q)| < \epsilon \text{ for all } \tau \in T(A). \tag{507}$$

Choose $n \geq 1$ such that

$$|m/n - r_1| < \epsilon/2 \text{ and } 1/n < \epsilon/2 \tag{508}$$

for some $1 \leq m < n$.

Note that $TR(pAp) \leq 1$. By [89], there are mutually orthogonal projections $q_0, p_1, p_2, \ldots, p_n$ with $[q_0] \leq [p_1]$ and $[p_1] = [p_i], i = 1, 2, \ldots, n$ and $\sum_{i=1}^n p_i + q_0 = p$. Put $q = \sum_{i=1}^m p_i$. We then compute that

$$|r_1\tau(p) - \tau(q)| < \epsilon \text{ for all } \tau \in T(A). \tag{509}$$

Theorem (2.3.15)[94]. Let A be a unital simple infinite dimensional C^*-algebra with $TR(A) = 1$. Then there exist unitaries $u, v \in U_0(A)$ with

$$\tau(u^k) = \tau(v^k) \text{ for all } \tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$$

such that $\Delta(u) \neq \Delta(v)$. In particular, u and v are not approximately unitarily equivalent.

Proof. Since we assume that $TR(A) = 1$, then, by Theorem (2.3.13), $\mathrm{Aff}(T(A)) \neq \overline{\rho_A(K_{0(A)})}$ and $U_0(A)/CU(A)$ are not trivial.

Let $\kappa_1, \kappa_2 : K_1(C(\mathbb{T})) \to U_0(A)/CU(A)$ be two different homomorphisms. Fix an affine continuous map $s : T(A) \to T_f(C(\mathbb{T}))$, where $T_f(C(\mathbb{T}))$ is the space of strictly positive normalized Borel measures on $\mathbb{T}$. Denote by $\gamma_0 : \mathrm{Aff}(T(C(\mathbb{T}))) \to \mathrm{Aff}(T(A))$ the positive affine continuous map induced by $\gamma_0(f)(\tau) = f(s(\tau))$ for all $f \in \mathrm{Aff}(T(C(T)))$ and $\tau \in T(A)$. Let

$$\gamma_0 : U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) = \mathrm{Aff}(T(C(\mathbb{T})))/Z \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$$
$$= U_0(A)/CU(A)$$

be the map induced by γ_0. Write

$$U(C(\mathbb{T}))/CU(C(\mathbb{T})) = U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) \oplus K_1(C(\mathbb{T})).$$

Define $\lambda_i : U(C(\mathbb{T}))/CU(C(\mathbb{T})) \to U_0(A)/CU(A)$ by

$$\lambda_i(x \oplus z) = \gamma_0(x) + \kappa_i(z)$$

for $x \in U_0(C(\mathbb{T}))/CU(C(\mathbb{T}))$ and $z \in K_1(C(\mathbb{T})), i = 1, 2$. That there are two unital monomorphisms $\varphi_1, \varphi_2 : C(\mathbb{T}) \to A$ such that

$$(\varphi_1)_{*i} = 0, \qquad \varphi_i^{\ddagger} = \lambda_i \quad \text{and} \quad \varphi_i^{\natural} = s, \tag{510}$$

$i = 1, 2$. Let z be the standard unitary generator of $C(S^1)$. Define $u = \varphi_1(z)$ and $v = \varphi_2(z)$.

Then $u, v \in U_0(A)$. The condition that $\varphi_i^{\natural} = s$ implies that $\tau(u^k) = \tau(v^k)$ for all $\tau \in T(A), k = 0, \pm 1, \pm 2, \dots, \pm n, \dots$.

But since $\lambda_1 \neq \lambda_2$,

$$\Delta(u) \neq \Delta(v).$$

Therefore u and v are not approximately unitarily equivalent.

Lemma (2.3.16)[94]. Let A be a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and let $h \in A$ be a self-adjoint element. Then h can be approximated by self-adjoint elements with finite spectrum if and only if $\widehat{h^n} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \dots$.

Proof. If h can be approximated by self-adjoint elements so can hn. By Proposition (2.3.14), $\overline{\rho_A(K_0(A))}$ is a closed linear subspace. Therefore $\widehat{h^n} \in \overline{\rho_A(K_0(A))}$ for all n.

Now we assume that $\widehat{h^n} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \dots$. The Stone-Weierstrass theorem implies that $\widehat{f(h)} \in \overline{\rho_A(K_0(A))}$ for all real-value functions $f \in C(sp(h))$. For any $\epsilon > 0$, by Lemma 2.4 of [89], there is $f \in C(sp(x))_{s.a}$ such that

$$\|f(h) - h\| < \epsilon$$

and $sp(f(h))$ consists of a union of finitely many closed intervals and finitely many points.

Thus, to simplify notation, we may assume that $X = sp(h)$ is a union of finitely many intervals and finitely many points. Let $\psi : C(X) \to A$ be the homomorphism defined by $\psi(f) = f(h)$. Let $s : T(A) \to T_f(C(X))$ be the affine map defined by $f(s(\tau)) = \psi(f)(\tau)$ for all $f \in \text{Aff}(C(X))$ and $\tau \in T(A)$.

Let B be a unital simple AH-algebra with real rank zero, stable rank one and

$$(K_0(B), K_0(B)+, [1_B], K_1(B)) \cong (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

In particular, $K_0(B)$ is weakly unperforated. The proof on [89] provides a unital homomorphism $\iota : B \to A$ which carries the above identification. This can be done by applying of [89] and the uniqueness Theorem of [89], or better by corollary 11.7 of [71] because $TR(B) = 0$, the map $\varphi^{\ddagger}$ is not needed since $U(B) = CU(B)$ and the map on traces is determined by the map on $K_0(B)$.

