Chapter 1
Primitively of Unital Products Homomorphisms in C*-Algebra

A C^*-algebra is called primitive if it admits a faithful and irreducible $*$-representation .Let A and B be unital separable simple amenable C^*-algebras which satisfy the Universal Coefficient Theorem. Suppose that A and B are $\mathcal{Z}$-stable and are of rationally tracial rank no more than one. We show that this holds if A is a rationally AH -algebra which is not necessarily simple. Moreover, for any strictly positive unit-preserving $\kappa \in KL(A, B)$, any continuous affine map $\lambda: \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ and any continuous group homomorphism $\gamma: U(A)/CU(A) \to U(B)/CU(B)$ which are compatible.

Section (1.1): Full Free Products of Residually Finite Dimensional C^*-Algebras

A C^*-algebra is called primitive if it admits a faithful and irreducible $*$-representation. Thus the simplest examples are matrix algebras. A nontrivial example, shown independently by Choi and Yoshizawa, is the full group C^*-algebra of the free group on n elements, $2 \le n \le \infty$, see [146] and [11]. In [17], Murphy gave numerous conditions for primitivity of full group C^*-algebras. More recently, $T.$Å. Omland showed in [27] that for G_1 and G_2 countable amenable discrete groups and σ a multiplier on the free product $G_1 * G_2$, the full twisted group C^*-algebra $C^* - (G_1 * G_2, \sigma)$ is primitive whenever $(|G_1| - 1)(|G_2| - 1) \ge 2$.

We prove that given two nontrivial, separable, unital, residually finite dimensional C^*-algebras $A_1$and A_2, their unital C^*-algebra full free product $A_1 * A_2$ is primitive except when $A_1 = \mathbb{C}^2 = A_2$. The methods used are essentially different from those in [17], [146], [2] and [105] but do rely on [40] that $A_1 * A_2$ is itself residually finite dimensional. Roughly speaking, we first show that if $(dim(A_1) - 1)(dim(A_2) - 1) \ge 2$, then there is an abundance of irreducible finite dimensional $*$–representations and later, by means of a sequence of approximations, we construct an irreducible and faithful $*$–representation.

Proposition (1.1.1)[30]: Let B be a finite dimensional C^*-algebra and assume B decomposes as

$$\bigoplus_{j=1}^{J} B_j$$

and there is a positive integer n such that all B_j are $*$–isomorphic to M_n. Fix $\{\beta_j : B_j \to M_n\}_{1 \le j \le J}$ a set of $*$–isomorphisms.

(i) For a permutation σ in S_J define $\psi_\sigma: B \to B$ by

$$\psi_\sigma(b_1, \dots, b_J) = (\beta_1^{-1} \circ \beta_{\sigma-1(1)}(b_{\sigma-1(1)}), \dots, \beta_J^{-1} \circ \beta_{\sigma-1(J)}(b_{\sigma-1(J)})).$$

Then ψ_σ lies in $\mathrm{Aut}(B)$ and the map $\sigma \mapsto \psi_\sigma$ defines a groupembedding of $S_J$into $Aut(B)$.

(ii) Every element α in $Aut(B)$ factors as

$$\left(\bigoplus_{j=1}^{J} Ad\, u_j\right) \circ \psi_\sigma$$

for some permutation σ in S_J and unitaries u_j in $\mathbb{U}(B_j)$.

(iii) There is a exact sequence

$$0 \to Inn(B) \to Aut(B) \to S_J \to 0.$$

So far we have considerC^*-algebras with only one type of block sub-algebra, so to speak. Next proposition shows that a $*$–automorphism cannot mix blocks of different dimensions. As a consequence, and along with Proposition (1.1.1), we get a general decomposition of $*$–automorphisms of finite dimensional C^*-algebras.

Proposition (1.1.2)[30]: Let B be a finite dimensional C^*-algebra. Decompose B as

$$\bigoplus_{i=1}^{1} \bigoplus_{j=1}^{J_i} B(i,j).$$

Where for each i, there is a positive integer n_i such that $B(i,j)$ is isomorphic to M_{n_i} for all$1 \leq j \leq J_i$, i.e. we group sub-algebras that are isomorphic to the same matrix algebra, and where $n_1 < n_2 < \cdots < n_I$..

Then any α in $Aut(B)$ factors as $\alpha = \bigoplus_{i=1}^{I} \alpha_i$ where

$$\alpha_i : \bigoplus_{j=1}^{J_i} B(i,j) \to \bigoplus_{j=1}^{J_i} iB(i,j)$$

is a $*$–isomorphism.

We summarize some result that, later on, will be repeatedly used. Definitions and proofs of results mentioned can be found in [56] and [53].

Theorem (1.1.3)[30]: Any closed subgroup of a Lie group is a Lie subgroup.

Theorem (1.1.4)[30]: Let G be a Lie group of dimension n and $H \subseteq G$ be a Lie subgroup of dimension k.

(i) Then the left coset space G/H has a natural structure of a manifold of dimension $n - k$ such that the canonical quotient map$\pi : G \to G/H$, is a fiber bundle, with fiber diffeomorphic to H.

(ii) If H is a normal Lie subgroup then G/H has a canonical structure of a Lie group.

Proposition (1.1.5)[30]: Let G denote a Lie group and assume it acts smoothly on a manifold M. For $m \in M$ let $\mathcal{O}(m)$ denote its orbit and $Stab(m)$ denote its stabilizer i.e.

$$\mathcal{O}(m) = \{g.m : g \in G\},$$
$$Stab(m) = \{g \in G : g.m = m\}.$$

The orbit $\mathcal{O}(m)$ is an immersed submanifold of M. If $\mathcal{O}(m)$ is compact, then the map $g \mapsto g.m$, is a diffeomorphism from $G/Stab(m)$ onto $\mathcal{O}(m)$. (In this case we say $\mathcal{O}(m)$ is an embedded submanifold of M.)

Corollary (1.1.6)[30]: Let G be a compact Lie group and let K and L be closed subgroups of G. The subspace $KL = \{kl : k \in K, l \in L\}$ is an embedded submanifold of G of dimension

$$dim\, K + dim\, L - dim(L \cap K).$$

Proof: First of all KL is compact. This follows from the fact that multiplication is continuous and both K and $L$are compact. Consider the action of $K \times L$ on G givenby $(k,l).g = kgl^{-1}$. Notice that the orbit of e is precisely KL. By Proposition (1.1.5), KL is an immersed sub-manifold diffeomorphic to $K \times L/Stab(e)$. Since it is compact, it is an embedded submanifold. But $Stab(e) = \{(x,x) : x \in K \cap L\}$ and we conclude

$$\dim KL = \dim(K \times L) - \dim \mathrm{Stab}(e) = \dim K + \dim L - \dim(K \cap L).$$

Proposition (1.1.7)[30]: Let G be a compact Lie group and let H be a closed subgroup. Let π denote the quotient map onto G/H.

There are:

$(i)\mathcal{N}_G$, a compact neighborhood of $e$in G,

(ii)$\mathcal{N}_H$, a compact neighborhood of e in H,

(iii) $\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(e)$ in G/H,

(iiii) a continuous function $s: \mathcal{N}_{G/H}(\pi(e)) \to G$ satisfying

(a) $s(\pi(e)) = e$and$\pi(s(y)) = y$ for all y in $\mathcal{N}_{G/H}(\pi(e))$,

(b) The map

$$\mathcal{N}_H \times \mathcal{N}_{G/H} \to \mathcal{N}_G, \qquad (h,y) \mapsto hs_g(y)$$

is a homeomorphism.

Notation (1.1.8)[30]: Whenever we take commutators they will be with respect to the ambient algebra M_N, in other words for a sub-algebra$A$ in $*$-SubAlg(M_N)

$$A' = \{x \in M_N : xa = ax, \text{for all } a \text{ } in \text{ } A\}.$$

Recall that $C(A)$ denotes the center of A i.e.

$$C(A) = A \cap A' = \{a \in A : xa = ax \text{for all } x \text{ } in \text{ } A\}.$$

Proposition (1.1.9)[30]: For any B_1 in $*$-SubAlg(M_N) and for any B in $*$-SubAlg(B_1), we have

$$\dim Stab(B_1, B) = \dim \mathbb{U}(B) + \dim \mathbb{U}(B_1 \cap B') - \dim \mathbb{U}(C(B)).$$

Proof: We'll find a normal subgroup of $Stab(B_1, B)$, for which we can compute its dimension and that partitions $Stab(B_1, B)$ into a finite number of cosets. Let G denote the subgroup of $Stab(B_1, B)$ generated by $\mathbb{U}(B_1 \cap B')$and $\mathbb{U}(B)$. Since the elements of $\mathbb{U}(B)$ commute with the elements of $\mathbb{U}(B_1 \cap B')$, a typical element of G looks like vw,where v lies in $\mathbb{U}(B)$ and w lies in $\mathbb{U}(B_1 \cap B')$. Taking into account compactness of $\mathbb{U}(B)$and $\mathbb{U}(B_1 \cap B')$, we deduced$G$ is compact.

Now we show G is normal in Stab(B_1, B). Take u an element in $Stab(B_1, B)$. For a unitary$v$ in $\mathbb{U}(B)$ it is immediate that uvu^* lies in $\mathbb{U}(B)$. For a unitary $w$in $\mathbb{U}(B_1 \cap B')$, the following computation shows uwu^* belongs to $\mathbb{U}(B_1 \cap B')$.

For any element b in B we have:

$$(uwu^*)b = uw(u^*bu)u^* = u(u^*bu)wu^* = b(uwu^*),$$

where in the second equality we used u^*bu lies in B. In conclusion uGu^* is contained in G for all u in $St(B_1, B)$i.e. G is normal in $Stab(B_1, B)$.

As a result $Stab(B_1, B)/G$ is a Lie group. The next step is to show $Stab(B_1, B)/G$ is finite. Decompose B as

$$B = \bigoplus_{i=1}^{I} \bigoplus_{j=1}^{J_i} B(i, j),$$

where for all i there is k_i such that for $1 \le j \le J_i, B(i, j)$ is $*$–isomorphic to M_{k_i} . For the rest of our proof we fix a family, $\beta(i, j) : B(i, j) \to M_{k_i}$, of $*$–isomorphisms.

An element u in $Stab(B_1, B)$ defines a $*$–automorphism of B by conjugation. As a consequence, Propositions (1.1.1) and (1.1.2) imply there are permutations σ_i in S_{J_i} and unitariesv_i in $\mathbb{U}(\bigoplus_{j=1}^{J_i} B(i, j))$ such that

$$\forall b \in B : ubu^* = v\psi(b)v^* \qquad (1)$$

Where $v = \bigoplus_{i=1}^{I} v_i$ is a uitary in $\mathbb{U}(B)$ and $\psi = \bigoplus_{i=1}^{I} \psi_{\sigma_i}$is a $*$–automorphism in $Aut(B)$ (the maps ψ depends on the family of $*$–isomorphisms$\beta(i, j)$ we fixed earlier). Equation (1) is telling us important information. Firstly, that ψ extends to an$*$–isomorphism of B_1 and most importantly, this extension is an inner $*$–automorphism. Fix a unitary$\mathbb{U}_\psi$ in $\mathbb{U}(B_1)$ such that $\psi(b) = Ad\mathbb{U}_\psi(b)$ for all b in B (note that $\mathbb{U}_\psi$ may not be unique but we just pick one and fix it for rest of the proof). From equation (1) we deduce there is a unitary w in $\mathbb{U}(B_1 \cap B')$ satisfying $u = v\mathbb{U}_\psi w$. Since the number of functions ψ, that may arise from (1), is at most $J_1! \ldots J_1!$, we conclude

$$|\text{Stab}(B_1, B)/G| \le J_1! \ldots J_1!.$$

Now that we know $Stab(B_1, B)/G$ is finite we have $\dim Stab(B_1, B) = \dim G$, and $*$-gives the result. From Proposition (1.1.9), we get the following corollary.

Corollary (1.1.10)[30]: For any B_1 in $*$-SubAlg(M_N) and any B in $*$-SubAlg(B_1), we have

$$\dim[B]_{B_1} = \dim \mathbb{U}(B_1) - \dim \mathbb{U}(B' \cap B_2) + \dim \mathbb{U}\big(C(B)\big) - \dim \mathbb{U}(B).$$

Now we focus our efforts on $Y(B_2;\ B)$.

Proposition (1.1.11)[30]: Assume $Y(B_2;\ B) \neq \emptyset$. Then $Y(B_2;\ B)$ is a finite disjoint union of embedded submanifolds of $\mathbb{U}(M_N)$. For each one of these submanifolds there is $u \in Y(B_2;\ B)$ such that the submanifold's dimension is

$$Stab(M_N, B) + \dim\mathbb{U}(B_2) - \dim Stab(B_2, u^*Bu).$$

Using Proposition (1.1.9) the later equals

$$\dim\mathbb{U}(B') + \dim\mathbb{U}(B_2) - \dim\mathbb{U}(B_2, u^*B'u). \qquad (2)$$

Proof: We'll define an action on $Y(B_2;\ B)$ which will partition $Y(B_2;\ B)$ into a finite number of orbits, each orbit an embedded sub-manifold of dimension (2) for a corresponding unitary. Define an action of $Stab(M_N, B) \times \mathbb{U}(B_2)$ on $Y(B_2;\ B)$ *via*

$$(w, v).u = wuv^*.$$

For $u \in Y(B_2;\ B)$ let $\mathcal{O}(u)$ denote the orbit of u and let $\mathcal{O}$ denote the set of all orbits. To prove $\mathcal{O}$ is finite consider the function

$$\varphi : \mathcal{O} \to * - SubAlg(B_2)/\sim_{B_2}, \varphi(\mathcal{O}(u)) = [u^*Bu]_{B_2}.$$

Firstly, we need to show φ is well defined. Assume $u_2 \in \mathcal{O}(u_1)$ and take $(w, v) \in Stab(M_N, B) \times \mathbb{U}(B_2)$ such that $u_2 = wu_1v^*$. From the identities

$$u_2^*Bu_2 = vu_1w^*Bwu_1v^* = vu_1Bu_1v^*$$

we obtain $[u_2Bu_2^*]_{B_2} = [u_1Bu_1^*]_{B_2}$. Hence φ is well defined.

The next step is to show φ is injective. Assume $\varphi(\mathcal{O}(u_1)) = \varphi(\mathcal{O}(u_2))$, for $u_1, u_2 \in Y(B_2;\ B)$. Since $[u_1Bu_1^*]_{B_2} = [u_2Bu_2^*]_{B_2}$, we have $u_2^*Bu_2 = vu_1Bu_1v^*$ for some $v \in \mathbb{U}(B_2)$. But this implies $u_1v^*u_2^* \in Stab(M_N, B)$so if $w = u_1v^*u_2^*$ we conclude $(w, v).u_2 = u_1$ which yields $\mathcal{O}(u_1) = \mathcal{O}(u_2)$. We conclude $|\mathcal{O}| \leq |* - SubAlg(B_2)/\sim_{B_2}| < \infty$.

Now we prove each orbit is an embedded submanifold of $\mathbb{U}(M_N)$ of dimension (2). Since $Stab(M_n, B) \times \mathbb{U}(B_2)$ is compact, every orbit $\mathcal{O}(u)$ is compact. Thus, Proposition (1.1.5) implies $\mathcal{O}(u)$ is an embedded submanifoldof $\mathbb{U}(M_N)$, diffeomorphic to

$$(Stab(M_N, B) \times \mathbb{U}(B_2))/Stab(u)$$

where

$$Stab(u) = \{(w, v) \in Stab(M_N, B) \times \mathbb{U}(B_2) : (w, v).u = u\}.$$

Since

$$(w, v).u = u \Leftrightarrow wuv^* = u \Leftrightarrow u^*wu = v,$$

we deduce the group $Stab(u)$ is isomorphic to

$$\mathbb{U}(B_2) \cap [u^*Stab(M_N, B)u],$$

via the map $(w, v) \longmapsto v$. A straightforward computation shows

$$u^*Stab(M_N, B)u = Stab(M_N, u^*Bu),$$

for any $u \in \mathbb{U}(M_N)$. Hence, for any $u \in Y(B_2;\ B)$, $\dim\mathcal{O}(u) = \dim Stab(M_N, B) + \mathbb{U}(B_2) - \dim\mathbb{U}(B_2) \cap Stab(M_N, u^*Bu)$. Lastly, one can check $\mathbb{U}(B_2) \cap Stab(M_N, u^*Bu) = Stab(B_2, u^*Bu)$.

Lemma (1.1.12)[30]: Suppose $\varphi : A_1 \to A_2$ is a unital $*$-homomorphism and A_i is isomorphic to $\bigoplus_{j=1}^{l_i} M_{k_i(j)}, (i = 1, 2)$. Then φ is determined, up to unitary in A_2, by on $l_2 \times l_1$ matrix, written $\mu = \mu(\phi) = \mu(A_2, A_1)$, having nonnegative integer entries such that

$$\mu \begin{bmatrix} k_1(1) \\ \vdots \\ k_1(l_1) \end{bmatrix} = \begin{bmatrix} k_2(1) \\ \vdots \\ k_2(l_2) \end{bmatrix}.$$

We call this the matrix of partial multiplicities. In the special case when φ is a unital $*$–representation of A_1 into M_N, μ is a row vector and this vector is called the multiplicity of the representation. One constructs μ as follows: decompose A_p as

$$A_p = \bigoplus_{j=1}^{l_p} A_p(j)$$

where each $A_p(j)$ is simple, $p = 1, 2, 1 \le j \le l_p$. Taking projections, π induces unital$*$–representations $\pi_i: A_1 \to A_2(i), 1 \le i \le l_2$. But up to unitary equivalence, π_i equals

$$\underbrace{\mathrm{id}_{A_1(1)} \oplus \dots \oplus \mathrm{id}_{A_1(1)}}_{m_{i,1}-times} \oplus \dots \oplus \underbrace{\mathrm{id}_{A_1(l_1)} \oplus \dots \oplus \mathrm{id}_{A_1(l_1)}}_{m_{i,l_1}-times}$$

for some nonnegative integer $m_{i,j}$, $1 \le j \le l_1$. Set $\mu[i,j] := m_{i,j}$. In particular, $\mu[i,j]$ equals the rank of $\pi_i(p) \in A_2(i)$, where p is a minimal projection in $A_1(j)$. Clearly, π is injective if and only if for all j there is i such that $\mu[i,j] \neq 0$.

Furthermore, the C^*-subalgebra

$$A_2 \cap \varphi(A_1)' = \{x \in A_2 : x\varphi(a) = \varphi(a)x \ \text{ for all } a \in A_1\}$$

is$*$–isomorphic to $\bigoplus_{i=1}^{l_2} \bigoplus_{j=1}^{l_1} M_{\mu[i,j]}$ and if we have morphisms $A_1 \to A_2 \to A_3$, then $\mu(A_3, A_2)\mu(A_2, A_1) = \mu(A_3, A_1)$ for the corresponding matrices.

Our next task is to show $d(B) < N^2$, for abelian $B \neq C$. We prove it by cases, so let us start.

Lemma (1.1.13)[30]: Assume B_i is $*$–isomorphic to $M_{k_1}, (i = 1, 2)$ and let $k = \gcd(k_1, k_2)$. Take B a unital C^*-subalgebra of B_1 such that it is unitarily equivalent to a C^*-subalgebra of B_2. Then there is an injective unital $*$–representation of B into M_k.

Proof: Take u in $Y(B_2; B)$ so that $u^*Bu \subseteq B_2$. Let $m_i := \mu(M_N, B_i)$, so that $m_i k_i = N, (i = 1, 2)$. Find positive integers p_1 and p_2 such that $k_1 = kp_1$ and $k_2 = kp_2$ Assume B is $*$–isomorphic to $\bigoplus_{j=1}^{l} M_{n_j}$.

To prove the result it is enough to show there are positive integers $(m(1), \dots m(l))$ such that

$$n_1 m(1) + \dots + n_l m(l) = k.$$

Let

$$\mu(B_1, B) = [m_1(1), \dots, m_1(l)]\mu(B_2, u^*Bu) = [m_2(1), \dots, m_2(l)].$$

Since $\mu(M_N, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^*Bu)$ we deduce that $m_1 m_1(j) = m_2 m_2(j)$ for all $1 \le j \le l$. Multiplying by k and using $N = m_1 k_1 = m_2 k_2$ we conclude

$$\frac{N}{p_1} m_1(j) = k m_1 m_1(j) = k m_2 m_2(j) = \frac{N}{p_2} m_2(j)$$

so $p_2 m_1(j) = p_1 m_2(j)$. Since $\gcd(p_1, p_2) = 1$, the number $\frac{m_1(j)}{p_1} = \frac{m_2(j)}{p_2}$ is a positive integer whose value we name $m(j)$. From

$$kp_1 = k_1 = \sum_{j=1}^{l} n_j\, m_1(j) = \sum_{j=1}^{l} n_j\, m(j) p_1,$$

we conclude $k = \sum_{j=1}^{l} n_j\, m(j) p_1$.

Lemma (1.1.14)[30]: Fix a positive integer n and let $r_1, \dots, r_n$ be positive real numbers. Then

$$\min\left\{\sum_{j=1}^{n}\frac{x_j^2}{r_j}\left|\sum_{j=1}^{n}x_j=1\right.\right\}=\frac{1}{\sum_{j=1}^{n}r_j},$$

where the minimum is taken over all $n-$tuples of real numbers that sum up to 1.

Proposition (1.1.15)[30]: Assume B_1 and B_2 are simple. Take $B \neq \mathbb{C}$ an abelian unital C^*-subalgebra of B_1, that is unitarily equivalent to a C^*-subalgebra of B_2. Then $d(B) < N^2$.

Lemma (1.1.16)[30]: For an integer $k \geq 2$ define

$$h(x,y)=2xy-\left(1+\frac{1}{k^2}\right)y^2-\frac{1}{2}x^2$$

Then

$$\max\{h(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1/2\}=\frac{1}{4}-\frac{1}{4k^2}$$

Proposition (1.1.17)[30]: Suppose $\dim C(B_1) \geq 2$ and $B_1$is $*$–isomorphic to

$$M_{N/\dim C(B_1)} \oplus \ldots \oplus M_{N/\dim C(B_1)}. \quad (3)$$

Assume one of the following cases holds:

(i) $\dim C(B_2)=1$,

(ii) $B_1$is$*$–isomorphic to

$$M_{N/2} \oplus M_{N/2}.$$

$B_2$is$*$–isomorphic to

$$M_{N/2} \oplus M_{N/(2\mathrm{k})}.$$

where$k \geq 2$.

(iii) $\dim C(B_2) \geq 3$andB_2 is $*$–isomorphic to

$$M_{N/\dim C(B_2)} \oplus \ldots \oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ an abelian unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebraof B_2, we have that $d(B) < N^2$.

Lemma (1.1.18)[30]: Take $B \neq \mathbb{C}$ a unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebra of B_2. If $\dim\mathbb{U}(B_1) + \dim\mathbb{U}(B_2) \leq N^2, B$ is simple and C in $*$-SubAlg(B) is $*$–isomorphic to $\mathbb{C}^2$, then $d(B) \leq d(C)$.

Proof: Assume B is $*$–isomorphic to M_k and let m denote the multiplicity of Bin M_N. Thus we must have $km = N$. Take a unitary u in the submanifold of maximum dimension in $Y(B_2;\, B)$, so that $d(B)$ is the sum of the terms

$S_1(B) := \dim\mathbb{U}(B_1) - \dim\mathbb{U}(B_1 \cap B')$,

$S_2(B) := \dim \mathbb{U}(B_2) - \dim\mathbb{U}(B_2 \cap u^*B'u)$,

$S_3(B) := \dim \mathbb{U}(B')$,

$S_4(B) := \dim \mathbb{U}(B \cap B') - \dim\mathbb{U}(B)$.

and let v lie in the submanifold of maximum dimension in $Y(B_2, C)$ so that $d(C)$ is the sum of the terms

$S_1(C) := \dim\mathbb{U}(B_1) - \dim\mathbb{U}(B_1 \cap C')$,

$S_2(C) := \dim \mathbb{U}(B_2) - \dim\mathbb{U}(B_2 \cap v^*C'v)$,

$S_3(C) := \dim \mathbb{U}(C')$.

Clearly, $S_4(B) = 1 - k^2$.We write

$$B_1 \simeq \bigoplus_{i=1}^{l_1} M_{k_1(i)},\; B_2 \simeq \bigoplus_{i=1}^{l_2} M_{k_2(i)},$$

and

$$\delta(B_1) = [k_1(1),\ldots,k_1(l_1)]^t, \delta(B_2) = [k_2(1),\ldots,k_2(l_2)]^t.$$

From definition of multiplicity and the fact that it is invariant under unitary equivalence we get

$$\mu(B_1, B)k = \delta(B_1), \tag{4}$$

$$\mu(B_2, u^*Bu)k = \delta(B_2),$$

$$\mu(M_N, B_1)\delta(B_1) = \mu(M_N, B_2)\delta(B_2) = N, \mu(M_n, B_1)\mu(B_1, B) = \mu(M_N, B_2)\mu(B_2, u^*Bu) = m.$$

From Lemma (1.1.12) and equation (4) we get

$$\dim\mathbb{U}(B_1 \cap B') = \frac{1}{k^2}\dim\mathbb{U}(B_1). \tag{5}$$

Hence

$$S_1(B) = \left(1 - \frac{1}{k^2}\right)\dim\mathbb{U}(B_1).$$

Similarly

$$S_2(B) = \left(1 - \frac{1}{k^2}\right)\dim\mathbb{U}(B_2).$$

Now it is the turn of C. To ease notation let

$$\mu(B, C) = [x_1, x_2].$$

Notice that $x_1, +x_2 = k$.We claim

$$S_1(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right)\dim\mathbb{U}(B_1).$$

Using $\mu(B_1, C) = \mu(B_1, B)\mu(B, C)$ we get

$$\dim\mathbb{U}(B_1 \cap C') = (x_1^2 + x_2^2)\dim\mathbb{U}(B_1 \cap B').$$

Furthermore using (5) we obtain

$$\dim\mathbb{U}(B_1 \cap C') = \frac{x_1^2 + x_2^2}{k^2}\dim\mathbb{U}(B_1).$$

Hence our claim follows from definition of $S_1(C)$. Similarly

$$S_2(C) = \left(1 - \frac{x_1^2 + x_2^2}{k^2}\right)\dim\mathbb{U}(B_2).$$

Lastly from $\mu(M_N, C) = [mx_1, mx_2]$ and $mk = N$ we get

$$S_3(C) = (x_1^2 + x_2^2)\frac{N^2}{k^2}\ S_3(B) = \frac{N^2}{k^2}.$$

To prove $d(B) \le d(C)$ we'll show

$$S_1(B) - S_1(C) + S_2(B) - S_2(C) + S_4(B) \le S_3(C) - S_3(B). \tag{6}$$

Using the description of each summand we have that left hand side of (6) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2}(\dim\mathbb{U}(B_1) + \dim\mathbb{U}(B_2)) + 1 - k^2.$$

The right hand side of (6) equals

$$\frac{x_1^2 + x_2^2 - 1}{k^2}N^2.$$

But x_1 and x_2 are strictly positive, because C is a unital subalgebra of B. Hence we can cancel $x_1^2 + x_2^2 - 1$ and finish the proof by using that $1 - \delta(B)^2 < 0$ and the assumption $\dim\mathbb{U}(B_1) + \dim\mathbb{U}(B_2) \le N^2$.

