Chapter 1
Quasi-Metric Measure Spaces

We show the validity of a Riesz–Thorin type interpolation theorem for linear operators acting from variable exponent Lebesgue spaces into variable exponent Morrey space in the framework of quasi-metric measure spaces.

The classical Riesz–Thorin interpolation theorem is a well-known result in harmonic analysis, where, loosely speaking, we obtain boundedness results of certain type of operators using the information on the endpoints. The geometric interpretation of this fact is that if the operator is of strong type $(p_0, q_0)$and of strong type (p_1, q_1), then it is of strong type (p_θ, q_θ) where the reciprocal of (p_θ, q_θ)belongsto the line joining the reciprocals of (p_0, q_0) and (p_1, q_1), viz.

$$\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}\right) = (1-\theta)\left(\frac{1}{p_\theta}, \frac{1}{q_\theta}\right) + \theta\left(\frac{1}{p_1}, \frac{1}{q_1}\right);$$

A generalization of this theorem when the target space is a Morrey–Campanato spacewas given by S.Campanato and M.Murth. It should be noted that it is not possible to prove Riesz–Thorin interpolation theorem when the domain space is a Mor-rey type space; for example, constructed an example of a bounded linear operator on $H^a$and L^2but not on $L^q > 2$, and BMO. Where there were given examples of operators bounded from L^{p_i, λ_i} to L^{q_i}, which are not bounded in the intermediate spaces. The Riesz–Thorin interpolation the-orem in the framework of variable exponent Lebesgue spaces was first proved using the abstract complex interpolation method of Calderón, and more recently. For interpolation results for positive operators in variable exponent Lebesgue spaces.
Morrey spaces first appeared relation to some problems in partial differential equations. During last decade, Morrey spaces were widely studied because of their proposal application in various allied fields of science.

Function spaces with variable exponents are a very active area of research nowadays and one of the reasons is a wide variety of applications of such spaces, in the modeling of the electro-rheological fluids as well as thermo-rheological fludis, in this study of image processing and in differential equations with non-standard growth. Lebesgue spaces with variable exponent in framework of quasi-measure space have also been studied by several atuthors.

We show a variant of Riesz-Thorin interpolation theorem when the domain space is the variable exponent Lebesgue space $L^{p(\cdot)}(X)$ and the target space is a variable exponent Morrey space $L^{q(\cdot),\lambda(\cdot)}(Y)$ where X and Y are a quasi-metric measure space (QMMS), wher we adopt techniques presented.

Constants (often different constants in the same series of in-equalities) will mainly be denoted by c or C; by symbol $p'(x)$ we denote the function $\frac{p(x)}{p(x)-1}, 1 < p(x) < \infty$; then the relation $a \approx b$ means that there are positive constants c_1 and c_2 such that $c_1 a \le b \le c_2 a$.

Let (X, d, μ) be a QMMS with a complete measure μ such that the space of compactly supported continuous functions are dense in $L^1_\mu(X)$.A quasi-metric d is a function $d: X \times X \to [0, \infty)$ which satisfies the following conditions:

(i) $d(x, y) = 0$ for all $x \in X$.
(ii) $d(x, y) > 0$ for all $x \neq y, x, y \in X$.
(iii) There is a constant $a_0 > 0$ such that $d(r, y) = a_0 d(y, x)$ for al $r, y \in X$.
(iv) There is a constant $a_1 > 0$ such that $d(x, y) \le a_1(d(x, z) + d(z, y))$ for all x, y, z ∈ X.

Let $d_X = diam\ (X) = \sup\{d(x, y): x, y \ \in X\}$. Let $B(x, r) = \{y \in X: d(r, y) < r\}$ be the ball with enter x and radius $r > 0$. We will assume that $0 < \mu\big(B(x, r)\big) < \infty$ for every $x \in X$ and $r > 0$. It is obvious that the condition $d_X < \infty$ and the assumption that all the balls have finite measure imply $\mu(X) < \infty$. For that further literature on the subject of quassi-metric measure spaces.

Let E be a measurable set in (X, μ) with positive measure. We denote

$$p^-(E) := \inf_E p\,, \qquad p^+(E) \coloneqq \sup_E p$$

For a measurable function p on E. suppose that $1 \le p^-(E) \le p^+(E) < \infty$. We say that a measurable function f on E belongs to $L^{P(\cdot)}(E)$ $\left(or\ \ to\ L^{p(x)}(E)\right)$ if

$$S_{p(\cdot),E}(f) = \int_E |f(x)|^{p(x)} d\mu(x) < \infty.$$

It is a Banach space with respect to the norm

$$\|f\|_{L^{p(\cdot)}(E)} = \left\{\eta > 0: S_{p(\cdot),E}\left(\frac{f}{\eta}\right) \le 1\right\}.$$

For the following propositions.

Preposition (1.1)[1]: Let E be a measurable subset of X. suppose that $1 \le p^-(E) \le p^+(E) < \infty$. Then

(i)

$$\|f\|^{p^+(E)}_{L^{p(\cdot)}(E)} \le S_{p(\cdot)E}(f) \le \|f\|^{p^-(E)}_{L^{p(\cdot)}}, \qquad \|f\|_{L^{p(\cdot)}(E)} \le 1;$$

$$\|f\|_{L^{p(\cdot)}(E)}^{p^-(E)} \le S_{p(\cdot)E}(f) \le \|f\|_{L^{p(\cdot)}}^{p^+(E)}, \qquad \|f\|_{L^{p(\cdot)}(E)} \le 1;$$

(ii) Hölder's inequality

$$\left|\int_E f(x)g(x)d\mu(x)\right| \le \left(\frac{1}{p^-(E)} + \frac{1}{\left(p^+(E)\right)'}\right)\|f\|_{L^{p(\cdot)}(E)}\|g\|_{L^{p'(\cdot)}(E)}$$

holds, where $f \in L^{p(\cdot)}(E)$ and $g \in L^{p'(\cdot)}(E)$.

Preposition (1.2)[1]: Let $1 \le r(x) \le p(x)$ and let E be a subset of X with $\mu(E) < \infty$. then the following inequality

$$\|f\|_{L^{r(\cdot)}(E)} \le (\mu(E) + 1)\|f\|_{L^{p(\cdot)}(E)}$$

Holds.

The following lemma can.

Lemma (1.3)[1]: Let E be a measurable subset of X. suppose that $1 \le p^-(E) \le p^+(E) \le \infty$. Then

$$\|f\|_{L^{p(\cdot)}(E)} \le S_{p(\cdot),E}(f) + 1$$

Holds.

Definition (1.4)[1]: We say that a μ-measurable function $p: X \to [1, \infty)$ belongs o the classes $\mathcal{P}_\mu^{\log}(X)$ if the inequality

$$|p(x) - p(y)| \le \frac{-A}{\ln \mu B\big((x, d(x, y)\big)}$$

Holds for all $x, y \in X$ such that $\mu B(x, d(x, y)) \le 1/2$.

For the next lemma.

Lemma (1.5)[1]: Let (X, d, μ) be QMMS with finite measure, and let $p \in \mathcal{P}_\mu^{\log}(X)$. Then

$$\left\|\chi_{B(x,r)}\right\|_{L^{p(\cdot)}} \le \mu\big(B(x, r)\big)^{\frac{1}{p(x)}}.$$

The Morrey spaces $L^{p(.),\lambda(.)}(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$ were introduced by several authors more or less simultaneously. Let $1 \le p(\cdot) < p^+(\mathrm{X}) < \infty$ and $0 \le \lambda(.) \le 1$ be μ-measurable function. We say that a function $f \in L^{p(\cdot)}(X)$ belongs to $f \in L^{p(\cdot),\lambda(\cdot)}(X)$ if

$$I_{p(\cdot),\lambda(\cdot)}(f) = \sup_{x\in X, r>0} \frac{1}{\mu\big(B(x, r)\big)^{\lambda(x)}} \int_{B(x,r)} |f(y)|^{p(y)} d\mu(y) < \infty.$$

The norm on variable exponent Morrey spaces can be introduced:

$$\|f\|_1 = \inf\{\eta > 0; I_{p(\cdot),\lambda(\cdot)}(f/\eta) \le 1\},$$

and

$$\|f\|_2 = \sup_{x\in X, R>0} \left\| \left(\mu B(x,r)\right)^{\frac{-\lambda(x)}{p\,(\cdot)}} f\chi_{B(x,r)} \right\|_{L^{p(\cdot)}(E)},$$

and

$$\|f\|_3 = \sup_{x\in X, R>0} \left(\mu B(x,r)\right)^{\frac{-\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot)}(B(x,r))}.$$

It can be verified that $\|f\|_1 = \|f\|_{2.}$ further, $p \in \mathcal{P}_\mu^{\log}(X)$ then $\|f\|_1$ and $\|f\|_2$ are equivalent to the norm $\|f\|_3$. Therefore it is possible to introduce the norm in several ways which are equivalent provided that the exponent satisfaction some condition. We define the norm on variable exponent Morrey space as:

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} = \|f\|_3.$$

It is easy to see that if $\lambda = 0$, then $L^{p(\cdot),0}(X) = L^{p(\cdot)}(X)$. When $p(x) \equiv const$ and $\lambda(x) \equiv const$ then $L^{p(\cdot),\lambda(\cdot)}(X)$ coincide with the classical Morrey space $L^{p,\lambda}(X)$.

The next lemma gives the embedding of variable Morrey spaces into variable lebesgue space in case $d_X < \infty$.Here we present the proof of this lemma for the sake of completeness.

Lemma (1.6) [1]: Let (X, d, μ) be a QMMS with $\mu(X) < \infty$. Suppose that $1 \le p(\cdot) < P^+(X) < \infty$ and $0 \le \lambda(\cdot) \le 1$. Then, $L^{p(\cdot),\lambda(\cdot)}, (X) \to L^{p(\cdot)}(X)$ and moreover for every $f \in L^{p(\cdot),\lambda(\cdot)}(X), x \in X$ and $r > 0$ e have

$$\|f\|_{L^{p(\cdot)}(B(x,r))} \le (\mu\left(B(x,r)\right)^{\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}.$$

Further the following inequity

$$\int_X f(y)g(y)d\mu(y) \le c\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \tag{1}$$

holds

Proof: Suppose that $f \in L^{p(\cdot),\lambda(\cdot)}(X)$. Let $x \in X$ and $r < 0$, then

$$\|f\|_{L^{p(\cdot)}(B(x,r))} = (\mu(B(x,r))^{\frac{\lambda(x)}{p(x)}} \frac{1}{\mu(B(x,r))^{\frac{\lambda(x)}{p(x)}}} \|f\|_{L^{p(\cdot)}(B(x,r))}$$
$$\leq \mu(B(x,r))^{\frac{\lambda(x)}{p(x)}} \|f\|_{L^{p(\cdot),\lambda(\cdot)}}(X).$$

Since p bounded, hence taking supermom with respect to $x \in X$ and $r < 0$ we have the following estimate

$$\|f\|_{L^{p(\cdot)}(X)} \leq \max\{1, \mu(X)\}^{\left(\frac{\lambda}{p}\right)+(X)} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \leq c_{p,\lambda,\mu} \|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)}$$

Consequently, via Holder's inequality, for $f \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$ there is a positive constant c such that

$$\int_X f(y)g(y)d\mu(y) \leq c\|f\|_{L^{p(\cdot),\lambda(\cdot)}(X)} \|g\|_{L^{p'(\cdot)}(X)} \quad (2)$$

Holds.

Lemma (1.7)[1]: Let $(X_k, d_k\mu_k)$ be QMMS for $k = 1,2 \; and \; \mu_2(X_2) < \infty$. Let $p_1 and \, p_2$ be bounded exponents and $0 \leq \lambda(\cdot) \leq 1$ and let $T: L^{p_1(\cdot)}(X_1) \to L^{p_2(\cdot),\lambda(\cdot)}(X_2)$ be linear and continuous. For $k = 1,2,$ suppose that there are positive real number $B_k \; and \; M_k$, measurable sets $A_k \; with \, \mu_1(A_1) < \infty$ and measurable functions $m_k, b_k: X_k \to \mathbb{R}$ such that $-M_k \leq m_k(x_k) \leq M_k$ and $0 \leq b_k(x_k) \leq B_k$, for almost every $x_k \in X_k$,for $z \in \mathbb{S}$, define

$$F(z) :\equiv \int_{X_2} T\left[a_1^{m_1(\cdot)z+b_1(\cdot)} \chi_{A_1}(.)\right](x_2) \, a_2^{m_2(x_2)z+b_2(x_2)} \chi_{A_2}(x_2) d\mu_2(x_2),$$

Where $a_k, k = 1,2$ are positive real numbers. Then F is continuous and bounded on the strip $\mathbb{S} = \{z: 0 \leq Re(z) \leq 1\}$ and analytic on int ($\mathbb{S}$).

Proof: Let $x_k \in X_k$ and $z \in \mathbb{C}$. Denote

$$\alpha_k(x_k, z) \coloneqq a^{m_k(x_k)z+b_k(x_k)} \chi_{A_k}(x_k)$$

and

$$Q_k(x_k, z, w) :\equiv \frac{\alpha_k(x_k, z) - \alpha(x_k, w)}{z - w} - \alpha_k(z_{k,} z) m_k(x_k) \log a_k.$$

Therefore, we may represent F as

$$F(z) :\equiv \int_{X_2} T[\alpha_1(\cdot, z)](x_2) \alpha_2(x_2, z) d\mu_2(x_2).$$

Notice that for almost every $x_k \in X_k$ the following point-wise holds

$$|\alpha_k(x_k, z)| = \left|a_k^{m_k(x_k)\,\mathrm{Re}(z)+b_k(x_k)} a_k^{\mathrm{i}m_k(x_k)\mathrm{Im}(z)} \chi_{A_k}(x_k)\right|$$
$$= a_k^{m_k(x_k)\,\mathrm{Re}(z)+b_k(x_k)} \chi_{A_k}(x_k) \le D_k \chi_{A_K}(x_k), \qquad (3)$$

Where $D_k :\equiv \max_{t\in[M_k, M_k+B_k]} a_k^t$. Further, by virtue of (3) we have the following estimates,

$$\lceil Q_k(x_k, z, w)\rceil = |\alpha_k(x_k, z)| \left|\frac{a_k^{m_k(x_k)(w-1)} - 1}{w-z} - m_k(x_k)\log\ a_k\right|$$
$$= |\alpha_k(x_k, z)| \left|\frac{e^{[m_k(x_k)\log\ a_k](w-z)} - 1}{w-z} - m_k(x_k)\log\ a_k\right|$$
$$= \left|\alpha_k\left(x_{k,}z\right)\right| \left|\frac{\left[\sum_{j=0}^{\infty} \frac{[m_k(x_k)\log\ a_k(w-z)]^j}{j^!}\right] - 1}{w-z}\right.$$
$$\left. - m_k(x_k)\log\ a_k\right|$$
$$= \left|\alpha_k\left(x_{k,}z\right)\right| \left|\sum_{j=2}^{\infty} \frac{[m_k(x_k)\log\ a_k]^j \lceil (w-1)\rceil^{j-1}}{j^!}\right|$$
$$\le D_k \chi_{A_k}(x_k) \sum_{j=2}^{\infty} \frac{(M_k|\log\ a_k|)^j \lceil (w-z)\rceil^{j-1}}{j^!}$$
$$\le D_k {M_k}^2 |\log\ a_k|^2 |z-w| \chi_{A_k}(x_k) \sum_{j-0}^{\infty} \frac{|M_k|\log\ a_k||w-1|^j}{j^!}$$
$$= D_k {M_k}^2 |\log\ a_k|^2 |z-w| e^{M_k|\log\ a_k||w-z|} \chi_{A_k}(x_k) \qquad (4)$$

for almost every $x_k \in X_k$. Now for sufficiently small $\lfloor z-w \rfloor$, we have that $D_k M_k^2 |\log\ a_k|^2 |z-w| e^{M_k|\log\ a_k||w-z|}$ is no greater than 1. Thus

$$S_{p_k'(\cdot),X_k}\left(Q_k(\cdot, z, w)\right) := \int_{X_k} |Q_k(x_k, z, w)|^{p_k'(x_k)} d\mu_k(x_k)$$
$$\le D_k M_k^2 |\log\ a_k|^2 |z-w| e^{M_k|\log\ a_k||w-z|} \mu_k(A_k) \to 0, \quad (5)$$

As $w \to z$.Since $(p_k')+ < \infty$, it follows that

$$\lim_{w\to z}\ \|Q_k(\cdot, z, w)\|_{L^{p_k'(\cdot)}(X_k)} = 0. \qquad (6)$$

Similarly,

$$\lim_{w\to z} \|Q_k(\cdot,z,w)\|_{L^{p_k(\cdot)}(X_k)} = 0. \tag{7}$$

Taking into account (6) and the following inequality

$$\begin{aligned}\|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p'_k}(X_k)} \\ &\le |z-w|\Big(\|Q_k(\cdot,z,w)\|_{L^{p'_k(\cdot)}(X_k)} \\ &+\|\alpha_k(\cdot,z)m_k(\cdot)\log\ a_k\|_{L^{p'_k(\cdot)}(X_k)}\Big),\end{aligned}$$

We have

$$\lim_{w\to z} \|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p'_k(\cdot)}(X_k)} = 0. \tag{8}$$

Analogously we have

$$\lim_{w\to z} \|\alpha_k(\cdot,z)-\alpha_k(\cdot,w)\|_{L^{p_k(\cdot)}(X_k)} = 0. \tag{9}$$

Now we prove the analyticity of F:

$$\begin{aligned}&\frac{F(z)-F(w)}{z-w}\\ &=\frac{\int_{X_2} T\left[\alpha_1(\cdot,z)(x_2)\alpha_2(x_2,z)d\mu_2-\int_{X_2} T[\alpha_1(\cdot,w)(x_2)\alpha_2(x_2,w)d\mu_2(x_2)]\right]}{z-w}\\ &=\int_{X_2}\frac{T[\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)-T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_z,z)}{z-w}\ d\mu_2(x_2)\\ &+\int_{X_2}\frac{T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,z)-T[\alpha_1(\cdot,w)](x_2)\alpha_2(x_2,w)}{z-w}\ d\mu_2(x_2)\\ &=\int_{X_2} T\left[\frac{\alpha_1(\cdot,z)-\alpha_1(\cdot,w)}{z-w}\right](x_2)\alpha_2(x_2,z)d\mu_2(x_2)\\ &+\int_{X_2} T[\,\alpha_1(\cdot,w)(x_2)\left[\frac{\alpha_2(x_2,z)-\alpha_2(x_2,w)}{z-w}\right]d\mu_2(x_2) =: 1+J.\end{aligned}$$

Denote

$$I' \coloneqq \int_{X_2} T[\,\alpha_1(\cdot,z)(m_1)(\cdot)\log a_1]\ (x_2)\alpha_2(x_2,z)d\mu_2(x_2)$$

and

$$J' \coloneqq \int_{X_2} T\ [\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2 d\mu_2(x_2).$$

We show that $I \to I'$ and $J \to J'$ as $w \to z$.Firstly, we show $I \to I'$; using linearity of the operator T along with the Lemma (1.6) we obtain

$$|I \to I'|$$
$$= \left| \int_{X_2} \left[\frac{\alpha_1(\cdot,z) - \alpha_1(\cdot,w)}{z-w} \right] (x_2)\alpha_2(x_2,z)d\mu_2(x_2) \right.$$
$$\left. - \int_{X_2} T[\alpha_1(\cdot,z)m_1(\cdot)\log\ \ a_1]\,(x_2)\alpha_2(x_2,z)d\mu_2(x_2) \right|$$
$$= \left| \int_{X_2} T[Q_1(\cdot,z,w)](x_2)\alpha_2(x_2,z)d\mu_2(x_2) \right|$$
$$\le c\|T[Q_1(\cdot,z,w)](\cdot)\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_2(\cdot,z)\|_{L^{p_2'}(X_2)}$$
$$\le c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|Q_1(\cdot,z,w)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)\|_{L^{p_2'}(X_2)}$$

where c is the constant appearing in Lemma (1.6). the desired result follows from Lemma (1.3), (3) and. To show $J \to J'$ again spilt $J - J'$ as two integrals:

$$J - J' = \int_{X_2} T[\alpha_1(\cdot,w)](x_2)\left[\frac{\alpha_2(x_2,z) - \alpha_2(x_2,w)}{z-w}\right] d\mu_2(x_2)$$
$$- \int_{X_2} T_1[\alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2\, d\mu_2(x_2)$$
$$= \int_{X_2} T[\alpha_1(\cdot,w)(x_2)\left[\frac{\alpha_2(x_2,z) - \alpha_2(x_2,w)}{z-w}\right.$$
$$\left. - \alpha(x_2,z)m_2(x_2)\log a_2 \right] d\mu_2(x_2)$$
$$+ \int_{X_2} T[a_1(\cdot,w)$$
$$- \alpha_1\,(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2\ \ d\mu_2(x_2)$$
$$= \int_{X_2} T[\,a_1(\cdot,w)]\,(x_2)Q_2(x_2,z,w)\,d\mu_2(x_2) + \int_{X_2} T[\,\alpha_1(.,w)$$
$$- \alpha_1(\cdot,z)](x_2)\alpha_2(x_2,z)m_2(x_2)\log a_2\ \ d\mu_2(x_2) =: S_1 + S_2.$$

By means of Lemma (1.6), the boundness of T and (3) we have the following estimates,

$$|S_1| \le c\|T[\alpha_1(\cdot,w)](\cdot)\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|T[Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)}$$

$$\leq c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_1(\cdot,w)\|_{L^{p_1(\cdot)}(X_1)}\|Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)}$$

$$\leq cD_1\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\left\|\chi_{A_1}(\cdot)\right\|_{L^{p_1(\cdot)}(X_1)}\|Q_2(\cdot,z,w)\|_{L^{p_2'(\cdot)}(X_2)} \tag{10}$$

Analogously,

$$|S_2| \leq c\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda}(X_2)}$$
$$\times \|\alpha_1(\cdot,w)-\alpha_1(\cdot,z)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)m_2(\cdot)\log\ a_2\|_{L^{p_2'(\cdot)}(X_2)} \tag{11}$$

Now, expression on the right-hand side in (10) tends 0 as $w \to z$ by virtue of Lemma (1.3) and (6). Therefore, $J \to J'$, as $w \to z$. Hence, F is analytic on in $\operatorname{int}(\mathbb{S})$; and $F' = I' + J'$. Now we show that F is continuous in the entire strip $\mathbb{S}$. In fact we use the same technique as above;

$$|F(z) - F(w)|$$

$$= \left| \int_{X_2} T[\alpha_1(\cdot,z) - \alpha_1(\cdot,w)](x_2)\alpha_2(x_2,z)d\mu_2(x_2) \right.$$

$$\left. + \int_{X_2} T[\alpha_1(\cdot,w)](x_2)[\alpha_2(x_2,z) - \alpha(x_2,w)]d\mu_2(x_2) \right|$$

$$\leq c\,\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\ \|\alpha_1(\cdot,z) - \alpha_1(\cdot,w)\|_{L^{p_1(\cdot)}(X_1)}\|\alpha_2(\cdot,z)\|_{L^{p'_2(\cdot)}(X_2)}$$

$$+ cD_1\,\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\left\|\chi_{A_1}(\cdot)\right\|_{L^{p_1(\cdot)}(X_1)}\ \|\alpha_2(\cdot,z)$$
$$- \alpha_2(\cdot,w)\|_{L^{p'_2(\cdot)}(X_2)}.$$

As $w \to z$ in $\mathbb{S}$, both terms in the above sum tends to 0 by virtue of Lemma (1.3), (8) and (9), proving the continuity of F in $\mathbb{S}$. Finally, F is bounded in $\mathbb{S}$. Indeed, by the boundedness of T and invoking Lemma (1.6), Lemma (1.3) and estimate (3) we have:

$$|F(z)| \leq c\|T[\alpha_1(\cdot,z)]\|_{L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\|\alpha_2(\cdot,z)\|_{L^{p'_2(\cdot)}(X_2)}$$
$$\leq c\,\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\ \|\alpha_1(\cdot,z)\|_{L^{p_1(\cdot)}}\ \|\alpha_2(\cdot,z)\|_{L^{p'_2(\cdot)}(X_2)}$$
$$\leq cD_1D_2\,\|T\|_{L^{p_1(\cdot)}(X_1)\to L^{p_2(\cdot),\lambda(\cdot)}(X_2)}\left\|\chi_{A_1}(\cdot)\right\|_{L^{p_1(\cdot)}(X_1)}\left\|\chi_{A_2}(\cdot)\right\|_{L^{p'_2(\cdot)}(X_2)} < \infty$$

Which ends the proof.