Note that $\text{Aff}(T(B)) = \overline{\rho_B(K_0(B))}$. By identifying B with a unital C^*-subalgebra of A, we may write $\overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $\psi^{\natural} : \text{Aff}(T(C(X))) \to \overline{\rho_A(K_0(A))}$ be the map induced by ψ. This gives an affine map $\gamma : \text{Aff}(T(C(X))) \to \overline{\rho_B(K_0(B))}$. That there exists a unital monomorphism $\varphi : C(X) \to B$ such that

$$\iota \circ \varphi_{*0} = \psi_{*0} \quad \text{and} \quad (\iota \circ \phi)^{\natural} = \psi^{\natural},$$

where $(\iota \circ \varphi)^{\natural} : \text{Aff}(T(C(X))) \to \text{Aff}(T(A))$ defined by $(\iota \circ \varphi)^{\natural}(a)(\tau) = \tau (\iota \circ \varphi)(a)$ for all $a \in A_{s.a}$.. It follows on [71] that ψ and $\iota \circ \varphi$ are approximately unitarily equivalent. On the hand, since B has real rank zero, φ can be approximated by

homomorphisms with finite dimensional range. It follows that h can be approximated by self-adjoint elements with finite spectrum.

Theorem (2.3.17)[94]. Let A be a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and let $u \in U_0(A)$. Then u can be approximated by unitaries with finite spectrum if and only if $u \in CU(A)$ and
$$\overline{u^n + (u^n)^*}, \iota(\overline{u^n - (u^n)^*}) \in \overline{\rho_A(K_0(A))}, n = 1, 2, \dots.$$

Proof. Suppose that there exists a sequence of unitaries $\{u_n\} \subset A$ with finite spectrum such that
$$\lim_{n\to\infty} u_n = u.$$

There are mutually orthogonal projections $p_{1,n}, p_{2,n}, \dots, p_{m(n),n} \in A$ and complex numbers $\lambda_{1,n}, \lambda_{2,n}, \dots, \lambda_{m(n),n} \in \mathbb{C}$ with $|\lambda_{i,n}| = 1, i = 1, 2, \dots, m(n,)$ and $n = 1, 2, \dots$, such that
$$\lim_{n\to\infty} \left\| u - \sum_{i=1}^{m(n)} \lambda_{i,n} p_{i,n} \right\| = 0.$$

It follows that
$$\lim_{n\to\infty} \left\| ((u^*)^n + u^n) - \sum_{i=1}^{m(n)} 2Re(\lambda_{i,n}) p_{i,n} \right\| = 0.$$

By Proposition (2.3.14),
$$\sum_{i=1}^{m(n)} 2Re(\lambda_{i,n}) \widehat{p_{i,n}} \in \overline{\rho_A(K_{0(A)})}.$$

Thus$\widehat{Re(u^n)} \in \overline{\rho_A(K_0(A))}$. Similarly, $\widehat{Im(u^n)} \in \overline{\rho_A(K_0(A))}$.

To show that $u \in CU(A)$, consider a unitary $v = \sum_{i=1}^m \lambda_i p_n$, where $\{p_1, p_2, \dots, p_m\}$ is a set of mutually orthogonal projections such that $\sum_{i=1}^m p_j = 1$, and where $|\lambda_i| = 1, i = 1, 2, \dots, m$. Write $\lambda_j = e^{i\theta_j}$ for some real number θ_j , $j = 1, 2, \dots$. Define
$$h = \sum_{j=1}^m \theta_j p_j.$$

Then
$$v = \exp(ih).$$

By Proposition (2.3.14), $\hat{h} \in \overline{\rho_A(K_0(A))}$. It follows from Theorem (2.3.7) that $v \in CU(A)$. Since u is a limit of those unitaries with finite spectrum, $u \in \overline{CU(A)}$.

Now assume $u \in CU(A)$ and $\overline{u^n + (u^n)^*}, i(\overline{u^n - (u^n)^*}) \in \overline{\rho_A(K_0(A))}$ for $n = 1, 2, \dots$. If $sp(u) \neq \mathbb{T}$, then the problem is reduced to the case in Lemma (2.3.16). So we now assume that $sp(u) = \mathbb{T}$. Define a unital monomorphism $\varphi: C(\mathbb{T}) \to A$ by $\varphi(f) = f(u)$. By the Stone-Weirestrass theorem and Proposition (2.3.14), every real valued funtion $f \in C(\mathbb{T}), [\phi(f) \in \overline{\rho_A(K_0(A))}$.

As in the proof of Lemma (2.3.16), one obtains a unital C^*-subalgebra $B \subset A$ which is a unital simple AH-algebra with tracial rank zero such that the embedding $\iota: B \to A$ gives an identification:
$$(K_0(B), K_0(B)+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$
Moreover, that there is a unital monomorphism $\psi: C(\mathbb{T}) \to B$ such that

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$$\psi_{*1} = 0 \quad \text{and} \quad (\iota \circ \psi)^{\natural} = \varphi^{\natural}.$$

Note also

$$(\iota \circ \psi)^{\ddagger} = \phi^{\ddagger}$$

(both are trivial, since $u \in CU(A)$).

It follows on (see [71]) that $\iota \circ \psi$ and φ are approximately unitarily equivalent. However, since $\psi_{*1} = 0$, in B, by [79], ψ can be approximated by homomorphisms with finite dimensional range. It follows that u can be approximated by unitaries with finite spectrum.

If A is a finite dimensional simple C^*-algebra, then $TR(A) = 0$. Of course, every unitary in A has finite spectrum. But $CU(A) \neq U_0(A)$. To unify the two cases, we note that $K_0(A) = Z$.

Instead of using $\overline{\rho_A(K_0(A))}$, one may consider the following definition:

Definition (2.3.18)[94]. Let A be a unital C^*-algebra. Denote by $V(\rho_A(K_0(A)))$, the closed $\mathbb{R}$-linear subspace of $\mathrm{Aff}(T(A))$ generated by $\rho_A(K_0(A))$. Let $\Pi: \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$ be the quotient map. Define the new determinant

$$\tilde{\Delta}: U_0(A) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$$

by

$$\tilde{\Delta}(u) = \Pi \circ \Delta(u) \text{ for all } u \in U_0(A).$$

Note that if A is a finite dimensional C^*-algebra $\mathrm{Aff}(T(A)) = V(\rho_A(K_0(A)))$. Thus $\tilde{\Delta} = 0$. If A is a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$, by Proposition (2.3.14),

$$V(\rho_A(K_0(A))) = \overline{\rho_A(K_0(A))}.$$

Definition (2.3.19)[94]. Suppose that $u \in A$ is a unitary with $X = sp(u)$. Then it induces a positive affine continuous map from $\gamma_0 : C(X)_{s.a.} \to \mathrm{Aff}(T(A))$ defined by

$$\gamma_0(f(u))(\tau) = \tau(f(u))$$

for all $f \in C(X)_{s.a.}$ and all $\tau \in T(A)$. Let $\Delta: \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$. Put $\Gamma(u) = \Pi \circ \gamma_0$. Then $\Gamma(u)$ is a map from $C(X)_{s.a.}$ into $\mathrm{Aff}(T(A))/V(\rho_A(K_0(A)))$.