We recall an important perturbation result that can be found in [27].

Lemma (1.1.19)[30]: Let A be a finite dimensional C^*-algebra. Given any positive number ε there is a positive number $\delta = \delta(\varepsilon)$ so that whenever B and C are unital C^*-subalgebras of A and such that C has a system of matrix units $\{e_C(s,i,j)\}_{s,i,j}$, satisfying $\text{dist}(e_C(s,i,j),B) < \delta$ for all s, i and j, then there is a unitary u in $\mathbb{U}(C^*(B,C))$ with $\|u - 1\| < \varepsilon$so that $uCu^* \subseteq B$.

Notation (1.1.20)[30]: For an element x in M_N and a positive number $\varepsilon, \mathcal{N}_\varepsilon(x)$ denotes the open ε −neighborhood around x (i.e. open ball of radius ε centered at x), where the distance is from the operator norm in M_N.

Lemma (1.1.21)[30]: Take B in $*$-SubAlg(B_1) and assume $Z(B_1, B_2; [B]_{B_1})$ is nonempty. Then the function

$$Z\left(B_1, B_2; [B]_{B_1}\right) \to [B]_{B_1} \quad (7)$$
$$u \longmapsto uB_2u^* \cap B_1$$

is continuous.

Proof: Assume B is $*$–isomorphic to

$$\bigoplus_{s=1}^{l} M_{k_s}.$$

First we recall that the topology of $[B]_{B_1}$ is induced by the bijection

$$\beta: [B]_{B_1} \to \frac{\mathbb{U}(B_1)}{\text{Stab}(B_1, B)}, \beta(uBu^*) = u\text{Stab}(B_1, B).$$

For convenience let $\pi : \mathbb{U}(B_1) \to \mathbb{U}(B_1)/\text{Stab}(B_1, B)$ denote the canonical quotient map. Pick $u_0$in $Z(B_1, B_2; [B]_{B_1})$. With no loss of generality we may assume $B = u_0B_2u_0^* \cap B_1$. We prove the result by contradiction. Suppose the function in (7) is not continuous at u_0. Then there is a sequence $(u_k)_{k\geq 1} \subset Z(B_1, B_2; [B]_{B_1})$ and an open neighborhood N of B in $[B]_{B_1}$ such that

(i) $\lim_k u_k = u_0$,

(ii) for all $k, u_kB_2u_k^* \cap B_1 \notin \mathcal{N}$.

On the other hand, let $\varepsilon > 0$ be such that $\pi(\mathcal{N}_\varepsilon(1_{B_1})) \subseteq \beta(\mathcal{N})$. Let$\{e_k(s,i,j)\}_{1\leq s\leq l, 1\leq i,j\leq k_s}$ denote a system of matrix units for $u_kB_2u_k^* \cap B_1$. Fix elements $f_k(s,i,j)$ in B_2such that $e_k(s,i,j) = u_kf_k(s,i,j)u_k^*$.Since B_2 is finite dimensional, passing to a subsequence if necessary, we may assume that $\lim_k f_k(s,i,j) = f(s,i,j)$, for all s, i and j. Using property (i) of the sequence $(u_k)_{k\geq 1}$, we deduce

$$\lim_k e_k(s,i,j) = \lim_k u_k f_k(s,i,j)u_k^* = u_0 f(s,i,j)u_0^*.$$

Hence the element $e(s,i,j) = u_0f(s,i,j)u^*$ belongs to $u_0B_1u_0^* \cap B_1 = B$. Use Lemma (1.1.13) and take δ_1 positive such that whenever C is a subal-gebra in $*$-SubAlg(B_1)having a system of matrix units$\{e_C(s,i,j)\}_{s,i,j}$ satisfying $\text{dist}(e_C(s,i,j),B) < \delta_1$,for all s, i and j, then there is a unitary Q in $U(B_1)$ such that $\left\|Q - 1_{B_1}\right\| < \varepsilon$ and $QCQ^* \subseteq B$. Take k such that $\|e_k(s,i,j) - e(s,i,j)\| < \delta_1$ for all s, i and j. This implies $\text{dist}(e_C(s,i,j),B) < \delta_1$ for all $s, i$and j. We conclude there is a unitary Q in $\mathbb{U}(B_1)$ such that $\left\|Q - 1_{B_1}\right\| < \varepsilon$and $Q^*(u_kB_2u_k^* \cap B_1)Q \subseteq B$. But $\dim B = \dim u_kB_2u_k^* \cap B_1 = \dim Q^*(u_kB_2u_k^* \cap B_1)Q$,

where in the first equality we used that u_k lies in $Z(B_1, B_2; [B]_{B_1})$. Hence $Q^*(u_kB_2u_k^* \cap B_1)Q = B$. As a consequence,

$$\beta(u_kB_2u_k^* \cap B_1) = \beta(QBQ^*) = \pi(Q) \in \beta(N).$$

But the latter contradicts property (ii) of $(u_k)_{k\geq 1}$.

Lemma (1.1.22)[30]: For B in $*$-SubAlg(B), the function $c: [B]_{B_1} \to [C(B)]_{B_1}$ given by $c(uBu^*) = uC(B)u^*$ is continuous.

Proof: First, we must show the function c is well defined. In other words we have to show $Stab(B_1, B) \subseteq Stab(B_1, C(B))$. But this follows directly from the fact that any u in $Stab(B_1, B)$ defines a $*$–automorphism of B and any $*$–automorphism leaves the center fixed. Since $[B]_{B_1}$ and $[C(B)]_{B_1}$ are homeomorphic to $\mathbb{U}(B_1)/Stab(B_1, B)$ and $\mathbb{U}(B_1)/Stab(B_1, C(B))$ respectively, it follows that c is continuous if and only if the function $\tilde{c}: \mathbb{U}(B_1)/Stab(B_1, B) \to \mathbb{U}(B_1)/Stab(B_1, C(B))$ given by $\tilde{c}(uStab(B_1, B)) = uStab(B_1, C(B))$ is continuous. But the spaces $\mathbb{U}(B_1)/Stab(B_1, B)$ and $\mathbb{U}(B_1)/Stab(B_1, C(B))$ have the quotient topology induced by the canonical projections

$$\pi_B : \mathbb{U}(B_1) \to Stab(B_1, B), \pi_C(B): \mathbb{U}(B_1) \to \mathbb{U}(B_1)/Stab(B_1, C(B)).$$

Thus $\tilde{c}$ is continuous if and only if $\pi_B \circ \tilde{c}$ is continuous. But $\pi_B \circ \tilde{c} = \pi_{C(B)}$, which is indeed continuous.

We are ready to find local parameterizations of $Z(B_1, B_2;\ [B]_{B_1})$.

Proposition (1.1.23)[30]: Take B a unital C^*-subalgebra in B_1 that is unitarily equivalent to a C^*-subalgebra of B_2. Fix an element $u_0$in $Z(B_1, B_2;\ [B]_{B_1})$. Then there is a positive number $r$and a continuous injective function

$$\Psi : N_r(u_0) \cap Z(B_1, B_2;\ [B]_{B_1}) \to \mathbb{R}^{d(C(B))}.$$

Proof: Using that $Z(B_1, B_2;\ [B]_{B_1}) = Z(B_1, B_2;\ [u_0 B_2 u_0^* \cap B_1]_{B_1})$, with no loss of generality we may assume $u_0 B_2 u_0^* \cap B_1 = B$. Now, we use the manifold structure of $[C(B)]_{B_1}$ and $Y(B_2;\ C(B))$ to construct Ψ. Note that if $Y(B_2, B)$ is nonempty then $Y(B_2, C(B))$ is nonempty as well. Let d_1 denote the dimension of $[C(B)]_{B_1}$ and let d_2 denote the dimension of the sub-manifold of $Y(B_2;\ C(B))$ that contains u_0. Of course, we have $d_1 + d_2 \leq d(C(B))$.

We use the local cross section result from previous section to parametrize$[C(B)]_{B_1}$. To ease notation take $G = \mathbb{U}(B_1), H = Stab(B_1, C(B))$ and let π denote the canonical quotient map from Gonto the left-cosets of H. By Proposition (1.1.7) there are

(i) $\mathcal{N}_G$, a compact neighborhood of 1 in G,

(ii) $\mathcal{N}_H$, a compact neighborhood of 1 in H,

(iii) $\mathcal{N}_{G/H}$, a compact neighborhood of $\pi(1)$ in G/H,

(iiii) a continuous function s : $\mathcal{N}_{G/H} \to \mathcal{N}_G$ satisfying

(a) $s(\pi(1)) = 1$and$\pi(s(\pi(g))) = \pi(g)$ whenever $\pi(g)$ lies in $\mathcal{N}_{G/H}$,

(b) the function

$$\mathcal{N}_H \times \mathcal{N}_{G/H} \to \mathcal{N}_G,$$
$$(h, \pi(g)) \longmapsto hs(\pi(g)),$$

is an homeomorphism.

Since G/H is a manifold of dimension d_1, we may assume there is a continuous injective map $\Psi_1 : \mathcal{N}_{G/H} \to \mathbb{R}^{d_2}$.

Parametrizing$Y(B_2, C(B))$ is easier. Since $u_0 B_2 u_0^* \cap B_1 = B, u_0$ belongs to $Y(B_2, B)$. Take r_1 positive and a diffeomorphismΨ_2 from$Y(B_2, C(B)) \cap \mathcal{N}_{r_1}(u_0)$ onto an open subset of $\mathbb{R}^{d_2}$.

Now that we have fixed parametrizations $\Psi_1$and Ψ_2, we can parametrize $Z(B_1, B_2; [B]_{B_1})$ around u_0. Recall $[C(B)]_{B_1}$has the topology induced by the bijection$\beta : [C(B)]_{B_1} \to G/H$, given by $\beta(uC(B)u^*) = \pi(u)$.The function

$$Z\big(B_1, B_2; [B]_{B_1}\big) \to [C(B)]_{B_1}, u \longmapsto c(uB_2u^* \cap B_1)$$

is continuous by Lemma (1.1.21) and Lemma (1.1.22). Hence there is $\delta_2$positive such that $\beta(c(uB_2u^* \cap B_1))$ belongs to $\mathcal{N}_{G/H}$, whenever u lies in the intersection $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$. For a unitary $u$in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$ define

$$q(u) := s(\beta(c(uB_2u^* \cap B_1)).$$

We note that $q(u_0) = 1, q(u)$ lies in G and that the map $u \longmapsto q(u)$ is continuous. The main property of $q(u)$ is that

$$(c(uB_2u^* \cap B_1) = q(u)c(B)q(u)^*. \tag{8}$$

Indeed, for u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_2}(u_0)$ there is a unitary v in G with the property $uB_2u^* \cap B_1 = vBv^*$. Hence $c(uB_2 \cap B_1) = vC(B)v^*$. Since $\|u - u_0\| < \delta_2, \beta(c(uB_2u^* \cap B_1))$lies in $\mathcal{N}_{G/H}$. Hence $\beta(c(uB_2u^* \cap B_1) = \pi(v)$ lies in $\mathcal{N}_{G/H}$. Using the fact that s is a local section on $\mathcal{N}_{G/H}$ (property (ia) above) we deduce $\pi(s(\pi(v))) = \pi(v)$.

On the other hand, by definition of $q(u)$ we have

$$\pi(s(\pi(v))) = \pi(s(\beta(uB_2u^* \cap B_1))) = \pi(q(u)).$$

As a consequence, $\pi(v) = \pi(q(u))$ i.e. $v^*q(u)$ belongs to $Stab(B_1, B)$ which is just another way to say (8) holds. At last we are ready to find r. Continuity of the map $u \longmapsto q(u)$ gives a positive δ_3, less that δ_2, such that $\|q(u) - 1\| < \frac{\delta_1}{2}$ whenever u lies in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta_3}(u_0)$. Define $r = \min\{\frac{\delta_1}{2}, \delta_3\}$. The first thing we notice is that $q(u)^*u$ belongs to $Y(B_2; C(B)) \cap \mathcal{N}_{\delta_1}(u_0)$ whenever u lies in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta}(u_0)$. Indeed, from

$$q(u)c(B)q(u)^* = c(uB_2u^* \cap B_1) \subseteq uB_2u^*$$

we obtain $q(u)^*u \in Y(B_2; c(B))$ and a standard computation, using $\|q(u) - 1\| < \frac{\delta_2}{2}$, shows $\|q(u)^*u - u_0\| < \delta_1$. Hence we are allowed to take $\Psi_2(q(u)^*u)$. Lastly, for u in $Z(B_1, B_2; [B]_{B_1}) \cap \mathcal{N}_{\delta}(u_0)$ define

$$\Psi(u) := (\Psi_1(\beta(c(uB_2u^* \cap B_1))), \Psi_2(q(u)^*u)).$$

It is clear that Ψ is continuous.

Now we show Ψ is injective. If $\Psi(u_1) = \Psi(u_2)$, for two element u_1 and u_2 in $Z(B_1, B_2; [B]_{B_1})$, then

$$\Psi_1\Big(\beta\big(c(u_1B_2u_1^* \cap B_1)\big)\Big) = \Psi_1\Big(\beta\Big(c\big((u_2B_2u_2^* \cap B_1)\big)\Big)\Big), \tag{9}$$

$$\Psi_2(q(u_1)u_1^*) = \Psi_2(q(u_2)u_2^*)). \tag{10}$$

From (9) and definition of $q(u)$ it follows that $q(u_1) = q(u_2)$ and from equation (10) we conclude $u_1 = u_2$.

Proposition (1.1.24)[30]: Take B a unital C^*-subalgebra of B_1 such that it is unitarily equivalent to a C^*-subalgebra of B_2. Fix an element $u_0$in $Z(B_1, B_2; [B]_{B_1})$.

There is a positive number r and a continuous injective function

$$\Psi : \mathcal{N}_r(u_0) \cap Z(B_1, B_2; [B]_{B_1}) \to \mathbb{R}^{d(B)}$$

The proof of Proposition (1.1.24) is similar to that of Proposition (1.1.23), so we omit it.

We now begin showing density in $\mathbb{U}(M_N)$ of certain sets of unitaries.

Lemma (1.1.25)[30]: Assume B_1 and B_2 are simple. If $B \neq C$ is a unital C^*-subalgebra of B_1 and it is unitarily equivalent to a C^*-subalgebra of B_2 then $Z(B_1, B_2; [B]_{B_1})^c$ is dense.

Proof: Firstly we notice that $\dim\mathbb{U}(B_1) + \dim\mathbb{U}(B_2) < N^2$. Indeed, if B_i is $*$–isomorphic to $M_{k_i}, i = 1,2$ and $m_i = \mu(M_N, B_i)$ then $\dim\mathbb{U}(B_1) + \dim\mathbb{U}(B_2) = N^2(1/m_2^2 + 1/m_2^2) < N^2$. Secondly we will prove that for any$u$ in $Z(B_1, B_2; [B]_{B_1})$ there is a natural number d_u, with $d_u < N^2$, a positive number ru and a continuous injective function $\Psi_u : \mathcal{N}_{r_u}(u) \cap Z(B_1, B_2; [B]_{B_1}) \rightarrow \mathbb{R}^{d_u}$. We will consider two cases.

Case (i): B is not simple. Take $d_u = d(C(B))$. Since $C(B) \neq \mathbb{C}$, Proposition (1.1.14) implies $d(C(B)) < N^2$. Take r_u and Ψ_u as required to exist by Proposition (1.1.23)

Case (ii): B is simple. Take $d_u = d(B)$. Since $B \neq \mathbb{C}, B$ contains a unital C^*-subalgebra isomorphic to $\mathbb{C}^2$, call it C. Lemma (1.1.12) implies $d(B) \leq d(C)$ and implies $d(C) < N^2$. Take r_u and Ψ_u the positive number and continuous injective function from Proposition (1.1.24)

We will show that $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$, for any nonempty open subset $U \subseteq \mathbb{U}(M_N)$.First notice that if the intersection $U \cap (\bigcup_{u \in Z(B_1,B_2;[B]_{B_1})} \mathcal{N}_{r_u}(u))^c$ is nonempty then we are done. Thus we may assume $U \subseteq (\bigcup_{u \in Z(B_1,B_2;[B]_{B_1})} \mathcal{N}_{r_u}(u))$. Furthermore, by making U smaller, if necessary, we may assume there is u in $Z(B_1, B_2; [B]_{B_1})$ such that $U \subseteq \mathcal{N}_{r_u}(u)$.

For sake of contradiction assume $U \subseteq Z(B_1, B_2; [B]_{B_1})$. We may take an open subset V, contained in U, small enough so that V is diffeomorphic to an open connected set $\mathcal{O}$ of $\mathbb{R}^{N^2}$.Let $\varphi : \mathcal{O} \rightarrow V$ be a diffeomorphism. It follows we have a continuous injective function

$$\mathbb{R}^{N^2} \supseteq \mathcal{O} \xrightarrow{\varphi} V \xrightarrow{\psi_u} \mathbb{R}^{d_u} \hookrightarrow \mathbb{R}^{N^2}$$

By the Invariance of Domain Theorem, the image of this map must be open in $\mathbb{R}^{N^2}$. But this is a contradiction since the image is contained in $\mathbb{R}^{d_u}$and $d_u < N^2$. We conclude $U \cap Z(B_1, B_2; [B]_{B_1})^c \neq \emptyset$

Lemma (1.1.26)[30] : Suppose $\dim C(B_1) \geq 2$ and B_1 is $*$–isomorphic to

$$M_{N/\dim C(B_1)} \oplus \ldots \oplus M_{N/\dim C(B_1)}.$$

Assume one of the following cases holds:

(i) $\dim C(B_2) = 1$,

(ii) $B_1$is$*$–isomorphic to

$$M_{N/2} \oplus M_{N/2}$$

and B_2 is $*$–isomorphic to

$$M_{N/2} \oplus M_{N/(2k)}$$

where $k \geq 2$.

(i) $\dim C(B_2) \geq 3$andB_2 is $*$–isomorphic to

$$(\text{ii}i) M_{N/\dim C(B_2)} \oplus \ldots \oplus M_{N/\dim C(B_2)}.$$

Then for any $B \neq \mathbb{C}$ unital i-subalgebra of B_1 such that it is unitarily equivalent to a i-subalgebra of $B_2, Z(B_1, B_2; [B]_{B_1})^c$ is dense.

Proof: The proof of Lemma (1.1.26) is exactly as the proof of (1.1.25) but using Lemma (1.1.17) instead of Lemma (1.1.14)

At this point if the sets $Z(B_1, B_2; [B]_{B_1})$ were closed one could conclude immediately that $\Delta(B_1, B_2)$ is dense. Unfortunately they may not be closed. What saves the day is the fact that we can control the closure of $Z(B_1, B_2; [B]_{B_1})$ with sets of the same form i.e. sets like $Z(B_1, B_2; [C]_{B_1})$ for a suitable finite family of subalgebras C.We make this statement clearer with the definition of an order on $*$-SubAlg(B_1).

Definition (1.1.27)[30]: On $*$-SubAlg$(B_1)/\sim_{B_1}$ we define a partial order as follows:

$$[B]_{B_1} \leq [C]_{B_1} \Leftrightarrow \exists D \ \in * -SubAlg(C) : D \sim_{B_1} B.$$

Lemma (1.1.28)[30]: Assume one of the conditions (i)–(iiii). Then for any $B \neq \mathbb{C}$, unital C^*-subalgebra of B_1 that is unitarily equivalent to a C^*-subalgebra of B_2, the set $\overline{Z(B_1, B_2;\ [B]_{B_1}\ )}^c$ is dense.

Proof: Assume $\overline{Z(B_1, B_2;\ [B]_{B_1}\ )}^c$ is not dense. There is $[C]_{B_1} > [B]_{B_1}$ such that $\overline{Z(B_1, B_2;\ [B]_{B_1}\ )}^c$ is not dense. We notice that again we are in the same condition to apply, since $[C]_{B_1} > [B]_{B_1} > [\mathbb{C}]_{B_1}$. In this way we can construct chains, in $*$-SubAlg$(B_1)/\sim_{B_1}$, of length arbitrarily large, but this cannot be since it is finite.

At last we can give a proof of Theorem (1.1.29)

Theorem (1.1.29)[30]: Assume one of the following conditions holds:

(i) $\dim C(B_1) = 1 = \dim C(B_2)$,

(ii)$\dim C(B_1) \geq 2, \dim C(B_2) = 1$ and B_1 is $*$–isomorphic to

$$M_{N/\,dim\,C(B_1)} \oplus \ldots \oplus M_{N/\,dim\,C(B_1)},$$

(iii) $\dim C(B_1) = 2 = \dim C(B_2), B_1$is $*$–isomorphic to

$$M_{N/2} \oplus M_{N/2},$$

and B_2 is $*$–isomorphic to

$$M_{N/2} \oplus M_{N/(2k)}.$$

Where $k \geq 2$,

(iiii)$\dim C(B_1) \geq 2, \dim C(B_2) \geq 3$and, for $i = 1, 2, B_i$ is $*$– isomorphic to

$$M_{N/\dim C(B_i)} \oplus \ldots \oplus M_{N/\dim C(B_i)}.$$

Then

$$\Delta(B_1, B_2) := \{u \in \mathbb{U}(M_N) : B_1 \cap uB_2u^* = \mathbb{C}\}$$

is dense in $\mathbb{U}(M_N)$.

Proof : A direct computation shows that

$$\Delta(B_1, B_2) = \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} Z(B_1, B_2, [B]_{B_1})^c$$

Thus

$$\Delta(B_1, B_2) \supseteq \bigcap_{[B]_{B_1} > [\mathbb{C}]_{B_1}} Z(B_1, B_2, [B]_{B_1})^c$$

Now whenever $[B]_{B_1} > [\mathbb{C}]_{B_1}$, the set $\overline{Z(B_1, B_2, [B]_{B_1})}^c$ is dense. Hence $\Delta(B_1, B_2)$ is dense.

We unless stated otherwise, $A_1 \neq \mathbb{C}$ and $A_2 \neq \mathbb{C}$ denote two nontrivial, separable, residually finite dimensional C$*$-algebras. Our goal is to prove $A_1 * A_2$ is primitive, except for the case $A_1 = \mathbb{C}^2 = A_2$. Two main ingredients are used. Firstly, the perturbation results from previous section. Secondly, the fact that $A_1 * A_2$has a separating family of finite dimensional $*$–representations, a result due to [40].

Before we start proving results about primitivity, we want to consider the case $\mathbb{C}^2 * \mathbb{C}^2$. This is a well studied$C^*$-algebra; see for in-stance [11], [107] and [118]. It is known that

$\mathbb{C}^2 * \mathbb{C}^2$ is $*$–isomorphic to the C$*$-algebra of continuous M_2-valued functions on the closed interval [0, 1], whose values at 0 and 1 are diagonal matrices. As a consequences its center is not trivial. Since the center of any primitive C^*-algebra is trivial, we conclude $\mathbb{C}^2 * \mathbb{C}^2$ is not primitive.

Definition (1.1.30)[30]: We denote by ι_j the inclusion $*$–homomorphism from A_j into $A_1 * A_2$. Given a unital$*$–representation $\pi: A_1 * A_2 \to \mathbb{B}(H)$, we define $\pi^{(1)} = \pi \circ \iota_1$ and $\pi^{(2)} = \pi \circ \iota_2$. Thus, with this notation, we have $\pi = \pi^{(1)} * \pi^{(2)}$. For a unitary u in $\mathbb{U}(H)$ we call the $*$representation $\pi^{(1)} * (Ad\, u \circ \pi^{(2)})$, a perturbation of π by u.

Definition (1.1.31)[30]: Assume A_1 and A_2 are finite dimensional and let $\rho: A_1 * A_2 \to \mathbb{B}(H)$ be a unital, finite dimensional representation. We say that ρ satisfies the Rank of Central Projections condition (or RCP condition) if for both $i = 1, 2$, the rank of $\rho(p)$ is the same for all minimal projections p of the center $C(A_i)$ of A_i, (but they need not agree for different values of i).

The RCP condition for ρ, of course, is really about the pair of representations $(\rho^{(1)}, \rho^{(2)})$. However, it will be convenient to express it in terms of $A_1 * A_2$. In any case, the following two lemmas are clear.

Lemma (1.1.32)[30]: Suppose A_1 and A_1 are finite dimensional, $\rho: A_1 * A_2 \to \mathbb{B}(H)$ is a finite dimensional representation that satisfies the RCP condition and $u \in \mathbb{U}(H)$. Then the representation $\rho^{(1)} * (Ad\, u \circ \rho^{(2)})$ of $A_1 * A_2$ also satisfies the RCP condition.

Lemma (1.1.33)[30]: Suppose A_1 and A_2 are finite dimensional, $\rho: A_1 * A_2 \to \mathbb{B}(H)$ and $\sigma : A_1 * A_2 \to \mathbb{B}(K)$ are finite dimensional representations that satisfy the RCP condition. Then $\rho \oplus \sigma : A_1 * A_2 \to \mathbb{B}(H \oplus K)$ also satisfies the RCP condition.

The following is clear from Lemma (1.1.12)

Lemma (1.1.34)[30]: Assume A is a finite dimensional C^*-algebra $*$–isomorphic to $\bigoplus_{j=1}^{l} M_{n(j)}$ and take $\pi: A \to \mathbb{B}(H)$ a unital finite dimensional $*$representation. Let $\mu(\pi) = [m(1), \ldots, m(l)]$ and let $\tilde{\pi}$ be the restriction of π to the center of A. Then

$$\mu(\tilde{\pi}) = [m(1)n(1), \ldots, m(l)n(l)].$$

The next lemma will help us to prove that the RCP condition is easy to get.