Finally, we prove the Riesz–Thorin theorem in the setting of variable Morrey spaces defined on quasi-metric measure spaces.

Theorem (1.8)[1]: Let (X,μ) and (Y,ν) be σ-finite, complete QMMS. For $k = 0,1$, assume that $1 \le p_k(\cdot), q_k(\cdot) < q_k^+(Y) < \infty$ and $0 \le \lambda_k \le 1$. Suppose that we have a linear operator $T: L^{p_k(\cdot)}(X) \to L^{q_k(\cdot),\lambda_k(\cdot)}(Y)$ such that for all $\|f\| \in L^{p_k(\cdot)}(X)$

$$\|Tf\|_{L^{q_k(\cdot),\lambda_k(\cdot)}(Y)} \le M_k \|f\|_{L^{p_k(\cdot)}(X)} \tag{12}$$

holds. For $z \in \mathbb{S} := \{z : 0 < \operatorname{Re}(z) < 1\}$, define p_z, q_z and λ_z by

$$\frac{1}{p_z(x)} = \frac{1-z}{p_0(x)} + \frac{z}{p_1(x)},$$

$$\frac{1}{q_z(x)} = \frac{1-z}{q_0(x)} + \frac{z}{q_1(x)},$$

and

$$\frac{\lambda_z(x)}{q_z(x)} = (1-z)\frac{\lambda_z(x)}{q_0(x)} + z\,\frac{\lambda_z(x)}{q_1(x)}$$

Then, given any $\theta \in (0,1)$, the inequality

$$\|Tf\|_{L^{q_\theta(\cdot),\lambda_\theta(\cdot)}(Y)} \le cM_0^{1-\theta}0\, M_1^{\theta}\, \|f\|_{L^{p_\theta(\cdot)}(X)}$$

holds for every $f \in L^{p_\theta(\cdot)}(X)$.

Proof: Since T is linear, we may assume that $f \neq 0$, otherwise the inequality holds for $f = 0$. By the homogeneity of norm and the scaling argument we may assume $\|f\|_{L^{q_\theta(\cdot)}(X)} \le 1$ andshow that

$$\|Tf\|_{L^{q_\theta(\cdot),\lambda_\theta(\cdot)}(Y)} \le cM_0^{1-\theta}M_1^{\theta}. \tag{13}$$

We will show (13) for simple functions in X and since simple function are dense in $L^{p(\cdot)}(X)$ we will have the estimate for all $f \in L^{p_\theta(\cdot)}(X)$.

Let us assume f, g are simple and complex valued function defined on X and Y, respectively, by

$$f(x) = \sum_{j-1}^{m} a_j e^{i\alpha_j}\, \chi_{A_j}(x),$$

$$g(x) = \sum_{k=1}^{n} b_k e^{i\beta_k} \chi_{B_k}(x),$$

Where the $a_j, b_k > 0$ and $\alpha_j, \beta_k \in \mathbb{R}$, $\mu(A_j), \mu(B_k) < \infty$, and the $\{A_j\}$ and $\{B_k\}$ are, respectively, pairwise disjoint. Now define

$$f_z(x) = \sum_{j-1}^{m} a_j^{\frac{p_\theta(x)}{p_z(x)}} e^{i\alpha_j} \chi_{A_j}(x),$$

$$g_z(x) = \sum_{k=1}^{n} b_k^{\frac{p'_\theta(y)}{p'_z(y)}} e^{i\beta_k} \chi_{B_k}(y).$$

Finally, for every $y \in Y$, $r > 0$ and $z \in \mathbb{C}$, we put

$$F(y,r,z) := \int_{B(y,r)} T(f_z(s)) g_z(s) dv(s).$$

Firstly note that for every $\theta \in (0,1)$, $p_\theta(y) \in [1, \infty)$. Further for almost every $x \in X$, $p_\theta(x) = \frac{p_0(x)p_1(x)}{(1-\theta)p_1(x)+\theta p_0(x)} \leq p_0^+ p_1^+ < \infty$ and hence $p_\theta(x) \in [1, p_0^+ p_1^+]$. Moreover,

$$-1 < \frac{1}{p_1^-} - 1 \leq \frac{1}{p_1(x)} - \frac{1}{p_0(x)} \leq 1 - \frac{1}{p_0} < 1,$$

for almost every $x \in X$. Let $\Phi_1(x) := p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right]$ and $\Phi_0(x) := \frac{p_\theta(x)}{p_0(x)}$. Hence, regarding

$$\frac{p_\theta(x)}{p_z(x)} = \Phi_1(x)z + \Phi_0(x)$$

as a linear polynomial in z and that Φ_1 maps X to the interval $[-p_0^+ p_1^+, p_0^+ p_1^+]$ while the map Φ_0 maps X to the interval $[0, p_0^+ p_1^+]$. Analogously for $\frac{q'_\theta(x)}{q'_z(x)}$, we have similar estimates. Since we can write F as

$$F(y,r,z) = \sum_{j-1}^{m} \sum_{k=1}^{n} \int_{B(y,r)} T\left[a_j^{\frac{p_\theta(\cdot)}{p_z(\cdot)}} \chi_{A_j}(\cdot)\right](s) b_k^{\frac{q'_\theta(s)}{q'_z(s)}} \chi_{B_k}(s) dv(s),$$

hence for almost every $y \in Y$, Lemma (1.7) ensures that F is analytic on $\mathrm{int}(\mathbb{S})$ and continuous and bounded on $\mathbb{S}$.

Since A_j are pairwise disjoint and $a_j > 0$, we have for $z = it$, with $t \in \mathbb{R}$

$$S_{p_0(\cdot),B(y,r)}(f_z) = \int_{B(y,r)} \left|\sum_{j=1}^{m} a_j^{\frac{p_\theta(x)}{p_z(x)}} e^{i\alpha_j} \chi_{A_j}(x)\right|^{p_0(x)}$$

$$= \int_{B(y,r)} \left| \sum_{j=1}^{m} a_j^{p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right] it + \frac{p_\theta(x)}{p_0(x)}} e^{i\alpha_j} \chi_{A_j}(x) \right|^{p_0(x)} d\mu(x)$$

$$= \int_{B(y,r)} \left| \sum_{j=1}^{m} a_j^{p_\theta(x)\left[\frac{1}{p_1(x)} - \frac{1}{p_0(x)}\right] it + \frac{p_\theta(x)}{p_0(x)}} e^{i\alpha_j} \chi_{A_j}(x) \right|^{p_0(x)} d\mu(x)$$

$$= \int_{B(y,r)} \sum_{j=1}^{m} a_j^{p_\theta(x)} \chi_{A_j}(x) d\mu(x) = \int_{B(y,r)} \left| \sum_{j=1}^{m} a_j e^{i\alpha_j} \chi_{A_j}(x) \right|^{p_0(x)} d\mu(x)$$

$$= S_{p_\theta(\cdot)B(y,r)}(f) \leq 1$$

since $\|f\|_{L^{p_\theta(\cdot)}(X)} \leq 1$. Hence $\|f_z\|_{L^{p_0(\cdot)}(B(y,r))} \leq 1$. A similar argument shows that $\|g_z\|_{L^{q_0'}(B(y,r))} \leq 1$ for $z = it$. Now by Hölder's inequality, Lemma (1.6) and (12) we have

$$|F(y,r,it)| \leq \left| \int_{B(y,r)} T\big(f_z(s)\big) g_z(s) dv(s) \right|$$

$$\leq c \|Tf_z\|_{L^{q_0(\cdot)}(B(y,r))} \|g_z\|_{L^{q_0'(\cdot)}(B(y,r))} \leq c \|Tf_z\|_{L^{q_0(\cdot)}(B(y,r))}$$

$$\leq cv\big(B(y,r)\big)^{\frac{\lambda_0(y)}{q_0(y)}} \|Tf_z\|_{L^{q_0(\cdot),\lambda_0(\cdot)}}(y)$$

$$\leq cv\big(B(y,r)\big)^{\frac{\lambda_0(y)}{q_0(y)}} M_0 \|f_z\|_{L^{p_0(\cdot)}(X)} \leq cv\big(B(y,r)\big)^{\frac{\lambda_0(y)}{q_0(y)}} M_0.$$

An analogous argument with $\mathrm{Re}(z) = 1$ and with the exponents p_1 and q_1 yields

$$|F(y,r,1+it)| \leq cv\big(B(y,r)\big)^{\frac{\lambda_1(y)}{q_1(y)}} M_1.$$

Finally, using Hadamard's three lines lemma gives

$$|F(y,r,\theta)| \leq cv(B(y,r))^{\frac{\lambda_\theta(y)}{q_\theta(y)}} M_0^1 0 \, M_1^\theta \, .$$

Also,

$$g^{sup}_{L^{q_\theta'} B(y,r)} {}_{\leq 1} |F(y,r,\theta)| = \|Tf\|_{L^{q_0(\cdot)}} Lq\theta(\cdot)(B(y,r))$$

Hence for almost every $y \in Y$ and $r > 0$ we have

$$\nu(B(y,r))^{\frac{-\lambda_\theta(y)}{q_\theta(y)}} \|Tf\|_{L^{q_\theta(\cdot)}(B(y,r))} \;\leq\; cM_0^{1-\theta}\, M_0^{\theta}\,,$$

which implies that

$$\|Tf\|_{L^{q_\theta(\cdot)}(B(y,r))} \leq cM_0^{1-\theta}\, M_0^{\theta}$$

This completes the proof.

Chapter 2

Sobolev Inequality on Non-Homogeneous Central Herz-Morrey-Orlicz Spaces

As an application, we give Sobolev's inequality for Riesz potentials.

Section (2.1): Boundedness of the Maximal Operator

Let $\boldsymbol{R}^N$ be the Euclidean space. Beurling introduced the space $B^p(\boldsymbol{R}^N)$ to extend Wiener's ideas which describes the behavior of functions at infinity. Feichtinger gave an equivalent norm on $B^p(\boldsymbol{R}^N)$, which is a special case of norms in Herz spaces $K_p^{\alpha,r}(\boldsymbol{R}^N)$ introduced by Herz. More precisely, $B^p(\boldsymbol{R}^N) = K_p^{-N/p,\infty}(\boldsymbol{R}^N)$. Alvarez, Guzm´an-Partida and Lakey defined the central Morrey spaces $B^{p,\lambda}(\boldsymbol{R}^N)$ to study the relationship with $\lambda-$central bounded mean oscillation spaces, where $B^{p,0}(\boldsymbol{R}^N) = B^p(\boldsymbol{R}^N)$.

García-Cuerva studied the boundedness of the maximal operator on the space $B^p(\boldsymbol{R}^N)$.Further, showed that the maximal operator is bounded on homogeneous Herz spaces and non-homogeneous Herz spaces. We introduce non-homogeneous central Herz–Morrey–Orlicz spaces $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ as an extension of $K_p^{\alpha,r}(\boldsymbol{R}^N)$, and study the boundedness of the Hardy–Littlewood maximal operator.

In classical Lebesgue spaces, we know Sobolev's inequality:

$$\| I_\alpha f \|_{L^{p^*}(\boldsymbol{R}^N)} \leq C \| f \|_{L^p} (\boldsymbol{R}^N)$$

For $f \in L^p(\boldsymbol{R}^N), 0 < \alpha < N$ and $1 < p < N/\alpha$, where Iα is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$. Fu, Lin and Lu showed Sobolev's inequality for $B^{p,\lambda}(\boldsymbol{R}^N)$ for non-homogeneous Herz spaces, for non-homogeneous Herz–Morrey spaces, for non-homogeneous central Morrey spaces. We give Sobolev's inequality for Riesz potentials of functions in non-homogeneous central Herz–Morrey–Orlicz spaces.

Suppose $f \in \mathcal{H}^{p,q,\omega}(\boldsymbol{R}^N)$, that is, it satisfies an L^p integrability such as

$$\int_1^\infty \{\omega(r) \| f \|_{L^p(A(0,r))}\}^q \frac{dr}{r} < \infty \qquad when\ 0 < q < \infty$$

$$\sup_{r>1} \omega(r) \| f \|_{L^p(A(0,r))} < \infty \qquad when\ q = \infty$$

Where ω is a doubling weight, $1 < p < \infty$ and $A(0,r) = B(0,2r) \setminus B(0,r)$ is the annulus with $B(x,r)$ denoting the open ball centered at x of radius r. Then we want to find p_1 and a weight τ such that $I_\alpha f \in \mathcal{H}^{p_1,q,\tau}(\boldsymbol{R}^N)$. In the borderline case $\alpha p = N$, instead of Trudinger's inequality, we show the weighted L^p integrability

$$\int_{\boldsymbol{R}^N} \left\{(1+|x|)^{-N/p}(\log(e+|x|))^{-1+\theta}|I_\alpha f(x)|\right\}^p dx \leq C \int_{\boldsymbol{R}^N} \left\{(\log(e+|y|))^{\theta}|f(y)|\right\}^p dy$$

Since it may happen that $I_\alpha|f| \equiv \infty$ for some $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$, we modify the Riesz kernel I_α by

$$I_{\alpha,k}(x,y) = \begin{cases} I_\alpha(x-y) & when\ |y| < 1, \\ I_\alpha(x-y) - \displaystyle\sum_{\{\mu:|\mu|\leq k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) & when\ |y| \geq 1, \end{cases}$$

for a nonnegative integer k; $I_{\alpha,0}$ is the usual Riesz kernel I_α of order α. Then our third task is to find k such that the generalized Riesz potential

$$I_{\alpha,k} f(x) = \int_{\boldsymbol{R}^N} I_{\alpha,k}(x,y) f(y)\ dy$$

is well defined for almost every $x \in \boldsymbol{R}^N$ and belongs to a suitable non-homogeneous central Herz–Morrey–Orliczspace.

Finally, following Gogatishvili–Mustafayev, we study the duality properties between $\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ and $\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ (for the definition of $\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$and $\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$).

Let us consider a function

$$\Phi(t) = t\phi(t) : [0,\infty) \to [0,\infty)$$

With ϕ satisfying the following conditions:

(Φ_1) ϕ is continuous in $[0,\infty)$ and $\phi(t) > 0$ for $t > 0$;

(Φ_2) ϕ is almost increasing in $[0,\infty)$; namely, there exists a constant $A_1 \geq 1$ such that

$$\phi(t) \leq A_1\phi(s) \text{ whenever } 0 \leq t < s;$$

(Φ_3) ϕ is doubling; namely, there exists a constant $A_2 \geq 1$ such that

$$A_2^{-1}\phi(t) \leq \phi(s) \leq A_2\phi(t) \text{ whenever } 0 < t/2 \leq s \leq 2t.$$

Example (2.1.1)[2]: For $p \geq 1$ and $\theta_j \in \boldsymbol{R}$ $(j = 1, 2)$, one sees that

$$\Phi(t) = t^p(\log(e + t))^{\theta_1}(\log(e + 1/t))^{\theta_2}$$

satisfies$(\Phi_1), (\Phi_2)$ and (Φ_3) when

(P_1) $1 < p < \infty$; or

(P_2) $p = 1, \theta_1 \geq 0$ and $\theta_2 \leq 0$.

From now on, we always assume that Φ satisfies (Φ_1), (Φ_2) and (Φ_3). Let $\tilde{\phi}(t) = \sup_{0 \le s \le t} \phi(s)$ and

$$\overline{\Phi}(t) = \int_0^t \tilde{\phi}(r)\ dr$$

for $t \geq 0$. Then $\overline{\Phi}$ is convex and

$$\frac{1}{2A_2}\Phi(t) \le \overline{\Phi}(t) \le A_1\Phi(t) \qquad (1)$$

for all $t \geq 0$. Moreover $\overline{\phi}(t) = t^{-1}\overline{\Phi}(t)$ is increasing in $(0, \infty)$.

For an open set Ωin $\boldsymbol{R}^N$, the associated Musielak–Orlicz space

$$L^{\Phi}(\Omega) = \left\{ f \in L^1_{loc}(\Omega);\ \int_{\Omega} \Phi(|f(y)|)\, dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\| f \|_{L^{\Phi}(\Omega)} = \inf\left\{\lambda > 0;\ \int_{\Omega} \overline{\Phi}(|f(y)|/\lambda)\, dy \le 1\right\}$$

We further consider a function $\omega : (0, \infty) \to (0, \infty)$ such that

(ω_1) ω is almost monotone in $(0, \infty)$; that is, ω or ω^{-1} is almost increasing in $(0, \infty)$;

(ω_2) $\omega$is doubling.

We set $A(0, r) = B(0, 2r) \setminus B(0, r)$ with the open ball $B(x, r)$ centered at x of radius r. For$0 < q \le \infty$ we denote by $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ the class of locally integrable functions f on $\boldsymbol{R}^N$ satisfying

$$\| f \|_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)} = \| f \|_{L^{\Phi}(B(0,2))} + \left(\int_1^{\infty} \left(\omega(r) \| f \|_{L^{\Phi}(A(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty$$

When $q < \infty$, and

$$\| f \|_{\mathcal{H}^{\Phi,\infty,\omega}(\boldsymbol{R}^N)} = \| f \|_{L^{\Phi}(B(0,2))} + \left(\sup_{1<r<\infty} \omega(r) \| f \|_{L^{\Phi}(A(0,r))} \right) < \infty$$

When $q = \infty$. The space $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ is referred to as a non-homogeneous central Herz–Morrey–Orlicz space. One sees the following: if $0 < q_1 < q_2 < \infty$, then

$$\mathcal{H}^{\Phi,q_1,\omega}(\boldsymbol{R}^N) \subset \mathcal{H}^{\Phi,q_2,\omega}(\boldsymbol{R}^N) \subset \mathcal{H}^{\Phi,\infty,\omega}(\boldsymbol{R}^N) \qquad (2)$$

When $\Phi(t) = t^p$, we sometimes write $\mathcal{H}^{p,q,\omega}(\boldsymbol{R}^N)$ for $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Moreover, let us consider the space $\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$of all locally integrable functions f on $\boldsymbol{R}^N$ satisfying

$$\|f\|_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)} = \left(\int_1^\infty \left(\omega(r) \parallel f \parallel_{L^\Phi(B(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty,$$

and the space $\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ of all locally integrable functions f on $\boldsymbol{R}^N$ satisfying

$$\|f\|_{\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)} = \parallel f \parallel_{L^\Phi(B(0,2))} + \left(\int_1^\infty \left(\omega(r) \parallel f \parallel_{L^\Phi(\boldsymbol{R}^N/B(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

when $q < \infty$; if $q = \infty$, then we need necessary modifications.

Lemma (2.1.2)[2]: Let $\Phi(t) = t^p(\log(e + t))^\theta$with $p > 1$ and $\theta \geq 0$.

(i) If $\parallel f \parallel_{L^\Phi(A(0,r))} \leq 1$ for $r > 0$, then

$$\int_{A(0,r)} \bar{\Phi}(|f(y)|)\ dy \leq \parallel f \parallel^p_{L^\Phi(A(0,r))}.$$

(ii) If $\parallel f \parallel_{L^\Phi(A(0,r))} \geq 1$ for $r > 0$, then

$$\int_{A(0,r)} \bar{\Phi}(|f(y)|)\ dy \geq \parallel f \parallel^p_{L^\Phi(A(0,r))}.$$

With the aid of (2) and Lemma (2.1.2), we have the following result.

Corollary (2.1.3)[2]: Let $\Phi(t) = t^p(\log(e + t))^\theta$ with $p > 1$ and $\theta \geq 0$. Let $\omega(r) = r^\nu$.

(i) If $\left(\int_1^\infty \left(r^\nu \parallel f \parallel_{L^\Phi(\boldsymbol{A}(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} \leq 1$, then there exists a constant $C > 0$ such that

$$\left(\int_1^\infty \left(\int_{A(0,r)} \Phi(r^\nu |f(y)|)\ dy\right)^{q/p} \frac{dr}{r}\right)^{1/q} \leq C,$$

which implies

$$\left(\int_1^\infty \left(r^{\nu p}\int_{A(0,r)} \Phi(|f(y)|)\ dy\right)^{q/p} \frac{dr}{r}\right)^{1/q} \le C \quad \text{when}\ \nu \ge 0;$$

(ii) If $\nu < 0$and $\left(\int_1^\infty \left(r^{\nu p}\int_{A(0,r)} \Phi(|f(y)|)\ dy\right)^{q/p} \frac{dr}{r}\right)^{1/q} \le 1$, then there exists a constant $C > 0$ such that

$$\left(\int_1^\infty \left(r^{\nu} \parallel f \parallel_{L^\Phi(\boldsymbol{A}(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} \le C$$

Let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 0$.

For a measurable set $E \subset \boldsymbol{R}^N$, we denote by χ_E the characteristic function of E and by $|E|$ the Lebesgue measure of E. Since $\overline{\Phi}$ is convex on $[0,\infty)$ and $0 < \Phi(t) < \infty$ for all $t \in (0,\infty)$, $\overline{\Phi}$ has the inverse which is denoted by $\overline{\Phi}^{-1}$ as usual. Let us begin with the following easy fact.

Lemma (2.1.4)[2]:

$$\parallel \chi_{B(0,r)} \parallel_{L^\Phi(\boldsymbol{R}^N)} \le |B(0,1)|\,\{\overline{\Phi}^{-1}(r^{-N})\}^{-1}$$

for all $r > 0$.

For a real number β, set

$$\kappa_\beta(x) = |x|^\beta.$$

Corollary (2.1.5)[2]: For a real number β, suppose

($\Phi\beta_0$) there exists a constant $C > 0$ such that

$$\int_0^r t^\beta\{\overline{\Phi}^{-1}(r^{-N})\}^{-1}\frac{dt}{t} \le Cr^\beta\{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \quad for\ all\ r > 0$$

Then there exists a constant $C > 0$ such that

$$\parallel \kappa_\beta \parallel_{L^\Phi(B(0,r))} \le Cr^\beta\{\overline{\Phi}^{-1}(r^{-N})\}^{-1}$$

for all $r > 0$.

Corollary (2.1.6)[2]: For a real number β, suppose

($\Phi\beta_\infty$) there exists a constant $C > 0$ such that

$$\int_0^\infty t^\beta\{\overline{\Phi}^{-1}(r^{-N})\}^{-1}\frac{dt}{t} \le Cr^\beta\{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \quad for\ all\ r > 0$$

Then there exists a constant $C > 0$ such that

$$\| \kappa_\beta \|_{L^\Phi(\boldsymbol{R}^N \setminus B(0,r))} \leq C r^\beta \{\bar{\Phi}^{-1}(r^{-N})\}^{-1}$$

for all $r > 0$.

Example (2.1.7)[2]: Let $\Phi(t)$ be as in Example (2.1.1). If $p > 1$and $\theta_j \in \boldsymbol{R}$ $(j = 1, 2)$, then

$$\bar{\Phi}^{-1}(t) \sim t^{1/p}(\log(e + t))^{-\theta_1/p}(\log(e + 1/t))^{-\theta_2/p}$$

For $t > 0$. Moreover

(i) If $\beta + N/p > 0$, then $(\Phi\beta_0)$ holds;

(ii) if $\beta + N/p < 0$, then $(\Phi\beta_\infty)$ holds.

Lemma (2.1.8)[2]: There is a constant $C > 0$ such that

$$\frac{1}{|A(0,r)|}\int_{A(0,r)} |f(y)|\ dy \leq C\bar{\Phi}^{-1}(r^{-N}) \| f \|_{L^\Phi(A(0,r))}$$

for all $r > 0$ and measurable functions f.