It is clear that, $\Gamma(u) = 0$ if and only if $u^n + \widehat{(u^n)}^*, i(u^n + \widehat{(u^n)}^*) \in V(\rho_A(K_0(A)))$ for all $n \geq 1$.

Thus, we may state the following:

Corollary (2.3.20)[94]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u \in U_0(A)$. Then u can be approximated by unitaries with finite spectrum if and only if

$$\tilde{\Delta}(u) = 0 \quad \text{and} \quad \Gamma(u) = 0.$$

Corollary (2.3.21)[94]. Suppose that $u = \exp(ih)$ for some self-adjoint element $h \in A$. If $u \in CU(A)$, then, by Theorem (2.3.7), $\tilde{\Delta}(u) = 0$, i.e., $\hat{h} \in V(\rho_A(K_0(A)))$. So one may ask if there are unitaries with $\tilde{\Delta}(u) = 0$ but $\Gamma(u) \neq 0$. Proposition (2.3.22) below says that this could happen.

Proposition (2.3.22)[94]. For any unital separable simple C^*-algebra A with $TR(A) = 1$, there is a unitary u with $\tilde{\Delta}(u) = 0$ (or $u \in CU(A)$) such that $\Gamma(u) \neq 0$ and which is not a limit of unitaries with finite spectrum.

Proof. Let $e \in A$ be a non-zero projection such that there is a projection $e_1 \in (1 - e)A(1 - e)$ such that $[e] = [e_1]$. Then $TR(eAe) \leq 1$. Since A does not have real rank zero, one has $TR(eAe) = 1$.

It follows from Theorem (2.3.13) that

$$\mathrm{Aff}(T(eAe)) \neq \overline{\rho_A(K_0(eAe))}^{\ddagger} = \overline{\rho_A(K_0(A))}.$$

Choose $h \in (eAe)_{s.a.}$ with $\|h\| \leq 1$ such that h is not a norm limit of self-adjoint elements with finite spectrum.

If $\hat{h} \in \overline{\rho_A(K_0(eAe))}$, then define
$$u = \exp(ih).$$
Then, $\Delta(u) = 0$ and by Theorem (2.3.7), $u \in CU(A)$. Since h can not be approximated by selfadjoint elements with finite spectrum, nor u can be approximated by unitaries with finite spectrum since $h = (1/i)\log(u)$ for a continuous branch of the logarithm (note that $sp(u) \neq \mathbb{T}$).

Now suppose that $\hat{h} \notin \overline{\rho_A(K_0(eAe))}$.

We also have, by Proposition (2.3.14), $2\pi\hat{h} \notin \overline{\rho_A(K_0(A))}$. We claim that there is a rational number $0 < r \leq 1$ such that $r\widehat{h^2} - 2\pi\hat{h} \notin \overline{\rho_A(K_0(eAe))}$.

In fact, if $\widehat{h^2} \in \overline{\rho_A(K_0(eAe))}$, then the claim follows easily. So we assume that $\widehat{h^2} \notin \overline{\rho_A(K_0(eAe))}$. Suppose that, for some $0 < r_1 < 1, r_1\widehat{h^2} - 2\pi\hat{h} \in \overline{\rho_A(K_0(eAe))}$. Then $(1 - r_1)\widehat{h^2} \notin \overline{\rho_A(K_0(eAe))}$. Hence
$$\widehat{h^2} - 2\pi\hat{h} = (1 - r_1)\widehat{h^2} + (r_1\widehat{h^2} - 2\pi\hat{h}) \notin \overline{\rho_A(K_0(eAe))}.$$
This proves the claim.

Now define $h_1 = rh + 2\pi e_1 - w^*rhw$, where $w \in A$ is a unitary such that $w^*ew = e_1$. Put
$$u = \exp(ih_1)$$
It follows from Proposition (2.3.14) that
$$2\pi\widehat{e_1} \in \overline{\rho_A(K_0(eAe))}.$$
Thus $\tau(h_1) = 2\pi\tau(e_1) \in \overline{\rho_A(K_0(eAe))}$. Therefore, by Theorem (2.3.7), $u \in CU(A)$. Since

$$\widehat{h_1^2} = \widehat{r^2h^2} + 4\pi^2\widehat{e^1} - 4\pi\widehat{rh} + \widehat{r^2h^2} \qquad (511)$$
$$= 2r(\widehat{rh^2} - 2\pi\hat{h}) - 4\pi^2\,\widehat{e_1} \notin \overline{\rho_A(K_0(A))}. \qquad (512)$$

Therefore, by Lemma (2.3.16), h_1 can not be approximated by self-adjoint elements with finite spectrum. It follows that u can not be approxiamted by unitaries with finite spectrum.

Another question is whether $\Gamma(u) = 0$ is sufficient for $\Delta(u) = 0$. For the case that $sp(u) \neq \mathbb{T}$, one has the following. But in general, Proposition (2.3.24) gives a negative answer.

Proposition (2.3.23)[94]. Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u \in U_0(A)$ with $sp(u) \neq \mathbb{T}$. If $\Gamma(u) = 0$, then $\tilde{\Delta}(u) = 0, u \in CU(A)$ and u can be approximated by unitaries with finite spectrum.

Proof. Since $sp(u) \neq \mathbb{T}$, there is a real valued continuous function $f \in C(sp(u))$ such that $u = \exp(if(u))$. Thus the condition that $\Gamma(u) = 0$ implies that $\widehat{f(u)} \in \overline{\rho_A(K_0(A))}$. By Theorem (2.3.7), $u \in CU(A)$.

Proposition (2.3.24)[94]. Let A be a unital infinite dimensional separable simple C^*-algebra with $TR(A) = 1$. Then there are unitaries $u \in U_0(A)$ with $\Gamma(u) = 0$ such that $u \notin CU(A)$. In particular, $\tilde{\Delta}(u) \neq 0$ and u can not be approximated by unitaries with finite spectrum.

