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Dynamical Systems on Clifford and $K\ddot{a}hler$ Manifolds

الأنظمة الديناميكية على متعددات طيات كليفورد وكهلر

Thesis Submitted for the Fulfillment of the Requirements for the Degree of PhD in Mathematics

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DEDICATION

To whom my words are not enough to express my deep indebtedness, are thanks and gratefulness. Those are my parents, the sustainable source of tenderness, kindness and endless support. Specifically my dedication also extends to my wife, my daughter, my son, sisters, brothers, colleagues, friends, relatives and teachers.

Acknowledgments

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ABSTRACT

Dynamical Systems is the study of the long-term behavior of evolving systems.

In this research we studied Lagrangian and Hamiltonian Dynamical systems using Clifford *Kähler* manifolds. The Clifford *Kähler* analogue of Lagrangian and Hamilton Dynamical systems is introduced. In fact a new dynamics on Clifford *Kähler* manifold has been constructed via some local canonical basis. This construction provides wide applications to Physical equations and their geometrical interpretation.

المستخلص

النظام الديناميكي هو دراسة السلوك طويل الأجل لتطوير الأنظمة.

لقد قمنا في هذا البحث بدراسة الأنظمة الديناميكية للأجرانج وهاملتون بإستخدام متعددات طيات كليفورد وكهلر. تم تقديم نظير كليفورد وكهلر لأنظمة لأجرانج وهاملتون الديناميكية. عملياً. هنالك ديناميكيات جديدة على متعددات طيات كليفورد وكهلر تم بناءها عبر بعض القانوني الأساسي الموضعي. يقدم هذا التركيب تطبيقات واسعة للمعادلات الفيزيائية والتفسيرات الهندسية.

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Chapter Three: Clifford $Kähler$ Manifolds

Chapter Four: Lagrangian Dynamical Systems on Clifford $Kähler$ Manifolds

Chapter Five: Hamiltonian Dynamical Systems on Clifford $Kähler$ Manifolds

INTRODUCTION

It is well-known that modern differential geometry expresses explicitly the dynamics of Lagrangians.

Therefore we explain that if M is an m-dimensional configuration manifold and $L : TM \to R$ is a regular Lagrangian function , then there is a unique vector field ξ on TM such that dynamics equations is determined by:

$$i_{\xi}\Phi_{l} = \mathrm{dE}_{l} \qquad \rightarrow \qquad (1)$$

Where Φ_{l} indicates the Symplectic form.

The Triple$(\mathrm{TM}, \Phi_{l}, \xi)$ is named Lagrangian system on tangent bundle TM.

It is known, there are many studies about Lagrangian mechanics, formalisms, system and equations such as real, complex, paracomplex and other analogues. So, it may be possible to produced different analogues in different spaces.

The goal of finding new dynamics equations is both a new expansion and contribution to science to explain physical events.

Also modern differential geometry explains explicitly the dynamics of Hamilton's. so, if Q is an m-dimensional configuration manifold and $H : T^{*}Q \to R$ is regular Hamilton function , then there is a unique vector field X on $T^{*}Q$ such that dynamic equations are determined by:

$$i_{\mathrm{X}}\Phi = \mathrm{dH} \qquad \rightarrow \qquad (2)$$

Where Φ indicates the symplectic form.

The triple $(\mathrm{T}^{*}\mathrm{Q} , \Phi, \mathrm{X})$ is called Hamilton system on cotangent bundle $T^{*}Q$.

Al last time, there are many studies and books about Hamilton mechanics, formalisms, systems and equations such as real , complex , paracomplex and other analogues.

Therefore it is possible to obtain different analogues in different spaces.

It is know that quaternion's were invented by sir William Rowan Hamilton as an extension to the complex numbers Hamilton's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = ijk = -1 \qquad \rightarrow \qquad (3)$$

If it is compared to the calculus of vectors, quaternion's have slipped in to the realm of obscurity.

They do however still fined use in the computation of rotations.

A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternion's. some physicists hope they will fined deeper understanding of the universe by restating basic principles in terms of quaternion algebra.

It is well-know that quaternion's are useful for representing rotations in both quantum and classical mechanics. It is well-know that Clifford manifold is a quaternion manifold.

So, all properties defined on quaternion manifold of dimension 8n also is valid for Clifford manifold. Hence we may construct mechanical equations on Clifford $K\ddot{a}hler$ manifold.

This is going to be our objective. We need to generalize all the physical concepts treated in quaternionic manifold. The most appropriate set up will be the Clifford $K\ddot{a}hler$ manifold.

This research consists of five chapters as follows:-

Chapter One: Review of differential geometry (revise important concepts in differential geometry that are related to dynamical systems. the applied concepts include Manifolds, vector fields, tensors).

Chapter Two: Calculus on manifolds (differential forms and derivatives on manifolds such as covariant derivative, Lie derivative and exterior derivatives. Grassman algebra for differential forms and the related algebraic operations will be considered. The chapter closes with integration theory on manifolds).

Chapter Three: Clifford $K\ddot{a}hler$ manifolds (complex manifolds, Clifford algebra, Almost Clifford structure, Almost Cliffordian manifolds, connection on almost Cliffordian manifolds, some formula, some theorems, Clifford $K\ddot{a}hler$ manifolds).

Chapter Four: Lagrangian Dynamical systems on Clifford $K\ddot{a}hler$ manifolds.

Chapter Five: Hamiltonian Dynamical systems on Clifford $K\ddot{a}hler$ manifolds.

Chapter One

Review of Differential Geometry

1.1 Topological Spaces and Smooth Manifolds:

Definition 1.1.1

1- A topological space $(X, \mathcal{T})$ consists of a set X together with a collection of subsets, referred to as open sets, such that the following conditions are satisfied:-

(i) the empty set $\emptyset$ and the whole set X are open sets .
(ii) the union of any collection of open sets is itself an open set .
(iii) the intersection of any finite collection of open sets is itself an open set.
2- A function $f: X \to Y$ from a topological space X to topological space Y is **said to be continuous** if $f^{-1}(v)$ is an open set in X for every open set V in Y , where :

$$f^{-1}(v) = \{x \in X: f(x) \in V\} \qquad \to \qquad (1.1.1)$$

Definition 1.1.2

A topological space $(X, \mathcal{T})$ is **said to be a Hausdorff space** if and only if it satisfies the following Hausdorff Axiom: if x and y are distinct points of X then there exist open sets U and V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 1.1.3

A topological space $(X, \mathcal{T})$ is second countable if there exists a countable sub collection $\mathcal{T}$ of $\mathcal{T}_0$ and any open set $U \in \mathcal{T}$ is a union of open sets in $\mathcal{T}_0$.

Definition 1.1.4

1- let $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ be topological spaces. A function $f: X \to Y$ is **said to be a homeomorphism** if and only if the following conditions are satisfied:
(i) the function $\mathrm{f}: \mathrm{X} \to \mathrm{Y}$ is invertible.
(ii) the function $\mathrm{f}: \mathrm{X} \to \mathrm{Y}$ is continuous.
(iii) the inverse function $f^{-1}: X \to Y$ is also continuous .
2 - A function $f: \mathrm{X} \to \mathrm{Y}$ between topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ is homeomorphism if it a one – to – one, onto map and both f and f^{-1} are continuous. Two topological spaces $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ are **homeomorphic** if there homeomorphism $f: X \to Y$ or $X \approx Y$.
3 - **A patch(chart)** on a topological space $(\mathrm{X}, \mathcal{T})$ is a pair (x, U), where U is an open subset of $\mathbb{R}^n$ and $x: \mathrm{U} \longrightarrow x(\mathrm{U}) \subset (\mathrm{X}, \mathcal{T})$ is a homeomorphism of U onto an open set x(U) of $(\mathrm{X}, \mathcal{T})$.

Here x is called the local homeomorphism of the patch, and X(U) the coordinate neighborhood. Frequently, we refer to ‘the patch x when the domain U is understood. Let

$$x_i = u_i \circ x^{-1} : x(U) \longrightarrow \mathbb{R} \qquad \longrightarrow \qquad (1.1.2)$$

for $i = 1, \ldots, n$. Then x_i is called the i^{th} coordinate function and $(x_1, \ldots, x_n)$ is called a system of local coordinates for $(X, \mathcal{T})$. The coordinate functions $x_1, \ldots, x_n$ contain the same information as the local homeomorphism x. Often, we write $x^{-1} = (x_1, \ldots, x_n)$, with the meaning that

$$x^{-1}(p) = (x_1(p), \ldots, x_n(p)), \quad \text{for all } p \in X(U).$$

Definition (Topological Manifolds) 1.1.5

1- A topological manifold of dimension n is a Hausdorff topological space $\mathcal{M}$ which is the union of a countable collection of open sets, where each of the open sets in the collection is homeomorphic to an open set in $n-$dimensional Euclidean space $\mathbb{R}^n$.
2- A topological space in which every point has a neighborhood homeomorphic to (topological disc) is called an n-dimensional (or $n-$ manifold) .

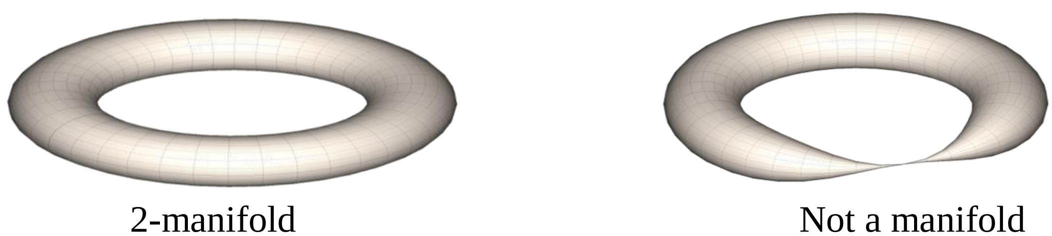


Fig (1.1)

Example 1.1.6

Earth is an example of a 2-manifold

Definition (Smoothness) 1.1.7

1- A real-valued function $f: U \to \mathbb{R}$ defined on an open subset U of a Euclidean space $\mathbb{R}^m$ is said to be smooth if the partial derivatives of f of all orders are defined throughout U.

A function $\varphi: U \to \mathbb{R}^n$ mapping an open subset U of $\mathbb{R}^m$ into $\mathbb{R}^n$ is said to be smooth if its components are smooth functions.

2- A regular parameterized manifold $\sigma : U \to \mathbb{R}^n$ which is A homeomorphism $U \to \sigma(U)$, is called an embedded parameterized manifold .

Definition (Coordinate Charts and Atlas) 1.1.8

1 – **A coordinate chart** on a set X is a subset $U \subseteq X$ together with a bijection

$$\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$$

onto an open set $\varphi(U)$ in $\mathbb{R}^n$.

2- A homeomorphism $\alpha : u_\alpha \to \mathbb{R}^n$ for neighborhood u_α of $x \in \mathcal{X}$ to $\mathbb{R}^n$ is called a chart.

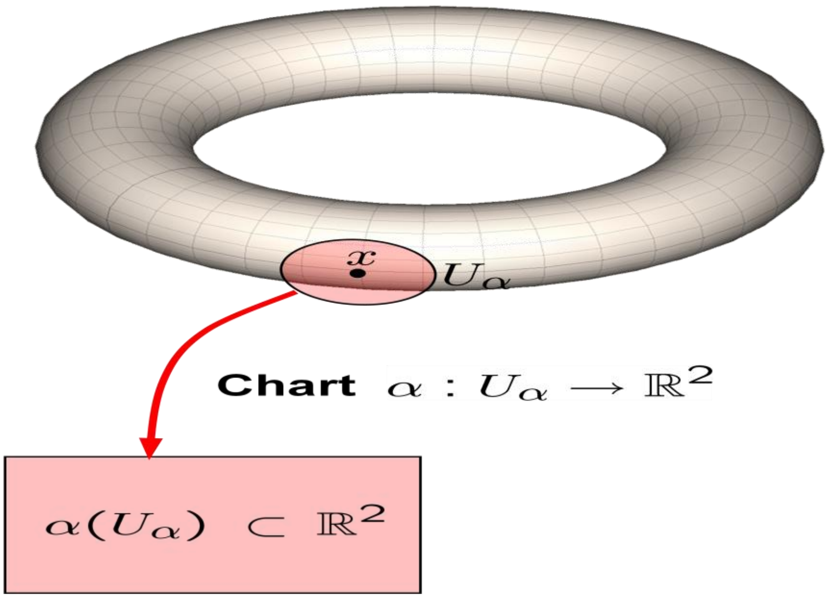


Fig (1.2)

3 – An n-dimensional atlas on X is collection of coordinate charts $\{U_\alpha, \varphi_\alpha\}_{\alpha\in I}$ such that:

* X is covered by the $\{U_\alpha, \}_{\alpha\in I}$.

* for each $\alpha, \beta \in I, \varphi_\alpha(U_\alpha \cap V_\beta)$ is open in $\mathbb{R}^n$.

* the map $\varphi_\alpha {\varphi_\beta}^{-1}: \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is C^∞ with C^∞ inverse.

4 – A collection of charts whose domains cover the manifold is called an atlas .

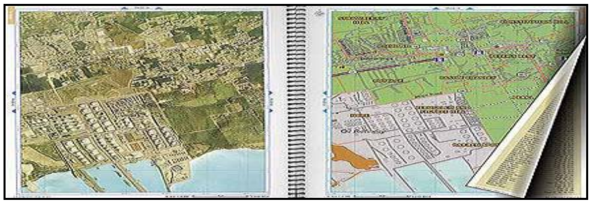
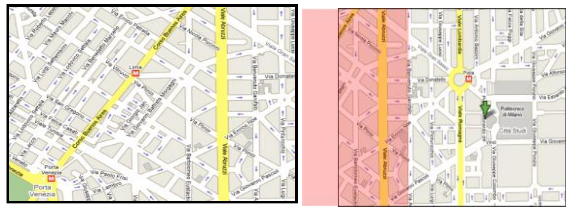


Fig (1.3)

Fig (1.4) chart and Atlases 

Definition 1.1.9

1- An atlas $\mathfrak{A}$ on a topological space $\mathcal{M}$ is a collection of patches on $\mathcal{M}$ such that all the patches map from open subsets of the same Euclidean space $\mathbb{R}^n$ into $\mathcal{M}$, and $\mathcal{M}$ is the union of all the $x(\mathcal{U})$ such that $(x, \mathcal{U}) \in \mathfrak{A}$.

A topological space $\mathcal{M}$ equipped with an atlas is called a topological manifold.

Let $\mathfrak{A}$ be an atlas on a topological space $\mathcal{M}$. Notice that if $(x, \mathcal{U})$ and $(y, \mathcal{V})$ are two patches in A such that $x(\mathcal{U}) \cap y(\mathcal{V}) = \mathcal{W}$ is a nonempty subset of M, then the map

$$x^{-1} \circ y : y^{-1}(W) \longrightarrow x^{-1}(W) \qquad \longrightarrow \qquad (1.1.3)$$

Is a homeomorphism between open subset of $\mathbb{R}^n$. We call $x^{-1} \circ y$ a change of coordinate.
2- Two atlases $\{(U_\alpha, \varphi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$ are compatible if their union is an atlas.
3- A differentiable structure on X is an equivalence class of atlases.
4- An n-dimensional differentiable manifold is a space X with a differentiable structure.

Definition (Smooth Atlases) 1.1.10

1-Let (V, φ) and (W, ψ) be continuous charts on a topological manifold $\mathcal{M}$ of dimension n, and let $\theta : \varphi(V \cap W) \to \psi(V \cap W)$ be the Homeomorphism from $\varphi(V \cap W)$ to $\psi(V \cap W)$ characterized by the requirement that $\theta(\varphi(m)) = (m)$ for $m \in V \cap W$.
The continuous charts (V, φ) and (W, ψ) are said to be smoothly compatible if and only if this transition function:
$\theta : \varphi(V \cap W) \to \psi(V \cap W)$ is a diffeomorphism.

$$\varphi(m) = \left(y^1(m), y^2(m), \dots, y^n(m)\right) \qquad \text{for all } m \in V$$

And

$$\psi(m) = \left(z^1(m), z^2(m), \dots, z^n(m)\right) \qquad \text{for all } m \in W$$

2- Let $\mathcal{M}$ be a topological manifold of dimension n. A smooth atlas on $\mathcal{M}$ is a collection of continuous charts on $\mathcal{M}$ where the domains of the charts cover $\mathcal{M}$, and where any two charts belonging to the atlas are smoothly compatible.

3 - Let M be a manifold of dimension n that is provided with some smooth atlas $\mathcal{A}$, let f: U $\to \mathbb{R}$ be a real-valued function defined over some open set U in $\mathcal{M}$, and let p ∈ U. The function f is said to be smooth around p with respect to the smooth atlas A if and only if there is a chart φ(V ∩ W) belonging to this smooth atlas and a smooth function F: φ(U ∩ V) → ℝ such that p ∈ V and f(m) = F(φ(m)) for all m ∈ U ∩ V.

1.2 Differentiable Manifolds

Definition (Chart and Local Coordinates) 1.2.1

1-A local chart on M is the pair (U_i, φ) consist of

(i) An open U_i of M .

(ii) A homeomorphism φ of U_i onto an open subset $\varphi(U_i\ )$ of $\mathbb{R}^n$ the open U_i is called domain of the chart .

2- The local chart X^i of a point p belonging to the domain U of a chart (U, φ) of M are coordinates of point we denote by $\varphi(p)$ of $\mathbb{R}^n$ $\quad \varphi(p) = (X^1, \dots, X^n)$.

Definition (Differentiable Manifold Structure) 1.2.2

1- An atlas of class C^q on M is the family of chart (U_i, φ_i) such that:

(i) The domain U_i of charts make up a covering of:

$$\bigcup_{i\epsilon I} U_i \supseteq \mathrm{M}$$

(ii) Any two chart (U_i, φ_i) , (U_j, φ_j) of A with $U_i \cap U_j \neq \emptyset\ are\ C^q$ compatible .

2- Suppose M is a Hausdorff space if for any $X \in M$ there exists a neighborhood $U\ of\ X$ such that U is homeomorphism to an open set in $\mathbb{R}^m$. Suppose the homeomorphism is given by :

$$\Phi_u = U \to \Phi_u(u) \subset \mathbb{R}^m$$

Where $\Phi_u(u)$ is open in $\mathbb{R}^m$ we call (U, Φ_u) a coordinate chart of M .

Since Φ_u is a homeomorphism , then for any $y \epsilon U$, we define the coordinates of $U = \Phi_u(y) \in \mathbb{R}^m$ i.e.

$$U^i = \left(\Phi_u(y)\right)^i \qquad i = 1, 2, . . . , m$$

The U^i , $i = 1, 2, . . . , m$ are called the local coordinate of point $y \in U$.

Suppose $(U, \Phi_u)\ and\ (V, \Phi_u)$ are two coordinates charts M .

If $U \cap V \neq \emptyset\ then\ \Phi_u(U \cap V) and\ \Phi_v(U \cap V)$ are two non-empty sets in $\mathbb{R}^m$.

And the map

$$\Phi_u \circ \Phi_v^{-1} : \Phi_u(U \cap V) \to \Phi_v(U \cap V) \qquad \to \qquad (1.2.1)$$

Define homeomorphism between these two open sets with inverse given by :

$$\Phi_u \circ \Phi_v^{-1}$$

These are maps between one set in Euclidean space.

Expressed in coordinate $\Phi_v \circ \Phi_u^{-1}\ and\ \Phi_u \circ \Phi_v^{-1}$ each represent m-real valued functions on an open set of Euclidean space . We may write

$$y^i = f^i(X^1, X^2, \dots, X^m) = \Phi_v \circ \Phi_u^{-1}(X^1, X^2, \dots, X^m)$$
$$X^i = g^i(X^1, X^2, \dots, X^m) = \Phi_u \circ \Phi_v^{-1}(X^1, X^2, \dots, X^m)$$

Say that the coordinate chart(U, Φ_u) and (V, Φ_v) are $C^r - compatible.$

This means that $\Phi_v \circ \Phi_u^{-1}$ and $\Phi_u \circ \Phi_v^{-1}$ are diffeomorphism

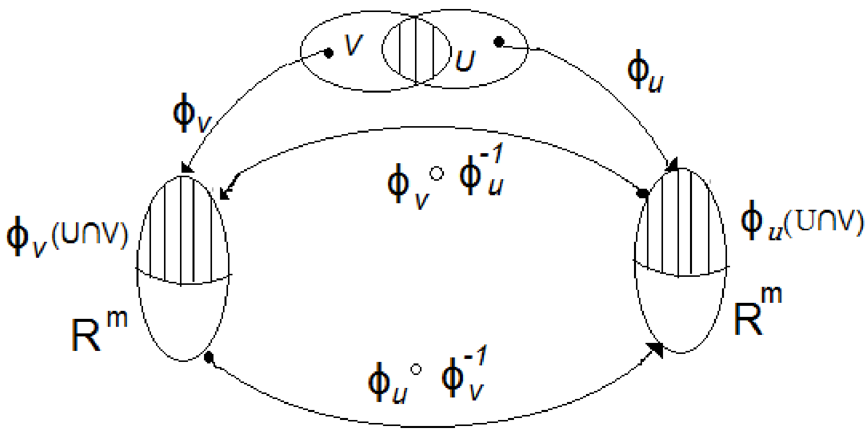


Fig (1.5)

3- Let $\mathcal{M}$ be a Hausdorff space. A differentiable structure of dimension m is a collection of open charts $(U_i, \phi_i)_{i\in\Lambda}$, where Λ is index set, on $\mathcal{M}$ and $\phi_i(U_i)$ is an open subset of $\mathbb{R}^m$ such that the following conditions are satisfied :

(i) $\mathcal{M} = \bigcup_{i\in\Lambda} U_i$

(ii) The mapping $\phi_j \circ \phi_i^{-1}$ is a differentiable mapping of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j) \neq \emptyset$ for each pair $i, j \in \Lambda$.

(iii) The collection$(U_i, \phi_i)_{i\in\Lambda}$ is a maximal family of open charts which satisfy the condition (i) and (ii).

The Housdorff topological space $\mathcal{M}$ with differentiable structure is called differentiable manifold (or manifold or smooth manifold) of dimension m In order to define **complex manifold** of (complex) dimension m, we replace $\mathbb{R}^m$ in the definition of differentiable manifold by m-dimensional complex number space C^m. The condition (ii) is replaced by the condition that the m-coordinates of $\phi_j \circ \phi_i(p)$ should be holomorphic functions of the coordinate of p.

Definition (Differentiable Manifolds) 1.2.3

A differentiable manifold is a pair consisting Hausdorff space with countable basis and every point of space there exists and admissible local chart (U, φ) $such$ $that$ $(U, \varphi) \subset R^n$.

Examples 1.2.4

1-The set M_{nm} of $n \times m$ matrices is a differentiable manifold .

Let $\varphi: M_{n\times m} \to R^{nm}$ be defined by :

$\varphi(A) = (a_{11}, a_{12}, \dots, a_{1m}\,; a_{21}, a_{22}, \dots, a_{2m}; \dots; a_{n1}, a_{n2}, \dots, a_{nm}) \in R^{nm}$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

The map φ is differentiable and φ^{-1} is also differentiable manifold thus $\{M, G\}$ is differentiable structure.

2- The set $GL(n, R)$ of all $n \times n$ matrices which are non-singular $(\det(A) \neq 0, \forall\ GL(n, R))$called the general linear group is a differentiable manifold , we need to proof that $GL(n, R)$ is open in M_{nm}. For let $f: M_{nm} \rightarrow \mathbb{R}$ be define by $f(A) = \det(A)$ is continuous function and therefore $f^{-1}(0)$ is closed in M_{nm} . Since M_{nm} is already a differentiable manifold.

Definition 1.2.5

A differentiable manifold is orientable if there is one atlas $(U_i, \varphi_i)_{i\in 1}$ such as the Jacobein of every coordinate transformation $\Phi_j \circ \Phi_{j-1}$ is positive at every point .

1.3 Differentiable Functions on Manifolds

Definition 1.3.1

Let f : W $\rightarrow \mathbb{R}$ be a function defined on an open subset $\mathcal{W}$ of a differentiable manifold $\mathcal{M}$. We say that f is **differentiable** at p $\in \mathcal{W}$, provided that for some patch x : U $\rightarrow \mathcal{M}$ with U $\subset \mathbb{R}^n$ and p $\in$ x(U) $\subset \mathcal{W}$, the composition $f \circ$ x: U $\rightarrow \mathbb{R}^n$ is differentiable (in the ordinary Euclidean sense) at $x^{-1}(p)$. If f is differentiable at all points of $\mathcal{W}$, we say that f is differentiable on $\mathcal{W}$.

Lemma 1.3.2

The definition of differentiability of a real-valued function on a differentiable manifold does not depend on choice of patch .

Definition 1.3.3

Let $\mathcal{M}$ be a differentiable manifold. We put

$$C^\infty = \mathfrak{I}(\mathcal{M}) = \{ f : \mathcal{M} \rightarrow \mathbb{R} \mid f \text{ is differentiable} \}$$

We call $\mathfrak{I}(\mathcal{M})$ the algebra of real valued differentiable functions $\mathcal{M}$.

For a, b $\in \mathbb{R}$ and $f, g \in \mathfrak{I}(\mathcal{M})$ the functions $af + bg$ and fg are defined by :

$(af + bg)(x) = af(x) + bg(x)$ and $(fg)(x) = f(x)g(x)$

for $x \in \mathcal{M}$.

Also, we identify any $a \in \mathbb{R}$ with the constant function a given by

$a(x) = a$ for $x \in \mathcal{M}$.

Let us note some of the algebraic properties of $\mathfrak{I}(\mathcal{M})$.

Lemma 1.3.4

1- Let $\mathcal{M}$ be a differentiable manifold. Then $\Im(\mathcal{M})$ is a commutative ring with identity and an algebra over the real numbers $\mathbb{R}$.

2- Let $\mathcal{W} \subset \mathcal{M}$ be an open neighborhood of $p \in \mathcal{M}$, and suppose that $f \in \Im(\mathcal{M})$. Then there exist $\tilde{f} \in \Im(\mathcal{M})$. and an open set $\mathcal{P}$ with $p \in \mathcal{P} \subseteq \mathcal{W}$ such that $\tilde{f}\,|\mathcal{P} = f\,|\mathcal{P}$. We call f a globalization of f.

Definition 1.3.5

Let $\mathcal{M}, \mathcal{N}$, be **differentiable manifolds**, and let: $\Psi : \mathcal{M} \to \mathcal{N}$ be a map. We say that Ψ is differentiable provided $y^{-1} \circ \Psi \circ x$ is differentiable for every patch $(x, \mathcal{U})$ in the atlas of $\mathcal{M}$ and every patch $(y, \mathcal{V})$ in the atlas of $\mathcal{N}$, where the compositions are defined. A **diffeomorphism** between manifolds $\mathcal{M}$and $\mathcal{N}$ is a differentiable map $\Phi : \mathcal{M} \to \mathcal{N}$ which has a differentiable inverse $\Phi^{-1} : \mathcal{N} \to \mathcal{M}$. If such a map Φ exists, $\mathcal{M}$ and $\mathcal{N}$ are said to be **diffeomorphic**.

A map $\Psi : \mathcal{M} \to \mathcal{N}$ is called a local diffeomorphism provided each $p \in \mathcal{M}$ has a neighborhood W such that $\Psi|\mathcal{W}: \mathcal{W} \to \Psi(\mathcal{W})$ is a diffeomorphism.

Lemma 1.3.6

Suppose $\mathcal{M} \overset{\Phi}{\to} \mathcal{N} \overset{\Psi}{\to} \mathcal{P}$ are differentiable maps between differentiable manifolds. Then the composition $\Psi \circ \Phi: \mathcal{M} \to \mathcal{P}$ is differentiable. If Φ and Ψ are diffeomorphisms, then so is $\Psi \circ \Phi$

$$(\Psi \circ \Phi)^{-1} = \Phi^{-1} \circ \Psi^{-1}$$

The coordinates (5) are examples of differentiable functions $x(U) \to \mathbb{R}$. Moreover.

Lemma 1.3.7

(i) Let $\mathcal{M}$ be an n-dimensional differentiable manifold, and let $x: \mathcal{U} \to \mathcal{M}$ be a patch. Write $x^{-1} = (x_1, x_2, \ldots, x_n)$: Then

(i) x is a differentiable mapping between the manifolds $\mathcal{U}$ and $\mathcal{M}$;

(ii) $x^{-1} : x(\mathcal{U}) \to \mathbb{R}^n$ is differentiable;

(ii) Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a differentiable mapping between manifolds. Then $f \in \Im(\mathcal{N})$. implies $f \circ \Phi \in \Im(\mathcal{M})$.

Definition (Maximal Atlases) 1.3.8

Let $\mathcal{M}$ be a manifold and $\mathcal{U}$ a smooth atlas on . then we define $\mathcal{D}(\mathcal{U})$ as the following set of charts on $\mathcal{M}$

$$\mathcal{D}(\mathcal{U}) = \left\{ \textbf{chart } \mathbf{y}: \mathcal{V} \to \bar{\mathcal{V}} \textbf{ on} \mathcal{M} \;\middle|\; \begin{array}{c} \textbf{for all charts } \mathbf{x}: \mathbf{u_1} \to \mathbf{\acute{u}_1} \textbf{ in the maps} \\ \mathbf{x} \circ \mathbf{y}^{-1} \mathbf{I}_{\mathbf{y}(\mathbf{u} \cup \mathbf{v})} \textbf{ and } \mathbf{y} \circ \mathbf{x}^{-1} \mathbf{I}_{\mathbf{y}(\mathbf{u} \cup \mathbf{v})} \quad \textbf{are smooth} \end{array} \right\}$$

Lemma 1.3.9
Let $\mathcal{M}$ be a manifold and $\mathcal{U}$ a smooth atlas on $\mathcal{M}$. Then $\mathcal{D}(\mathcal{U})$ is a differentiable atlas.

Definition 1.3.10
A smooth structure on a topological manifold is a maximal smooth atlas. a smooth manifold $(\mathcal{M}, \mathcal{U})$ is a topological manifold $\mathcal{M}$ equipped with a smooth structure $\mathcal{U}$. A differentiable manifold is a topological manifold for which there exist differential structures.

1.4 Smooth Manifolds

Definition 1.4.1
A smooth or C^∞ manifold is a topological manifold $\mathcal{M}$ together with a maximal atlas. The maximal atlas is also called a differentiable structure on $\mathcal{M}$. A manifold is said to have dimension n if all of its connected components have dimension n. A manifold of dimension n is also called an n-manifold.

Proposition 1.4.2

Any atlas $\mathfrak{A} = \{U_\alpha, \phi_\alpha\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Examples (of Smooth Manifolds) 1.4.3

Example 1

The Euclidean space$\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, r^2, \dots, r^n)$ where $r^1, r^2, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$

Example 2

Any open subset $\mathcal{V}$ of a manifold $\mathcal{M}$ is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $\mathcal{M}$, then $\{(U_\alpha \cap \mathcal{V}, \phi_\alpha | U_\alpha \cap \mathcal{V}\}$ is an atlas for ,where $\phi_\alpha | U_\alpha \cap \mathcal{V} : U_\alpha \cap \mathcal{V} \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap \mathcal{V}$.

1.5 Smooth Maps on a Manifold

Smooth Functions and Maps 1.5.1

Let $\mathcal{M}$ be a smooth manifold of dimension n. A function $f : \mathcal{M} \to \mathbb{R}$ is said to be C^∞ or smooth at a point p in $\mathcal{M}$ if there is a chart (U, φ) containing p in the atlas of $\mathcal{M}$ such that $f \circ \phi^{-1}$, which is defined on the open subset $\varphi(U)$ of $\mathbb{R}^n$,is C^∞ at $\varphi(p)$.