Proof: Fix $r > 0$. Let f be a nonnegative measurable function on $A(0,r)$ satisfying $\| f \|_{L^\Phi(A(0,r))} \leq 1$.Then we have

$$\begin{aligned} &\frac{1}{|A(0,r)|}\int_{A(0,r)} |f(y)|\ dy \\ &\leq \bar{\Phi}^{-1}(r^{-N}) + \frac{1}{|A(0,r)|}\int_{A(0,r)} |f(y)|\ dy\, f(y)\frac{\bar{\phi}(f(y))}{\bar{\phi}(\bar{\Phi}^{-1}(r^{-N}))} dy \\ &= \bar{\Phi}^{-1}(r^{-N}) + \frac{\bar{\Phi}^{-1}(r^{-N})}{|A(0,r)|}\int_{A(0,r)} \bar{\Phi}(f(y))\{\bar{\Phi}(\bar{\Phi}^{-1}(r^{-N}))\}^{-1} dy \\ &\leq \bar{\Phi}^{-1}(r^{-N}) + C\bar{\Phi}^{-1}(r^{-N})\int_{A(0,r)} \bar{\Phi}(f(y)) \leq C\bar{\Phi}^{-1}(r^{-N}) \end{aligned}$$

as required.

For a locally integrable function $f$on $\boldsymbol{R}^N$, the Hardy–Littlewood maximal function $M\, f$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} (f(y))\, dy.$$

The mapping $f \longmapsto M\, f$ is called the maximal operator.

To assure the boundedness of the maximal operator, we need the condition:

$(\Phi_2;\ \varepsilon_0)$ $t^{-\varepsilon_0}\phi(t)$is almost increasing in $(0,\infty)$ for some $\varepsilon_0 > 0$;

more precisely, there exists a constant $A_{1,\varepsilon_0} \geq 1$ such that

$$t^{-\varepsilon_0}\,\phi(t) \leq A_{1,\varepsilon_0} s^{-\varepsilon_0}\phi(s) \qquad whenever\; 0 \leq t < s.$$

Then we find for $A > 1$

$$A\phi(t) \leq \phi\left(\left(AA_{1,\varepsilon_0}\right)^{\frac{1}{\varepsilon_0}} t\right) \quad whenever\; t > 0. \tag{3}$$

We have the following result.

Lemma (2.1.9)[2]: Suppose$(\Phi_2;\ \varepsilon_0)$ holds. Then the maximal operator M is bounded from $L^{\Phi}(\boldsymbol{R}^N)$ into itself, namely, there is a constant $C > 0$ such that

$$\| Mf \|_{L^{\Phi}(\boldsymbol{R}^N)} \leq C \| f \|_{L^{\Phi}(\boldsymbol{R}^N)}$$

for all $f \in L^{\Phi}(\boldsymbol{R}^N)$.

Let $\eta : (0,\infty) \to (0,\infty)$ satisfy (ω_1) and (ω_2). Then, for $0 < q < \infty$ and $1/2 < a < 1 < b < 2$ with $2a \geq b$, there exists a constant $C > 0$ such that

$$\int_{at}^{bt} (\eta(r) \| f \|_{L^{\Phi}(A(0,r))})^q \frac{dr}{r} \geq C\left(\eta(t) \| f \|_{L^{\Phi}(A(0,t))}\right)^q \tag{4}$$

for all $t > 0$.

Let

$$\frac{1}{q'} = \begin{cases} 0 & \text{when } 0 < q \leq 1, \\ (q-1)/q & \text{when } 1 < q < \infty, \\ q & \text{when } q = \infty \end{cases}$$

For a nonnegative function $f \in L^1_{loc}(\boldsymbol{R}^N)$ and a real number β, set

$$H^{\infty}_{\beta} f(r) = r^{\beta} \int_{\boldsymbol{R}^N \setminus B(0,2r)} |y|^{-N-\beta} f(y)\, dy.$$

Lemma (2.1.10)[2]: for a real number β, suppose

$(\Phi\omega_1;\ \beta) t^{\varepsilon_1-\beta}\omega(t)^{-1}\overline{\Phi}^{-1}(r^{-N})$is almost decreasing in $[1,\infty)$ for some $\varepsilon_1 > 0$.

If $0 < \varepsilon < \varepsilon_1$, then there exists a constant $C > 0$ such that

$$H^{\infty}_{\beta} f(r) \leq C r^{\varepsilon} \omega(t)^{-1} \overline{\Phi}^{-1}(r^{-N}) \left(\int_r^{\infty} \left(t^{-\varepsilon}\omega(t)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

When $0 < q < \infty$ and

$$H_\beta^\infty f(r) \le C\omega(t)^{-1}\overline{\Phi}^{-1}(r^{-N}) \sup_{t>r}\left(\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)$$

When $q = \infty$, for all $r \ge 1$ and nonnegative functions $f \in L_{loc}^1(\boldsymbol{R}^N)$.

Proof : We treat only the case $1 < q < \infty$, since the remaining case is easily treated. Let $f \in L_{loc}^1(\boldsymbol{R}^N)$ be a nonnegative function on $\boldsymbol{R}^N$. Let $r \ge 1$ and $0 < \varepsilon < \varepsilon_1$. Then we have by Lemma (2.1.9), Hölder's inequality,$(\Phi\omega_1;\ \beta)$ and (4)

$$H_\beta^\infty f(r) = r^\beta \sum_{j=1}^{\infty} \int_{A(0,2^j r)} |y|^{-N-\beta} f(y)dy \le Cr^\beta \sum_{j=1}^{\infty} (2^j r)^{-\beta} \frac{1}{|A(0,2^j r)|} \int_{A(0,2^j r)} f(y)dy$$

$$\le Cr^\beta \sum_{j=1}^{\infty} (2^j r)^{-\beta}\, \overline{\Phi}^{-1}\left((2^j r)^{-N}\right) \|f\|_{L^\Phi\left(A(0,2^{-j}r)\right)}$$

$$\le Cr^\beta \left(\sum_{j=1}^{\infty}\left((2^j r)^{\varepsilon-\beta}\omega(2^j r)^{-1}\overline{\Phi}^{-1}\left((2^j r)^{-N}\right)\right)^{q'}\right)^{\frac{1}{q'}} \left(\sum_{j=1}^{\infty}\left((2^j r)^{-\varepsilon}\omega(2^j r)\|f\|_{L^\Phi\left(A(0,2^j r)\right)}\right)^{q'}\right)^{\frac{1}{q}}$$

$$\le\ Cr^{-\varepsilon}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\left(\int_r^\infty \left(t^{-\varepsilon}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

Which proves the case $1 < q < \infty$.

For a nonnegative function $f \in L_{loc}^1(\boldsymbol{R}^N)$ and a real number β, set

$$H_\beta^0 f(r) = r^\beta \int_{B(0,r)\setminus B(0,1)} |y|^{-N-\beta} f(y)dy$$

Lemma (2.1.11)[2]: for a real number β, suppose

$(\Phi\ \omega_2;\ \beta)$ $t^{-\varepsilon 2-\beta}\omega(t)^{-1}\overline{\Phi}^{-1}(t^{-N})$ is almost increasing in $[1,\infty)$ for some $\varepsilon_2 > 0$.

If $0 < \varepsilon < \varepsilon_2$, then there exists a constant $C > 0$ such that

$$H_\beta^0 f(r)\ \le\ Cr^{-\varepsilon}\omega(t)^{-1}\overline{\Phi}^{-1}(t^{-N})\left(\int_{1/2}^r \left(t^\varepsilon\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

When $0 < q < \infty$ and

$$H_\beta^0 f(r) \le C\omega(t)^{-1}\overline{\Phi}^{-1}(t^{-N}) \sup_{\frac{1}{2}<t<r}\left(\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)$$

When $q = \infty$, for all $r \ge 1$ and nonnegative functions $f \in L_{loc}^1\ (\boldsymbol{R}^N)$.

Proof: We show only the case $1 < q < \infty$. Let $f \in L_{loc}^1\ (\boldsymbol{R}^N)$ be a nonnegative function on $\boldsymbol{R}^N$. Let $r \ge 1$ and $0 < \varepsilon < \varepsilon_2$. Let j_0 be the largest integer such that $2^{-j_0+1}r \ge 1$.We have by Lemma (2.1.9), Hölder's inequality, $(\Phi\omega_2;\ \beta)$ and (4)

$$
\begin{aligned}
H_\beta^\infty f(r) &= r^\beta \sum_{j=1}^{j_0} \int_{A(0,2^{-j}r)\setminus B(0,1)} |y|^{-N-\beta} f(y)dy \\
&\le Cr^\beta \sum_{j=1}^{j_0} (2^{-j}r)^{-\beta} \frac{1}{|A(0,2^{-j}r)|} \int_{A(0,2^{-j}r)\setminus B(0,1)} f(y)dy \\
&\le Cr^\beta \sum_{j=1}^{j_0} (2^{-j}r)^{-\beta}\, \overline{\Phi}^{-1}\left((2^{-j}r)^{-N}\right) \|f\chi_{\boldsymbol{R}^N\setminus B(0,1)}\|_{L^\Phi(A(0,2^j r))} \\
&\le Cr^\beta \left(\sum_{j=1}^{j_0} \left((2^{-j}r)^{-\varepsilon-\beta} \omega(2^j r)^{-1} \overline{\Phi}^{-1}\left((2^{-j}r)^{-N}\right)\right)^{q'} \right)^{\frac{1}{q'}} \\
&\times \left(\sum_{j=1}^{j_0} \left((2^{-j}r)^{\varepsilon} \omega(2^{-j}r) \|f\chi_{\boldsymbol{R}^N\setminus B(0,1)}\|_{L^\Phi(A(0,2^j r))}\right)^{q} \right)^{\frac{1}{q}} \\
&\le Cr^{-\varepsilon}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \left(\sum_{j=1}^{j_0} \left((2^{-j}r)^{\varepsilon} \omega(2^{-j}r) \|f\chi_{\boldsymbol{R}^N\setminus B(0,1)}\|_{L^\Phi(A(0,2^j r))}\right)^{q} \right)^{\frac{1}{q}} \\
&\le Cr^{-\varepsilon}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \left(\int_{1/4}^{r} \left(t^\varepsilon \omega(t) \parallel f\chi_{\boldsymbol{R}^N\setminus B(0,1)} \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\le Cr^{-\varepsilon}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \left(\int_{1/2}^{r} \left(t^\varepsilon \omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}},
\end{aligned}
$$

Which gives the required result.

We present the boundedness of the maximal operator in $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Theorem (2.1.12)[2]: In addition to $(\Phi_2;\ \varepsilon_0)$, assume $(\Phi\omega_1;\ 0)$ and $(\Phi\omega_2;\ -N)$ hold. Then the maximal operator M is bounded from $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$ to itself, that is,

$$\parallel Mf \parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)} \le C \parallel f \parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Proof: we show only the case $1 < q < \infty$, because the remaining case is easily treated. Let f be a nonnegative measurable function on $\boldsymbol{R}^N$ such that $\parallel f \parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)} \le 1$.First we show

$$\int_2^\infty \left(\omega(r) \parallel Mf \parallel_{L^\Phi(A(0,r))}\right)^q \frac{dr}{r} \le C.$$

For $r \ge 2$, set

$$f = f\chi_{B(0,r/2)} + f\chi_{B(0,4r)\setminus B(0,r/2)} + f\chi_{\boldsymbol{R}^N\setminus B(0,4r)} = f_{1,r} + f_{2,r} + f_{3,r}.$$

We have by Lemma (2.1.9)

$$\| Mf_{2,r} \|_{L^{\Phi}(A(0,r))} \leq C \| Mf_{2,r} \|_{L^{\Phi}(\boldsymbol{R}^N)} \leq C \| f \|_{L^{\Phi}(B(0,4r))\setminus B(0,r/2)}.$$

so that

$$\int_2^{\infty} \left(\omega(r) \| Mf_{2,r} \|_{L^{\Phi}(A(0,r))}\right)^q \frac{dr}{r} \leq C \int_2^{\infty} \left(\omega(t) \| f \|_{L^{\Phi}(B(0,4r))\setminus B(0,r/2)}\right)^q \frac{dr}{r}$$
$$\leq C \int_1^{\infty} \left(\omega(r) \| f \|_{L^{\Phi}(A(0,r))}\right)^q \frac{dr}{r} \leq C$$

For $f_{1,r}$, we find for $x \in A(0,r)$

$$Mf_{1,r}(x) \leq C|x|^{-N} \int_{B(0,r/2)} f(y)\, dy = C_{\kappa-N}(x) \int_{B(0,r/2)} f(y)\, dy.$$

For fixed $r \geq 2$, note from Lemma (2.1.4) that

$$\| \kappa - N \|_{L^{\Phi}(A(0,r))} \leq C^{r-N} \{\overline{\Phi}^{-1}(r^{-N})\}^{-1}.$$

We have for $0 < \varepsilon_2' < \varepsilon_2$

$$\| Mf_{1,r} \|_{L^{\Phi}(A(0,r))} \leq C^{r-N} \{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \int_{B(0,r/2)} f(y)\, dy$$
$$\leq C \Bigg[\{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \left(r^{-N} \int_{B(0,r/2)\setminus B(0,1)} f(y)\, dy \right)$$
$$+ r^{-N} \{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \int_{B(0,1)} f(y)\, dy \Bigg]$$
$$\leq C \Bigg[r^{-\varepsilon_2'} \omega(r)^{-1} \left(\int_{1/2}^{r} \left(t^{\varepsilon_2'} \omega(t) \| f \|_{L^{\Phi}(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$
$$+ r^{-N} \{\overline{\Phi}^{-1}(r^{-N})\}^{-1} \Bigg]$$

by Lemma (2.1.11). Since $t^{\varepsilon_2'} \omega(t) \{\overline{\Phi}^{-1}(r^{-N})\}^{-1}$ is almost decreasing in $[1, \infty)$ by $(\Phi\omega_2;\, -N)$, we have

$$\int_2^{\infty} [r^{-N} \omega(t) \{\overline{\Phi}^{-1}(r^{-N})\}^{-1}]^q \frac{dr}{r} \leq C$$

Hence we obtain

$$\int_2^\infty \left(\omega(t) \parallel M f_{1,r} \parallel_{L^\Phi(A(0,r))}\right)^q \frac{dr}{r}$$

$$\leq C\left\{\int_2^\infty r^{-\varepsilon_2' q}\left\{\int_{\frac{1}{2}}^r \left(t^{\varepsilon_2'}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right\}\right.$$

$$\left.+\int_2^\infty [r^{-N}\omega(t)\{\overline{\Phi}^{-1}(r^{-N})\}^{-1}]^q \frac{dr}{r}\right\}$$

$$\leq C\left\{\int_{1/2}^\infty \left(t^{\varepsilon_2'}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \left(\int_t^\infty r^{-\varepsilon_2' q}\frac{dr}{r}\right)\frac{dt}{t}+1\right\}$$

$$\leq \ C\left\{\int_{1/2}^\infty \left(\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}+1\right\} \leq \ C.$$

For $f_{3,r}$, we find for $x \ \in \ A(0,r)$

$$M\, f_{3,r}(x) \ \leq C\int_{\boldsymbol{R}^N\setminus B(0,4r)} f(y)|y|^{-N}\,dy$$

and by Lemmas (2.1.4) and (2.1.10)

$$\parallel M\, f_{3,r} \parallel_{L^\Phi(A(0,r))} \leq C\left\{\overline{\Phi}^{-1}(r^{-N})\right\}^{-1}\int_{\boldsymbol{R}^N\setminus B(0,4r)} f(y)|y|^{-N}\,dy$$

$$\leq C r^{\varepsilon_1'}\omega(r)^{-1}\left(\int_1^\infty \left(t^{\varepsilon_1'}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

For $0 < \varepsilon_1' < \varepsilon_1$. Hence we have

$$\int_2^\infty \left(\omega(r) \parallel M f_{3,r} \parallel_{L^\Phi(A(0,r))}\right)^q \frac{dr}{r}$$

$$\leq C\int_2^\infty r^{\varepsilon_1' q}\left(\int_1^\infty \left(t^{\varepsilon_1'}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)\frac{dr}{r}$$

$$\leq C\int_2^\infty \left(t^{\varepsilon_1'}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \left(\int_1^t r^{\varepsilon_1' q}\frac{dr}{r}\right)\frac{dt}{t}$$

$$\leq C\int_2^\infty \left(\omega(r) \parallel f \parallel_{L^\Phi(A(0,r))}\right)^q \frac{dt}{t} \leq C$$

Finally we show

$$\parallel Mf \parallel_{L^\Phi(B(0,4))} \leq C$$

Since

$$\| Mf \|_{L^{\Phi}(B(0,2))} \leq \left(\int_1^2 \left(\omega(t) \| Mf \|_{L^{\Phi}(A(0,r))} \right)^q \frac{dt}{t} \right)^{1/q} \leq C \| Mf \|_{L^{\Phi}(B(0,4))}.$$

Set

$$f = f\chi_{B(0,8)} + f\chi_{\boldsymbol{R}^N \setminus B(0,8)} = f_4 + f_5.$$

The above arguments yield

$$\| M f_4 \|_{L^{\Phi}(B(0,4))} \leq C \| f \|_{L^{\Phi}(B(0,8))} \leq C$$

and

$$\| M f_5 \|_{L^{\Phi}(B(0,4))} \leq C \left(\int_1^{\infty} \left(\omega(t) \| f \|_{L^{\Phi}(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q} \leq C,$$

as required.

In the same manner we can prove the following results.

Theorem (2.1.13)[2]: in addition to $(\Phi_2;\ \varepsilon_0)$, assume that $(\Phi\omega_1;\ 0)$ holds. Then the maximal operator M is bounded from $\mathcal{H}^{\Phi,\mathrm{q,w}}(\boldsymbol{R}^N)$ to itself.

Theorem (2.1.14)[2]: In addition to $(\Phi_2;\ \varepsilon_0)$, assume that $(\Phi\omega_2;\ -N)$ holds. Then the maximal operator M is bounded from $\overline{\mathcal{H}}^{\Phi,\mathrm{q,w}}(\boldsymbol{R}^N)$ to itself.

Here, noting that

$$\int_r^{\infty} t^{-N} \{\overline{\Phi}^{-1}(t^{-N})\}^{-1} \frac{dt}{t} \leq C r^{-N} \{\overline{\Phi}^{-1}(r^{-N})\}^{-1}$$

by $(\Phi_2;\ \varepsilon_0)$, and hence we can prove the last theorem.

Section (2.2): Sobolev Inequality and Generalized Potentials with Associate Space

The Riesz potential is defined by

$$I_\alpha f(x) = \int_{\boldsymbol{R}^N} I_\alpha(x - y) f(y)\, dy$$

for a locally integrable function f on $\boldsymbol{R}^N$.

Lemma (2.2.1)[2]: Assume that $(\Phi\omega_1;\ -\alpha)$ holds for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0, 2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\boldsymbol{R}^N)$,

$$\left|I_\alpha(f\,\chi_{\boldsymbol{R}^N\setminus B(0,4r)})(x)\right|$$
$$\leq\ Cr^{\varepsilon+\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\left(\int_r^\infty t^{-\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)^{1/q}$$

when $0<q<\infty$ and

$$\left|I_\alpha\left(f\,\chi_{\boldsymbol{R}^N\setminus B(0,4r)}\right)(x)\right|\leq Cr^\alpha\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\sup_{t>r}\left(\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)$$

When $q=\infty$.

Proof: We treat only the case $1<q<\infty$, as before. Let $f\in L^1_{loc}(\boldsymbol{R}^N)$ be a nonnegative function on $\boldsymbol{R}^N$. Let $r\geq 1, x\in B(0,2r)$ and $0<\varepsilon<\varepsilon_1$.First note from Lemma (2.1.10) with $\beta=-\alpha$ that

$$\left|I_\alpha\left(f\,\chi_{\boldsymbol{R}^N\setminus B(0,4r)}\right)(x)\right|\leq C\int_{\boldsymbol{R}^N\setminus B(0,4r)}|y|^{\alpha-N}f(y)dy$$
$$\leq\ Cr^{\varepsilon+\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\left(\int_r^\infty\left(t^{-\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

as required.

Lemma (2.2.2)[2]: Assume $(\Phi\omega_2;\,-N)$ holds for $\varepsilon_2>0$. Then, for $0<\varepsilon<\varepsilon_2$, there exists a constant $C>0$ such that for all $x\in\boldsymbol{R}^N\setminus B(0,r)$ with $r\geq 1$ and nonnegative functions $f\in L^1_{loc}(\boldsymbol{R}^N)$,

$$\left|I_\alpha\left(f\,\chi_{\boldsymbol{R}^N\setminus B(0,r/2)\setminus B(0,1)}\right)(x)\right|$$
$$\leq C(|x|/r)^{\alpha-N}r^{-\varepsilon+\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\times\left(\int_{1/2}^r\left(t^\varepsilon\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}$$

When $0<q<\infty$ and

$$\left|I_\alpha\left(f\,\chi_{\boldsymbol{R}^N\setminus B(0,r/2)\setminus B(0,1)}\right)(x)\right|$$
$$\leq C(|x|/r)^{\alpha-N}r^\alpha\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\times\sup_{\frac{1}{2}<t<r}\left(\omega(t)\|f\|_{L^\Phi(A(0,t))}\right)$$

When $q=\infty$.

Proof: We show only the case $1<q<\infty$. Let $f\in L^1_{loc}(\boldsymbol{R}^N)$be a nonnegative function on $\boldsymbol{R}^N$. Let $r\geq 1, x\in\boldsymbol{R}^N\setminus B(0,r)$ and $0<\varepsilon<\varepsilon_2$.First note that

$$\left|I_\alpha\left(f\,\chi_{\boldsymbol{R}^N\setminus B(0,r/2)\setminus B(0,1)}\right)(x)\right|\leq C|x|^{\alpha-N}\int_{\boldsymbol{R}^N\setminus B\left(0,\frac{r}{2}\right)\setminus B(0,1)}f(y)dy$$
$$\leq\ C\left(\frac{|x|}{r}\right)^{\alpha-N}r^{\alpha-N}\int_{B(0,r/2)\setminus B(0,1)}f(y)dy$$

Hence Lemma (2.1.11) with $\beta = -N$ gives the first case.

To obtain the Sobolev type inequality, we use a function

$$\Psi(t) = t\psi(t) : [0,\infty) \to [0,\infty)$$

With ψ satisfying the following conditions:

(Ψ_1) ψ is continuous in $[0,\infty)$;

(Ψ_2) $\psi$is almost increasing in $[0,\infty)$.

Setting $\tilde{\psi}(r) = \sup_{0\le s\le t} \psi(s)$ in the same manner as ϕ, we define

$$\overline{\Psi}(r) = \int_0^r \tilde{\psi}(t)\, dt$$

and note that $\bar{\psi}(r) = r^{-1}\overline{\Psi}(r)$ is increasing in $(0,\infty)$.

Moreover we need the following conditions:

(Φ_α) $r \longmapsto r^{\varepsilon+\alpha}\bar{\Phi}^{-1}(r^{-N})$is almost decreasing in $(0,\infty)$ for some $\varepsilon > 0$;

$(\Psi\Phi_\alpha)$there exists a constant $A_3 \ge 1$ such that

$$\overline{\Psi}\left(t\bar{\Phi}(t)^{-\alpha/N}\right) \le A_3\bar{\Phi}(t)$$

For all $t > 0$.

Lemma (2.2.3)[2]: Suppose $(\Phi_2;\ \varepsilon_0), (\Phi_\alpha)$ and $(\Psi\Phi_\alpha)$ hold. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^{\Psi}(\boldsymbol{R}^N)} \le C\|f\|_{L^{\Phi}(\boldsymbol{R}^N)}$$

For all $f \in L^{\Phi}(\boldsymbol{R}^N)$.