Proof. There exists a unital C^*-subalgebra $B \subset A$ with tracial rank zero such that the embedding gives the following identification:
$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$
Note that $\text{Aff}(T(B)) = \overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $w \in U_0(B)$ be a unitary with $sp(w) = \mathbb{T}$. Thus $\Gamma(w) = 0$. Let $\gamma : \text{Aff}(T(C(\mathbb{T}))) \to Aff(T(A))$ be given by $\gamma(f)(\tau) = \tau(f(u))$ for $f \in C(\mathbb{T})_{s.a.}$ and $\tau \in T(A)$. Since $TR(A) = 1$, by Theorem (2.3.7), there are unitaries $u_0 \in U_0(A) \setminus CU(A)$. By the proof of Theorem (2.3.15), there is a unitary $u \in U_0(A)$ such that
$$u = u_0 \quad \text{and}$$
$$\tau(f(u)) = \tau(f(w)) \text{ for all } \tau \in T(A)$$
and for all $f \in C(\mathbb{T})_{s.a.}$. Thus $\tilde{\Delta}(u) \neq 0$ and $\Gamma(u) = \Gamma(w) = 0$. By Theorem (2.3.17), u can not be approximated by unitaries with finite spectrum.

Corollary (2.3.25)[147]. Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u^2 \in U_0(A)$. Then the following are equivalent:

(i) $u^2 \in CU(A)$;

(ii) $\Delta(u^2) = 0$;

(iii) for some piecewise continuous path of unitaries $\{u^2(t) : t \in [0,1]\} \subset A$ with $u^2(0) = u^2$ and $u^2(1) = 1_A$,
$$R(\{u^2(t)\}) \in \overline{\rho_A(K_0(A))},$$

(iv) for any piecewise continuous path of unitaries $\{u^2(t) : t \in [0,1]\} \subset A$ with $u^2(0) = u^2$ and $u^2(1) = 1_A$,
$$R(\{u^2(t)\}) \in \overline{\rho_A(K_0(A))}.$$

(v) there are $h_1^2, h_2^2, \ldots, h_m^2 \in A_{s.a^2}$. such that
$$u^2 = \prod_{j=1}^{m} \exp(ih_j^2) \text{ and } \sum_{j=1}^{m} \widehat{h_j^2} \in \overline{\rho_A(K_0(A))}.$$

(vi) $\sum_{j=1}^{m} \widehat{h_j^2} \in \overline{\rho_A(K_0(A))}$ for any $h_1^2, h_2^2, \ldots, h_m^2 \in A_{s.a^2}$. for which
$$u^2 = \prod_{j=1}^{m} \exp(ih_j^2)$$

Proof. Equivalence of (ii), (iii), (iv), (v) and (vi) follows from the definition of the determinant and follows from the Bott periodicy. The equivalence of (i) and (ii) follows on [133].

The following is a consequence.

Corollary (2.3.26)[147]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let $\gamma : C(\mathbb{T})_{s.a^2} \to \text{Aff}(T(A))$ be a (positive) affine continuous map.

For any $\epsilon > 0$, there exists $\delta > 0$ and there exists a finite subset $\mathcal{F} \subset C(\mathbb{T})_{s.a^2}$ satisfying the following: If $u^2 + \epsilon \in U_0(A)$ with
$$|\tau(f(u^2)) - \gamma(f)(\tau)| < \delta, \quad \text{for all } f \in \mathcal{F} \text{ and } \tau \in T(A), \text{and} \tag{513}$$
$$\text{dist}(\overline{u^2}, \overline{u^2 + \epsilon}) < \delta \text{ in } U_0(A)/CU(A). \tag{514}$$

Then there exists a unitary $W \in U(A)$ such that
$$\|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \tag{515}$$
Proof. The lemma follows immediately on [64]. See [71] and [64]. Note that of [64], we can replace the given map h_1^2 (in this case a given unitary) by a given map γ.

Corollary (2.3.27)[147]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$ and let $u^2 \in U_0(A)$ be a unitary. For any $\epsilon > 0$, there exists $\delta > 0$ and there exists an integer $N \geq 1$ satisfying the following: If $(u^2 + \epsilon) \in U_0(A)$ with
$$\left|\tau(u^{2k}) - \tau\left((u^2 + \epsilon)^k\right)\right| < \delta, k = 1, 2, \ldots, N \text{ for all } \tau \in T(A) \text{ and} \tag{516}$$
$$\mathrm{dist}\left(\overline{u^2}, \overline{u^2 + \epsilon}\right) < \delta \text{ in } U_0(A)/CU(A). \tag{517}$$
Then there exists a unitary $W \in U(A)$ such that
$$\|u^2 - W^*(u^2 + \epsilon)W\| < \epsilon. \tag{518}$$
Proof. Note that (516),
$$\left|\tau(u^{2k}) - \tau((u^2 + \epsilon)^k)\right| < \delta \quad k = \pm 1, \pm 2, \ldots, \pm N. \tag{519}$$
For any subset $\mathcal{G} \subset C(S^1)$ and any $\eta > 0$, there exists $N \geq 1$ and $\delta > 0$ such that
$$\left|\tau\left(g(u^2)\right) - \tau\left(g(u^2 + \epsilon)\right)\right| < \eta \quad \text{for all} \quad \tau \in T(A)$$
if (519) holds.

Then the lemma follows from ([64])

Corollary (2.3.28)[147]. Let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u^2 \in U_0(A)$, then, for any $\epsilon > 0$, there exists a selfadjoint element $a^2 \in A_{s.a^2}$ such that
$$\|u^2 - \exp(ia^2)\| < \epsilon. \tag{520}$$
Proof. Since $u^2 \in U_0(A)$, we may write
$$u^2 = \prod_{j=1}^{k} \exp(ih_j^2). \tag{521}$$
Let $M = \max\{\|h_j^2\| : j = 1, 2, \ldots, k\} + 1$. Let $\delta > 0$ and N be given in 3.2 for u^2. We may assume that $\delta < 1$ and $N \geq 3$. We may also assume that $\delta < \epsilon$. Since $TR(A) \leq 1$, there exists a projection $p^2 \in A$ and a C^*-subalgebra $B \in A$ with $1_B = p^2$ such that $B \cong \bigoplus_{i=1}^{m} C(X_i, M_{r(i)})$, where $X_i = [0, 1]$ or a point, and
$$\|p^2 u^2 - u^2 p^2\| < \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}, \tag{522}$$
$$\left\| (1 - p^2)u^2(1 - p^2) - (1 - p^2)\prod_{j=1}^{k} \exp(i((1 - p^2)h_j^2(1 - p^2)) \right\|$$
$$< \frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}, \tag{523}$$
$$p^2 u^2 p^2 \in_{\frac{\delta}{16\tilde{N}\tilde{M}\tilde{k}}} B \text{ and } \tau(1 - p^2) < \frac{\delta}{2\tilde{N}\tilde{M}\tilde{k}} \quad \text{for all} \quad \tau \in T(A). \tag{524}$$
There exist unitary $u_1^2 \in B$ such that

$$\|p^2 u^2 p^2 - u_1^2\| < \frac{\delta}{8\widetilde{N}\widetilde{M}\widetilde{k}} \tag{525}$$