Lemma (1.1.35)[30]: Assume A is a finite dimensional C^*-algebra and $\pi: A \to \mathbb{B}(H)$ is a unital finite dimensional $*$–representation. Let

$$\mu(\pi) = [m(1), \ldots, m(l)].$$

For any nonnegative integers $q(1), \ldots, q(l)$ there is a finite dimensional unital $*$–representation $\rho : A \to \mathbb{B}(K)$ such that

$$\mu(\pi \oplus \rho) = [m(1) + q(1), \ldots, m(l) + q(l)].$$

Proof: Write A as

$$A = \bigoplus_{i=1}^{l} A(i)$$

where $A(i) = \mathbb{B}(V_i)$ for V_i finite dimensional. For $1 \le i \le l$, let $p_i : A \to A(i)$ denote the canonical projection onto $A(i)$. Notice that p_i is a unital $*$–representation of A. Define

$$\rho := \bigoplus_{i=1}^{l} \underbrace{(p_i \oplus \ldots \oplus p_i)}_{q(i)-times} : A \to \bigoplus_{i=1}^{l} A(i)^{q(i)} \subseteq \mathbb{B}(K).$$

Where $K = \bigoplus_{i=1}^{l} (V_i^{\oplus q_i})$. Then ρ is a unital$*$ –representation of A on K and

$$\mu(\pi \oplus \rho) = [m(1) + q(1),\ldots,m(l) + q(l)].$$

The next lemma takes slightly more work and is essential to our construction.

Lemma (1.1.36)[30]: Assume A_1 and A_2 are finite dimensional. Given a unital finite dimensional $*$–representation $\pi : A_1 * A_2 \to \mathbb{B}(H)$, there is a finite dimensional Hilbert space $\widehat{H}$ and a unital $*$–representation

$$\hat{\pi} : A_1 * A_2 \to \mathbb{B}(\widehat{H})$$

such that $\pi \oplus \hat{\pi}$ satisfies the RCP condition.

Proof: For $i = 1, 2$, let $l_i = dim\, C(A_i)$, let A_i be $*$–isomorphic to $\bigoplus_{j=1}^{l_i} M_{n_i(j)}$and write

$$\mu(\pi^{(i)}) = [m_i(1),\ldots,m_i(l_i)].$$

Take $n_i = lcm(n_i(1),\ldots,n_i(l_i))$ and integers $r_i(j)$, such that $r_i(j)n_i(j) = n_i$, for $1 \le j \le l_i$. Take a positive integer s such that $sr_i(j) \ge m_i(j)$for all $i = 1, 2$and $1 \le j \le l_i$. Use Lemma (1.1.36) to find a unital finite dimensional $*$–representation $\rho_i : A_i \to \mathbb{B}(K_i), i = 1, 2$ such that

$$\mu(\pi^{(i)} \oplus \rho_i) = [sr_i(1),\ldots,sr_i(l_i)].$$

Letting κ_i denote the restriction of $\pi^{(i)} \oplus \rho_i$to $C(A_i)$, from Lemma (1.1.36) we have

$$\mu(\kappa_i) = [sr_i(1)n_i(1),\ldots,sr_i(l_i)n_i(l_i)] = [sn_i, sn_i,\ldots,sn_i].$$

The $*$–representations $(\pi^{(1)} \oplus \rho_1)$ and $(\pi^{(2)} \oplus \rho_2)$ are almost what we want, but they may take values in Hilbert spaces with different dimensions. To take care of this, we take multiples of them. Let $N = \mathrm{lcm}(\dim(H \oplus K_1), \dim(H \oplus K_2))$, find positive integers k_1 and k_2 such that

$$N = K_1 \dim(H \oplus K_1) = K_2 \dim(H \oplus K_2)$$

and consider the Hilbert spaces $(H \oplus K_i)^{\oplus k_i}$, whose dimensions agree for $i = 1, 2$. Then

$$\dim(K_1 \oplus (H \oplus K_1)^{\oplus(K_1-1)}) = \dim(K_2 \oplus (H \oplus K_2)^{\oplus(K_2-1)})$$

and there is a unitary operator

$$U : K_2 \oplus (H \oplus K_2)^{\oplus(K_2-1)} \to K_1 \oplus (H \oplus K_1)^{\oplus(K_1-1)} .$$

Take

$$\widehat{H} := K_1 \oplus (H + K_1)^{\oplus(K_1-1)}$$

$$\hat{\pi}_1 := \rho_1 \oplus (\pi^{(1)} \oplus \rho)^{\oplus(K_1-1)},$$

$\sigma_1 := \pi^{(1)} \oplus \hat{\pi}_1,$

$\hat{\pi}_2 := Ad\, U \circ (\rho_2 \oplus (\pi^{(2)} \oplus \rho)^{\oplus(K_2-1)}),$

$\sigma_2 := \pi^{(2)} \oplus \hat{\pi}_2,$

$\hat{\pi} := \hat{\pi}_1 * \hat{\pi}_2.$

Then $\sigma_1 * \sigma_2 = (\pi^{(1)} \oplus \hat{\pi}_1) * (\pi^{(2)} \oplus \hat{\pi}_2) = \pi \oplus \hat{\pi}$. We have $\mu(\sigma_i) = [k_i sr_i(1),\ldots,k_i sr_i(l_i)]$. Let $\tilde{\sigma}_i$ denote the restriction of $\sigma_i$to $C(A_i)$.

From Lemma (1.1.35) we have

$$\mu(\tilde{\sigma}_i) = [k_i sr_i(1)n_i(1),\ldots,k_i sr_i(l_i)n_i(l_i)] = [k_i sn_i,\ldots,k_i sn_i].$$

The purpose of the next definition and lemma is to emphasize an important property about $*$–representations satisfying the RCP.

Definition (1.1.37)[30]: A $*$–representation $\pi: A_1 * A_2 \to \mathbb{B}(H)$ is said to be densely perturbable to an irreducible $*$-representation, abbreviated DPI, if the set

$$\Delta(\pi) := \{u \in \mathbb{U}(H) : \pi^{(1)}(A_1)' \cap (u\pi^{(2)}(A_2)'u *) = \mathbb{C}\}$$

is norm dense in $\mathbb{U}(H)$. Here the commutants are taken with respect to $\mathbb{B}(H)$.

The next lemma shows that any $**$–representation satisfying the R.C.P is DPI.

Lemma (1.1.38)[30]: Assume A_1 and A_2 are finite dimensional C^*-algebras and $(dim(A_1)-1)(dim(A_2) - 1) \geq 2$. If $\rho: A_1 * A_2 \to \mathbb{B}(H)$, with Hfinite dimensional, satisfies the Rank of Central Projections condition, then ρ is DPI.

Proof: Since $(\dim(A_1)-1)(\dim(A_2) - 1) \geq 2$, and after interchanging A_1 and A_2, if necessary, one of the following must hold:

(i) $A_1$and A_2 are simple,

(ii) $\dim C(A_1) \geq 2$ andA_2 is simple,

(iii) for $i = 1,2, A_i = M_{n_{i(1)}} \oplus M_{n_{i(2)}}$, with $n_2(2) \geq 2$,

(iiii) $\dim C(A_1) \geq 2$, $\dim C(A_2) \geq 3$.

In case (1), take $B_i = \rho^{(i)}(A_i)', i = 1,2$.

In case (2), let $B_1 = \rho^{(1)}(C(A_1))'$and $B_2 = \rho^{(2)}(A_2)'$. Notice that $\dim C(B_2) = 1, \dim C(B_1) = \dim C(A_1) \geq 2$ and, by the R.C.P assumption, B_1 is $*$–isomorphic to $M_{dim\, H/\, dim\, C(B_1)} \oplus ... \oplus M_{dim\, H/\, dim\, C(B_1)}$.

In case (iii), let $B_1 = \rho^{(1)}(C(A_1))'$and $B_2 = \rho^{(2)}(\mathbb{C} \oplus M_{n_{2(2)}})'$. By the RCP assumption, B_1 is $*$–isomorphic to

$$M_{dim\, H/2} \oplus M_{dim\, H/2}$$

andB_2 is $*$–isomorphic to

$$M_{dim\, H/2} \oplus M_{dim\, H/(2n_2(2))}.$$

In case (iiii), let $B_i = \rho^{(i)}(C(A_i))'$ for $i = 1,2$. Then $\dim C(B_1) = \dim C(A_1) \geq 2, \dim C(B_2) = \dim C(A_2) \geq 3$ and, for $i = 1,2$, RCP implies Bi is $*$–isomorphic to

$$M_{dim\, H/\, dim\, C(B_i)} \oplus ... \oplus M_{dim\, H/\, dim\, C(B_i)}$$

Now define

$$\Delta(B_1, B_2) := \{u \in \mathbb{U}(H) : B_1 \cap\, Ad\, u(B_2) = \mathbb{C}\}.$$

and notice that in all four cases $\Delta(B_1, B_2) \subseteq \Delta(\rho)$. By Theorem (1.1.29), the set $\Delta(B_1, B_2)$ is dense in all the four cases.

A downside of the DPI property is that it is not stable under direct sums. However, it is stable under perturbations.

We obtain the following.

Lemma (1.1.39)[30]: For any unital finite dimensional $*$-representation $\pi: A_1 * A_2 \to \mathbb{B}(H)$, there is a unital finite dimensional $*$-representation$\hat{\pi}: A_1 * A_2 \to \mathbb{B}(\widehat{H})$ such that $\pi \oplus \hat{\pi}$ is DPI.

Proof: The assumption $(\dim(A_1)-1)(\dim(A_2)-1) \geq 2$ implies there is a unital finite dimensional $*$-representation $\vartheta : A_1 * A_2 \to \mathbb{B}(H_0)$, such that $(\dim(\vartheta^{(1)}(A_1)) - 1)(\dim(\vartheta^{(2)}(A_2)) - 1) \geq 2$. Consider the unital$C^*$-subalgebras of $\mathbb{B}(H \oplus H_0), D_i = (\pi \oplus \vartheta)^{(i)}(A_i), i = 1,2$, and notice that $(\dim(D_1) - 1)(\dim(D_2) - 1) \geq 2$. Let $\theta: D_1 * D_2 \to \mathbb{B}(H \oplus H_0)$ be the unital $*$-representation induced by the universal property of $D_1 * D_2$via the unitalinclusions $D_i \subseteq \mathbb{B}(H \oplus H_0)$. Lemma 5.8 implies there is a unital finite dimensional $*$-representation $\rho: D_1 * D_2 \to \mathbb{B}(K)$ such that $\theta \oplus \rho$ satisfies the RCP condition, so by is DPI.

Let $j_i: D_i \to D_1 * D_2, i = 1,2$, be the inclusion $*$-homomorphism from the definition of unital full free product. Now consider the unital$*$-homomorphism $\sigma = (j_1 \circ (\pi \oplus \vartheta)^{(1)}) * (j_2 \circ (\pi \oplus \vartheta)^{(2)})) : A_1 * A_2 \to D_1 * D_2$.

Now just take $\widehat{H} = H_0 \oplus K$and $\hat{\pi} = \vartheta \oplus (\rho \circ \sigma)$. In order to show$\pi \oplus \hat{\pi}$ is DPI we just need to show that, for $i = 1, 2, (\pi \oplus \hat{\pi})^{(i)}(A_i) = (\theta \oplus \rho)^{(i)}(D_i)$, but this is a direct computation.

The proof of next lemma is a standard approximation argument and we omit it.

Proposition (1.1.40)[30]: Let A_1 and A_2 be two unital C^*-algebras. Given a non zero element x in $A_1 * A_2$ and a positive number ε, there is a positive number $\delta = \delta(x, \varepsilon)$ such that for any u and v in $\mathbb{U}(H)$ satisfying $\|u - v\| < \delta$ and any unital$*$-representations $\pi : A_1 * A_2 \to \mathbb{B}(H)$, we have

$$\left\|(\pi^{(1)} * (Ad\, u \circ \pi^{(2)}))(x) - (\pi^{(1)} * (Ad\, u \circ \pi^{(2)}))(x)\right\| < \varepsilon.$$

Here is our main theorem.

Theorem (1.1.41)[30]: Assume A_1 and A_2 are unital, separable, residually finite dimensional C^*-algebras with $(\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2$. Then $A_1 * A_2$ is primitive.

Proof: By the result of [40], there is a separating sequence $(\pi_j : A_1 * A_2 \to \mathbb{B}(H_j))_{j\geq 1}$, of finite dimensional unital$*$-representations. For later use in constructing an essential representation of $A_1 * A_2$, i.e., a $*$-representation with the property that zero is the only compact operator in its image, we modify $(\pi_j)_{j\geq 1}$, if necessary, so that that each $*$-representation is repeated infinitely many times.

By recursion and using Lemma (1.1.39), we define a sequence

$$\hat{\pi}_j : A_1 * A_2 \to \mathbb{B}(\widehat{H}_j), (j \geq 1)$$

of finite dimensional unital$*$-representations such that, for all $k \geq 1$, $\bigoplus_{j=1}^{k} (\pi_j \oplus \hat{\pi}_j)$ is D.P.I. Let $\pi := \bigoplus_{j\geq 1} \pi_j \oplus \hat{\pi}_j$and $H := \bigoplus_{j\geq 1} H_j \oplus \widehat{H}_j$. To ease notation, for $k \geq 1$, let $\pi_{[k]} = \bigoplus_{j=1}^{k} \pi \oplus \hat{\pi}$. Note that we have $\pi(A_1 * A_2) \cap \mathbb{K}(H) = \{0\}$. Indeed, if $\pi(x)$ is compact then $\lim_j \left\|(\pi_j \oplus \hat{\pi}_j)(x)\right\| = 0$, since each representation is repeated infinitely many times and we are considering a separating family we get $x = 0$.

We will show that given any positive number ε, there is a unitary u on $\mathbb{U}(H)$ such that $\|u - \mathrm{id}_H\| < \varepsilon$ and $\pi^{(1)} * (Ad\, u \circ \pi^{(2)})$ is both irreducible and faithful. To do this, we will to construct a sequence $(u_k, \theta_k, F_k)_{k\geq 1}$ where:

(i) For all k, u_k is a unitary in $\mathbb{U}(\bigoplus_{j=1}^{k} (H_j \oplus \widehat{H}_j))$ satisfying

$$\left\|u - \mathrm{id}_{\bigoplus_{j=1}^{k} H_j \oplus \widehat{H}_j}\right\| < \frac{\varepsilon}{2^{k+1}} \quad (11)$$

(ii) Letting

$$u_{(j,k)} = u_j \oplus \mathrm{id}_{H_{j+1} \oplus \widehat{H}_{j+1}} \oplus \dots \oplus \mathrm{id}_{H_k \oplus \widehat{H}_k}$$

and

$$U_k = u_k u_{(k-1,k)} u_{(k-2,k)} \dots u_{(1,k)}, \quad (12)$$

theunital$*$-representation of $A_1 * A_2$onto $\mathbb{B}(\bigoplus_{j=1}^{k} H_j \oplus \widehat{H}_j)$, given by

$$\theta_k = \pi_{[k]}^{(1)} * \left(\mathrm{Ad} U_k \circ \pi_{[k]}^{(2)}\right), \quad (13)$$

is irreducible.

(iii) F_k is a finite subset of the closed unit ball of $A_1 * A_2$ and for all y in the closed unit ball of $A_1 * A_2$ there is an element x in F_k such that

$$\|\theta_k(x) - \theta_k(y)\| < \frac{1}{2^{k+1}}. \quad (14)$$

(iv) If $k \geq 2$, then for any element x in the union $\cup_{j=1}^{k-1} F_j$, , we have

$$\|\theta_k(x) - (\theta_{k-1} \oplus \pi_k \oplus \hat{\pi}_k(x)\| < \frac{1}{2^{k+1}}. \quad (14)$$

We construct such a sequence by recursion.

Step 1: Construction of (u_1, θ_1, F_1). Since $\pi \quad \oplus \hat{\pi}$ is DPI, there is a unitary u_1 in $H_1 \oplus \widehat{H}_1$ such that $\|u_1 - \mathrm{id}_{H\oplus\widehat{H}}\| < \frac{\varepsilon}{2^2}$and $\pi^{(1)}_{[1]} * \mathrm{Ad}u_1 \circ \pi^{(2)}_{[1]}$is irreducible.Hence condition (11) and (13) trivially hold. Since $H_1 \oplus \widehat{H_1}$is finite dimensional, there is a finite set F_1 contained in the closed unit ball of $A_1 * A_2$ satisfying condition (14). At this stage there is no condition (15).

Step 2: Construction of $(u_{k+1}, \theta_{k+1}, F_{k+1})$ from $(u_j, \theta_j, F_j), 1 \le j \le k$.First, we are prove there exists a unitary u_{k+1} in $\mathbb{U}(\bigoplus_{j=1}^{k+1} H_j \oplus \widehat{H}_j)$ such that $\left\|u_{k+1} - \mathrm{id}_{\bigoplus_{j=1}^{k+1} H_j \oplus \widehat{H}_j}\right\| < \frac{\varepsilon}{2^{k+2}}$, the unital $*$-representation of $A_1 * A_2$ into $\mathbb{B}(\bigoplus_{j=1}^{k+1} H_j \oplus \widehat{H}_j)$ defined by

$$\theta_{k+1} \coloneqq (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1})^{(1)} * (\mathrm{Ad}\, u_{k+1}) \circ (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1})^{(2)} \quad (16)$$

is irreducible and for any element x in the union $\cup_{j=1}^{k} F_j$, the inequality $\|\theta_{k+1}(x) - (\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1}(x)\| < \frac{1}{2^{k+1}}$, holds , $\theta_k \oplus \pi_{k+1} \oplus \hat{\pi}_{k+1}$ is D.P.I so Proposition (1.1.40) assures the existence of suchunitary u_{k+1}. Notice that, from construction, conditions (11) and (15) are satisfied. A consequence of (13) and (12) is $\theta_{k+1} = \pi^{(1)}_{[k+1]} * \left(\mathrm{Ad}U_{k+1} \circ \pi^{(2)}_{[k+1]}\right)$,.

Finite dimensionality of $\bigoplus_{j=1}^{k+1} H_j \oplus \widehat{H}_j$ guarantees the existence of a finite set F_{k+1} contained in the closed unit ball of $A_1 * A_2$ satisfying condition (14). This completes Step 2.

Now consider the $*$-representations

$$\sigma_k = \theta_k \bigoplus_{j \ge k+1} \pi_j \oplus \hat{\pi}_j. \quad (17)$$

We now show there is a unital $*$-representation of $\sigma : A_1 * A_2 \to \mathbb{B}(H)$, such that for all x in $A_1 * A_2, \lim_k \|\sigma_k(x) - \sigma(x)\| = 0$. If we extend the unitariesu_k to all of H via $\tilde{u}_k = u_k \bigoplus_{j\ge k+1} \mathrm{id}_{H_j \oplus \widehat{H}_j}$, then we obtain

$$\sigma_k = \pi^{(1)} * \left(Ad\, \widetilde{U}_k \circ \pi^{(2)}\right), \quad (18)$$

Where $\widetilde{U}_k = \tilde{u}_k \dots \tilde{u}_1$. Thanks to condition (11), we have

$$\left\|\widetilde{U}_k - \mathrm{id}_H\right\| \le \sum_{j=1}^{k} \|\tilde{u}_k - \mathrm{id}_H\| < \sum_{j=1}^{k} \frac{\varepsilon}{2^{k+1}}$$

,

and for $l \ge 1$

$$\left\|\widetilde{U}_{k+l} - \widetilde{U}_k\right\| = \|\tilde{u}_{k+l} \dots \tilde{u}_{k+1} - \mathrm{id}_H\| \le \sum_{j=k+1}^{k+l} \frac{\varepsilon}{2^{j+1}}.$$

Hence, Cauchy's criterion implies there is a unitary u in $\mathbb{U}(H)$ such that the sequence $(\widetilde{U}_k)_{k\ge 1}$ converges in norm to u and $\|u - \mathrm{id}_H\| < \frac{\varepsilon}{2}$.

Define

$$\sigma = \pi^{(1)} * \left(Ad\, u \circ \pi^{(2)}\right). \quad (19)$$

From Proposition (1.1.40) we have that for all x in $A_1 * A_2$,

$$\lim_k \|\sigma_k(x) - \sigma(x)\| = 0. \tag{20}$$

Our next goal is to show σ is irreducible. To ease notation let $A = A_1 * A_2$. We will show $\overline{\sigma(A)}^{SOT} = \mathbb{B}(H)$. Take Tin $\mathbb{B}(H)$. With no loss of generality we may assume $\|T\| \leq \frac{1}{2}$. Recall that a neighborhood basis for the SOT topology around T is given by the sets

$$\mathcal{N}_T(\xi_1, \dots, \xi_n;\ \varepsilon) = \{S \in \mathbb{B}(H) : \|S\xi_i - T\xi_i\| < \varepsilon, i = 1, \dots, n\}$$

where $\varepsilon > 0, n \in \mathbb{N}$, and $\xi_1, \dots, \xi_n \in H$ are unit vectors. We show that for any $\varepsilon > 0$ and any unit vectors $\xi_1, \dots, \xi_n, \mathcal{N}_T(\xi_1, \dots, \xi_n;\ \varepsilon) \cap \sigma(A)$ is nonempty. Let P_k denote the orthogonal projection from H onto $\bigoplus_{j=1}^{k} H_j \oplus \widehat{H}_j$. Take $k_1 \geq 1$ such

$$\sum_{k \geq k_1} \frac{1}{2^k} < \frac{\varepsilon}{2^3}$$

and for $k \geq k_1, 1 \leq i \leq n$,

$$\|(\mathrm{id}_H - P_k)(\xi_i)\| < \frac{\varepsilon}{2^3}, \tag{21}$$

$$\|(\mathrm{id}_H - P_k)(T\xi_i)\| < \frac{\varepsilon}{2^3}, \tag{22}$$

Since P_k has finite rank and θ_k is irreducible, there is a in A, with $\|a\| \leq 1$ such that

$$P_{k_1} T\, P_{k_1}(\xi_i) = \theta_{k_1}(a)\left(P_{k_1}(\xi_i)\right) \tag{23}$$

for$i = 1, \dots, n$. We have

$$\theta_{k_1}(a)\left(P_{k_1}(\xi_i)\right) = \sigma_{k_1}(a)\left(P_{k_1}(\xi_i)\right). \tag{24}$$

Take x in F_{k_1} such that

$$\left\|\theta_{k_1}(a) - \theta_{k_1}(x)\right\| < \frac{1}{2^{k_1+1}}. \tag{25}$$

We will show $\sigma(x) \in \mathcal{N}_T(\xi_1, \dots, \xi_n;\ \varepsilon)$. To ease notation let $\xi_i = \xi$. From (21), (22), (23) and (24), we deduce

$$\begin{aligned}\|T\xi - \sigma(x)\xi\| &\leq \left\|T\xi - P_{k_1} TP_{k_1}\xi\right\| + \left\|P_{k_1} T\, P_{k_1}\xi - \sigma_{k_1}(a)\xi\right\| \\ &< \frac{3\varepsilon}{2^{\varepsilon}} + \left\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\right\| + \left\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\right\|.\end{aligned}$$

For any $p \geq 1$ we have

$$\begin{aligned}&\sigma_{k_1}(a)\xi - \sigma(x)\xi \\ &= \sigma_{k_1}(a)\xi - \sigma_{k_1}(x)\xi + \sum_{j=k_1}^{k_1+p}\left(\sigma_j(x)\xi - \sigma_{j+1}(x)\xi\right) + \sigma_{k_1+p+1}(x)\xi \\ &- \sigma(x)\xi.\end{aligned}$$

Thus, from (21), (24), (25), (17) and (15) we deduce

$$\left\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\right\| < \frac{\varepsilon}{2} + \left\|\sigma_{k_1+p+1}(x)\xi - \sigma(x)\xi\right\|$$

hence

$$\left\|\sigma_{k_1}(a)\xi - \sigma(x)\xi\right\| \leq \frac{\varepsilon}{2}$$

We conclude $\sigma(x)$ lies in $\mathcal{N}_T(\xi_1, \dots, \xi_n;\ \varepsilon)$.

An application of Choi's technique will give us faithfulness of σ. Indeed, from construction, for all x in $A, \sigma(x) = \lim_k \sigma_k(x)$. Thus if each σ_k is faithful then so is σ. But faithfulness of $\sigma_k$follows from the commutativity of the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi} & \mathbb{B}(H) \\ \pi \downarrow & & \downarrow \pi_C \\ \mathbb{B}(H) & \xrightarrow{\pi_C} & \mathbb{B}(H)/\mathbb{K}(H) \end{array}$$

(whereπ_C denotes the quotient map onto the Calkin algebra), which in turn is implied by (17).

To obtain the following corollary, see [2].

Corollary (1.1.42)[30]: Assume A_1 and A_2 are nontrivial residually finite dimensional C^*-algebraswith $(dim(A_1) - 1)(dim(A_2) - 1) \geq 2$. Then $A_1 * A_2$ is antiliminal and has an uncountable family of pairwise in-equivalent irreducible faithful $*$representations.

We finish with a corollary derived in [28].

Corollary (1.1.43)[30]: Assume A_1 and A_2 are nontrivial residually finite dimensional C^*-algebras with $(dim(A_1) - 1)(dim(A_2) - 1) \geq 2$. Then pure states of $A_1 * A_2$ are W^*-dense in the state space.

Section (1.2): Homomorphisms into Z-Stable C*-Algebra

Let X and Y be two compact Hausdorff spaces, and denote by $C(X)$ (or $C(Y)$) the C^*-algebra of complex-valued continuous functions on X (or Y). Any continuous map $\lambda: Y \to X$ induces a homomorphism ϕ from the commutative C^*-algebra $C(X)$ into the commutative C^*-algebra $C(Y)$ by $\phi(f) = f\lambda$, and any homomorphism from $C(X)$ to $C(Y)$ arises this way (by homomorphisms or isomorphisms between C^*-algebras, we mean $*$-homomorphismsn or $*$-isomorphisms). It should be noted that, by the Gelfand-Naimark theorem, every unital commutative C^*-algebra has the form $C(X)$ as above.

For non-commutative C^*-algebras, one also studies homomorphisms. Let A and B be two unital C^*-algebras and let $\phi, \psi : A \to B$ be two homomorphisms. A fundamental problem in the study of C^*-algebras is to determine when ϕ and ψ are (approximately) unitarily equivalent.

The last two decades saw the rapid development of classification of amenable C^*-algebras, or otherwise known the Elliott program. For instance, all unital simple AH-algebras with slow dimension growth are classified by their Elliott invariant ([36]). In fact, the class of classifiable simple C^*-algebras includes all unital separable amenable simple C^*-algebras with the tracial rank at most one which satisfy the Universal Coefficient Theorem (the UCT) (see [88]). One of the crucial problems in the Elliott program is the so-called uniqueness theorem which usually asserts that two monomorphisms are approximately unitarily equivalent if they induce the same K-theory related maps under certain assumptions on C^*-algebras involved.