Now we show the Sobolev type inequality for Riesz potentials of functions in $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Theorem (2.2.4)[2]: Suppose$(\Phi_2;\ \varepsilon_0), (\Phi_\alpha)$ and $(\psi\Phi_\alpha)$ are fulfilled. Further, assume that $(\Phi\ \omega_1;\ -\alpha)$ and $(\Phi\omega_2;\ -N)$ hold. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{H}^{\Psi,q,\omega}(\boldsymbol{R}^N)} \le C\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

For all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Proof. We show only the case $1 < q < \infty$. Let f be a nonnegative measurable function on $\boldsymbol{R}^N$ such that $\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)} \le 1$. For $r \ge 1$ and $x \in A(0,r)$, set

$$\begin{aligned} f &= f\,\chi_{B(0,1)} + f\,\chi_{\boldsymbol{R}^N\setminus B(0,r/2)\setminus B(0,1)} + f\,\chi_{\boldsymbol{R}^N\setminus B(0,4r)\setminus B(0,r/2)} + \chi_{\boldsymbol{R}^N\setminus B(0,4r)} \\ &= f_0 + f_{1,r} + f_{2,r} + f_{3,r}. \end{aligned}$$

First we treat $f_{1,r}$. Note from $(\Psi\Phi_\alpha)$ that

$$\sup_{t>0} t^\alpha\, \overline{\Phi}^{-1}(t^{-N})\{\overline{\Psi}^{-1}(t^{-N})\}^{-1} \le C \tag{5}$$

$$\begin{aligned}
&\left\|I_\alpha f_{1,r}\right\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}} \\
&\qquad \le C r^{-\varepsilon_2'}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \\
&\qquad \times \left(\int_{1/2}^{r}\left(r^{-\varepsilon_2'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right)^{1/q} \|1\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}} \\
&\qquad \le C r^{-\varepsilon_2'}\omega(r)^{-1}\left(\int_{1/2}^{r}\left(r^{-\varepsilon_2'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right)^{1/q}
\end{aligned}$$

for some $0 < \varepsilon_2' < \varepsilon_2$, since Lemma (2.1.3) holds for $\overline{\Psi}$. Hence

$$\begin{aligned}
\int_1^\infty \left(\omega(r)\left\|I_\alpha f_{1,r}\right\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}}\right)^q \frac{dr}{r} &\le C\int_1^\infty r^{-\varepsilon_2' q}\left(\int_{1/2}^{r}\left(t^{-\varepsilon_2'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right)\frac{dr}{r} \\
&\le C\int_{1/2}^{\infty}\left(t^{-\varepsilon_2'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q\left(\int_t^\infty r^{-\varepsilon_2' q}\frac{dr}{r}\right)\frac{dr}{t} \\
&\le C\int_{1/2}^{\infty}\left(\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{t}
\end{aligned}$$

Similarly, for $f_{3,\mathrm{r}}$

$$\begin{aligned}
&\left\|I_\alpha f_{3,r}\right\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}} \\
&\qquad \le C r^{-\varepsilon_1'+\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \\
&\qquad \times \left(\int_r^\infty\left(t^{-\varepsilon_1'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right)^{1/q} \|1\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}} \\
&\qquad \le C r^{-\varepsilon_1'}\omega(r)^{-1}\left(\int_r^\infty\left(t^{-\varepsilon_1'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right)^{1/q}
\end{aligned}$$

for some $0 < \varepsilon_1' < \varepsilon_1$, so that

$$\begin{aligned}
&\int_1^\infty \left(\omega(r)\left\|I_\alpha f_{3,r}\right\|_{L^{\Psi(\mathrm{A}(0,\mathrm{r}))}}\right)^q \frac{dr}{r} \\
&\qquad \le C\int_1^\infty r^{-\varepsilon_1' q}\left(\int_r^\infty\left(t^{-\varepsilon_1'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{t}\right)\frac{dr}{r} \\
&\qquad = C\int_1^\infty\left(t^{-\varepsilon_1'}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)\left(\int_t^\infty r^{-\varepsilon_1' q}\frac{dr}{r}\right)\frac{dr}{t} \\
&\qquad \le C\int_1^\infty\left(\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{t} \le C
\end{aligned}$$

For $f_{2,r}$, we have by Lemma (2.2.4)

$$\left\|I_\alpha f_{2,r}\right\|_{L^{\Psi(A(0,r))}} \leq C\|f\|_{L^{\Phi}(B(0,4r)\backslash B(0,r/2))},$$

So that

$$\int_1^\infty \left(\omega(r)\left\|I_\alpha f_{2,r}\right\|_{L^{\Psi(A(0,r))}}\right)^q \frac{dr}{r} \leq C\int_{1/2}^\infty \left(\omega(r)\|f\|_{L^{\Phi}(A(0,r))}\right)^q \frac{dr}{t} \leq C$$

Next we show that

$$\int_2^\infty \left(\omega(r)\|I_\alpha f_0\|_{L^{\Psi(A(0,r))}}\right)^q \frac{dr}{r} \leq C$$

Since

$$I_\alpha f_0(x) \ = \int_{B(0,1)} |x \ - \ y|^{\alpha-N} \ f(y)\, dy \ \leq \ C|x|^{\alpha-N}\int_{B(0,1)} f(y)\, dy$$
$$\leq C_{\kappa\alpha-N}(x)$$

when$|x| \ \geq \ 2$ and

$$\| \ \kappa\alpha - N \ \|_{L^{\Psi}(\mathbf{R}^N\backslash B(0,r))} \leq C\sum_{j=1}^{\infty}\left(2^{j-1}r\right)^{\alpha-N}\left\{\overline{\Psi}^{-1}\left(\left(2^{j-1}r\right)^{-N}\right)\right\}^{-1}$$
$$\leq C\sum_{j=1}^{\infty}\left(2^{j-1}r\right)^{-N}\left\{\overline{\phi}^{-1}\left(\left(2^{j-1}r\right)^{-N}\right)\right\}^{-1} \leq Cr^{-N}\{\overline{\phi}^{-1}(r^{-N})\}^{-1}$$

by Lemma (2.1.3), (5) and $(\Phi_2;\ \varepsilon_0)$, we have

$$\int_2^\infty \left(\omega(r)\|I_\alpha f_0\|_{L^{\Psi}(A(0,r))}\right)^q \frac{dr}{r} \leq \int_2^\infty [\{r^{-N}\omega(r)\ \overline{\Phi}^{-1}(r^{-N})\}^{-1}]^q \frac{dr}{r} \leq \ C.$$

Finally, the above arguments and Lemma (2.2.3) yield

$$\|I_\alpha f_0\|_{L^{\Psi}(B(0,2))} + \left(\int_1^2 \left(\omega(r)\|I_\alpha f_0\|_{L^{\Psi}(A(0,r))}\right)^q \frac{dr}{r}\right)^{1/q} \leq C\|I_\alpha f_0\|_{L^{\Psi}(B(0,4))}$$
$$\leq C\left(\|I_\alpha f_0\|_{L^{\Phi}(B(0,8))} + \int_1^2 \left(\omega(r)\|I_\alpha f_0\|_{L^{\Phi}(A(0,t))}\right)^q \frac{dr}{r}\right) \leq C$$

Which proves the theorem.

Theorem (2.2.5)[2]: Suppose$(\Phi_2;\ \varepsilon_0), (\Phi_\alpha)$ and $(\Psi\Phi_\alpha)$ are satisfied. Further, assume that $(\Phi\omega_1;\ -\alpha)$ holds. Then there exists a constant $C\ >\ 0$ such that

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{\Psi,q,\Phi}(\boldsymbol{R}^N)} \leq C\|f\|_{\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

for all $f \in \overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Theorem (2.2.6)[2]: Suppose$(\Phi_2;\ \varepsilon_0)$, (Φ_α) and $(\Psi\Phi_\alpha)$ are satisfied. Further assume that $(\Phi\omega_2;\ -N)$ holds. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{\Psi,q,\Phi}(\boldsymbol{R}^N)} \le C\|f\|_{\overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

For all $f \in \overline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

In fact, set

$$f = f\,\chi_{B(0,1)} + f\,\chi_{\boldsymbol{R}^N\backslash B(0,r/2)\backslash B(0,1)} + \chi_{\boldsymbol{R}^N\backslash B(0,r/2)} = f_0 + f_{1,r} + f_{2,r}.$$

For a locally integrable function f on $\boldsymbol{R}^N$. Then Lemma (2.2.2) yields

$$\left|I_\alpha f_{1,r(x)}\right|$$
$$\le C|x|^{\alpha-N} r^{-\varepsilon+N}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\left(\int_{1/2}^{r}\left(t^{\varepsilon}\omega(r)\|f\|_{L^{\Phi}(A(0,t))}\right)^{q}\frac{dr}{r}\right)^{1/q}$$

Since the Orlicz case is rather complicated as was seen by Corollary 2.3, we here restrict ourselves to treat the Lebesgue's L^p case. Consider a measurable function f on $\boldsymbol{R}^N$ such that

$$\int_{\boldsymbol{R}^N}(1 + |y|)^{vp}\,|f(y)|^p dy < \infty;$$

namely, $f \in \mathcal{H}^{p,p,\omega}(\boldsymbol{R}^N)$. Then Theorem (2.2.5) gives

$$\int_0^\infty\left\{(1 + r)^v\|I_\alpha f\|_{L^{p^*}(A(0,r))}\right\}^p\frac{dr}{r} < \infty.$$

If $p \le p_1 \le p^*$, then

$$\int_0^\infty\left\{(1 + r)^v|A(0,r)|^{1/p^*-1/p_1}\|I_\alpha f\|_{L^{p_1}(A(0,r))}\right\}^p\frac{dr}{r} < \infty.$$

which gives

$$\int_0^\infty\left\{(1 + r)^v|A(0,r)|^{1/p^*-1/p_1}\|I_\alpha f\|_{L^{p_1}(A(0,r))}\right\}^{p_1}\frac{dr}{r} < \infty.$$

so that

$$\int_{\boldsymbol{R}^N}(1 + |y|)^{(v+N/p^*-N/p_1)p_1}\,|I_\alpha f(x)|^{p_1}dx < \infty. \qquad (6)$$

When $\alpha - N/p = v$, we modify condition $(\Phi\omega_1;\ \beta)$ to obtain a weak version of inequality (6), by adding logarithmic terms.

Further, inequality (6) is shown to hold for a wider range of exponents when f is radially symmetric. In fact, if $f \geq 0$ is radially symmetric and $0 < \alpha < 1$, then, by polar coordinates, we have

$$\int_{\boldsymbol{R}^N} |x-y|^{\alpha-N} f(|y|)dy$$
$$= \int_{B(0,2|x|)} |x-y|^{\alpha-N} f(|y|)dy$$
$$+ \int_{\boldsymbol{R}^N \backslash B(0,2|x|)} |x-y|^{\alpha-N} f(|y|)dy$$
$$\leq \int_0^{2|x|} f(t) \left(\int_{\partial B(0,1)} |x-t\sigma|^{\alpha-N}\, dS(\sigma) \right) t^{N-1}$$
$$+ C \int_{\boldsymbol{R}^N \backslash B(0,2|x|)} |y|^{\alpha-N} f(|y|)dy$$
$$\leq C \left\{ \int_0^{2|x|} \big||x|-t\big|^{\alpha-1} f(t)dt + \int_{2|x|}^{0} t^{\alpha-1} f(t)dt \right\}$$
$$\leq C \int_0^{\infty} \big||x|-t\big|^{\alpha-1} f(t)dt,$$

and apply the Sobolev inequality in the one dimensional case. In this case Theorem (2.2.4) can be extended for a wider class of Orlicz functions.

Next we consider the case that (Φ_α) does not hold.

Lemma (2.2.7)[2]: Suppose Φ satisfies $(\Phi_2;\ \varepsilon_0)$. Then there exists a constant $C > 0$ such that

$$r^{-\alpha} \|I_\alpha f_r\|_{L^\Phi(B(0,r))} \leq C \|f\|_{L^\Phi(B(0,r))}$$

for all $f \in L^\Phi_{loc}(\boldsymbol{R}^N)$ and $r > 0$, where $f_r = f\chi_{B(0,r)}$.

Proof: Let $f \in L^\Phi_{loc}(\boldsymbol{R}^N)$ be a nonnegative measurable function on $\boldsymbol{R}^N$. We have

$$I_\alpha f_r(x) \leq \int_{B(x,2r)} |x-y|^{\alpha-N} f_r(y)\, dy \leq Cr^\alpha M f_r(x)$$

for $x \in B(0,r)$. Hence we find by above Lemma

$$r^{-\alpha} \|I_\alpha f_r\|_{L^\Phi(B(0,r))} \leq C \|M f_r\|_{L^\Phi(B(0,r))} \leq C \|f_r\|_{L^\Phi(\boldsymbol{R}^N)} \leq C \|f\|_{L^\Phi(B(0,r))},$$

As required.

We obtain the following result.

Theorem (2.2.8)[2]: Suppose Φ satisfies $(\Phi_2;\ \varepsilon_0)$. Further, assume that $(\Phi\omega_1;\ -\alpha)$ and $(\Phi\omega_2;\ -N)$ hold. Set $\tau(r)\ = r^{-\alpha}\omega(r)$.Then there exists a constant $C\ >\ 0$ such that

$$\|I_\alpha f\|_{\mathcal{H}^{\Phi,q,\tau}(\boldsymbol{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

Even if neither (Φ_α) nor $(\Phi\omega_1;\ -\alpha)$ are not true, we still establish the following result in the same manner.

Theorem (2.2.9)[2]: Suppose Φsatisfies $(\Phi_2;\ \varepsilon_0)$. Further, assume that $(\Phi\omega_2;\ -N)$ holds and $(\Phi\omega_{\alpha q})$ $(\log(1\ +\ t))^{\varepsilon_3+1/q'}t^\alpha\omega(t) - 1\overline{\Phi}^{-1}(t^{-N})$ is almost decreasing in $[1,\infty)$ for some $\varepsilon_3 >\ 0$.

Set

$$\tau(r) = \begin{cases} (\log(1\ +\ r))^{-1/q}r^{-\alpha}\omega(r) & when\ 0 <\ q\ \leq 1, \\ (\log(1\ +\ r))^{-1}r^{-\alpha}\omega(r) & when\ 1 <\ q\ \leq \infty, \end{cases}$$

Then there exists a constant $C\ >\ 0$ such that

$$\|I_\alpha f\|_{\mathcal{H}^{\Phi,q,\tau}(\boldsymbol{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R}^N)$.

For this purpose we prepare the next result instead of Lemma (2.2.1)

Lemma (2.2.10)[2]: Suppose$(\Phi\omega_{\alpha q})$ holds. If $0 < \varepsilon < \varepsilon_3$, then there exists a constant $C\ >\ 0$ such that

$$\begin{aligned} &\left|I_\alpha\big(f\,\chi_{\boldsymbol{R}^N\setminus B(0,4r)}\big)(x)\right| \\ &\qquad \leq C(\log(1\ +\ r))^{\varepsilon+1/q'}r^\alpha\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N}) \\ &\qquad \times\left(\int_{1/2}^{r}\Big((\log(1+t))^{-\varepsilon}\omega(t)\|f\|_{L^\Phi(A(0,t))}\Big)^q\frac{dt}{t}\right)^{\frac{1}{q}} \end{aligned}$$

When $q\ <\ \infty$

and for all $x \in B(0{,}2r)$ with $r \geq 1$and nonnegative functions $f \in L^1_{loc}(\boldsymbol{R}^N)$

When $1/p^*\ =\ 1/p\ - \alpha/N\ =\ 0$and $p_1 =\ p$, we can modify (6) by

$$\int_{\boldsymbol{R}^N}(1\ +\ |x|)^{-N}\,(\log(2\ +\ |x|))^{(\theta-1)p}|I_\alpha f(x)|^p\,dx <\ \infty$$

for all functions f satisfying

$$\int_{\boldsymbol{R}^N} (\log(2 + |x|))^{\theta p} |f(x)|^p \, dx < \infty$$

With $\theta > 1/p'$.

This is the best possible in the following sense: if ω is a nondecreasing positive function on $[0, \infty)$ and

$$\int_{\boldsymbol{R}^N} \omega(|x|)(1 + |x|)^{-N} (\log(2 + |x|))^{(\theta-1)p} |I_\alpha f(x)|^p \, dx \int_{\boldsymbol{R}^N} (\log(2 + |x|))^{\theta p} |f(x)|^p \, dx \quad (7)$$

Holds for all nonnegative measurable functions f on $\boldsymbol{R}^N$, then ω is bounded. In fact, for $r > 1$ and $\varepsilon > 0$, consider the function

$$f_r(y) = (\log |y|)^{-\theta-\varepsilon} |y|^{-\alpha} \chi_{\boldsymbol{R}^N \setminus B(0,r)}.$$

Then note that

$$\int_{\boldsymbol{R}^N} (\log(2 + |y|))^{\theta p} |f_r(y)|^p \, dy \leq C \int_{\boldsymbol{R}^N \setminus B(0,r)} (\log |y|)^{-\theta-\varepsilon} |y|^{-\alpha} \, dy$$
$$\leq C(\log r)^{-\varepsilon p+1}$$

When $\varepsilon > 1/p$. Further, for $x \in \boldsymbol{R}^N \setminus B(0, r)$ we have

$$I_\alpha f(x) \geq \int_{\boldsymbol{R}^N \setminus B(0,|x|)} (2|y|)^{\alpha-N} |f_r(y)| \, dy$$
$$\geq C \int_{\boldsymbol{R}^N \setminus B(0,|x|)} (\log |y|)^{-\theta-\varepsilon} |y|^{-N} \, dy \geq C(\log |x|)^{-\theta-\varepsilon+1}$$

Since $\varepsilon > 1/p > -\theta + 1$, so that

$$\int_{\boldsymbol{R}^N \setminus B(0,r)} \omega(|x|)(1 + |x|)^{-N} (\log(2 + |x|))^{(\theta-1)p} |I_\alpha f(x)|^p \, dx$$
$$\geq C\omega(r) \int_{\boldsymbol{R}^N \setminus B(0,r)} |x|^{-N} (\log |x|)^{-\varepsilon p} \, dx \geq C\omega(r)(\log r)^{-\varepsilon p+1}$$

Hence it follows from (7) that

$$\omega(r) \leq C,$$

which implies that ω is bounded.

For an integer $k \geq 0$, let us remind that $I_{\alpha,k} f$ is the generalized potential of a locally integrable function f on $\boldsymbol{R}^N$, which is defined in the Introduction. For the sake of convenience, set

$$\tilde{I}_{\alpha,k}f(x) = \int_{\boldsymbol{R}^N \setminus B(0,1)} I_{\alpha,k}(x,y)f(y)\, dy$$

for a locally integrable function f on $\boldsymbol{R}^N$. Note here that

$$\tilde{I}_{\alpha,k}f(x) = \int_{\boldsymbol{R}^N \setminus B(0,1)} I_{\alpha,k}(x,-y)f(y)\, dy$$

when $k = 0$.

The following estimates for $I_{\alpha,k}$ are fundamental.

Lemma (2.2.11)[2]: Let $k \geq 0$ be an integer.

(i) If $2|x| < |y|$, then

$$\left| I_\alpha(x - y) - \sum_{\{\mu:|\mu|\leq k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \leq C|x|^k |y|^{\alpha-N-k}.$$

(ii) If $|x|/2 \leq |y| \leq 2|x|$, then

$$\left| I_\alpha(x - y) - \sum_{\{\mu:|\mu|\leq k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \leq C|x-y|^{\alpha-N}.$$

(iii) If $|y| \leq |x|/2$ and $\alpha - N - (k - 1) \leq 0$, then

$$\left| I_\alpha(x - y) - \sum_{\{\mu:|\mu|\leq k-1\}} \frac{x^\mu}{\mu!}(D^\mu I_\alpha)(-y) \right| \leq C|x|^{k-1} |y|^{\alpha-N-(k-1)}.$$

Lemma (2.2.12)[2]: Assume that $(\Phi\omega_1;\, k - \alpha)$ holds for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0,2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\boldsymbol{R^N})$,

$$\left|\tilde{I}_{\alpha,k}(f\chi_{\boldsymbol{R}^N \setminus B(0,4r)})(x)\right| \leq Cr^{\varepsilon+\alpha}\omega(r)^{-1}\bar{\Phi}^{-1}(r^{-N})\left(\int_r^\infty \left(t^{-\varepsilon}\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)^q \frac{dt}{t}\right)^{1/q}$$

When $0 < q < \infty$ and

$$\left|\tilde{I}_{\alpha,k}(f\chi_{\boldsymbol{R}^N \setminus B(0,4r)})(x)\right| \leq Cr^{\alpha}\omega(r)^{-1}\bar{\Phi}^{-1}(r^{-N}) \sup_{t>r}\left(\omega(t) \parallel f \parallel_{L^\Phi(A(0,t))}\right)$$

When $q = \infty$.

Lemma (2.2.13)[2]: Let $k \geq 1$ be an integer. Assume $(\Phi\omega_2;\, k - 1 - \alpha)$ holds for $\varepsilon_2 > 0$. Then, for $0 < \varepsilon < \varepsilon_2$, there exists a constant $C > 0$ such that for all $x \in B(0,2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\boldsymbol{R^N})$,

$$\left|\tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})(x)\right| \leq Cr^{-\varepsilon+\alpha}\omega(r)^{-1}\bar{\Phi}^{-1}(r^{-N})\left(\int_{1/2}^{r}\left(t^{\varepsilon}\omega(t) \parallel f \parallel_{L^{\Phi}(A(0,t))}\right)^{q}\frac{dt}{t}\right)^{1/q}$$

When $0 < q < \infty$ and

$$\left|\tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})(x)\right| \leq Cr^{\alpha}\omega(r)^{-1}\bar{\Phi}^{-1}(r^{-N}) \sup_{1/2 < t < r}\left(\omega(t) \parallel f \parallel_{L^{\Phi}(A(0,t))}\right)$$

when $q = \infty$.

Now, using Lemmas (2.2.11) → (2.2.14), we give the Sobolev type inequality for generalized Riesz potentials of functions in $\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})$.

Theorem (2.2.14)[2]: Suppose $(\Phi_2;\ \varepsilon_0)$, $(\Phi\alpha)$ and $(\Psi\Phi\alpha)$ are fulfilled. Further, for an integer $k \geq 1$, assume that $(\Phi\omega_1;\ k-\alpha)$ and $(\Phi\ \omega_2;\ k-1-\alpha)$ hold. Then there exists a constant $C > 0$ such that

$$\parallel I_{\alpha,k}f \parallel_{\mathcal{H}^{\Psi,q,\omega}(\boldsymbol{R^N})} \leq C \parallel f \parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})}$$

For all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})$.

In fact, set

$$f = f_0 + f_{1,r} + f_{2,r} + f_{3,r}$$

Theorem (2.2.15)[2]: Suppose $(\Phi_2;\ \varepsilon_0)$, (Φ_α) and $(\Psi\Phi_\alpha)$ are fulfilled. Further, for an integer $k \geq 1$, assume that $(\Phi\omega_1;\ k-\alpha)$ and $(\Phi\omega_2;\ k-1-\alpha)$ hold. Then there exists a constant $C > 0$ such that

$$\parallel I_{\alpha,k}f \parallel_{\underline{\mathcal{H}}^{\Psi,q,\omega}(\boldsymbol{R^N})} \leq C \parallel f \parallel_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R^N})}$$

For all $f \in \underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R^N})$.

Theorem (2.2.16)[2]: Suppose Φ satisfies $(\Phi_2;\ \varepsilon_0)$. Further, for an integer $k \geq 1$, assume that $(\Phi\ \omega_1;\ k-\alpha)$ and $(\Phi\ \omega_2;\ k-1-\alpha)$ hold. Set $\tau(r) = r^{-\alpha}\omega(r)$. Then there exists a constant $C > 0$ such that

$$\parallel I_{\alpha,k}f \parallel_{\mathcal{H}^{\tau,q,\omega}(\boldsymbol{R^N})} \leq C \parallel f \parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})}$$

For all $f \in \mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})$.

Lemma (2.2.17)[2]: Suppose

$(\Phi\omega\alpha q)(\log(1+t))^{\varepsilon_4+1/q'}t^{\alpha-k}\omega(t)^{-1}\bar{\Phi}^{-1}(t-N)$is almost decreasing in $[1,\infty)$ for some $\varepsilon_4 > 0$.