Put $u_2^2 = (1 - p^2)\prod_{j=1}^k \exp(i(1 - p^2)h_j^2(1 - p^2))$. Since $u_1^2 \in B$, it is well known that there exists a selfadjoint element $b^2 \in B_{s.a^2}$ such that

$$\|u_1^2 - p^2 \exp(ib^2)\| < \frac{\delta}{16\widetilde{N}\widetilde{M}\widetilde{k}}. \tag{526}$$

Let $u_0^2 + \epsilon = (1 - p^2) + p^2 \exp(ib^2)$ and $u_0^2 = p^2 \exp(ib^2) + u_2^2$. Then, by (522), (523), (525) and (526),

$$\|u_0^2 - u^2\| < \|u^2 - p^2 u^2 p^2 - (1 - p^2)u^2(1 - p^2)\| \tag{527}$$

$$+\|(p^2 u^2 p^2 - p^2 \exp(ib^2)) + ((1 - p^2)u^2(1 - p^2) - u_2^2)\| \tag{528}$$

$$< \frac{3\delta}{16\widetilde{N}\widetilde{M}\widetilde{k}} + \frac{\delta}{8\widetilde{N}\widetilde{M}\widetilde{k}} + \frac{\delta}{16\widetilde{N}\widetilde{M}\widetilde{k}} = \frac{3\delta}{8\widetilde{N}\widetilde{M}\widetilde{k}}. \tag{529}$$

and

$$u_0^2(u_0^{*2} + \epsilon) = \prod_{j=1}^k \exp(i(1 - p^2)h_j^2(1 - p^2)). \tag{530}$$

Note that

$$\left|\tau\left(\sum_{j=1}^k (1 - p^2)h_j^2(1 - p^2)\right)\right| \leq \sum_{j=1}^k \left|\tau\left((1 - p^2)h_j^2(1 - p^2)\right)\right| \tag{531}$$

$$= k\tau(1 - p^2)\max\{\|h_j^2\| : j = 1, 2, \ldots, k\} < \delta/16\widetilde{N} \tag{532}$$

for all $\tau \in T(A)$. It follows that

$$\text{dist}(\overline{u^2}, \overline{u_0^2} + \epsilon) < \delta/16\widetilde{N} \text{ in } U_0(A)/CU(A). \tag{533}$$

It follows from that

$$\text{dist}(\overline{u^2}, \overline{u_0^2} + \epsilon) < \delta/8\widetilde{N}. \tag{534}$$

On the other hand, for each $s = 1, 2, \ldots, N$, by (530), (529) and (524)

$$|\tau(u^{2s}) - \tau(u_0^2 + \epsilon)^s| \leq |\tau(u^{2s}) - \tau(u_0^{2s})| + |\tau(u_0^{2s}) - \tau(u_0^2 + \epsilon)^s| \tag{535}$$

$$\leq \|u^{2s} - u_0^{2s}\| + \left|\tau\left((1 - p^2) - (1 - p^2)\prod_{j=1}^k \exp\left(i(1 - p^2)sh_j^2(1 - p^2)\right)\right)\right| \tag{536}$$

$$\leq \widetilde{N}\|u^2 - u_0^2\| + 2\tau(1 - p^2) \tag{537}$$

$$< \frac{3\delta}{8\widetilde{M}\widetilde{k}} + \frac{\delta}{\widetilde{M}\widetilde{N}\widetilde{k}} < \delta \tag{538}$$

for all $\tau \in T(A)$. From the above inequality and (534) and applying Corollary (2.3.27), one obtains a unitary $W \in U(A)$ such that

$$\|u^2 - W^*(u_0^2 + \epsilon)W\| < \epsilon. \tag{539}$$

Put $a^2 = W^*((1 - p^2) + b^2)W$. Then

$$\|u^2 - \exp(ia^2)\| < \epsilon. \tag{540}$$

Note that Corollary (2.3.28) does not assume that A is amenable, in particular, it may not be a simple AH-algebra. The proof used a kind of uniqueness theorem for unitaries in a

unital simple C^*-algebra A with $TR(A) \leq 1$. This bring us to the following theorem which is an immediate consequence of Corollary (2.3.27).

Corollary (2.3.29)[147]. Let A be a unital simple infinite dimensional C^*-algebra with $TR(A) \leq 1$. If $a^2 \in \overline{\rho_A(K_0(A))}$, then

$$ra^2 \in \overline{\rho_A(K_0(A))} \tag{541}$$

for all $r \in \mathbb{R}$. In fact, $\overline{\rho_A(K_0(A))}$ is a closed $\mathbb{R}$-linear subspace of $\mathrm{Aff}(T(A))$.

Proof. Note that $\overline{\rho_A(K_0(A))}$ is an additive subgroup of $\mathrm{Aff}(T(A))$. It suffices to prove the following: Given any projection $p^2 \in A$, any real number $0 < r_1 < 1$ and $\epsilon > 0$, there exists a projection $p^2 + \epsilon \in A$ such that

$$|r_1\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \quad \text{for all} \quad \tau \in T(A). \tag{542}$$

Choose $n \geq 1$ such that

$$|m/n - r_1| < \epsilon/2 \text{ and } 1/n < \epsilon/2 \tag{543}$$

for some $1 \leq m < n$.

Note that $TR(p^2Ap^2) \leq 1$. By [89], there are mutually orthogonal projections $p_0^2 + \epsilon, p_1^2, p_2^2, \ldots, p_n^2$ with $[p_0^2 + \epsilon] \leq [p_1^2]$ and $[p_1^2] = [p_i^2], i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} p_i^2 + p_0^2 + \epsilon = p^2$.