Definition 1.5.2

Let $F: N \to M$ a map and h a function on M the pull-back of h by F , denoted by F^*h , is the composite function $h \circ F$ be in this terminology, a function f on $\mathcal{M}$ is C^∞ on a chart (U, Φ) if its pullback by $\Phi^{-1}$is C^∞ on the subset $\varphi(U)$ of a Euclidean space.

Definition 1.5.3

The pull-back of the function h by f is :

$$f^*h = h \circ f \qquad \text{where } h \circ f \text{ is composite function .}$$

The mapping f* is so defined :

$$C^\infty(N_m, R) \to C^\infty(M_n, R): h \to f^*h$$

Therefore from a mapping $f: M_n N_m$ we have constructed an induced

Mapping $f^*:C^\infty(N_m , R) \to C^\infty(M_n, R)$.

1.6 The Tangent Structure

Rough Ideas II 1.6.1

Let us suppose that we have two coordinate systems $x = (x^1, x^2, \ldots , x^n)$ and $y = (y^1, y^2, \ldots , y^n)$ defined on some common open set of a differentiable Manifold M .

Let us also suppose that we have two lists of numbers $v^1, v^2, \ldots , v^n$ and $\bar{v}^1, \bar{v}^2, \ldots , \bar{v}^n$ somehow coming from the respective coordinate systems and associated to a point p in the common domain of the two coordinate systems . Suppose that the lists are related to each other by :

$$v^i = \sum_{k=1}^{n} \frac{\partial x^i}{\partial y^k} \bar{v}^k \qquad \rightarrow \qquad (1.6.1)$$

Where the derivatives $\frac{\partial x^i}{\partial y^k}$ are evaluated at the coordinates $y^1(p), y^2(p), \dots, y^n(p)$.

Now if f is a function also defined in a neighborhood of p then the representative functions for f in the respective systems are related by:

$$\frac{\partial f}{\partial x^i} = \sum_{k=1}^{n} \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial f}{\partial \bar{x}^k} \qquad \rightarrow \quad (1.6.2)$$

The chain rule then implies that :

$$\frac{\partial f}{\partial x^i} v^i = \frac{\partial f}{\partial \bar{x}^i} \bar{v}^i \qquad \rightarrow \quad (1.6.3)$$

Thus if we had a list $v^1, v^2, \dots, v^n$ for every coordinate chart on the manifold whose domains contain the point p and related to each other as above then we say that we have **a tangent vector $\boldsymbol{v}$** at $p \in$ M . It then follows that if we define the **directional derivative** of a function f at p in the direction of v by:

$$vf := \frac{\partial f}{\partial x^i} v^i \qquad \rightarrow \qquad (1.6.4)$$

A differentiable curve though $p \in$ M is a map $c: (-a, a) \rightarrow$ M with $c(0) = p$ such that the coordinate expressions for the curve $x^i(t) = \left(x^i \circ c\right)(t)$ are all differentiable.

We then take :

$$v^i := \frac{dx^i}{dt}(0) \qquad \rightarrow \qquad (1.6.5)$$

For each coordinate system X $= (x^1, x^2, \dots, x^n)$ with p in its domain .

This gives a well defined tangent vector $v\ at\ p$ called the velocity of c at $t = 0$.

We denote this by $\acute{c}(0)\ \ or\ by\ \frac{dc}{dt}(0)$.

Of course we could have done this for each $t \in (-a, a)$ by defining $v^i := \frac{dx^i}{dt}(t)$ and we would get a smoothly varying family of velocity vectors $\acute{c}(t)$ defined at the points $c(t) \in M$.

Definition (Tangent Vectors) 1.6.2

1-**We define the Tangent vector via charts as follows** : consider the set of all admissible charts $(\mathrm{x}_\alpha, \mathrm{u}_\alpha)_{\alpha \epsilon A}$ *on* M indexed by some set A for convenience .

Next consider the set T of all triples (p, v, α) such that p∈ U_α . Define an equivalence relation so that $(p, v, \alpha) \sim (q, w, \beta)\ iff\ p = q$ and

$$D\left(\mathrm{x}_\beta \circ \mathrm{x}_\alpha^{-1}\right)\Big|_{\mathrm{x(p)}} . \mathrm{v} = \mathrm{w} \qquad \rightarrow \qquad (1.6.6)$$

In other words ,the derivative at X(p) of the coordinate change $\mathrm{x}_\beta \circ \mathrm{x}_\alpha^{-1}$ " identifies '' v with w .

Tangent Vectors are then equivalence classes with the tangent vector at a point p being those equivalence classes represented by triples with first slot occupied by p The set of all tangent vectors at p is written as $\mathrm{T}_p\mathrm{M}$.

The tangent bundle TM is the disjoint union of all the tangent spaces for all points in M .

$$\mathrm{TM} := \bigsqcup_{p \in \mathrm{M}} \mathrm{T}_p\mathrm{M} \qquad \rightarrow \qquad (1.6.7)$$

This viewpoint take son a more familiar appearance in finite dimensions if we use a more classical notation ; let (x , u)and (y , v) two charts containing p in there domains . If an n-tuple $(v^1, v^2, \ldots, v^n)$ represents a tangent vector at p from the point of view of (x , u) and if the n-tuple $(w^1, w^2, \ldots, w^n)$ represents the same vector from the point of view of (y , v) then :

$$w^i = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j}\Big|_{\mathrm{x}(p)} v^i \qquad \rightarrow \qquad (1.6.8)$$

Where we write the change of coordinates as $y^i = y^i(x^1, \ldots, x^n)$ with $1 \le i \le n$.

We can get a similar expression in the infinite dimensional case by just letting $\mathrm{D}(\mathrm{y} \circ \mathrm{x}^{-1})|_{\mathrm{x}(p)}$ be denoted by $\left.\frac{\partial \mathrm{y}}{\partial \mathrm{x}}\right|_{\mathrm{x}(p)}$ then we write :

$$w = \left.\frac{\partial \mathrm{y}}{\partial \mathrm{x}}\right|_{\mathrm{x}(p)} \mathrm{v} \ .$$

2- **We define the Tangent Vector Via curves as follows** : Let p be a point in a C^r manifold with $k > 1$.

Suppose that we have C^r curves c_1 *and* c_2 mapping into manifold M , each with open domains containing $0 \in R$ and with $c_1(0) = c_2(0) = p$.

We say that c_1 tangent to c_2 at p if for all C^r functions $f\colon \mathrm{M} \to R$ we have

$$\left.\frac{d}{dt}\right|_{t=0} f \circ c_1 = \left.\frac{d}{dt}\right|_{t=0} f \circ c_2 \qquad \rightarrow \qquad (1.6.9)$$

This is an equivalence relation on the set of all such curves. Define a tangent vector at p to be an equivalence class $\mathrm{X}_p = [c]$under this relation. In this case we will also write $\acute{c}(0) = \mathrm{X}_p$.

The tangent space $\mathrm{T}_p\mathrm{M}$ is defined to be the set of all tangent vectors at $p \in \mathrm{M}$. The tangent bundle TM is the disjoint union of all the tangent spaces for all points in M . in equation (1.6.7)

The tangent bundle is actually a differentiable manifold itself as we shall soon see.

If $\mathrm{X}_p \in \mathrm{T}_p\mathrm{M}$ for p in the domain of an admissible chart $(\mathrm{U}_\alpha , \mathrm{X}_\alpha)$.

In this chart X_p is represented by a triple (p , v , α) . We denote by $[\mathrm{X}_p]_\alpha$ the principle part v of the representative of X_p .

Equivalence, $[\mathrm{X}_p]_\alpha = \mathrm{D}(\mathrm{X}_\alpha \circ \mathrm{c})|_0$ for any c with $\acute{c}(0) = \mathrm{X}_p$ i.e. $\mathrm{X}_p = [c]$.

3- **Let f be the germ of a function** f :: M → R . let us define the differential of f at p to be a map $df(p)\colon \mathrm{T}_p\mathrm{M} \to \mathrm{R}$ by simply composing a curve c representing a given vector $\mathrm{X}_p = [c]$ with f to get the

$$f \circ c :: \mathrm{R} \to \mathrm{R} \qquad \rightarrow \qquad (1.6.10)$$

Then defined$df(p).X_p = \left.\frac{d}{dt}\right|_{t=0} f \circ c \in R$.

Clearly we get the same answer if we use another function with the same germ at p.The differential at p is also often written as $df|_p$.More generally , if $f :: M \to E$ for some Euclidean space E then $df(p): T_pM \to E$ is defined by same formula .

It is easy to check that $df(p): T_pM \to E$ the composition of the tangent map T_pf defined below and the canonical map $T_yE \cong E$ where $y = f(p)$.

Diagrammatically we have :

$$df(p): T_pM \overset{Tf}{\to} TE = E \times E \overset{pr_1}{\longrightarrow} E \qquad \to \qquad (1.6.11)$$

4- **A derivation of the algebra** $\mathcal{F}_p$ is a map $\mathcal{D} : \mathcal{F}_p \to R$ such that $\mathcal{D}(\widetilde{fg}) = \check{f}(p)\mathcal{D}\breve{g} + \breve{g}(p)\mathcal{D}\check{f}$ for all $\check{f}, \breve{g} \in \mathcal{F}_p$.

We note that the set of all derivations on $\mathcal{F}_p$ is easily seen to be a real vector space and we will denote this by $Der(\mathcal{F}_p)$.

5- **Let** $\mathcal{D}X_p: \mathcal{F}_p \to R$ be given by the rule $\mathcal{D}X_p f = df(p).X_p$.

6- **We define the tangent vectors as derivations** : Let M be a smooth manifold of dimension $n < \infty$.

Consider the set of all (germs of) smooth functions $\mathcal{F}_p$ $at\ p \in M$.

A tangent vector at p is linear map $X_p: \mathcal{F} \to R$ which is also a derivation in the sense that for $f, g \in \mathcal{F}_p$:

$$X_p(fg) = g(p)X_p f + f(p)X_p g \ .$$

Once again the tangent space at p is the set of all tangent vectors at p and the tangent bundle is define by disjoint union as before .

In any event , even in the general case of a C^r Banach manifold with $r \geq 1$ a tangent vector determines a unique derivation written $X_p: f \to X_p f$.

Remark (Very Useful Notation) 1.6.3

This use of the " differential " notation for maps into vector spaces is useful for coordinates expressions .

Let $p \in U$ where (X, U) is a chart and consider again a tangent vector v at p . then the local representative of v in this chart is exactly $dX(v)$.

Interpretations 1.6.4

For simplicity let us assume that M is a smooth (c^{∞}) n-manifold .

1- Suppose that we think of a tangent vector X_p as an equivalence class of curves represented by: $\mathrm{I} \to \mathrm{M}$ with $\mathrm{c}(0) = p$.

We obtain a derivation by defining

$$\mathrm{X}_p f := \frac{d}{dt}\Big|_{t=0} f \circ c$$

2- If X_p is a derivation at p and $\mathrm{U}_\alpha, \mathrm{X}_\alpha = (\mathrm{x}^1, \mathrm{x}^2, \dots, \mathrm{x}^n)$ an admissible

chart with domain containing p , then X_p as a tangent vector ,is represented by the triple (p, v, α) where $\mathrm{v} = (v^1, v^2, \dots, v^n)$ is given by

$$v^i = \mathrm{X}_p x^i \text{ (acting as a derivation)}$$

3- a vector X_p at $p \in \mathrm{M}$ represented by (p, v, α) where $\mathrm{v} \in R^n$ *and* α name's the chart $(\mathrm{X}_\alpha, \mathrm{U}_\alpha)$.

We obtain a derivation by defining :

$$\mathrm{X}_p f = \mathrm{D}(f \circ \mathrm{X}_\alpha^{-1})|_{\mathrm{X}_\alpha(p)} \cdot \mathrm{v}$$

In case the manifold if modeled on R^n then we have more traditional notation :

$$\mathrm{X}_p f = \sum v^i \frac{\partial}{\partial x^i}\Big|_p f \quad . \text{ for } v = (v^1, v^2, \dots, v^n)$$

Definition 1.6.5

For a chart $\mathrm{x} = (x^1, x^2, \dots, x^{\mathrm{n}})$ with domain U containing a point p we define a tangent vector $\left.\frac{\partial}{\partial x^i}\right|_p \in \mathrm{T}_p\mathrm{M}$ by:

$$\left.\frac{\partial}{\partial x^i}\right|_p f = D_i(f \circ \mathrm{x}^{-1})(\mathrm{x}(p)) \qquad \rightarrow \qquad (1.6.12)$$

Alternatively, we may take $\left.\frac{\partial}{\partial x^i}\right|_p$ to be the equivalence class of a coordinate curve. In other words, $\left.\frac{\partial}{\partial x^i}\right|_p$ is the velocity at $\mathrm{x}(p)$ of the curve $t \mapsto \mathrm{x}^{-1}\big(x^1(p), \dots, x^i(p) + t, \dots, x^n(p)\big)$ defined for sufficiently small t.

We may also identify $\left.\frac{\partial}{\partial x^i}\right|_p$ as the vector represented by the triple (p, e_i, α) where e_i is the i^{th} member of the standard basis for R^{n} and α refers to the current chart $\mathrm{x} = \mathrm{x}_\alpha$.

Definition (The Tangent Map) 1.6.6

The first definition of the tangent map of a map$f\colon M_p \rightarrow Nf(p)$ will be considered our main definition but the others are actually equivalent at least for finite dimensional manifolds. Given $\boldsymbol{f}$ and p as above wish to define a linear map $T_pf\colon T_pM \rightarrow T_{f(p)}N$.

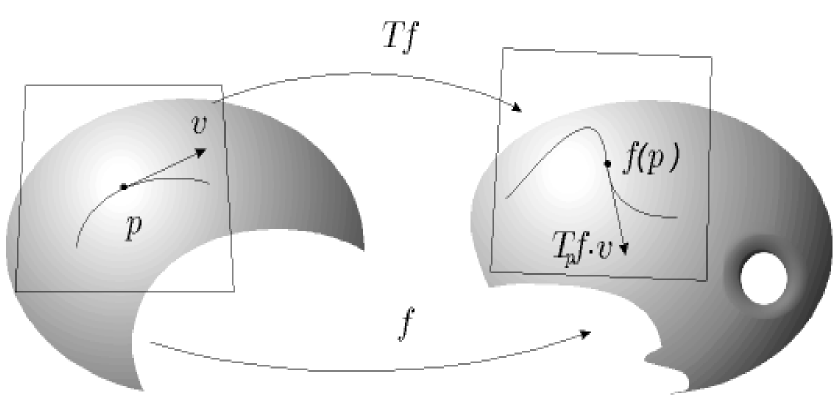


Fig (1.6) The Tangent map as understood via curve transfer.

We have three definition of the Tangent map as the following :

1- **Definition of Tangent Map I**: If we have a smooth function between manifolds :

$$f : \mathrm{M} \to \mathrm{N}$$

and we consider a point $p \in \mathrm{M}$ and its image $q = f(p) \in N$ then we define the tangent map at p by choosing any chart (x, U) containing p and a chart (y, v) containing $q = f(p)$ and then for any $v \in T_p M$ we have the representative $d\mathrm{x}(v)$ with respect to (x, U).

Then the representative of $\mathrm{T}pf \cdot v$ is given by :

$$dy(\mathrm{T}pf \cdot v) = \mathrm{D}(y \circ f \circ x^{-1}) \cdot d\mathrm{x}(v) \qquad \to \quad (1.6.13)$$

This uniquely determines $\mathrm{T}pf \cdot v$ and the chain rule guarantees that this is well defined (independent of the choice of charts).

2- **Definition of Tangent Map II** : If we have a smooth function between manifolds :

$$f : \mathrm{M} \to \mathrm{N}$$

and we consider a point $p \in \mathrm{M}$ and its image $q = f(p) \in N$ then we define the tangent map at p :

$$\mathrm{T}pf : \mathrm{T}p\,\mathrm{M} \to \mathrm{T}q\,\mathrm{N} \qquad \to \quad (1.6.14)$$

in the following way: Suppose that $v \in \mathrm{T}p\,\mathrm{M}$ and we pick a curve c with $c(0) = p$ so that v = [c], then by definition :

$$\mathrm{T}pf \cdot v = [f \circ c] \in \mathrm{T_q N}$$

where $[f \circ c] \in \mathrm{T_q N}$ is the vector represented by the curve $f \circ c$.

An alternative definition for finite dimensional smooth manifolds in terms of derivations is the following.

3- **Definition of Tangent Map III** : Let M be a smooth n-manifold. View tangent vectors as derivation as explained above. Then continuing our set up above and letting g be a smooth germ at $q = f(p) \in N$ we define the derivation $Tpf \cdot v$ by :

$$(Tpf \cdot v)g = v(f \circ g) \qquad \rightarrow \quad (1.6.15)$$

Thus we get a map Tpf called the tangent map (at p).

1.7 The Tangent Bundle

We have defined the tangent bundle of a manifold as the disjoint union of the tangent spaces $TM = \sqcup_{p\in M} T_p M$.

Definition 1.7.1

Give a smooth map $f: M \rightarrow N$ as above then the tangent maps on the individual tangent spaces combine to give a map $Tf: TM \rightarrow TN$
On the tangent bundles that is linear on each fiber called the tangent lift.

Definition 1.7.2

The map $\tau M : TM \rightarrow M$ defined by $\tau M(v)=p$ for every $p \in T_p M$ is called the (tangent bundle) projection map. The TM together with the map $\tau M : TM \rightarrow M$ is an example of a vector bundle.

Proposition1.7.3

TM is a differentiable manifold and $\tau M : TM \rightarrow M$ is a smooth map.

Furthermore, for a smooth map $f: M \rightarrow N$ the tangent map is smooth and the following diagram commutes.

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ \tau M \downarrow & & \downarrow \tau N \\ M & \xrightarrow{f} & N \end{array}$$

Now for every chart (x, U) let $TU = \tau M^{-1}(U)$. The charts on TM are defined using charts from M are as follows :

$$Tx: TU \rightarrow Tx(TU) \cong x(U) \times R^n \qquad \rightarrow \quad (1.7.1)$$

$$Tx: \xi \mapsto (x \circ \tau M(\xi), v) \qquad \rightarrow \quad (1.7.2)$$

Where $v = dx(\xi)$ is the principal part of ξ in the x chart. The chart Tx, TU is then described by the composition

$$\xi \mapsto (\tau M(\xi), \xi) \mapsto (x \circ \tau M(\xi), dx(\xi)) \quad \rightarrow \quad (1.7.3)$$

but $x \circ \tau M(\xi)$ it is usually abbreviated to just x so we may write the chart in the handy form (x, dx).

$$\begin{array}{ccc} TU & \rightarrow & x(U) \times R^n \\ \downarrow & & \downarrow \\ U & \rightarrow & x(U) \end{array}$$

For a finite dimensional manifold and with and a chart $x = (x^1, ..., x^n)$, any vector $\xi \in \tau M^{-1}(U)$ can be written :

$$\xi = \sum v^i(\xi) \frac{\partial}{\partial x^i}\Big|_{\tau M(\xi)} \quad \rightarrow \quad (1.7.4)$$

for some $v^i(\xi) \in R$ depending on ξ . So in the finite dimensional case the chart is just written $(x^1, \dots, x^n, v^1, \dots, v^n)$.

1.8 The Cotangent Bundle

Each T_pM has a dual space T^*M. In case M is modeled on a Euclidean R^n we have $T_pM \approx R^n$ and so we want to assume that $T_p^*M \approx R^{n*}$.

Definition 1.8.1

Let us define the cotangent bundle of a manifold M to be the set

$$T^*M = \sqcup_{p \in M} T^*{}_p M \quad \rightarrow \quad (1.8.1)$$

And define the map $\pi := \pi M: \sqcup_{p \in M} T^*{}_p M \rightarrow M$ to be the obvious projection taking elements in each space $T^*{}_p M$ to the corresponding point . Let $\{U, x\}_{\alpha \epsilon A}$ be an atlas of admissible charts on M . Now endow T^*M with the smooth structure given by the charts:

$$T^*U = \pi M^{-1}(U) \rightarrow T^*x(T^*U) \cong x(U) \times (R^n)^* \quad \rightarrow \quad (1.8.2)$$

where the map $(Tx^{-1})^t$ the contragradient of Tx.

If M is a smooth n-dimensional manifold and $x^1, \dots, x^n$ are coordinate functions coming from some chart on M then the "differentials"

$dx^1|_p , \ldots , dx^n|_p$ are a basis of $T^*_p M$ basis dual to $\frac{\partial}{\partial x^1}\Big|_p , \ldots , \frac{\partial}{\partial x^n}\Big|_p$.
Let$\alpha \in T^*U$. Then we can write :

$$\alpha = \sum a_i(\alpha) dx^i\big|_{\pi M(\alpha)} \qquad \rightarrow \quad (1.8.3)$$

For some numbers $a_i(\alpha)$ depending on .
In fact $a_i(\alpha) = \alpha\left(\frac{\partial}{\partial x^i}\Big|_{\pi(\alpha)}\right)$. So if U , $x = (x^1, \ldots, x^n)$ is a chart on an n-manifold M , then the natural chart (TU, T^*x) defined above is given by :

$$\alpha \mapsto \left(x^1 \circ \pi(\alpha), \ldots, x^n \circ \pi(\alpha) , a_1(\alpha), \ldots, a_n(\alpha)\right)$$

And abbreviated to $(x^1, \ldots, x^n, a_1, \ldots, a_n)$.
Suppose that $(x^1, \ldots, x^n, a_1, \ldots, a_n)$ and $(\overline{x}^1, \ldots, \overline{x}^n , \overline{a}_1, \ldots, \overline{a}_n)$ are two such charts constructed in this way from two charts on U and U' respectively with $U \backslash \acute{U} \neq \emptyset$. Then $T^*U \backslash T^*\acute{U} \neq \emptyset$ and on the overlap we have the coordinate transitions

$$(T\acute{x}^{-1})^* \circ (Tx)^* : x\left(U \backslash \acute{U}\right) \times R^{n*} \rightarrow \acute{x}\left(U \backslash \acute{U}\right) \times (R^n)^* \quad \rightarrow \ (1.8.4)$$

Or write equation (1.8.4) by :

$$(Tx \circ T\acute{x}^{-1})^* : x\left(U \backslash \acute{U}\right) \times R^{n*} \rightarrow \acute{x}\left(U \backslash \acute{U}\right) \times (R^n)^* \quad \rightarrow \ (1.8.5)$$

Notation 1.8.2

The contra-gradient of $D(\acute{x} \circ x^{-1})$ *at* $x \in x\left(U \backslash \acute{U}\right)$ is the map

$$\frac{\partial^* x}{\partial \acute{x}}(x) : \ (R^n)^* \ \rightarrow (R^n)^*$$

Defined by

$$\frac{\partial^* x}{\partial \acute{x}}(x). a = \left(D(x \circ \acute{x}^{-1})\right)^* . a \qquad \rightarrow \qquad (1.8.6)$$

When convenient we also write $\frac{\partial^* x}{\partial \acute{x}}\Big|_{x(p)} . a$.

With this notation we can write coordinate change maps as

$$(x, a) \mapsto \left((\acute{x} \circ x^{-1})(x) , \ \frac{\partial^* x}{\partial \acute{x}}(x)\right) \quad \rightarrow \ (1.8.7)$$

Write $(\acute{x} \circ x^{-1})^i := pr_i \circ (\acute{x} \circ x^{-1})$ and then

$$\acute{x}^i = (\acute{x} \circ x^{-1})^i (x^1 \circ \pi, \ldots, x^n \circ \pi)$$

$$\acute{a}_i = \sum_{k=1}^{n} \left(D(x \circ \acute{x}^{-1})\right)_i^k a_k \quad \rightarrow \quad (1.8.8)$$

And classically abbreviated even further to

$$\acute{x}^i = \acute{x}^i(x^1, \ldots \ x^n)$$

$$\acute{p}_i = p_k \frac{\partial x^k}{\partial \acute{x}^i} \qquad \rightarrow \quad (1.8.9)$$

This is the called "index notation" and does not generalize well to infinite dimensions. The following version is index free and makes sense even in the infinite dimensional case:

$$\acute{\mathrm{x}} = \acute{\mathrm{x}} \circ \mathrm{x}^{-1}(\mathrm{x})$$

$$\acute{\mathrm{a}} = \frac{\partial^* \mathrm{x}}{\partial \acute{\mathrm{x}}}\bigg|_{\mathrm{x}(p)} . \mathrm{a}$$

1-9 Lie Groups and Fiber Bundles:

Definition(The Lie Groups)1.9.1

A nonempty subset $G \subset \mathbb{R}^{n\times n}$ is called a Lie group if it is a submanifold of $\mathbb{R}^{n\times n}$ and a subgroup of $GL(n,\mathbb{R})$, i.e.

$$g\,,h \in G \qquad \Rightarrow \quad gh \in G$$

(where gh denotes the product of the matrices g and h) and

$$g \in G \qquad \Rightarrow \det(g) \neq 0 \; and \; g^{-1} \in G$$

(since $G \neq \emptyset$ it follows from these conditions that the identity matrix$\mathbb{1}$ is an element of G)

Definition1.9.2

A C^∞ differentiable manifold G is called a Lie group if it is a group (abstract group) such that the multiplication map $\mu: G\times G \to G$ and the inverse map $v: G \to G \; given \; by \; \mu(g,h) = gh \; and \; v(g) = g^{-1} \; are \;\; C^\infty$ functions.

If $G \; and \; H$ are Lie groups then so is the product group $G \times H$ where multiplication is $(g_1,h_1).(g_2,h_2) = (g_1g_2,h_1h_2)$. Also, if H is a subgroup of a Lie group G that is also a regular closed sub-manifold then H is a Lie group itself and we refer to H as a (regular) Lie subgroup.

Definition1.9.3

A left action of a Lie group G on a manifold M is a smooth map

$\lambda: G\times M \to M$ such that$\lambda\big(g_1,\lambda(g_2,m)\big) = \lambda(g_1g_2,m) for \; all \; g_1,g_2 \in G$

Define the partial map $\lambda_g: M \to M \; by \; \lambda_g(m) = \lambda(g, m)$ and then the requirement is that $\tilde{\lambda}: g \longmapsto \lambda_g$ is a group homomorphism $G \longrightarrow Diff(M)$. We often write $\lambda(g\,, m) as\; g.m$.

Definition (Lie Group Homomorphism)1.9.4

A smooth map $f: G \to H$ is called a lie group homomorphism if

$$f(g_1 g_2) = f(g_1) f(g_2) \;\; for\; all\; g_1\,, g_2 \in G\; and$$

$$f(g^{-1}) = f(g)^{-1} \;\; for\; all\; g \in G.$$

and an isomorphism in case it has an inverse which is also a lie group homomorphism. A Lie group isomorphism $G \to G$ is called a Lie group automorphism.

Definition (Fiber Bundles)1.9.5

A general $C^r - bundle\; is\; a\; triple\; \xi = (E, \pi, X)$ where $\pi: E \to M$ is a surjective $C^r - map\; of\; C^r - spaces$ (called the bundle projection).

For each $p \in X\; the\; subgroup\; E_p \coloneqq \pi^{-1}(p)$ is called the fiber over p. The space E is called the total space and X is the base space. If $S \subset X$ is a subspace we can always form the restricted bundle $(E_S\,, \pi_S\,, S)$ where $E_S = \pi^{-1}(S)\; and\; \pi_S = \pi|_S$ is the restriction.

Definition1.9.6

A (C^r-) section of a general bundle $\pi_E: E \to M\; is\; a\; (C^r-)$ map $s: M \to E$ such that $\pi_E \circ s = id_M$. In other words, the following diagram must commute:

$$\begin{array}{cccc} s & & E & \\ & \nearrow & \downarrow & \pi_E \\ M & \longrightarrow & M & \\ & id & & \end{array}$$

The set of all $C^r - sections$ of a general bundle $\pi_E: E \to M$ is denoted by $\Gamma^k(M, E)$. We also define the notion of a section over an open set U in M is the obvious way and these are denoted by $\Gamma^k(U, E)$.

Notation1.9.7

We shall often abbreviate to just $\Gamma(U,E) or\ even\ \Gamma(E)$ whenever confusion is unlikely. This is especially true in case $k=\infty$ (smooth case) or $k=0$ (continuous case).

Now there are two different ways to treat bundles as a category:

The Category Bundle.1.9.8

Actually, we should define the Categories Bun_k ; $k=0,1,\dots,\infty$ and then abbreviate to just "Bun" in cases where a context has been establish and confusion is unlikely. The objects of $Bun_k\ are\ C^r-fiber\ bundles$

Definition1.9.9

A morphism from $Hom_{Bun_k}(\xi_1,\xi_2)$, also called a bundle map from a $C^r-fiber\ bundle\ \xi_1 \coloneqq (E_1,\pi_1,X_1)$ to another fiber bundle $\xi_2 \coloneqq (E_2,\pi_2,X_2)$ is a pair of $C^r-maps(\bar{f},f)$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \overset{\bar{f}}{\to} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \overset{f}{\to} & X_2 \end{array}$$

If both maps are $C^r-isomorphisms\ we\ call\ the\ map\ a$ (C^r-) bundle isomorphism.

Definition1.9.10

Two fiber bundle $\xi_1 \coloneqq (E_1,\pi_1,X_1)\ and\ \xi_2 \coloneqq (E_2,\pi_2,X_2)$ are equivalent in Bun_k or isomorphic if there exists a bundle isomorphism from $\xi_1\ to\ \xi_2$.

Definition1.9.11

A morphism from $Hom_{Bun_k(X)}(\xi_1,\xi_2)$, also called a bundle map over X from $a\ C^r-fiber\ bundle\ \xi_1 \coloneqq (E_1,\pi_1,X_1)$ to another fiber bundle $\xi_2 \coloneqq (E_2,\pi_2,X_2)$ is $a\ C^r-map\ \bar{f}$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \overset{\bar{f}}{\rightarrow} & E_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

If both maps are $C^r - isomorphisms$ we call the map a (C^r-) bundle isomorphism over X (also called a bundle equivalence).

Definition1.9.12

Two fiber bundles $\xi_1 \coloneqq (E_1, \pi_1, X_1)$ and $\xi_2 \coloneqq (E_2, \pi_2, X_2)$ are equivalent in $Bun_k(X)$ or isomorphic if there exists a $(C^r -)$ bundle isomorphism over X $from$ ξ_1 to ξ_2.

1-10 Vector Fields and 1-Forms

Definition of (Vector Fields and 1-Forms)1.10.1

1- A smooth vector field is a smooth map $X: M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$. We often write $X(p) = X_p$. In other words, a vector field on M is a smooth section of the tangent bundle $\mathcal{T}_M: TM \to M$.
The map X being smooth is equivalent to the requirement that $Xf: M \to \mathbb{R}$ $given$ by $p \mapsto X_pf$ is smooth whenever $f: M \to \mathbb{R}$ is smooth.
If (x, U) is a $chart$ and X a vector field defined on U then the local representation of X is $\mathrm{x} \longmapsto (\mathrm{x}, \mathrm{X}(\mathrm{x}))$ where the principal representative(or principal part) X is given by projecting $T\mathrm{x} \circ X \circ \mathrm{x}^{-1}$ onto the second factor in $TE = E \times E$:

$$\mathrm{x} \longmapsto \mathrm{x}^{-1}(\mathrm{x}) = p \longmapsto \mathrm{X}(\mathrm{p}) \longmapsto \mathrm{Tx.X(p)}$$
$$= \left(\mathrm{x}(p), \mathrm{X}\big(\mathrm{x}(p)\big)\right) = (\mathrm{x}, \mathrm{X}(\mathrm{x})) \longmapsto \mathrm{X}(\mathrm{x})$$

In finite dimensions one can write $X(\mathrm{x}) = (v_1(\mathrm{x}), \dots, v_n(\mathrm{x}))$.