If $0 < \varepsilon < \varepsilon_4$, then there exists a constant $C > 0$ such that

$$\left|\tilde{I}_{\alpha,k}(f\chi_{\boldsymbol{R^N}\setminus B(0,4r)})(x)\right|$$
$$\leq\ Cr^{\varepsilon+\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\left(\int_r^\infty\left(t^{-\varepsilon}\omega(t)\parallel f\parallel_{L^{\Phi}(A(0,t))}\right)^q\frac{dt}{t}\right)^{1/q}$$

And

$$\left|\tilde{I}_{\alpha,k}(f\chi_{\boldsymbol{R^N}\setminus B(0,4r)})(x)\right|$$
$$\leq C(\log(1+r))r^{\alpha}\omega(r)^{-1}\overline{\Phi}^{-1}(r^{-N})\ \sup_{t>r}\ \left(\omega(t)\parallel f\parallel_{L^{\Phi}(A(0,t))}\right)\ when\ q=\infty$$

For all $x\ \in\ B(0,2r)$ with $r\ \geq\ 1$ and nonnegative functions $f\in L^1_{loc}(\boldsymbol{R^N})$.

Theorem (2.2.18)[2]: Suppose Φ satisfies $(\Phi_2;\ \varepsilon_0)$. Further, for an integer $k\ \geq\ 1$, assume that $(\Phi\omega_2;\ k\ -\ 1\ -\ \alpha)$and $(\Phi\omega_{\alpha qk})$ hold. Set

$$\tau(r)=\begin{cases}(\log(1\ +\ r))^{-1/q}r^{-\alpha}\omega(r) & when\ 0<\ q\leq 1,\\ (\log(1\ +\ r))^{-1}r^{-\alpha}\omega(r) & when\ 1<\ q\leq\ \infty\end{cases}$$

Then there exists a constant $C\ >\ 0$ such that

$$\parallel I_{\alpha,k}f\parallel_{\mathcal{H}^{\Phi,q,\tau}(\boldsymbol{R^N})}\leq C\parallel f\parallel_{\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})}$$

for all $f\in\mathcal{H}^{\Phi,q,\omega}(\boldsymbol{R^N})$.

When $1/p^*\ =\ 1/p\ -\alpha/N\ =\ 0$ and $p_1\ =\ p$, we can modify (8) by

$$\int_{\boldsymbol{R^N}}(1\ +\ |x|)^{-kp-N}\,(\log(2\ +\ |x|))^{(\theta-1)p}|I_{\alpha,k}f(x)|^p\,dx\ <\ \infty$$

for all functions f satisfying

$$\int_{\boldsymbol{R^N}}(1\ +\ |x|)^{-kp}\,(\log(2\ +\ |x|))^{\theta p}|f(x)|^p\,dx\ <\ \infty$$

With $\theta\ >\ 1/p'$.

Associate space of $\overline{\mathcal{H}}^{\Phi,q,\omega}(R^N)$ and $\underline{\mathcal{H}}^{\Phi,q,\omega}(R^N)$

We assume that Φ satisfies $(\Phi_2;\ \varepsilon_0)$ for some $\varepsilon_0>0$. We consider a complementary function Φ^* of $\overline{\Phi}$.For this purpose, set

$$\Phi^*(t)=\int_0^t\bar{\phi}^{-1}(s)ds$$

And

$$\phi^*(t)\ =t^{-1}\Phi^*(t)$$

For $t > 0$.

Lemma (2.2.19)[2]: (i) $\bar{\phi}^{-1}(t)$ is doubling in $(0, \infty)$;

(ii) $\Phi(\phi^*(t)) \sim \Phi^*(t)$ For all $t \geq 0$.

Proof: We first prove (i). Let $A > 1$. First note from (1), (3) and (Φ_2) that

$$A\bar{\phi}(t) \leq AA_1\phi(t) \leq \phi((AA_1A_{1,\varepsilon_0})^{1/\varepsilon_0}t)$$

And

$$\bar{\phi}(t) \geq (2A_2)^{-1}\phi(t) \geq \phi((2A_2A_{1,\varepsilon_0})^{-1/\varepsilon_0}t)$$

For $t > 0$. Hence we find

$$A\bar{\phi}(t) \leq \bar{\phi}(2A_2A_{1,\varepsilon_0})^{1/\varepsilon_0}((AA_1A_{1,\varepsilon_0})^{1/\varepsilon_0}t)$$

which implies

$$\bar{\phi}^{-1}(At) \leq (2A_2A_{1,\varepsilon_0})^{1/\varepsilon_0}((AA_1A_{1,\varepsilon_0})^{1/\varepsilon_0}\bar{\phi}^{-1}(t). \tag{9}$$

This yields (i).

Noting that $\phi^* \sim \bar{\phi}^{-1}$, we obtain assertion (ii).

Lemma (2.2.20)[2]: There exists $C > 1$ such that

$$st \leq \bar{\Phi}(s) + C\Phi^*(t)$$

For all $s, t \geq 0$.

Proof: If $t \leq \bar{\phi}(s)$, then

$$st \leq s\bar{\phi}(s) = \bar{\Phi}(s)$$

If $t > \bar{\phi}(s)$, then above Lemma gives

$$st \leq \bar{\phi}^{-1}(t)t\, C\Phi^*(t),$$

Which proves the lemma.

Lemma (2.2.21)[2]: readily yields the following result.

Lemma (2.2.22)[2]: For all locally integrable functions f and $g$on $\boldsymbol{R}^N$,

$$\int_{\boldsymbol{R}^N} |f(x)g(x)|\, dx \leq C\|f\|_{L^\Phi(\boldsymbol{R}^N)}\|g\|_{L^{\Phi^*}(\boldsymbol{R}^N)}.$$

Let X be a family of measurable functions on $\boldsymbol{R}^N$ with a norm $\|\cdot\|_X$. Then the associate space X' of X is defined as the family of all measurable functions f on $\boldsymbol{R}^N$ such that

$$\| f \|_{X'} = \sup_{g \in X: \| g \|_X \le 1} \int_{\boldsymbol{R}^N} |f(x)g(x)|\, dx < \infty.$$

Further, we denote by X^* the dual space of X.

The following is an easy consequence of above Lemma.

Lemma (2.2.23)[2]: Let Ω be an open set in $\boldsymbol{R}^N$. Then there exists a constant

$C > 0$ such that

$$\|g\|_{L^{\Phi^*}(\Omega)} \le C \sup_f \int_\Omega |f(x)g(x)|\; dx = C\|g\|_{\left(L^{\Phi}(\Omega)\right)'}$$

For all measurable functions g on Ω, where the supremum is taken over all measurable functions f on Ω such that $\|f\|_{L^{\Phi}(\Omega)} \le 1$.

We discuss the associate space of non-homogeneous central Herz–Morrey–Orlicz spaces. We recall that ω satisfies (ω_1) and (ω_2).

Theorem (2.2.24)[2]: Let $1 \le q \le \infty$. Assume:

(ω_0) there exists a constant $a > 0$ such that $t^{-a}\omega(t)^{-1}$ is almost decreasing in $[1, \infty)$;

(ω_4) there exists a constant $b > 0$ such that $t^{-b}\omega(t)^{-1}$ is almost increasing in $[1, \infty)$.

Then

$$\left(\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)\right)' = \overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)$$

And

$$\left(\overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)\right)' = \underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$$

Corollary (2.2.24)[2]: Let $1 < q < \infty$. If (ω_3) and (ω_4) hold, then

$$\left(\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)\right)^* = \overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)$$

And

$$\left(\overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)\right)^* = \underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$$

Theorem (2.2.24) can be proved if one notes the following lemmas which can be obtained.

Lemma (2.2.25)[2]: Let $1 \leq q \leq \infty$. If (ω_3) holds, then there exists a constant $C > 0$ such that

$$\int_{\boldsymbol{R}^N} |f(x)g(x)|\, dx \leq C\|f\|_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)} \|g\|_{\overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)}.$$

for all measurable functions f and g on $\boldsymbol{R}^N$.

Lemma (2.2.26)[2]: Let $1 \leq q \leq \infty$. Set $X = \underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)$. If $(\omega 4)$ holds, then there exists a constant $C > 0$ such that

$$\|g\|_{\overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)} \leq C \sup_f \int_{\boldsymbol{R}^N} |f(x)g(x)|\, dx = C\|g\|_{X'}$$

for all measurable functions g on $\boldsymbol{R}^N$, where the supremum is taken over all measurable functions f on $\boldsymbol{R}^N$ such that $\|f\|_X \leq 1$.

Lemma (2.2.27)[2]: Let $1 \leq q \leq \infty$. Set $Y = \overline{\mathcal{H}}^{\Phi^*,q',1/\omega}(\boldsymbol{R}^N)$. If (ω_4) holds, then there exists a constant $C > 0$ such that

$$\|f\|_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\boldsymbol{R}^N)} \leq C \sup_g \int_{\boldsymbol{R}^N} |f(x)g(x)|\, dx = C\|f\|_{Y'}$$

for all measurable functions f on $\boldsymbol{R}^N$, where the supremum is taken over all measurable functions g on $\boldsymbol{R}^N$ such that $\|g\|_Y \leq 1$.

Chapter 3

Continuity for Riesz Potentials of Functions in Musielak-Orlicz-Morrey Spaces on Metric Measure Spaces

We concerned with Trudinger's inequality and continuity for Riesz potentials of functions in Musielak–Orlicz–Morrey spaces on metric measure spaces.

Section (3.1): Musielak-Orliez-Morrey Spaces

A famous Trudinger inequality insists that Sobolev functions in $W^{1,N}(G)$ satisfy finite exponential integrability, where G is an open bounded set in R^N. For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on R^N by

$$U_\alpha f(x) = \int_{\boldsymbol{R}^N} |x-y|^{\alpha-N} f(y)dy .$$

Great progress on Trudinger type inequalities has been made for Riesz potentials of order α in the limiting case $\alpha p = N$. Trudinger type exponential integrability was studied on Orlicz spaces, on generalized Morrey spaces $L^{1,\varphi}$,and on Orlicz–Morrey spaces.For Morrey spaces, which were introduced to estimate solutions of partial differential equations.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a surve,. Trudinger type exponential integrability was investigated on variable exponent Lebesgue spaces $L^{p(\cdot)}$ and on two variable exponent spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. For two variable exponent spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$.

For $x \in \boldsymbol{R}^N$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r and $d_\Omega = sup\{d(x,y) : x,y \in \Omega\}$ for a set $\Omega \subset \boldsymbol{R}^N$. For bounded measurable functions $\nu(\cdot): \boldsymbol{R}^N \to (0,N]$ and $\beta(\cdot): \boldsymbol{R}^N \to \boldsymbol{R}$, let $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$ be the set of all measurable functions f on G such that $\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)} < \infty$, where

$$\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)}$$
$$= \inf\Bigg\{\lambda$$
$$> 0: \sup_{x\in G, 0<r\le d_G} \frac{r^{\nu(x)}\left(\log\left(e + \frac{1}{r}\right)\right)^{\beta(x)}}{|B(x,r)|}$$
$$\times \int_{B(x,r)} \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} \left(\log\left(e + \frac{|f(y)|}{\lambda}\right)\right)^{q(y)} dy \le 1\Bigg\};$$

We set $f = 0$ outside G. Mizuta, proved Trudinger type exponential integrability for two variable exponent Morrey spaces $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$ when $p(\cdot)$ and $q(\cdot)$ are variable exponents satisfying the log-Hölder and loglog-Hölder conditions on G, respectively. The result is animprovement,. In fact we proved the following:

Theorem (3.1.1)[3]: Suppose $\inf_{x\in\boldsymbol{R}^N} \nu(x) > 0$ and $\inf_{x\in\boldsymbol{R}^N}(\alpha - \nu(x)/p(x)) \ge 0$ hold. Let ε be a constant such that

$$\inf_{x\in\boldsymbol{R}^N}\left(\frac{\nu(x)}{p(x)} \& - \&_{\varepsilon}\right) > 0 \text{ and } 0 < \varepsilon < \alpha.$$

Then there exist constants $C_1, C_2 > 0$ such that

(i) In case $\sup_{x\in\boldsymbol{R}^N} (q(x) + \beta(x))/p(x) < 1$,

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{B(z,r)} \exp\left(\frac{|U_\alpha f(x)|^{p(x)/(p(x)-q(x)-\beta(x))}}{C_1}\right) dx \le C_2;$$

(ii) In case $\inf_{x\in\boldsymbol{R}^N} (q(x) + \beta(x))/p(x) < 1$,

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{B(z,r)} \exp\left(\exp\left(\frac{|U_\alpha f(x)|}{C_1}\right)\right) dx \le C_2;$$

For all $z \in G, 0 < r < d_G$ and f satisfying $\|f\|_{L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)} \le 1$.

We give a general version of Trudinger type exponential integrability for Riesz potentials $I_\alpha f$ of functions in Musielak–Orlicz–Morrey spaces $L^{\Phi,\kappa}(X)$ on metric measure spaces X as an extension of the above results (The definitions of Φ and κ for the definition of $I_\alpha f$). Since we discuss the Morrey version, our strategy is to find an estimate of Riesz potentials by use of Riesz potentials of order ε, which plays a role of the maximal functions.

Beginning with Sobolev embedding theorem, continuity properties of Riesz potentials or Sobolev functions have been studied by many authors. Continuity of Riesz potentials of functions in Orlicz spaces was studied. Then such continuity was investigated on generalized Morrey spaces $L^{1,\varphi}$, on Orlicz–Morrey spaces, on variable exponent Lebesgue spaces and on variable exponent Morrey spaces, Mizuta, Nakai and the authors also proved continuity for Riesz potentials of functions in two variable exponent Morrey spaces $L^{p(\cdot),q(\cdot),\nu(\cdot),\beta(\cdot)}(G)$.

These results have been extended to Musielak–Orlicz–Morrey spaces. We give a general version of continuity for Riesz potentials $I_\alpha f$ of functions in Musielak–Orlicz–Morrey spaces $L^{\Phi,\kappa}(X)$ on metric measure spaces as an extension of the above results.

We established Trudinger type exponential integrability for Musielak–Orlicz spaces in the Euclidean setting by use of the maximal functions, which are a crucial tool as. We give a general version of Trudinger type exponential integrability for Riesz potentials $I_\alpha f$ of functions in Musielak–Orlicz spaces $L^\Phi(X)$ on metric measure spaces as an extension. To obtain our results, we need the boundedness of the maximal operator on $L^\Phi(X)$.

We show the continuity for Riesz potentials Iαf of functions in Musielak–Orlicz spaces $L^\Phi(X)$ on metric measure spaces.

Let C denote various constants independent of the variables in question.

We denote by (X, d, μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ). For $x \in X$and $r > 0$, we denote by $B(x, r)$ the open ball centered at x with radius r and $d_\Omega = \sup\{d(x, y): x, y \in \Omega\}$ for a set $\Omega \subset X$.

We say that the measure μ is a doubling measure if there exists a constant $c_0 > 0$ such that $\mu(B(x, 2r)) \leq c_0\mu\big(B(x, r)\big)$ for every $x \in X$ and $0 < r < dX$. We say that X is a doubling space if μ is a doubling measure.

We assume that X is a bounded set and a doubling space, that is $dX < \infty$. This implies that $\mu(X) < \infty$. We consider a function

$$\Phi(x, t) = t\phi(x, t): X \times [0, \infty) \to [0, \infty)$$

Satisfying the following conditions (Φ_1)– (Φ_4):

(Φ_1) $\phi(\cdot, t)$ is measurable on X for each $t \geq 0$ and $\phi(x,\cdot)$ is continuous on$[0, \infty)$ for each $x \in X$;

(Φ_2) There exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x,1) \leq A_1 for\ all\ x \in X;$$

(Φ_3) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geq 1$ such that

$\phi(x,t) \leq A_2\phi(x,s)$ for all $x \in X$ whenever $0 \leq t < s$;

(Φ_4) There exists a constant $A_3 \geq 1$ such that

$$\phi(x,2t) \leq A_3\phi(x,t) for\ all\ x \in X\ and\ t > 0.$$

Note that $(\Phi_2), (\Phi_3)$ and (Φ_4) imply

$$0 < \inf_{x \in X} \phi(x,t) \leq \sup_{x \in X} \phi(x,t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in X$, then (Φ_3) holds with $A_2 = 1$; namely $\phi(x,\cdot)$ is non-decreasing for each $x \in X$.

Let$\bar{\phi}(x,t) = \sup_{0 \leq s \leq t} \phi(x,s)\ and$

$$\bar{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)dr \tag{1}$$

for $x \in X$ and $t \geq 0$. Then $\bar{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \leq \bar{\Phi}(x,t) \leq A_2\Phi(x,t) \tag{2}$$

for all $x \in X$ and $t \geq 0$.

We shall also consider the following condition:

(Φ_5) for every $\gamma_1, \gamma_2 > 0$, there exists a constant $B_{\gamma_1,\gamma_2} \geq 1$ such that

$$\phi(x,t) \leq B_{\gamma_1,\gamma_2}\phi(y,t)$$

whenever $d(x,y) \leq \gamma_1 t^{-1/\gamma_2}\ and\ t \geq 1$.

Example (3.1.2)[3]: Let $p(\cdot)$ and $q_j(\cdot), j = 1, \dots, k$, be measurable functions on X such that

$(P_1) 1 < p^- \coloneqq \inf_{x \in X} p(x) \leq \sup_{x \in X} p(x) =: p^+ < \infty$

And

$(Q_1) -\infty < q_j^- \coloneqq \inf_{x \in X} q_j(x) \leq \sup_{x \in X} q_j(x) =: q_j^+ < \infty$

For all $j = 1, \dots, k$.

Set $L_c(t) = \log(c+t)$ for $c \geq e$ and $t \geq 0, L_c^{(1)}(t) = L_c(t), L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}$$

Then, $\Phi(x,t)$ satisfies $(\Phi_1), (\Phi_2), (\Phi_3)$ and (Φ_4).

Moreover, we see that $\Phi(x,t)$ satisfies (Φ_5) if $(P_2)\, p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq \frac{C_p}{L_e(1/d(x,y))}$$

With a constant $C_p \geq 0$ and

$(Q_2)\ q_j(\cdot)\ is\ j+$ 1-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \leq \frac{C_{qj}}{L_e^{(j+1)}(1/d(x,y))}$$

with constants $C_{qj} \geq 0, j = 1, \ldots, k$.

Example (3.1.3)[3]: Let $p(\cdot)$ be a measurable function on X satisfying (P_1) and (P_2). Let $q_1(\cdot)$ be a measurable function on X satisfying (Q_1) and (Q_2) and let $q_2(\cdot)$ be a measurable function on X satisfying (Q_1). Then

$$\Phi(x,t) = t^{p(x)}(\log(e+t))^{q_1(x)}(\log(e+1/t))^{q_2(x)}$$

Satisfies $(\Phi_1), (\Phi_2), (\Phi_3), (\Phi_4)$ and (Φ_5).

In view of (2), given $\Phi(x,t)$ as above, the associated Musielak–Orlicz space

$$L^{\Phi}(X) = \left\{ f \in L^1_{loc}(X); \int_X \Phi(\, y, |f(y)|) d\mu(y) < \infty \right\}$$

Is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(X)} = \inf\left\{ \lambda > 0; \int_X \bar{\Phi}(\, y, |f(y)|/\lambda) d\mu(y) \leq 1 \right\}$$

We also consider a function $\kappa(x,r): X \times (0, d_X] \to (0, \infty)$ satisfying the following conditions:

(κ_1) $\kappa(x,\cdot)$ is measurable for each $x \in X$;

(κ_2) $\kappa(x,\cdot)$ is uniformly almost increasing on $(0, d_X]$, namely there exists a constant $Q_1 \geq 1$ such that

$$\kappa(x,r) \leq Q_1\kappa(x,s)$$

For all $x \in X$ whenever $0 < r < s \leq d_X$;

(κ_3) There are constants $Q > 0$ and $Q_2 \geq 1$ such that

$$Q_2^{-1}\min(1,r^Q) \leq \kappa(x,r) \leq Q_2$$

For all $x \in X$ and $0 < r \leq d_X$.

Example (3.1.4)[3]: For $Q > 0$, let $\nu(\cdot)$ and $\beta_j(\cdot), j = 1,\dots,k$ be measurable functions on X such that $\inf_{x\in X}\nu(x) > 0$,

$\sup_{x\in X}\nu(x) \leq Q$ and $-c(Q-\nu(x)) \leq \beta_j(x) \leq c$ for all $x \in X, j = 1,\dots,k$ and some constant $c > 0$. Then

$$\kappa(x,r) = r^{\nu(x)}\prod_{j=1}^{k}(L_e^{(j)}(1/r))^{\beta_j(x)}$$

Satisfies $(\kappa_1),(\kappa_2)$ and (κ_3).

For a locally integrable function f on X, define the $L^{\Phi,\kappa}$ norm

$$\|f\|_{L^{\Phi,\kappa}(X)}$$
$$= \inf\left\{\lambda > 0: \sup_{x \in X, 0<r\leq d_X}\frac{\kappa(x,r)}{\mu(B(x,r))}\int_{X\cap B(x,r)}\overline{\Phi}(y,|f(y)|/\lambda)\,d\mu(y) \leq 1\right\}.$$

For the definition of $\overline{\Phi}$. Let $L^{\Phi,\kappa}(X)$ denote the set of all functions f such that $\|f\|_{L^{\Phi,\kappa}(X)} < \infty$, which we call a Musielak–Orlicz–Morrey space. Note that $L^{\Phi,\kappa}(X) = L^{\Phi}(X)$ if $\mu(B(x,r)) \sim \kappa(x,r)$ for all $x \in X$ and $0 < r \leq d_X$. (Here $h_1(x,s) \sim h_2(x,s)$means that $C^{-1}h_2(x,s) \leq h_1(x,s) \leq Ch_2(x,s)$for a constant $C > 0$.)

Set

$$\Phi^{-1}(x,s) = \sup\{t > 0;\ \Phi(x,t) < s\}$$

For $x \in X$ and $s > 0$.

Lemma (3.1.5)[3]: $\Phi^{-1}(x,\cdot)$ is non-decreasing;

$$\Phi^{-1}(x,\lambda s) \leq A_2\lambda\Phi^{-1}(x,s) \tag{3}$$

For all $x \in X$, $s > 0$ and $\lambda \geq 1$ and

$$\min\left\{1,\frac{s}{A_1A_2}\right\} \leq \Phi^{-1}(x,s) \leq \max\{1,A_1A_2s\} \tag{4}$$

For all $x \in X$ and $s > 0$, where A_1 and A_2 are the constants appearing in (Φ_2) and (Φ_3).

Lemma (3.1.6)[3]: There exists a constant $C > 0$ such that

$$C^{-1} \leq \Phi^{-1}(x, \kappa(x,r)^{-1}) \leq Cr^{-Q} \tag{5}$$

for all $x \in X$ and $0 < r \leq d_X$.

Proof: By (κ_3),

$$Q_2^{-1} \leq \kappa(x,r)^{-1} \leq Q_2 \max(1, r^{-Q})$$

For $x \in X$ and $0 < r \leq d_X$. Hence, by (4), we obtain (5).

We can prove the following result.

Lemma (3.1.7)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Then there exists a constant $C > 0$ such that

$$\int_{X \cap B(x,r)} f(y)\, d\mu(y) \leq C\mu(B(x,r))\Phi^{-1}(x, \kappa(x,r)^{-1})$$

for all $x \in X$, $0 < r \leq d_X$ and $f \geq 0$ satisfying $\|f\|L^{\Phi,\kappa}(X) \leq 1$.

For $\alpha > 0$, we define the Riesz potential of order α for a locally integrable function f on X by

$$I_\alpha f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x, d(x,y)))} d\mu(y)$$

Set

$$\Gamma(x,s) = \int_{1/s}^{dX} \rho^\alpha \Phi^{-1}(x, \kappa(x,\rho)^{-1}) \frac{d\rho}{\rho}$$

for $s \geq 2/d_X$ and $x \in X$. For $0 \leq s < 2/d_X$ and $x \in X$, we set $\Gamma(x,s) = \Gamma(x, 2/d_X)(d_X/2)s$. Then note that $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in X$.