Put $p^2 + \epsilon = \sum_{i=1}^{m} p_i^2$. We then compute that

$$|r_1\tau(p^2) - \tau(p^2 + \epsilon)| < \epsilon \text{ for all } \tau \in T(A). \tag{544}$$

Corollary (2.3.30)[147]. Let A be a unital simple infinite dimensional C^*-algebra with $TR(A) = 1$. Then there exist unitaries $u^2, u^2 + \epsilon \in U_0(A)$ with

$$\tau(u^{2k}) = \tau(u^{2k}) \quad \text{for all} \quad \tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$$

such that $\Delta(u^2) \neq \Delta(u^2 + \epsilon)$. In particular, u^2 and $u^2 + \epsilon$ are not approximately unitarily equivalent.

Proof. Since we assume that $TR(A) = 1$, then, by Theorem (2.3.13), $\mathrm{Aff}(T(A)) \neq \overline{\rho_A(K_{0(A)})}$ and $U_0(A)/CU(A)$ are not trivial.

Let $\kappa_1, \kappa_2 : K_1(C(\mathbb{T})) \to U_0(A)/CU(A)$ be two different homomorphisms. Fix an affine continuous map $s : T(A) \to T_f(C(\mathbb{T}))$, where $T_f(C(\mathbb{T}))$ is the space of strictly positive normalized Borel measures on $\mathbb{T}$. Denote by $\gamma_0 : \mathrm{Aff}(T(C(\mathbb{T}))) \to \mathrm{Aff}(T(A))$ the positive affine continuous map induced by $\gamma_0(f)(\tau) = f(s(\tau))$ for all $f \in \mathrm{Aff}(T(C(T)))$ and $\tau \in T(A)$. Let

$$\gamma_0 : U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) = \mathrm{Aff}(T(C(\mathbb{T})))/Z \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$$
$$= U_0(A)/CU(A)$$

be the map induced by γ_0. Write

$$U(C(\mathbb{T}))/CU(C(\mathbb{T})) = U_0(C(\mathbb{T}))/CU(C(\mathbb{T})) \oplus K_1(C(\mathbb{T})).$$

Define $\lambda_i : U(C(\mathbb{T}))/CU(C(\mathbb{T})) \to U_0(A)/CU(A)$ by

$$\lambda_i(x \oplus x + 2\epsilon) = \gamma_0(x) + \kappa_i(x + 2\epsilon)$$

for $x \in U_0(C(\mathbb{T}))/CU(C(\mathbb{T}))$ and $x + 2\epsilon \in K_1(C(\mathbb{T})), i = 1, 2$. That there are two unital monomorphisms $\varphi_1, \varphi_2 : C(\mathbb{T}) \to A$ such that

$$(\varphi_1)_{*i} = 0, \qquad \varphi_i^{\ddagger} = \lambda_i \quad \text{and} \quad \varphi_i^{\natural} = s, \qquad (545)$$

$i = 1, 2$. Let $x + 2\epsilon$ be the standard unitary generator of $C(S^1)$. Define $u^2 = \varphi_1(x + 2\epsilon)$ and $u^2 + \epsilon = \varphi_2(x + 2\epsilon)$.

Then $u^2, u^2 + \epsilon \in U_0(A)$. The condition that $\varphi_i^{\natural} = s$ implies that $\tau(u^{2k}) = \tau((u^2 + \epsilon)^k)$ for all $\tau \in T(A), k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$.

But since $\lambda_1 \neq \lambda_2$,

$$\Delta(u^2) \neq \Delta(u^2 + \epsilon).$$

Therefore u^2 and $u^2 + \epsilon$ are not approximately unitarily equivalent.

Corollary (2.3.31)[147]. Let A be a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and let $h^2 \in A$ be a self-adjoint element. Then h^2 can be approximated by self-adjoint elements with finite spectrum if and only if $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \ldots$.

Proof. If h^2 can be approximated by self-adjoint elements so can h^2n. By 3.6, $\overline{\rho_A(K_0(A))}$ is a closed linear subspace. Therefore $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}$ for all n.

Now we assume that $\widehat{h^{2n}} \in \overline{\rho_A(K_0(A))}, n = 1, 2, \ldots$. The Stone-Weierstrass theorem implies that $\widehat{f(h^2)} \in \overline{\rho_A(K_0(A))}$ for all real-value functions $f \in C(sp(h^2))$. For any $\epsilon > 0$, by [89], there is $f \in C(sp(x))_{s.a^2}$. such that

$$\|f(h^2) - h^2\| < \epsilon$$

and $sp(f(h^2))$ consists of a union of finitely many closed intervals and finitely many points.

Thus, to simplify notation, we may assume that $X = sp(h^2)$ is a union of finitely many intervals and finitely many points. Let $\psi : C(X) \to A$ be the homomorphism defined by $\psi(f) = f(h^2)$. Let $s : T(A) \to T_f(C(X))$ be the affine map defined by $f(s(\tau)) = \psi(f)(\tau)$ for all $f \in \text{Aff}(C(X))$ and $\tau \in T(A)$.

Let B be a unital simple AH-algebra with real rank zero, stable rank one and

$$\left(K_0(B), K_0(B)+, [1_B], K_1(B)\right) \cong \left(K_0(A), K_0(A)_+, [1_A], K_1(A)\right).$$

In particular, $K_0(B)$ is weakly unperforated. The proof of Theorem 10.4 of [89] provides a unital homomorphism $\iota : B \to A$ which carries the above identification. This can be done by [89] and the uniqueness Theorem of [89], or better by corollary 11.7 of [71] because $TR(B) = 0$, the map $\varphi^{\ddagger}$ is not needed since $U(B) = CU(B)$ and the map on traces is determined by the map on $K_0(B)$.