2- Let $f: M \to \mathbb{R}$ be a smooth function with $r \geq 1$. The map $df: M \to T^*M$ defined by $p \longmapsto df(p)$ where $df(p)$ is differential at p as defined in 3.3. is a 1-form called the differential of f.

Notation1.10.2

The set of all smooth vector fields on M is denoted by $\Gamma(M, TM)$ or by the common notation $\mathfrak{X}(M)$. Smooth vector fields may at times be defined only on some open set so we also have the notation $\mathfrak{X}(U) = \mathfrak{X}_M(U)$ for these fields. The map $U \mapsto \mathfrak{X}_M(U)$ is a presheaf (in fact a sheaf).

A (smooth) section of the cotangent bundle is called a co-vector field or also a smooth 1-form. The set of all $C^r\ 1 - forms$ is denoted by $\mathfrak{X}^{r^*}(M)$ with the smooth 1-forms denoted by $\mathfrak{X}^*(M)$.

$\mathfrak{X}^*(M)$ is module over the ring of function $C^\infty(M)$ with a similar statement for the C^rcase.

Pull back and Push Forward of Functions and 1-Forms 1.10.3

If $\phi: N \to M$ is a C^r map with $r \geq 1\ and\ f: M \to \mathbb{R}\ a\ C^r$ function we define the pullback of $f\ by\ \phi$ as

$$\phi^* f = f \circ \phi$$

and the pullback of a 1-form $\alpha \in \mathfrak{X}^*(M)$ by $\phi^*\alpha = \alpha \circ T\phi$. To get a clear picture of what is going on we could view things at a point and we have $\phi^*\alpha|_p. v = \alpha|_{\phi\ (p)}. (T_p\phi. v)$.

The pull-back of a function or 1-form is defined whether $\phi: N \to M$ happens to be a diffeomorphism or not. On the other hand, when we define the pull-back of a vector field in a later section we will only be able to do this if the map that we are using is a diffeomorphism. Push-forward is another matter.

Definition1.10.4

If $\phi: N \to M\ is\ a\ C^r$ diffeomorphism with $r \geq 1$. The push-forward of a function $\phi_* f\ by\ \phi_* f(p) \coloneqq f(\phi^{-1}(p))$. We can also define the push-forward of a 1-form as $\phi_*\alpha = \alpha \circ T\phi^{-1}$.

It should be clear that the pull-back is the more natural of two when it comes to forms and functions but in the case of vector fields this is not true.

Lemma1.10.5

The differential is natural with respect to pullback. In other words, if $\phi : N \to M$ *is a* C^r *map with* $r \geq 1$ *and* $f: M \to \mathbb{R}$ *a* C^r function with $r \geq 1$ *then* $d(\phi^* f) = \phi^* df$. Consequently, differential is also natural with respect to restrictions

Proof

Let v be a curve such that $\dot{c}(0) = v$. Then

$$d(\phi^* f)(v) = \frac{d}{dt}\Big|_0 \phi^* f\big(c(t)\big) = \frac{d}{dt}\Big|_0 f(\phi\big(c(t)\big))$$

$$= df \frac{d}{dt}\Big|_0 \phi\big(c(t)\big) = df(T\phi . v)$$

As for the second statement (besides being obvious from local coordinate expressions) notice that if U is open in M *and* $\iota: U \hookrightarrow M$ is the inclusion map (identity map id_M) restricted to U then $f|_U = \iota^* f$ *and* $df|_U = \iota^* df$ this part follows from the first part.

We also have the following familiar looking formula in the finite dimensional case

$$df = \sum \frac{\partial f}{\partial x^i} dx^i$$

which means that at each $p \in U_\alpha$

$$df(p) = \sum \frac{\partial f}{\partial x^i}\Big|_p dx^i\Big|_p.$$

In general, if we have a chart U, x then we may write

$$df = \frac{\partial f}{\partial \mathrm{x}} d\mathrm{x}$$

We have seen this before. All that has happened is that p is allowed to vary so we have a field.

For any open set $U \subset M$, the set of smooth functions defined $C^\infty(U)$ on U is an algebra under the obvious linear structure $(af + bg)(p) := af(p) + bg(p)$ and obvious multiplication; $(fg) := f(p)g(p)$. When we think of $C^\infty(U)$ in this way we sometimes denote it by $C^\infty(U)$. The assignment $U \mapsto \mathfrak{X}_M(U)$ is a presheaf of modules over C^∞.

Frame Fields1.10.6

If U ,x is a chart on a smooth n-manifold then writing x $= (x^1, \dots, x^n)$ we have vector fields defined on U by

$$\frac{\partial}{\partial x^i} : p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

Such that the together the $\frac{\partial}{\partial x^i}$ form a basis at each tangent space at point in U. We call the set of fields $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ a holonomic frame field over U. If X is a vector field defined on some set including this local chart domain U then for some smooth functions X^i defined on U we have

$$X(p) = \sum X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

Or in other words

$$X|_U = \sum X^i \frac{\partial}{\partial x^i}.$$

Notice also that $dx^i : p \to \left.dx^i\right|_p$ defines a field of co-vectors such that $dx^1|_p, \dots, dx^n|_p$ forms a basis of T_p^*M for each $p \in U$. The fields form what is called a holonomic co-frame over U. In fact, the functions X^i are given by $dx^i(X) : p \to \left.dx^i\right|_p(X_p)$.

Notation1.10.7

We will not usually bother to distinguish X from its restrictions and so we just write $X = \sum X^i \frac{\partial}{\partial x^i}$ or using the Einstein summation convention X= $X^i \frac{\partial}{\partial x^i}$.

It is important to realize that it is possible to have family of fields that are linearly independent at each point in their mutual domain and yet are not necessarily of the form $\frac{\partial}{\partial x^i}$ for any coordinate chart.

Definition1.10.8

Let $F_1, F_2, \dots, F_n$ be smooth vector fields defined on some open subset U of a smooth n-manifold M. If $F_1(p), F_2(p), \dots, F_n(p)$ from a basis for T_pM for each $p \in U$ then we say that $F_1, F_2, \dots, F_n$ is a (non-holonomic) frame field over U.

If $F_1, F_2, \dots, F_n$ is frame field over $U \subset M \; and \; X$ is a vector field defined on U then we may write

$$X = \sum X^i F_i \quad on \;\; U$$

For some functions $X^i \;\; defined \; on \; U$. Taking the dual basis in T_p^*M for each $p \in U$ we get a (non-holonomic) co-frame field $F^1, \dots, F^n$ and then $X^i = F^i(X)$.

Definition1.10.9

A derivation on $C^\infty(U)$ is a linear map $\mathcal{D}: C^\infty(U) \to C^\infty(U)$ such that

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g)$$

A C^∞ vector field on U may be considered as a derivation on $\mathfrak{X}(U)$ where we view $\mathfrak{X}(U)$ as a module over the ring of smooth functions $C^\infty(U)$.

Definition1.10.10

To a vector field $X \; on \; U$ we associate the map $\mathcal{L}_X: \mathfrak{X}_M(U) \to \mathfrak{X}_M(U)$ defined by

$$(\mathcal{L}_X f)(p) \coloneqq X_p . f$$

and called the Lie derivative on functions.

It is easy to see, based on the Leibnitz rule established for vector X_p in individual tangent spaces, that $\mathcal{L}_X$ is a derivation on $C^\infty(U)$. We also define the symbolism $''Xf''$, where $X \in \mathfrak{X}(U)$, to be an abbreviation for the function $\mathcal{L}_X f$. We often leave out parentheses and just write $Xf(p)$ instead of the more careful $(Xf)(p)$ and so, for example the derivation law (Leibnitz rule) reads.

$$X(fg) = fXg + gXf.$$

1.11 Vectors Fields and Flows

Definition (Vector Field) 1.11.1

1- Let $M \subset R^k$ be a smooth m-manifold. A (smooth) vector field on M is a smooth map $X : M \to R^k$ such that $X(p) \in T_p M$ for every $p \in M$. The set of smooth vector fields on M will be denoted by :

$$\text{Vect}(M) \coloneqq \{X : M \to R^k \mid X \text{ is smooth}, X(p) \in T_p M \ \forall p \in M\} \to (1.11.1)$$

2- A vector field on a smooth manifold M is a smooth family of tangent vectors $v(x) \in T_x M$ parameterized by the points x of the manifold.

Locally a vector field is written in the form:

$$v = v_1(x)\partial_{x_1} + \cdots + v_n(x)\partial_{x_n}, \quad \partial_{x_i} = \frac{\partial}{\partial_{x_i}} \ , \to \ (1.11.2)$$

Where $v_1, \ldots, v_n$ are smooth functions.

The derivative $f \mapsto vf = \sum v_i \frac{\partial f}{\partial x_i}$ along the vector field determines an operation on the ring of smooth functions which is ℝ-linear and satisfies the Leibniz rule:

$$v(fg) = f(vg) + g(vf) \qquad \to \qquad (1.11.3)$$

Any such operation (i.e. linear and satisfying the Leibniz rule) is called derivation .

3- The phase curve of a vector field v is a curve $\gamma : I \to M$ $(I \subset \mathbb{R}$ is an open interval) which is tangent to the field at any point . Finding phase curves is equivalent to solving ODE

$$\dot{\gamma} = v(\gamma(t)), \gamma = (\gamma_1(t), \ldots, \gamma_n(t)), v = (v_1(x), \ldots, v_n(x)), \ \dot{} = \frac{d}{dt} \to (1.11.4)$$

According to the main of ODE , for every initial point $x \in M$ there exists a unique phase curve passing through this point (and defined for small values of t) . The phase curve through the point x is denoted by: $t \mapsto g_v^t x$.

If M is compact then the phase curve extends to the whole range $\mathbb{R}$ of values of t . In general , this is not always possible . If every phase curve is defined for all values of t then these curves define the phase flow, a one-parameter family of diffeomorphisms:

$$g_v^t : M \to M \ , \ t \in \mathbb{R} \qquad \to \qquad (1.11.5)$$

Theorem 1.11.2

1- The flows g_u^t , g_v^s commute for all t, s iff the commutator $[u, v]$ of the fields u , v vanishes identically on.

2- For any fields u , v , initial point $x \in M$, and small values t , s one has

$$g_v^{-s} \circ g_u^{-t} \circ g_v^s \circ g_u^t x = x - ts[u, v] + \cdots,$$

Where the dots denote the terms of higher order in t , s .

Corollary 1.11.3

Let $\xi_1 , \dots , \xi_n$ be a collection of vector fields which are linearly independent in a neighborhood of some point on an n-dimensional manifold. The following properties are equivalent:

(i) there exists a coordinate system $(x_1, \dots, x_n)$ such that $\xi_i = \frac{\partial}{\partial x_i}$;

(ii) the fields ξ_i commute pairwise ,$[\xi_i , \xi_j] \equiv 0$.

Theorem 1.11.4

Let $M \subset \mathbb{R}^k$ be smooth m-manifold and $X \in Vect(M)$ be a smooth vector field on M. Fix a point $p_0 \in M$. Then the following holds.

(i) There is an open interval $I \subset \mathbb{R}$ containing 0 and a smooth curve $\gamma: I \to M$ satisfying the equation:

$$\dot{\gamma}(t) = X(\gamma(t)) \quad , \quad \gamma(0) = p_0 \qquad \to \quad (1.11.6)$$

For every$t \in I$.

(ii) If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$are two solutions of (1.11.6) on open intervals I_1 and I_2 containing 0, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.

Definition(The Flow of a Vector Field)1.11.5

Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in Vect(M)$ be a smooth vector field on M . For $p_0 \in M$ the maximal existence interval of p_0 is the open interval

$$\boldsymbol{I(p_0) := \cup \left\{ I \middle| \begin{array}{c} I \subset \mathbb{R} \text{ is an open interval containing } 0 \\ \text{and there is a solution } x: I \to M \text{ of } (1.11.6) \end{array} \right\}}$$

By theorem (1.11.4) equation (1.11.6) has a solution $\gamma: I(\boldsymbol{p_0}) \to M$.

The flow of X is the map:

$$\emptyset : D \to M$$

Defined by

$$D := \{(t, p_0) | p_0 \in M , t \in I(p_0)\}$$

And

$\emptyset(t, p_0) := \gamma(t)$, where $\gamma: I(p_0) \to M$ is the unique solution of (1.11.6).

Theorem1.11.6

Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in Vect(M)$ be a smooth vector field on M . Let $\emptyset : D \to M$ be the flow of X.

Then the following holds:

(i) D is an open subset of $\mathbb{R} \times M$.

(ii) the map $\emptyset : D \to M$ is smooth .

(iii) Let $p_0 \in M$ and $s \in I(p_0)$. Then

$$I(\emptyset(s, p_0)) = I(p_0) - s \qquad \to (1.11.7)$$

And for every $t \in \mathbb{R}$ with $s + t \in I(p_0)$, we have

$$\emptyset(s + t, p_0) = \emptyset(t, \emptyset(s, p_0)) \quad \to (1.11.8)$$

The Lie Bracket 1.11.7

Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be smooth m-manifolds and $X \in Vect(M)$ be smooth vector field on M . If $\psi: N \to M$ is a diffeomorphism ,the pullback of X under ψ is the vector field on N defined by :

$$(\psi^* X)(q) := d\psi(q)^{-1} X(\psi(q)) \quad \to \quad (1.11.9)$$

for $q \in N$. If $\emptyset: M \to N$ is a diffeomorphism then the pushforward of X under ψ is the vector field on N defined by :

$$(\emptyset_* X)(q) := d\emptyset(\emptyset^{-1}(q)) X(\emptyset^{-1}(q)) \quad \to (1.11.10)$$

For $q \in N$.

Definition 1.11.8

Let $M \subset \mathbb{R}^k$ be a smooth manifold and $X , Y \in Vect(M)$ be a smooth vector fields on M . the Lie bracket of X and Y is the vector field $[X, Y] \in Vect(M)$ defined by :

$$[X, Y](p) := dX(p)Y(p) - dY(p)X(p) \quad \to \quad (1.11.11)$$

Lemma 1.12.9

Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be a smooth manifolds . Let X, Y, Z be smooth vector field on M and let $\psi: N \to M$ be a diffeomorphism . Then

$$\psi^*[X,Y] = [\psi^* X, \psi^* Y] \qquad \rightarrow \quad (1.11.12)$$

$$[X,Y] + [Y,X] = 0 \qquad \rightarrow \quad (1.11.13)$$

$$\big[X,[Y,Z]\big] + \big[Y,[Z,X]\big] + \big[Z,[X,Y]\big] = 0 \quad \rightarrow \quad (1.11.14)$$

The last equation is called the **Jacobi identity.**

Definition 1.11.10

A **Lie algebra** is a real vector space g equipped with skew symmetric bilinear map $g \times g \rightarrow g : (\xi\ , \eta) \ \mapsto [\xi\ , \eta]$ that satisfies the Jacobi identity.

Remark 1.11.11

There is a linear map

$$\mathbb{R}^{m \times m} \rightarrow Vect(\mathbb{R}^m) : \ \xi \mapsto X_\xi$$

Which assigns to a matrix $\xi \in gl(m, \mathbb{R})$ the linear vector field $X_\xi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ given by $X_\xi(x) := \xi x$ for $x \in \mathbb{R}^m$. This map preserves the Lie bracket, i.e. $\big[X_\xi, X_\eta\big] = X_{[\xi,\eta]}$, hence is a **Lie algebra homomorphism** .

1.12 Tensors

Definition(Tensors and Tensor Products)1.12.1

1- Let $V_1, \dots, V_r$, and W be real vector spaces .

A mapping $T : V_1 \times \dots \times V_r \rightarrow W$ is **called a multilinear mapping** if :

$$T(\mathrm{v}_1, \dots, \lambda \mathrm{v}_i + \mu \acute{\mathrm{v}}_i, \dots \mathrm{v}_r) =$$
$$\lambda T(\mathrm{v}_1, \dots, \mathrm{v}_i, \dots, \mathrm{v}_r) + \ \mu T(\mathrm{v}_1, \dots, \acute{\mathrm{v}}_i, \dots, \mathrm{v}_r) \ , \ \forall\, i \quad \rightarrow \quad (1.12.1)$$

and for all $\lambda, \mu \in \mathbb{R}$ i.e. f is linear in each variable v_i separately .

- Now consider the special case that $W = \mathbb{R}$, $then$ T becomes a multilinear function , or form , and a generalization of linear functions .

- If in addition $V_1 = \cdots = V_r = V$, then

$$T : V \times \dots \times V \rightarrow \mathbb{R}$$

Is a multi-linear function on$V$. And is **called a covariant r-tensor** on V.

The number of copies **r** is called the rank of T. The space of covariant r-tensors on V is denoted by $T^r(V)$, which clearly is real vector space using the multi-linearity property in equation (1.12.1).

- In particular we have that $T^0(V) \cong \mathbb{R}$, $T^1(V) = V^*$ and $T^2(V)$ is the space of bilinear forms on V. If we consider the case $V_1 = \cdots = V_r = V^*$, then

$$T: V^* \times \ldots \times V^* \to \mathbb{R}$$

is a multi-linear function on V^*,and is **called a contra-variant r-tensor** on V . The space of contravariant r-tensors on V is denoted by $T_r(V)$. Here we have that $T_0(V) \cong \mathbb{R}$, and $T_1(V) = V^{**} \cong V$.

- Since multi-linear functions on V can be multiplied, i.e. given vector spaces V, W and tensors $T \in T^r(V)$ and $S \in T^s(W)$, the multilinear function

$$R(v_1, \ldots, v_r, w_1, \ldots, w_s) = T(v_1, \ldots, v_r)S(w_1, \ldots, w_s)$$

is well defined and is a multi-linear function on $V^r \times W^s$.

This brings us to the following definition. Let $T \in T^r(V)$,and $S \in T^s(W)$, then

$$T \otimes S: V^r \times W^s \to \mathbb{R}$$

is given by

$$T \otimes S(v_1, \ldots, v_r\,,\, w_1, \ldots, w_s) = T(v_1, \ldots, v_r)S(w_1, \ldots, w_s)\ .$$

This product is called the **tensor product.** By taking $V = W$,$T \otimes S$ is a covariant $(r+s)$-tensor on V , which is a element of the space $T^{r+s}(V)$ and$\otimes: T^r(V) \times T^s(V) \to T^{r+s}(V)$.

2- **The tensor product** of V and W is the real vector space of (finite) linear combinations

$$V \otimes W := \{\lambda^{ij} v_i \otimes w_j : \lambda^{ij} \in \mathbb{R}\} = \left[\{v_i \otimes w_j\}_{ij}\right]$$

Where $v_i \otimes w_j(v^*, w^*) \coloneqq v^*(v_i) w^*(w_j)$, using the identification $v_i(v^*) := v^*(v_i)$, and $w_j(w^*) := w^*(w_j)$, with $(v^*, w^*) \in V^* \times W^*$.

- To get a feeling of what the tensor product of two vector spaces represents consider the tensor product of the dual spaces V^* and W^*. We obtain the real vector space of (finite) linear combinations

$$V^* \otimes W^* := \{\lambda_{ij} \theta^i \otimes \sigma^j : \lambda_{ij} \in \mathbb{R}\} = \left[\{\theta^i \otimes \sigma^j\}_{ij}\right]$$

Where $\theta^i \otimes \sigma^j(v, w) = \theta^i(v)\sigma^j(w)$ for any $(v, w) \in V \times W$.

One can show that $V^* \otimes W^*$ is isomorphic to space of bilinear maps from $V \times W$ to $\mathbb{R}$. In particular elements $v^* \otimes w^*$ all lie in

$V^*\otimes W^*$, but not all elements in $V^*\otimes W^*$ are of this form . The isomorphism is easily seen as follows. Let $\mathrm{v} = \xi_i v_i \; and \; w = \eta_j w_j$, then for a given bilinear form b it holds that $b(\mathrm{v}, w) = \xi_i \eta_j b\left(v_i , w_j\right)$. By definition of dual basis we have that $\xi_i \eta_j = \theta^i(v)\sigma^j(w) = \theta^i \otimes \sigma^j(v, w)$, which shows the isomorphism by setting $\lambda_{ij} = b\left(v_i , w_j\right)$.

- In the case $V^*\otimes W$ the tensor represent linear maps from $V \; to \; W$. Indeed , from the previous we know that elements in $V^*\otimes W$ represent bilinear maps from $V\otimes W^*$ to $\mathbb{R}$. For an element $b \in V^*\otimes W$ this means that $(v , .) : W^* \to \mathbb{R}$, and thus $b(v , .) \in (W^*)^* \cong W$.

Examples 1.12.2

1- The cross product on $\mathbb{R}^3$ is an example of a multilinear (bilinear) function mapping not to $\mathbb{R}$ to $\mathbb{R}^3$. Let $x, y \in \mathbb{R}^3$, then

$$T(x, y) = x \times y \in \mathbb{R}^3$$

which clearly is a bilinear function on $\mathbb{R}^3$.

2- The last property can easily be seen by the following example. Let $\mathrm{V} = \mathbb{R}^2$, and $T, S \in T^1(\mathbb{R}^2)$, given by

$T(v) = v_1 + v_2 \; and \; S(w) = w_1 - w_2$, then

$$T\otimes S(1 , 1 , 1 , 0) = 2 \neq 0 = S\otimes T(1 , 1 , 1 , 0)$$

which shows that $\otimes$ is not commutative in general .

3- consider vectors $a \in V \; and \; b^* \in W \; , then \; a^*\otimes(b^*)^*$ can be identified with a matrix , i.e.

$a^*\otimes(b^*)^*(v, .) = a^*(v)(b^*)^*(.) \cong a^*(v)b$. For example Let $a^*(v) = a_1 v_1 + a_2 v_2 + a_3 v_3$ and

$$Av = a^*(v)b = \begin{pmatrix} a_1 b_1 v_1 + a_2 b_1 v_2 + a_3 b_1 v_3 \\ a_1 b_2 v_1 + a_2 b_2 v_2 + a_3 b_2 v_3 \end{pmatrix} =$$

$$\begin{pmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} .$$

Symbolically we can write

$$A = a^*\otimes b = \begin{pmatrix} a_1 b_1 & a_2 b_1 & a_3 b_1 \\ a_1 b_2 & a_2 b_2 & a_3 b_2 \end{pmatrix} = (a_1 \quad a_2 \quad a_3)\otimes \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which shows how a vector and convector can be 'tensored' to become a matrix. Note that it also holds that $A = (a.b^*)^* = b.a^*$.

4- The inner product on a vector space V is an example of a covariant 2-tensor . This is also an example of a symmetric tensor.

5- The determinant of n-vectors in $\mathbb{R}^n$ is an example of covariant n-tensor on $\mathbb{R}^n$. The determinant is skew symmetric , and an example of an alternating tensor .

Lemma 1.12.3

Let $T \in T^r(V)$, $S, S' \in T^s(V)$ and $R \in T^t(V)$, then

(i) $(T \otimes S) \otimes R = T \otimes (S \otimes R)$ (associative) ,

(ii) $T \otimes (S + S') = T \otimes S + T \otimes S'$ (distributive) ,

(iii) $T \otimes S \neq S \otimes T$ (non-commutative).

The tensor product is also defined for contra-variant tensors and mixed tensors .

Theorem 1.12.4

Let $\{v_1, \dots, v_n\}$ be a basis for V, and let $\{\theta^1, \dots, \theta^n\}$ be the dual basis for V^* . Then the set

$$\mathcal{B} = \{\theta^{i_1} \otimes \dots \otimes \theta^{i_r} : 1 \leq i_1 , \dots, i_r \leq n\}$$

Is a basis for the $n^r - dimensional$ vector space $T^r(V)$.

proof :compute

$$T_{i_1 \dots i_r} \theta^{i_1} \otimes \dots \otimes \theta^{i_r}(v_{j_1}, \dots, v_{j_r}) = T_{i_1 \dots i_r} \theta^{i_1}(v_{j_1}) \dots \theta^{i_r}(v_{j_r})$$

$$= T_{i_1 \dots i_r} \delta_{j_1}^{j_1} \dots \delta_{j_r}^{j_r} = T_{j_1 \dots j_r}$$

$$= T(v_{j_1} , \dots, v_{j_r})$$

which shows by using the multi-linearity of tensors that T can be expanded in the basis $\mathcal{B}$ as follows

$$T = T_{i_1 \dots i_r} \theta^{i_1} \otimes \dots \otimes \theta^{i_r}$$

where$T_{i_1 \dots i_r} = T(v_{i_1} , \dots, v_{i_r})$,the components of the tensor T.

Lemma 1.12.5

We have that

(i) $V \otimes W$ and $W \otimes V$ are isomorphic .

(ii) $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ are isomorphic .

with the notion of tensor product of vector spaces at hand we now conclude that the above describe tensor spaces $T^r(V)$ and $T_r(V)$ are given as follows :

$$T^r(V) = \underbrace{V^*\otimes \dots \otimes V^*}_{r\ times} \ , \quad T_r(V) = \underbrace{V\otimes \dots \otimes V}_{r\ times} \ .$$

By considering tensor products of V's and V^*'s we obtain the tensor space of mixed tensor:

$$T_s^r(V) := \underbrace{V^*\otimes \dots \otimes V^*}_{r\ times} \otimes \underbrace{V\otimes \dots \otimes V}_{s\ times} \qquad \rightarrow \qquad (1.12.2)$$

Elements in this space are called (r,s)-mixed tensors on $V - r$ copies of V^* , and s copies of V. Of course the tensor product described above is defined in general for tensor $T \in T_s^r(V)$, and $T \in T_{s'}^{r'}(V)$:

$$\otimes : T_s^r(V) \times T_{s'}^{r'}(V) \to T_{s+s'}^{r+r'}(V) \qquad \rightarrow \qquad (1.12.3)$$

Notice1.12.6

If $f : V \to W$ is a linear mapping between vector spaces and T is an covariant tensor on W we can define concept of pullback of T . Le T$\in T^r(W)$, then $f^*T \in T^r(V)$ is defined as follows:

$$f^*T(v_1 , \dots , v_r) = T\big(f(v_1) , \dots , f(v_r)\big) \ \rightarrow \qquad (1.12.4)$$

and $f^* : T^r(W) \to T^r(V)$ is a linear mapping .

Indeed , $\quad f^*(T + S) = T \circ f + S \circ f = f^*T + f^*S$, and

$f^*\lambda T = \lambda T \circ f = \lambda f^*T$. If we represent f by a matrix A with respect to bases $\{v_i\}$ and $\{w_j\}$ for V and W respectively, then the matrix for the linear f^* is given by:

$$\underbrace{A^*\otimes \dots \otimes A^*}_{r\ times}$$

with respect to the bases $\{\theta^{i_1}\otimes \dots \otimes \theta^{i_r}\}$ and $\{\sigma^{j_1}\otimes \dots \otimes \sigma^{j_r}\}$ for $T^r(W)$ and $T^r(V)$ respectively.

Remark 1.12.7

The direct sum

$$T^*(V) = \bigoplus_{r=0}^{\infty} T^r(V)$$

Consisting of finite sums of covariant tensors is called the covariant tensor algebra of V with multiplication given by the tensor product

$\otimes$. Similarly , one defines the **contra-variant tensor algebra**

$$T_*(V) = \bigoplus_{r=0}^{\infty} T_r(V)$$

For mixed tensor we have

$$T(V) = \bigoplus_{r,s=0}^{\infty} T_s^r(V)$$

which is called the **tensor algebra of mixed tensor** of V . Clearly , $T^*(V)$ and $T_*(V)$ subalgebras of $T(V)$.

Definition(Symmetric and Alternating Tensors)1.12.8

1-**A covariant r-tensor *T*** on a vector space V is called symmetric if:

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_r) = T(v_1, \dots, v_j, \dots, v_i, \dots, v_r)$$

for any pair of indices $i \leq j$. The set of symmetric covariant r-tensors on V is denoted by $\Sigma^r(V) \subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

- If $\mathrm{a} \in S_r$ is a permutation, then define

$${}^{\mathrm{a}}T(v_1, \dots, v_r) = T(v_{\mathrm{a}(1)}, \dots, v_{\mathrm{a}(r)})$$

where $\mathrm{a}(\{1, \dots, r\}) = \{\mathrm{a}(1), \dots, \mathrm{a}(r)\}$. From this notation we have that for two permutationsa, $\mathrm{b} \in S_r$, ${}^{\mathrm{b}}({}^{\mathrm{a}}T) = {}^{\mathrm{ba}}T$.Define

$$Sym\, T = \frac{1}{r!} \sum_{\mathrm{a} \in S_r} {}^{\mathrm{a}}T$$

It is straightforward to see that for any tensor$T \in T^r(V)$,

$Sym\, T$ is a symmetric . Moreover a tensor T is symmetric if and only if $SymT = T$. For that reason $Sym\, T$ is called **(tensor) symmetrization**.

2- **A covariant r-tensor *T*** on a vector space V is called alternating if :

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_r) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_r)$$

for any pair of indices $i \leq j$. The set of alternating covariant r-tensors on V is denoted by $\Lambda^r(V) \subset T^r(V)$, which is a (vector) subspace of $T^r(V)$.

- As before we define

$$Alt\ T = \frac{1}{r!}\sum_{a\in S_r}(-1)^{a}\ {}^{a}T$$

where $(-1)^{a}$ is $+1$ for even permutations , and -1 for odd permutations . We say that $Alt\ T$ is the alternating projection of a tensor T, and $Alt\ T$ is of course a alternating tensor.

Examples 1.12.9

1- Let $T, T' \in T^2(\mathbb{R}^2)$ be defined as follows: $T(x,y) = x_1y_2$, and $T'(x,y) = x_1y_1$. Clearly, T is not symmetric and T' . We have that

$$Sym\ T(x,y) = \frac{1}{2}T(x,y) + \frac{1}{2}T(y,x)$$
$$= \frac{1}{2}x_1y_2 + \frac{1}{2}y_1x_2$$

Which clearly is symmetric . If we do the same thing for T' we obtain:

$$Sym\ T'(x,y) = \frac{1}{2}T'(x,y) + \frac{1}{2}T'(y,x)$$
$$= \frac{1}{2}x_1y_1 + \frac{1}{2}y_1x_1 = T'(x,y)$$

Showing that operation Sym applied to symmetric tensors produces the same tensor again .

Using symmetrization we can define the symmetric product.

Let $S \in \Sigma^r(V)$ and $T \in \Sigma^s(V)$ be symmetric tensor , then

$$S.T = Sym\ (S\otimes T)$$

The symmetric product of symmetric tensors is commutative which follows directly from the definition:

$$S.T(v_1, \dots, v_{r+s}) =$$
$$\frac{1}{(r+s)!}\sum_{a\in S_{r+s}} S(v_{a(1)}, \dots, v_{a(r)})T\big(v_{a(r+1)}, \dots, v_{a(r+s)}\big)$$

2- Consider the 2-tensors $\mathrm{T}(x) = x_1 + x_2$, and $(y) = y_2$.

Now$S\otimes T(x,y) = x_1y_2 + x_2y_2andT\otimes S(x,y) = x_2y_1 + x_2y_2$

which clearly gives that $S\otimes T \neq T\otimes S$. Now compute

$$Sym\ (S\otimes T)(x,y) = \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_2 + \frac{1}{2}y_1x_2 + \frac{1}{2}x_2y_2$$
$$= \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + x_2y_2 = S.T(x,y).$$

Similarly

$$Sym\ (T\otimes S)(x,y) = \frac{1}{2}x_2y_1 + \frac{1}{2}x_2y_2 + \frac{1}{2}y_2x_1 + \frac{1}{2}x_2y_2$$

$$= \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + x_2y_2 = T.S(x,y)$$

Which gives that S.$T = T.S$.