Lemma (3.1.8)[3]: There exists a positive constant C' such that $\Gamma(x, 2/d_X) \geq C' > 0$ for all $\in X$.

Lemma (3.1.9)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Then there exists a constant $C > 0$ such that

$$\int_{X \setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x, d(x,y)))} d\mu(y) \leq C\Gamma\left(x, \frac{1}{\delta}\right)$$

for all $x \in X$, $0 < \delta \leq d_X/2$ and nonnegative $f \in L^{\Phi,\kappa}(X)$ with $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Proof. Let j_0 be the smallest positive integer such that $2^{j_0}\delta \geq d_X$. we have

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu\Big(B\big(x,d(x,y)\big)\Big)} d\mu(y)$$

$$= \sum_{j=1}^{j_0} \int_{X\cap\left(B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)\right)} \frac{d(x,y)^\alpha f(y)}{\mu\Big(B\big(x,d(x,y)\big)\Big)} d\mu(y)$$

$$\leq \sum_{j=1}^{j_0} \left(2^j\delta\right)^\alpha \frac{1}{\mu(B(x,2^{j-1}\delta))} \int_{X\cap B(x,2^j\delta)} f(y)d\mu(y)$$

$$\leq c_0 \sum_{j=1}^{j_0} \left(2^j\delta\right)^\alpha \frac{1}{\mu(B(x,2^j\delta))} \int_{X\cap B(x,2^j\delta)} f(y)d\mu(y)$$

$$\leq C\left(\sum_{j=1}^{j_0-1} (2^j\delta)^\alpha \Phi^{-1}(x,\kappa(x,2^j\delta)^{-1}) + d_X^\alpha \Phi^{-1}(x,\kappa(x,d_X)^{-1})\right).$$

By (κ_2) and (3), we have

$$2j\delta \int_{2^{j-1}\delta}^{2^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1}) \frac{dt}{t} \geq (2^{j-1}\delta)^\alpha \Phi^{-1}(x,Q_1^{-1}\kappa(x,2^j\delta)^{-1}) \log 2$$

$$\geq \frac{\left(2^j\delta\right)^\alpha \log 2}{2^\alpha A_2 Q_1} \Phi^{-1}\left(x,\kappa\big(x,2^j\delta\big)^{-1}\right)$$

$$= C(2^j\delta)^\alpha \Phi^{-1}\left(x,\kappa\big(x,2^j\delta\big)^{-1}\right)$$

And

$$\int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1}) \frac{dt}{t} \geq \frac{d_X^\alpha \log 2}{2^\alpha A_2 Q_1} \Phi^{-1}(x,\kappa(x,d_X)^{-1})$$

$$= C d_X^\alpha \Phi^{-1}(x,\kappa(x,d_X)^{-1}).$$

Hence, we obtain

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))} d\mu(y) \leq C\left(\sum_{j=1}^{j_0-1}\int_{2^{j-1}\delta}^{2^j\delta} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\frac{dt}{t} + \int_{d_X/2}^{d_X} t^\alpha \Phi^{-1}(x,\kappa(x,t)^{-1})\frac{dt}{t}\right) \leq \ C\Gamma\left(x,\frac{1}{\delta}\right),$$

As required

Lemma (3.1.10)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Let $\varepsilon\ >0$ and define

$$\lambda_\varepsilon(z,r) = \frac{1}{1+\int_r^{d_X} \rho^\varepsilon \Phi^{-1}(z,\kappa(z,\rho)^{-1})\frac{d\rho}{\rho}}$$

for $z\in\ X$. Then there exists a constant $C_{I,\varepsilon} > 0$ such that

$$\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))}\int_{X\cap B(z,r)} I_\varepsilon f(x) d\mu(x) \leq C_{I,\varepsilon}$$

for all $z\ \in\ X$, $0 <\ r\ \leq\ d_X$ and $f\ \geq 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Proof: Let $z \in X$. Write

$$I_\varepsilon f(x) = \int_{X\cap B(z,2r)} \frac{d(x,y)^\varepsilon f(y)}{\mu\left(B\left(x,d(x,y)\right)\right)} d\mu(y) + \int_{X/B(z,2r)} \frac{d(x,y)^\varepsilon f(y)}{\mu\left(B\left(x,d(x,y)\right)\right)} d\mu(y) = \ I_1(x)\ +\ I_2(x)$$

For $x \in X$. By Fubini's theorem,

$$\int_{X\cap B(z,r)} I_1(x)\, d\mu(x) = \int_{X\cap B(z,2r)}\left(\int_{X\cap B(z,r)} \frac{d(x,y)^\varepsilon}{\mu\left(B\left(x,d(x,y)\right)\right)} d\mu(x)\right) f(y)\, d\mu(y)$$

$$\leq \int_{X\cap B(z,2r)} \left(\int_{X\cap B(y,3r)} \frac{d(x,y)^{\varepsilon}}{\mu\Big(B\big(x,d(x,y)\big)\Big)} d\mu(x) \right) f(y)\, d\mu(y)$$

$$\leq \int_{X\cap B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{X\cap (B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r))} \frac{\left(2^{-j+2}r\right)^{\varepsilon}}{\mu\Big(B\big(x,d(x,y)\big)\Big)} d\mu(x) \right) f(y)\, d\mu(y)$$

Since μ is a doubling measure, we have

$$\int_{X\cap B(z,r)} I_1(x) d\mu(x)$$

$$\leq c_0^2 \int_{X\cap B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{X\cap \left(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r)\right)} \frac{(2^{-j+2}r)^{\varepsilon}}{\mu\big(B(x,2^{-j+3}r)\big)} d\mu(x) \right) f(y)\, d\mu(y)$$

$$\leq c_0^2 \int_{X\cap B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{X\cap \left(B(y,2^{-j+2}r)\setminus B(y,2^{-j+1}r)\right)} \frac{\left(2^{-j+2}r\right)^{\varepsilon}}{\mu\big(B(x,2^{-j+2}r)\big)} d\mu(x) \right) f(y)\, d\mu(y)$$

$$\leq c_0^2 \int_{X\cap B(z,2r)} \left(\sum_{j=0}^{\infty} \left(2^{-j+2}r\right)^{\varepsilon} \right) f(y)\, d\mu(y)$$

$$\leq C8^{\varepsilon} \int_{X\cap B(z,2r)} \left(\sum_{j=0}^{\infty} \int_{2^{-j-1}r}^{2^{-j}r} t^{\varepsilon} \frac{dt}{t} \right) f(y)\, d\mu(y)$$

$$\leq C \int_{X\cap B(z,2r)} \left(\int_0^r t^{\varepsilon} \frac{dt}{t} \right) f(y)\, d\mu(y)$$

$$= \frac{C}{\varepsilon} r^{\varepsilon} \int_{X\cap B(z,2r)} f(y)\, d\mu(y)\,.$$

Now (κ_2) and (3), we have

$$r^{\varepsilon} \int_{X\cap B(z,2r)} f(y)\, dy \ \leq\ Cr^{\varepsilon} \mu(B(z,2r)) \Phi^{-1}(z,\kappa(z,2r)^{-1})$$

$$\leq\ C\mu(B(z,2r)) \int_r^{2r} \rho^{\varepsilon} \Phi^{-1}(z,\kappa(z,\rho)^{-1}) \frac{d\rho}{\rho}$$

if $0 < r \le d_X/2$ and, by Lemma (3.1.6) and (5), we have

$$r^{\varepsilon}\int_{X\cap B(z,2r)} f(y)\,dy \ = r^{\varepsilon}\int_{B(z,2r)} f(y)\,dy$$

$$\le C{d_X}^{\varepsilon}\mu(B(z,d_X))\Phi^{-1}(z,\kappa(z,d_X)^{-1}) \ \le \ C\mu(B(z,r))$$

if $d_X/2 < r \le d_X$. Therefore

$$\int_{X\cap B(z,2r)} I_1(x)\,d\mu(x) \le \frac{C}{\varepsilon}\frac{\mu(B(z,r))}{\lambda_{\varepsilon}(z,r)}$$

For all $0 < r \le d_X$.

For I_2, first note that $I_2(x) = 0$ if $x \in X$ and $r \ \ge \ d_X/2$. Let $0 < r < d_X/2$. Let j_0 be the smallest positive integer such that $2^{j_0}r \ge d_X$. Since

$$d(z,y)\varepsilon f\ (y)$$

$$I_2(x) \le C\int_{X\setminus B(z,2r)} \frac{d(z,y)^{\varepsilon}f(y)}{\mu\Big(B\big(z,d(z,y)\big)\Big)}d\mu(y) \quad \text{for } x \in X\cap B(z,r),$$

We have

$$I_2(x) \le C\sum_{j=1}^{j_0-1}\int_{B(z,2^{j+1}r)\setminus B(z,2^{j}r)} \frac{d(z,y)^{\varepsilon}}{\mu\Big(B\big(z,d(z,y)\big)\Big)} f(y)d\mu(y)$$

$$\le C\sum_{j=1}^{j_0-1}(2^{j+1}r)^{\varepsilon}\frac{1}{\mu\big(B(z,2^{j}r)\big)}\int_{X\cap B(z,2^{j+1}r)} f(y)d\mu(y)$$

$$\le C\sum_{j=1}^{j_0-1}(2^{j+1}r)^{\varepsilon}\frac{1}{\mu\big(B(z,2^{j+1}r)\big)}\int_{X\cap B(z,2^{j+1}r)} f(y)d\mu(y)$$

$$\le \ C\left(\sum_{j=1}^{j_0-2}(2^{j+1}r)^{\varepsilon}\,\Phi^{-1}(x,\kappa(x,2^{j+1}r)^{-1}) + d_X^{\varepsilon}\Phi^{-1}(x,\kappa(x,d_X)^{-1})\right).$$

As in the proof of Lemma (3.1.9), we obtain

$$I_2(x) \le C\sum_{j=1}^{j_0-2}\int_{2^{j}r}^{2^{j+1}r}\rho^{\varepsilon}\Phi^{-1}(x,\kappa(x,\rho)^{-1})\frac{d\rho}{\rho} + \int_{d_X/2}^{d_X}\rho^{\varepsilon}\Phi^{-1}(x,\kappa(x,\rho)^{-1})\frac{d\rho}{\rho}$$

$$\le C\int_{r}^{d_X}\rho^{\varepsilon}\Phi^{-1}(z,\kappa(z,\rho)^{-1})\frac{d\rho}{\rho} \le \frac{C}{\lambda_{\varepsilon}(z,r)}$$

For all $x \in X \cap B(z,r)$. Hence

$$\int_{X\cap B(z,r)} I_2(x)d\mu(x) \leq \ C\frac{\mu(B(z,r))}{\lambda_\varepsilon(z,r)} \ .$$

Thus this lemma is proved.

Section (3.2): Trudinger Inequality for Musielak–Orlicz–Morrey Spaces

We deal with the case $\Gamma(x,t)$ satisfies the uniform log-type condition:

(Γ_{Log}) There exists a constant $c_\Gamma > 0$ such that

$$\Gamma\ (x,t^2) \leq c_\Gamma \Gamma\ (x,t)$$

For all $x \in X$ and $t \geq\ 1$.

Example (3.2.1)[3]: Let Φ and κ be as in Examples (2.1) and (2.3) ,respectively. Then

$$\Gamma(x,t) \sim \int_{\frac{1}{t}}^{d_X} \rho^{\alpha-\nu(x)/p(x)} \prod_{j=1}^{k} [L_e^{(j)}(1/\rho)]^{-(q_j(x)+\beta_j(x))/p(x)} \frac{d\rho}{\rho} (t \geq 2/d_X),$$

so that it satisfies $(\Gamma_{\log})$ if and only if

$\alpha_{p(x)} \geq\ \nu(x)$ for all $x \in X$.

By $(\Gamma_{\log})$, together with Lemma (3.1.8), we see that $\Gamma(x,t)$ satisfies the uniform doubling condition in t:

Lemma (3.2.2)[3]: Suppose $\Gamma(x,t)$ satisfies $(\Gamma_{\log})$. For every $a > 1$, there exists $b > 0$ such that $\Gamma(x,at) \leq b\Gamma(x,t)$ for all $x\ \in\ X$ and $t\ > 0$.

Theorem (3.2.3)[3]: Assume that $\Phi(x,t)$ satisfies $(\Phi_5), \Gamma(x,t)$satisfies $(\Gamma_{\log})$. For each $\in X$, let $\gamma(x) = \sup_{s>0} \Gamma(x,s)$ suppose $\Psi\ (x,t){:}\, X \times\ [0,\infty) \rightarrow\ [0,\infty]$ satisfies the following conditions:

$(\Psi_1)\ \Psi\ (\cdot,t)$ is measurable on X for each $t \in [0,\infty); \Psi\ (x,\cdot)$ is continuous on$[0,\infty)$ for each $x \in X$;

(Ψ_2) there is a constant $A_1' \geq 1$such that $\Psi\ (x,t)\ \leq\ \Psi\ (x,A_1's)$ for all $x \in X$ whenever $0 < t < s$; $(\Psi_3)\ \Psi\ (x,\Gamma\ (x,t)/A_2')\ \leq A_3't$ for all $x \in X$ and $t > 0$ with constants $A_2', A_3' \geq 1$ independent of x.

Then, for $0 <\ \varepsilon\ <\ \alpha$, there exists a constant $C^* > 0$ such that $I_\alpha f(x)/C^* < \gamma\ (x)$ for a.e. $x\ \in\ X$ and

$$\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))}\int_{X\cap B(z,r)} \Psi\left(x,\frac{I_\alpha f(x)}{C^*}\right) d\mu(x) \leq 1$$

For all $z \in X$, $0 < r \leq d_X$ and $f \geq 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Proof: Let $f \geq 0$ and $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$. Fix $x \in X$. For $0 < \delta \leq d_X/2$.

$$\begin{aligned} I_\alpha f(x) &\leq \int_{X\cap B(x,\delta)} \frac{d(x,y)^{\varepsilon} f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\ &= \int_{X\cap B(x,\delta)} d(x,y)^{\alpha-\varepsilon} \frac{d(x,y)^{\varepsilon} f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) + C\Gamma\left(x, \frac{1}{\delta}\right) \\ &\leq C\left\{\delta^{\alpha-\varepsilon} I_\varepsilon f(x) + \Gamma\left(x, \frac{1}{\delta}\right)\right\} \end{aligned}$$

With constants $C > 0$ independent of x.

If $I_\varepsilon f(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma (3.1.8)

$$I_\varepsilon f(x) \leq C\Gamma\left(x, \frac{2}{d_X}\right).$$

There exists $C_1^* > 0$ independent of x such that

$$I_\alpha f(x) \leq C_1^* \Gamma\left(x, \frac{1}{2A_3'}\right) if\ I_\varepsilon f(x) \leq \frac{2}{d_X}. \tag{6}$$

Next, suppose $2/d_X < I_\varepsilon f(x) < \infty$. Let $m = \sup_{s\geq 2/d_X, x\in X} \Gamma(x,s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define δ by

$$\delta^{\alpha-\varepsilon} = \frac{(d_X/2)^{\alpha-\varepsilon}}{m} \Gamma(x, I_\varepsilon f(x))(I_\varepsilon f(x))^{-1}.$$

Since $\Gamma(x, I_\varepsilon f(x))(I_\varepsilon f(x))^{-1} \leq m$, $0 < \delta \leq d_X/2$. Then by Lemma (3.1.8)

$$\begin{aligned} \frac{1}{\delta} &\leq C\Gamma(x, I_\varepsilon f(x))^{-1/(\alpha-\varepsilon)} (I_\varepsilon f(x))^{1/(\alpha-\varepsilon)} \\ &\leq C\Gamma(x, 2/d_X)^{-1/(\alpha-\varepsilon)} (I_\varepsilon f(x))^{1/(\alpha-\varepsilon)} \leq C(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)}. \end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma (3.2.2), we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \leq \Gamma\left(x, C(I_\varepsilon f(x))^{1/(\alpha-\varepsilon)}\right) \leq C\Gamma(x, I_\varepsilon f(x)).$$

By Lemma (3.2.2) again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$I_\alpha f(x) \leq C_2^* \Gamma\left(x, \frac{1}{2C_{I,\varepsilon} A_3'} I_\varepsilon f(x)\right) if\ 2/d_X < I_\varepsilon f(x) < \infty,$$

where $C_{I,\varepsilon}$ is the constant given in Lemma (3.1.10).

Now, let $C^* = A_1' A_2' \max(C_1^*, C_2^*)$. Then, by (6) and (7),

$$\frac{I_\alpha f(x)}{C^*} \le \frac{1}{A_1' A_2'} \max\left\{\Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\varepsilon} A_3'}\right) I_\varepsilon f(x)\right\} \qquad (7)$$

Whenever $I_\varepsilon f(x) < \infty$. Since $I_\varepsilon f(x) < \infty$ for a.e. $x \in X$ by Lemma (3.1.10), $I_\alpha f(x)/C^* < \gamma\,(x)$ a.e. $x \in X$, and by (Ψ_2) and (Ψ_3),

We have

$$\Psi\left(x, \frac{I_\alpha f(x)}{C^*}\right) \le \max\left\{\Psi\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Psi\left(x, \Gamma\left(x, \frac{1}{2C_{I,\varepsilon} A_3'}\right) I_\varepsilon f(x)/A_2'\right)\right\}$$
$$\le \frac{1}{2} + \frac{1}{2C_{I,\varepsilon}} I_\varepsilon f(x)$$

For a.e. $x \in X$. Thus, noting that $\lambda_\varepsilon(z,r) \le 1$ and using Lemma (3.1.10), we have

$$\frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))} \int_{X\cap B(z,r)} \Psi\left(x, \frac{I_\alpha f(x)}{C^*}\right) d\mu(x)$$
$$\le \frac{1}{2}\lambda_\varepsilon(z,r) + \frac{1}{2C_{I,\varepsilon}} \frac{\lambda_\varepsilon(z,r)}{\mu(B(z,r))} \int_{X\cap B(z,r)} I_\alpha f(x) d\mu(x)$$
$$\le \frac{1}{2} + \frac{1}{2} = 1$$

For all $z \in X$ and $0 < r \le d_X$.

Corollary (3.2.4)[3]: Let Φ and κ be as in Examples (3.1.1) and (3.1.3).

Assume that

$$\alpha - \nu(x)/p(x) = 0 \text{ for all } x \in X.$$

(i) Suppose there exists an integer $1 \le j_0 \le k$ such that

$$\inf_{x\in X}(p(x) - q_{j_0}(x) - \beta_{j_0}(x)) > 0$$

and

$$\sup_{x\in X}(p(x) - q_j(x) - \beta_j(x)) > 0$$

for all $j \le j_0 - 1$ in case $j_0 \ge 2$. Then for $0 < \varepsilon < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{B(z,r)} \int_{X\cap B(z,r)} E_+^{(j_0)}\left(\left(\frac{I_\alpha f(x)}{C^*}\right)^{p(x)/p(x)-q_{j_0}(x)-\beta_{j_0}(x)}\right.$$

$$\times \prod_{j=1}^{k-j_0} L_e^{(j)} \left(\frac{I_\alpha f(x)}{C^*} \right)^{q_{j_0+j}(x)+\beta_{j_0+j}(x)/p(x)-q_{j_0}(x)-\beta_{j_0}(x)} \Bigg) d\mu(x) \le C^{**}$$

for all $z \in X$, $0 < r \le d_X$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}}(X) \le 1$, where $E^{(1)}(t) = e^t - e, E^{(j+1)}(t) = exp(E^j(t)) - e$ and $E_+^{(j)}(t) = \max(E^{(j)}(t), 0)$.

(ii) If

$$\sup_{x\in X} \left(p(x) - q_j(x) - \beta_j(x) \right) \le 0$$

for all $j = 1, \ldots, k$, then for $0 < \varepsilon < \alpha$ there exist constants $C^* > 0$ and $C^{**} > 0$ such that

$$\frac{r^{\nu(z)/p(z)-\varepsilon}}{|B(z,r)|} \int_{X\cap B(z,r)} E^{(k+1)} \left(\frac{I_\alpha f(x)}{C^*} \right) d\mu(x) \le C^{**}$$

for all $z \in X$, $0 < r \le d_X$ and $f \ge 0$ satisfying $\|f\|_{L^{\Phi,\kappa}(X)} \le 1$.

We discuss the continuity of Riesz potentials $I_\alpha f$ of functions in Musielak–Orlicz–Morrey spaces under the condition: there are constants $\theta > 0$ and $C_0 > 0$ such that

$$\left| \frac{d(x,y)^\alpha}{\mu\left(B\left(x, d(x,y)\right)\right)} - \frac{d(z,y)^\alpha}{\mu\left(B\left(z, d(z,y)\right)\right)} \right| \le C_0 \left(\frac{d(x,z)}{d(x,y)} \right)^\theta \frac{d(x,y)^\alpha}{\mu\left(B\left(x, d(x,y)\right)\right)} \qquad (8)$$

Whenever $d(x,z) \le d(x,y)/2$.

We consider the functions

$$\omega(x,r) = \int_0^r \rho^\alpha \Phi^{-1}(x, \kappa(x,\rho)^{-1}) \frac{d\rho}{\rho}$$

and

$$\omega_\theta(x,r) = r^\theta \int_0^{d_X} \rho^{\alpha-\theta} \Phi^{-1}(x, \kappa(x,\rho)^{-1}) \frac{d\rho}{\rho}$$

for $\theta > 0$ and $0 < r \le d_X$.

Lemma (3.2.5)[3]: Let $E \subset X$. If $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$, then $\omega_\theta(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in E$.

There exists a constant $C > 0$ such that

$$\omega(x, 2r) \leq C\omega(x, r)$$

for all $x \in X$ and $0 < r \leq d_X/2$.

Theorem (3.2.6)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Then there exists a constant $C > 0$ such that

$$|I_\alpha f(x) - I_\alpha f(z)| \leq C\{\omega(x, d(x,z)) + \omega(z, d(x,z)) + \omega_\theta(x, d(x,z))\}$$

for all $x, z \in X$ with $d(x,z) \leq d_X/4$ and nonnegative $f \in L^{\Phi,\kappa}(X)$ with $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$.

Before giving a proof of Theorem (3.2.6), we prepare two more lemmas.

Proof: Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^{\Phi,\kappa}(X)} \leq 1$ and $x, z \in X$ with $d(x,z) \leq d_X/4$. Write

$$\begin{aligned} I_\alpha f(x) - I_\alpha f(z) &= \int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) \\ &- \int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu\left(B\left(z, d(z,y)\right)\right)} d\mu(y) \\ &+ \int_{X\setminus B(x,2d(x,z))} \left(\frac{d(x,y)^\alpha}{\mu\left(B\left(x, d(x,y)\right)\right)} - \frac{d(z,y)^\alpha}{\mu\left(B\left(z, d(z,y)\right)\right)}\right) f(y) d\mu(y) \end{aligned}$$

Using Lemmas (3.2.7) and (3.2.9), we have

$$\int_{X\cap B(x,2d(x,z))} \frac{d(x,y)^\alpha f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) \leq C\omega(x, 2d(x,z)) \leq C\omega(x, d(x,z))$$

And

$$\int_{X\cap B(x,2d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu\left(B\left(z, d(z,y)\right)\right)} d\mu(y) \leq \int_{X\cap B(x,3d(x,z))} \frac{d(z,y)^\alpha f(y)}{\mu\left(B\left(z, d(z,y)\right)\right)} d\mu(y)$$

$$\leq C\omega(z, 3d(x,z)) \leq C\omega(z, d(x,z)).$$

On the other hand, by (8) and Lemma (3.2.4), we have

$$\int_{X\setminus B(x,2d(x,z))} \left|\frac{d(x,y)^\alpha}{\mu\left(B\left(x, d(x,y)\right)\right)} - \frac{d(z,y)^\alpha}{\mu\left(B\left(z, d(z,y)\right)\right)}\right| f(y) d\mu(y)$$

$$\leq Cd(x,z)^{\theta}\int_{X\setminus B(x,2d(x,z))}\frac{d(x,y)^{\alpha-\theta}f(y)}{\mu\left(B\big(x,d(x,y)\big)\right)}d\mu(y)$$

$$\leq\ C\omega_{\theta}(x,2d(x,z))\ \leq\ C\omega_{\theta}(x,d(x,z)).$$

Then we have the conclusion.