Recently, W. Winter's method ([141]) greatly advances the Elliott classification program. The class of amenable separable simple C^*-algebras that can be classified by the Elliott invariant has been enlarged so that it contains simple C^*-algebras which no longer are assumed to have finite tracial rank. In fact, with [141], [86], [99] and [73], the classifiable C^*-algebras now include any unital separable simple Z-stable C^*-algebra A satisfying the UCT such that $A \otimes U$ has the tracial rank no more than one for some UHF-algebra U (it has recently been shown, for example, $A \otimes U$ has tracial rank at most one for all UHF-algebras U of infinite type, if $A \otimes C$ has tracial rank at most one for one of infinite dimensional unital simple AF-algebra (see [95])). This class of C^*-algebras is strictly larger than the class of AH-algebras without dimension growth. For example, it contains the Jiang-Su algebra Z itself which is projectionless and all simple unital inductive limits of so-called generalized dimension drop algebras (see [85]).

Recall that the Elliott invariant for a stably finite unital simple separable C^*-algebra A is

$$Ell(A) := \Big(\big(K_0(A), K_0(A)_+, [1_A], T(A)\big), K_1(A)\Big),$$

where $\big(K_0(A), K_0(A)_+, [1_A], T(A)\big)$ is the quadruple consisting of the K_0-group, its positive cone, the order unit and tracial simplex together with their pairing, and $K_1(A)$ is the K_1-group.

Denote by C the class of all unital simple C^*-algebras A for which $A \otimes U$ has tracial rank no more than one for some UHF-algebra U of infinite type. Suppose that A and B are two unital separable amenable C^*-algebras in C which satisfy the UCT. The classification theorem in [73] states that if the Elliott invariants of A and B are isomorphic, i.e.

$$Ell(A) \cong Ell(B),$$

then there is an isomorphism $\phi: A \to B$ which carries the isomorphism above.

However, the question when two isomorphisms are approximately unitarily equivalent was still left open. A more general question is: for any two such C^*-algebras A and B, and, for any two homomorphisms $\phi, \psi : A \to B$, when are they approximately unitarily equivalent?

If ϕ and ψ are approximately unitarily equivalent, then one must have,

$$[\phi] = [\psi] \; in \; KL(A, B) \;\; and \;\; \phi_\# = \psi_\#,$$

where $\phi_\#, \psi_\#$: Aff$(T(A)) \to$ Aff$(T(B))$ are the affine maps induced by ϕ and ψ, respectively. Moreover, as shown in [71], one also has

$$\phi^\ddagger = \psi^\ddagger,$$

where $\phi^\ddagger, \psi^\ddagger : U(A)/CU(A) \to U(B)/CU(B)$ are homomorphisms induced by ϕ, ψ, and $CU(A)$ and $CU(B)$ are the closures of the commutator subgroups of the unitary groups of A and B, respectively.

We will show that the above conditions are also sufficient, that is, the maps ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$, $\phi_\# = \psi_\#$ and $\phi^\ddagger = \psi^\ddagger$.

The proof of this uniqueness theorem is based on the methods developed in the proof of the classification result mentioned above, which can be found in [73], [82], [71], [99] and [74]. Most technical tools are developed in this research, either directly or implicitly. We will collect them and then assemble them into production.

In [103], it is shown that, for any partially ordered simple weakly unperforated rationally Riesz group G_0 with order unit u, any countable abelian group G_1, any metrizable Choquet simple S, and any surjective affine continuous map r : S ⟶ $Su(G_1)$ (the state space of G_0) which preserves extremal points, there exists one (and only one up to isomorphism) unital separable simple amenable C^*-algebra $A \in C$ which satisfies the UCT so that $Ell(A) = (G_0, (G_0)_+, u, G_1, S, r)$.

Then a natural question is: Given two unital separable simple amenable C^*-algebras $A, B \in C$ which satisfy the UCT, and a homomorphism Γ from $Ell(A)$ to $Ell(B)$, does there exist a unital homomorphism $\phi : A \to B$ which induces Γ? We will give an answer to this question. Related to the uniqueness theorem discussed earlier and also related to the question above, one may also ask the following: Given an element $\kappa \in KL(A,B)$ which preserves the unit and order, an affine map $\lambda : Aff(T(A)) \to Aff(T(B))$ and a homomorphism $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ which are compatible, does there exist a unital homomorphism $\phi : A \to B$ so that $[\varphi] = \kappa, \phi_{\#} = \lambda$ and $\phi^{\ddagger} = \gamma$? We will, at least, partially answer this question.

Let A be a unital stably finite C^*-algebra. Denote by $T(A)$ the simplex of tracial states of A and denote by $Aff(T(A))$ the space of all real affine continuous functions on $T(A)$. Suppose that $\tau \in T(A)$ is a tracial state. We will also denote by τ the trace $\tau \otimes Tr$ on $M_k(A) = A \otimes M_k(\mathbb{C})$ (for every integer $k \geq 1$), where Tr is the standard trace on $M_k(\mathbb{C})$. A trace τ is faithful if $\tau(a) > 0$ for any $a \in A_+ \setminus \{0\}$. Denote by $T_f(A)$ the convex subset of $T(A)$ consisting of all faithful tracial states.

Denote by $M_\infty(A)$ the set $\bigcup_{k=1}^{\infty} M_k(A)$, where Mk(A) is regarded as a C^*-subalgebra of $M_{k+1}(A)$ by the embedding $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. For any projection $p \in M_\infty(A)$, the restriction $\tau \mapsto \tau(p)$ defines a positive affine function on $T(A)$. This induces a canonical positive homomorphism $\rho_A: K_0(A) \to Aff(T(A))$.

Denote by $U(A)$ the unitary group of A, and denote by $U(A)_+$ the connected component of $U(A)$ containing the identity. Let C be another unital C^*-algebra and let $\phi: C \to A$ be a unital $*$-homomorphism. Denote by $\phi_T: T(A) \to T(C)$ the continuous affine map induced by ϕ, i.e.,

$$\phi_T(\tau)(c) = \tau \circ \phi(c)$$

for all $c \in C$ and $\tau \in T(A)$. Denote by $\phi_{\#}: Aff(T(C)) \to Aff(T(A))$ the map defined by

$$\phi_{\#}(f)(\tau) = f\big(\phi_T(\tau)\big) \quad \text{for all } \tau \in T(A).$$

Definition (1.2.1)[98]:

Let A be a unital C^*-algebra. Denote by $CU(A)$ the closure of the subgroup generated by commutators of $U(A)$. If $u \in U(A)$, its image in the quotient $U(A)/CU(A)$ will be denoted by u. Let B be another unital C^*-algebra and let $\phi : A \to B$ be a unital homomorphism. it is clear that ϕ maps $CU(A)$ into $CU(B)$. Let $\phi^{\ddagger}$ denote the induced homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$.

Let $n \geq 1$ be any integer. Denote by $U_n(A)$ the unitary group of $M_n(A)$, and denote by $CU(A)_n$ the closure of commutator subgroup of $U_n(A)$. Regard $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the embedding $u \longmapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and denote by $U_\infty(A)$ the union of all $U_n(A)$. Consider the union $CU_\infty(A) := \bigcup_n C\,U_n(A)$. It is then a normal subgroup of $U_\infty(A)$, and the quotient $U(A)_\infty/CU_\infty(A)$ is in fact isomorphic to the inductive limit of $U_n(A)/CU_n(A)$ (as abelian groups). We will use $\phi^{\ddagger}$ for the homomorphism induced by ϕ from $U_\infty(A)/CU_\infty(A)$ into $U_\infty(B)/CU_\infty(B)$.

Definition (1.2.2)[98]:

Let A be a unital C^*-algebra, and let $u \in U(A)_0$. Let $u(t) \in C([0,1],A)$ be a piecewise-smooth path of unitaries such that $u(0) = u$ and $u(1) = 1$. Then the de la Harpe–Skandalis determinant of $u(t)$ is defined by

$$Det\,(u(t))(\tau) = \frac{1}{2\pi i}\int_0^1 \tau\left(\frac{du(t)}{dt}u(t)^*\right)dt \quad for\ all\ \tau \in T(A),$$

which induces a homomorphism

$$Det: U(A)_0 \to Aff\,(T(A))/\overline{\rho_A\big(K_0(A)\big)}.$$

The determinant Det can be extended to a map from $U_\infty(A)_0$ into $Aff(T(A))/\rho_A(K_0(A))$. It is easy to see that the determinant vanishes on the closure of commutator subgroup of $U_\infty(A)$. In fact, by a result of K. Thomsen ([133]), the closure of the commutator subgroup is exactly the kernel of this map, that is, it induces an isomorphism $Det: U_\infty(A)_0/CU_\infty(A) \to Aff(T(A))/\overline{\rho_A(K_0(A))}$. Moreover, by ([133]), one has the following short exact sequence

$$0 \to Aff(T(A))/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \xrightarrow{\Pi} K_1(A) \to 0 \qquad (26)$$

which splits (with the embedding of $Aff(T(A))/\overline{\rho_A(K_0(A))}$ induced by $(\overline{Det})^{-1}$). We will fix a splitting map $s_1: K_1(A) \to U_\infty(A)/CU_\infty(A)$. The notation Π and s_1 will be used late without further warning.

For each $\bar{u} \in s_1(K_1(A))$, select and fix one element $u_c \in \bigcup_{n=1}^{\infty} M_n(A)$ such that $u_c = \bar{u}$. Denote this set by $U_c(A)$.

In the case that A has tracial rank at most one .

$$U_\infty(A)_0/CU_\infty(A) = U(A)_0/CU(A)$$

and thus the following splitting short exact sequence:

$$0 \to Aff(T(A))/\overline{\rho_A(K_0(A))} \to U(A)/CU(A) \to K_1(A) \to 0. \qquad (27)$$

Definition (1.2.3)[98]:

Let A be a unital C^*-algebra and let C be a separable C^*-algebra which satisfies the Universal Coefficient Theorem. Recall that $KL(C,A)$ is the quotient of $K(C,A)$ modulo pure extensions. By a result of D˘ad˘arlat and Loring in [82], one has

$$KL(C,A) = Hom_A\left(\underline{K}(C), \underline{K}(A)\right), \qquad (28)$$

where

$$\underline{K}(B) = \left(K_0(B, K_1(B))\right) \oplus \bigoplus_{n=2}^{\infty} \left(\mathrm{K}_0(\mathrm{B}, \mathbb{Z}/\mathrm{n}\mathbb{Z})\right) \oplus K_1(B, K_1(B))$$

for any C^*-algebra B. Then, we will identify $KL(C,A)$ with $Hom_A\left(\underline{K}(C), \underline{K}(A)\right)$. Denote by $\kappa_i: K_i(C) \to K_i(A)$ the homomorphism given by κ with $i = 0, 1$, and denote by $KL(C,A)^{++}$ the set of those $\kappa \in Hom_A\left(\underline{K}(C), \underline{K}(A)\right)$ such that

$$k_0(K_0^+(C)\{0\}) \subseteq K_0^+(A)\backslash\{0\}.$$

Denote by $KL_e(C,A)^{++}$ the set of those elements $\kappa \in KL(C,A)^{++}$ such that $\kappa_0([1_C]) = [1_A]$. Suppose that both A and C are unital, $T(C) \neq \emptyset$ and $T(A) \neq \emptyset$. Let $\lambda_T: T(A) \to T(C)$ be a continuous affine map. Let $h_0: K_0(C) \to K_0(A)$ be a positive homomorphism. We say λ_T is compatible with h_0 if for any projection $p \in M_\infty(C), \lambda_T(\tau)(p) = \tau(h_0([p]))$ for all $\tau \in T(A)$. Let $\lambda: Aff(T_f(C)) \to Aff(T(A))$ be an affine continuous map. We say λ and h_0 are compatible if h_0 is compatible to λ_T, where $\lambda_T: T(A) \to T_f(C)$ is the map $\lambda_T(\tau)(a) = \lambda(a^*)(\tau)$, $\forall a \in C^+$ and $\tau \in T(A)$, where $a^* \in Aff(T_f(C))$ is the affine function induced by a. We say κ and λ (or λ_T) are compatible, if κ is positive and κ_0 and λ are compatible.

Denote by $KLT_e(C,A)^{++}$ the set of those pairs (κ, λ_T) (or, (κ, λ)), where $\kappa \in KL_e(C,A)^{++}$ and $\lambda_T: T(A) \to T_f(C)$ (or, $\lambda: Aff(T_f(C)) \to Aff(T(A))$) is a continuous affine map which is compatible with κ. If λ is compatible with κ, then λ maps $\rho_C(K_0(C))$ $into$ $\rho_A(K_0(A))$. Therefore λ induces a continuous homomorphism $\bar{\lambda}: Aff(T_f(C))/\overline{\rho_C(K0(C))} \to Aff(T(A))/\overline{\rho_A(K0(A))}$. Suppose that $\gamma: U_\infty(C)/CU_\infty(C) \to U_\infty(A)/CU_\infty(A)$ is a continuous homomorphism and $h_i: K_i(C) \to K_i(A)$ are homomorphisms for which h_0 is positive. We say that γ and h_1 are compatible if $\gamma(U_\infty(C)_0/CU_\infty(C)) \subset v(A)_0/CU_\infty(A)$ and $\gamma \circ s_1 = s_1 \circ h_1$, we say that h_0, h_1, λ and γ are compatible, if λ and h_1 are compatible, γ and h_1 are compatible and

$$\overline{Det_A} \circ \gamma|_{U_\infty(C)_0/CU_\infty(C)} = \bar{\lambda} \circ \overline{Det_C},$$

and we also say that κ, λ and γ are compatible, if $\kappa_0, \kappa_1, \lambda$ and γ are compatible.

For each prime number p, let ϵ_p be a number in $\{0, 1, 2, \ldots, +\infty\}$. Then a supernatural number is the formal product $p = \prod_p p^{\epsilon_p}$. Here we insist that there are either infinitely many p in the product, or, one of ϵ_p is infinite. Two supernatural numbers $p = \prod_p p^{\epsilon_p(p)}$ and $q = \prod_p p^{\epsilon_p(q)}$ are relatively prime if for any prime number p, at most one of $\epsilon_p(p)$ and $\epsilon_p(q)$ is nonzero. A supernatural number p is called of infinite type if for any prime number, either $\epsilon_p(p) = 0$ or $\epsilon_p(p) = +\infty$. For each supernatural number p, there is a UHF-algebra M_p associated to it, and the UHF-algebra is unique up to isomorphism (see [124]).

Denote by Q the UHF-algebra with $(K_0(Q), K_0(Q)_+, [1_A]) = (\mathbb{Q}, \mathbb{Q}_+, 1)$ (the supernatural number associated to Q is $\prod_p p^{+\infty}$), and let M_p and M_q be two UHF-algebras with $M_p \otimes M_p \cong Q$ and $p = \prod_p p^{\epsilon_p(p)}$ and $q = \prod_p p^{\epsilon_p(q)}$ relatively prime. Then it follows that p and q are of infinite type. Denote by

$$\mathbb{Q}_p = \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \ldots\right] \subseteq \mathbb{Q},\ \ where\ \epsilon_{p_n}(p) = +\infty\ \ and$$

$$\mathbb{Q}_q = \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_n}, \ldots\right] \subseteq \mathbb{Q},\ \ where\ \epsilon_{p_n}(q) = +\infty.$$

Note that $(K_0(M_p), K_0(M_p)_+, [1_{M_p}]) = (\mathbb{Q}_p, (\mathbb{Q}_p)_+, 1)$ and $(K_0(M_q), K_0(M_q)_+, [1_{M_p}]) = (\mathbb{Q}_q, (\mathbb{Q}_q)_+, 1)$. Moreover, $\mathbb{Q}_p \cap \mathbb{Q}_q = \mathbb{Z}\ and\ \mathbb{Q} = \mathbb{Q}_q + \mathbb{Q}_q$

For any pair of relatively prime supernatural numbers p and q, define the C^*-algebra $\mathcal{Z}_{p,q}$ by

$$\mathcal{Z}_{p,q} = \left\{f: [0,1] \to M_p \otimes M_q; f(0) \in M_p \otimes 1_{M_q}\ \ and\ \ f(1) \in 1_{M_q} \otimes M_q\right\}.$$

The Jiang-Su algebra $\mathcal{Z}$ is the unital inductive limit of dimension drop interval algebras with unique trace, and $(K_0(\mathcal{Z}), K_0(\mathcal{Z}), [82]) = (\mathbb{Z}, \mathbb{Z}^+, 1)$ (see [55]). For any pair of relatively prime supernatural numbers p and q of infinite type, the Jiang-Su algebra $\mathcal{Z}$ has a stationary inductive limit decomposition:

$$\mathcal{Z}_{p,q} \to \mathcal{Z}_{p,q} \to \cdots \to \mathcal{Z}_{p,q} \to \cdots \to \mathcal{Z}.$$

The C^*-algebra $\mathcal{Z}_{p,q}$ absorbs the Jiang-Su algebra: $\mathcal{Z}_{p,q} \otimes \mathcal{Z} \cong \mathcal{Z}_{p,q}$. A C^*-algebra A is said to be $\mathcal{Z}$ -stable if $A \otimes \mathcal{Z} \cong A$.

Definition (1.2.4)[98]:

A unital simple C^*-algebra A has tracial rank at most one, denoted by $TR(A) \leq 1$, if for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, and any nonzero $a \in A^+$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra $I \cong \bigoplus_{i=1}^m C(X_i) \otimes M_{r(i)}$ with $1_I = p$ for some finite CW complexes X_i with dimension at most one such that

(i) $\|[x, p]\| \leq \epsilon\ \ for\ any\ x \in \mathcal{F}$,

(ii) for any $x \in \mathcal{F}$, there is $x' \in I$ such that $\|pxp - x'\| \leq \epsilon$, and

(iii) $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Moreover, if the C^*-subalgebra I above can be chosen to be a finite dimensional C^*-algebra, then A is said to have tracial rank zero, and in such case, we write $TR(A) = 0$. It is a theorem of Guihua Gong [51] that every unital simple AH-algebra with no dimension growth has tracial rank at most one. It has been proved in [73] that every $\mathcal{Z}$ -stable unital simple AH-algebra has tracial rank at most one.

Definition (1.2.5)[98]:

Denote by $\mathcal{N}$ the class of all separable amenable C^*-algebras which satisfy the Universal Coefficient Theorem (UCT). Denote by C the class of all simple C^*-algebras A for which $TR(A \otimes M_p) \leq 1$ for some UHF-algebra M_p, where p is a supernatural number of infinite type. Note, by [103], that, if $TR(A \otimes M_p) \leq 1$ for some supernatural number p then $TR(A \otimes M_p) \leq 1$ for all supernatural number p.

Denote by C_0 the class of all simple C^*-algebras A for which $TR(A \otimes M_p) = 0$ for some supernatural number p of infinite type (and hence for all supernatural number p of infinite type).

Theorem (1.2.6)[98]:

Let C be a unital AH-algebra and let A be a unital simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\phi, \psi: C \to A$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if

$$[\phi] = [\psi] \ \ in\ KL(C, A),$$
$$\phi_{\#} = \psi_{\#} \ \ and\ \ \phi^{\ddagger} = \psi^{\ddagger}.$$

Let A and B be two unital C^*-algebras. Let $h: A \to B$ be a homomorphism and $v \in U(B)$ be such that

$$[h(g), v] = 0 \ \ for\ any\ g \in A.$$

We then have a homomorphism $\bar{h}: A \otimes C(\mathbb{T}) \to B$ defined by $f \otimes g \mapsto h(f)g(v)$ for any $f \in A$ and $g \in C(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)}: K_0(A) \to K_1(A \otimes C(\mathbb{T})) \ \ and\ \ \beta^{(1)}: K_1(A) \to K_0(A \otimes C(\mathbb{T})).$$

The second one is the usual Bott map. Note that, in this way, one writes

$$K_i(A \otimes C(\mathbb{T})) = K_i(A) \oplus \beta^{(i-1)}(K_{i-1}(A)).$$

Let us use $\hat{\beta}^{(i)}: K_i(A \otimes C(\mathbb{T})) \to \beta^{(i-1)}(K_{i-1}(A))$ to denote the quotient map.

For each integer $k \geq 2$, one also has the following injective homomorphisms:

$$\beta_k^{(i)}: K_i(A, k\mathbb{Z}) \to K_{i-1}(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), \qquad i = 0,1.$$

Thus, we write

$$K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) = K_i(A, \mathbb{Z}/k\mathbb{Z}) \otimes \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z}).$$

Denote by $\hat{\beta}_k^{(i)}: K_i\left(A \otimes C(\mathbb{T}), \frac{\mathbb{Z}}{k\mathbb{Z}}\right) \to \beta^{(i-1)}\,(K_{i-1}(A),\ \mathbb{Z}/k\mathbb{Z})$ the map analogous to $\hat{\beta}^{(i)}$. If $x \in \underline{K}(A)$, we use $\beta(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\beta : K(A) \to K(A \otimes C(\mathbb{T}))$ as well as $\hat{\beta}: \underline{K}(A \otimes C(\mathbb{T})) \to \beta(\underline{K})$. Therefore, we may write $\underline{K}(A \otimes C(\mathbb{T})) = \underline{K}(A) \oplus \beta(\underline{K}(A))$. On the other hand, $\bar{h}$ induces homomorphisms

$$\bar{h}_{*i,k}: K_i(A \otimes C(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}),$$

$k = 0, 2, \ldots, and\ i = 0, 1.$

We use $Bott(h, v)$ for all homomorphisms $\bar{h}_{*i,k} \circ \beta_k^{(i)}$, and we use $bott_1(h, v)$ for the homomorphism $\bar{h}_{1,0} \circ \beta^{(1)}: K_1(A) \to K_0(B)$, and $bott_0(h, v)$ for the homomorphism $h_{0,0}\, \beta^{(0)}: K_0(A) \to K_1(B)$. $Bott(h, v)$ as well as $bott_i(h, v)(i = 0, 1)$ may be defined for a unitary v which only approximately commutes with h. In fact, given a finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$Bott\ (h, v)|_{\mathcal{P}}$$

is well defined if

$$\|[h(a), v]\| < \delta_0$$

for all $a \in \mathcal{F}$.

We have the following generalized Exel's formula for the traces of Bott elements.

Theorem (1.2.7)[98]:

There is $\delta > 0$ satisfying the following: Let A be a unital separable simple C^*-algebra with $TR(A) \leq 1$ and let $u, v \in U(A)$ be two unitaries such that $\|uv - vu\| < \delta$. Then $bott_1(u, v)$ is well defined and

$$\tau\big(bott_1(u, v)\big) = \frac{1}{2\pi i}\big(\tau(\log(vuv^*u^*))\big)$$

for all $\tau \in T(A)$.

we collect several facts on the rotation map which are going to be used frequently in this essay. Most of them can be found in the literature.

Definition (1.2.8)[98]:

Let A and B be two unital C^*-algebras, and let ψ and ϕ be two unital monomorphisms from B to A. Then the mapping torus $M_{\phi,\psi}$ is the C^*-algebra defined by

$$M_{\phi,\psi} := \{f \in C([0,1]); f(0) = \phi(b)\ \ and\ \ f(1) = \psi(b)\ \ for\ some\ b \in B\}.$$

For any $\psi, \phi \in Hom(B, A)$, denoting by π_0 the evaluation of $M_{\phi,\psi}$ at 0, we have the short exact sequence

$$0 \longrightarrow S(A) \overset{i}{\to} M_{\phi,\psi} \overset{\pi_0}{\to} B \longrightarrow 0,$$

where $S(A) = C_0((0,1), A)$. If $\phi_{*i} = \psi_{*i} (i = 0,1)$, then the corresponding six-term exact sequence breaks down to the following two extensions:

$$\eta_i\big(M_{\phi,\psi}\big): 0 \longrightarrow K_{i+1}(A) \longrightarrow K_i\big(M_{\phi,\psi}\big) \longrightarrow K_i(B) \longrightarrow 0, \qquad (i = 0,1).$$

Suppose that, in addition,

$$\tau \circ \phi = \tau \circ \psi\ \ for\ all\ \ \tau \in T(A). \tag{29}$$

For any continuous piecewise smooth path of unitaries $u(t) \in M_{\phi,\psi}$, consider the path of unitaries $w(t) = u^*(0)u(t)$ in A. Then it is a continuous and piecewise smooth path with $w(0) = 1$ and $w(1) = u^*(0)u(1)$. Denote by $R_{\phi,\psi}(u) = Det(w)$ the determinant of $w(t)$. It is clear with the assumption that $R_{\phi,\psi}(u)$ depends only on the homotopy class of $u(t)$. Therefore, it induces a homomorphism, denoted by $R_{\phi,\psi}$, from $K_1(M_{\phi,\psi})$ to $Aff(T(A))$.

Definition (1.2.9)[98]:

Fix two unital C^*-algebras A and B with $T(A) \neq \emptyset$. Define $\mathcal{R}_0$ to be the subset of $Hom(K_1(B), Aff(T(A)))$ consisting of those homomorphisms $h \in Hom(K_1(B), Aff(T(A)))$ for which there exists a homomorphism $d: K_1(B) \to K_0(A)$ such that

$$h = \rho_A \circ d.$$

It is clear that R0 is a subgroup of $Hom(K_1(B), Aff(T(A)))$.

If $[\phi] = [\psi]$ in $KK(B, A)$, then the exact sequences $\eta_i(M_{\phi,\psi})$ $(i = 0,1)$ above split. In particular, there is a lifting $\theta: K_1(B) \to K_1(M_{\phi,\psi})$. Consider the map

$$R_{\phi,\psi} \circ \theta: K_1(B) \to Aff\big(T(A)\big).$$

If a different lifting θ' is chosen, then, $\theta - \theta'$ maps $K_1(B)$ into $K_0(A)$. Therefore

$$R_{\phi,\psi} \circ \theta - R_{\phi,\psi} \circ \theta' \in \mathcal{R}_0.$$

Then define

$$\bar{R}_{\phi,\psi} = [R_{\phi,\psi} \circ \theta] \in Hom(K_1(B), Aff(T(A)))/\mathcal{R}_0.$$

If $[\phi] = [\psi]$ in $KL(B,A)$, then the exact sequences $\eta_i(M_{\phi,\psi})(i = 0,1)$ are pure, i.e., any finitely generated subgroup in the quotient groups has a lifting. In particular, for any finitely generated subgroup $G \subseteq K_1(B)$, one has a map

$$R_{\phi,\psi} \circ \theta_G: G \to Aff\big(T(A)\big),$$

where $\theta_G: G \to K_1(M_{\phi,\psi})$ is a lifting. Let $G \subset K_1(B)$ be a finitely generated subgroup. Denote by $\mathcal{R}_{0,G}$ the set of those elements h in $Hom(G, Aff(T\ (A)))$ such that there exists a homomorphism $d_G: G \to K_0(A)$ such that $h|_G = \rho_A \circ d_G$.