Note that $\text{Aff}(T(B)) = \overline{\rho_B(K_0(B))}$. By identifying B with a unital C^*-subalgebra of A, we may write $\overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $\psi^{\natural} : \text{Aff}(T(C(X))) \to \overline{\rho_A(K_0(A))}$ be the map induced by ψ. This gives an affine map $\gamma : \text{Aff}(T(C(X))) \to \overline{\rho_B(K_0(B))}$. It follows that there exists a unital monomorphism $\varphi : C(X) \to B$ such that

$$\iota \circ \varphi_{*0} = \psi_{*0} \quad \text{and} \quad (\iota \circ \phi)^{\natural} = \psi^{\natural},$$

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where $(\iota \circ \varphi)^{\sharp}: \text{Aff}(T(C(X))) \to \text{Aff}(T(A))$ defined by $(\iota \circ \varphi)^{\sharp}(a^2)(\tau) = \tau \ (\iota \circ \varphi)(a^2)$ for all $a^2 \in A_{s.a^2}$. It follows from Corollary 11.7 of [71] that ψ and $\iota \circ \varphi$ are approximately unitarily equivalent. On the other hand, since B has real rank zero, φ can be approximated by homomorphisms with finite dimensional range. It follows that h^2 can be approximated by self-adjoint elements with finite spectrum.

Corollary (2.3.32)[147]. Let A be a unital separable simple infinite dimensional C^*-algebra with $TR(A) \leq 1$ and let $u^2 \in U_0(A)$. Then u^2 can be approximated by unitaries with finite spectrum if and only if $u^2 \in CU(A)$ and

$$u^{2n} \widehat{+ (u^{2n})^*}, \iota(u^{2n} \widehat{- (u^{2n})^*}) \in \overline{\rho_A(K_0(A))}, n = 1, 2, \ldots.$$

Proof. Suppose that there exists a sequence of unitaries $\{u_n^2\} \subset A$ with finite spectrum such that

$$\lim_{n \to \infty} u_n^2 = u^2.$$

There are mutually orthogonal projections $p_{1,n}^2, p_{2,n}^2, \ldots, p_{m(n),n}^2 \in A$ and complex numbers $\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{m(n),n} \in \mathbb{C}$ with $|\lambda_{i,n}| = 1, i = 1, 2, \ldots, m(n,)$ and $n = 1, 2, \ldots$, such that

$$\lim_{n \to \infty} \left\| u^2 - \sum_{i=1}^{m(n)} \lambda_{i,n} p_{i,n}^2 \right\| = 0.$$

It follows that

$$\lim_{n \to \infty} \left\| ((u^*)^{2n} + u^{2n}) - \sum_{i=1}^{m(n)} 2Re(\lambda_{i,n}) p_{i,n}^2 \right\| = 0.$$

By Corollary (2.3.29),

$$\sum_{i=1}^{m(n)} 2Re(\lambda_{i,n}) \widehat{p_{i,n}^2} \in \overline{\rho_A(K_0(A))}.$$

Thus $\widehat{Re(u^{2n})} \in \overline{\rho_A(K_0(A))}$. Similarly, $\widehat{Im(u^{2n})} \in \overline{\rho_A(K_0(A))}$.

To show that $u^2 \in CU(A)$, consider a unitary $u^2 + \epsilon = \sum_{i=1}^{m} \lambda_i p_n^2$, where $\{p_1^2, p_2^2, \ldots, p_m^2\}$ is a set of mutually orthogonal projections such that $\sum_{i=1}^{m} p_j^{2} = 1$, and where $|\lambda_i| = 1, i = 1, 2, \ldots, m$. Write $\lambda_j = e^{i\theta_j^2}$ for some real number $\theta_j^2, j = 1, 2, \ldots$. Define

$$h^2 = \sum_{j=1}^{m} \theta_j^2 p_j^2.$$

Then

$$u^2 + \epsilon = \exp(ih^2).$$

By Corollary (2.3.29), $\widehat{h^2} \in \overline{\rho_A(K_0(A))}$. It follows from Theorem (2.3.7) that $u^2 + \epsilon \in CU(A)$. Since u^2 is a limit of those unitaries with finite spectrum, $u^2 \in CU(A)$.

Now assume $u^2 \in CU(A)$ and $u^{2n} \widehat{+ (u^{2n})^*}, i(u^{2n} \widehat{- (u^{2n})^*}) \in \overline{\rho_A(K_0(A))}$ for $n = 1, 2, \ldots$. If $sp(u^2) \neq \mathbb{T}$, then the problem is reduced to the case in Corollary (2.3.31). So

we now assume that $sp(u^2) = \mathbb{T}$. Define a unital monomorphism $\varphi: C(\mathbb{T}) \to A$ by $\varphi(f) = f(u^2)$. By the Stone-Weirestrass theorem and Corollary (2.3.29), every real valued funtion $f \in C(\mathbb{T}), [\varphi(f) \in \overline{\rho_A(K_0(A))}$.

As in the proof of Corollary (2.3.31), one obtains a unital C^*-subalgebra $B \subset A$ which is a unital simple AH-algebra with tracial rank zero such that the embedding $\iota: B \to A$ gives an identification:

$$(K_0(B), K_0(B)+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Moreover, by Lemma 5.1 of [8] that there is a unital monomorphism $\psi: C(\mathbb{T}) \to B$ such that

$$\psi_{*1} = 0 \quad \text{and} \quad (\iota \circ \psi)^\natural = \varphi^\natural.$$

Note also

$$(\iota \circ \psi)^\ddagger = \varphi^\ddagger$$

(both are trivial, since $u^2 \in CU(A)$).

It follows from ([71]) that $\iota \circ \psi$ and φ are approximately unitarily equivalent. However, since $\psi_{*1} = 0$, in B, by [79], ψ can be approximated by homomorphisms with finite dimensional range. It follows that u^2 can be approximated by unitaries with finite spectrum.

If A is a finite dimensional simple C^*-algebra, then $TR(A) = 0$. Of course, every unitary in A has finite spectrum. But $CU(A) \neq U_0(A)$. To unify the two cases, we note that $K_0(A) = Z$.

Instead of using $\overline{\rho_A(K_0(A))}$, one may consider the following definition:

Corollary (2.3.33)[147]. For any unital separable simple C^*-algebra A with $TR(A) = 1$, there is a unitary u^2 with $\tilde{\Delta}(u^2) = 0$ (or $u^2 \in CU(A)$) such that $\Gamma(u^2) \neq 0$ and which is not a limit of unitaries with finite spectrum.

Proof. Let $e^2 \in A$ be a non-zero projection such that there is a projection $e_1^2 \in (1 - e^2)A(1 - e^2)$ such that $[e^2] = [e_1^2]$. Then $TR(e^2Ae^2) \leq 1$ by 5.3 of [4]. Since A does not have real rank zero, one has $TR(e^2Ae^2) = 1$.