In view of Lemma (3.2.6), we obtain the following corollary.

Lemma (3.2.7)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\kappa}(X)}\leq 1$. Then there exists a constant $C\ >0$ such that

$$\int_{X\cap B(x,\delta)}\frac{d(x,y)^{\alpha}f(y)}{\mu\left(B\big(x,d(x,y)\big)\right)}d\mu(y)\leq\ C\omega(x,\delta)$$

For all $x\ \in\ X$ and $0<\ \delta\leq\ d_X$.

Proof. Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^{\Phi,\kappa}(X)}\leq 1$. As usual we start by decomposing $B(x,\delta)$ dyadically:

$$\int_{X\cap B(x,\delta)}\frac{d(x,y)^{\alpha}f(y)}{\mu\left(B\big(x,d(x,y)\big)\right)}d\mu(y)=\sum_{j=1}^{\infty}\int_{X\cap B(x,2^{-j+1}\delta)\setminus B(x,2^{-j+1}\delta))}\frac{d(x,y)^{\alpha}f(y)}{\mu\left(B\big(x,d(x,y)\big)\right)}d\mu(y)$$

$$\leq\sum_{j=1}^{\infty}\left(2^{-j+1}\delta\right)^{\alpha}\frac{1}{\mu\big(B(x,2^{-j}\delta)\big)}\int_{B(x,2^{-j+1}\delta)}f(y)\,d\mu(y)$$

$$\leq c_0\sum_{j=1}^{\infty}\left(2^{-j+1}\delta\right)^{\alpha}\frac{1}{\mu\big(B(x,2^{-j+1}\delta)\big)}\int_{B(x,2^{-j+1}\delta)}f(y)\,d\mu(y)$$

By Lemma (3.1.4), we have

$$\int_{X\cap B(x,\delta)}\frac{d(x,y)^{\alpha}f(y)}{\mu\left(B\big(x,d(x,y)\big)\right)}d\mu(y)\leq C\sum_{j=1}^{\infty}\left(2^{-j+1}\delta\right)\Phi^{-1}(x,\kappa(x,2^{-j+1}\delta)^{-1})$$

$$\leq C\int_0^{\delta}\rho^{\alpha}\Phi^{-1}(x,\kappa(x,\rho)^{-1})\frac{d\rho}{\rho}$$

$$=\ C\omega(x,\delta).$$

The following lemma can be proved in the same manner as Lemma (3.1.7).

Lemma (3.2.8)[3]: Assume that $\Phi(x,t)$ satisfies (Φ_5). Let$\theta\in$ **R**. Let f be a nonnegative function on X such that $\|f\|_{L^{\Phi,\kappa}(X)}\leq 1$. Thenthere exists a constant $C\ >0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha-\theta} f(y)}{\mu\Big(B\big(x,d(x,y)\big)\Big)} d\mu(y) \le C\delta^{-\theta}\omega_\theta$$

for all $x \in X$ and $0 < \delta \le d_X/2$.

Corollary (3.2.9)[3]: Assume that $\Phi(x,t)$satisfies (Φ_5).

(i) Let $x_0 \in X$ and suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in X \cap B(x_0,\delta)$ for some $\delta > 0$. Then $I_\alpha f$ is continuous at x_0 for every $f \in L^{\Phi,\kappa}(X)$.

(ii) Suppose $\omega(x,r) \to 0$ as $r \to 0+$ uniformly in $\in X$. Then $I_\alpha f$ is uniformly continuous on X for every $f \in L^{\Phi,\kappa}(X)$.

For a measurable function $Q(\cdot)$ satisfying

$$0 < Q^- := \inf_{x\in X} Q(x) \le \sup_{x\in X} Q(x) =: Q^+ < \infty, \tag{9}$$

we say that a measure μ is lower Ahlfors $Q(x)$-regular if there exists a constant $c_1 > 0$ such that

$$\mu(B(x,r)) \ge c_1 r^{Q(x)}$$

for all $x \in X$ and $0 < r < d_X$. Recall that we say that the measure μ is a doubling measure if there exists a constant $c_0 > 0$ such that $\mu(B(x,2r)) \le c_0\mu(B(x,r))$ for every $x \in X$ and $0 < r < d_X$. Here note that if μ is a doubling measure and $d_X < \infty$, then μ is lower Ahlfors $\log_2 c_0$-regular since

$$\frac{\mu(B(x,r))}{\mu(B(x,d_X))} \ge c_0^{-2}\left(\frac{r}{d_X}\right)^{\log_2 c_0}$$

for all $x \in X$ and $0 < r < d_X$.

For a locally integrable function f on X, the Hardy–Littlewood maximal function Mf is defined by

$$Mf\,(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)\cap X} |f(y)|\, d\mu(y)\,.$$

We can show the following boundedness of the maximal operator on $L^\Phi(X)$.

Lemma (3.2.10)[3]: Suppose that $\Phi(x,t)$ satisfies (Φ_5) and further assume : $(\Phi 3^*)$ $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some

$$\varepsilon_0 > 0.$$

Then the maximal operator M is bounded from $L^\Phi(X)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^\Phi(x)} \le C\|f\|_{L^\Phi(x)}$$

For all $f \in L^{\Phi}(X)$.

We consider the function

$$\gamma\,(x,t) : X \times (0, d_X) \to (0, \infty)$$

Satisfying the following conditions:

(γ_1) $\gamma(\cdot, t)$ is measurable on X for each $0 < t < d_X$ and $\gamma\,(x, \cdot)$ is continuous on $(0, d_X)$ for each $x \in X$;

(γ_2) There exist constants $\gamma_0 > 0$ and $B_0 \geq 1$ such that

$B_0^{-1} \leq \gamma\,(x,t) \leq B_0 t^{-\gamma_0}$ for all $x \in X$ whenever $0 < t < d_X$.

(γ_3) there exists a constant $B_1 \geq 1$ such that

$B_1^{-1}\gamma\,(x,s) \leq \gamma\,(x,t) \leq B_1\gamma\,(x,s)$ for all $x \in X$ and $1 \leq t/s \leq 2$.

Further we consider the function

$$\tilde{\Gamma}\,(x,t): X \times [0,\infty) \to [0,\infty)$$

Satisfying the following conditions $(\Gamma_1), (\Gamma_2)$ and (Γ_3):

(Γ_1) $\tilde{\Gamma}(\cdot, t)$ is measurable on X for each $t \geq 0$ and $\tilde{\Gamma}(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

(Γ_2) $\tilde{\Gamma}(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $B_2 \geq 1$ such that

$\tilde{\Gamma}\,(x,t) \leq B_2\tilde{\Gamma}\,(x,s)$ for all $x \in X$ whenever $0 \leq t < s$;

(Γ_3) For a measurable function Q(·) satisfying (9), there exist constants $\alpha_0 > 0, B_3 \geq 1$ and $B_4 \geq 1$ such that

$$t^{\alpha - Q(x)}\phi(x, \gamma\,(x,t))^{-1} \leq B_3\tilde{\Gamma}(x, 1/t)$$

For all $x \in X$ and $\alpha \geq \alpha_0$ whenever $0 < t < d_X$ and

$$\int_t^{d_X} \rho^{\alpha}\gamma\,(x,\rho)\frac{d\rho}{\rho} \leq B_4\tilde{\Gamma}(x, 1/t)$$

For all $x \in X, 0 < t \leq d_X/2$ and $\alpha \geq \alpha_0$.

Example (3.2.13)[3]: Let Φ be as in Example (3.1.1).

(i) Suppose there exists an integer $1 \leq j_0 \leq k$ such that

$$\inf_{x \in X}(p(x) - q_{j_0}(x) - 1) > 0$$

And

$$\sup_{x\in X}(p(x) - q_j(x) - 1) \le 0$$

For all $j \le j_0 - 1$ in case $j_0 \ge 2$. Set

$$\gamma\,(x,t) = t^{-Q(x)/p(x)}\left(\prod_{j=1}^{j_0-1}\left[L_e^{(j)}(1/t)\right]^{-1}\right)\left[L_e^{(j)}(1/t)\right]^{-q_{j_0}(x)+1/p(x)}\left(\prod_{j=j_0+1}^{k}\left[L_e^{(j)}(1/t)\right]^{-q_j(x)/p(x)}\right)$$

And

$$\tilde{\Gamma}(x,t) = [L_e^{(j_0)}(t)]^{(p(x)-q_{j_0}(x)-1)/p(x)}\left(\prod_{j=j_0+1}^{k}\left[L_e^{(j)}(t)\right]^{-q_j(x)/p(x)}\right)$$

Then $\gamma\,(x,t)$ satisfies $(\gamma_1),(\gamma_2)$ and (γ_3) and $\tilde{\Gamma}(x,t)$ satisfies $(\Gamma_1),(\Gamma_2)$ and $(\Gamma\ 3)$ for all $\alpha \ge Q^+/p^-$. (2) Suppose that

$$\sup_{x\in X}(p(x) - q_j(x) - 1) \le 0$$

For all $j = 1,\ldots,k$. Set

$$\gamma\,(x,t) = t^{-Q(x)/p(x)}\left(\prod_{j=1}^{k}\left[L_e^{(j)}(1/t)\right]^{-1}\right)\left[L_e^{(k+1)}(1/t)\right]^{-1/p(x)}$$

And

$$\tilde{\Gamma}\,(x,t) = \left[L_e^{(k+1)}(1/t)\right]^{1-1/p(x)}.$$

Then $\gamma\,(x,t)$ satisfies $(\gamma_1),(\gamma_2)$ and (γ_3) and $\tilde{\Gamma}(x,t)$ satisfies $(\Gamma_1),(\Gamma_2)$ and (Γ_3) for all $\alpha \ge Q^+/p^-$. In fact.

Lemma (3.2.12)[3]. Assume thatμis lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies (Φ_5). Let $\alpha \ge \alpha_0$. Then there exists aconstant $C > 0$ such that

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha} f(y)}{\mu\left(B\left(x,d(x,y)\right)\right)}\,d\mu(y) \le C\tilde{\Gamma}\left(x,\frac{1}{\delta}\right)$$

For all $x \in X, 0 < \delta \le d_X/2$ and nonnegative $f \in L^{\Phi}(X)$ with $\|f\|_{L^{\Phi}(X)} \le 1$.

Proof: Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^{\Phi}(X)} \le 1$. Let j_0 be the smallest integer j_0 such that

$2^{j_0}\delta \geq d_X$. Since

$$B_0^{-1} \leq \gamma\ (x, d(x,y)) \leq\ B_0 d(x,y)^{-\gamma_0}$$

in view of (γ_2), we have

$$d(x,y) \leq B_0^{2/\gamma_0}\left(B_0\gamma\left(x, d(x,y)\right)\right)^{-1/\gamma_0}.$$

Hence, by $(\Phi_3), (\Phi_4)$ and (Φ_5), we obtain

$$\phi(y, \gamma\ (x, d(x,y)))^{-1} \leq B'\phi(x, \gamma\ (x, d(x,y)))^{-1}$$

with some constant $B' >\ 0$. By $(\gamma_3), (\Phi_3), (\Gamma_2)$ and (Γ_3), we have

$$\begin{aligned}
&\int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) \\
&\leq \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha \gamma\ (x, d(x,y))}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) \\
&\quad +\ A_2 \int_{X\setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu\left(B\left(x, d(x,y)\right)\right)} \frac{\phi(y, f(y))}{\phi(y, \gamma\ (x, d(x,y)))} d\mu(y) \\
&\leq \sum_{j=1}^{j_0} \int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)}^{\delta} \frac{d(x,y)^\alpha \gamma(x, d(x,y))}{\mu\left(B\left(x, d(x,y)\right)\right)} d\mu(y) \\
&\quad + c_0^{-1} A_2 B' \int_{X\setminus B(x,\delta)} d(x,y)^{\alpha - Q(x)}\ \phi(x, \gamma\ (x, d(x,y)))^{-1} \Phi(y, f(y)) d\mu(y) \\
&\leq\ 2^\alpha B_1 \sum_{j=1}^{j_0} (2^{j-1}\delta)^\alpha \gamma\ (x, 2^{j-1}\delta) \int_{B(x,2^j\delta)\setminus B(x,2^{j-1}\delta)} \frac{1}{\mu(B(x, 2^{j-1}\delta))} d\mu(y) \\
&\quad + c_0^{-1} A_2 B_2 B_3 B' \tilde{\Gamma}(x, 1/\delta) \int_{X\setminus B(x,\delta)} \Phi(y, f(y)) d\mu(y) \\
&\leq 2^\alpha c_2 B_1 \sum_{j=1}^{j_0} (2^{j-1}\delta)^\alpha\ \gamma(x, 2^{j-1}\delta) + c_0^{-1} A_2 B_2 B_3 B' \tilde{\Gamma}(x, 1/\delta).
\end{aligned}$$

since

$$\int_\delta^{d_X} \rho^\alpha \gamma\ (x,\rho) \frac{d\rho}{\rho} \geq \sum_{j=1}^{j_0-1} \int_{2^{j-1}\delta}^{2^j\delta} \rho^\alpha \gamma\ (x,\rho) \frac{d\rho}{\rho} \geq \frac{\log 2}{B_1} \sum_{j=1}^{j_0-1} \left(2^{j-1}\delta\right)^\alpha \gamma(x, 2^{j-1}\delta)$$

and

$$\int_{\delta}^{d_X} \rho^{\alpha}\gamma\,(x,\rho)\frac{d\rho}{\rho} \geq \int_{d_X/2}^{d_X} \rho^{\alpha}\gamma\,(x,\rho)\frac{d\rho}{\rho} \geq \frac{\log 2}{2^{\alpha}B_1}\left(2^{j_0-1}\delta\right)^{\alpha}\gamma(x,2^{j_0-1}\delta)$$

we have

$$\sum_{j=1}^{j_0-1}\left(2^{j-1}\delta\right)^{\alpha}\gamma(x,2^{j-1}\delta) \leq \frac{B_1}{\log 2}(2^{\alpha}+1)\int_{\delta}^{d_X} \rho^{\alpha}\gamma\,(x,\rho)\frac{d\rho}{\rho}$$

Hence, we obtain

$$\int_{X\setminus B(x,\delta)} \frac{d(x,y)^{\alpha}f(y)}{\mu\left(B\left(x,d(x,y)\right)\right)}d\mu(y)$$

$$\leq (\log 2)^{-1}2^{\alpha}(2^{\alpha}+1)c_2B_1^2\int_{\delta}^{d_X} \rho^{\alpha}\gamma\,(x,\rho)\frac{d\rho}{\rho} + c_0^{-1}A_2B_2B_3B'\tilde{\Gamma}(x,1/\delta)$$

$$\leq (\log 2)^{-1}2^{\alpha}(2^{\alpha}+1)c_2B_1^2B_4\tilde{\Gamma}(x,1/\delta) + c_0^{-1}A_2B_2B_3B'\tilde{\Gamma}(x,1/\delta)$$

$$= ((\log 2)^{-1}2^{\alpha}(2^{\alpha}+1)c_2B_1^2B_4 + c_0^{-1}A_2B_2B_3B')\tilde{\Gamma}(x,1/\delta)$$

as required.

Lemma (3.2.13)[3]: Let $\alpha \geq \alpha_0$. Then there exists a constant $C' > 0$such that $\tilde{\Gamma}(x,2/d_X) \geq C'$for all $\in X$.

Lemma (3.2.14)[3]: Suppose $\tilde{\Gamma}(x,t)$ satisfies the uniform log-type condition:

$(\tilde{\Gamma}_{\log})$ there exists a constant $c_{\Gamma} > 0$ such that

$$c_{\Gamma}^{-1}\tilde{\Gamma}(x,t) \leq \tilde{\Gamma}(x,t^2) \leq c_{\Gamma}\tilde{\Gamma}(x,t)$$

for all $x \in X \; and \; t > 0$.

Then, for every $a > 1$, there exists $b > 0$ such that $\tilde{\Gamma}(x,at) \leq b\tilde{\Gamma}(x,t)$ for all $x \in X$ and $t > 0$.

Theorem (3.2.15)[3]. Suppose that μ is lower Ahlfors $Q(x)$-regular. Assume that $\Phi(x,t)$satisfies (Φ_5) and (Φ_3). Further, assume

that $\tilde{\Gamma}(x,t)$ satisfies $(\tilde{\Gamma}_{\log})$. For each $\in X$, let $\tilde{\gamma}(x) = \sup_{s>0}\tilde{\Gamma}(x,s)$. Suppose $\tilde{\Psi}(x,t): X \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:

$(\tilde{\Psi}_1)\tilde{\Psi}(\cdot,t)$ is measurable on X for each $t \in [0,\infty)$ and $\tilde{\Psi}(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in X$;

$(\tilde{\Psi}_2)$ there is a constant $B_5 \geq 1$ such that $\tilde{\Psi}(x,t) \leq \tilde{\Psi}(x,B_5s)$ for all $x \in X$ whenever $0 < t < s$;

$(\widetilde{\Psi}_3)$ there are constants $B_6, B_7 \geq 1$ and $t_0 > 0$ such that $\widetilde{\Psi}(x, \tilde{\Gamma}\,(x,t)/B_6) \leq B_7 t$ for all $x \in X$ and $t \geq t_0$.

Then there exist constants $c_1, c_2 > 0$ such that $I_\alpha f(x)/c_1 \leq \gamma\,(x)$ for μ-a.e. $x \in X$ and

$$\int_X \widetilde{\Psi}\,\left(x, \frac{I_\alpha f(x)}{c_1}\right) d\mu(x) \leq c_2$$

for all $\alpha \geq \alpha_0$ and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

Proof: Let f be a nonnegative μ-measurable function on X with $\|f\|_{L^\Phi(X)} \leq 1$. Note from Lemma (3.2.10) that

$$\int_X Mf(x)\, d\mu(x) \leq \mu(X) + A_1 A_2 \int_X \Phi\big(x, Mf(x)\big) d\mu(x) \leq C_M. \quad (10)$$

Fix $x \in X$. For $0 < \delta \leq d_X/2$, Lemma (3.2.14)[3] implies

$$I_\alpha f(x) = \int_{B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu\Big(B\big(x, d(x,y)\big)\Big)} d\mu(y) + \int_{X \setminus B(x,\delta)} \frac{d(x,y)^\alpha f(y)}{\mu\Big(B\big(x, d(x,y)\big)\Big)} d\mu(y)$$

$$\leq C\left\{\delta^\alpha Mf(x) + \tilde{\Gamma}\left(x, \frac{1}{\delta}\right)\right\}$$

With a constant $C > 0$ independent of x.

If $(x) \leq 2/d_X$, then we take $\delta = d_X/2$. Then, by Lemma (3.2.10)

$$I_\alpha f(x) \leq C\tilde{\Gamma}\left(x, \frac{2}{d_X}\right)$$

By Lemma (3.2.16)[3] and $(\Gamma\ 2)$, there exists $C_1^* > 0$ independent of x such that

$$I_\alpha f(x) \leq C_1^* \tilde{\Gamma}(x, t_0) \text{ if } Mf(x) \leq 2/d_X. \quad (11)$$

Next, suppose $2/d_X < Mf(x) < \infty$. Let $m = \sup_{s \geq 2/d_X, x \in X} \tilde{\Gamma}(x,s)/s$. By $(\tilde{\Gamma}_{\log})$, $m < \infty$. Define δ by

$$\delta^\alpha = \frac{(d_X/2)^\alpha}{m} \tilde{\Gamma}(x, Mf(x))(Mf(x))^{-1}.$$

Since $\tilde{\Gamma}\big(x, Mf(x)\big)\big(Mf(x)\big)^{-1} \leq m, 0 \leq \delta \leq d_X/2$ Then by Lemma (3.2.13) and (Γ_2)

$$\frac{1}{\delta} = \frac{m^{1/\alpha}}{d_X/2} \tilde{\Gamma}\,(x, Mf(x))^{-1/\alpha}(Mf(x))^{1/\alpha}$$

$$\leq \frac{m^{1/\alpha}}{d_X/2} B_2^{1/\alpha} \tilde{\Gamma}\ (x, 2/d_X)^{-1/\alpha} (Mf(x))^{1/\alpha} \leq C(Mf(x))^{1/\alpha}$$

Hence, using (Γ_2), $(\tilde{\Gamma}_{\log})$ and Lemma (3.2.14), we obtain

$$\tilde{\Gamma}\left(x, \frac{1}{\delta}\right) \leq B_2 \tilde{\Gamma}\left(x, C(Mf\ (x))^{1/\alpha}\right) \leq C\tilde{\Gamma}(x, Mf(x)).$$

By Lemma (3.2.14) again, we see from (Γ_2) that there exists a constant $C_2^* > 0$ independent of x such that

$$I_\alpha f(x) \leq\ C_2^* \tilde{\Gamma}\left(x, \tfrac{t_0 d_X}{2} Mf(x)\right) \text{ if } 2/d_X < Mf(x) < \infty \qquad (12)$$

Now, let $c_1 = B_5 B_6 \max(C_1^*, C_2^*)$. Then, by (11) and (12),

$$\frac{I_\alpha f(x)}{c_1} \leq \frac{1}{B_5 B_6} \max\left\{\tilde{\Gamma}(x, t_0), \tilde{\Gamma}\left(x, \frac{t_0 d_X}{2} Mf(x)\right)\right\}$$

Whenever $(x)\ < \infty$. Since $Mf(x) < \infty$ for μ-a.e. $x \in X$ by Lemma(3.2.10), $I_\alpha f(x)/c_1 \leq \tilde{\gamma}\ (x)$ μ-a.e.$x \in X$, and by $(\widetilde{\Psi}_2)$ and $(\widetilde{\Psi}_3)$, we have

$$\widetilde{\Psi}\left(x, \frac{I_\alpha f(x)}{c_1}\right) \leq \max\left\{\widetilde{\Psi}\left(x, \tilde{\Gamma}(x, t_0)/B_6\right), \widetilde{\Psi}\left(x, \tilde{\Gamma}\left(x, \frac{t_0 d_X}{2} Mf(x)\right)/B_6\right)\right\}$$

$$\leq B_7 t_0 + \frac{B_7 t_0 d_X}{2} Mf(x)$$

for μ-a.e. $x \in X$. Thus, we have by (10)

$$\int_X \widetilde{\Psi}\left(x, \frac{I_\alpha f(x)}{c_1}\right) d\mu(x) \leq B_7 t_0 \mu(X) + \frac{B_7 t_0 d_X}{2} \int_X Mf(x) d\mu(x)$$

$$\leq\ B_7 t_0 \mu(X) + \frac{B_7 t_0 d_X C_M}{2} = c_2.$$

We obtain the following corollary applying Theorem (3.2.15) to special Φ given in Example (3.1.1).

Corollary (3.2.16)[3]: Let Φ be as in Example (3.1.1). Assume that μ is lower Ahlfors $Q(x)$-regular.(i) Suppose there exists an integer $1 \leq j_0 \leq k$ such that

$$\inf_{x \in X} (p(x) - q_{j_0}(x) - 1)\ >\ 0 \qquad (13)$$

And

$$\sup_{x \in X} (p(x) - q_j(x) - 1) \leq 0 \qquad (14)$$

For all $j \leq j_0 - 1$ in case $j_0 \geq 2$. Then there exist constants $c_1, c_2 > 0$ such that

$$\int_X E_+^{(j_0)}\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-q_{j_0}(x)-1)} \prod_{j=1}^{k-j_0}\left(L_e^{(j)}\left(\frac{I_\alpha f(x)}{c_1}\right)\right)^{q_{j_0+j}(x)/(p(x)-q_{j_0}(x)-1)}\right) d\mu(x) \leq c_2$$

For all $\alpha \geq Q^+/p^-$ and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

(ii) If

$$\sup_{x\in X}(p(x) - q_j(x) - 1) \leq 0$$

For all $j = 1, \ldots, k$, then there exist constants $c_1, c_2 > 0$ such that

$$\int_X E^{(k+1)}\left(\left(\frac{I_\alpha f(x)}{c_1}\right)^{p(x)/(p(x)-1)}\right) d\mu(x) \leq c_2$$

For all $\alpha \geq Q^+/p^-$and nonnegative functions $f \in L^\Phi(X)$ satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

For a measurable function $Q(\cdot)$ satisfying (9), we consider the functions

$$\widetilde{\omega}(x,r) = \int_0^r \rho^\alpha \Phi^{-1}\left(x, \rho^{-Q(x)}\right)\frac{d\rho}{\rho}$$

And

$$\widetilde{\omega}_\theta(x,r) = r^\theta \int_r^{d_X} \rho^{\alpha-\theta} \Phi^{-1}\left(x, \rho^{-Q(x)}\right)\frac{d\rho}{\rho}$$

for $\theta > 0$ and $0 < r \leq d_X$.