3- Let $T, T' \in T^2(\mathbb{R}^2)$ be defined as follows : $T(x,y) = x_1y_2$, and $T'(x,y) = x_1y_2 - x_2y_1$. Clearly , T is not alternating and $T'(x,y) = -T'(y,x)$ is alternating . We have that

$$Alt\ T(x,y) = \frac{1}{2}T(x,y) - \frac{1}{2}T(y,x)$$

$$= \frac{1}{2}x_1y_2 - \frac{1}{2}y_1x_2 = \frac{1}{2}T'(x,y)$$

which clearly is alternating . If we do the same thing for T' we obtain:

$$Alt\ T'(x,y) = \frac{1}{2}T'(x,y) - \frac{1}{2}T'(y,x)$$

$$= \frac{1}{2}x_1y_2 - \frac{1}{2}x_2y_1 - \frac{1}{2}y_1x_2 + \frac{1}{2}y_2x_1 = T'(x,y)$$

showing that operation Alt applied to alternating tensors produces the same tensor again .Notice that $T'(x,y) = det(x,y)$

This brings us the fundamental product of alternating tensors called the wedge product. Let $S \in \wedge^r(V)$ and $T \in \wedge^s(V)$ be symmetric tensors , then

$$S \wedge T = \frac{(r+s)!}{r!s!} Alt(S \otimes T)$$

The wedge product of alternating tensors is anti-commutative which follows directly from the definition :

$$S \wedge T(v_1, \dots, v_{r+s}) =$$

$$\frac{1}{(r+s)!} \sum_{\mathrm{a} \in S_{r+s}} (-1)^{\mathrm{a}} S\big(v_{\mathrm{a}(1)}, \dots, v_{\mathrm{a}(r)}\big) T\big(v_{\mathrm{a}(r+1)}, \dots, v_{\mathrm{a}(r+s)}\big)$$

In special case of the wedge of two co-vectors $\theta, \omega \in V^*$gives

$$\theta \wedge \omega = \theta \otimes \omega - \omega \otimes \theta$$

In particular we have that :

(i) $(T \wedge S) \wedge R = T \wedge (S \wedge R)$;

(ii) $(T + T') \wedge S = T \wedge S + T' \wedge S$;

(iii) $\mathrm{T} \wedge S = (-1)^{rs} S \wedge T\ for\ T \in \wedge^r(V) and\ S \in \wedge^s(V)$;

(iv) $\mathrm{T} \wedge T = 0$;

Lemma 1.12.10

Let $\{v_1, \dots, v_n\}$ be a basis for V and let $\{\theta^1, \dots, \theta^n\}$ be the dual basis for V^* . Then the set

$$\mathcal{B}_\Sigma = \left\{\theta^{i_1} \dots \theta^{i_r} : 1 \le i_1 \le \dots \le i_r \le n\right\}$$

is a basis for the (sub)space $\Sigma^r(V)$ of symmetric r-tensors .

Moreover , $dim\Sigma^r(V) = \binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}$.

Remark1.12.11

$$S\overline{\wedge}T = Alt(S\otimes T)$$

Which is in accordance with the definition of the symmetric product. This definition is usually called the alt convention for the wedge product , and our definition is usually referred to as the determinant convention. For computational purpose the determinant convention is more appropriate.

- If $\{e_1^* , \dots, e_n^*\}$ is the standard dual basis for $(\mathbb{R}^n)^*$, then for vectors $a_1 , \dots, a_n \in \mathbb{R}^n$,

$$det(a_1 , \dots, a_n) = e_1^* \wedge \dots \wedge e_n^*(a_1 , \dots, a_n)$$

Using the multi-linearity the more general statement reads

$$\beta^1 \wedge \dots \wedge \beta^n(a_1 , \dots, a_n) = det\left(\beta^i\left(a_j\right)\right) \rightarrow (1.12.5)$$

where β^i are co-vector. The alternating tensor $det = e_1^* \wedge \dots \wedge e_n^*$ is called the determinant function on $\mathbb{R}^n$.

If $f : V \to W$ is a linear map between vector spaces then the pullback $f^*T \in \wedge^r(V)$ of any alternating tensor $T \in \wedge^r(W)$ is given via the relation:

$$f^*T(v_1 , \dots, v_r) = T\left(f(v_1), \dots, f(v_r)\right) \ , f^* : \wedge^r(W) \to \wedge^r(V).$$

In particular $f^*(T \wedge S) = (f^*T) \wedge f^*(S)$. As a special case we have that if $f: V \to V$, linear , and $dimV = n$, then

$$f^*T = \det(f)\, T$$

For any alternating tensor T$\in \wedge^n(V)$. This can be seen as follows. By multilinearity we verify the above relation for the vectors $\{e_i\}$. We have that

$$f^*T(e_1 , \dots, e_n) = T\left(f(e_1), \dots, f(e_n)\right)$$
$$= T(f_1 , \dots, f_n) = cdet(f_1, \dots, f_n) = cdet(f)$$

where we use the fact that $\wedge^n(V) \cong \mathbb{R}$. On the other hand

$$\det(f)\, T(e_1, \dots, e_n) = \det(f)\, c.det(e_1, \dots, e_n)$$
$$= \mathrm{c}\det(f)$$

Lemma 1.12.12

Let $\{\theta^1, \dots, \theta^n\}$ be a basis for V^* then the set

$$\mathcal{B}_\wedge = \{\theta^{i_1} \wedge \dots \wedge \theta^{i_r} : 1 \le i_1 < \dots < i_r \le n\}$$

is a basis for $\wedge^r(V)$, and $dim\, \wedge^r(V) = \frac{n!}{(n-r)!r!}$. In particular $dim\, \wedge^r(V) = 0\ for\ \ r > n.$

Proof : From Theorem (1-12-4) we know that any alternating tensor $T \in \wedge^r(V)$ can be written as

$$T = T_{j_1 \dots j_r} \theta^{j_1} \otimes \dots \otimes \theta^{j_r} .$$

We have that Alt $T = T$, and so

$$T = T_{j_1 \dots j_r} Alt(\theta^{j_1} \otimes \dots \otimes \theta^{j_r}) = \frac{1}{r!} T_{j_1 \dots j_r} \theta^{j_1} \wedge \dots \wedge \theta^{j_r}$$

In the expansion the terms with $j_k = j_\ell$ are zero since $\theta^{j_k} \wedge \theta^{j_\ell} = 0.$ If we order the indices in increasing order we obtain

$$T = \pm \frac{1}{r!} T_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \theta^{i_r}$$

which show that $\mathcal{B}_\wedge$ spans $\wedge^r(V)$.

Linear independence can be proved as follows.

Let $0 = \lambda_{i_1 \dots i_r} \theta^{i_1} \wedge \dots \wedge \theta^{i_r}$ and thus

$\lambda_{i_1 \dots i_r} = \theta^{i_1} \wedge \dots \wedge \theta^{i_r} (v_{i_1}, \dots, v_{i_r}) = 0$, which proves linear independence.

It is immediately clear that $\mathcal{B}_\wedge$ consists of $\binom{n}{r}$ elements.

Lemma 1.12.13

Some of the basic properties can be listed as follows :

(i) Sym and Alt are projections on $T^r(V)$, i.e. $Sym^2 = Sym$ and $Alt^2 = Alt$;

(ii) T is symmetric if and only if $Sym\ T = T$, and T is alternating if and only if $Alt\ T = T$;

(iii) $Sym(T^r(V)) = \Sigma^r(V)$, and $Alt(T^r(V)) = \wedge^r(V)$;

(iv) Sym$\circ\, Alt = Alt \circ Sym = 0$ $, i.e. if\ T \in \wedge^r(V)$, then

$Sym\, T = 0$, and if T$\in \Sigma^r(V)$, $then\ \ Alt\, T = 0$;

(v) let $f: V \to W$ then Sym and Alt commute with $f^*: T^r(W) \to T^r(V)$, i.e. $Sym \circ f^* = f^* \circ Sym$ and $Alt \circ f^* = f^* \circ Alt$.

Definition(Tensor Bundles and Tensor Fields)1.12.14

Generalizations of tangent spaces and cotangent spaces are given by tensor spaces

$$T^r(T_pM) \quad , \quad T_s(T_pM) \quad and \quad T_s^r(T_pM)$$

where $T^r(T_pM) = T_r(T_p^*M)$. As before we can introduce the tensor bundles:

$$T^rM = \sqcup_{p\in M} T^r(T_pM) \qquad \to \qquad (1.12.6)$$

$$T_sM = \sqcup_{p\in M} T_s(T_pM) \qquad \to \qquad (1.12.7)$$

$$T_s^rM = \sqcup_{p\in M} T_s^r(T_pM) \qquad \to \qquad (1.12.8)$$

Equation (1.12.6) **is called the covariant r-tensor bundle** on M, the equation (1.12.7) **is called the contra-variant s-tensor bundle** on M, the equation (1.12.8) **is called the mixed** **($\boldsymbol{r,s}$)-tensor bundle on $\boldsymbol{M}$**.

Lemma 1.12.15

A covariant tensor field σ is smooth at $p \in U$ if and only if

(i) The coordinate functions $\sigma_{i_1,\dots,i_r} : U \to \mathbb{R}$ are smooth , or equivalently if and only if .

(ii) For smooth vector fields $X_1 , \dots , X_r$ defined on any open set $\subset M$, then the function $\sigma(X_1 , \dots , X_r): U \to \mathbb{R}$, given by

$$\sigma(X_1 , \dots , X_r)(p) = \sigma_p(X_1(p), \dots , X_r(p))$$

Is smooth.

The same equivalences hold for contra-variant and mixed tensor fields.

Lemma 1.12.16

Let $f: N \to M$, $g: M \to P$ be smooth mappings , and let $h \in C^{\infty}(M)$, $\sigma \in \mathcal{F}^r(M), and\ \tau \in \mathcal{F}^r(N)$, then :

(i) $f^*: \mathcal{F}^r(M) \to \mathcal{F}^r(N)$ is linear ;

(ii) $f^*(h\sigma) = (f \circ h) f^*\sigma$;

(iii) $f^*(\sigma \otimes \tau) = f^*\sigma \otimes f^*\tau$;

(iv) $f^*\sigma$ is a smooth covariant tensor field ;

(v) $(g \circ f)^* = f^* \circ g^*$;

(vi) $\mathrm{id}_{\mathrm{M}}^*\sigma = \sigma$;

Chapter Two

Calculus on Manifolds

2-1 Differential Forms:

Definition (Permutation Group)2.1.1

1- A group is a set G with an associative binary operation, $\cdot : G \times G \to G$ with identity, called the multiplication, such that each element has an inverse. That is, the following conditions are satisfied

1. for any three elements $g, h, k \in G$, the associativity law holds :

$(gh)k = g(hk)$;

2. there exists an identity element $e \in G$ such that for any $g \in G$, $ge = eg = g$;

3. each element $g \in G$ has an inverse g^{-1}, such that $gg^{-1} = g^{-1}g = e$.

2- Let X be a set . A transformation of the set X is a bijective map

$g: X \to X$.

3- The set of all transformations of a set X forms a group $Aut(X)$, with composition of maps as group multiplication.

4- Any subgroup of $Aut(X)$ is a transformation group of the set X.

5- The transformations of a finite set X are called permutations .

6- The group S_p of permutations of the set $\mathbb{Z}_p = \{1, \dots, p\}$ is called the symmetric group of order p.

Theorem 2.1.2

The order of the symmetric group S_p is $|S_p| = p!$.

Remarks 2.1.3

1- Any subgroup of S_p is called a permutation group.

2- A permutation $\varphi: \mathbb{Z}_p \to \mathbb{Z}_p$ can be represented by

$$\begin{pmatrix} 1 & \cdots & p \\ \varphi(1) & \cdots & \varphi(p) \end{pmatrix}$$

3- The identity permutation is

$$\begin{pmatrix} 1 & \cdots & p \\ 1 & \cdots & p \end{pmatrix}$$

4- The inverse $\varphi^{-1}: \mathbb{Z}_p \to \mathbb{Z}_p$ is represented by

$$\begin{pmatrix} \varphi(1) & \cdots & \varphi(p) \\ 1 & \cdots & p \end{pmatrix}$$

5- The product of permutations is then defined in an obvious manner .

6- An elementary permutation is a permutation that exchanges the order of only two elements.

7- Every permutation can be realized by an even number of elementary permutations is called an even permutation.

8- A permutation can be realized by an odd number of elementary permutations is called an odd permutation.

Proposition 2.1.4

1- The parity of a permutation does not depend on the representation of a permutation by a product of the elementary ones .

2- That is , each representation of an even permutation has even number of elementary permutations, and similarly for odd permutations .

3- The sign of a permutation , denoted by $sign(\varphi)$ (or simply $(-1)^{\varphi}$), is defined by :

$$sign(\varphi) = (-1)^{\varphi} = \begin{cases} +1, if\ \varphi\ is\ even \\ -1\ , if\ \varphi\ is\ odd \end{cases}$$

Definition (Permutations of Tensors) 2.1.5

1- Let S_p be the symmetric group of order p. Then every permutation $\varphi \in s_p$ defines a map:

$$\varphi :\ T_p \to T_p$$

Which assigns to every tensor T of type (0 , p) a new tensor $\varphi(T)$, called a permutation of the tensor T , of type (0 , p) by : $\forall$ $\mathrm{v}_1, \dots \mathrm{v}_p$

$$\varphi(T)\big(\mathrm{v}_1\ , \dots , \mathrm{v}_p\big) = T\big(\mathrm{v}_{\varphi(1)}\ , \dots , \mathrm{v}_{\varphi(p)}\big) \quad \to \quad (2.1.1)$$

2- Let $(i_1, \dots , i_p)$ be $a\ p - tuple$ of integers. Then a permutation $\varphi : \mathbb{Z}_p \to \mathbb{Z}_p$ defines an action

$$\varphi\big(i_1, \dots , i_p\big) = \big(i_{\varphi(1)}, \dots , i_{\varphi(p)}\big) \quad \to \quad (2.1.2)$$

3- The components of the tensor $\varphi(T)$ are obtained by the action of the permutation φ on the indices of the tensor T

$$\varphi(T)_{i_1,\dots,i_p} = T_{i_{\varphi(1)},\dots,i_{\varphi(p)}} \quad \to \quad (2.1.3)$$

4- The symmetrization of the tensor T of the type (0, p) is defined by

$$Sym(T) = \frac{1}{p!}\sum_{\varphi \in S_p} \varphi(T) \quad \to \quad (2.1.4)$$

5- The symmetrization is also denoted by parenthesis. The components of the symmetrized tensor $Sym(T)$ are given by

$$T_{(i_1,\dots,i_p)} = \frac{1}{p!}\sum_{\varphi \in S_p} T_{i_{\varphi(1)},\dots i_{\varphi(p)}} \quad \to \quad (2.1.5)$$

6- The anti-symmetrization of the tensor T of the type (0,p) is defined by

$$Alt(T) = \frac{1}{p!}\sum_{\varphi \in S_p} sign(\varphi)\varphi(T) \quad \to \quad (2.1.6)$$

7- The anti-symmetrization is also denoted by square brackets. The components of the anti-symmetrized tensor $Alt(T)$ are given by

$$T_{[i_1,\dots,i_p]} = \frac{1}{p!}\sum_{\varphi\in S_p} sign(\varphi) T_{i_{\varphi(1)},\dots,i_{\varphi(p)}} \rightarrow \qquad (2.1.7)$$

Examples 2.1.6

1- A tensor T of type (0,p) is called symmetric if for any permutation $\varphi \in S_p$:

$$\varphi(T) = T$$

2- A tensor T of type (0,p) is called anti-symmetric if for any permutation $\varphi \in S_p$:

$$\varphi(T) = sign(\varphi)T$$

3- An anti-symmetric tensor of type (0,p) is called a p-form .

Remarks 2.1.6

1- permutation , symmetrization , anti-symmetrization of tensors of type (p,0) .

2- completely symmetric and completely anti-symmetric tensor of type (p,0) .

3- An anti-symmetric tensor of type (p,0) is called a p-vector .

Definition (Alternating Tensors) 2.1.7

- Let $(i_1, \dots, i_p)$ and $(j_1, \dots, j_p)$ be two $p - tuples$ of integers

$1 \le i_1, \dots, i_p , j_1, \dots, j_p \le n$. The generalized Kronecker symbol is defined by :

$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = \begin{cases} 1 \ if \ (i_1, \dots, i_p) is \ an \ even \ permutation \ of (j_1, \dots, j_p) \\ -1 \ if \ (i_1, \dots, i_p) is \ an \ odd \ permutation \ of (j_1, \dots, j_p) \\ 0 \quad otherwise \end{cases}$$

- One can easily check that

$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = det\begin{bmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_p}_{j_p} \\ \vdots & \ddots & \vdots \\ \delta^{i_p}_{j_1} & \cdots & \delta^{i_p}_{j_p} \end{bmatrix}$$

- Thus, the Kronecker symbols $\delta^{i_1 \dots i_p}_{j_1 \dots j_p}$ are the components of the tensors

$$p!\, Alt\, \underbrace{(I \otimes \dots \otimes I)}_{p}$$

Of type (p , p) , which are anti-symmetric separately in upper indices and the lower indices .

- Thus, the anti-symmetrization can also be written as

$$T_{[i_1 \dots i_p]} = \frac{1}{p!} \delta^{i_1 \dots i_p}_{j_1 \dots j_p} T_{j_1 \dots j_p}$$

Notation 2.1.8

Obviously, the Kronecker symbols vanish for $p > n$

$$\delta^{i_1 \dots i_p}_{j_1 \dots j_p} = 0 \qquad if\ p > n$$

Theorem 2.1.9

For any $p, q \in \mathbb{N}$,$1 \leq p$, $q \leq n$, there holds

$$\delta^{i_1 \dots i_p l_1 \dots l_q}_{j_1 \dots j_p l_1 \dots l_q} = \frac{(n-p)!}{(n-q)!} \delta^{i_1 \dots i_p}_{j_1 \dots j_p}$$

Corollary 2.1.10

For any $q \in \mathbb{N}, 1 \leq q \leq n$ we have

$$\delta^{i_1 \dots i_q}_{i_1 \dots i_q} = \frac{n!}{(n-q)!}$$

In particular

$$\delta^{i_1 \dots i_n}_{i_1 \dots i_n} = n!$$

Lemma 2.1.11

There holds

$$\delta^{i_1\dots i_p j_1\dots j_r}_{l_1\dots l_p m_1\dots m_r}\delta^{k_1\dots k_r}_{j_1\dots j_r} = r!\,\delta^{i_1\dots i_p k_1\dots k_r}_{l_1\dots l_p m_1\dots m_r}$$

- Let $(i_1, \dots, i_p) be\ an\ n-tuple$ of integers $1 \leq i_1, \dots i_n \leq n$. The completely anti-symmetric (Alternating) Levi-Civita symbols are defined by

$$\varepsilon_{i_1\dots i_n} = \delta^{1\dots n}_{i_1\dots i_n} \quad , \quad \varepsilon^{i_1\dots i_n} = \delta^{i_1\dots i_n}_{1\dots n}$$

So that

$$\varepsilon^{i_1\dots i_n} = \varepsilon_{i_1\dots i_n} = \begin{cases} 1\ if (i_1, \dots, i_n) is\ an\ even\ permutation\ of (1, \dots, n) \\ -1\ if (i_1, \dots, i_n)\ is\ an\ odd\ permutation\ of (1, \dots, n) \\ 0\ \ otherwies \end{cases}$$

Theorem 2.1.12

There holds the identity

$$\varepsilon^{i_1\dots i_n}\varepsilon_{j_1\dots j_n} = \sum_{\varphi\in S_n} sign(\varphi)\delta^{i_1}_{j_{\varphi(1)}} \dots \delta^{i_n}_{j_{\varphi(n)}}$$

$$= n!\,\delta^{i_1}_{[j_1} \dots \delta^{i_n}_{j_n]}$$

$$= \delta^{i_1\dots i_n}_{j_1\dots j_n}$$

The contraction of this identity over k indices gives

$$\varepsilon^{i_1\dots i_{n-k} m_1\dots m_k}\varepsilon_{j_1\dots j_{n-k} m_1\dots m_k} = k!\,(n-k)!\,\delta^{i_1}_{[j_1} \dots \delta^{i_{n-k}}_{j_{n-k}]}$$

$$= k!\,\delta^{i_1\dots i_{n-k}}_{j_1\dots j_{n-k}}$$

In particular

$$\varepsilon^{m_1\dots m_n}\varepsilon_{m_1\dots m_n} = n!$$

- It is easy to see that there holds also

$$\delta^{i_1 \ldots i_{n-p}}_{l_1 \ldots l_{n-p}} \varepsilon^{j_1 \ldots j_p l_1 \ldots l_{n-p}} = (n-p)!\, \varepsilon^{j_1 \ldots j_p i_1 \ldots i_{n-p}}$$

- The set of all $n \times n$ real matrices is denoted by $Mat(n, \mathbb{R})$.

- The determinant is a map $det: Mat(n, \mathbb{R}) \rightarrow \mathbb{R}$ that assigns to each matrix $A = (A^i_j)$ a real number $detA$ defined by

$$detA = \sum_{\varphi \in S_n} sign(\varphi) A^1_{\varphi(1)} \ldots A^n_{\varphi(n)}$$

Theorem 2.1.13

1- The determinant of the product of matrices is equal to the product of the determinants:

$$\det(AB) = \det A \det B$$

2- The determinants of a matrix A and of its transpose A^T are equal:

$$detA = detA^T$$

3- The determinant of the inverse A^{-1} of an invertible matrix A is equal to the inverse of the determinant of A :

$$detA^{-1} = (detA)^{-1}$$

4- A matrix is invertible if and only if its determinant is non-zero.

- The determinant of a matrix $A = (A^i_j)$ can be written as:

$$\begin{aligned} detA &= \varepsilon^{i_1 \ldots i_n} A^1_{i_1} \ldots A^n_{i_n} \\ &= \varepsilon_{j_1 \ldots j_n} A^{j_1}_1 \ldots A^{j_n}_n \\ &= \frac{1}{n!} \varepsilon^{i_1 \ldots i_n} \varepsilon_{j_1 \ldots j_n} A^{j_1}_{i_1} \ldots A^{j_n}_{i_n} \end{aligned}$$

Definition(Differential Forms)2.1.14

A special class of covariant tensor bundles and associated bundle sections are the so-called alternating tensor bundles .

Let $\wedge^r(T_pM) \subset T^r(T_pM)$ be the space alternating tensors on T_pM . We have that a basis for $\wedge^r(T_pM)$ is given by

$$\{dx^{i_1}\wedge \dots \wedge dx^{i_r} : 1 \le i_1 , \dots , i_r \le m\}$$

and $\wedge^r(T_pM) = \frac{m!}{r!(m-r)!}$. The associated tensor bundles of alternating covariant tensors is denoted by $\wedge^r M$. Smooth sections in $\wedge^r M$ are called **differential r-forms**, and the space of smooth sections is denoted by $\Gamma^r(M) \subset \mathcal{F}^r(M)$.

In particular $\Gamma^0(M) = C^\infty(M)and\Gamma^1(M) = \mathcal{F}^*(M)$.

In terms of components a differential r-form , or r-form for short , is given by :

$$\sigma_{i_1 \dots i_r} dx^{i_1}\wedge \dots \wedge dx^{i_r}$$

and the components $\sigma_{i_1 \dots i_r}$ are smooth functions . An r-form σ acts on vector fields $X_1 , \dots , X_r$ as follows :

$$\sigma(X_1 , \dots , X_r) = \sum_{a \in S_r} (-1)^a \sigma_{i_1 \dots i_r} dx_{i_1}(X_{a(1)}) \dots dx_{i_r}(X_{a(r)})$$
$$= \sum_{a \in S_r} (-1)^a \sigma_{i_1 \dots i_r} X^{i_1}_{a(1)} \dots X^{i_r}_{a(r)} .$$

Remark 2.1.15

An important notion that comes up in studying differential forms is the notion contracting an r-form . Given an r-form $\sigma \in \Gamma^r(M)$ and a vector field $X \in \mathcal{F}^r(M)$, then

$$i_X\sigma \coloneqq \sigma(X , . , \dots , .) \qquad \rightarrow \quad (2.1.8)$$

is **called the contraction** with X , and is a differential $(r-1)$-form on . Another notation for this is$i_X\sigma = X \lrcorner \sigma$.
Contraction is a linear mapping

$$i_X : \Gamma^r(M) \to \Gamma^{r-1}(M) \qquad \rightarrow \quad (2.1.9)$$

Contraction is also linear in X, i.e. for vector fields X, Y it holds that

$$i_{X+Y\sigma} = i_X\sigma + i_Y\sigma \ , \ \ i_{\lambda X}\sigma = \lambda . i_X\sigma$$

Lemma 2.1.16

Let $\sigma \in \Gamma^{r-1}(M)$ and $X \in \mathcal{F}(M)$ a smooth vector field , then

(i) $i_X\sigma \in \Gamma^{r-1}(M)$ (smooth(r-1)-form);

(ii) $i_X \circ i_X = 0$;

(iii) i_X is an anti-derivation, i.e. for$\sigma \in \Gamma^r(M)$and$\omega \in \Gamma^s(M)$

$$i_X(\sigma\wedge\omega) = (i_X\sigma)\wedge\omega + (-1)^r\sigma\wedge(i_X\omega)$$

A direct consequence of (iii) is that $\sigma = \sigma_1\wedge\ldots\wedge\sigma_r$ where $\sigma_i \in \Gamma^1(M) = \mathcal{F}^*(M)$, then

$$i_X\sigma = (-1)^{i-1}\sigma_i(X)\sigma_1\wedge\ldots\wedge\widehat{\sigma}_i\wedge\ldots\wedge\sigma_r \ \rightarrow (2.1.10)$$

where the hat indicates that σ_i is to be omitted , and we use the summation convention.

Examples 2.1.17

1- Let M= $\mathbb{R}^3$ $and\ \sigma = dx\wedge dz$. Then the vector fields

$$X_1 = X_1^1\frac{\partial}{\partial x} + X_1^2\frac{\partial}{\partial y} + X_1^3\frac{\partial}{\partial z}$$

and

$$X_2 = X_2^1\frac{\partial}{\partial x} + X_2^2\frac{\partial}{\partial y} + X_2^3\frac{\partial}{\partial z}$$

We have that

$$\sigma(X_1\ ,X_2) = X_1^1X_2^3 - X_1^3X_2^1\ .$$

2-Let $\sigma = x_2dx^1\wedge dx^3$ be a 2-form on $\mathbb{R}^3$, and

$X = x_1^2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + (x_1+x_2)\frac{\partial}{\partial x_3}$ a given vector field on $\mathbb{R}^3$. If $Y = Y^1\frac{\partial}{\partial y_1} + Y^2\frac{\partial}{\partial y_2} + Y^3\frac{\partial}{\partial y_3}$ is an arbitrary vector fields then

$$\begin{aligned}(i_X\sigma)(Y) = \sigma(X,Y) &= dx^1(X)dx^3(Y) - dx^1(Y)dx^3(X)\\ &= x_1^2Y^3 - (x_1+x_2)Y^1\end{aligned}$$

which gives that

$$i_X\sigma = x_1^2dx^3 - (x_1+x_2)dx^1$$

3- Let $\sigma = dx^1\wedge dx^2\wedge dx^3$ be a 3-form on $\mathbb{R}^3$, and X the vector field as given in the previous example . By linearity

$$i_X\sigma = i_{X_1}\sigma + i_{X_2}\sigma + i_{X_3}\sigma$$

where $X_1 = x_1^2 \frac{\partial}{\partial x_1}$, $X_2 = x_3 \frac{\partial}{\partial x_2}$ and $X_3 = (x_1 + x_2) \frac{\partial}{\partial x_3}$.

This composition is chosen so that X is decomposed in Vector fields in the basis directions. Now

$$i_{X_1}\sigma = dx^1(X_1)dx^2 \wedge dx^3 = x_1^2 dx^2 \wedge dx^3$$
$$i_{X_2}\sigma = -dx^2(X_2)dx^1 \wedge dx^3 = -x_3 dx^1 \wedge dx^3$$
$$i_{X_2}\sigma = dx^3(X_3)dx^1 \wedge dx^2 = (x_1 + x_2)dx^1 \wedge dx^2$$

which gives

$$i_X\sigma = x_1^2 dx^2 \wedge dx^3 - x_3 dx^1 \wedge dx^3 + (x_1 + x_2)dx^1 \wedge dx^2$$

4- Consider $\sigma = dx \wedge dy$ on $\mathbb{R}^2$, and mapping f given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The map f the identity mapping that maps $\mathbb{R}^2$ in Cartesian coordinates to $\mathbb{R}^2$ in polar coordinates(consider the chart $U = \{(r, \theta): r > 0, 0 < \theta < 2\pi\}$) .

As before we can compute the pullback of σ to $\mathbb{R}^2$ with polar coordinates:

$$\begin{aligned}\sigma &= dx \wedge dy = d(rcos(\theta)) \wedge d(rsin(\theta)) \\ &= (\cos(\theta)\,dr - rsin(\theta)d\theta) \wedge (\sin(\theta)\,dr + rcos(\theta)d\theta) \\ &= rcos^2(\theta)dr \wedge d\theta - rsin^2(\theta)d\theta \wedge dr \\ &= rdr \wedge d\theta.\end{aligned}$$

Remark 2.1.18

For completeness we recall that for a smooth mapping $f: N \to M$, the pullback of a r-form σ is given by

$(f^*\sigma)_p(X_1, \dots, X_r) = f^*\sigma_{f(p)}(X_1, \dots, X_r) = \sigma_{f(p)}(f_*X_1, \dots, f_*X_r)$

We recall that for a mapping $h: M \to \mathbb{R}$, then pushforward , or differential of h $dh_p = h_* \in T_p^*M$.

In coordinates $dh_p = \frac{\partial \hat{h}}{\partial x_i} dx^i \Big|_p$, and thus the mapping

$p \mapsto dh_p$ is a smooth section in $\Lambda^1(M)$,and therefore a differential 1-form , with component $\sigma_i = \frac{\partial \hat{h}}{\partial x_i}$ (in local coordinates).

If $f: N \to M$ is a mapping between m-dimensional manifolds with chart (U, φ) , and (V, ψ) respectively , and $f(U) \subset V$.

Set $x = \varphi(p)$, and $y = \psi(q)$, then

$$f^*(\sigma dy^1 \wedge \dots \wedge dy^m) = (\sigma \circ f) det(J\tilde{f}|_x) dx^1 \wedge \dots \wedge dx^m \ \rightarrow (2.1.11)$$

This can be proved as follows . From definition of the wedge product and lemma 1.12.16 it follows that $f^*(dy^1 \wedge \dots \wedge dy^m) = f^* dy^1 \wedge \dots \wedge f^* dy^m$, and

$f^* dy^j = \frac{\partial \tilde{f}_j}{\partial x_i} dx^i = dF^j$,where $F = \psi \circ f and \tilde{f} = \psi \circ f \circ \varphi^{-1}$

Now

$$\begin{aligned} f^*(dy^1 \wedge \dots \wedge dy^m) &= f^* dy^1 \wedge \dots \wedge f^* dy^m \\ &= dF^1 \wedge \dots \wedge dF^m \end{aligned}$$

and furthermore , using eq(1.12.5)

$$dF^1 \wedge \dots \wedge dF^m \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right) = det \left(dF^i \left(\frac{\partial}{\partial x_j} \right) \right) = det \left(\frac{\partial \tilde{f}_j}{\partial x_j} \right)$$

Which proves the above claim.