If $[\phi] = [\psi]$ in $KL(B,A)$ and $R_{\phi,\psi}(K_1(M_{\phi,\psi})) \subset \rho_A(K_0(A))$, then $\theta_G \in \mathcal{R}_{0,G}$ for any finitely generated subgroup $G \subset K_1(B)$ and any lifting θ_G. In this case, we will also write

$$\bar{R}_{\phi,\psi} = 0.$$

Lemma (1.2.10)[98]:

Let C and A be unital C^*-algebras with $T(A) \neq \emptyset$. Suppose that $\phi, \psi: C \to A$ are two unital homomorphisms such that

$$[\phi] = [\psi]\ \ in\ KL(C,A),\ \ \phi_{\#} = \psi_{\#},\ \ and\ \phi^{\ddagger} = \psi^{\ddagger}.$$

Then the image of $R_{\phi,\psi}$ is in the $\overline{\rho_A(K_0(A))} \subseteq Aff(T(A))$.

Proof:

Let $z \in K_1(C)$. Suppose that $u \in U_n(C)$ for some integer $n \geq 1$ such that $[u] = z$. Note that $\psi(u)^*\phi(u) \in CU_n(A)$. Thus, by (28), for any continuous and piecewise smooth path of unitaries $\{w(t): t \in [0, 1]\} \subset U(A)\ with\ w(0) = \psi(u)^*\phi(u)\ and\ w(1) = 1$,

$$Det(w)(\tau) = \int_0^1 \tau\left(\frac{dw(t)}{dt} w(t)^*\right) dt \in \overline{\rho_A(K_0(A))}. \tag{30}$$

Suppose that $\{(v)(t): t \in [0,1]\}$ is a continuous and piecewise smooth path of unitaries in $U_n(A)$ with $v(0) = \phi(u)$ and $v(1) = \psi(u)$. Define $w(t) = \psi(u)^*v(t)$. Then $w(0) = \psi^*(u)\phi(u)$ and $w(1) = 1$. Thus, by (3),

$$R_{\phi,\psi}(z)(\tau) = \int_0^1 \tau\left(\frac{dv(t)}{dt} v(t)^*\right) dt \tag{31}$$

$$= \int_0^1 \tau\left(\psi(u)^* \frac{dv(t)}{dt} v(t)^* \psi(u)\right) dt \tag{32}$$

$$= \int_0^1 \tau\left(\frac{dw(t)}{dt} w(t)^*\right) dt \in \overline{\rho_A(K_0(A))}. \tag{33}$$

Let A be a unital C^*-algebra and let u and v be two unitaries with $\|u^*v - 1\| < 2$. Then $h = \frac{1}{2\pi i}\ log(u^*v)$ is a well-defined self-adjoint element of A, and $w(t) := uexp(2\pi iht)$ is a smooth path of unitaries connecting u and v. It is a straightforward calculation that for any $\tau\ \in\ T(A)$,

$$Det(w(t))(\tau) = \frac{1}{2\pi i} \tau\ (log(u^*v)).$$

Let A be a unital C^*-algebra, and let u and w be two unitaries. Suppose that $w \in U_0(A)$. Then $w = \prod_{k=0}^{m} exp\ (2\pi i h_k)$ for some self-adjoint elements $h_0, \ldots, h_m$. Define the path

$$w(t)=\left(\prod_{k=0}^{l-1} exp\,(2\pi i h_k)\right)\exp(2\pi i h_l m t),\ \ if\ t\in[(l-1)/m, l/m],$$

and define $u(t)=w^*(t)uw(t)$ for $t\in[0,1]$. Then, $u(t)$ is continuous and piecewise smooth, and $u(0) = u$ and $u(1) = w*uw$. A straightforward calculation shows that $Det\big(u(t)\big)=0$.

In general, if w is not in the path-connected component containing the identity, one can consider unitaries $diag(u,1)$ and $diag(w,w^*)$. Then, the same argument as above shows that there is a piecewise smooth path $u(t)$ of unitaries in $M_2(A)$ such that $u(0) = diag(u,1), u(1) = diag(w^*uw,1)$, and

$$Det\big(u(t)\big)=0.$$

Lemma (1.2.11)[98]:

Let B and C be two unital C^*-algebras with $T(B)\neq\emptyset$. Suppose that $\phi,\psi: C\to B$ are two unital monomorphisms such that $[\phi] = [\psi]$ in $KL(C,B)$ and

$$\tau\circ\phi=\tau\circ\psi$$

for all $\tau\in T(B)$. Suppose that $u\in Ul(C)$ is a unitary and $w\in Ul(B)$ such that

$$\left\|\big(\phi\otimes id_{M_l}\big)(u)w^*\big(\psi\otimes id_{M_l}\big)(u^*)w-1\right\|<2.$$

Then, for any unitary $U\in U_l(M_{\phi,\psi})$ with $U(0)=(\phi\otimes id_{M_l})(u)\ and\ U(1)=(\psi\otimes id_{M_l})(u)$, one has that

$$\frac{1}{2\pi i}\tau\left(\log\left(\big(\phi\otimes id_{M_l}\big)(u^*)w^*\big(\psi\otimes id_{M_l}\big)(u)w\right)\right)-R_{\phi,\psi}([U])(\tau)\in\rho_B\big(K_0(B)\big). \quad (34)$$

Proof:

Without loss of generality, one may assume that $u\in C$. Moreover, to prove the lemma, it is enough to show that (34) holds for one path of unitaries $U(t)$ in $M_2(B)$ with $U(0)=diag(\phi(u),1)$ and $U(1)=\ diag(\psi(u),1)$.

Let U_1 be the path of unitaries specified with $U_1(0)=diag(\phi(u),1)$ and $U_1(1/2) = diag(w^*\psi(u)w,1)$, and let U_2 be the path specified with $U_2(1/2) = diag(w^*\psi(u)w,1)$ and $U_2(1)=diag(\psi(u),1)$.

Set U the path of unitaries by connecting U_1 and U_2. Then $U(0)=diag(\phi(u),1)$ and $U(1) = diag(\psi(u),1)$, for any $\tau\in T(B)$, one computes that

$$R_{\phi,\psi}([U])=Det(U(t))(\tau)=Det(U_1(t))(\tau)+Det(U_2(t))(\tau)$$
$$=\frac{1}{2\pi i}\tau(\phi(u^*)w^*\psi(u)w),$$

as desired.

Definition (1.2.12)[98]:

Let A be a unital C^*-algebra. In the following, for any invertible element $x\in A$, let $\langle x\rangle$ denote the unitary $x(x^*x)^{-\frac{1}{2}}$, and let $\bar{x}$ denote the element $\langle\bar{x}\rangle$ in $U(A)/CU(A)$. Consider a subgroup $\mathbb{Z}^k\subseteq K_1(A)$, and write the unitary $\{u_1,\dots,u_k\}\subseteq U_c(A)$ the unitary corresponding to the standard generators $\{e_1,e_2,\dots,e_k\}$ of $\mathbb{Z}^k$. Suppose that $\{u_1,u_2,\dots,u_k\}\subset M_n(A)$ for some integer $n\geq 1$. Let $\Phi: A\to B$ be a unital positive linear map and $\Phi\otimes id_{M_n}$ is at least $\{u_1,u_2,\dots,u_k\}-1/4$-multiplicative (hence each $\Phi\otimes id_{M_n}(u_i)$ is invertible), then the map $\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)}:\mathbb{Z}^k\to\ U(B)/CU(B)$ is defined by

$$\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)}(e_i)=\overline{\langle\Phi\otimes id_{M_n}(u_i)\rangle},\ \ 1\leq i\leq k.$$

Thus, for any finitely generated subgroup $G \subset U_c(A)$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, for any unital $\delta - \mathcal{G}$-multiplicative completely positive linear map $L: A \to B$ (for any unital C^*-algebra B), the map $L^{\ddagger}$ is well defined on $s_1(G)$. (Please see 2.1 for $U_c(A)$ and s_1.)

The following theorems are taken from [97].

Theorem (1.2.13)[98]**:**

Let $= PM_n(C(X))P$, where X is a compact subset of a finite CW-complex and P a projection in $M_n(C(X))$ with an integer $n \geq 1$. Let Δ: (0,1) → (0,1) be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$, $\eta > 0$, $\gamma > 0$, a finite subsets $\mathcal{G} \subseteq C$, $\mathcal{P} \subseteq \underline{K}(C)$, a finite subset $Q = \{x_1, x_2, \ldots, x_k\} \subset K_0(C)$ which generates a free subgroup and $x_i = [\mathcal{P}_i] - [q_i]$, where $p_i, q_i \in M_m(C)$ (for some integer $m \geq 1$) are projections, satisfying the following:

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi: C \to A$ is a unital homomorphism and $u \in A$ is a unitary, and suppose that

$$\|[\phi(c), u]\| < \delta, \ \ \forall c \in \mathcal{G} \ and \ \ Bott(\phi, u)|_{\mathcal{P}} = 0,$$

and

$$\mu_{\tau \circ \phi}(O_a) \geq \Delta(a) \ \ \forall \tau \in T(A \otimes D),$$

where O_a is any open ball in X with radius $\eta \leq a < 1$ and $\mu_{\tau \circ \phi}$ is the Borel probability measure defined by $\tau \circ \phi$. Moreover, for each $1 \leq i \leq k$, there is $v_i \in CU(M_m(A))$ such that

$$\|\langle (1_m - \phi(p_i) + \phi(p_i)u)(1_m - \phi(q_i) + \phi(q_i)u^*) \rangle - v_i\| < \gamma.$$

Then there is a continuous path of unitaries $\{u(t) : t \in [0,1]\}$ in A such that

$$u(0) = u, u(1) = 1, \ \ and \ \ \|[\phi(c), u(t)]\| < \epsilon$$

for any $c \in \mathcal{F}$ and for any $t \in [0,1]$.

Theorem (1.2.14)[98]:

Let $C = PM_n(C(X))P$, where X is a compact subset of a finite CW-complex and P a projection in $M_n(C(X))$ for some integer $n \geq 1$. Let $G \subset K_0(C)$ be a finitely generated subgroup. Write $G = \mathbb{Z}^k \oplus Tor(G)$ with $\mathbb{Z}^k$ generated by

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], \ldots, x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_m(C)$ (for some integer $m \geq 1$) are projections, $i = 1, \ldots, k$.

Let A be a simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\phi: C \to A$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subseteq C$ and $P \subseteq \underline{K}(C)$, any $\epsilon > 0$ and $\gamma > 0$, any homomorphism

$$\Gamma: \mathbb{Z}^k \to U_0(A)/CU(A),$$

there is a unitary $w \in A$ such that

$$\|[\phi(c), w]\| < \epsilon \ \ \forall f \in \mathcal{F}$$
$$Bott(\phi, w)|_p = 0,$$

and

$$dist\left(\overline{\langle (1_m - \phi(p_i) + \phi(p_i)w)(1_m - \phi(q_i) + \phi(q_i)w^*) \rangle}, \Gamma(x_i)\right) < \gamma, \forall 1 \leq i \leq k,$$

where $U_0(A)/CU(A)$ is identified as $U_0(M_m(A))/CU(M_m(A))$, and the distance above is understood as the distance in $U_0(M_m(A))/CU(M_m(A))$.

Lemma (1.2.15)[98]:

Let A be a simple C^*-algebra with $TR(A) \leq 1$, and let C be a unital AH-algebra. If there are monomorphisms $\phi, \psi: C \to A$ such that

$$[\phi] = [\psi] \; in \; KL(C, A), \quad \phi_{\#} = \psi_{\#}, \; and \; \phi^{\ddagger} = \psi^{\ddagger},$$

then, for any $2 > \epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$, any finite subset of unitaries $\mathcal{P} \subset U_n(C)$ for some $n \geq 1$, there exist a finite subset $\mathcal{G} \subset K_1(C)$ with $\bar{\mathcal{P}} \subseteq \mathcal{G}$ (where $\bar{\mathcal{P}}$ is the image of $\mathcal{P}$ in $K_1(C)$) and $\delta > 0$ such that, for any map $\eta : G(\mathcal{G}) \to Aff(T(A))$ with $|\eta(x)(\tau)| < \delta$ for all $\tau \in T(A)$ and $\eta(x) - \bar{R}_{\phi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, there is a unitary $u \in A$ such that

$$\|\phi(f) - u^*\psi(f)\| < \epsilon \quad \forall f \in \mathcal{F},$$

and

$$\tau\left(\frac{1}{2\pi i} \; log\left(\left(\phi \otimes id_{M_n}(x^*)\right)\left(u \otimes 1_{M_n}\right)^* \left(\psi \otimes id_{M_n}(x)\right)\left(u \otimes 1_{M_n}\right)\right)\right) = \tau(\eta([x]))$$

for all $x \in \mathcal{P}$ and for all $\tau \in T(A)$

Proof:

Without loss of generality, one may assume that any element in $\mathcal{F}$ has norm at most one. Let $\epsilon > 0$. Choose $\epsilon > \theta > 0$ and a finite subset $\mathcal{F} \subset \mathcal{F}_0 \subset C$ satisfying the following: For all $x \in \mathcal{P}$, $\tau\left(\frac{1}{2\pi i} log\left(\phi(x^*)w_j^*\psi(x)w_j\right)\right)$ is well defined and

$$\tau\left(\frac{1}{2\pi i} log\left(\phi(x^*)w_j^*\psi(x)w_j\right)\right) \tag{35}$$

$$= \tau\left(\frac{1}{2\pi i} log(\phi(x^*)v_1^*\psi(x)v_1)\right) + \cdots$$

$$+ \tau\left(\frac{1}{2\pi i} log\left(\phi(x^*)v_j^*\psi(x)v_j\right)\right) \quad for \; all \; \tau \in T(A), \tag{36}$$

whenever

$$\left\|\phi(f) - v_j^*\psi(f)v_j\right\| < \theta \quad for \; all \; f \in \mathcal{F}_0,$$

where v_j are unitaries in A and $w_j = v_1, \cdots, v_j, j = 1, 2, 3$. In the above, if $x \in U_n(C)$, we denote by ϕ and ψ the extended maps $\phi \otimes id_{M_n}$ and $\psi \otimes id_{M_n}$, and replace w_j, and v_j by $diag(w_j, \dots, w_j)$ and $diag(v_j, \dots, v_j)$, respectively.

Let $C', l: C' \to C$, $\delta' > 0$ (in the place of δ) and $\mathcal{G}' \subseteq K_1(C')$ (in the place of Q) the constant and finite subset with respect to C (in the place of C), $\mathcal{F}_0$ (in the place of $\mathcal{F}$), $\mathcal{P}$ (in the place of $\mathcal{P}$), and ψ (in the place of h). Put $\delta = \delta'/2$.

Fix a decomposition $(l)_{*1}(C') = \mathbb{Z}^k \oplus Tor((l)_{*1}(C'))$ (for some integer $k \geq 0$), and let $\mathcal{G}$ be a set of standard generators of $\mathbb{Z}^k$. Let $\mathcal{G}'' \subset U_m(C)$ be a finite subset containing a representative for each element of $\mathcal{G}$. Without loss of generality, one may assume that $\mathcal{P} \subseteq \mathcal{G}''$, the maps ϕ and ψ are approximately unitary equivalent. Hence, for any finite subset Q and any δ_1, there is a unitary $v \in A$ such that

$$\|\phi(f) - v^*\psi(f)v\| < \delta_1, \;\; \forall f \in Q.$$

By choosing $Q \supseteq \mathcal{F}_0$ sufficiently large and $\delta_1 < \eta/2$ sufficiently small, the map

$$[x] \mapsto \tau\left(\frac{1}{2\pi i} \log(\phi^*(x)v^*\psi(x)v)\right), x \in \mathcal{G}'',$$

induces a homomorphism $\eta_1 : (l)_{*1}\left(K_1\,(C')\right) \to Aff(T\,(A))$ (note that $\eta_1(Tor(((l)_{*1}(K_1(C')))) = \{0\})$, and moreover, $\|\eta_1(x)\| < \delta$ for all $x \in \mathcal{G}$.

By Lemma (1.2.11), the image of $\eta_1 - \bar{R}_{\phi,\psi}$ *is in* $\rho(K_0(A))$. Since $\eta(x) - \bar{R}_{\phi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, the image $(\eta - \eta_1)\left((l)_{*1}(K_1(C'))\right)$ is also in $\rho_A(K_0(A))$. Since $\eta - \eta_1$ factors through $\mathbb{Z}^k$, there is a map $h: (l)_{*1}(K_1(C')) \to K_0(A)$ such that $\eta - \eta_1 = \rho_A \circ h$. Note that $|\tau\,(h(x))| < 2\delta = \delta'$ for all $\tau \in T(A)$ and $x \in \mathcal{G}$.

By the universal multi-coefficient theorem, there is $\kappa \in Hom_\Lambda(\underline{K}(C' \otimes C(\mathbb{T})), \underline{K}(A))$ with

$$k \circ \beta|_{K_1(C')} = h \circ ((l)_{*1}.$$

Applying, there is a unitary w such that

$$\|[w, \psi(f)]\| < \theta/2, \qquad \forall f \in \mathcal{F}_0,$$

and $Bott(w, \psi \circ \iota) = \kappa$. In particular, $bott_1(w,\psi)(x) = h(x)$ for all $x \in \mathcal{P}$.

Set $u = wv$. One then has

$$\|\phi(f) - u^*\psi(f)u\| < \theta, \qquad \forall f \in \mathcal{F}_0,$$

and for any $x \in \mathcal{P}$ and any $\tau \in T(A)$,

$$\begin{aligned}
\tau\left(\frac{1}{2\pi i}\log(\phi(x^*)u^*\psi(x)u)\right) &= \tau\left(\frac{1}{2\pi i}\log(\phi(x)v^*w^*\psi(z)wv)\right)\\
&= \tau\left(\frac{1}{2\pi i}\log(\phi(x^*)v^*\psi(x)vv^*\psi(x^*)w^*\psi(x)wv)\right)\\
&= \tau\left(\frac{1}{2\pi i}\log(\phi(x^*)v^*\psi(x)v)\right) + \tau\left(\frac{1}{2\pi i}\log\psi(x^*)w^*\psi(x)w\right)\\
&= \eta_1([x])(\tau) + h([x])(\tau) = \eta([x])(\tau).
\end{aligned}$$

Corollary (1.2.16)[98]:

Let C be a unital AH-algebra and let A be a unital separable simple Z-stable C^*-algebra in C. Let $\phi, \psi : C \to A$ be two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that

$$\lim_{n\to\infty} u_n^*\,\psi(c)u_n = \phi(c) \quad for\ all\ c \in C,$$

if and only if

$$[\phi] = [\psi] \quad in\ KL(C,A), \quad \phi_\# = \psi_\# \ \ and \ \ \phi^\ddagger = \psi^\ddagger.$$

Proof:

We only show the "if" part. Suppose that ϕ and ψ satisfy the condition. Let $\epsilon > 0$, and let $\mathcal{F} \subset C$ be a finite subset. Then exists a unitary $v \in A \otimes$ Z such that

$$\|v^*(\psi(a) \otimes 1)v - \phi(a) \otimes 1\| < \frac{\epsilon}{3} \quad for\ all\ a \in \mathcal{F}. \tag{37}$$

Let $l: A \to A \otimes Z$ be defined by $l(a) = a \otimes 1$ for $a \in A$. There exists an isomorphism $j : A \otimes Z \to A$ such that $j \circ l$ is approximately inner. So there is a unitaries $w \in A$ such that

$$\|j(\psi(a)\otimes 1)-w^*\psi(a)w\|<\frac{\epsilon}{3}\ \ and\ \ \|w^*\phi(a)w-j(\phi(a)\otimes 1)\|<\frac{\epsilon}{3} \quad (38)$$

for all $a\ \in \mathcal{F}$. Let $u\ =\ wj(v)w^*\in A$; then, for $a\ \in \mathcal{F}$,

$$\|u^*\psi(a)u-\phi(a)\|=\|wj(v)^*\psi(a)wj(v)w^*-\phi(a)\| \quad (39)$$

$$\le \|wj(v)^*w^*\psi(a)wj(v)w^*-wj(v)^*j(\psi(a)\otimes 1)j(v)w^*\| \quad (40)$$

$$+\left\|wj(v)^*\big(j\big(\psi(a)\big)\otimes 1\big)j(v)w^*-w\big(j(\phi(a)\otimes 1)\big)w^*\right\| \quad (41)$$

$$+\left\|w\big(j(\phi(a)\otimes 1)\big)w^*-\phi(a)\right\| \quad (42)$$

$$<\frac{\epsilon}{3}+\frac{\epsilon}{3}\frac{\epsilon}{3}=\epsilon\ \ for\ all\ a\in \mathcal{F}. \quad (43)$$

A version of the following is also obtained by H. Matui.

Corollary (1.2.17)[98]:

Let C be a unital AH-algebra and let A be a unital separable simple C^*-algebra in C_0 which is Z-stable. Suppose that $\phi,\psi\colon C\to A$ are two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\}\subset A$ such that

$$\lim_{n\to\infty} u_n^*\,\phi(c)u_n=\psi(c)\ \ \ for\ all\ c\in C,$$

if and only if

$$[\phi]=[\psi]\quad in\ KL(C,A),\quad \phi_\#=\psi_\#\ \ and\ \ \phi^\ddagger=\psi^\ddagger.$$

Lemma (1.2.18)[98]:

Let A be a unital C^*-algebra such that $\mathrm{A}\otimes \mathrm{M_r}$ is an AH-algebra for any supernatural number r of infinite type. Let $\mathrm{B}\ \in \mathrm{C}$ be a unital separable C^*-algebra, and let $\phi,\psi\colon \mathrm{A}\to \mathrm{B}$ be two unital monomorphisms. Suppose that

$$[\phi]=[\psi]\quad in\ KL(A,B), \quad (44)$$

$$\phi_\#=\psi_\#\ \ and\ \ \phi^\ddagger=\psi^\ddagger. \quad (45)$$

Let p and q be two relatively prime supernatural numbers of infinite type with $\mathrm{M_p}\otimes \mathrm{M_q}=\mathrm{Q}$. Then, for any $\epsilon>0$ and any finite subset $\mathcal{F}\subset \mathrm{A}\otimes \mathrm{Z_{p,q}}$, there exists a unitary $\mathrm{v}\in \mathrm{B}\otimes \mathrm{Z_{p,q}}$ such that

$$\left\|v^*\big((\phi\otimes id)(a)\big)v-(\psi\otimes id)(a)\right\|<\epsilon\quad for\ all\ a\ \in \mathcal{F} \quad (46)$$

The proof of this lemma will be lengthy and technical in nature. Using homotopy lemmas, one could find a certain path of unitaries in $B\otimes Q$ such that it implements the approximate equivalence above when it is regarded as a unitary in $B\otimes Z_{p,q}$. But since the domain C^*-algebra A is only assumed to be rational tracial rank at most one, in order to apply the homotopy lemmas, one also needs to interpolate paths in $A\otimes Z_{p,q}$, and this increases the technical difficulty of the proof.

Proof:

Let r be a supernatural number. Denote by $l_r\colon A\to A\otimes M_r$ the embedding defined by $l_r(a)=a\otimes 1$ for all $a\in A$. Denote by $j_r\colon B\to B\otimes M_r$ the embedding defined by $j_r(b)=b\otimes 1$ for all $b\in B$. Without loss of generality, one may assume that $\mathcal{F}=\mathcal{F}_1\otimes \mathcal{F}_2$, where $\mathcal{F}_1\subseteq A$ and $\mathcal{F}_2\subseteq Z_{p,q}$ are finite subsets and $1_A\in \mathcal{F}$ and $1_{Z_{p,q}}\in \mathcal{F}_2$. Moreover, one may assume that any element in $\mathcal{F}_1$ or $\mathcal{F}_2$has norm at most one.

Let $0=t_0<t_1<\cdots<t_m=1$ be a partition of $[0,1]$ such that

$$\|b(t)-b(t_i)\|<\frac{\epsilon}{4}\quad \forall b\in \mathcal{F}_2,\forall t\in [t_{i-1},t_i],i=1,\dots,m. \quad (47)$$

Consider

$$\varepsilon = \{a \otimes b(t_i); a \in \mathcal{F}_1, b \in \mathcal{F}_2, i = 0, \dots, m\} \subseteq A \otimes Q,$$
$$\varepsilon_p = \{a \otimes b(t_0); a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_p \subset A \otimes Q \text{ and} \quad (48)$$
$$\varepsilon_q = \{a \otimes b(t_m); a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_q \subset A \otimes Q. \quad (49)$$

Since $A \otimes Q$ is an AH-algebra, without loss of generality, one may assume that the finite subset E is in a C^*-subalgebra of $A \otimes Q$ which is isomorphic to $C := PM_n(C(X))P$ (for some $n \geq 1$) for some compact metric space X. Since $PM_n(C(X))P = lim_{m\to\infty}(P_m M_n(C(X_m))P_m)$, where X_m are closed subspaces of finite CW-complexes, then, without loss of generality, one may assume further that X is a closed subset of a finite CW-complex.

Fix a metric on X, and for any $a \in (0,1)$, denote by
$\Delta(a) = \inf\{\mu_{\tau\circ(\phi\otimes id)}(O_a); \tau \in T(B), O_a \text{ an open ball of radius } a \text{ in } X\}$.
Since B is simple, one has that $0 < \Delta(a) \leq 1$.

Let $\mathcal{H} \subset C$, $\mathcal{P} \subseteq \underline{K}(C)$, $Q = \{x_1, x_2, \dots, x_m\} \subset K_0(C)$ which generates a free subgroup of $K_0(C), \delta > 0, \gamma > 0$, and $d > 0$ (in the place of η) be the constants of Theorem (1.2.13) with respect to E, $\epsilon/8$, and Δ. We may assume that $x_i = [pi] - [q_i]$, where $p_i, q_i \in M_n(C)$ are projections (for some integer $n \geq 1$), $i = 1,2,\dots,m$. Moreover, we may assume that $\gamma < 1$. Denote by ∞ the supernatural number associated with $\mathbb{Q}$. Let $P_i = P \cap K_i(A \otimes Q), i = 0, 1$. There is a finitely generated free subgroup $G(\mathcal{P})_{i,0} \subset K_i(A)$ such that if one sets

$$G(\mathcal{P})_{i,\infty,0} = G\big(\{gr: g \in (l_\infty)_{*i}\big(G(\mathcal{P})_{i,0}\big) \text{ and } r \in D_0\}\big), \quad (50)$$

where $1 \in D_0 \subset \mathbb{Q}$ is a finite subset, then $G(\mathcal{P})_{i,\infty,0}$ contains the subgroup generated by $\mathcal{P}_i$, $i = 0,1$. Moreover, we may assume that, if $r = k/m$, where k and m are nonzero integers, and $r \in D_0$, then $1/m \in D_0$. Let $\mathcal{P}_i' \subset K_i(A)$ be a finite subset which generates $G(\mathcal{P})_{i,0}, i = 0, 1$.Also denote by $\mathcal{P}' = \mathcal{P}_0' \cup \mathcal{P}_1'$.