It follows from Theorem (2.3.13) that

$$\text{Aff}(T(e^2Ae^2)) \neq \overline{\rho_A(K_0(e^2Ae^2))} = \overline{\rho_A(K_0(A))}.$$

Choose $h^2 \in (e^2Ae^2)_{s.a^2}$ with $\|h^2\| \leq 1$ such that h^2 is not a norm limit of self-adjoint elements with finite spectrum.

If $\widehat{h^2} \in \overline{\rho_A(K_0(e^2Ae^2))}$, then define

$$u^2 = \exp(ih^2).$$

Then, $\Delta(u^2) = 0$ and by Theorem 2.9, $u^2 \in CU(A)$. Since h^2 can not be approximated by selfadjoint elements with finite spectrum, nor u^2 can be approximated by unitaries with finite spectrum since $h^2 = (1/i)\log(u^2)$ for a continuous branch of the logarithm (note that $sp(u^2) \neq \mathbb{T}$).

Now suppose that $\hat{h} \notin \overline{\rho_A(K_0(e^2Ae^2))}$.

We also have, by Corollary (2.3.29), $2\pi\widehat{h^2} \notin \overline{\rho_A(K_0(A))}$. We claim that there is a rational number $0 < r \leq 1$ such that $r\widehat{h^4} - 2\pi\widehat{h^2} \notin \overline{\rho_A(K_0(e^2Ae^2))}$.

In fact, if $\widehat{h^4} \in \overline{\rho_A(K_0(e^2Ae^2))}$, then the claim follows easily. So we assume that $\widehat{h^4} \notin \overline{\rho_A(K_0(e^2Ae^2))}$. Suppose that, for some $0 < r_1 < 1, r_1\widehat{h^4} - 2\pi\widehat{h^2} \in \overline{\rho_A(K_0(e^2Ae^2))}$. Then $(1 - r_1)\widehat{h^4} \notin \overline{\rho_A(K_0(e^2Ae^2))}$. Hence

$$\widehat{h^4} - 2\pi\widehat{h^2} = (1 - r_1)\widehat{h^4} + (r_1\widehat{h^4} - 2\pi\widehat{h^2}) \notin \overline{\rho_A(K_0(e^2Ae^2))}.$$

This proves the claim.

Now define $h_1^2 = rh^2 + 2\pi e_1^2 - w^*rh^2w$, where $w \in A$ is a unitary such that $w^*e^2w = e_1^2$. Put

$$u^2 = \exp(ih_1^2)$$

It follows from Corollary (2.3.29) that

$$2\pi\widehat{e_1^2} \in \overline{\rho_A(K_0(e^2Ae^2))}.$$

Thus $\tau(h_1^2) = 2\pi\tau(e_1^2) \in \overline{\rho_A(K_0(e^2Ae^2))}$. Therefore, by 2.9, $u^2 \in CU(A)$. Since

$$\widehat{h_1^4} = \widehat{r^2h^4} + 4\pi^2\widehat{e_1^2} - 4\pi r\widehat{h^2} + \widehat{r^2h^4} \tag{546}$$

$$= 2r(\widehat{rh^4} - 2\pi\widehat{h^2}) - 4\pi^2\widehat{e_1^2} \notin \overline{\rho_A(K_0(A))}. \tag{547}$$

Therefore, by Corollary (2.3.31), h_1^2 can not be approximated by self-adjoint elements with finite spectrum. It follows that u^2 can not be approxiamted by unitaries with finite spectrum.

Another question is whether $\Gamma(u^2) = 0$ is sufficient for $\Delta(u^2) = 0$. For the case that $sp(u^2) \neq \mathbb{T}$, one has the following. But in general, Corollary (2.3.35) gives a negative answer.

Corollary (2.3.34)[147]. Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$. Suppose that $u^2 \in U_0(A)$ with $sp(u^2) \neq \mathbb{T}$. If $\Gamma(u^2) = 0$, then $\tilde{\Delta}(u^2) = 0, u^2 \in CU(A)$ and u^2 can be approximated by unitaries with finite spectrum.

Proof. Since $sp(u^2) \neq \mathbb{T}$, there is a real valued continuous function $f \in C(sp(u^2))$ such that $u^2 = \exp(if(u^2))$. Thus the condition that $\Gamma(u^2) = 0$ implies that $\widehat{f(u^2)} \in \overline{\rho_A(K_0(A))}$. By Theorem (2.3.7), $u^2 \in CU(A)$.

Corollary (2.3.35)[147]. Let A be a unital infinite dimensional separable simple C^*-algebra with $TR(A) = 1$. Then there are unitaries $u^2 \in U_0(A)$ with $\Gamma(u^2) = 0$ such that $u^2 \notin CU(A)$. In particular, $\tilde{\Delta}(u^2) \neq 0$ and u^2 can not be approximated by unitaries with finite spectrum.

Proof. There exists a unital C^*-subalgebra $B \subset A$ with tracial rank zero such that the embedding gives the following identification:

$$(K_0(B), K_0(B)+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A)).$$

Note that $\text{Aff}(T(B)) = \overline{\rho_B(K_0(B))} = \overline{\rho_A(K_0(A))}$.

Let $w^2 \in U_0(B)$ be a unitary with $sp(w^2) = \mathbb{T}$. Thus $\Gamma(w^2) = 0$. Let $\gamma: \text{Aff}(T(C(\mathbb{T}))) \rightarrow \text{Aff}(T(A))$ be given by $\gamma(f)(\tau) = \tau(f(u^2))$ for $f \in C(T)_{s.a^2}$ and $\tau \in T(A)$. Since $TR(A) = 1$, by Theorem (2.3.7), there are unitaries $u_0^2 \in U_0(A) \setminus CU(A)$. By the proof of Corollary (2.3.30) (see also Theorem (2.3.8)), there is a unitary $u^2 \in U_0(A)$ such that

$$u^2 = u_0^2 \quad \text{and}$$
$$\tau(f(u^2)) = \tau(f(w^2)) \text{ for all } \tau \in T(A)$$

and for all $f \in C(T)_{s.a^2}$. Thus $\tilde{\Delta}(u^2) \neq 0$ and $\Gamma(u^2) = \Gamma(w^2) = 0$. By Corollary (2.3.32), u^2 can not be approximated by unitaries with finite spectrum.