As a consequence of this a change of coordinates $\tilde{f} = \psi \circ \varphi^{-1}$ yields

$$f^*(dy^1 \wedge \dots \wedge dy^m) = det(J\tilde{f}|_x) dx^1 \wedge \dots \wedge dx^m \ \rightarrow (2.1.12)$$

Remark 2.1.19

If we define

$$\Gamma(M) = \bigoplus_{r=0}^{\infty} \Gamma^r(M)$$

which is an associative , anti-commutative graded algebra , then $f^*: \Gamma(\mathrm{N}) \rightarrow \Gamma(\mathrm{M})$ is a algebra homomorphism.

Definition (Pullback of A differential Form)2.1.20

Let $\eta \in \Omega^k(N)$. For vectors $v_1, \dots, v_k \in T_p M$ define

$$(f^*\eta)(p)(v_1, \dots, v_k) = \eta_{f(p)}(T_p v_1, \dots, T_p v_k)$$

then the map $f^*\eta: p \rightarrow (f^*\eta)(p)$ is a differential form on M. $f^*\eta$ is called the pullback of $\eta \ by \ f$.

Proposition 2.1.21

With $f: M \to N$ smooth map and $\eta_1, \eta_2 \in \Omega(N)$ we have

$$f^*(\eta_1 \wedge \eta_2) = f^*\eta_1 \wedge f^*\eta_2$$

Remark 2.1.22

Notice the space $\Omega_M^0(U)$ is just the space of smooth functions $C^\infty(U)$ and so unfortunately we have several notations for the same set:

$$C^\infty(U) = C_M^\infty(U) = \mathcal{F}_M(U) = \Omega_M^0(U).$$

All that follows and much of what we have done so far works well for $\Omega_M(U)$ whether $U = M$ or not and will also respect restriction map. Thus we will simply write Ω_M instead of $\Omega_M(U) or\ \Omega_M(M)$ and $\mathfrak{X}_M$ instead of $\mathfrak{X}(U)$ so forth. In fact, the exterior derivative $d$commutes with restrictions and so is really presheaf map.

The algebra of smooth differential forms Ω(U) is an example of a $\mathbb{Z}$ graded algebra over the ring $C^\infty(U)$ and is also a graded vector space over $\mathbb{R}$. We have for each $U \subset M$

1) the direct sum decomposition

$$\Omega(U) = \cdots \oplus \Omega^{-1}(U) \oplus \Omega^0(U) \oplus \Omega^1(U) \oplus \Omega^2 \ldots$$

Where $\Omega^k(U) = 0\ if\ k < 0\ or\ if\ k >$ dim (U);

2) The exterior product is a graded product

$$\alpha \wedge \beta \in \Omega^{k+l}(U)\ for\ \alpha \in \Omega^k(U)\ and\ \beta \in \Omega^l(U)$$

3) graded commutative:

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha\ \ for\ \alpha \in \Omega^k(U)\ \ and\ \ \beta \in \Omega^l(U).$$

Definition (Exterior $p-Forms$) 2.1.23

- In particular

$$\wedge_0 = \mathbb{R} \quad \text{and} \quad \wedge_1 = T_1 = E^*$$

- In other words, a 0-form is a smooth function, and a 1-form is a covector filed.

- The components of the $p-form\ \alpha$ are

$$\alpha_{i_1 \ldots i_p} = \alpha(e_{i_1}, \ldots e_{i_p})$$

Definition (Exterior Product) 2.1.24

1- Since the tensor product of two skew-symmetric tensors is not a skew-symmetric tensor is define the algebra of anti-symmetric tensors we need to define the anti-symmetric tensor product is called the exterior (or wedge) product .

2- If α is an p-form and β is an q-form then the exterior product of $\alpha\ and\ \beta$ is an $(p+q)-form\ \alpha\wedge\beta$ defined by :

$$\alpha\wedge\beta = \frac{(p+q)!}{p!q!} Alt(\alpha\otimes\beta) \quad \rightarrow \quad (2.1.13)$$

- In components

$$(\alpha\wedge\beta)_{i_1 \ldots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \ldots i_p} \beta_{i_{p+1} \ldots i_{p+q}]} \rightarrow \quad (2.1.14)$$

3- Let $\alpha \in \wedge_p\ be\ a\ p-form$. Then

$$p = \deg(\alpha)$$

Is called the degree (or rank) of α .

Theorem 2.1.25

The exterior product has the following properties

$$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma) \quad \text{(associativity)}$$

$$\alpha\wedge\beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta\wedge\alpha \quad \text{(anticommutativity)}$$

$(\alpha+\beta)\wedge\gamma=\alpha\wedge\gamma+\beta\wedge\gamma$ (distributivity)

2-2 The Lie Derivative :

Definition (Flows on Manifolds) 2.2.1

A local flow is a smooth map $f: \mathcal{N} \to \mathcal{M}$, $(t, m) \to f^t(m)$, where $\mathcal{N}$ is a balanced neighborhood of $\{0\} \times \mathcal{M}$ in $\mathbb{R} \times \mathcal{M}$, such that

(a) $f^0(m) = m$, $\forall m \in \mathcal{M}$.

(b) $f^t\left(f^s(m)\right) = f^{t+s}(m) for\ all\ s, t \in R$, $m \in \mathcal{M}$ such that

$$(s, m), (s+t, m), (t, f^s(m)) \in \mathcal{N}$$

When $\mathcal{N} = \mathbb{R} \times \mathcal{M}$, f is called a flow.

Definition 2.2.2

Let $f: \mathcal{N} \to \mathcal{M}$ be a local flow on $\mathcal{M}$. The infinitesimal generator of f is the vector field X on M defined by

$$X(p) = X_f(p) \coloneqq \left.\frac{d}{dx}\right|_{t=0} f^t(p) \quad , \quad \forall p \in \mathcal{M} \qquad \to \qquad (2.2.1)$$

Definition 2.2.3

The Lie derivative of the vector field Y with respect to the vector field X is the vector field $L_{\mathrm{x}}Y$ defined at a point x by

$$(L_{\mathrm{x}}Y)_x = \lim_{t\to 0}\frac{1}{t}\left(Y_{\varphi_t(x)} - \varphi_{t^*}Y_x\right) \qquad \to \qquad (2.2.2)$$

Remark 2.2.4

Notice that the equation (2.2.2) can be written as

$$(L_{\mathrm{x}}Y)_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_{-t^*}Y_{\varphi_t(x)} - Y_x\right)$$

Or

$$(L_{\mathrm{x}}Y)_x = \left.\frac{d}{dt}\left(\varphi_{-t^*}Y_{\varphi_t(x)}\right)\right|_{t=0}$$

Proposition 2.2.5

$$L_{\mathrm{x}}Y = [X, Y]$$

Proof

We compute in local coordinates

$$(L_{\mathrm{x}}Y)^i = \frac{d}{dt}\left(\varphi_{-t^*}Y_{\varphi_t(x)}\right)^i\Big|_{t=0}$$

$$= \frac{d}{dt}\left[(\varphi_{-t^*}))^i{}_j Y^j(\varphi_t(x))\right]\Big|_{t=0}$$

$$= \frac{d}{dt}(\varphi_{-t^*})^i{}_j\Big|_{t=0} Y^j(x) + \delta^i_j \frac{d}{dt} Y^j(\varphi_t(x))\Big|_{t=0}$$

$$= -\frac{\partial X^i}{\partial x^j} Y^j(x) + \frac{\partial X^i}{\partial x^j} X^j$$

$$= [X, Y]^i$$

We notice that in particular $L_X X = 0$.

Definition (Lie Derivative of Forms) 2.2.6

1- Let f be a function (0-form) on M . Then the Lie derivative of f with respect to X is a function $L_X f$ defined by

$$(L_X f)_X = \frac{d}{dt}(\varphi_t^* f)_x\Big|_{t=0}$$

$$= \frac{d}{dt} f(\varphi_t(x))\Big|_{t=0} \quad \rightarrow \quad (2.2.3)$$

2- Let α be a 1-form on M . The Lie derivative of α with respect to X is a 1-form $L_X\alpha$ defined by

$$(L_X\alpha)_x = \lim_{t\to 0}\frac{1}{t}(\ \varphi_t^*\alpha_{\varphi_t(x)} - \alpha_x\ )$$

$$= \frac{d}{dt}(\varphi_t^*\alpha)_x\Big|_{t=0} \quad \rightarrow \quad (2.2.4)$$

3- Let α be a $p - form$ on M . The Lie derivative of α with respect to X

is a $p-form\ L_X\alpha$ defined by

$$(L_X\alpha)_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_t^*\alpha_{\varphi_t(x)} - \alpha_x\right)$$

$$= \frac{d}{dt}(\varphi_t^*\alpha)_x\Big|_{t=0} \qquad \rightarrow \qquad (2.2.5)$$

Proposition 2.2.7

The Lie derivative of a function f with respect to a vector field X is equal to

$$L_X f = X(f) \qquad \rightarrow \qquad (2.2.6)$$

We notice that in local coordinates

$$L_X f = X^i \partial_i f \qquad \rightarrow \qquad (2.2.7)$$

Proposition 2.2.8

The Lie derivative of a $p-form\ \alpha$ with respect to a vector field X is given by

$$(L_X\alpha)_{i_1\dots i_p} = X^j\partial_j\alpha_{i_1\dots i_p} + \alpha_{j i_2\dots i_p}\partial_{i_1}X^j + \dots + \alpha_{i_1\dots i_{p-1}j}\partial_{i_p}X^j$$

Definition (Lie Derivative of Tensors) 2.2.9

Let T be a tensor field of type (p , q) on M . The Lie derivative of T with respect to X is a tensor field $L_X T$ of type (p , q) defined by

$$(L_X T)_x = \lim_{t\to 0}\frac{1}{t}\left(\varphi_{-t}^* T_{\varphi_t(x)} - T_x\right)$$

$$= \frac{d}{dt}(\varphi_t^* T)_x\Big|_{t=0} \qquad \rightarrow \qquad (2.2.8)$$

Proposition 2.2.10

The Lie derivative of a tensor field T of type (p , q) with respect to a vector field X is given in local coordinates by

$$(L_X T)^{k_1\dots k_p}_{i_1\dots i_q} = X^j\partial_j T^{k_1\dots k_p}_{i_1\dots i_q} + T^{k_1\dots k_p}_{j i_2\dots i_q}\partial_{i_1}X^j + \dots + T^{k_1\dots k_p}_{i_1\dots i_{q-1}j}\partial_{i_q}X^j$$

$$- \; T^{j k_2\dots k_p}_{i_1\dots i_q}\partial_j X^{k_1} - \dots - T^{k_1\dots k_{p-1}j}_{i_1\dots i_q}\partial_{k_p}X^{k_p}$$

Theorem 2.2.11

For any two tensor T and R and a vector field X the Leibnitz rule holds

$$L_X(T \otimes R) = (L_X T) \otimes R + T \otimes (L_X R)$$

Theorem 2.2.12

Let α be a $p-form\ ,\beta\ be\ a\ q-form\ and\ X$ be a vector field on M . Then the Leibnitz rule holds

$$L_X(\alpha \wedge \beta) = (L_X\alpha)\wedge\beta + \alpha\wedge(L_X\beta)$$

Theorem 2.2.13

For any 1-form ω and vector fields $X\ and\ Y$ the Leibnitz rule holds

$$L_X\big(\omega(Y)\big) = (L_X\omega)(Y) + \omega(L_X Y)$$

Theorem 2.2.14

Let $X\ and\ Y_1, \dots, Y_p$ be vector fields on a manifold M and $\alpha \in \wedge_p$ be a

p-form . Then

$$L_X\left(\alpha\big(Y_1, \dots, Y_p\big)\right) = (L_X\alpha)\big(Y_1, \dots, Y_p\big) + \sum_{i=1}^{p} \alpha(Y_1, \dots, L_X Y_i, \dots, Y_p)$$

Theorem 2.2.15

Let $X\ and\ Y$ be any two vector fields and c ∈ ℝ . Then

1- $L_{X+Y} = L_X + L_Y$

2- $L_{cX} = cL_X$

3- $L_X Y = -L_Y X$

4- $[L_X\ , L_Y] = L_{[X\ ,Y]}$

2-3 Exterior Derivative:

From now on , if not specified otherwise , we will denote the derivatives by :
$\partial_i = \frac{\partial}{\partial x^i}$

Definition (Exterior Derivative) 2.3.1

1- The Exterior derivative of a 0-form (that is , a function) f is a 1-form df defined by : for any vector V

$$(df)(V) = V(f) \qquad \rightarrow \quad (2.3.1)$$

- In local coordinates

$$df = \partial_j f dx^j \qquad \rightarrow \quad (2.3.2)$$

2- The Exterior derivative of a 1-form (that is , a covector) A is a 2-form dA defined by : for any vectors V , W

$$(dA)(V,W) = V\big(A(W)\big) - W\big(A(V)\big) - A([V,W]) \rightarrow \quad (2.3.3)$$

- In local coordinates

$$dA = \frac{1}{2}\big(\partial_i A_j - \partial_j A_i\big) dx^i \wedge dx^j \qquad \rightarrow \quad (2.3.4)$$

3- Let α be a p-form

$$\alpha = \alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \qquad \rightarrow \quad (2.3.5)$$

The Exterior derivative of α is a (p+1)-form $d\alpha$ defined by

$$d\alpha = \frac{1}{p!} d\alpha_{i_1 \dots i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

$$= \frac{1}{p!} \partial_{i_1} \alpha_{i_2 \dots i_{p+1}} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{p+1}} \rightarrow \quad (2.3.6)$$

- In components

$$(d\alpha)_{i_1 i_2 \dots i_{p+1}} = (p+1) \partial_{i_1} \alpha_{i_2 \dots i_{p+1}}$$

$$= \sum_{k=1}^{p+1} (-1)^{k-1} \partial_{i_k} \alpha_{i_1 \dots i_{k-1} i_{k+1} \dots i_{p+1}} \rightarrow (2.3.7)$$

Theorem 2.3.2

The exterior derivative is a linear map

$$d\colon \Lambda_p \to \Lambda_{p+1}$$

Theorem 2.3.3

Let $\alpha \in \Lambda_p$ be a p-form and $\{\mathrm{v}_1, \dots, \mathrm{v}_{p+1}\}$ be a collection of $(p+1)$ vectors . Then

$$(d\alpha)(\mathrm{v}_1, \dots, \mathrm{v}_{p+1}) = \sum_{k=1}^{p+1}(-1)^{k-1}\mathrm{v}_k(\alpha(\mathrm{v}_1, \dots, \mathrm{v}_{k-1}, \mathrm{v}_{k+1}, \dots \mathrm{v}_{p+1})) - \sum_{k=1}^{p+1}\sum_{i=1}^{k-1}(-1)^{i+k-1}\alpha([\mathrm{v}_i, \mathrm{v}_k], \mathrm{v}_1, \dots, \mathrm{v}_i, \mathrm{v}_{i+1}, \dots, \mathrm{v}_{k-1}, \mathrm{v}_{k+1}, \dots, \mathrm{v}_{p+1})$$

Theorem 2.3.4

For any $p - form$

$$d^2 = 0$$

Theorem 2.3.5

The exterior derivative $d : \Lambda \to \Lambda$ is an anti-derivation on the exterior algebra .

That is , for any $p - form\ \alpha \in \Lambda_p$ and any $q - form\ \beta \in \Lambda_q$ there holds

$$d(\alpha \wedge \beta) = (d\alpha)\wedge\beta + (-1)^p \alpha \wedge (d\beta)$$

2-4 Covariant Derivatives:

Definition (Covariant Derivative) 2.4.1

The covariant derivative ∇ of a vector field Y in the direction of a vector field X is a vector field $\nabla_X Y$ that is uniquely defined by the following properties

1- $\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1} + f_2\nabla_{X_2}$

2- $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$

3- $\nabla_X(fY) = f\nabla_X Y + (Xf)Y$

Note that a linear combination of covariant derivatives is also a covariant derivative.

Definition 2.4.2

Let $F: R^3 \rightarrow R^3$ be a vector field , let $v_p \in T_pR^3$, then the covariant derivative of F with respect to v_p is :

$$\nabla_v F := \frac{d}{dt} F(p+tv)(0) \qquad \rightarrow \qquad (2.4.1)$$

Definition 2.4.3

A vector field Y in (M, ∇) along a curve γ is said to be parallel if $\nabla_{\dot{\gamma}} Y = 0 , \forall t . If \ \nabla_X Y = 0 , \forall X , then \ Y$ is said to be parallel . This is equivalent to $\sum_j Y^j \Gamma_{ij}^k \partial_k + \partial_i Y^k = 0 \quad \forall i \ \forall k.$

Definition 2.4.4

A curve $\gamma \ on \ (M , \nabla)$ is a geodesic if $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \ for \ all \ t$.

Definition 2.4.5

Let (M , ∇) be a manifold with a certain coordinate system ξ. If the vector fields $\left\{\partial_i = \frac{\partial}{\partial \xi^i}\right\}$ are parallel, we say that ξ is an affine coordinate system (with respect to ∇) (this is equivalent to the requirement that $\nabla_{\partial_i} \partial_j = 0, \forall i, j \ or \ \Gamma_{ij}^k = 0 , \forall i, j, k$).

If an affine coordinate system exists for (M , ∇) we say that ∇ (or M with respect to∇) is flat.

2-5 Grassman Algebra for Differential Forms (Exterior Algebra):

Definition (Grassman Algebra Λ) 2.5.1

The exterior algebra Λ (or Grassmann algebra) is the set of all forms of all degrees , that is

$$\Lambda = \Lambda_0 \oplus \ldots \oplus \Lambda_n \qquad \rightarrow \qquad (2.5.1)$$

- The dimension of the exterior algebra is

$$dim\Lambda = \sum_{p=0}^{n} \binom{n}{p} = 2^n \quad \rightarrow \quad (2.5.2)$$

- A basis of the space Λ_p is

$$\sigma^{i_1} \wedge \dots \wedge \sigma^{i_p} \quad , \quad 1 \le i_1 \le \dots \le i_p \le n \quad \rightarrow \quad (2.5.3)$$

An p-form α can be represented in one of the following ways

$$\alpha = \alpha_{i_1 \dots i_p} \sigma^{i_1} \otimes \dots \otimes \sigma^{i_p}$$

$$= \frac{1}{p!} \alpha_{i_1 \dots i_p} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p}$$

$$= \sum_{i_1 \le \dots \le i_p} \alpha_{i_1 \dots i_p} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_p} \rightarrow \quad (2.5.4)$$

- The exterior product of a p-form α and a q-form β can be represented as

$$\alpha \wedge \beta = \frac{1}{p!q!} \alpha_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]} \sigma^{i_1} \wedge \dots \wedge \sigma^{i_{p+q}}$$

Theorem 2.5.2

Let $\sigma^j \in \Lambda_1 , 1 \le j \le n \; and \; \alpha^j \in \Lambda_1 , 1 \le j \le n$, be two collections of n 1-forms releated by a linear transformation

$$\alpha^j = \sum_{i=1}^{n} A^j{}_i \sigma^j \quad , \quad 1 \le j \le n$$

Then

$$\alpha^1 \wedge \dots \wedge \alpha^n = detA_j^i \sigma^1 \wedge \dots \wedge \sigma^n$$

Theorem 2.5.3

Let $\alpha^j \in \Lambda_1 = \mathrm{E}^*$, $1 \le j \le p$, be a collections of p 1-forms and $\mathrm{v}_i \in \mathrm{E}$, $1 \le \mathrm{i} \le \mathrm{p}$, be a collection of p vectors . Let

$$A_{\mathrm{i}}^{j} = \alpha^j(\mathrm{v_i}) \quad , \quad 1 \le \mathrm{i}, j \le \mathrm{p}$$

Then

$$(\alpha^1 \wedge \dots \wedge \alpha^{\mathrm{p}})(\mathrm{v}_1, \dots, \mathrm{v_p}) = detA_j^{\mathrm{i}}$$

Theorem 2.5.4

A collections of p 1-forms $\alpha^j \in \Lambda_1 = \mathrm{E}^*$, $1 \leq j \leq p$ is linearly dependent if and only if

$$\alpha^1 \wedge \ldots \wedge \alpha^{\mathrm{p}} = 0$$

Corollary 2.5.5

Let $\mathrm{x}^{\mathrm{i}} = \mathrm{x}^{\mathrm{i}}(\acute{\mathrm{x}})$, $\mathrm{i} = 1, \ldots,$ n be a local diffeomorphism . Then

$$\mathrm{dx}^1 \wedge \ldots \wedge \mathrm{dx}^{\mathrm{n}} = \det\left(\frac{\partial \mathrm{x}^{\mathrm{i}}}{\partial \acute{\mathrm{x}}^{\mathrm{m}}}\right) \mathrm{d}\acute{\mathrm{x}}^1 \wedge \ldots \wedge \mathrm{d}\acute{\mathrm{x}}^{\mathrm{n}}$$

2-6 Integration Theory on Manifolds:

One-dimensional Integrals 2.6.1

A one-dimensional manifold C is described by a single coordinate t . Consider an interval on the manifold bounded by $t = a \; and \; t = b$. There are two possible orientations of this manifold , from $t = a \;\; to \; t = b$ or from $t = b \; to \; t = a$.

Suppose for the sake of definiteness that the manifold has the first orientation .

Then the differential form $f(t)dt$ has the integral

$$\int_C f(t)\, dt = \int_{t=a}^{t=b} f(t)\, dt \qquad \rightarrow \qquad (2.6.1)$$

If s is another coordinate , then t is related to s by $t = g(s)$. Futhermore , there are numbers $p , q \; such \; that \; a = g(p) and \; b = g(q)$.

The differential form is thus $f(t)dt = f\big(g(s)\big)\acute{g}(s)ds$. The end points of the manifold are $s = p \; and \; s = q$. Thus

$$\int_C f(t)\, dt = \int_{s=p}^{s=q} f\big(g(s)\big)\, \acute{g}(s) ds \qquad \rightarrow \qquad (2.6.2)$$

The value of the integral thus does not depend on which coordinate is used .

Notice that this calculation depends on the fact that $\frac{dt}{ds} = \acute{g}(s)$ is non-zero . However we could also consider a smooth function u on the manifold that is not a coordinate . Several points on the manifold could give the same value of u , and $\frac{du}{ds}$ could be zero at various places .

However we can express $u = h(s)$ and $\frac{du}{ds} = \acute{h}(s)$ and define an integral

$$\int_C f(u)\,du = \int_{s=p}^{s=q} f\big(h(s)\big)\,\acute{h}(s)ds \qquad \rightarrow \qquad (2.6.3)$$

Thus the differential form $f(u)du$ also has a well-defined integral on the manifold , even though u is not a coordinate .

Integration on Manifolds 2.6.2

Next look at the two dimensional case . Say that we have a coordinate system x , y in a two-dimensional oriented manifold . Consider a region R bounded by curves $x = a , x = b , and\ by\ y = c,\ y = d$. Suppoes that the orientation is such that one goes around the region in the order a , b then c , d $then$ b , a $then$ d , c . Then the differential form $f(x, y)dxdy$ has integral

$$\int_R f(x,y)\,dxdy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx \qquad \rightarrow \qquad (2.6.4)$$

The limits are taken by going around the region in the order given by the orientation , first a , b $then$ c , d. We could also have taken b , a $then$ d, c and obtained the same result .

Notice , by the way , that we could also define an integral with $dydx$ in place of $dxdy$. This would be

$$\int_R f(x,y)dydx = \int_b^a \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_b^a f(x,y)dx\right] dy \qquad \rightarrow \qquad (2.6.5)$$

The limits are taken by going around the region in the order given by the orientation , first c , d $then$ b, a. We could also have taken d , c then a, b and obtained the same result . This result is precisely the negative of the previous result. This is consistent with the fact that $dydx = -dxdy$.

These formula have generalizations . Say that the region is given letting $go\ from\ a\ to\ b\ and\ y\ from\ h(x)\ to\ k(x)$. Alternatively , it might

be given by letting $y\ go\ from\ c\ to\ d\ and\ x\ from\ p(y)\ to\ q(y)$.

This is a more general region than a rectangle , but the same kind of formula applies :

$$\int_R f(x,y)dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx\right] dy = \int_a^b \left[\int_{h(x)}^{k(x)} f(x,y)dy\right] dx \quad \rightarrow \ (2.6.6)$$

There is yet one more generalization , to the case where the differential form is $f(u,v)dudv\ , but\ \ u\ , v$ do not form a coordinate system .

Thus , for instance , the $1 - form\ du$ might be a multiple of dv at a certain point , so that $dudv$ would be zero at the point . However we can define the integral by using the customary change of variable formula :

$$\int_R f(u,v)dudv = \int_R f(u,v)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) dxdy \quad \rightarrow (2.6.7)$$

In fact , since $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\ \ and\ dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$, this is just saying that the same differential form has the same integral .

In fact , we could interpret this integral directly as a limit of sums involving only the $u\ , v$ increments . partition the manifold by curves of constant x and constant y .

This divides the manifold into small regions that look something like parallelograms . Then we could write this sum as

$$\int_R f(u,v)\, dudv \approx \sum f(u,v)(\Delta u_x \Delta v_x - \Delta v_x \Delta u_y) \quad \rightarrow (2.6.8)$$

Here the sum is over the parallelograms . The quantity Δu_x is the increment in u from x to $x + \Delta x$, keeping y fixed , along one side of the parallelogram . The quantity Δv_y is the increment in v from y to $y + \Delta y$, keeping y fixed , along one side of the parallelogram. The other quantities are defined similarly . The $u\ , v$ value is evaluated somewhere inside the parallelogram . The minus sign seems a bit surprising , until one realizes that going around the oriented boundary of the parallelogram the proper orientation makes a change from $x\ to\ x + \Delta x$ followed by a change from $y\ to\ y + \Delta y$, or a change from $y\ to\ y + \Delta y$ followed by a change from $x + \Delta x\ to\ x$ .So both terms have the form $\Delta u \Delta v$, where the changes are now taken along two sides in the proper orientation , first the change in u, then the change in v .

The Fundamental Theorem 2.6.3

The fundamental theorem of calculus says that for every scalar function s we have

$$\int_C \; ds = s(Q) - s(P) \qquad \rightarrow \quad (2.6.9)$$

Here C is an oriented path from point P to point Q.

Notice that the result does not depend on the choice of path.

This is because ds is an exact form.

As an example , we can take a path in space.

Then $ds = \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy + \frac{\partial s}{\partial z} dz$, so

$$\int_C \; ds = \int_C \; \frac{\partial s}{\partial x} dx + \frac{\partial s}{\partial y} dy + \frac{\partial s}{\partial z} dz = \int_C \; \left(\frac{\partial s}{\partial x}\frac{dx}{dt} + \frac{\partial s}{\partial y}\frac{dy}{dt} + \frac{\partial s}{\partial z}\frac{dz}{dt}\right) dt \quad \rightarrow \quad (2.6.10)$$

By the chain rule this is just

$$\int_C \; ds = \int_C \; \frac{ds}{dt} dt = s(Q) - s(P) \qquad \rightarrow \quad (2.6.11)$$

Green's Theorem 2.6.4

The next integral theorem is Green's theorem .it say that

$$\int_R \; \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy = \int_{\partial R} pdx + qdy \qquad \rightarrow \quad (2.6.12)$$

Here R is an oriented region in two dimensional space , and ∂R is the curve that is its oriented boundary . Notice that this theorem may be stated in the succinct form

$$\int_R \; d\alpha = \int_{\partial R} \alpha \qquad \rightarrow \quad (2.6.13)$$

The proof of Green's theorem just amounts to applying the fundamental theorem of calculus to each term.

Thus for the second term one applies the fundamental theorem of calculus in the x variable for fixed y.

$$\int_R \frac{\partial q}{\partial x} dxdy = \int_c^d \left[\int_{C_y} qdx\right] dy = \int_c^d \left[q\left(C_y^+\right) - q\left(C_y^-\right)\right] dy \rightarrow (2.6.14)$$

This is

$$\int_c^d q\left(C_y^+\right) dy + \int_d^c q\left(C_y^-\right) dy = \int_{\partial R} q\, dy \quad \rightarrow \quad (2.6.15)$$

The other term is handled similarly , except that the fundamental theorem of calculus is applied with respect to the x variable for fixed y.

Then such regions can be pieced together to give the general Green's theorem.

Stokes's Theorem 2.6.5

The most common version of Stokes's theorem says that for a oriented two dimensional surface S in a three dimensional manifold with oriented boundary curve ∂S we have

$$\int_S \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) dydz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right) dzdx + \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dxdy =$$

$$\int_{\partial S} (pdx + qdy + rdz) \quad \rightarrow \quad (2.6.16)$$

Again this has the simple form

$$\int_S d\alpha = \int_{\partial S} \alpha \quad \rightarrow \quad (2.6.17)$$

This theorem reduces to Green's theorem. The idea is to take coordinates u , v on the surface S and apply Green's theorem in the u , v coordinates.

In the theorem the left hand side is obtained by taking the form

$pdx + qdy + rdz$ and applying d to it. The key observation is that when the result of this is expressed in the u , v coordinates, it is the same as if the form $pdx + qdy + rdz$ were first expressed in the u , v coordinates and then d were applied to it. In this latter form Green's theorem applies directly.

Here is the calculation. To make it simple, consider only the pdx term.

Then taking d gives

$$d(pdx) = \left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right)dx = \frac{\partial p}{\partial z}dzdx - \frac{\partial p}{\partial y}dxdy \quad \rightarrow \quad (2.6.18)$$

In u , v coordinates this is

$$d(pdx) = \left[\frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right) - \frac{\partial p}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)\right]dudv \quad \rightarrow \quad (2.6.19)$$

There are four terms in all.

Now we do it in the other. In u , v coordinates we have

$$pdx = p\frac{\partial x}{\partial u}du + p\frac{\partial x}{\partial v}dv \quad \rightarrow \quad (2.6.20)$$

Taking d of this gives

$$d\left(p\frac{\partial x}{\partial u}du + p\frac{\partial x}{\partial v}dv\right) = \left[\frac{\partial}{\partial u}\left(p\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(p\frac{\partial x}{\partial u}\right)\right]dudv \quad \rightarrow \quad (2.6.21)$$

The miracle is that the second partial derivatives cancel.

So in this version

$$d\left(p\frac{\partial x}{\partial u}du + p\frac{\partial x}{\partial v}dv\right) = \left[\frac{\partial p}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial p}{\partial v}\frac{\partial x}{\partial u}\right]dudv \quad \rightarrow \quad (2.6.22)$$

Now we can express $\frac{\partial p}{\partial u}$ and $\frac{\partial p}{\partial v}$ by the chain rule. This gives at total of six terms. But two of them cancel, so we get the same result as before.

Gauss's Theorem 2.6.6

Let W be an oriented three dimensional region, and let ∂W be the oriented surface that forms its boundary.

Then Gauss's theorem states that

$$\int_W \left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z}\right)dxdydz = \int_{\partial W} adydz + bdzdx + cdxdy \rightarrow (2.6.23)$$

Again this has the form

$$\int_W d\sigma = \int_{\partial W} \sigma \quad \rightarrow \quad (2.6.24)$$

Where now σ is a 2-form.

The proof of Gauss's theorem is similar to the proof of Green's theorem.

The generalized Stokes's Theorem 2.6.7

The generalized Stokes's theorem says that

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega \qquad \rightarrow \qquad (2.6.25)$$

Here ω is $a(k-1) - form, and\ d\omega\ is\ a\ k - form.$ Furthermore, Ω is a k-dimensional region , and $\partial\Omega$ is its (k-1)-dimensional oriented boundary.

The forms may be expressed in arbitrary coordinate systems.