Denote by $j: C \to A \otimes Q$ the embedding.

Write the subgroup generated by the image of Q in $K_0(A \otimes Q)$ as $\mathbb{Z}^k$ (for some integer $k \geq 1$). Choose $\{x_1', \dots, x_k'\} \subseteq K_0(A)$ and $\{r_{ij}; 1 \leq i \leq m, 1 \leq j \leq k\} \subseteq \mathbb{Q}$ such that

$$j_{*0}(x_i) = \sum_{j=1}^{k} r_{ij}\, x_j', \quad 1 \leq i \leq m, 1 \leq j \leq k,$$

and moreover, $\{x_1', \dots, x_k'\}$ generates a free subgroup of $K_0(A)$ of $rank\, k$. Choose projections $p_j', q_j' \in M_n(A)$ such that $x_j' = [p_j'] - [q_j'], 1 \leq j \leq k$. Choose an integer M such that Mr_{ij} are integers for $1 \leq i \leq m$ and $1 \leq j \leq k$. In particular Mx_i is the linear combination of x_j' with integer coefficients.

Also noting that the subgroup of $K_0(A \otimes Q)$ generated by $\{(l_\infty)_{*i}(x_1'), \dots, (l_\infty)_{*i}(x_k')\}$ is isomorphic to $\mathbb{Z}^k$ and the subgroup of $K_0(A \otimes M_r)$ generated by $\{(l_r)_{*i}(x_1'), \dots, (l_r)_{*i}(x_k')\}$ has to be isomorphic to $\mathbb{Z}^k$, where $r = p \text{ or } r = q$.

Since $A \otimes M_r$ is an AH-algebra, one can choose a C^*-subalgebra C_r of $A \otimes M_r$ which is isomorphic to $P_r M_{n_r}(C(X_r))P_r$ (for some $n_r \geq 1$) such that $E_r \subseteq C_r$ and projections $\{p_{1,r}', \dots, p_{k,r}', q_{1,r}', \dots, q_{k,r}'\} \subseteq M_n(C_r)$ such that for any $1 \leq j \leq k$,

$$\left\|p_j' \otimes 1_{M_r} - p_{j,r}'\right\| < \gamma / \left(32 \left(1 + \sum_{i,j'} \left| Mr_{i,j'} \right| \right) \right) < 1, \tag{51}$$

and

$$\left\|q_j' \otimes 1_{M_r} - q_{j,r}'\right\| < \gamma / \left(32 \left(1 + \sum_{i,j'} \left| Mr_{i,j'} \right| \right) \right) < 1, \tag{52}$$

where X_r is a closed subset of a finite CW-complex, and $r = p \; or \; r = q$.

Denote by $x_{j,r}' = [p_{j,\tau}'] - [q_{j,r}']$, $1 \le j \le k$, and denote by G_r the subgroup of $K_0(C_r)$ generated by $\{x_{1,r}', \ldots, x_{k,r}'\}$, and write $G_r = \mathbb{Z}^k \oplus Tor(G_r)$. Since G_r is generated by k elements, one has that $r \le k$ and $r = k$ if and only if G_r is torsion free. Note that the image of G_r in $K_0(A \otimes M_r)$ is the group generated by $\{[p_1' \otimes 1_{M_r}] - [q_1' \otimes 1_{M_r}], \ldots, [p_k' \otimes 1_{M_r}] - [q_k' \otimes 1_{M_r}\}$, which is isomorphic to $\mathbb{Z}^k$ (with $\{[p_j' \otimes 1_{M_r}] - [q_j' \otimes 1_{M_r}];\; 1 \le j \le k\}$ as the standard generators). Hence G_r is torsion free and $r = k$.

Without loss of generality, one may assume that $l_r(\mathcal{P}') \subseteq K(C_r)$. Assume that $\mathcal{H}$ is sufficiently large and δ is sufficiently small such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary $z_j (j = 1, 2, 3, 4)$, the map $Bott(h, z_j)$ and $Bott(h, w_j)$ are well defined on the subgroup generated by $\mathcal{P}$ and

$$Bott(h, z_j) = Bott(h, z_1) + \cdots + Bott(h, z_j)$$

on the subgroup generated by $\mathcal{P}$, if $\left\|[h(x), z_j]\right\| < \delta$ for any $x \in \mathcal{H}$, where $w_j = z_1, \ldots, z_j, j = 1, 2, 3, 4$.

By choosing larger $\mathcal{H}$ and smaller δ, one may also assume that

$$\left\|h(p_i), z_j\right\| < \frac{1}{16} \;\; and \;\; \left\|h(q_i), z_j\right\| < \frac{1}{16}, \;\; 1 \le i \le m, j = 1,2,3,4, \tag{53}$$

and for any $1 \le i \le m$,

$$dist\left(\zeta_{i,z_1}^M, \prod_{j=1}^{k} \left(\zeta_{i,z_1}'\right)^{Mr_{i,j}} \right) < \gamma/8, \tag{54}$$

where

$$\zeta_{i,,z_1} = \overline{\langle (1_n - h(p_i) + h(p_i)) z_1)(1_n - h(p_i) + h(p_i)) z_1^*) \rangle},$$

and

$$\zeta_{i,z_1}' = \overline{\langle \left(1_n - h\left(p_j' \otimes 1_{A\otimes Q}\right) + h\left(p_j' \otimes 1_{A\otimes Q}\right)\right) z_1 \big) \left(1_n - h\left(q_j' \otimes 1_{A\otimes Q}\right) + h\left(q_j' \otimes 1_{A\otimes Q}\right)\right) z_1^* \big) \rangle}.$$

By choosing even smaller δ, without loss of generality, we may assume that

$$\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^p \otimes \mathcal{H}^q,$$

where $\mathcal{H}^0 \subset A$, $\mathcal{H}^p \subset M_p$ and $\mathcal{H}^q \subset M_q$ are finite subsets, and $1 \in \mathcal{H}^0$, $1 \in \mathcal{H}^p$ and $1 \in \mathcal{H}^q$.

Moreover, choose $\mathcal{H}^0$, $\mathcal{H}^p$ and $\mathcal{H}^q$ even larger and δ even smaller so that for any homomorphism $h_r: A \otimes M_r \to B \otimes M_r$ and unitaries $z_1, z_2 \in B \otimes M_r$ with $\|h_r(x), z_i\| < \delta$ for any $x \in \mathcal{H}_0 \otimes \mathcal{H}_r$, one has

$$\|h_r(p'_{i,r}), z_j\| < \frac{1}{16} \; and \; \|h_r(q'_{i,r}), z_j\| < \frac{1}{16}, \;\; 1 \le i \le k, j = 1,2, \tag{55}$$

and

$$dist\left(\zeta_{i,z_1,z_2}, \overline{\left(1_{B\otimes M_r}\right)_n}\right) < dist\left(\zeta_{i,z_1^*}, \zeta_{i,z_2}\right) + \gamma/\left(32\left(1+\sum_{i\prime,j}\left|Mr_{i\prime,j}\right|\right)\right),$$

where

$$\zeta_{i,,z\prime} = \overline{\langle\left(1_n - h_r(p'_{i,r}) + h_r(p'_{i,r})\right)z'\right)\left(1_n - h_r(q'_{i,r}) + h(q'_{i,r})\right)(z')^*)\rangle}, z' = z_1z_2, z_1^*, z_2.$$

Denote by $C' = P'M_n(C(\tilde{X}))P', l: C' \to A \otimes Q$, δ_2 (in the place of δ) the constant, $G \subseteq K_1(C(\tilde{X}))$ (in the place of Q) the finite subset with respect to $A \otimes Q$ (in the place of C), $B \otimes Q$ (in the place of A), $\phi \otimes idQ$ (in the place of h), $\delta/4$ (in the place of ϵ), $\mathcal{H}$ (in the place of $\mathcal{F}$) and $\mathcal{P}$. Note that $\tilde{X}$ is a finite CW-complex.

Let $\mathcal{H}' \subseteq A \otimes Q$ be a finite subset and assume that δ_2 is small enough such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary $z_j (j = 1,2,3,4)$, the map $Bott(h, z_j)$ and $Bott(h, w_j)$ is well defined on the subgroup $[l](\underline{K}(C'))$ and

$$Bott(h, w_j) = Bott(h, z_1) + \cdots + Bott(h, z_j)$$

on the subgroup $[l](K(C'))$, if $\|[h(x), z_j]\| < \delta_2$ for any $x \in H'$, where $w_j = z_1, z_j$, $j = 1,2,3,4$. Furthermore, as above, one may assume, without loss of generality, that

$$\mathcal{H}' = \mathcal{H}^{0\prime} \otimes \mathcal{H}^{p\prime} \otimes \mathcal{H}^{q\prime},$$

where $\mathcal{H}^0 \subseteq \mathcal{H}^{0\prime} \subset A, \mathcal{H}^p \subseteq \mathcal{H}^{p\prime} \in M_q$ and $\mathcal{H}^q \subseteq \mathcal{H}^{q\prime} \subset M_q$ are finite subsets.

Let $\delta_2' > 0$ be a constant such that for any unitary with $\|u - 1\| < \delta_2'$, one has that $\|log\, u\| < \delta_2/4$. Without loss of generality, one may assume that $\delta_2' < \delta_2/4 < \epsilon/4$ and $\delta_2' < \delta$.

Let $C_r' := P_rM_nC(X_r')P_r$ (in the place of C'), $l_r': C_r \to A \otimes M_r$ (in the place of l), $R_r \subset K_1(C_r'))$ (in the place of Q) and δ_r (in the place of δ) be the finite subset and constant with respect to $A \otimes M_r$ (in the place of C), $B \otimes M_r$ (in the place of A), $\phi \otimes idM_r$ (in the place of h), $\mathcal{H}^{0\prime} \otimes \mathcal{H}^{r\prime}$ (in place of $\mathcal{F}$) and $(l_r)_{*0}(\mathcal{P}_0') \cup (l_r)_{*1}(\mathcal{P}_1')$ (in the place of $\mathcal{P}$) and $\delta_2'/8$ (in place of ϵ) ($r = p$ or $r = q$). Note that X_r' is a finite CW-complex with $K_1(C_1') = \mathbb{Z}^{k_r} \oplus Tor(K_1(C_r'))$. Let $R_r^{(i)} = (l_r')_{*i}(K_i(C_r')), i = 0,1$. There is a finitely generated subgroup $G_{i,0,r} \subset K_i(A)$ and a finitely generated subgroup $D_{0,r} \subseteq \mathbb{Q}_r$ so that

$$G'_{i,0,r} \coloneqq G\left(\{gr: g \in (l_r)_{*i}(G_{i,0,r}) \; and \; r \in D_{0,r}\}\right)$$

contains the subgroup $R_r^{(i)}, i = 0,1$. Without loss of generality, one may assume that $D_{0,p} = \{\frac{k}{m_p}; k \in Z\}$ and $D_{0,q} = \{\frac{k}{m_q}; k \in Z\}$ for an integer m_p divides p and an integer m_q divides q. Let $R \subset \underline{K}(A \otimes Q)$ be a finite subset which generates a subgroup containing

$$\frac{1}{m_pm_q}\left(\left(l_{p,\infty}\right)_*\left(G'_{0,0,p} \cup G'_{1,0,p}\right) \cup \left(l_{q,\infty}\right)_*\left(G'_{0,0,q} \cup G'_{1,0,q}\right)\right)$$

in $\underline{K}(A \otimes Q)$, where $l_{r,\infty}$ is the canonical embedding $A \otimes M_r \to A \otimes Q, r = p, q$. Without loss of generality, one may also assume that $R \supseteq l_{/1}(\mathcal{G}). Let\; \mathcal{H}_r \subset A \otimes M_r$ be a finite subset and $\delta_3 > 0$ such that for any homomorphism h from $A \otimes M_r$ to $B \otimes M_r$ ($r = p \; or \; r = q$) any unitary z_j ($j = 1,2,3,4$), the map $Bott(h, z_j)$ and $Bott(h, w_j)$ are well defined on the subgroup $[l_r'](\underline{K}(C_r'))$ and

$$Bott(h, w_j) = Bott(h, z_1) + \cdots + Bott(h, z_j)$$

on the subgroup generated by $[l'_r](\underline{K}(C'_r))$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}_r$, where $w_j = z_1, \ldots, z_j$, $j = 1, 2, 3, 4$. Without loss of generality, we assume that $\mathcal{H}^0 \otimes \mathcal{H}^p \subset \mathcal{H}_p$ and $\mathcal{H}^0 \otimes \mathcal{H}^q \subset \mathcal{H}_q$. Furthermore, we may also assume that

$$\mathcal{H}_r = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,r}$$

for some finite subsets $\mathcal{H}_{0,0}$ and $\mathcal{H}_{0,r}$ with $\mathcal{H}^{0'} \subset \mathcal{H}_{0,0} \subset A, \mathcal{H}^{p'} \subset \mathcal{H}_{0,p} \subset M_p$ *and* $\mathcal{H}^{q'} \subset \mathcal{H}_{0,q}$. In addition, we may also assume that $\delta_3 < \delta_2/2$.

Furthermore, one may assume that δ_3 is sufficiently small such that, for any unitaries z_1, z_2, z_3 in a C^*-algebra with tracial states, $\tau\left(\frac{1}{2\pi i} log(z_i z_j^*)\right)$ $(i, j = 1, 2, 3)$ is well defined and

$$\tau\left(\frac{1}{2\pi i} log(z_1 z_2^*)\right) = \tau\left(\frac{1}{2\pi i} log(z_1 z_3^*)\right) + \tau\left(\frac{1}{2\pi i} log(z_3 z_2^*)\right)$$

for any tracial state τ, whenever $\|z_1 - z_3\| < \delta_3$ and $\|z_2 - z_3\| < \delta_3$.

To simply notation, we also assume that, for any unitary z_j, $(j = 1, 2, 3, 4)$ the map $Bott(h, z_j)$ and $Bott(h, w_j)$ are well defined on the subgroup generated by $\mathcal{R}$ and

$$Bott(h, w_j) = Bott(h, z_1) + \cdots + Bott(h, z_j)$$

on the subgroup generated by $\mathcal{R}$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}''$, where $w_j = z_1, \ldots, z_j$, $j = 1, 2, \ldots, 4$, and assume that

$$\mathcal{H}'' = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,p} \otimes \mathcal{H}_{0,q}.$$

Let $R_i = R \cap K_i(A \otimes Q)$. There is a finitely generated subgroup $G_{i,0}$ of $K_i(A)$ and there is a finite subset $D'_0 \subset \mathbb{Q}$ such that

$$G_{i,\infty} := G\left(\{gr: g \in (l_r)_{*i}(G_{i,0}) \text{ and } r \in D'_0\}\right)$$

contains the subgroup generated by $R^i, i = 0, 1$. Without loss of generality, we may assume that $G_{i,\infty}$ is the subgroup generated by R^i. Note that we may also assume that $G_{i,0} \supset G(\mathcal{P})_{i,0}$ and $1 \in D'_0 \supset D_0$. Moreover, we may assume that, if $r = k/m$, where m, k are relatively prime non-zero integers, and $r \in D'_0$, then $1/m \in D'_0$. We may also assume that $G_{i,0,r} \subseteq G_{i,0}$ for $r = p, q$ and $i = 0,1$. Let $R^{i'} \subset K_i(A)$ be a finite subset which generates $G_{i,0}, i = 0,1$. Choose a finite subset $U \subset U_n(A)$ for some n such that for any element of $R^{1'}$, there is a representative in U. Let S be a finite subset of A such that if $(z_{i,j}) \in U$, then $z_{i,j} \in S$.

Denote by δ_4 and $Q_r \subset K_1(A \otimes M_r) \cong K_1(A) \otimes Q_r$the constant and finite subset of Lemma (1.2.15) corresponding to $\mathcal{E}_r \cup \mathcal{H}_r \otimes 1 \cup l_r(S)$ (in the place of $\mathcal{F}$), $l_r(\mathcal{U})$ (in the place of $\mathcal{P}$) and $\frac{1}{n^2} min\{\delta'_2/8, \delta_3/4\}$ (in the place of ϵ) ($r = p$ *or* $r = q$). We may assume that $Q_r = \{x \otimes r: x \in Q' \text{ and } r \in D''_r\}$, where $Q' \subset K_1(A)$ is a finite subset and $D''_r \subset \mathbb{Q}_r$ is also a finite subset. Let $K = max\{|r|: r \in D''_p \cup D''_q\}$. Since $[\phi] = [\psi]$ in $KL(A, B)$, $\phi_\# = \psi_\#$ and $\phi^\ddagger = \psi^\ddagger$, by Lemma (1.2.10), $\bar{R}_{\phi,\psi}(K_1(A)) \subseteq \overline{\rho_B(K_0(B))} \subset Aff(T(B))$. Therefore, there is a map $\eta : G(Q') \to \overline{\rho_B(K_0(B))} \subset Aff(T(B))$ such that

$$(\eta - \bar{R}_{\phi,\psi})([z]) \in \rho_B(K_0(B)) \text{ and } \|\eta(z)\| < \frac{\delta_4}{1 + K} \text{ for all } z \in Q' \quad (56)$$

Consider the map $\phi_r = \phi \otimes id_{M_r}$ and $\psi_r = \psi \otimes id_{M_r}$ $(r = p \; or \; r = q)$. Since η vanishes on the torsion part of $G(Q')$, there is a homomorphism $\eta_r: G((l_r)_{*1}(Q')) \to \overline{\rho_{B\otimes M_r}(K_0(B \otimes M_r))} \subset Aff(T(B \otimes M_r))$ such that

$$\eta_r \circ (l_r)_{*1} = \eta. \tag{57}$$

Since $\overline{\rho_{B\otimes M_r}(K_0(B \otimes M_r))} = \overline{\mathbb{R}_{\rho_B}(K_0(B))}$ is divisible, one can extend η_r so it defines on $K_1(A) \otimes \mathbb{Q}_r$. We will continue to use η_r for the extension. It follows from (50) that $\eta_r(z) - \bar{R}_{\phi,\psi}(z) \in \rho_{B\otimes M_r}(K_0(B \otimes M_r))$ and $\|\eta_r(z)\| < \delta_4$ for all $z \in Q_r$. By Lemma 1.2.17, there exists a unitary $u_p \in B \otimes M_p$ such that

$$\left\| u_p^* \left(\phi \otimes id_{M_p}\right)(z) u_p - \left(\psi \otimes id_{M_p}\right)(z) \right\| < \frac{1}{n^2} \min\{\delta_2'/8, \delta_3/4\}, \forall c \in \mathcal{E}_p \cup \mathcal{H}_p \cup l_p(S). \tag{58}$$

Note that

$$\left\| u_p^* \left(\phi \otimes id_{M_p}\right)(z) u_p - \left(\psi \otimes id_{M_p}\right)(z) \right\| < \delta_3 \quad for\ any\ z \in \mathcal{U}.$$

Therefore $\tau(\frac{1}{2\pi i} log(u_p^*(\phi \otimes id_p)(z)u_p(\psi \otimes id_p)(z))) = \eta_p([z^*])(\tau)$ for all $z \in l_p(U)$, where we identify ϕ and ψ with $\phi \otimes id_{M_n}$ and $\psi \otimes id_{M_n}$, and up with $u_p \otimes 1_{M_n}$, respectively.

The same argument shows that there is a unitary $u_q \in B \otimes M_q$ such that

$$\left\| u_q^* \left(\phi \otimes id_{M_q}\right)(z) u_q - \left(\psi \otimes id_{M_q}\right)(z) \right\| < \frac{1}{n^2} \min\{\delta_2'/8, \delta_3/4\}, \forall c \in \mathcal{E}_q \cup \mathcal{H}_q \cup l_q(S). \tag{59}$$

and $\tau(\frac{1}{2\pi i} log(u_q^*(\phi \otimes id_q)(z)u_q(\psi \otimes id_q)(z))) = \eta_q([z^*])(\tau)$ for all $z \in l_q(U)$, where we identify ϕ and ψ with $\phi \otimes id_{M_n}$ and $\psi \otimes id_{M_n}$, and uq with $u_q \otimes 1_{M_n}$, respectively. We will also identify u_p with $u_p \otimes 1_{M_q}$ and u_q with $u_q \otimes 1_{M_q}$ respectively. Then $u_p u_q^* \in A \otimes Q$ and one estimates that for any $c \in \mathcal{H}_{00} \otimes \mathcal{H}_{0,p} \otimes \mathcal{H}_q$,

$$\left\| u_q u_p^* (\phi \otimes 1_Q(c))(z) u_p u_q^* - (\phi \otimes 1_Q)(c) \right\| < \delta_3, \tag{60}$$

and hence $Bott(\phi \otimes id_Q, u_p u_q^*)(z)$ is well defined on the subgroup generated by R. Moreover, for any $z \in U$, by the Exel formula by applying (83),

$$\begin{aligned}
&\tau\big(bott_1(\phi \otimes id_Q, u_p u_q^*)(l_\infty)_{*1}([z])\big) && (61)\\
&= \tau\left(bott_1(\phi \otimes id_Q, u_p u_q^*)(l_\infty(z))\right) && (62)\\
&= \tau\left(\frac{1}{2\pi i} log\big(u_p u_q^* (\phi \otimes id_q)(l_\infty(z)) u_q u_p^* (\psi \otimes id_q)(l_\infty(z))^*\big)\right) && (63)\\
&= \tau\left(\frac{1}{2\pi i} log\left(u_q^* (\phi \otimes id_q)(l_\infty(z)) u_q (\psi \otimes id_q)(l_\infty(z^*))\right)\right) && (64)\\
&- \tau\left(\frac{1}{2\pi i} log\left(u_p^* (\phi \otimes id_q)(l_\infty(z)) u_p (\psi \otimes id_q)(l_\infty(z^*))\right)\right) && (65)\\
&= \eta_q\left(\left(l_q\right)_{*1}([z])\right)(\tau) - \eta_p\left(\left(l_p\right)_{*1}([z])\right)(\tau) && (66)\\
&= \eta([z])(\tau) - \eta([z])(\tau) = 0 \; for\ all\ \tau \in T(B), && (67)
\end{aligned}$$

where we identify ϕ and ψ with $\phi \otimes id_{M_n}$ and $\psi \otimes id_{M_n}$, and u_p and u_q with $u_p \otimes 1_{M_n}$ and u_q with $u_q \otimes 1_{M_n}$, respectively.

Now suppose that $g \in G_{1,\infty}$. Then $g = (k/m)(l_\infty)_{*1}([z])$ for some $z \in U$, where k, m are non-zero integers. It follows that

$$\tau\big(bott_1(\phi \otimes id_Q, u_p u_q^*)(mg)\big) = k\tau\big(bott_1(\phi \otimes id_Q, u_p u_q^*)([z])\big) = 0 \qquad (68)$$

for all $\tau \in T(B)$. Since $Aff(T(B))$ is torsion free, it follows that

$$\tau\big(bott_1(\phi \otimes id_Q, u_p u_q^*)(g)\big) = 0 \qquad (69)$$

for all $g \in G_{1,\infty}$ and $\tau \in T(B)$. Therefore, the image of R^1 under $bott_1(\phi \otimes id_Q, u_p u_q^*)$ is in $ker\,\rho_{B\otimes Q}$. One may write

$$G_{1,0} = \mathbb{Z}^r \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/p_s\mathbb{Z}.$$

where r is a non-negative integer and $p_1, \ldots, p_s$ are powers of primes numbers. Since p and q are relatively prime, one then has the decomposition

$$G_{1,0} = \mathbb{Z}^r \oplus Tor_p\big(G_{1,0}\big) \oplus Tor_q\big(G_{1,0}\big) \subseteq K_1(A),$$

where $Tor_p\big(G_{1,0}\big)$ consists of the torsion-elements with their orders divide p and $Tor_q\big(G_{1,0}\big)$ consists of the torsion-elements with their orders divide q. Fix this decomposition. Note that the restriction of $\big(l_p\big)_{*1}\, to\, \mathbb{Z}^r \oplus Tor_q\big(G_{1,0}\big)$ is injective and the restriction to $Tor_p\big(G_{1,0}\big)$ is zero, and the restriction of $\big(l_q\big)_{*1}\, to\, \mathbb{Z}^r \oplus Tor_p\big(G_{1,0}\big)$ is injective and the restriction to $Tor_q\big(G_{1,0}\big)$ is zero.

Moreover, using the assumption that p and q are relatively prime again, for any element $k \in \big(l_q\big)_{*1}\, to\, \mathbb{Z}^r \oplus Tor_p\big(G_{1,0}\big)$ and any nonzero integer q which divides q, the element $\frac{k}{q}$ is well defined in $K_1\big(A \otimes M_q\big)$; that is, there is a unique element $s \in K_1\big(A \otimes M_q\big)$ such that $qs = k$.

Denote by $e_1, \ldots, e_r$ the standard generators of $\mathbb{Z}^r$. It is also clear that

$$(l_\infty)_{*1}\left(Tor_p\big(G_{1,0}\big)\right) = (l_\infty)_{*1}\left(Tor_p\big(G_{1,0}\big)\right)$$

Recall that $D_{0,p} = \{k/m_p; k \in \mathbb{Z}\} \subset \mathbb{Q}_p$ and $D_{0,q} = \{k/m_q; k \in \mathbb{Z}\} \subset \mathbb{Q}_{qqp}$ for an integer m_p dividing p and an integer m_q dividing q. Put $m_\infty = m_p m_q$.

Consider $\frac{1}{m_\infty}\mathbb{Z}^r \in K_1(A \otimes Q)$, and for each $e_i, 1 \le i \le r$, consider

$$\frac{1}{m_\infty} bott_1\big(\phi \otimes id_Q, u_p u_q^*\big)\big((l_\infty)_{*1}(e_i)\big) \in \ker \rho_{B\otimes Q}.$$

Since $ker\,\rho_{B\otimes Q} \cong (ker\,\rho_B) \otimes \mathbb{Q}$, ker $ker\,\rho_{B\otimes M_p} \cong (ker\,\rho_B) \otimes \mathbb{Q}_p$, and $ker\,\rho_{B\otimes M_q} \cong (ker\,\rho_B) \otimes \mathbb{Q}_q$, there are $g_{i,p} \in ker\ \rho_{B\otimes M_p}$ and $g_{i,q} \in ker\ \rho_{B\otimes M_q}$ such that

$$bott_1\big(\phi \otimes id_Q, u_p u_q^*\big)\left(\frac{1}{m_\infty}(l_\infty)_{*1}(e_i)\right) = \big(j_p\big)_{*0}\big(g_{i,p}\big) + \big(j_q\big)_{*0}\big(g_{i,q}\big),$$

where $g_{i,p}$ and $g_{i,q}$ are identified as their images in $K_0(A \otimes Q)$.