As in the proof of Theorem (3.2.9), we can obtain the continuity of Riesz potentials $I_\alpha f$ of functions in Musielak–Orlicz spaces under the condition (8).

Theorem (3.2.17)[3]: Assume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies(Φ_5). Suppose that (8) holds.Thenthere exists a constant $C > 0$ such that

$$|I_\alpha f(x) - I_\alpha f(z)| \leq C\{\widetilde{\omega}(x, d(x,z)) + \widetilde{\omega}(z, d(x,z)) + \widetilde{\omega}_\theta(x, d(x,z))\}$$

for all $x, z \in X$ with $0 < d(x,z) \leq d_X/2$ whenever $f \in L^\Phi(X)$ is a nonnegative function on X satisfying $\|f\|_{L^\Phi(X)} \leq 1$.

Corollary (3.2.18)[3]. Assume that μ is lower Ahlfors $Q(x)$-regular. Suppose that $\Phi(x,t)$ satisfies (Φ_5). Suppose that (8) holds.

(i) $Let\, x_0 \in X$ and suppose $\widetilde{\omega}(x,r) \to 0$ as $r \to 0+$ uniformly in $x \in B(x_0,\delta) \cap X$ for some $\delta > 0$. Then $I_\alpha f$ is continuous at x_0 for every $f \in L^{\Phi}(X)$.

(ii) Suppose $\widetilde{\omega}(x,r) \to 0$ as $r \to 0+$ uniformly in $\in X$. Then $I_\alpha f$ is uniformly continuous on X for every $f \in L^{\Phi}(X)$.

Chapter 4

Riesz Potentials of Functions in Generalized Grand Morrey Spaces

We result will imply the boundedness of the Riesz potential operator from a grand Morrey space to a Morrey space.

Section (4.1): Grand Morrey Spaces

Let $\boldsymbol{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x, r)$ the open ball centered at x of radius r and denote by $|E|$ the Lebesgue measure of a measurable set $|E| \subset \boldsymbol{R}^n$. Let G be a bounded open set in $\boldsymbol{R}^n$. We denote by d_G the diameter of G.

Morrey considered the integral growth condition on derivatives over balls, in order to study the existence and regularity for partial differential equations. A family of functions with the integral growth condition is then called a Morrey space after his name. A systematical study for Morrey spaces was done, where the Morrey space $L^{p,\nu}(G)$ is a family of $f \in L^1_{loc}(G)$ satisfying the Morrey condition.

$$\sup_{x \in G, 0 < r < dG} \frac{r^\nu}{|B(x,r)|} \int_{G \cap B(x,r)} |f(y)|^p dy < \infty$$

For $p \geq 1$ and $\nu > 0$. Grand Lebesgue spaces were introduced for the sake of study of the integrability of the Jacobian.

For $0 < \alpha < n$ and a locally integrable function f on G, we define the Riesz potential $U_\alpha f$ of order α by

$$U_\alpha f(x) = \int_G |x - y|^{\alpha - n} f(y)\, dy;$$

For fundamental properties of Riesz potentials.

Meskhi investigated the boundedness for several integral operators, including the Riesz potential operator, in the grand Morrey spaces $L^{p),\nu,\theta}(G)$ which consists of all functions $\|f\| \in L^1_{loc}(G)$ satisfying the grand Morrey condition

$$\sup_{x\in G, 0<r<dG, 0<\varepsilon<p-1} \varepsilon^{\theta} \frac{r^{v}}{|B(x,r)|} \int_{G\cap B(x,r)} |f(y)|^{p-\varepsilon} dy < \infty$$

For $p > 1, v > 0$ and $\theta > 0$; in what follows, let $f = 0$ outside G. We establish Trudinger's exponential integrability for Riesz potentials of functions in generalized grand Morrey spaces which will be mentioned below.

In view of Fusco, Lions and Sbordone, we see that if f is a measurable function on G satisfying the grand Lebesgue condition

$$\lim_{\varepsilon\to 0+} \varepsilon^{\theta} \int_{G} |f(y)|^{n-\varepsilon}\, dy = 0$$

Then

$$\int_{G} exp\left(|U_1 f(x)|^{n/(n-1+\theta)}\right) dx < \infty$$

We also obtain Trudinger's exponential integrability for Riesz potentials of functions in grand Lebesgue spaces.

Let C denote various constants independent of the variables in question, and $C(a, b, \dots)$ a constant that depends on $a, b, \dots$. The symbol $g \sim h$ means that $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$.

Let φ be a positive nondecreasing function on $(0, \infty)$ satisfying the following conditions:

(φ_1) there exists a constant $A_1 \geq 1$ suchthat

$$A_1^{-1}\, r^n \leq \varphi(r) \leq A_1 \text{ for } 0 < r < 1;$$

(φ_2) φ is doubling on $(0, \infty)$, namely there exists a constant $A_2 \geq$ 1suchthat

$$\varphi(2t) \leq A_2 \varphi(t) \text{ for } t > 0.$$

For $\beta > 0$, set

$$\psi_{\beta}(r) = \int_{1/r}^{2d_G} t^{\beta}\, \varphi(t)^{-1/p} \left(\log(2d_G/t)\right)^{\theta/p} \frac{dt}{t}$$

When $r \geq 1/d_G$; and set

$$\psi_{\beta}(r) = d_G \psi_{\beta}(1/d_G) r$$

When $0 < r < 1/d_G$.

Let us begin with the following result, which is easily proved by (φ_2).

Lemma (4.1.1)[4]: For $\beta > 0, \psi_\beta$ is increasing and doubling on $[0, \infty)$.

Now, for $p > 1$ and $\theta > 0$, we introduce the generalized grand Morrey space $L^{p),\varphi,\theta}(G)$ which consists of all measurable functions f on G such that

$$\|f\|_{L^{p),\varphi,\theta}(\mathrm{G})} = \inf\{\lambda > 0: \sup_{x\in \mathrm{G}, 0<r<d\mathrm{G}, 0<\varepsilon<p-1} \varepsilon^\theta \frac{r^v}{|\mathrm{B}(\mathrm{x},\mathrm{r})|} \int_{\mathrm{G}\cap \mathrm{B}(\mathrm{x},\mathrm{r})} |f(y)|^{p-\varepsilon} dy < \infty$$

recall here that $f = 0$ outside G.

In case $\varphi(r) = r^\nu, L^{p),\varphi,\theta}(G)$ is denoted by $L^{p),\nu,\theta}(G)$ for simplicity; in particular, $L^{p),n,\theta}(G)$ is usually written as $L^{p),\theta}(G)$.

We establish the following exponential integrability for Riesz potentials of functions in $L^{p),\varphi,\theta}(G)$.

Theorem (4.1.2)[4]: For $0 < \beta < \alpha$ there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{\mathrm{B}(\mathrm{z},\mathrm{r})|} \int_{\mathrm{B}(\mathrm{z},\mathrm{r})} \{\psi_\alpha^{-1} (\mathrm{c}_1 \mathrm{U}_\alpha f(x)\}^{\alpha-\beta} \mathrm{d}x \le \mathrm{c}_2 \psi_\beta(1/\mathrm{r})$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

Proof: let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}} \le 1$. Then,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy \le C\varphi(r)^{-\frac{1}{p}} (\log(2d_G/r))^{\theta/p}$$

For all $x \in G$ and $0 < r < d_G$.

For $x \in G$ and $0 < \delta \le d_G$, write

$$U_a f(x) = \int_{B(x,r)} |x-y|^{\alpha-n} f(y) dy + \int_{G./B(r,\delta)} |x-y|^{\alpha-n} f(y) dy$$
$$= U_1(r) + U_2(r).$$

For $0 < \beta < \alpha$, we have

$$U_1(x) \le \delta^{\alpha-\beta} \int_{B(x,\delta)} |x-y|^{\beta-n} f(y) dy \le \delta^{\alpha-\beta} U_\beta f(x)$$

And

$$U_2(x) \le C\int_{\delta}^{2d_G} t^{\alpha-N}\left(\int_{B(x,t)} f(y)dy\right)\frac{dt}{t} \le C\int_{\delta}^{2d_G} t^{\alpha}\,\varphi(t)^{-\frac{1}{p}}\left(\log\left(\frac{2dG}{t}\right)\right)^{\frac{\theta}{p}}\frac{dt}{t}$$
$$\le C\psi_\alpha\,(\delta^{-1})$$

Since

$$\int_{\delta}^{2d_G} t^{\alpha-N}\left(\int_{B(x,t)} f(y)dy\right)\frac{dt}{t} \le C \le C\int_{\delta}^{2d_G} t^{\alpha}\,\varphi(t)^{-\frac{1}{p}}\left(\log\left(\frac{2dG}{t}\right)\right)^{\frac{\theta}{p}}\frac{dt}{t}$$

Hence it follows that

$$U_\alpha f(x) \le C\left\{\delta^{\alpha-\beta}\,U_\beta f(x) + \psi_\alpha\,(\delta^{-1})\right\}.$$

If $U_\beta f(x)^{-\frac{1}{\alpha-\beta}} \le dG$, then we take $\delta = U_\beta f(x)^{-\frac{1}{\alpha-\beta}}$ to obtain

$$U_\beta f(x) \le \left(\{C\psi_\alpha\,U_\beta f(x)\}^{\frac{1}{\alpha-\beta}}\right);$$

if $\left\{U_\beta f(x)\right\}^{-\frac{1}{\alpha-\beta}} \ge d_G$, then we take $\delta = d_G$ to obtain

$$U_\alpha f(x) \le C.$$

Hence

$$U_\alpha f(x) \le C_1\psi_\alpha\left(\left\{1 + U_\beta f(x)\right\}^{1/(\alpha-\beta)}\right);$$

Which together with Lemma (4.1.7) gives

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \{\psi_\alpha^{-1}U_\alpha f(x)/C_1\}^{\alpha-\beta}\,dx \le C\frac{1}{|B(z,r)|}\int_{B(z,r)}\left\{1+U_\beta f(x)\right\}dx$$
$$\le C\left\{1+\beta^{-1}\,\psi_\beta(1/r)\right\}$$
$$\le C(\beta)\psi_\beta(1/r)$$

For $z \in G$ and $0 < r < d_G$. The proof is now completed.

Example (4.1.3)[4]: *Let* $\varphi(r) = r^{\alpha p}(\log(c_0 + r^{-1}))^{\tau_1}\{\log(\log(c_0 + r^{-1}))\}^{\tau_2}$, where τ_1, τ_2 are constants and $c_0 > 1$ is chosen so that φ is decreasing on $(0,\infty)$.

(i) If $\tau_1 < p + \theta$, then

$$\psi_\alpha(r) \sim (\log(c_0 + r))^{(p+\theta-\tau_1)/p}\{\log(\log(c_0 + r))\}^{-\tau_2/p}$$

And

$$(\psi_\alpha)^{-1}(r) \sim exp\left(r^{p/(p+\theta-\tau_1)}(\log(c_0 + r))^{\tau_2/(p+\theta-\tau_1)}\right);$$

(ii) if $\tau_1 = p + \theta$ and $\tau_2 < p$, then

$$\psi_\alpha(r) \sim \{\log(\log(c_0 + r))\}^{1-\tau_2/p}$$

And

$$(\psi_\alpha)^{-1}(r) \sim \exp(\exp\left(r^{p/(p-r_2)}\right));$$

(iii) if $\tau_1 = p + \theta$ and $\tau_2 = p$, then

$$\psi_\alpha(r) \sim \log(\log(\log(c_0 + r)))$$

And

$$\psi_\alpha(r) \sim \exp(\exp(\exp(r));$$

(iv) if $\tau_1 = p + \theta$ and $\tau_2 > p$, then $\psi_\alpha(\infty) < \infty$, so that

$$\psi_\alpha(r) \sim 1$$

For large $r > 0$.

Corollary (4.1.4)[4]: Let $\varphi(r) = r^{\alpha p}(\log(c_0 + r^{-1}))^{\tau_1}\{\log(\log(c_0 + r^{-1}))\}^{\tau_2}$ as a bove. If $0 < \eta < \alpha$, then there exit constants $c_1, c_2 > 0$ (depending on η) such that

(i) In case $\tau_1 > p + \theta$,

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[c_1 U_\alpha f(x)^{1/(1+(\theta-\tau_1)/p)}(\log(c_0 + U_\alpha f(f)))^{\tau_2/(p+\theta-\tau_1}\right]dx \le c_2 r^{-\eta};$$

(ii) In case $\tau_1 = p + \theta$ and $\tau_2 < p$,

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[\exp(c_1 U_\alpha f(x)^{1/(1-\tau_2/P)})\right] dx \le c_2 r^{-\eta}$$

for all $z \in G, 0 < r < d_G$ and $f \ge 0$ with $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

In fact, to prove (i), letting $0 < \alpha - \beta < \eta < \alpha$, we see from Theorem (4.1.2) and example (4.1.3) that

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} \exp\left[c_1(\alpha-\beta)U_\alpha f(x)^{1/(1+(\theta-\tau_1/P))}\right)(\log(c_0 + U_\alpha f(x)))^{\tau_2/(p+\theta-\tau_1)}\Big]\,dx$$
$$\leq c_2 r^{-(\alpha-\beta)}\log(c_0+r)^{(\theta-\tau_1)/p}\{\log(\log(c_0+r))\}^{-\tau_2/p}$$

For all $z \in G, 0 < r < d_G$ and $f \geq 0$ with $\|f\|_{L^{p),\varphi.\theta}(G)} \leq 1$. Hence it suffices to note that

$$c_2 r^{-(a-\beta)}(\log(c_0+r))^{(\theta-\tau_1)/p}\{\log(\log(c_0+r))\}^{-\tau_2/p} \leq C(\eta)r^{-\eta}$$

When $0 < r < d_G$. Assertion (ii) can be proved similarly.

For a proof of Theorem (4.1.2), we prepare some lemmas.

Lemma (4.1.5)[4]: There exists a constant $C > 1$ such that

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \leq C\varphi(r)^{-\frac{1}{p}}(\log\left(\frac{2d_G}{r}\right))^{\frac{\theta}{p}} \tag{1}$$

For all $x \in G, 0 < r < d_G$ and $f \geq 0$ with $\|f\|_{L^{p),\varphi.\theta}(G)} \leq 1$.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{L^{p),\varphi.\theta}(G)} \leq 1$.Then note that

$$\varepsilon^{\theta}\frac{\varphi(r)}{|B(x,r)|}\int_{B(x,r)} f(y)^{p-\epsilon}dy \leq 1$$

For all $x \in G, 0 < r < d_G$ and $0 < \varepsilon < p-1$. By Jensen's inequality, we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \leq \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p-\varepsilon}\,dy\right)^{1/(p-\varepsilon)}$$
$$\leq \varepsilon^{-\theta/(p-\epsilon)}\varphi(r)^{-1/(p-\varepsilon)}$$

Here, taking $\varepsilon = \min\{(p-1)/2, (\log(2d_G/r))^{-1}\}$, we find by (φ_1)

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)dy \leq C\varphi(r)^{-\frac{1}{p}}(\log(2d_G/r))^{\theta/p},$$

Since $r^{-1/(\log(2\mathrm{d_G}/\mathrm{r}))}$ is bounded above when $0 < r < d_G$. This proves the lemma.

Lemma (4.1.6)[4]: let $0 < \beta < \alpha$. Then there exist a constant $C > 0$ such that

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} U_\beta(x)dx \le C\beta^{-1}\psi_\beta(1/r)$$

For all $z \in G, 0 < d_G$ and $f \ge 0$ satisfying (1).

Proof: Let $z \in G, 0 < r < d_G$ and $0 < \beta < \alpha$, for $f \ge 0$ satisfying (1), write

$$U_\beta f(r) = \int_{|B(z,2r)|} |x-y|^{\beta-n} f(y)dy + \int_{G\backslash B(z,2r)} |x-y|^{\beta-n} f(y)dy$$
$$= U_1(x) + U_2(x).$$

By Fubini's theorem, (1) and (φ_2), we have

$$\frac{1}{|B(z,r)|}\int_{B(z,r)} U_1(x)d_x = \frac{1}{|B(z,r)|}\int_{B(z,2r)}\left(\int_{B(z,r)} |x-y|^{\beta-n}dx\right)f(y)dy$$
$$\le C\beta^{-1}r^\beta \frac{1}{|B(z,r)|}\int_{B(z,2r)} f(y)dy$$
$$\le C\beta^{-1}r^\beta\varphi(2r)^{-1/p}(\log(2d_G/r))^{\theta/p} \le C\beta^{-1}\psi_\beta(1/r)$$

Since

$$\psi_\beta(1/r) \ge \int_r^{3r/2} t^\beta\varphi(t)^{-\frac{1}{p}}(\log(2\,d_G/t))^{\frac{\theta}{p}}\frac{dt}{t} \ge r^\beta\varphi(r)^{-\frac{1}{p}}(\log(2d_G/r))^{\frac{\theta}{p}}.$$

For U_2, note that

$$U_2(r) \le C\int_{G/B(z,2r)} |z-y|^{\beta-n} f(y)dy$$

For $x \in B(z,r)$. Hence we have only to consider the case $0 < 1 < d_G/2$ since $U_2(r) = 0$ for $r \le d_G/2$. Hence we obtain

$$U_2(r) \le C\int_{2r}^{2d_G} t^{\beta-n}\left(\int_{B(z,t)} f(y)dy\right)\frac{dt}{t} \le C\int_{2r}^{2d_G} t^\beta\,\varphi(t)^{-\frac{1}{p}}(\log(\,2d_G/t))^{\theta/p}\frac{dt}{t}$$
$$\le C\psi_\beta(1/r)$$

by lemma (4.1.1), which proves the present lemma.

Section (4.2): Grand Lebesgue Spaces

In view of Fusco, Lions and Sbordone, we see that if

$$\lim_{\varepsilon\to 0+} \varepsilon^{\theta} \int_G |f(y)|^{n-\varepsilon}\, dy = 0,$$

Then

$$\int_G \exp\left(|U_1 f(x)|^{n/(n-1+\theta}\right) dx = \infty$$

In connection with their result, we can prove the following result.

Theorem (4.2.1)[4]: Let $\alpha p = n$. Then for $0 < \eta < \alpha$ there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{|B(z,r)|} \int_{B(z,r)} \exp\left(c_1\{U_\alpha f(x)\}^{1/(1+(\theta-1)/p)}\right) dx \le c_2 r^{-\eta}$$

for all $z \in G, 0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\theta}(G)} \le 1$.

Proposition (4.2.2)[4]: Let $\alpha p = n$. Then for $0 < \beta < \alpha$ there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{|\mathrm{B(z,r)}|} \int_{\mathrm{B(z,r)}} \exp\left(c_1\{U_\alpha f(x)\}^{1/(1+(\theta-1)/p)} dx \le c_2 \psi_\beta(1/r)\right)$$

for all $z \in G$, $0 < r < d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\varphi,\theta}(\mathrm{G})} \le 1$.

To prove Proposition (4.2.2), we prepare the next lemma.

Proof: Let f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}(\mathrm{G})} \le 1$.

Then for $0 < \beta < \alpha$, we have by above Lemma

$$U_\alpha f(x) = \int_{\mathrm{B}(x,\delta)} |x - \mathrm{y}|^{\alpha-\mathrm{n}} f(\mathrm{y})\mathrm{dy} + \int_{\mathrm{G}\setminus\mathrm{B}(x,\mathrm{r})} |x - \mathrm{y}|^{\alpha-\mathrm{n}} f(y)\, dy$$

$$\le \delta^{\alpha-\beta} U_\beta f(x) + \mathrm{C}\left(\log(2d_G/r)\right)^{1-(1-\theta)/\mathrm{p}}$$

Here, as in the proof of Theorem (4.1.2), we have the inequality

$$U_\alpha f(x) \le C_1 \left(\log\left(e + U_\beta f(x)\right)\right)^{1-(1-\theta)/p}$$

Hence we find

$$\frac{1}{|B(z,r)|}\int_{B(z,\delta)} \exp\left(\{U_\alpha f(x)/ C_1\}^{1/(1-(1-\theta)/p)}\right)dx$$

$$\le C\frac{1}{|B(z,r)|}\int_{B(z,\delta)} \{1+U_\beta f(x)\}dx$$

$$\le C\psi_\beta(1/r)$$

For all $z \in G$ and $0 < r < d_G$, in view of Lemma (4.1.7). Now we obtain the present result.

Lemma (4.2.3)[4]: Let $\alpha p = n$. Then there exists a constant $C > 0$ such that

$$\int_{G\backslash B(z,r)} |x-y|^{\alpha-n} f(y)\, dy \le C\,(\log(2d_G/r))^{1-(1-\theta)/p}$$

for all $x \in G$, $0 < r \le d_G$ and $f \ge 0$ satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$.

Proof: Let $p = n/\alpha$ and f be a nonnegative measurable function on G satisfying $\|f\|_{L^{p),\varphi,\theta}(G)} \le 1$. Then note that

$$\int_{G\backslash B(x,r)} f(y)^{p-\varepsilon} dy \le \varepsilon^{-\theta}$$

For all $0 < \varepsilon < p-1$. For $x \in G$, $0 < r \le d_G$ and $0 < \varepsilon < p-1$, we have by Hölder's inequality

$$\int_{G\backslash B(x,r)} |x-y|^{\alpha-n} f(y)\, dy$$

$$\le \left(\int_{G\backslash B(x,r)} |x-y|^{(\alpha-n)(p-\varepsilon)'}\, dy\right)^{1/(p-\varepsilon)'} \left(\int_{G\backslash B(x,r)} (y)^{p-\varepsilon}\, dy\right)^{1/(p-\varepsilon)}$$

$$\le C\left(\int_r^{d_G} t^{(\alpha-n)(p-\varepsilon)'+n-1} dt\right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)}$$

$$\leq C\left(\int_{r}^{d_G} t^{-\alpha\varepsilon/(p-\varepsilon-1)-1}dt\right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)}$$

$$\leq G\left(\frac{r^{-\alpha\varepsilon/(p-\varepsilon-1)}}{\alpha\varepsilon/(p-\varepsilon-1)}\right)^{1/(p-\varepsilon)'} \varepsilon^{-\theta/(p-\varepsilon)}$$

$$\leq Cr^{-\alpha\varepsilon/(p-\varepsilon)}\varepsilon^{-1/(p-\varepsilon)'-\theta/(p-\varepsilon)}$$

$$\leq Cr^{-\alpha\varepsilon/(p-\varepsilon)}\varepsilon^{-1/p'-\theta/p}$$

Now, taking $\varepsilon = \min\{(p-1)/2, (\log(2d_G/r))^{-1}\}$, we find

$$\int_{G\backslash B(x,r)} |x-y|^{\alpha-n} f(y)\, dy \leq C\left(\log(2d_G/r)\right)^{1/p'+\theta/p}$$

which gives the result.

List of symbols

Symbols		Page
L^q	Duel Lebesgue Space	1
L^2	Hilbert Space	1
BOM	Bounded meem Oscillation	1
L^{p_i,λ_i}	Lebesgue Space	1
$QMMS$	Quasi-Metric Measure Space	1
L^1_μ	Lebesgue space on the Line	2
diam	Diameter	2
sup	Supremum	2
inf	Infimum	2
Re	Real	5
max	Maximum	6
loc	Local	16
min	Minimum	45

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