Chapter Three

Clifford $K\ddot{a}hler$ Manifolds

3-1 Complex Manifolds:

Some Complex Linear Algebra 3.1.1

The set of all $n - tuples\ of\ complex$ $\mathbb{C}^n$ numbers is a complex vector space and by choice of a basis, every complex vector space of finite dimension (over $\mathbb{C}$) is linearly isomorphic to $\mathbb{C}^n$ $for\ some$.

Now multiplication by $i \coloneqq \sqrt{-1}$ is a complex linear map $\mathbb{C}^n \rightarrow \mathbb{C}^n$ and since $\mathbb{C}^n$ is also a real vector space $\mathbb{R}^{2n}$ under the identification

$$(x^1 + iy^1, \dots, x^n + iy^n) \rightleftharpoons (x^1, y^1, \dots, x^n, y^n)$$

We obtain multiplication by i as a real linear map $J_0 :\ \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ given by the matrix

$$\begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{bmatrix}$$

Conversely, if V is a real vector space of dimension $2n$ and there is a map $J : V \rightarrow V\ with\ J^2 = -1$ then we can define the structure of a complex vector space on V by defining the scalar multiplication by complex numbers via the formula

$$(x + iy)v \coloneqq xv + yJv \quad for\ v \in V \qquad \rightarrow \quad (3.1.1)$$

Denote this complex vector space by V_J. Now if $e_1, \dots, e_n$ is a basis for V_J (over $\mathbb{C}$) then we claim that $e_1, \dots, e_n, Je_1, \dots, Je_n$ is a basis for $V\ for\ \mathbb{R}$.

We only need to show that $e_1, \dots, e_n, Je_1, \dots, Je_n$ span. For this let $v \in V$ and then for some complex numbers $c^i = a^i + \mathrm{i}b^j$ we have

$$\sum c^i e_i = \sum (a^j + \mathrm{i} b^j) e_j = \sum a^j e_j + \sum b^j J e_j \quad \rightarrow \quad (3.1.2)$$

Next we consider the complexification of V which is $V_{\mathbb{C}} := \mathbb{C} \otimes V$. Now any real basis $\{f_j\}$ of V is also a basis for $V_{\mathbb{C}}$ iff we identify f_j with $1 \otimes f_j$.

Furthermore, the linear map $J : V \rightarrow V$ extends to a complex linear map $J : V_{\mathbb{C}} \rightarrow V_{\mathbb{C}}$ and still satisfies $J^2 = -1$. Thus this extension has eigen values i and $-\mathrm{i}$. Let $V^{1,0}$ be the i eigenspace and $V^{0,1}$ be the – i eigen space. Of course we must have $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. The reader may check that the set of vectors $\{e_1 - \mathrm{i} J e_1, \dots, e_n - \mathrm{i} J e_n\}$ $span\ V^{1,0}$ while $\{e_1 + \mathrm{i} J e_1, \dots, e_n + \mathrm{i} J e_n\}$ $span\ V^{0,1}$.

Lemma 3.1.2

There is a natural complex linear isomorphism $V_J \cong V^{1,0}$ given by $e_i \longmapsto e_i - \mathrm{i} J e_i$. Furthermore, the conjugation map on $V_{\mathbb{C}}$ interchanges the spaces $V^{1,0}$ and $V^{0,1}$.

Let us apply these considerations to the simple case of the complex plane $\mathbb{C}$. The realification is $\mathbb{R}^2$ $and\ the\ map\ J\ is$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If we identify the tangent space of $\mathbb{R}^{2n}$ $at\ 0\ with$ $\mathbb{R}^{2n}$ itself then $\left\{ \left.\frac{\partial}{\partial x^i}\right|_0, \left.\frac{\partial}{\partial y^i}\right|_0 \right\}_{1 \le i \le n}$ $is\ basis\ for\ \mathbb{R}^{2n}$. A complex basis for $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J_0)$ is for instance $\left\{ \left.\frac{\partial}{\partial x^i}\right|_0 \right\}_{1 \le i \le n}$. A complex basis for $\mathbb{R}^2_J \cong \mathbb{C}$ is $e_1 = \left.\frac{\partial}{\partial x}\right|_0$ and so $\left.\frac{\partial}{\partial x}\right|_0, J \left.\frac{\partial}{\partial x}\right|_0$ is a basis for $\mathbb{R}^2$. This is clear any way since $J \left.\frac{\partial}{\partial x}\right|_0 = \left.\frac{\partial}{\partial y}\right|_0$. Now the complexification of $\mathbb{R}^2$ is $\mathbb{R}^2_{\mathbb{C}}$ which has basis consisting of $e_1 - \mathrm{i} J e_1 = \left.\frac{\partial}{\partial x}\right|_0 - \mathrm{i} \left.\frac{\partial}{\partial y}\right|_0$ and $e_1 + \mathrm{i} J e_1 = \left.\frac{\partial}{\partial x}\right|_0 + \mathrm{i} \left.\frac{\partial}{\partial y}\right|_0$. These are usually denoted by $\left.\frac{\partial}{\partial z}\right|_0$ and $\left.\frac{\partial}{\partial \bar{z}}\right|_0$. More generally, we see that if $\mathbb{C}^n$ is reified to $\mathbb{R}^{2n}$ which is then complexified to $\mathbb{R}^{2n}_{\mathbb{C}} \coloneqq \mathbb{C} \otimes \mathbb{R}^{2n}$ then a basis for $\mathbb{R}^{2n}_{\mathbb{C}}$ is given by

$$\left\{ \left.\frac{\partial}{\partial z^1}\right|_0, \dots, \left.\frac{\partial}{\partial z^n}\right|_0, \left.\frac{\partial}{\partial \bar{z}^1}\right|_0, \dots, \left.\frac{\partial}{\partial \bar{z}^n}\right|_0 \right\}$$

Where

$$2\frac{\partial}{\partial z^i}\bigg|_0 := \frac{\partial}{\partial x^i}\bigg|_0 - \mathrm{i}\frac{\partial}{\partial y^i}\bigg|_0 \quad and \quad 2\frac{\partial}{\partial \bar{z}^i}\bigg|_0 := \frac{\partial}{\partial x^i}\bigg|_0 + \mathrm{i}\frac{\partial}{\partial y^i}\bigg|_0$$

Now if we consider the tangent bundle $U \times \mathbb{R}^{2n}$ of an open set $U \subset \mathbb{R}^{2n}$ then we have the vector fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$. We can complexify the tangent bundle of $U \times \mathbb{R}^{2n}$ to get $U \times \mathbb{R}^{2n}_{\mathbb{C}}$ and then following the ideas above we have that the fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$ also span each tangent space $T_pU := \{p\} \times \mathbb{R}^{2n}_{\mathbb{C}}$.
On the other hand, so do the fields $\left\{\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}\right\}$.

Now if $\mathbb{R}^{2n}$ had a complex vector space structure, say $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J_0)$, then J_0 defines a bundle map $J_0: T_pU \to T_pU \text{ } given \text{ } by \text{ } (p, v) \longmapsto (p, J_0 v)$ This can be extended to a complex bundle map $J_0: TU_{\mathbb{C}} = \mathbb{C}\otimes TU \to TU_{\mathbb{C}} = \mathbb{C}\otimes TU$ and we get a bundle decomposition

$$\mathbb{C}\otimes TU = T^{1,0}U \oplus T^{0,1}U$$

Where $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}$ spans $T^{1,0}U$ at each point and $\frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n}$ spans$T^{0,1}U$.

Now the symbols $\frac{\partial}{\partial z^1}$ etc., already have a meaning a differential operators.

Let us now that this view is at least consistent with what we have done above. For a smooth complex valued function $f: U \subset \mathbb{C}^n \to \mathbb{C}$ we have for $p = (z_1, \dots, z_n) \in U$

$$\frac{\partial}{\partial z^i}\bigg|_p f = \frac{1}{2}\left(\frac{\partial}{\partial x^i}\bigg|_p f - \mathrm{i}\frac{\partial}{\partial y^i}\bigg|_p f\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}\bigg|_p u - \mathrm{i}\frac{\partial}{\partial y^i}\bigg|_p u - \frac{\partial}{\partial x^i}\bigg|_p \mathrm{i}v - \mathrm{i}\frac{\partial}{\partial y^i}\bigg|_p \mathrm{i}v\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x^i}\bigg|_p + \frac{\partial v}{\partial y^i}\bigg|_p\right) + \frac{\mathrm{i}}{2}\left(\frac{\partial u}{\partial y^i}\bigg|_p - \frac{\partial v}{\partial x^i}\bigg|_p\right) \quad \to \quad (3.1.3)$$

And

$$\left.\frac{\partial}{\partial \bar{z}^i}\right|_p f = \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p f + \mathrm{i}\left.\frac{\partial}{\partial y^i}\right|_p f\right)$$

$$= \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p u + \mathrm{i}\left.\frac{\partial}{\partial y^i}\right|_p u + \left.\frac{\partial}{\partial x^i}\right|_p \mathrm{i}v + \mathrm{i}\left.\frac{\partial}{\partial y^i}\right|_p \mathrm{i}v\right)$$

$$= \frac{1}{2}\left(\left.\frac{\partial u}{\partial x^i}\right|_p - \left.\frac{\partial v}{\partial y^i}\right|_p\right) + \frac{\mathrm{i}}{2}\left(\left.\frac{\partial u}{\partial y^i}\right|_p + \left.\frac{\partial v}{\partial x^i}\right|_p\right) \quad \rightarrow \quad (3.1.4)$$

Definition 3.1.3

A function $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ is called holomorphic if

$$\frac{\partial}{\partial \bar{z}^i} f \equiv 0 \qquad (all\ \ i\ )$$

On U. A function f is called antiholomorphic if

$$\frac{\partial}{\partial z^i} f \equiv 0 \qquad (\ all\ \ i\ )$$

Definition 3.1.4

A map $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}^m$ given by functions $f_1, \dots, f_m$ *is called* holomorphic (resp. antiholomorphic) if each component function $f_1, \dots, f_m$ is holomorphic (resp. antiholomorphic).

Now if $f: U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ is holomorphic then by definition $\left.\frac{\partial}{\partial \bar{z}^i}\right|_p f \equiv 0$ for all $p \in U$ and so we have the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x^i} = \frac{\partial v}{\partial y^i} \qquad \text{(Cauchy-Riemann)}$$

$$\frac{\partial v}{\partial x^i} = -\frac{\partial u}{\partial y^i} \qquad \rightarrow \qquad (3.1.5)$$

And from this we see that for holomorphic f

$$\frac{\partial f}{\partial z^i} = \frac{\partial u}{\partial x^i} + i\frac{\partial v}{\partial x^i} = \frac{\partial f}{\partial x^i} \quad \rightarrow \quad (3.1.6)$$

Which means that as derivations on the sheaf $\mathcal{O}$ of locally defined holomorphic functions on $\mathbb{C}^n$, the operators $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial x^i}$ are equal. This corresponds to the complex isomorphism $T^{1,0}U \cong TU$,J_0 which comes from the isomorphism in lemma??. In fact, if one looks at a function $f\colon \mathbb{R}^{2n} \to \mathbb{C}$ as a differentiable map of real manifolds then with J_0 given the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$, out map f is holomorphic iff

$$Tf \circ J_0 = J_0 \circ Tf$$

Or in other words

$$\begin{pmatrix} \frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial y^1} & \\ \frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial y^1} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial y^1} & \\ \frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial y^1} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix}$$

This last matrix equation is just the Cauchy-Riemann equations again.

Definition (Complex Structure)3.1.5

A manifold M is said to be an almost complex manifold if there is a smooth bundle map $J\colon TM \to TM$, called an almost complex structure, having the property that $J^2 = -1$.

Definition 3.1.6

A complex manifold M is a manifold modelled on $\mathbb{C}^n$ for some n, together with an atlas for M such that the transition functions are all holomorphic maps. The charts from this atlas are called holomorphic charts. We also use the phrase ‘‘holomorphic coordinates’’.

Lemma 3.1.7

Let $\psi\colon U \to \mathbb{C}^n$ be a holomorphic chart with $p \in U$. Then writing $\psi = (z^1, \ldots, z^n)$ and $z^k = x^k + \mathrm{i}y^k$ we have that the map $J_p\colon T_pM \to T_pM$ defined by

$$J_p \frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial}{\partial y^i}\bigg|_p$$

$$J_p \frac{\partial}{\partial y^i}\bigg|_p = -\frac{\partial}{\partial x^i}\bigg|_p$$

Is well defined independent of the choice of coordinates.

The maps J_p combine to give a bundle map $J: TM \to TM$ and so an almost complex structure on M called the almost complex structure induced by the holomorphic atlas.

Definition 3.1.8

An almost complex structure $J\ on\ M$ is said to be integrable if there it has a holomorphic atlas giving the map J as the induced almost complex structure. That is if there is an family of admissible charts $\psi_\alpha: U_\alpha \to \mathbb{R}^{2n}$ such that after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the charts form a holomorphic atlas with J the induced almost complex structure. In this case, we call J a complex structure.

Complex Tangent Structures 3.1.9

Let $\mathcal{F}_p(\mathbb{C})$ denote the algebra germs of complex valued smooth functions at p on a complex $n - manifld\ M$ thought of as a smooth real 2n-manifold with real tangent bundle TM. Let $Der_p(\mathcal{F})$ be the space of derivations this algebra. It is not hard to see that this space is isomorphic to the complexified tangent space $T_pM_{\mathbb{C}} = \mathbb{C} \otimes T_pM$.

The (complex) algebra of germs of holomorphic functions at a point p in a complex manifold is denoted $\mathcal{O}_p$ and the set of derivations of this algebra denoted $Der_p(\mathcal{O})$. We also have the algebra of germs of antiholomorphic functions at p which is $\bar{\mathcal{O}}_p$ and also $Der_p(\bar{\mathcal{O}})$.

If $\psi : U \to \mathbb{C}^n$ is a holomorphic chart then writing $\psi = (z^1, \dots, z^n)$ and $z^k = x^k + \mathrm{i}y^k$ we have the differential operators at $p \in U$:

$$\left\{ \frac{\partial}{z^i}\bigg|_p ,\ \frac{\partial}{\partial \bar{z}^i}\bigg|\ p \right\}$$

(now transferred to the manifold). To be pedantic about it, we now denote the coordinates on $\mathbb{C}^n$ by $w_i = u_i + \mathrm{i}v_i$ and then

$$\left.\frac{\partial}{z^i}\right|_p f := \left.\frac{\partial f\circ\psi^{-1}}{\partial w^i}\right|_{\psi(p)}$$

$$\left.\frac{\partial}{\bar{z}^i}\right|_p f := \left.\frac{\partial f\circ\psi^{-1}}{\partial \bar{w}^i}\right|_{\psi(p)}$$

Thought of derivations these span $Der_p(\mathcal{F})$ but we have also seen that they span the complexified tangent space at p. In fact, we have the following:

$$T_pM_{\mathbb{C}} = span_{\mathbb{C}}\left\{\left.\frac{\partial}{\partial z^i}\right|_p, \left.\frac{\partial}{\partial \bar{z}^i}\right|_p\right\} = Der_p(\mathcal{F})$$

$$T_pM^{1,0} = span_{\mathbb{C}}\left\{\left.\frac{\partial}{\partial z^i}\right|_p\right\}$$

$$= \left\{v \in Der_p(\mathcal{F}): vf = 0 \ for\ all\ f \in \bar{\mathcal{O}}_p\right\}$$

$$T_pM^{0,1} = span_{\mathbb{C}}\left\{\left.\frac{\partial}{\partial \bar{z}^i}\right|_p\right\}$$

$$= \left\{v \in Der_p(\mathcal{F}): vf = 0 \ for\ all\ f \in \mathcal{O}_p\right\}$$

$and\ of\ course$

$$T_pM = span_{\mathbb{R}}\left\{\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_p\right\}$$

The reader should go back and check that the above statements are consistent with our definitions as long as we view the $\left.\frac{\partial}{\partial z^i}\right|_p, \left.\frac{\partial}{\partial \bar{z}^i}\right|_p$ not only as the algebraic objects constructed above but also as derivations. Also, the definitions of $T_pM^{1,0}\ and\ T_pM^{0,1}$ are independent of the holomorphic coordinates since we also have

$$T_pM^{1,0} = ker\{J_p: T_pM \to T_pM\}$$

The Holomorphic Tangent Map 3.1.10

We leave it to the reader to verify that the construction that we have at each tangent space globalize to give natural vector bundles $TM_{\mathbb{C}}$, $TM^{1,0}$ and $TM^{0,1}$ (all with M as base space).

Let M and N be complex manifolds and let $f: M \to N$ be a smooth map.

The tangent map extend to a map of the complexified bundles $Tf: TM_{\mathbb{C}} \to TN_{\mathbb{C}}$.

Now $TM_{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1}$ and similarly $TM_{\mathbb{C}} = TN^{1,0} \oplus TN^{0,1}$.if f is holomorphic then $Tf\left(T_pM^{1,0}\right) \subset T_{f(p)}N^{1,0}$. In fact since it is easily verified that $Tf\left(T_pM^{1,0}\right) \subset T_{f(p)}N^{1,0}$ equivalent to the Cauchy-Riemann equations being satisfied by the local representative on F in any holomorphic chart we obtain the following.

Proposition 3.1.11

$Tf\left(T_pM^{1,0}\right) \subset T_{f(p)}N^{1,0}$ if and only if f is a holomorphic map.

The map given by the restriction $T_pf: T_pM^{1,0} \to T_{f(p)}N^{1,0}$ is called the holomorphic tangent map at p. Of course, these maps concatenate to give a bundle map.

Dual Spaces 3.1.12

Let M , J be complex manifold. The dual of $T_pM_{\mathbb{C}}\ is\ T_p^*M_{\mathbb{C}} = \mathbb{C} \otimes T_p^*M$. Now the map J has a dual bundle map $J^*: T^*M_{\mathbb{C}} \to T^*M_{\mathbb{C}}$ which must also satisfy $J^* \circ J^* = -1$and so we have the at each $p \in M$ the decomposition by eigen spaces

$$T_p^*M_{\mathbb{C}} = T_p^*M^{1,0} \oplus T_p^*M^{0,1}$$

Corresponding to the eigen values $\pm$i.

Definition 3.1.13

The space $T_p^*M^{1,0}$ is called the space of holomorphic co-vectors at p while $T_p^*M^{0,1}$ is the space of antiholomorphic co-vector at p.

We now choose a holomorphic chart $\psi: U \to \mathbb{C}^n\ at\ p$. Writing $\psi = (z^1, \dots, z^n)\ and\ z^k = x^k + \mathrm{i}y^k$ we have the 1-forms

$$dz^k = dx^k + \mathrm{i}dy^k$$

And

$$d\bar{z}^k = dx^k - \mathrm{i}dy^k$$

Equivalently, the point wise definitions are $dz^k\big|_p = dx^k\big|_p + \mathrm{i}dy^k\big|_\mathrm{p}$ and $d\bar{z}^k\big|_p = dx^k\big|_p - \mathrm{i}dy^k\big|_\mathrm{p}$. Notice that we have the expected relations:

$$dz^k\left(\frac{\partial}{\partial z^i}\right) = (dx^k + \mathrm{i}dy^k)\left(\frac{1}{2}\frac{\partial}{\partial x^i} - \mathrm{i}\frac{1}{2}\frac{\partial}{\partial y^i}\right)$$

$$= \frac{1}{2}\delta_j^k + \frac{1}{2}\delta_j^k = \delta_j^k$$

$$dz^k\left(\frac{\partial}{\partial \bar{z}^i}\right) = (dx^k + \mathrm{i}dy^k)\left(\frac{1}{2}\frac{\partial}{\partial x^i} + \mathrm{i}\frac{1}{2}\frac{\partial}{\partial y^i}\right)$$

$$= \frac{1}{2}\delta_j^k - \frac{1}{2}\delta_j^k = 0$$

And similarly

$$d\bar{z}^k\left(\frac{\partial}{\partial \bar{z}^i}\right) = \delta_j^k\ and\ d\bar{z}^k\left(\frac{\partial}{\partial z^i}\right) = \delta_j^k.$$

Let us check the action of J^* on these forms:

$$J^*(dz^k)\left(\frac{\partial}{\partial z^i}\right) = J^*(dx^k + \mathrm{i}dy^k)\left(\frac{\partial}{\partial z^i}\right)$$

$$= (dx^k + \mathrm{i}dy^k)\left(J\frac{\partial}{\partial z^i}\right)$$

$$= \mathrm{i}(dx^k + \mathrm{i}dy^k)\frac{\partial}{\partial z^i}$$

$$= \mathrm{i} dz^k \left(\frac{\partial}{\partial z^i}\right) = \mathrm{i}\delta_j^k$$

And

$$J^*(dz^k)\left(\frac{\partial}{\partial \bar{z}^i}\right) = (dz^k)\left(J\frac{\partial}{\partial \bar{z}^i}\right)$$

$$= -\mathrm{i}(dz^k)\left(\frac{\partial}{\partial \bar{z}^i}\right) = 0$$

$$= \mathrm{i} dz^k \left(\frac{\partial}{\partial \bar{z}^i}\right).$$

Thus we conclude that $dz^k\big|_p \in T_p^*M^{1,0}$. A similar calculation shows that $d\bar{z}^k\big|_p \in T_p^*M^{0,1}$ and in fact

$$T_p^*M^{1,0} = span\left\{dz^k\big|_p : k = 1, \dots, n\right\}$$

$$T_p^*M^{0,1} = span\left\{d\bar{z}^k\big|_p : k = 1, \dots, n\right\}$$

And $\left\{dz^1|_p, \dots, dz^n|_p, d\bar{z}^1|_p, \dots, d\bar{z}^n|_p\right\}$ is a basis for $T_p^*M_{\mathbb{C}}$.

Remark 3.1.14

If we don't specify base points then we are talking about fields (over some open set) which form a basis for each fiber separately. These are called frame fields $(e.g.\ \frac{\partial}{\partial z^i}\ ,\ \frac{\partial}{\partial \bar{z}^i})$ or co-frame fields $(e.g.\, dz^k, d\bar{z}^k)$.

Definition (Almost Complex Manifolds) 3.1.15

An almost complex structure on a differentiable manifold X is a differentiable endomorphism on the tangent bundle.

$$\mathrm{I} : \mathrm{T}_{\mathbb{R}}\mathrm{X} \to \mathrm{T}_{\mathbb{R}}\mathrm{X} \quad \text{with} \quad \mathrm{I}^2 = -\mathrm{id}$$

A differentiable manifold with some fixed almost complex structure is called an almost complex manifold.

Almost complex manifolds must be even dimensional.

Definition (Complex Manifolds) 3.1.16

Let M be configuration manifold of real dimension m . A tensor field J on TM is called an almost complex structure on TM if at every point p of M, J is endomorphism of the tangent space $T_p(TM)$ such that $J^2 = -1$.

A manifold TM with fixed almost complex structure J is called almost complex manifold.

Assume that (x_i) be coordinates of M and $(x_i\, , y_i)$ be a real coordinate system on a neighborhood U of any point p of TM. Also , let us to be $\left\{\left(\frac{\partial}{\partial x_i}\right)_p \, , \left(\frac{\partial}{\partial y_i}\right)_p\right\}$ and $\left\{\left(dx^i\right)_p ,\; \left(dy^i\right)_p\right\}$ to natural bases over R of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM , respectively.

Let TM be an almost complex manifold with fixed almost complex structure J. The manifold TM is called complex manifold if there exists an open covering $\{U\}\ of\ TM$ satisfying the following condition : There is a local coordinate system $(x_i\, , y_i)$ on each U , such that

$$J\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i} \quad , \qquad J\left(\frac{\partial}{\partial y_i}\right) = -\frac{\partial}{\partial x_i} \qquad \to \quad (3.1.7)$$

For each point of U . Let $z_i = x_i + \mathrm{i}y_i$, $\mathrm{i} = \sqrt{-1}$, be a complex local coordinate system on a neighborhood U of any point p of TM .

We define the vector fields $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial \bar{z}^i}$ by:

$$\left(\frac{\partial}{\partial z^i}\right)_p = \frac{1}{2}\left\{\left(\frac{\partial}{\partial x_i}\right)_p - \mathrm{i}\left(\frac{\partial}{\partial y_i}\right)_p\right\}, \left(\frac{\partial}{\partial \bar{z}^i}\right)_p = \frac{1}{2}\left\{\left(\frac{\partial}{\partial x_i}\right)_p + \mathrm{i}\left(\frac{\partial}{\partial y_i}\right)_p\right\} \to (3.1.8)$$

And the dual co-vector fields :

$$\left(dz^i\right)_p = \left(dx^i\right)_p + \mathrm{i}\left(dy^i\right)_p \;, d\bar{z}^i = \left(dx^i\right)_p - \mathrm{i}\left(dy^i\right)_p \quad \to \quad (3.1.9)$$

Which represent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)\ of\ TM$, respectively.

Then the endomorphism J is shown as :

$$J\left(\frac{\partial}{\partial z_i}\right) = \mathrm{i}\frac{\partial}{\partial z_i} \ \ , \ J\left(\frac{\partial}{\partial \bar{z}_i}\right) = -\mathrm{i}\frac{\partial}{\partial \bar{z}_i} \qquad \rightarrow \quad (3.1.10)$$

The dual endomorphism J^* of the cotangent space $T_p^*(TM)$ at any point p of manifold TM satisfies ${J^*}^2 = -1$ and is defined by

$$J^*(dz_i) = \mathrm{i}dz_i \ \ , \ J^*(d\bar{z}_i) = -\mathrm{i}d\bar{z}_i \qquad \rightarrow \qquad (3.1.11)$$

3-2 Clifford Algebra:

Recall that $\mathcal{C}\ell_{03}$ denotes the Clifford algebra with three generators $\{e_1, e_2, e_3\}$. It is a real unital associative 8-dimensional algebra for which there exists a special basis $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ such that

$$e_0 e_i = e_i e_0 = e_i \ , \quad i = 0,1,\dots,7,$$

$$e_i^2 = -e_0\ , \ e_7^2 = e_0 \ , i = 1,\dots,6\ ,$$

$$e_i e_j + e_j e_i = 0 \quad , \qquad i \neq j, i, j = 1,2,\dots,6 \ , i + j \neq 7\ ,$$

$$e_i e_j = e_j e_i \quad , \qquad i = 0,1,\dots,7, \ i \neq j\ , \ i + j = 7\ ,$$

$$e_1 e_2 = e_4 \ , \qquad e_1 e_3 = e_5 \ , \ e_2 e_3 = e_6.$$

For our comfort , we denote $\mathcal{O} = \mathcal{C}\ell_{03}$ and name its elements octons . The before introduced basis $\mathcal{B}_a = (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is called the canonical (or , natural) basis of $\mathcal{O}$.

The center of $\mathcal{O}$ is $Z(\mathcal{O}) = \mathbb{R}e_0 + \mathbb{R}e_7$.

It must be remarked that $Z(\mathcal{O}) \simeq \mathcal{D} where\ \mathcal{D}$ denotes the real (associative and commutative) algebra of the so-called double numbers.

A new basis $\mathcal{B}_a = (E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7)$, defined by

$$E_0 = \frac{1}{2}(e_0 + e_7)\ , E_1 = \frac{1}{2}(e_1 - e_6), E_2 = -\frac{1}{2}(e_2 + e_5), E_3 = \frac{1}{2}(e_3 - e_4),$$

$$E_4 = \frac{1}{2}(e_0 - e_7)\ , E_5 = \frac{1}{2}(e_1 + e_6),\ E_6 = \frac{1}{2}(e_2 - e_5),\ E_7 = \frac{1}{2}(e_3 + e_4),$$

Is associated with the canonical basis $\mathcal{B}_c$; it is named the adapted basis corresponding to $\mathcal{B}_c$. The multiplication table for , in the adapted basis $\mathcal{B}_a$ is :

$$E_0 E_i = E_i E_0 = E_i \,,\; i \in \{1,2,3\} \quad , \quad E_4 E_\alpha = E_\alpha E_0 = E_\alpha \;,\; \alpha \in \{5,6,7\} \,,$$

$$E_i^2 = -E_0 \,, \quad i \in \{1\,,2\,,3\} \quad , \qquad E_\alpha^1 = -E_4 \;\;,\; \alpha \in \{5\,,6\,,7\} \,,$$

$$E_i E_j = -E_j E_i = E_k \quad , \qquad\qquad E_\alpha E_\beta = -E_\beta E_\alpha = E_\gamma \,,$$

Where $(i\,,j\,,k)$ and $(\alpha\,,\beta\,,\gamma)$ are cyclic permutations of $\{1,2,3\}$ and $\{5\,,6\,,7\}$, respectively. Consequently , $\mathcal{O}$ can be naturally identified with $\mathbb{H}\oplus\mathbb{H}$.

Thus , the conjugation in $\mathbb{H}$ induces a conjugation in $\mathcal{O}$ defined by :

$$\mathcal{O} \ni a = a_0 e_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 \rightarrow$$

$$\rightarrow \bar{a} = a_0 e_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 - a_4 e_4 - a_5 e_5 - a_6 e_6 + a_7 e_7 \in \mathcal{O}.$$

Since $\overline{(\bar{a})} = a$ and $\overline{a.b} = \bar{b}.\bar{a}$, it results that this conjugation is an anti-involution . Moreover,

$$a\bar{a} = \left(\sum_{i=0}^{7} a_i^2\right)e_0 + (a_0 a_7 - a_1 a_6 + a_2 a_5 - a_3 a_4)e_7 \in \mathcal{D} \;\rightarrow (3.2.1)$$

So that the following two quadratic forms $h_1\,,h_2 \colon \mathcal{O} \rightarrow \mathbb{R}$ arise naturally

$$h_1(a) = \sum_{i=0}^{7} a_i^2 \quad , \; h_2(a) = a_0 a_7 - a_1 a_6 + a_2 a_5 - a_3 a_4 \;,\; \forall\, a \in \mathcal{O}.$$

h_1 is a positive defined quadratic form , while h_2 is nonsingular one having the signature (4 , 4).

The linear group preserving both these quadratic forms is isomorphic to $O(4,\mathbb{R}) \times O(4,\mathbb{R})$.

The presence of a natural conjugation on $\mathcal{O}$ suggests the possibility to define an (quasi-) inner product on it. Actually, (3.2.1) claims to consider a structure of $\mathcal{D} - module$ on M. Such a structure arise naturally since every element $a = a_0 e_0 + a_1 e_1 + \cdots + a_7 e_7 \in M$ has the form

$$a = (a_0 e_0 + a_7 e_7)e_0 + (a_1 e_0 - a_6 e_7)e_1 + (a_2 e_0 + a_5 e_7)e_2 + (a_3 e_0 - a_4 e_7)e_3 \;\rightarrow (3.2.2)$$

We define now the following bilinear $\mathcal{D}-form$ on the $\mathcal{D}-module\ \mathcal{O}$ by

$$\mathcal{O}\times\mathcal{O}\ni(a,b)\to\langle a,b\rangle\in\mathcal{D},$$

$$\langle a,b\rangle=\frac{1}{2}\left(a.\bar{b}+b.\bar{a}\right)\in\mathcal{D}\,,\quad\forall\, a,b\in\mathcal{O}\qquad\to\qquad(3.2.3)$$

Since it is a $\mathcal{D}-bilinear$ symmetric $\mathcal{D}-form$ satisfying the condition

$$\langle a,a\rangle=0\ \Leftrightarrow a=0$$

It will be called a quasi-inner product.

The set $G_{\mathcal{O}}$ consisting in all regular elements of $\mathcal{O}$ is a group. It is the product of two subgroups , namely $G_{\mathcal{O}}=\mathcal{O}(1).\mathcal{D}^*$, where

$\mathcal{O}(1)=\{a\in\mathcal{O}|a.\bar{a}=e_0\}$ and $\mathcal{D}^*$ is the set of all invertible elements from $Z(\mathcal{O})\cong\mathcal{D}.\mathcal{O}(1)$ and $\mathcal{D}^*$ are normal subgroup of $G_{\mathcal{O}}$ with

$\mathcal{O}(1)\cap\mathcal{D}^*=\{\pm e_0\,,\pm e_7\}$.