Note that the subgroup $(l_p)_{*1}(G_{1,0})\, in\, K_0(A \otimes M_p)$ is isomorphic to $\mathbb{Z}^r \oplus Tor_q$ and $\frac{1}{m_q}(\mathbb{Z}^r \oplus Tor_q)$ is well defined in $K_0(A \otimes M_p)$, and the subgroup $(l_q)_{*1}(G_{1,0})$ in $K_0(B \otimes M_p)$ is isomorphic to $\mathbb{Z}^r \oplus Tor_p$ and $\frac{1}{m_q}(\mathbb{Z}^r \oplus Tor_p)$ is well defined in $K_0(A \otimes M_q)$.

One then defines the maps $\theta_p\colon \frac{1}{m_p}\,(l_p)_{*1}(G_{1,0}) \to ker\, \rho_{B\otimes M_p}$ and $\theta_q\colon \frac{1}{m_q}\,(l_q)_{*1}(G_{1,0}) \to ker\, \rho_{B\otimes M_q}$ by

$$\theta_p\left(\frac{1}{m_p}\,(l_p)_{*1}(e_i)\right) = m_q g_{i,p} \;\; and \;\; \theta_q\left(\frac{1}{m_q}\,(l_q)_{*1}(e_i)\right) = m_p g_{i,q}$$

for $1 \le i \le r$ and

$$\theta_p|_{Tor((l_p)_{*1}(G_{1,0}))} = 0 \;\; and \;\; \theta_q|_{Tor((l_q)_{*1}(G_{1,0}))} = 0.$$

Then, for each e_i, one has

$$\begin{aligned}(j_p)_{*0}\circ\theta_p\circ(l_p)_{*1}(e_i) &+ (j_q)_{*0}\circ\theta_q\circ(l_q)_{*1}(e_i)\\ &= m_p\left(\frac{1}{m_p}(j_p)_{*0}\circ\theta_p\circ(l_p)_{*1}(e_i)\right) + m_q\left(\frac{1}{m_q}(j_q)_{*0}\circ\theta_q\circ(l_q)_{*1}(e_i)\right)\\ &= m_p m_q\left((j_p)_{*0}(g_{i,p}) + (j_q)_{*0}(g_{i,q})\right)\\ &= m_\infty bott_1(\phi\otimes id_Q, u_p u_q^*)\circ\left((l_\infty)_{*1}(e_i/m_\infty)\right)\\ &= bott_1(\phi\otimes id_Q, u_p u_q^*)\circ\left((l_\infty)_{*1}(e_i)\right).\end{aligned}$$

Since the restriction of $\theta_p\circ(l_p)_{*1}, \theta_q\circ(l_q)_{*1}$ and $bott_1(\phi\otimes id_Q, u_p u_q^*)\circ((l_\infty)_{*1})$ to the torsion part of $G_{1,0}$ is zero, one has

$$bott_1(\phi\otimes id_Q, u_p u_q^*)\circ((l_\infty)_{*1}) = (j_p)_{*1}\circ\alpha_p\circ(l_p)_{*0} + (j_q)_{*1}\circ\alpha_q\circ(l_q)_{*0}.$$

The same argument shows that there also exist maps $\alpha_p : \frac{1}{m_p}((l_p)_{*1}(G_{0,0})) \to K_1(B\otimes M_p)$ and $\alpha_q : \frac{1}{m_q}((l_q)_{*1}(G_{0,0})) \to K_1(B\otimes M_q)$ such that

$$bott_0(\phi\otimes id_Q, u_p u_q^*)\circ((l_\infty)_{*0}) = (j_p)_{*1}\circ\alpha_p\circ(l_p)_{*0} + (j_q)_{*1}\circ\alpha_q\circ(l_q)_{*0}.$$

On $G_{0,0}$.

Note that $G_{i,0,r} \subseteq G_{i,0}, i = 0,1, r = p,q$. In particular, one has that $(l_r)_{*i}\left(G_{i,0,r}\right) \subseteq (l_r)_{*i}\left(G_{i,0}\right)$, and therefore $G'_{1,0,p} \subseteq \frac{1}{m_p}(l_p)_{*i}\left(G_{1,0}\right)$ and $G'_{1,0,q} \subseteq \frac{1}{m_q}(l_q)_{*i}\left(G_{1,0}\right)$. Then the maps θ_p and θ_q can be restricted to $G'_{1,0,p}$ and $G'_{1,0,q}$ respectively. Since the group $G'_{i,0,r}$ contains $(l'_r)_{*i}(K_i(C'_r))$, the maps θ_p and θ_q can be restricted further to $(l'_p)_{*1}\left(K_1(C'_p)\right)$ and $(l'_q)_{*1}\left(K_1(C'_q)\right)$ respectively.

For the same reason, the maps α_p and α_q can be restricted to $(l'_p)_{*0}\left(K_0(C'_p)\right)$ and $(l'_q)_{*0}\left(K_0(C'_q)\right)$ respectively. We keep the same notation for the restrictions of these maps α_p, α_q, θ_p, and θ_q.

By the universal multi-coefficient theorem, there is $k_p \in Hom_\Lambda(\underline{K}(C'_p\otimes C(\mathbb{T})), \underline{K}(B\otimes M_p))$ such that

$$k_p|_{\beta(K_1(C'_p))} = -\theta_p\circ(l'_p)_{*1}\circ\beta^{-1} \;and\;\; k_p|_{\beta(K_1(C'_p))} = -\alpha_p\circ(l'_p)_{*0}\circ\beta^{-1}.$$

Similarly, there exists $k_q \in Hom_\Lambda(\underline{K}(C'_q\otimes C(\mathbb{T})), \underline{K}(B\otimes M_q))$ such that

$$k_q|_{\beta\left(K_1(C_q')\right)} = -\theta_q \circ \left(l_q'\right)_{*1} \circ \beta^{-1} \ and \ k_q|_{\beta\left(K_1(C_q')\right)} = -\alpha_q \circ \left(l_q'\right)_{*0} \circ \beta^{-1}.$$

Note that since $g_{i,r} \in ker\rho_{A\otimes M_r}, k_r(\beta(K_1(C_r'))) \subseteq ker\rho_{B\otimes M_r}, r = p \ or \ r = q$. By Theorem (1.2.15), there exist unitaries $w_p \in B \otimes M_p$ and $w_q \in B \otimes M_q$ such that

$$\left\|\left[w_p, \left(\phi \otimes id_{M_p}\right)(x)\right]\right\| < \delta_2'/8, \qquad \left\|\left[w_p, \left(\phi \otimes id_{M_q}\right)(y)\right]\right\| < \delta_2'/8,$$

for any $x \in \mathcal{H}^{0'} \otimes \mathcal{H}^{p'}$ and $y \in \mathcal{H}^{0'} \otimes \mathcal{H}^{q'}$, and

$$Bott\left(\phi \otimes id_{M_p}, w_p\right) \circ [l_p'] = k_p \circ \beta \ \ and \ \ Bott\left(\phi \otimes id_{M_q}, w_q\right) \circ [l_q'] = k_q \circ \beta.$$

For $r = p$ or $r = q$ and each $1 \leq j \leq k$, define

$$\zeta_{j,w_r u_r} = \overline{\langle\left(1_n - (\phi \otimes id_{M_r})(p_{j,r}') + \left((\phi \otimes id_{M_r})(p_{j,r}')\right) w_r u_r\right)\left(1_n - (\phi \otimes id_{M_r})(q_{j,r}') + \left((\phi \otimes id_{M_r})(q_{j,r}')\right) u_r^* w_r^*\right)\rangle}.$$

It is element in $U(B \otimes M_r)/CU(B \otimes M_r)$.

Define the map $\Gamma_r: \mathbb{Z}^K \to U(B \otimes M_P)/CU(B \otimes M_P)$ by

$$\Gamma_r(x'_{j,r}) = \zeta_{j,w_r,u_r}, \qquad 1 \leq j \leq k.$$

C_r (in the place of C), $G(x'_{1,r}, \ldots, x'_{K,r})$ (in the place of G), $B \otimes M_r$ (in the place of A), and $\left(\phi \otimes id_{M_r}\right)\big|_{C_r}$ (in the place of ϕ), there is a unitary $c_r \in B \otimes M_r$ such that

$$\left\|c_r, \left(\phi \otimes id_{M_r}\right)(x)\right\| < \delta_2'/16$$

for any $x \in \mathcal{H}^{0'} \otimes \mathcal{H}^{r'}$,

$$Bott\left(\phi \otimes id_{M_r}, c_r\right)\big|_{\iota_r}(\mathcal{P}') = 0,$$

$$dist\left(\zeta_{j,c_r^*}, \Gamma_r\left(x_{j,r}\right)\right) \leq \gamma/(32(1 + \sum_{i,j}|Mr_{ij}|)), \quad 1 \leq j \leq k \qquad (70)$$

where

$$\zeta_{j,c_r^*} = \overline{\langle(1_n - \left(\phi \otimes id_{M_r}\right)\left(p'_{j,r}\right) + \left(\left(\phi \otimes id_{M_r}\right)\left(p'_{j,r}\right)\right) c_r^*)\left(1_n - \left(\phi \otimes id_{M_r}\right)\left(q'_{j,r}\right) + ((\phi \otimes id_{M_r})(q'_{j,r}))c_r^*)\right\rangle}$$

Put $v_r = c_r w_r u_r$. Then, by (81) and (70), for $1 \leq j \leq k$

$$dist(\zeta_{j,v_r}, \overline{(1_{B\otimes M_r})n}) < dist(\zeta_{j,c_r^*}, \zeta_{j,w_r u_r}) + \gamma/(32(1 + \sum_{i,j}|Mr_{ij}|))$$
$$< \gamma/(16(1 + \sum_{i,j}|Mr_{ij}|)), \qquad (71)$$

where

$$\zeta_{j,v_r} = \overline{\langle(1_n - \left(\phi \otimes id_{M_r}\right)\left(p'_{j,r}\right) + \left(\left(\phi \otimes id_{M_r}\right)\left(p'_{j,r}\right)\right) v_r)\left(1_n - \left(\phi \otimes id_{M_r}\right)\left(q'_{j,r}\right) + ((\phi \otimes id_{M_r})(q'_{j,r}))v_r)\right\rangle}.$$

Recall that $[x'_j] = [p'_j] - [q'_j]$. Define

$$\zeta_{x'_j,v_r} = \overline{\langle(1_n - \phi(p'_j) \otimes 1_{M_r} + (\phi(p'_j) \otimes 1_{M_r})v_r)(1_n - \phi\left(q'_j\right) \otimes 1_{M_r} + \left(\phi\left(q'_j\right) \otimes 1_{M_r}\right) v_r^*)\rangle}.$$

one has

$$dist\left(\zeta_{x'_j,v_r}, \zeta_{j,v_r}\right) < \gamma/(16(1 + \sum_{i,j'}|Mr_{ij'}|)),$$

and hence by (39),

$$\operatorname{dist}\left(\zeta_{x'_j,v_r}, \overline{(1_{B\otimes M_\mathfrak{r}})n)}\right) < \gamma/(8(1+\sum_{i,j'}|Mr_{ij'}|)).$$

Regard $\zeta_{x'_j,v_r}$ as its image in $B\otimes Q$, one has

$$\operatorname{dist}\left(\zeta_{x'_j,v_r}, \overline{(1_{B\otimes Q})n)}\right) < \gamma/(8(1+\sum_{i,j'}|Mr_{ij'}|)),$$

and hence for any $1 \le i \le m$,

$$\operatorname{dist}(\prod_{j=1}^{k}(\zeta_{x'_j,v_r})^{Mr_{ij}}, \overline{(1_{B\otimes Q})n)} < \gamma/8.$$

One has

$$\operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_r)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)) +(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_\mathfrak{r}^*)\rangle^\mathrm{M}}, \overline{(1_{B\otimes Q})n)} < \gamma/4,$$

$$\operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_r)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)) +(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_\mathfrak{r}^*)\rangle}, \overline{(1_{B\otimes Q})n)} < \gamma/(4M) < \gamma/4.$$

In particular,

$$\operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_qv_p^*)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)) +(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_pv_q^*)\rangle}, \overline{(1_{B\otimes Q})n)}$$
$$\le \operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_q)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)) +(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_q^*)\rangle}, \overline{(1_{B\otimes Q})n)} + \operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+ (\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_p)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_p^*)\rangle}, \overline{(1_{B\otimes Q})n)} < \gamma/2$$

That is

$$\operatorname{dist}\left(\zeta_{i,v_qv_p^*}, \overline{1_n}\right) < \gamma/2, \tag{72}$$

where

$$\zeta_{i,v_qv_p^*} = \operatorname{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)+(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{p}_1)v_qv_p^*)(1-(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)) +(\phi\otimes \mathrm{id}_\mathrm{Q})(\mathrm{q}_1)v_pv_q^*)\rangle}$$

Moreover, one also has

$$\|\psi\otimes \mathrm{id}_\mathrm{Q}(x) - v_p^*(\phi\otimes \mathrm{id}_\mathrm{Q}(x))v_p\| < \delta_2'/4, \quad \forall\, x\in \mathcal{H}^{0'}\otimes\mathcal{H}^{p'}\otimes\mathcal{H}^{q'} \text{ and}$$
$$\|\psi\otimes \mathrm{id}_\mathrm{Q}(x) - v_q^*(\phi\otimes \mathrm{id}_\mathrm{Q}(x))v_q\| < \delta_2'/4, \quad \forall\, x\in \mathcal{H}^{0'}\otimes\mathcal{H}^{p'}\otimes\mathcal{H}^{q'}$$

Hence

$$\|v_pv_q^*, \phi(x)\otimes 1_Q\| < \delta_2'/2, \quad \forall\, x\in\mathcal{H}'$$

Thus $\mathrm{Bott}(\phi\otimes \mathrm{id}_\mathrm{Q}, v_pv_q^*)$is well defined on the subgroup generated by $\mathcal{P}$.

Moreover, a direct calculation shows that

$\mathrm{bott}(\phi\otimes \mathrm{id}_\mathrm{Q}, v_pv_q^*)\circ(\ell_\infty)_{*1}(z)$

$= \mathrm{bott}_1(\phi\otimes \mathrm{id}_\mathrm{Q}, c_p)\circ(\ell_\infty)_{*1}(z) + \mathrm{bott}_1(\phi\otimes \mathrm{id}_\mathrm{Q}, w_p)\circ(\ell_\infty)_{*1}(z)$

$+\mathrm{bott}(\phi\otimes \mathrm{id}_\mathrm{Q}, u_pu_q^*)\circ(\ell_\infty)_{*1}(z) + \mathrm{bott}_1(\phi\otimes \mathrm{id}_\mathrm{Q}, w_q^*)\circ(\ell_\infty)_{*1}(z)$

$+\mathrm{bott}_1(\phi\otimes \mathrm{id}_\mathrm{Q}, c_q^*)\circ(\ell_\infty)_{*1}(z)$

$= (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_p}, \quad c_p) \circ \left(\ell_p\right)_{*1}(z) + (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_p}, \quad w_p) \circ \left(\ell_p\right)_{*1}(z) + \mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, \quad u_p u_q^*) \circ (\ell_\infty)_{*1}(z) + (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_q}, \quad w_q^*) \circ \left(\ell_p\right)_{*1}(z) + (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_q}, c_q^*) \circ \left(\ell_p\right)_{*1}(z)$

$= (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_p}, w_p) \circ \left(\ell_p\right)_{*1}(z) + \mathrm{bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, u_p u_q^*) \circ (\ell_\infty)_{*1}(z)$
$+ (\mathrm{j}_p)_{*0} \circ \mathrm{bott}_1(\phi \otimes \mathrm{id}_{M_q}, w_q^*) \circ \left(\ell_q\right)_{*1}(z)$

$= -(\mathrm{j}_p)_{*0} \circ \theta_p \circ \left(\ell_p\right)_{*1}(z) + \left(\left(\mathrm{j}_p\right)_{*0} \circ \theta_p \circ \left(\ell_p\right)_{*1} + \left(\mathrm{j}_q\right)_{*0} \circ \theta_q \circ \left(\ell_q\right)_{*1}\right) - \left(\mathrm{j}_q\right)_{*0} \circ \theta_q \circ \left(\ell_q\right)_{*1}(z)$

$= 0$ for all $z \in G(\mathcal{P})_{1,0}$.

The same argument shows that $\mathrm{bott}_0(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*) = 0$ on $G(\mathcal{P})_{0,0}$ Now, for any $g \in G(\mathcal{P})_{1,\infty,0}$ there is $z \in G(\mathcal{P})_{1,0}$ and integers k, m such that $(k/m)z = g$. From the above,

$$\mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)(mg) = k\mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)(z) = 0. \tag{73}$$

Since $K_0(B \otimes Q)$ is torsion free, it follows that $\mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)(g) = 0$.
for all $g \in G(\mathcal{P})_{1,\infty,0}$ So it vanishes on $\mathcal{P} \cap K_1(A \otimes Q)$. Similarly,
$\mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)|_{\mathcal{P} \cap K_1(A \otimes Q)} = 0$ on $\mathcal{P} \cap K_0(A \otimes Q)$.

Since $K_i(B \otimes Q, \mathbb{Z}/m\mathbb{Z}) = \{0\}$ for all $m \geq 2$, we conclude that $\mathrm{Bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)\,|_{\mathcal{P}} = 0$ on the subgroup generated by $\mathcal{P}$

Since $[\phi] = [\psi]$ in $KL(A, B), \phi_\# = \psi_\#$ *and* $\phi^\ddagger = \psi^\ddagger$, one has that

$$[\phi \otimes \mathrm{id}_{\mathrm{Q}}] = [\psi \otimes \mathrm{id}_{\mathrm{Q}}] \text{ in } KL(A \otimes Q, B \otimes Q), \tag{74}$$

$$(\phi \otimes \mathrm{id}_{\mathrm{Q}})_\# = (\psi \otimes \mathrm{id}_{\mathrm{Q}})_\# \ \ and \ \ (\phi \otimes \mathrm{id}_{\mathrm{Q}})^\ddagger = (\psi \otimes \mathrm{id}_{\mathrm{Q}})^\ddagger \tag{75}$$

Therefore, $\phi \otimes \mathrm{id}_{\mathrm{Q}}$ and $\psi \otimes \mathrm{id}_{\mathrm{Q}}$ are approximately unitarily equivalent. Thus there exists a unitary $u \in B \otimes Q$ such that

$$\left\|u^*(\phi \otimes \mathrm{id}_{\mathrm{Q}})(c)u - (\psi \otimes \mathrm{id}_{\mathrm{Q}})(c)\right\| < \delta_2'/8 \quad for\ all\ \ c \in \varepsilon \cup \mathcal{H}' \tag{76}$$

It follows that

$$\left\|u v_q^*(\phi(c) \otimes 1_{\mathrm{Q}}) v_p u^* - \psi(c) \otimes 1_{\mathrm{Q}}\right\| < \delta_2'/2 + < \delta_2'/8 \quad \forall\, c \in \mathcal{G}'$$

By the choice of δ_2' and $\mathcal{H}'$, $\mathrm{Bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)$ is well defined on $[\iota](K(C'))$, and

$$\left|\tau\, \mathrm{bott}_1(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p v_q^*)(z)\right| < \delta_2/2 \quad \forall\, \tau \in T(B), \forall z \in \mathcal{G}.$$

There exists a unitary $y_p \in B \otimes Q$ such that

$$\left\|[y_p, (\phi \otimes \mathrm{id}_{\mathrm{Q}})(h)]\right\| < \delta/2, \qquad \forall h \in \mathcal{H},$$

and $\mathrm{Bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, y_p) = \mathrm{Bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, v_p u^*)$ on the subgroup generated by $\mathcal{P}$.

For each $1 \leq i \leq m$, define

$$\zeta_{i,y_p,uv_p^*} = \overline{\langle (1_n - (\phi \otimes \iota d_{\mathrm{Q}})(p_\iota) + ((\phi \otimes \iota d_{\mathrm{Q}})(p_\iota) y_p u v_p^*)(1_n - (\phi \otimes \iota d_{\mathrm{Q}})(q_\iota) + (\phi \otimes \iota d_{\mathrm{Q}})(q_\iota)) v_p u^* y_p^*) \rangle},$$

and define the map $\Gamma : Z^m \to U(B \otimes Q)/CU(B \otimes Q)$ by $\Gamma(x_i) = \zeta_{i,y_p,uv_q^*}$.

Applying Corollary (1.2.15)to C and $G(Q)$, there is a unitary $c \in B \otimes Q$ such that

$$\left\|[c, (\phi \otimes \mathrm{id}_{\mathrm{Q}})(h)]\right\| < \delta/4, \qquad \forall h \in \mathcal{H}$$

$\mathrm{Bott}(\phi \otimes \mathrm{id}_{\mathrm{Q}}, c)\,|_{\mathcal{P}} = 0$
and for any $1 \leq i \leq k$,

$$\mathrm{dist}(\zeta'_{i,c^*}, \Gamma(x_i)) \leq \gamma/2,$$

$$\zeta'_{i,c^*} = \overline{\langle(1_n - (\phi \otimes \iota d_Q)(p_\iota) + (\phi \otimes \iota d_Q)(p_\iota)c^*)(1_n - (\phi \otimes \iota d_Q)(q_\iota)} \overline{+(\phi \otimes \iota d_Q)(q_\iota))c)\rangle},$$

Consider the unitary $v = cy_p u$, one has that

$$\|[v, (\phi \otimes \mathrm{id}_Q)(h)]\| < \delta/4, \quad \text{for all} \quad h \in \mathcal{H} \quad \mathrm{Bott}(\phi \otimes \mathrm{id}_Q, vv_p^*) = 0$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \leq i \leq m$,

$$\mathrm{dist}\left(\zeta'_{i,vv_p^*}, \overline{1_n}\right) \leq \gamma/2, \tag{77}$$

where

$$\zeta'_{i,vv_p^*} = \overline{\langle(1_n - (\phi \otimes \iota d_Q)(p_\iota) + (\phi \otimes \iota d_Q)(p_\iota)vv_p^*)(1_n - (\phi \otimes \iota d_Q)(q_\iota)} \overline{+(\phi \otimes \iota d_Q)(q_\iota))v_p v^*)\rangle},$$

By the construction of Δ, it is clear that

$$\mu_{\tau\circ(\psi\otimes 1)}(O_a) \geq \Delta(a)$$

for all a, where O_a is any open ball of X with radius a; in particular, it holds for all $a \geq d$. Applying Theorem (1.2.13) to C and $\mathrm{Bott}(\phi \otimes \mathrm{id}_Q)|_c$, one obtains a continuous path of unitaries $v(t)$ in $B \otimes Q$ such that $v(0) = 1$ and $v(t_1) = vv_p^*$ and

$$\|[z_p(t), (\phi \otimes \mathrm{id}_Q)(c)]\| < \epsilon/2, \quad \forall x \in \varepsilon, \quad \forall t \in [0, t_1]. \tag{78}$$

Note that

$$\mathrm{Bott}(\phi \otimes \mathrm{id}_Q, v_q v^*) = \mathrm{Bott}(\phi \otimes \mathrm{id}_Q, v_q v_p^* v_p v^*) \tag{79}$$

$$= \mathrm{Bott}(\phi \otimes \mathrm{id}_Q, v_q v_p^*) + \mathrm{Bott}(\phi \otimes \mathrm{id}_Q, v_p v^*) \tag{80}$$

$$= 0 + 0 = 0 \tag{81}$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \leq i \leq m$,

$$\mathrm{dist}\left(\zeta'_{i,v_q v^*}, \overline{1}\right) \tag{82}$$

$$\leq \mathrm{dist}\left(\zeta'_{i,v_q v_p^*}, \overline{1}\right) + \mathrm{dist}\left(\zeta'_{i,v_p v^*}, \overline{1}\right) \tag{83}$$

$$= \gamma, \qquad (\text{by } (98) \text{ and } (127)) \tag{84}$$

where

$$\zeta'_{i,v_q v^*} = \overline{\langle(1 - (\phi \otimes \iota d_Q)(p_\iota) + (\phi \otimes \iota d_Q)(p_\iota)v_q v^*)(1 - (\phi \otimes \iota d_Q)(q_\iota)} \overline{+(\phi \otimes \iota d_Q)(q_\iota))vv_q^*)\rangle},$$

Theorem (1.2.13) implies that there is a path of unitaries $z_q(t) : [t_{m-1}, 1] \to U(A \otimes Q)$ such that $z_q(t_{m-1}) = vv_q^*$, $z_q(1) = 1$ and

$$\|[z_p(t), \phi \otimes \mathrm{id}_Q(c)]\| < \epsilon/8, \quad \forall t \in [t_{m-1}, 1] \quad \forall c \in \varepsilon. \tag{85}$$

Consider the unitary

$$v(t) = \begin{cases} z_p(t)v_p, & \text{if} \quad 0 \leq t \leq t_1, \\ v, & \text{if} \quad t_1 \leq t \leq t_{m-1}, \\ z_p(t)v_p, & \text{if} \quad t_{m-1} \leq t \leq t_m. \end{cases}$$

Then, for any $t_i, 0 \leq i \leq m$, one has that

$$\|v^*(t_i)(\phi \otimes \mathrm{id}_Q)(c)v(t_i) - (\psi \otimes \mathrm{id}_Q)(c)\| < \epsilon/2, \quad \forall c \in \varepsilon. \tag{86}$$

Then for any $t \in [t_i, t_{i+1}]$ with $1 \le j \le m - 2$, one has

$$\|v^*(t)(\phi \otimes \mathrm{id}(a \otimes b(t)))v(t) - \psi \otimes \mathrm{id}(a \otimes b(t))\| \tag{87}$$

$$= \|v^*(\phi(a) \otimes b(t)))v - \psi(a) \otimes b(t))\| \tag{88}$$

$$< \|v^*(\phi(a) \otimes b(t_j)))v - \psi(a) \otimes b(t_j))\| + \epsilon/4 \tag{89}$$

$$< \epsilon/4 + \epsilon/4 = \epsilon/2. \tag{90}$$

For any $t \in [0, t_1]$, one has that for any $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$,

$$\|v^*(t)(\phi \otimes \mathrm{id}(a \otimes b(t)))v(t) - \psi \otimes \mathrm{id}_Q(a \otimes b(t))\| \tag{91}$$

$$= \|v_p^* z_p^*(\phi(a) \otimes b(t)))z_p(t)v_p - \psi(a) \otimes b(t))\| \tag{92}$$

$$< \|v_p^* z_p^*(\phi(a) \otimes b(t_0)))z_p(t)v_p - \psi(a) \otimes b(t_0))\| + \epsilon/2 \tag{93}$$

$$< \|v_p^*(\phi(a) \otimes b(t_0))v_p - \psi(a) \otimes b(t_0))\| + 3\epsilon/2 \tag{94}$$

$$3\epsilon/2 + \epsilon/4 = \epsilon. \tag{95}$$

The same argument shows that for any $t \in [t_{m-1}, 1]$, one has that for any $a \in \mathcal{F}_1$ and $b \in \mathcal{F}_2$

$$\|v^*(t)(\phi \otimes \mathrm{id}(a \otimes b(t)))v(t) - \psi \otimes \mathrm{id}(a \otimes b(t))\| < \epsilon. \tag{96}$$

Therefore, one has

$$\|v(\phi \otimes \mathrm{id}(f))v - \psi \otimes \mathrm{id}(f)\| < \epsilon \qquad \text{for all} \quad f \in \mathcal{F}.$$

$$[\phi] = [\psi] \text{ in } KL(A,B), \phi_\# = \psi_\# \; and \; \phi^\ddagger = \psi^\ddagger. \tag{97}$$

Theorem (1.2.19)[98]:

Let A be a $\mathcal{Z}$-stable C^*-algebra such that $A \otimes M_R$ is an AH-algebra for any supernatural number $\mathfrak{r}$ of infinite type, and let $B \in C$ be a unital separable $\mathcal{Z}$-stable C^*-algebras.

If ϕ and ψ are two monomorphisms from A to B with

$$[\phi] = [\psi] \text{ in } KL(A,B), \phi_\# = \psi_\# \; and \; \phi^\ddagger = \psi^\ddagger. \tag{98}$$

then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subseteq A$, there exists a unitary $u \in B$ such that

$$\|u^*\phi(a) - \psi(a)\| < \epsilon \quad \text{for all} \quad a \in \mathcal{F}. \tag{99}$$

Proof :

Let $\alpha : A \to A \otimes Z$ and $\beta : \mathcal{Z} \to \mathcal{Z} \otimes \mathcal{Z}$ be isomorphisms. Consider the map

$$\Gamma_A : A \xrightarrow{\alpha} A \otimes \mathcal{Z} \xrightarrow{id \otimes \beta} A \otimes \mathcal{Z} \otimes \mathcal{Z} \xrightarrow{\alpha^- \otimes id} A \otimes \mathcal{Z}.$$

Then Γ is an isomorphism. However, since β is approximately unitarily equivalent to the map

$$\mathcal{Z} \ni a \mapsto a \otimes 1 \in \mathcal{Z} \otimes \mathcal{Z},$$

the map Γ_A is approximately unitarily equivalent to the map

$$A \ni a \mapsto a \otimes 1 \in A \otimes \mathcal{Z}.$$

Hence the map $\Gamma_B \circ \phi \circ \Gamma_A$ is approximately unitarily equivalent to $\phi \otimes \mathrm{id}_{\mathcal{Z}}$. The same argument shows that $\Gamma_B \circ \psi \circ \Gamma_A$is approximately unitarily equivalent to $\psi \otimes \mathrm{id}_{\mathcal{Z}}$. Thus, in order to prove the theorem, it is enough to show that $\phi \otimes \mathrm{id}_{\mathcal{Z}}$ is approximately unitarily equivalent to $\psi \otimes \mathrm{id}_{\mathcal{Z}}$.

Since $\mathcal{Z}$ is an inductive limit of C^*-algebras $\mathcal{Z}_{p,q}$,it is enough to show that $\phi \otimes \mathrm{id}_{\mathcal{Z}_{p,q}}$ isapproximately unitarily equivalent to $\psi \otimes \mathrm{id}_{\mathcal{Z}_{p,q}}$, and this follows from Lemma (1.2.18).

The range of approximate equivalence classes of homomorphisms.

Now let A and B be two unital C^* −algebras in $N \cap C$. States that two unital monomorphisms are approximately unitarily equivalent if they induce the same element in $KLT_e(A,B)^{++}$ and the same map on $U(A)/CU(A)$. In this section, we will discuss the following problem: Suppose that one has $k \in KLT_e(A,B)^{++}$ and a continuous homomorphism $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ which is compatible with k. Is there always a unital monomorphism $\phi : A \to B$ such that ϕ induces k and $\phi^{\ddagger} = \gamma$? At least in the case that $K_1(A)$ is free, states that such ϕ always exists.

Lemma (1.2.20)[98]:

Let A and B be two unital infinite dimensional separable stably finite C^* −algebras whose tracial simplexes are non-empty. *Let* $\gamma : U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ be a continuous homomorphism, $h_i: K_i(A) \to K_i(B)$ $(i = 0,1)$ be homomorphisms for which h_0 is positive, and let $\lambda : \mathrm{Aff(T(A))} \to \mathrm{Aff(T(B))}$ be an affine map so that $(h_0, h_1, \lambda, \gamma)$ are compatible. Let p be a supernatural number. Then γ induces a unique homomorphism $\gamma_p : U_\infty(A_p)/CU_\infty(A_p) \to U_\infty(B_p)/CU_\infty(B_p)$ which is compatible with $(h_p)_i(i = 0,1)$ and γ_p, where $A_p = A \otimes M_p$ and $B_p = B \otimes M_p$, and $(h_p)_i$: $K_i(A) \otimes \mathbb{Q}_p \to K_i(B) \otimes \mathbb{Q}_p$ is induced by h_i $(i = 0,1)$. Moreover, the diagram

$$\begin{array}{ccc} U_\infty(A)/CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B)/CU_\infty(B) \\ \downarrow_{\iota_p^{\ddagger}} & & \downarrow_{(\iota_p')^{\ddagger}} \\ U_\infty(A_p)/CU(A_p) & \xrightarrow{\gamma_P} & U_\infty(B_P)/CU_\infty(B_P) \end{array}$$

commutes, where $\iota_p : A \to A_p$ and $\iota_p: B \to B_P$ are the maps induced by $a \mapsto a \otimes 1$ and $b \mapsto b \otimes 1$,respectively.

Proof. Denote by $A_0 = A$, $A_p = A \otimes M_p$, $B_0 = B$ and $B_P = B \otimes M_P$. By a result of K.

Thomsen ([133]), using the de la Harpe and Skandalis determinant, one has the following short exact sequences:

$$0 \to \mathrm{Aff(T(A_i))}/\overline{\rho_A(K_0(\mathrm{A_i}))} \to U_\infty(\mathrm{A_i})/CU_\infty(\mathrm{A_i}) \to K_1(\mathrm{A_i}) \to 0, i = 0, \mathfrak{p},$$

and

$$0 \to \mathrm{Aff(T(B_i))}/\overline{\rho_A(K_0(\mathrm{B_i}))} \to U_\infty(\mathrm{B_i})/CU_\infty(\mathrm{B_i}) \to K_1(\mathrm{B_i}) \to 0, i = 0, \mathfrak{p}.$$

Note that, in all these cases, $\mathrm{Aff(T(A_i))}/\overline{\rho_A(K_0(\mathrm{A_i}))}$ and $\mathrm{Aff(T(B_i))}/\overline{\rho_A(K_0(\mathrm{B_i}))}$ are divisible groups, $i = 0, \mathfrak{p}$. Therefore the exact sequences above splits. Fix splitting maps $s'_i: K_1(\mathrm{A_i}) \to U_\infty(\mathrm{A})/CU_\infty(\mathrm{A_i})$ and $s'_i: K_1 U_\infty(\mathrm{B})/CU_\infty(\mathrm{B_i})$, $i = 0, \mathfrak{p}$, for the above two splitting short exact sequences. Let $\iota_p : A \to A_p$ be the homomorphism defined by $\iota_p(a) = a \otimes 1$ for all $a \in A$ and $\iota_p : B \to B_P$ be the homomorphism defined by $\iota_p(b) = b \otimes 1$ for all $b \in B$. Let $(\iota_p')^{\ddagger}$: $U_\infty(A)/CU_\infty(A) \to U_\infty(A_p)/CU_\infty(A)$ and $(\iota_p')^{\ddagger}: U_\infty(\mathrm{B})/CU_\infty(B) \to U_\infty(B_p)/CU_\infty(B_p)$ be the induced maps. The map ι_p induces the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \mathrm{Aff(T(A))}/\overline{\rho_A(K_0(\mathrm{A}))} & \to & U_\infty(A)/CU_\infty(A) & \to K_1(A) & \to 0 \\ & \downarrow_{\overline{(\iota_p)_\#}} & & \downarrow_{\iota_p^{\ddagger}} & \downarrow_{(\iota_p)_{*1}} & \\ & \mathrm{Aff(T(A_p))}/\overline{\rho_A(K_0(\mathrm{A_p}))} & & U_\infty(A_I)/CU_\infty(\mathrm{A_p}) & K_1(\mathrm{A_p}) & \end{array}$$

Since there is only one tracial state on M_p, one may identify $T(A)$ with $T(\mathrm{A_p})$ and $T(B)$ with $\mathrm{T(B_p)}$. One may also identify $\overline{\rho_{\mathrm{A_p}}(K_0(\mathrm{A_p}))}$ with $\overline{\mathbb{R}_{\rho_\mathrm{A}}(K_0(\mathrm{A}))}$which is the closure of those elements $r[\widehat{p}]$ with $r \in R$. Note that $(h_p)_i$: $K_i(A \otimes M_p) \to K_i(B \otimes M_p)$ $(i = 0, 1)$ is given by the K¨unneth formula. Since γ is compatible with λ, γ maps $\overline{\mathbb{R}_{\rho_A}(K_0(A))}/\overline{\rho_A(K_0(A))}$ into $\overline{\mathbb{R}_{\rho_B}(K_0(B))}/\overline{\rho_B(K_0(B))}$. Note that

$$\ker(\iota_p)_{*1} = \{x \in K_1(A): px = 0\, for\, (6.1)\text{some factor}\, p\, of\, p\} \quad (100)$$

and

$$\ker(\iota'_p)_{*1} = \{x \in K_1(B): px = 0\, for\, (6.1)\text{some factor}\, p\, of\, p\}. \quad (101)$$

Therefore

$$\ker\left(\iota_p^{\ddagger}\right) = \{x + s_0(y) : x \in \overline{\mathbb{R}_{\rho_A}\left(K_0(A)\right)}/\overline{\rho_A\left(K_0(A)\right)}, y \in ker((\iota_p)_{*1})\} \quad (102)$$

and

$$\ker(\iota_p^{'})^{\ddagger} = \{x + s_0^{'}(y) : x \in \overline{\mathbb{R}_{\rho_A}\left(K_0(B)\right)}/\overline{\rho_B\left(K_0(B)\right)}, y \in ker(\left(\iota_p\right)_{*1})\} \quad (103)$$

If $y \in \ker((\iota_p)_{*1})$, then, for some factor p of $p, py = 0$. It follows that $p\gamma(s_0(y)) = 0$. Therefore $\gamma(s_0(y))$ must be in $\ker((\iota_p^{'})^{\ddagger})$ It follows that

$$\gamma\left(\ker\left(\iota_p^{\ddagger}\right)\right) \subset \ker(\iota'_p)^{\ddagger} \quad (104)$$

This implies that γ induces a unique homomorphism γ_P such that the following diagram commutes:

$$\begin{array}{ccc} U_\infty(A)/CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B)/CU_\infty(B) \\ \downarrow_{\iota_p^{\ddagger}} & & \downarrow_{(\iota'_p)^{\ddagger}} \\ U_\infty(A_p)/CU(A_p) & \xrightarrow{\gamma_P} & U_\infty(B_P)/CU_\infty(B_P) \end{array}$$

The lemma follows.

Lemma (1.2.21)[98]:

Let A and B be two unital infinite dimensional separable stably finite C^* −algebras whose tracial simplexes are non-empty. Let $\gamma : U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ be a continuous homomorphism, h_i: $K_i(A) \to K_i(B)$ $(i = 0, 1)$ be homomorphisms and λ : Aff(T(A)) → Aff(T(B)) be an affine homomorphism which are compatible. Let p and q be two relatively prime supernatural numbers such that $M_p \otimes M_q = Q$. Denote by ∞ the supernatural number associated with the product ꝑ and ꝗ. Let $E_B: B \to B \otimes \mathcal{Z}_{p,q}$ be the embedding defined by $E_B(b) = b \otimes 1, \forall b \in B$. Then

$$(\pi_t \circ E_B)^{\ddagger} \circ \gamma = \gamma_\infty \circ \iota_\infty^{\ddagger} \quad \text{for all } t \in (0,1) \quad (105)$$

$$(\pi_0 \circ E_B)^{\ddagger} \circ \gamma = \gamma_p \circ \iota_p^{\ddagger} \quad \text{and} \quad (106)$$

$$(\pi_1 \circ E_B)^{\ddagger} \circ \gamma = \gamma_q \circ \iota_q^{\ddagger} \quad (107)$$

with the notation of (1.2.20) where $\pi_t: \mathcal{Z}_{p,q} \to Q$ is the point-evaluation at t.

Proof:

Fix $z \in U_\infty(B)/CU_\infty(B)$. Let $u \in U_n(B)$ for some integer $n \geq 1$ such that $\bar{u} = z$ in $U_\infty(B)/CU_\infty(B)$. Then

$$E_B^{\ddagger}(z) = \overline{u \otimes 1} \quad (108)$$

In other words, $E_B^{\ddagger}(z)$ is represented by $w(t) \in M_n(B \otimes \mathcal{Z}_{p,q})$ for which

$$w(t) = u \otimes 1 \text{ for all } t \in [0,1]. \tag{109}$$

Therefore, for any $t \in (0,1)$, $\pi_t \circ E_B^{\ddagger}(z)$ may be written as

$$\pi_t \circ E_B^{\ddagger}(z) = \overline{u \otimes 1} \quad \text{in} \quad U_\infty(\mathrm{B} \otimes \mathrm{Q})/CU_\infty(\mathrm{B} \otimes \mathrm{Q}). \tag{110}$$

This implies that

$$\pi_t \circ E_B^{\ddagger}(z) = (\iota_\infty)^{\ddagger}(z) \quad \text{for all} \quad z \in U_\infty(\mathrm{B}\,)/CU_\infty(\mathrm{B}\,). \tag{111}$$

where $\iota_\infty : B \rightarrow B \otimes Q$ is defined by $\iota_\infty(b) = b \otimes 1$ for all $b \in B$.

$$(\pi_t \circ E_B)^{\ddagger} \circ \gamma = \gamma_\infty \circ \iota_\infty^{\ddagger} \qquad \text{for all} \quad \mathrm{t} \in (0,1) \tag{112}$$

The identities (106) and (107) for end points exactly follow from the same arguments.

Lemma (1.2.22)[98]:

Let A be a unital AH-algebra and let B be a unital separable simple amenable C^* −algebra with $TR(B) \leq 1$. Suppose that $\phi_1, \phi_2 : A \rightarrow B$ are two monomorphisms such that

$$[\phi_1] = [\phi_2] \text{ in } KK(A,B), (\phi_1)_\# = (\phi_2)_\# \, and \;\; \phi_1^{\ddagger} = \phi_2^{\ddagger}. \tag{113}$$

Then there exists a monomorphism $\beta: \phi_2(A) \rightarrow B$ such that $[\beta \circ \phi_2] = [\phi_2]$in $KK(A,B), (\beta \circ \phi_2)_\# = \phi_{2_\#}, (\beta \circ \phi_2)^{\#} = \phi_2^{\ddagger}$ and $\beta \circ \phi_2$ is asymptotically unitarily equivalent to ϕ_1. Moreover, if $H_1(K_0(A), K_1(B)) = K_1(B)$, they are strongly asymptotically unitarily equivalent, where $H_1(K_0(A), K_1(B))$ $= \{x \in K_1(B): \psi([1_A]) = x \text{ for some } \psi \in \mathrm{Hom}(K_0(A), K_1(B))\}$.

Proof:

There is a monomorphism $\beta \in \overline{\mathrm{Inn}}(\phi_2(A), B)$ such that $[\beta] = [\iota]$ in $KK(\phi_2(A), B)$ and

$$\bar{R}_{\iota,\beta} = -\bar{R}_{\phi_1,\phi_2}$$

where l is the embedding of $\phi_2(A)$ to B and $\bar{R}_{\iota,\beta}$ is viewed as a homomorphism from $K_1(A) = K_1(\phi_2(A))$ *to* Aff(T(B)). In other words

$$\bar{R}_{\phi_2,\beta\circ\phi_2} = -\bar{R}_{\phi_1,\phi_2}\,. \tag{114}$$

One also has that

$$[\phi_2] = [\beta \circ \phi_2] in\, KK(A,B), \tag{115}$$

$$(\beta \circ \phi_2)_\# = (\phi_2)_\# \quad \text{and} \quad (\beta \circ \phi_2)^{\#} = \phi_2^{\ddagger} \tag{116}$$

$$[\phi_1] = [\beta \circ \phi_2]]\; in\, KK(A,B), \tag{117}$$

$$(\phi_1)_\# = (\beta \circ \phi_2)_\# and\; \phi_1^{\ddagger} = (\beta \circ \phi_2)^{\#} \tag{118}$$

It follows from (100) and (115) that

$$c = \bar{R}_{\phi_1,\phi_2} = \bar{R}_{\phi_2,\beta\circ\phi_2} = 0. \tag{119}$$

Therefore, it follows from Theorem (1.2.13) of [97] that the map ϕ_1 and $\beta \circ \phi_2$ are asymptotically unitarily equivalent.

In the case that $H_1(K_0(A), K_1(B)) = K_1(B)$ of [97] that $\beta \circ \phi_2$ and ϕ_1 are strongly asymptotically unitarily equivalent.

Lemma (1.2.23)[98]:

Let C and A be two unital separable stably finite C^* −algebras. Suppose that $,\psi : C \to A$ are two unital monomorphisms such that

$$[\phi] = [\psi] \text{ in } KL(C,A), \phi_{?} = \psi_{?} \ and\ \bar{R}_{\phi,\psi} = 0.$$

Suppose that $\{U(t): t \in [0,1)\}$ is a piecewise smooth and continuous path of unitaries in A with $U(0) = 1$ such that

$$\lim_{t\to 1} U^*(t)\phi(u)U(t) = \psi(u) \tag{120}$$

for some $u \in U(C)$ and suppose that there exists $w \in U(A)$ such that $\psi(u)\, w^* \in U_0(A)$. Let

$$Z = Z(t) = U^*(t)\phi(u)U(t)\, w^* \quad if \ \ t \in [0,1)$$

and $Z(1) = \psi(u)\, w^*$. Suppose also that there is a piecewise smooth continuous path of unitaries $\{z(s): s \in [0,1]\}$ in A such that $z(0) = \phi(u)\, w^*$ and $z(1) = 1$. Then, for any piecewise smooth continuous path $\{Z(t,s): s \in [0,1]|\} \subset C([0,1],A)$ of unitaries such that $Z(t,0) = Z(t)$ and $Z(t,1) = 1$, there is $f \in \rho_A(K_0(A))$ such that

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\,(\frac{dZ(t,s)}{ds}Z(t,s)^*)ds = \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\,(\frac{dZ(s)}{ds}Z(s)^*)ds + f(\tau) \tag{121}$$

for all $t \in [0,1]$ and $\tau\, T(A)$.

Proof:

Define

$$Z_1(t,s) = \begin{cases} U^*(t-2s)\phi(u)U(t-2s)\, w^* & \text{for s } \in [0, \text{t}/2) \\ \phi(u)\, w^* & \text{for s } \in [\text{t}/2, 1/2) \\ z(2s-1) & \text{for s } \in [1/2, 1] \end{cases} \tag{122}$$

For $t \in [0,1)$ and define

$$Z_1(t,s) = \begin{cases} \psi(u)\, w^* & \text{for s} = 0 \\ U^*(1-2s)\phi(u)U(1-2s)\, w^* & \text{for s } \in [0, 1/2) \\ z(2s-1) & \text{for s } \in [1/2, 1] \end{cases} \tag{123}$$

Thus $\{Z_1(t,s): s \in [0,1]\} \subset C([0,1],A)$ is a piecewise smooth continuous path of unitaries such that $Z_1(t,0) = Z(t) and\ Z_1(t,1) = 1$. Thus, there is an element $f_1 \in \rho_A(k_0(A))$, such that

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\left(\frac{dZ(t,s)}{ds}Z(t,s)^*\right)ds - \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\,(\frac{d\,Z_1(t,s)}{ds}\ Z_1(t,s)^*)ds$$

for all $\tau \in T(A)$ an for all $t \in [0,1]$.

On the other hand, let $V(t) = U^*(t)\phi(u)U(t)$ for $t \in [0,1)$ and $V(1) = \psi(u)$. For any $s \in [0,1)$, since $U(0) = 1$, $U(t) \in U(C([0,s],A))_0$(for $t \in [0,s]$). There there are $a_1, a_2, \dots, a_k \in U([0,s],A)_{s.a}$. such that

$$f_1(\tau) = \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\left(\frac{dZ(t,s)}{ds}Z(t,s)^*\right)ds - \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\left(\frac{d\,Z_1(t,s)}{ds}\,Z_1(t,s)^*\right)ds \qquad (124)$$

for all $\tau \in T(A)$ an for all $t \in [0,1]$.

On the other hand, let $V(t) = U^*(t)\phi(u)U(t)$ for $t \in [0,1)$ and $V(1) = \psi(u)$. For any $s \in [0,1)$, since $U(0) = 1, U(t) \in U(C([0,s],A))_0$ (for $t \in [0,s]$). There there are $a_1, a_2, \dots, a_k \in U([0,s],A)_{s.a}$such that

$$U(t) = \prod_{j=1}^{k} \exp(ia_j(t)) \quad \text{for all} \quad t \in [0,s]$$

Then a straightforward calculation shows that

$$\int_0^s \frac{dV(t)}{dt}V^*(t)dt = 0 \qquad (125)$$

we also have

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\frac{dV(t)}{dt}V^*(t)dt = R_{\phi,\psi}([V])(\tau) \coloneqq f(\tau) \in \rho_A(k_0(A))$$

for all $\tau \in T(A)$.

Then

$$\frac{1}{2\pi\sqrt{-1}}\int_0^{1/2} \tau\left(\frac{d\,Z_1(1,s)}{ds}\,Z_1(1,s)^*\right)ds = \frac{1}{2\pi\sqrt{-1}}\int_0^{1/2} \tau\left(\frac{dV(2s-1)}{ds}V(2s-1)^*\right)ds \qquad (126)$$

$$R_{\phi,\psi}([V])(\tau) = f(\tau) \quad \text{for all} \qquad \tau \in T(A). \qquad (127)$$

One computes that, for any $\tau \in T(A)$ and for any $t \in [0,1)$, by applying (126),

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau\left(\frac{d\,Z_1(t,s)}{ds}\,Z_1(t,s)^*\right)ds \qquad (128)$$

$$= \frac{1}{2\pi\sqrt{-1}}[\int_0^{t/2} \tau(\frac{\mathrm{d}(U^*(\mathrm{t}-2\mathrm{s})\phi(\mathrm{u})\mathrm{U}(\mathrm{t}-2\mathrm{s})w^*)}{ds}\ (U^*(\mathrm{t}-2\mathrm{s})\phi(\mathrm{u})\mathrm{U}(\mathrm{t}-2\mathrm{s})w^*)^*)\,ds] \quad (129)$$

$$\int_{t/2}^{1/2} \tau\left(\frac{d\,Z_1(t,s)}{ds}\,Z_1(t,s)^*\right)ds + \int_{1/2}^1 \tau\left(\frac{dz(s-1)}{ds}z(2s-1)^*\right)ds] \qquad (130)$$

$$= \frac{1}{2\pi\sqrt{-1}}[\int_0^{t/2} \tau\left(\frac{dV(t-2s)}{ds}V(t-2s)^*\right)ds + \int_{1/2}^1 \tau\left(\frac{dz(s-1)}{ds}z(2s-1)^*\right)ds] \qquad (131)$$

$$= 0 + \frac{1}{2\pi\sqrt{-1}}\int_{1/2}^1 \tau\left(\frac{dz(2s-1)}{ds}z(2s-1)^*\right)ds] \qquad (132)$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau\,(\frac{dz(s)}{ds} z(s)^*)\, ds \qquad (133)$$

It then follows from (126) that

$$= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau\,(\frac{d\,Z_1(1,s)}{ds}\; Z_1(1,s)^*) ds \qquad (134)$$

$$= \frac{1}{2\pi\sqrt{-1}} [\int_0^{1/2} \tau \left(\frac{d\,Z_1(1,s)}{ds}\; Z_1(1,s)^*\right) ds + \int_{1/2}^1 \tau \left(\frac{dz(2s-1)}{ds} z(2s-1)^* ds\right)] \qquad (135)$$

$$= f(\tau) + \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau\,(\frac{dz(s)}{ds} z(s)^*) ds \qquad (136)$$

The lemma follows.

Lemma (1.2.24)[98]:

Let A be a unital C^* −algebra satisfying that $A \otimes M_{\mathfrak{r}}$ is an AH-algebra for all supernatural number r with infinite type (in particular, all AH-algebra satisfies this property), and let B be a unital simple C^* −algebra in $\mathcal{N} \cap \mathcal{C}$. Let $\kappa \in KL_e(A,B)^{++}$ and $\lambda :$ Aff(T(A)) $\to Aff(T(B))$ be an affine homomorphism which are compatible (see Definition 1.2.3). Then there exists a unital homomorphism $\phi: A \to B$ such that

$$[\phi] = \kappa \; and \; (\phi)_{\sharp} = \lambda.$$

Moreover, if $\gamma \in U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ is a continuous homomorphism which is compatible with κ and λ, then one may also require that

$$\phi^{\ddagger}|_{U_\infty(A)_0/CU_\infty(A)} = \gamma|_{U_\infty(A)_0/CU_\infty(A)}\; \phi^{\ddagger} \circ s_1 = \gamma \circ s_1 - \bar{h}, \qquad (137)$$

where $s_1 : k_1(A) \to U_\infty(A)/CU_\infty(A)$ is a splitting map (see 2.3), and

$$\bar{h}: k_1(A) \to \overline{\mathbb{R}\rho_B(k_0(B))}/\overline{\rho_B(k_0(B))}$$

is a homomorphism. Moreover,

$$(\phi \otimes \mathrm{id}_{z_{p,q}})^{\ddagger} \circ s_1 = E_B \circ \gamma \circ s_1 - \bar{h}, \qquad (138)$$

where $E_B$is as defined in (101).