The $\mathcal{D}-module\ \mathcal{O}^m$ can be endowed with an quasi-inner product defined by

$$\langle p,q\rangle=\frac{1}{2}\sum_{i=1}^{m}(p_i.\bar{q}_i+q_i.\bar{p}_i)\ \ ,for\ all$$

$$p=(p_1,p_2,\dots,p_m),q=(q_1,q_2,\dots,q_m)\in\mathcal{O}^m\ .\quad\to\quad(3.2.4)$$

As it is usual , we define the group $Op(m)$ as being the consisting in all matrices $\sigma\in\mathcal{M}_m(\mathcal{O})$ such that $\langle\sigma p,\sigma q\rangle=\langle p,q\rangle\,,for\ all\ p,q\in\mathcal{O}^m$.

It is easily to prove that $\mathcal{O}(1)$ can be identified , via an isomorphism , with $Op(1)$.

The LIE algebra $\mathcal{O}^-$ associated by means of the usual bracket to the associative algebra $\mathcal{O}$ is isomorphic to $su(2)\oplus su(2)\oplus\mathcal{D}^-$.

It prove that $GL_n(\mathcal{O})$ can be isomorphically identified with a subgroup of $(8n\,,\mathbb{R})$, namely

$$GL_n(\mathcal{O})=\{\mathcal{T}\in GL(8n\,,\mathbb{R})|\mathcal{T}F_i=F_i\mathcal{T}\,,i=1,2,\dots,6\}\,;$$

Here $F_i(i=1,2,\dots,6)$ is the matrix of linear transformation $\mathcal{O}^n\to\mathcal{O}^n$, $q=(q_1,q_2,\dots,q_n)\to qe_i=(q_1e_i,q_2e_i,\dots,q_ne_i)where\ e_i$ is an element of canonical basis of $\mathcal{C}\ell_{03}$ in an admissible frame of $\mathcal{O}^n$. The Lie algebra $gl_n(\mathcal{O}) of\ GL_n(\mathcal{O})$ can be isomorphically identified with a subalgebra of $(8n\ ,\mathbb{R})$, namely

$$gl_n(\mathcal{O}) = \{\theta \in gl(8n\ ,\mathbb{R}) |\ \theta F_i = F_i\theta\ ,\ i = 1,2,\dots,6\}.$$

On the other hand, the Lie algebra g of $\mathcal{O}p_1.GL_n(\mathcal{O})$ can be isomorphically identified with a subalgebra of $gl(8n\ ,\mathbb{R})$, namely

$$g = \left\{ \theta \in gl(8n\ ,\mathbb{R}) \left| \begin{array}{c} \theta F_1 - F_1\theta = dF_2 + eF_3 - bF_4 - cF_5\ , \\ \theta F_2 - F_2\theta = -dF_1 + fF_3 + aF_4 - cF_6, \\ \theta F_3 - F_3\theta = -eF_1 - fF_2 + aF_5 + bF_6, \\ \theta F_4 - F_4\theta = bF_1 - aF_2 + fF_5 - eF_6\ , \\ \theta F_5 - F_5\theta = cF_1 - aF_3 - fF_4 + dF_6\ , \\ \theta F_6 - F_6\theta = cF_2 - bF_3 + eF_4 - dF_5\ , \end{array} \right. \right\}$$

3-3 Almost Clifford Structures:

Let (M^{8n}, Q) be an almost Clifford manifold. The tensor fields (J_a) and $(J'_a\ ) with\ a \in \{1\ ,2\ ,\dots,6\}$ defining canonical basis of Q on coordinate neighborhoods $U\ ,U'$, respectively , are related on $U \cap U'$ by:

$$J'_a = \sum_{b=1}^{6} S_{ab}J_b \qquad \rightarrow \qquad (3.3.1)$$

Where $[S_{ab}]$ belongs to a subgroup of $SO(6)$ isomorphic with $SO(3) \times SO(3)$.

The structural group of $(M^{8n}, Q)\ is\ G = GL(n, \mathcal{O}).Sp(1)$ where $GL(n, \mathcal{O})$ is the real representation of the linear group of square non-degenerate matrices of order n with entries octons; it can be identified with a subgroup of $GL(8n\ ,\mathbb{R})$.

In this section we consider the situation when there exists three compatible (almost) complex structures $I_1\ ,I_2\ ,I_3$ such that $I_1 \neq \pm I_2\ ,\ I_1 \neq \pm I_3\ ,I_j \neq I_k \circ I_\ell\ \ for\ all\ (j\ ,k\ ,\ell) \in S_3$ (here S_3 denote the symmetric group with three elements) which are globally defined on M. We can always choose a basis $\mathrm{H} = (J_\alpha)\ ,\alpha = 1,2,\dots 6\ such\ that\ I_1 = J_1\ ,\ I_2 = a_1J_1 + a_2J_2 + a_3J_3 + a_4J_4 + a_5J_5 + a_6J_6\ ,$ $I_3 = b_1J_1 + b_2J_2 + b_3J_3 + b_4J_4 + b_5J_5 + b_6J_6\ ,$ where $\sum_{i=1}^{6} a_i^2 = 1\ ,\ \sum_{i=1}^{6} b_i^2 = 1$ and $a_1a_6 - a_2a_5 + a_3a_4 = 0\ \ ,\ b_1b_6 - b_2b_5 + b_3b_4 = 0$. Then we have $\langle I_1\ ,I_2\rangle = a_1\ ,\langle I_1\ ,I_3\rangle = b_1\ ,\langle I_2\ ,I_3\rangle = a_1b_1 + \cdots + a_6b_6$. We are interested in the globally defined angle functions $a_1 , b_1 : M \to [-1 , 1]$ given by $a_1(p) = \langle I_1 , I_2\rangle(p) , b_1(p) = \langle I_1 , I_3\rangle(p)$, respectively.

Remark 3.3.1

(1) If $a_1(p) = \pm 1\ then\ I_1 = \pm I_2\ \ at\ p\ and\ I_1 ,\ I_2$ commute at p; conversely , if $a_1 \neq 0\ and\ I_1 , I_2\ commute\ at\ p , then\ necessarily\ \ a_1 \pm 1\ and\ I_1 = \pm I_2$.

(2) The anti-commutator $\{I_1 , I_2\} = \frac{1}{2}(I_1I_2 + I_2I_1)$ satisfies the following identity

$$\{I_1 , I_2\} = -a_1 id + (a_4 - a_5 + a_6)\, J_7.$$

Therefore $a_1(p) = 0$ and $a_6 = -a_4 + a_5\ if\ and\ only\ if\ \ I_1\ and\ I_2$ anti-commute in p, or equivalently, $I_2 = a_2(J_2 + \lambda J_3) + a_4(J_4 + \lambda J_5 + (\lambda - 1)J_6)$. Similar result holds regarding the anti-commutator $\{I_1 , I_3\}$. Actually , the smooth functions $a_1 , b_1 : M \to [-1 , 1]$ measure the angles of $I_1\ with\ I_2\ and\ I_3$ respectively.

(3) The commutator of two endomorphisms P_1 , P_2 is defined by $[P_1, P_2] = \frac{1}{2}(P_1P_2 - P_2P_1)$.

By a straightforward computation we get:

$$[I_1 , I_2] = -a_4J_2 - a_5J_3 + a_2J_4 + a_3J_5$$

$$[I_1 , I_3] = -b_4J_2 - b_5J_3 + b_2J_4 + a_3J_5$$

$$\left[I_1 , [I_1 , I_2]\right] = -a_2J_2 - a_3J_3 - a_4J_4 - a_5J_5$$

$$\left[I_1 , [I_1 , I_3]\right] = -b_2J_2 - b_3J_3 - b_4J_4 - b_5J_5$$

They belong to Q and therefore , at each point where $I_1 \neq \pm I_2 , I_1 \neq \pm I_3$, we can fined

$$\begin{matrix} J_2 = \alpha_1[I_1 , I_2] + \alpha_2[I_1 , I_3] + \alpha_3 I_1 \circ [I_1 , I_2] + \alpha_4 I_1 \circ [I_1 , I_2] \\ J_3 = \beta_1[I_1 , I_2] + \beta_2[I_1 , I_3] + \beta_3 I_1 \circ [I_1 , I_2] + \beta_4 I_1 \circ [I_1 , I_2] \end{matrix} \quad \to (3.3.2)$$

Future , we get

$$J_4 = I_1 \circ J_2 ,\ J_5 = I_1 \circ J_3 ,\ J_6 = J_2 \circ J_3 ,\ J_7 = I_1 \circ J_6 \quad \to \quad (3.3.3)$$

Conversely , given the complex structures I_1 , I_2 , I_3 as before we consider $J_1 = I_1\ and\ define\ J_2\ and\ J_3\ by\ formula$ (3.3.2) with appropriately locally defined functions α_i , β_i , $i = 1 , 2 , 3 ,4$; then J_1 , J_2 , J_3 are two by two anti-commuting and $J_1^2 = J_2^2 = J_3^2 = -Id$.

We recall some definitions and results on the relation between hyper-complex and Clifford structures.

Let $H = (J_\alpha)$ be an almost hyper-complex structure on M. Recall that the Nijenhuis tensor $N(A , B)$ of two endomorphisms A, B is defined by

$$N(A,B)(X,Y) = N(AX,BY) - AN(BX,Y) - BN(X,AY) + N(BX,AY) - BN(AX,Y) - AN(X,BY) + (AB + BA)[X,Y]$$

for any vector fields X and Y. We denote $N_{ab} = N(J_a,J_b) for\ a,b = 1,2,\dots,6$.

Let us define the structure tensor N of H by

$$N = \sum_{a=1}^{6} N_{aa}$$

and the structure $1 - forms\ \alpha_a$ ($a = 1,2,\dots,6$) by

$$\alpha_a(X) = \frac{1}{8n-2} trace\big(Y \to J_\alpha N(X,Y)\big).$$

Setting $N_X Y \coloneqq N(X,Y)$ we get

$$N_X = \sum_{a=1}^{6} J_a N_X J_a Y \qquad \rightarrow \qquad (3.3.4)$$

It must be remarked that

$$\sum_{a=1}^{6} \alpha_a(J_a X) = 0.$$

The Nijenhuis tensor of an almost complex structure J is of particular importance because , by the Newlander-Niremberg Theorem, $N(J , J) = 0$ is a necessary and sufficient condition for the integrability of J.

Taking into account that the ordered triples $(J_1 ,J_2 ,J_4), (J_1 ,J_3 ,J_5)$, $(J_2 ,J_3 ,J_6)\ and\ (J_4 ,J_5 ,J_6)$ generate subalgebras of $\mathcal{O}$ isomorphic to the real algebra of quaternions and by using the well known results of K. YANO &M. AKO on the integrability of the almost quaternionic manifolds we get following result.

Proposition 3.3.2

Let (M, Q) be an almost Clifford manifold. Then

a) if two of the six Nijenhuis tensor fields $N_{11}, N_{22}, N_{44}, N_{12}, N_{14}, N_{24}$ vanish, then the others vanish, too;

b) if two of the six Nijenhuis tensor fields $N_{11}, N_{33}, N_{55}, N_{13}, N_{15}, N_{35}$ vanish, then the others vanish, too;

c) if two of the six Nijenhuis tensor fields $N_{22}, N_{33}, N_{66}, N_{23}, N_{26}, N_{36}$ vanish, then the others vanish, too;

d) if two of the six Nijenhuis tensor fields $N_{44}, N_{55}, N_{66}, N_{45}, N_{46}, N_{56}$ vanish, then the others vanish, too.

Then we can prove the Theorem.

Theorem 3.3.3

If $N_{11} = N_{22} = N_{33} = 0,\ then\ N_{ij} = 0\ for\ i, j \in \{1, 2, \dots, 6\}$.

This result can be improved relative to some three vanishing tensor fields N_{ab} that are not belong to one of the four series a), b), c), d).

Theorem 3.3.4

In order that there exists in an almost Clifford manifold a symmetric affine connection such that $\nabla J_1 = \nabla J_2 = \nabla J_3 = 0$, it is necessary and sufficient that $N_{11} = N_{22} = N_{33} = 0$.

Following the argumentation used in, one obtains that the structure tensor T^Q of a Clifford structure Q which is locally generated by $H = (J_\alpha)$ is given by

$$T^Q = N + \sum_{a=1}^{6} \partial(\alpha_a \otimes J_a) \qquad \rightarrow \quad (3.3.5)$$

Where ∂ denotes the (SPENCER's) operator of alternation. Obviously, T^Q depends only on Q. Furthermore, $T^Q = 0$ if and only if Q is a Clifford structure, i.e. there is a torsionless connection preserving Q.

Proposition 3.3.5

The almost Clifford structure $Q\ on\ M^{8n}$ is a Clifford structure if and only if in a neighborhood of any point there exists a local admissible basis $H = (J_\alpha)$ such that

$$N(X,Y) = \sum_{a=1}^{6}(\alpha_a(X)J_a(Y) - \alpha_a(Y)J_a(X)) \quad \to (3.3.6)$$

This result was used in proving of the main result of this Note.

Theorem 3.3.6

Let Q be an almost Clifford structure on M_{8n} . If there exist three compatible complex structure $I_1 , I_2 , I_3\ such\ that\ I_1 \neq \pm I_2 , I_1 \neq \pm I_3$
$I_j \neq I_k \circ I_\ell\ for\ all\ (j , k , \ell) \in S_3$ (here S_3denote the symmetric group with three elements), then Q is a Clifford structure.

3-4 Almost Cliffordian Manifolds:

Let M be a real smooth manifold of dimension , and let assume that there is a $6 - dimensional$ vector bundle V consisting of tensors of $type(1,1)$ over M such that in any coordinate neighborhood $U\ of\ M$, there exists a local basis $(F_1 , F_2 , \dots , F_6)\ of\ V$ whose elements behave under the usual composition like the similar labeled elements of the natural basis of the Clifford algebra $\mathcal{C}\ell_{03}$.

Such a local basis $(F_1 , F_2 , \dots , F_6)$ is called a canonical basis of the bundle $in\ U$. Then the bundle V is called an almost Cliffordinan structure on $M\ and\ (M,V)$ is called an almost Cliffordian manifold. Thus, any almost Cliffordian manifold is necessarily of dimension $m = 8n$.

An almost Cliffordian structure on M is given by reduction of the structural group of the principal frame bundle of $M\ to\ Op(n).Op(1)$. That is why the tensor fields $(F_1 , F_2 , \dots , F_6)$ can be defined only locally. In the almost Cliffordian manifold (M,V) we take the intersecting coordinate neighborhood $U\ and\ U'\ and\ let\ (F_1 , F_2 , \dots , F_6)\ and\ (F_1', F_2', \dots , F_6')$ be the canonical local bases of $V\ in\ U\ and\ U'$,respectively. Then $F_1', F_2', \dots , F_6'$are linear combinations of $F_1, F_2, \dots , F_6 on\ U \cap U'$,that is

$$F_i' = \sum_{j=1}^{6} s_{ij}F_j \quad , \quad i = 1 , 2 , \dots 6 , \quad \to \quad (3.4.1)$$

Where s_{ij} $(i, j = 1,2, \ldots ,6)$ are functions defined on $U \cap U'$.

The coefficients s_{ij} appearing in (3.4.1) form an element $s \cup U' = (s_{ij})$ of a proper subgroup, of dimension 6, of the special orthogonal group $SO(6)$. Consequently, any almost Cliffordian manifold is orientable.

If there exists on $(M, V) a\ global\ basis\ (F_1, F_2, \ldots, F_6), then\ (M, V)$ is called an almost Clifford manifold; the basis $(F_1, F_2, \ldots, F_6)$ is named a global canonical basis for V.

Example 3.4.1

The Clifford module $\mathcal{O}^n$ is naturally identified with $\mathbb{R}^{8n}$. It supplies the simplest example of Clifford manifold. Indeed, if we consider the Cartesian coordinate map with the coordinates $(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ \ldots, x_{2n}, \ldots, x_{7n+1}, \ldots, x_{8n})$, then the standard almost Clifford structure on $\mathbb{R}^{8n}$ is defined by means of the three anti-commuting operators J_1, J_2, J_3 defined by:

$$J_1 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{2n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{3n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_i} \quad , \quad J_2 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_{4n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{n+i}} = -\frac{\partial}{\partial x_{5n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{2n+i}} = \frac{\partial}{\partial x_{4n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{2n+i}} = -\frac{\partial}{\partial x_i} \quad , \quad J_3 \frac{\partial}{\partial x_{2n+i}} = -\frac{\partial}{\partial x_{6n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{3n+i}} = \frac{\partial}{\partial x_{5n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{3n+i}} = \frac{\partial}{\partial x_{6n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{3n+i}} = -\frac{\partial}{\partial x_i} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{4n+i}} = -\frac{\partial}{\partial x_{2n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{4n+i}} = \frac{\partial}{\partial x_{n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{4n+i}} = \frac{\partial}{\partial x_{n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{5n+i}} = -\frac{\partial}{\partial x_{3n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{5n+i}} = -\frac{\partial}{\partial x_{7n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{5n+i}} = \frac{\partial}{\partial x_{n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{6n+i}} = \frac{\partial}{\partial x_{7n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{6n+i}} = -\frac{\partial}{\partial x_{3n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{6n+i}} = \frac{\partial}{\partial x_{2n+i}} \quad ,$$

$$J_1 \frac{\partial}{\partial x_{7n+i}} = -\frac{\partial}{\partial x_{6n+i}} \quad , \quad J_2 \frac{\partial}{\partial x_{7n+i}} = \frac{\partial}{\partial x_{5n+i}} \quad , \quad J_3 \frac{\partial}{\partial x_{7n+i}} = -\frac{\partial}{\partial x_{4n+i}},$$

Example 3.4.2

The tangent bundle of any quaternionic-like manifold endowed with a linear connection can be naturally endowed with an almost Cliffordian structure.

3-5 Connection on Almost Cliffordian Manifolds:

An almost Cliffordian connection on the almost Cliffordian manifold (M, V) is a linear connection on M which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local or global) of the bundle V, then $\nabla_X \Phi$ is also a cross-section (local or global, respectively) of V , X being an arbitrary vector field of M.

Proposition 3.5.1

The linear connection ∇ on the almost Cliffordian manifold (M, V) is an almost Cliffordian connection on M if and only if the covariant derivatives of the local canonical base are expressed as follows

$$\begin{cases} \nabla J_1 = \eta_4 \otimes J_2 + \eta_5 \otimes J_3 - \eta_2 \otimes J_4 - \eta_3 \otimes J_5 \\ \nabla J_2 = -\eta_4 \otimes J_1 + \eta_6 \otimes J_3 + \eta_1 \otimes J_4 - \eta_3 \otimes J_6 \\ \nabla J_3 = -\eta_5 \otimes J_1 - \eta_6 \otimes J_2 + \eta_1 \otimes J_5 + \eta_2 \otimes J_6 \\ \nabla J_4 = \eta_2 \otimes J_1 - \eta_1 \otimes J_2 + \eta_6 \otimes J_5 - \eta_5 \otimes J_6 \\ \nabla J_5 = \eta_3 \otimes J_1 - \eta_1 \otimes J_3 - \eta_6 \otimes J_4 + \eta_4 \otimes J_6 \\ \nabla J_6 = \eta_3 \otimes J_2 - \eta_2 \otimes J_3 + \eta_5 \otimes J_4 - \eta_4 \otimes J_5 \end{cases} \rightarrow \quad (3.5.1)$$

Where $\eta_1, \eta_2, \dots, \eta_6$ are locally 1-forms defined on the dimension of $J_1, J_2, \dots, J_6$.

Let $\eta_1, \eta_2, \dots, \eta_6$ be the 1-forms defined by the connection ∇ with respect to the canonical base $J_1, J_2, \dots, J_6$. Then using the relations (3.4.1) we get the following change formulae

$$\eta'_a = \sum_{b=1}^{6} s_{ab} \eta_b + \lambda_a \quad , \quad a = 1,2, \dots, 6$$

Where λ_a are linear combinations of s_{ab} *and* ds_{ab}.

Definition (Clifford Hermitian Manifolds) 3.5.2

The triple $(M, g, V), where\ (M, V)$ is an almost Cliffordian manifold endowed with the Riemannian structure g, is called an almost Cliffordian Hermitian manifold or a metric Cliffordian manifold if for any canonical basis $J_1, J_2, \dots, J_6\ of\ V$ in coordinate neighborhood U , the identities

$$g(J_k X\ , J_k Y) = g(X\ , Y) \qquad \forall\ X\ , Y \in \mathcal{X}(M)$$

hold. Since each $J_i (i = 1,2, \dots ,6)$ is almost Hermitian with respect to g, putting

$$\Phi_i(X\ , Y) = g(J_i X\ , Y), \quad \forall\ X\ , Y \in \mathcal{X}(M)\ , \ \ i = 1,2, \dots .6 \ \ \rightarrow \ \ (3.5.2)$$

One gets 6 local 2-forms on U. However, by means of (3.4.1), it results that the 4-form

$$\Omega = \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2 + \Phi_3 \wedge \Phi_3 + \Phi_4 \wedge \Phi_4 + \Phi_5 \wedge \Phi_5 + \Phi_6 \wedge \Phi_6 \ \rightarrow (3.5.3)$$

is globally define on M.

By using (3.4.1) we easily see that

$$\Lambda = J_1 \otimes J_1 + J_2 \otimes J_2 + J_3 \otimes J_3 + J_4 \otimes J_4 + J_5 \otimes J_5 + J_6 \otimes J_6 \qquad \rightarrow \ \ (3.5.4)$$

is also a global tensor field of $type (2\ , 2)\ on\ M$.

If the Levi-Civita-connection $\nabla = \nabla^g\ on\ (M, g, V)$ preserves the vector bundle V by parallel transport, then $(M, g, V)\ is\ called\ a\ Clifford - Kähler\ manifold$. Consequently, for any $Clifford - Kähler$ manifold , the formulae (3.5.1) hold (with $\nabla = \nabla^g$).

Actually, a Riemannian manifold is a Clifford- $Kähler$ manifold if and only if its holonomy group is a subgroup is a subgroup of $Op(n).Op(1)$. Then , one can prove the formulae

$$\nabla \Omega = 0 \ \ , \nabla \Lambda = 0 \qquad \qquad \rightarrow \qquad (3.5.5)$$

Conversely, if one of the equations (3.5.5) hold then (M, g, V) is a $Clifford - Kähler\ manifold$. Thus we get the following result.

Theorem 3.5.3

An almost Clifford Hermitian manifold is a $Clifford - Kähler$ manifold if and only if either $\nabla\Omega = 0 \quad or \quad \nabla\Lambda = 0$.

3-6 Some Formulae

Let (M, V, g) be a $Clifford - Kähler\ manifold$ with $dim\ M = 8n$. In a coordinate neighborhood $(U, x^h)\ of\ M$ we denote by g_{ij} the components of g and by $\overset{k}{J_i^j}$ the components of J^k $, with\ k = 1,2, \dots ,6$ (here and in what follows we shall put the label of any element of a local basis in V above it , i.e. $(J^1, J^2, \dots J^6)$ is a canonical local basis of $V\ in\ U$) . Then formulae (3.5.1) become

$$\begin{cases} \nabla {J_i^1}^h = \eta_j^4 {J_i^2}^h + \eta_j^5 {J_i^3}^h - \eta_j^2 {J_i^4}^h - \eta_j^3 {J_i^5}^h \\ \nabla {J_i^2}^h = -\eta_j^4 {J_i^1}^h + \eta_j^6 {J_i^3}^h + \eta_j^1 {J_i^4}^h - \eta_j^3 {J_i^6}^h \\ \nabla {J_i^3}^h = -\eta_j^5 {J_i^1}^h - \eta_j^6 {J_i^2}^h + \eta_j^1 {J_i^5}^h + \eta_j^2 {J_i^6}^h \\ \nabla {J_i^4}^h = \eta_j^2 {J_i^1}^h - \eta_j^1 {J_i^2}^h + \eta_j^6 {J_i^5}^h - \eta_j^5 {J_i^6}^h \\ \nabla {J_i^5}^h = \eta_j^3 {J_i^1}^h - \eta_j^1 {J_i^3}^h - \eta_j^6 {J_i^4}^h + \eta_j^4 {J_i^6}^h \\ \nabla {J_i^6}^h = \eta_j^3 {J_i^2}^h - \eta_j^2 {J_i^3}^h + \eta_j^4 {J_i^4}^h - \eta_j^4 {J_i^5}^h \end{cases} \rightarrow \quad (3.6.1)$$

Where η_j^i are the components of η^i $(i = 1,2, \dots ,6) in\ (U, x^h)$.

Using Ricci formula, form (3.6.1) one gets:

$$\begin{cases} K_{kjs}^h {J_i^1}^s - K_{kji}^s {J_s^1}^h = \omega_{kj}^4 {J_i^2}^h + \omega_{kj}^5 {J_i^3}^h - \omega_{kj}^2 {J_i^4}^h - \omega_{kj}^3 {J_i^5}^h \\ K_{kjs}^h {J_i^2}^s - K_{kji}^s {J_s^2}^h = -\omega_{kj}^4 {J_i^1}^h + \omega_{kj}^6 {J_i^3}^h + \omega_{kj}^1 {J_i^4}^h - \omega_{kj}^3 {J_i^6}^h \\ K_{kjs}^h {J_i^3}^s - K_{kji}^s {J_s^3}^h = -\omega_{kj}^5 {J_i^1}^h - \omega_{kj}^6 {J_i^2}^h + \omega_{kj}^1 {J_i^5}^h + \omega_{kj}^2 {J_i^6}^h \\ K_{kjs}^h {J_i^4}^s - K_{kji}^s {J_s^4}^h = \omega_{kj}^2 {J_i^1}^h - \omega_{kj}^1 {J_i^2}^h + \omega_{kj}^6 {J_i^5}^h - \omega_{kj}^5 {J_i^6}^h \\ K_{kjs}^h {J_i^5}^s - K_{kji}^s {J_s^5}^h = \omega_{kj}^3 {J_i^1}^h - \omega_{kj}^1 {J_i^3}^h - \omega_{kj}^6 {J_i^4}^h + \omega_{kj}^4 {J_i^6}^h \\ K_{kjs}^h {J_i^6}^s - K_{kji}^s {J^6}_s^h = \omega_{kj}^3 {J_i^2}^h - \omega_{kj}^2 {J_i^3}^h + \omega_{kj}^4 {J_i^4}^h - \omega_{kj}^4 {J_i^5}^h \end{cases} \rightarrow (3.6.2)$$

Where K_{kjs}^{h} are the components of the curvature tensor K of the Clifford-$Kähler\ manifold\ (M, V, g) and\ \omega^1, \omega^2, \dots, \omega^6$ are defined by

$$\begin{cases} \omega^1 = d\eta^1 + \eta^2 \wedge \eta^6 + \eta^3 \wedge \eta^5 \\ \omega^2 = d\eta^2 + \eta^4 \wedge \eta^1 + \eta^6 \wedge \eta^3 \\ \omega^3 = d\eta^3 + \eta^5 \wedge \eta^1 + \eta^6 \wedge \eta^2 \\ \omega^4 = d\eta^4 + \eta^1 \wedge \eta^2 + \eta^5 \wedge \eta^6 \\ \omega^5 = d\eta^5 + \eta^1 \wedge \eta^3 + \eta^6 \wedge \eta^4 \\ \omega^6 = d\eta^6 + \eta^2 \wedge \eta^3 + \eta^4 \wedge \eta^5 \end{cases} \quad \rightarrow \quad (3.6.3)$$

And

$$\omega_{ij}^{k} = -\omega_{ji}^{k} \quad \omega^k = \frac{1}{2}\omega_{ij}^{k} dx^i \wedge dx^j \ , \ k = 1,2, \dots, 6 \ \rightarrow \ (3.6.4)$$

Thus ω^i , $i = 1,2, \dots, 6$, are local 2-forms defined on U.

From (3.6.2) we get

$$\begin{cases} [K(X,Y), J^1] = \omega^4(X,Y)J^2 + \omega^5(X,Y)J^3 - \omega^2(X,Y)J^4 - \omega^3(X,Y)J^5 \\ [K(X,Y), J^2] = -\omega^4(X,Y)J^1 + \omega^6(X,Y)J^3 + \omega^1(X,Y)J^4 - \omega^3(X,Y)J^6 \\ [K(X,Y), J^3] = -\omega^5(X,Y)J^1 - \omega^6(X,Y)J^2 + \omega^1(X,Y)J^5 + \omega^2(X,Y)J^6 \\ [K(X,Y), J^4] = \omega^2(X,Y)J^1 - \omega^1(X,Y)J^2 + \omega^6(X,Y)J^5 - \omega^5(X,Y)J^6 \\ [K(X,Y), J^5] = \omega^3(X,Y)J^1 - \omega^1(X,Y)J^3 - \omega^6(X,Y)J^4 + \omega^4(X,Y)J^6 \\ [K(X,Y), J^6] = \omega^3(X,Y)J^2 - \omega^2(X,Y)J^3 + \omega^5(X,Y)J^4 - \omega^4(X,Y)J^5 \end{cases} \quad (3.6.5)$$

In a coordinate neighborhood (U, x^h) , $X\ and\ Y$ being arbitrary vector fields in M. In another coordinate neighborhood $\left(U', x'^h\right)$ we get

$$\begin{cases} \left[K'(X,Y), J'^1\right] = \omega'^4(X,Y)J'^2 + \omega'^5(X,Y)J'^3 - \omega'^2(X,Y)J'^4 - \omega'^3(X,Y)J'^5 \\ \left[K'(X,Y), J'^2\right] = -\omega'^4(X,Y)J'^1 + \omega'^6(X,Y)J'^3 + \omega'^1(X,Y)J'^4 - \omega'^3(X,Y)J'^6 \\ \left[K'(X,Y), J'^3\right] = -\omega'^5(X,Y)J'^1 - \omega'^6(X,Y)J'^2 + \omega'^1(X,Y)J'^5 + \omega'^2(X,Y)J'^6 \\ \left[K'(X,Y), J'^4\right] = \omega'^2(X,Y)J'^1 - \omega'^1(X,Y)J'^2 + \omega'^6(X,Y)J'^5 - \omega'^5(X,Y)J'^6 \\ \left[K'(X,Y), J'^5\right] = \omega'^3(X,Y)J'^1 - \omega'^1(X,Y)J'^3 - \omega'^6(X,Y)J'^4 + \omega'^4(X,Y)J'^6 \\ \left[K'(X,Y), J'^6\right] = \omega'^3(X,Y)J'^2 - \omega'^2(X,Y)J'^3 + \omega'^5(X,Y)J'^4 - \omega'^4(X,Y)J'^5 \end{cases}$$

$$\rightarrow \quad (3.6.6)$$

Where $(J'^1, J'^2, \ldots, J'^6)$ form a canonical local basis of $V\ in\ U'$.

Since $S_{U,U'} = (s_{ij}) \in SO(6, \mathbb{R})$, by means of (3.4.1) we find in $U \cap U'$

$$\omega'^i = s_{i1}\omega^1 + s_{i2}\omega^2 + \cdots + s_{i6}\omega^6 \quad , \quad i = 1,2, \ldots, 6 \quad \rightarrow \quad (3.6.7)$$

Using (3.6.7) we see that the local 4-form

$$\Sigma = \omega^1 \wedge \omega^1 + \omega^2 \wedge \omega^2 + \omega^3 \wedge \omega^3 + \omega^4 \wedge \omega^4 + \omega^5 \wedge \omega^5 + \omega^6 \wedge \omega^6 \quad (3.6.8)$$

Determines in M a global 4-form , which is denoted also by Σ. This Σ is, in some sense, the curvature tensor of a linear connection defined in the bundle V by means of (3.5.1).

Now, using (3.6.3) we can prove.

Lemma 3.6.1

Let (M, V, g) be a Clifford-$K\ddot{a}hler$ 8n-dimensional real manifold.

A necessary and sufficient condition for the 4-form Σ to vanish on M, is that in each coordinate neighborhood U to exist a canonical local basis $(J^1, J^2, \ldots, J^6)\ of\ V$ satisfying

$$\nabla J^i = 0 \qquad i = 1,2, \ldots, 6$$

i.e., that the bundle V be locally parallelizable.

Assuming that a Clifford-$K\ddot{a}hler$ manifold satisfies the conditions stated in Lemma 3.6.1 we see that the functions s_{ij} appearing in (3.4.1) are constant in a connected component of $U \cap U'$, $U\ and\ U'$ being coordinate neighborhoods, if we take $(J^1, J^2, \ldots, J^6)$ such that $\nabla J^i = 0,\ i = 1,2, \ldots, 6\ \ in\ each\ U$.

In a Clifford-$k\ddot{a}hler$ manifold with M a simply connected manifold and the bundle V is locally paralelizable, then V has a canonical global basis.

Transvecting the 6 equations of (3.6.2) by $J^i_{hu} = J^{t\,i}_h g_{tu}$ $(i = 1,2, \ldots, 6)$ and changing indices, we find respectively

$$\begin{cases} -K_{kjts}J_i^{t^1}J_h^{s^1} + K_{kjih} = \omega_{kj}^4 J_{ih}^4 + \omega_{kj}^5 J_{ih}^5 + \omega_{kj}^2 J_{ih}^2 + \omega_{kj}^3 J_{ih}^3 \\ -K_{kjts}J_i^{t^2}J_h^{s^2} + K_{kjih} = \omega_{kj}^4 J_{ih}^4 + \omega_{kj}^6 J_{ih}^6 + \omega_{kj}^1 J_{ih}^1 + \omega_{kj}^3 J_{ih}^3 \\ -K_{kjts}J_i^{t^3}J_h^{s^3} + K_{kjih} = \omega_{kj}^5 J_{ih}^5 + \omega_{kj}^6 J_{ih}^6 + \omega_{kj}^1 J_{ih}^1 + \omega_{kj}^2 J_{ih}^2 \\ -K_{kjts}J_i^{t^4}J_h^{s^4} + K_{kjih} = \omega_{kj}^2 J_{ih}^2 + \omega_{kj}^1 J_{ih}^1 + \omega_{kj}^6 J_{ih}^6 + \omega_{kj}^5 J_{ih}^5 \\ -K_{kjts}J_i^{t^5}J_h^{s^5} + K_{kjih} = \omega_{kj}^3 J_{ih}^3 + \omega_{kj}^1 J_{ih}^1 + \omega_{kj}^6 J_{ih}^6 + \omega_{kj}^4 J_{ih}^4 \\ -K_{kjts}J_i^{t^6}J_h^{s^6} + K_{kjih} = \omega_{kj}^3 J_{ih}^3 + \omega_{kj}^2 J_{ih}^2 + \omega_{kj}^5 J_{ih}^5 + \omega_{kj}^4 J_{ih}^4 \end{cases} \rightarrow (3.6.9)$$

Where $K_{kjih} = K_{kji}^s g_{sh}$ *and* $J_{ih}^k = J_i^{s^k} g_{sh}$ $(k = 1,2,\dots,6)$ are the components of Φ^k defined by (3.5.2).

Transvecting the second equation (3.6.9) with $J^{1^{ih}} = g^{ip}J_p^{1^h}$ we get

$$-K_{kjts}J_i^{2^t}J_h^{2^s}J^{1^{ih}} + K_{kjih}J^{1^{ih}} = \omega_{kj}^4 J_{ih}^4 J^{1^{ih}} + \omega_{kj}^5 J_{ih}^5 J^{1^{ih}} + \omega_{kj}^2 J_{ih}^2 J^{1^{ih}} + \omega_{kj}^3 J_{ih}^3 J^{1^{ih}}$$

But

$$-K_{kjts}J_i^{2^t}J_h^{2^s}g^{ip}J_p^{1^h} = K_{kjts}J_i^{2^t}g^{ip}J_p^{4^s} = -K_{kjts}J_i^{2^t}g^{sp}J_p^{4^i} = -K_{kjts}J_p^{1^t}g^{sp} = K_{kjts}J_p^{1^s}g^{tp} = K_{kjts}J^{1^{ts}}$$

So that

$$2K_{kjih}J^{1^{ih}} = 8m\omega_{kj}^1 \qquad \Leftrightarrow \qquad \omega_{kj}^1 = \frac{1}{4m}K_{kjih}J^{1^{ih}}$$

Similarly we obtain

$$\omega_{kj}^s = \frac{1}{4n}K_{kjih}J^{s^{ih}} \quad s = 1,2,\dots,6. \qquad \rightarrow \quad (3.6.10)$$

Using (3.6.10) and identity $K_{kjth} + K_{jtkh} + K_{tkjh} = 0$, one gets

$$K_{ktsh}J^{ts^i} = -n\omega_{kh}^i \quad i = 1,2,\dots,6. \qquad \rightarrow \quad (3.6.11)$$

On the other hand, taking into account of (3.6.11) and transvecting successively (3.6.9) with g^{ij} it results:

$$\begin{cases} K_{kh} = -2n\omega_{ks}^1 J_h^{1^s} - \omega_{ks}^2 J_h^{2^s} - \omega_{ks}^3 J_h^{3^s} - \omega_{ks}^4 J_h^{4^s} - \omega_{ks}^5 J_h^{5^s} \\ K_{kh} = -\omega_{ks}^1 J_h^{1^s} - 2n\omega_{ks}^2 J_h^{2^s} - \omega_{ks}^3 J_h^{3^s} - \omega_{ks}^4 J_h^{4^s} - \omega_{ks}^6 J_h^{6^s} \\ K_{kh} = -\omega_{ks}^1 J_h^{1^s} - \omega_{ks}^2 J_h^{2^s} - 2n\omega_{ks}^3 J_h^{3^s} - \omega_{ks}^5 J_h^{5^s} - \omega_{ks}^6 J_h^{6^s} \\ K_{kh} = -\omega_{ks}^1 J_h^{1^s} - \omega_{ks}^2 J_h^{2^s} - 2n\omega_{ks}^4 J_h^{4^s} - \omega_{ks}^5 J_h^{5^s} - \omega_{ks}^6 J_h^{6^s} \\ K_{kh} = -\omega_{ks}^1 J_h^{1^s} - \omega_{ks}^3 J_h^{3^s} - \omega_{ks}^4 J_h^{4^s} - 2n\omega_{ks}^5 J_h^{5^s} - \omega_{ks}^6 J_h^{6^s} \\ K_{kh} = -\omega_{ks}^2 J_h^{2^s} - \omega_{ks}^3 J_h^{3^s} - \omega_{ks}^4 J_h^{4^s} - \omega_{ks}^5 J_h^{5^s} - 2n\omega_{ks}^6 J_h^{6^s} \end{cases} \quad (3.6.12)$$

Here, $K_{kh} = K_{kjih} g^{ji}$ are the components of the Ricci tensor $S\ of\ (M, V, g)$.

From these equations it follows that

$$K_{kh} = -2(n+2)\omega_{ks}^i J_h^{s^i} \quad i = 1,2,\dots,6. \ \rightarrow \quad (3.6.13)$$

Formulae (3.6.13) give

$$\omega_{kh}^i = \frac{1}{2(n+2)} K_{ks} J_h^{s^i} \qquad i = 1,2,\dots,6 \quad \rightarrow \quad (3.6.14)$$

Substituting (3.6.14) in (3.6.9) we get

$$\begin{cases} -K_{kjts} J_i^{t^1} J_h^{s^1} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^2} J_{ih}^2 + J_j^{t^3} J_{ih}^3 + J_j^{t^4} J_{ih}^4 + J_j^{t^5} J_{ih}^5) \\ -K_{kjts} J_i^{t^2} J_h^{s^2} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^1} J_{ih}^1 + J_j^{t^3} J_{ih}^3 + J_j^{t^4} J_{ih}^4 + J_j^{t^6} J_{ih}^6) \\ -K_{kjts} J_i^{t^3} J_h^{s^3} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^1} J_{ih}^1 + J_j^{t^2} J_{ih}^2 + J_j^{t^5} J_{ih}^5 + J_j^{t^6} J_{ih}^6) \\ -K_{kjts} J_i^{t^4} J_h^{s^4} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^1} J_{ih}^1 + J_j^{t^2} J_{ih}^2 + J_j^{t^5} J_{ih}^5 + J_j^{t^6} J_{ih}^6) \\ -K_{kjts} J_i^{t^5} J_h^{s^5} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^1} J_{ih}^1 + J_j^{t^3} J_{ih}^3 + J_j^{t^4} J_{ih}^4 + J_j^{t^6} J_{ih}^6) \\ -K_{kjts} J_i^{t^6} J_h^{s^6} + K_{kjih} = \frac{1}{2(m+2)} (J_j^{t^2} J_{ih}^2 + J_j^{t^3} J_{ih}^3 + J_j^{t^4} J_{ih}^4 + J_j^{t^5} J_{ih}^5) \end{cases} \quad (3.6.15)$$

Since ω_{ks}^i ($i = 1,2,\dots,6$) are all skew-symmetric, using (3.6.15) we find

$$K_{ts} J_k^{t^i} J_j^{s^i} = K_{kj} \qquad i = 1,2,\dots,6 \quad \rightarrow \quad (3.6.16)$$

Using (3.6.3) we get the identities

$$\begin{cases} d\omega^1 = \eta^4 \wedge \omega^2 + \eta^5 \wedge \omega^3 - \eta^2 \wedge \omega^4 - \eta^3 \wedge \omega^5 \\ d\omega^2 = -\eta^4 \wedge \omega^1 + \eta^6 \wedge \omega^3 + \eta^1 \wedge \omega^4 - \eta^3 \wedge \omega^6 \\ d\omega^3 = -\eta^5 \wedge \omega^1 - \eta^6 \wedge \omega^2 + \eta^1 \wedge \omega^5 + \eta^2 \wedge \omega^6 \\ d\omega^4 = \eta^2 \wedge \omega^1 - \eta^1 \wedge \omega^2 + \eta^6 \wedge \omega^5 - \eta^5 \wedge \omega^6 \\ d\omega^5 = \eta^5 \wedge \omega^1 - \eta^1 \wedge \omega^3 - \eta^6 \wedge \omega^4 + \eta^4 \wedge \omega^6 \\ d\omega^6 = \eta^3 \wedge \omega^2 - \eta^2 \wedge \omega^3 + \eta^5 \wedge \omega^4 - \eta^4 \wedge \omega^5 \end{cases} \rightarrow \quad (3.6.17)$$

(3.6.1) gives

$$\nabla_k\left(K_{js} J_i^{s1}\right) = \left(\nabla_k K_{js}\right) J_i^{s1} + K_{js}\left(\eta_k^4 J_i^{s2} + \eta_k^5 J_i^{s3} - \eta_k^2 J_i^{s4} - \eta_k^3 J_i^{s5}\right);$$

Taking into account that $\left(\nabla_k K_{js}\right) J_i^{s1} + (\nabla_k K_{is}) J_j^{s1} = 0$, one gets

$$\nabla_k K_{ij} = (\nabla_k K_{ts}) J_i^{t1} J_j^{s1}.$$

The following identity holds:

$$\nabla_k K_{ij} = (\nabla_k K_{ts}) J_i^{tp} J_j^{sp} \qquad p = 1,2,\dots,6 \quad \rightarrow \quad (3.6.18)$$

3-7 Some Theorems:

Lemma 3.7.1

For any Clifford-$K\ddot{a}hler\ manifold\ (M, V, g)$ the Ricci tensor is parallel.

Proof. By means of formulae (3.5.1) and (3.6.14) and the first identity (3.6.17) it follows

$$\left(\nabla_k K_{js}\right) J_i^{ps} + \left(\nabla_j K_{is}\right) J_k^{ps} + (\nabla_i K_{ks}) J_j^{ps} = 0, \;\; p = 1,2,\dots,6 \;\; \rightarrow (3.7.1)$$

Transvecting (3.5.1) with J_h^{1i} one gets

$$\left(\nabla_k K_{js}\right) J_i^{1s} J_h^{1i} + \left(\nabla_j K_{is}\right) J_k^{1s} J_h^{1i} + (\nabla_i K_{ks}) J_j^{1s} J_h^{1i} = 0\,,$$

i.e.

$$-\nabla_k K_{jh} + \left(\nabla_j K_{ts}\right) J_k^{1s} J_h^{1t} + (\nabla_t K_{ks}) J_h^{1t} J_j^{1s} = 0\,.$$

Substituting in this equation $\left(\nabla_j K_{ts}\right) J_h^{1\,t} J_k^{1\,s} = \nabla_j K_{kh}$ (which is a consequence of (3.6.18)), one obtains

$$-\nabla_k K_{jh} + \nabla_j K_{kh} = -(\nabla_t K_{ks}) J_h^{1\,t} J_j^{1\,s}.$$

If we substitute in this equation $\nabla_t K_{ks} = (\nabla_t K_{ba}) J_k^{2\,b} J_s^{2\,a}$ which is obtained in a similar way as (3.6.18), then we find

$$\nabla_j K_{kh} - \nabla_k K_{jh} = (\nabla_c K_{ba}) J_h^{1\,c} J_k^{2\,b} J_j^{4\,a}.$$

Similarly, we get

$$\nabla_j K_{kh} - \nabla_k K_{jh} = -(\nabla_c K_{ba}) J_h^{2\,c} J_k^{4\,b} J_j^{1\,a} = -(\nabla_c K_{ba}) J_h^{4\,c} J_k^{1\,b} J_j^{2\,a}.$$

Combining the last two equations gives

$$(\nabla_c K_{ba}) J_k^{1\,c} J_j^{2\,b} J_i^{4\,a} = (\nabla_c K_{ba}) J_k^{2\,c} J_j^{4\,b} J_i^{1\,a} = (\nabla_c K_{ba}) J_k^{4\,c} J_j^{1\,b} J_i^{2\,a} \rightarrow (3.7.2)$$

In particular, one gets

$$(\nabla_c K_{ba}) J_k^{1\,c} J_j^{2\,b} J_i^{4\,a} = (\nabla_c K_{ba}) J_k^{2\,c} J_j^{4\,b} J_i^{1\,a}$$

Form which, by transvecting with $J_r^{4\,k} J_q^{1\,j} J_p^{2\,i}$ it follows

$$-(\nabla_c K_{ba}) J_r^{2\,c} J_q^{4\,b} J_p^{1\,a} = (\nabla_c K_{ba}) J_r^{1\,c} J_q^{2\,b} J_p^{4\,a} \qquad \rightarrow \qquad (3.7.3)$$

Thus, by combining (3.7.2) and (3.7.3) it follows

$$(\nabla_c K_{ba}) J_k^{1\,c} J_j^{2\,b} J_i^{4\,a} = 0,$$

Which implies

$$\nabla_c K_{ba} = 0 \qquad \rightarrow \qquad (3.7.4)$$

Lemma 3.7.1 allow us to prove.

Theorem 3.7.2

Any Clifford-$K\ddot{a}hler$ manifold is an Einstein space.

Theorem 3.7.3

The restricted holonomy group of a Clifford-$K\ddot{a}hler$ manifold 8m-dimensional manifold is a subgroup of $Op(m)$ if and only if the Ricci tensor vanishes identically.

Proof. From (3.6.10) and (3.6.14) we get

$$K_{kjih}J^{p^{ih}} = \frac{4m}{2(m+2)} K_{ks}J_j^{p^s}, \quad p = 1,2,\dots,6 \quad \rightarrow (3.7.5)$$

If Ricci tensor vanish identically, then we obtain for successive covariant derivatives of the curvature tensor the identities

$$K_{kjih}J^{p^{ih}} = 0, \quad p = 1,2,\dots,6$$

$$(\nabla_\ell K_{kjih})J^{p^{ih}} = 0, \quad p = 1,2,\dots,6,$$

$$(\nabla_s \dots \nabla_\ell K_{kjih})J^{p^{ih}} = 0, \quad p = 1,2,\dots 6, \quad \rightarrow (3.7.6)$$

Therefore, by Ambrose-Singer theorem, the restricted holonomy group of (M, g, V) is a subgroup of $Op(m)$.

Conversely, if the restricted holonomy group is a subgroup of $Op(m)$, then (3.7.6) hold and hence $K_{ij} = 0$ (by taking account of (3.7.4)).

Taking into account of Lemma (3.6.1), we have:

Theorem 3.7.4

For a Clifford-$K\ddot{a}hler\ manifold$ (M, V, g) the bundle V is locally paralelizable if and only if the Ricci tensor vanishes identically.

3-8 Clifford $Kähler$ Manifolds:

Let M be a real smooth manifold of dimension m. Suppose that there is a 6-dimensional vector bundle V consisting of $F_i(i = 1,2,\dots,6)$ tensors of type (1,1) over M. Such a local basis $\{F_1, F_2, \dots, F_6\}$ is named a canonical local basis of the bundle V in a neighborhood U of M. Then V is called an almost Clifford structure in M. The pair $(M, V)$is named an almost Clifford manifold with V. Thus, an almost Clifford manifold M is of $dimension\ m = 8n$. If there exists on (M, V) a global basis $\{F_1, F_2, \dots, F_6\}$, then (M, V) is called an almost Clifford manifold; the basis $\{F_1, F_2, \dots, F_6\}$ is said to be a global basis for V.

An almost Clifford connection on the almost Clifford manifold (M, V) is a linear connection $\nabla\ on\ M$ which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local-global) of the bundle V. Then $\nabla_x\Phi$ is also a cross-section (local-global, respectively) of V, X being an arbitrary vector field of M.

If for any canonical basis $\{J_i\}, i = \overline{1,6}\ of\ V$ in a coordinate neighborhood U, the identities

$$g(J_iX, J_iY) = g(X, Y),\ \forall X, Y \in \chi(M), i = 1,2,\dots,6 \quad \rightarrow \quad (3.8.1)$$

Hold, the triple (M, g, V) is called an almost Clifford Hermitian manifold or metric Clifford manifold denoting by V an almost Clifford structure V and by g a Riemannian metric and by (g, V) an almost Clifford metric structure.

Since each $J_i(i = 1,2,\dots,6)$ is almost Hermitian structure with respect to g, setting

$$\Phi_i(X, Y) = g(J_iX, Y), \quad i = 1,2,\dots,6 \quad \rightarrow \quad (3.8.2)$$

For any vector fields $X\ and\ Y$, we see that Φ_i are 6-local 2-forms.

If the Levi-Civita connection $\nabla = \nabla^g\ on\ (M, g, V)$ preserves the vector bundle V by parallel transport, then (M, g, V) is named a Clifford $Kähler$ manifold, and an almost Clifford structure Φ_i of M is said to be a Clifford $Kähler$ structure. Suppose that let

$$\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\}, i = \overline{1,n}$$

be a real coordinate system on (M,V). Then we denote by

$$\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}, \frac{\partial}{\partial x_{4n+i}}, \frac{\partial}{\partial x_{5n+i}}, \frac{\partial}{\partial x_{6n+i}}, \frac{\partial}{\partial x_{7n+i}}\right\}, \quad \to (3.8.3)$$
$$\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}, dx_{4n+i}, dx_{5n+i}, dx_{6n+i}, dx_{7n+i}\}$$

The natural bases over R of the tangent space $T(M)$ and the cotangent space $T^*(M)$ $of\ M$, respectively.

By structure $\{J_1, J_2, J_3, J_4, J_5, J_6\}$ the following expressions are given

$$J_1\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{n+i}} \qquad J_2\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{2n+i}} \qquad J_3\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{3n+i}}$$

$$J_1\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_i} \quad J_2\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}} \quad J_3\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{5n+i}}$$

$$J_1\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{4n+i}} \quad J_2\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_i} \quad J_3\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{6n+i}}$$

$$J_1\left(\frac{\partial}{\partial x_{i+3n}}\right) = \frac{\partial}{\partial x_{5n+i}} \quad J_2\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{6n+i}} \quad J_3\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_i}$$

$$J_1\left(\frac{\partial}{\partial x_{4n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}} \quad J_2\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \quad J_3\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{7n+i}} \quad (3.8.4)$$

$$J_1\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}} \quad J_2\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{7n+i}} \quad J_3\left(\frac{\partial}{\partial x_{5n+i}}\right) = \frac{\partial}{\partial x_{n+i}}$$

$$J_1\left(\frac{\partial}{\partial x_{6n+i}}\right) = \frac{\partial}{\partial x_{7n+i}} \quad J_2\left(\frac{\partial}{\partial x_{6n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}} \quad J_3\left(\frac{\partial}{\partial x_{6n+i}}\right) = \frac{\partial}{\partial x_{2n+i}}$$

$$J_1\left(\frac{\partial}{\partial x_{7n+i}}\right) = -\frac{\partial}{\partial x_{6n+i}} \quad J_2\left(\frac{\partial}{\partial x_{7n+i}}\right) = \frac{\partial}{\partial x_{5n+i}} \quad J_3\left(\frac{\partial}{\partial x_{7n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{4n+i}} \qquad J_5\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{5n+i}} \qquad J_6\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{6n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}} \quad J_5\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}} \quad J_6\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{7n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \quad J_5\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{7n+i}} \quad J_6\left(\frac{\partial}{\partial x_{2n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_{7n+i}} \quad J_5\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{n+i}} \quad J_6\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_{2n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{4n+i}}\right) = -\frac{\partial}{\partial x_{i}} \quad J_5\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{6n+i}} \quad J_6\left(\frac{\partial}{\partial x_{4n+i}}\right) = \frac{\partial}{\partial x_{5n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{5n+i}}\right) = \frac{\partial}{\partial x_{6n+i}} \quad J_5\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{i}} \quad J_6\left(\frac{\partial}{\partial x_{5n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}}$$

$$J_4\left(\frac{\partial}{\partial x_{6n+i}}\right) = -\frac{\partial}{\partial x_{5n+i}} \quad J_5\left(\frac{\partial}{\partial x_{6n+i}}\right) = -\frac{\partial}{\partial x_{4n+i}} \quad J_6\left(\frac{\partial}{\partial x_{6n+i}}\right) = -\frac{\partial}{\partial x_{i}}$$

$$J_4\left(\frac{\partial}{\partial x_{7n+i}}\right) = \frac{\partial}{\partial x_{3n+i}} \quad J_5\left(\frac{\partial}{\partial x_{7n+i}}\right) = \frac{\partial}{\partial x_{2n+i}} \quad J_6\left(\frac{\partial}{\partial x_{7n+i}}\right) = \frac{\partial}{\partial x_{n+i}}$$

A canonical local basis $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*, J_6^*\}$ $of\ V^*$ of the cotangent space $T^*(M)$ of manifold M satisfies the following condition:

$$J_1^{*2} = J_2^{*2} = J_3^{*2} = J_4^{*2} = J_5^{*2} = J_6^{*2} = -I\,, \qquad \rightarrow \qquad (3.8.5)$$

Being

$$J_1^*(dx_i) = dx_{n+i} \quad J_2^*(dx_i) = dx_{2n+i} \quad J_3^*(dx_i) = dx_{3n+i}$$

$$J_1^*(dx_{n+i}) = -dx_i \quad J_2^*(dx_{n+i}) = -dx_{4n+i} \quad J_3^*(dx_{n+i}) = -dx_{5n+i}$$

$$J_1^*(dx_{2n+i}) = dx_{4n+i} \quad J_2^*(dx_{2n+i}) = -dx_i \quad J_3^*(dx_{2n+i}) = -dx_{6n+i}$$

$$J_1^*(dx_{3n+i}) = dx_{5n+i} \quad J_2^*(dx_{3n+i}) = dx_{6n+i} \quad J_3^*(dx_{3n+i}) = -dx_i$$

$$J_1^*(dx_{4n+i}) = -dx_{2n+i} \quad J_2^*(dx_{4n+i}) = dx_{n+i} \quad J_3^*(dx_{4n+i}) = dx_{7n+i}$$

$$J_1^*(dx_{5n+i}) = -dx_{3n+i} \quad J_2^*(dx_{5n+i}) = -dx_{7n+i} \quad J_3^*(dx_{5n+i}) = dx_{n+i}$$

$$J_1^*(dx_{6n+i}) = dx_{7n+i} \quad J_2^*(dx_{6n+i}) = -dx_{3n+i} \quad J_3^*(dx_{6n+i}) = dx_{2n+i}$$

$$J_1^*(dx_{7n+i}) = -dx_{6n+i} \quad J_2^*(dx_{7n+i}) = dx_{5n+i} \quad J_3^*(dx_{7n+i}) = -dx_{4n+i}$$

$$J_4^*(dx_i) = dx_{4n+i} \quad J_5^*(dx_i) = dx_{5n+i} \quad J_6^*(dx_i) = dx_{6n+i}$$

$$J_4^*(dx_{n+i}) = -dx_{2n+i} \quad J_5^*(dx_{n+i}) = -dx_{3n+i} \quad J_6^*(dx_{n+i}) = -dx_{7n+i}$$

$$J_4^*(dx_{2n+i}) = dx_{n+i} \quad J_5^*(dx_{2n+i}) = -dx_{7n+i} \quad J_6^*(dx_{2n+i}) = -dx_{3n+i}$$

$$J_4^*(dx_{3n+i}) = -dx_{7n+i} \quad J_5^*(dx_{3n+i}) = dx_{n+i} \quad J_6^*(dx_{3n+i}) = dx_{2n+i}$$

$$J_4^*(dx_{4n+i}) = -dx_i \qquad J_5^*(dx_{4n+i}) = dx_{6n+i} \qquad J_6^*(dx_{4n+i}) = dx_{5n+i}$$

$$J_4^*(dx_{5n+i}) = dx_{6n+i} \qquad J_5^*(dx_{5n+i}) = -dx_i \qquad J_6^*(dx_{5n+i}) = -dx_{4n+i}$$

$$J_4^*(dx_{6n+i}) = -dx_{5n+i} \qquad J_5^*(dx_{6n+i}) = -dx_{4n+i} \qquad J_6^*(dx_{6n+i}) = -dx_i$$

$$J_4^*(dx_{7n+i}) = dx_{3n+i} \qquad J_5^*(dx_{7n+i}) = dx_{2n+i} \qquad J_6^*(dx_{7n+i}) = dx_{n+i}$$

$$\rightarrow \qquad (3.8.6)$$

Chapter Four

Lagrangian Dynamical Systems on Clifford $K\ddot{a}hler$ Manifolds

4-1 Lagrangian Mechanics:

In this section, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of canonical local basis $\{J_i\}$, $i=\overline{1\,,6}$ $of\ V$on Clifford $K\ddot{a}hler$ manifold (M,V). We say that the Euler-Lagrange equations using basis $\{J_1,J_2,J_3\}$ of V on (R^{8n},V) are introduce. In this study, we obtain that they are the same as the obtained by operators J_1 ,J_2 ,J_3 $of\ V$ on Clifford $K\ddot{a}hler$ manifold (M,V).

If we express them, they are respectively:

First:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{n+i}}=0\,,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_i}=0\,,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{4n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0\,,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_{7n+i}}=0\,,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)-\frac{\partial L}{\partial x_{6n+i}}=0.$$

Second:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{2n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_i}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{6n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0.$$

Third:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{3n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{5n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{6n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{7n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0\,,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0.$$

Here, only we derive Euler-Lagrange equations using operators J_4 ,J_5,J_6 of V on Clifford $K\ddot{a}hler$ manifold (M,V).

Fourth, let take a local basis component on Clifford $K\ddot{a}hler$ manifold (M,V), and $\{x_i\,,x_{n+i}\,,x_{2n+i}\,,x_{3n+i}\,,x_{4n+i}\,,x_{5n+i}\,,x_{6n+i}\,,x_{7n+i}\}\,,i=\overline{1\,,n}$ be its coordinate functions.

Let semisparay be the vector field ξ defined by:

$$\xi=X^i\frac{\partial}{\partial x_i}+X^{n+i}\frac{\partial}{\partial x_{n+i}}+X^{2n+i}\frac{\partial}{\partial x_{2n+i}}+X^{3n+i}\frac{\partial}{\partial x_{3n+i}}+X^{4n+i}\frac{\partial}{\partial x_{4n+i}}$$

$$+X^{5n+i}\frac{\partial}{\partial x_{5n+i}}+X^{6n+i}\frac{\partial}{\partial x_{6n+i}}+X^{7n+i}\frac{\partial}{\partial x_{7n+i}}\qquad\rightarrow\qquad(4.1.1)$$

Where

$$X^i=\dot{x}_i\,,X^{n+i}=\dot{x}_{n+i}\,,X^{2n+i}=\dot{x}_{2n+i}\,,X^{3n+i}=\dot{x}_{3n+i}\,,X^{4n+i}=\dot{x}_{4n+i}$$

$$X^{5n+i}=\dot{x}_{5n+i}\,,X^{6n+i}=\dot{x}_{6n+i}\,,X^{7n+i}=\dot{x}_{7n+i}.$$

This equation (4.1.1) can be written concise manner

$$\xi=\sum_{a=0}^{7}X^{an+i}\frac{\partial}{\partial x_{an+i}}\qquad\rightarrow\qquad(4.1.2)$$

And the dot indicates the derivative with respect to time t. The vector fields determined by

$$V_{J_4}=J_4(\xi)=X^i\frac{\partial}{\partial x_{4n+i}}-X^{n+i}\frac{\partial}{\partial x_{2n+i}}+X^{2n+i}\frac{\partial}{\partial x_{n+i}}-X^{3n+i}\frac{\partial}{\partial x_{7n+i}}$$

$$-X^{4n+i}\frac{\partial}{\partial x_i}+X^{5n+i}\frac{\partial}{\partial x_{6n+i}}-X^{6n+i}\frac{\partial}{\partial x_{5n+i}}+X^{7n+i}\frac{\partial}{\partial x_{3n+i}}\rightarrow(4.1.3)$$

Is named Liouvile vector field on Clifford $K\ddot{a}hler$ manifold (M,V).

The maps explained by $T\,,P\colon M\rightarrow R$ such that:

$$T=\frac{1}{2}m_i\left(\dot{x}_i^2+\dot{x}_{n+i}^2+\dot{x}_{2n+i}^2+\dot{x}_{3n+i}^2+\dot{x}_{4n+i}^2+\dot{x}_{5n+i}^2+\dot{x}_{6n+i}^2+\dot{x}_{7n+i}^2\right)$$

$$\therefore \quad T=\frac{1}{2}m_i\sum_{a=0}^{7}\dot{x}_{an+i}^2 \qquad , \quad P=m_igh$$

Are said to be the kinetic energy and the potential energy of the system, respectively. Here m_i , g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on Clifford $K\ddot{a}hler$ manifold (M,V), respectively.

Then $L: M \rightarrow R$ is a map that satisfies the conditions:

i) $L=T-P$ is a Lagrangian function.

ii) the function given by $E_L^{J_4}=V_{J_4}(L)-L$, is energy function.

The operator i_{J_4} induced by J_4 and defined by:

$$i_{J_4}\omega(X_1,X_2,\dots,X_r)=\sum_{i=1}^{r}\omega\,(X_1,\dots,J_4X_i,\dots,X_r) \qquad \rightarrow \qquad (4.1.4)$$

Is called vertical derivation, where $\omega \in \Lambda^r M$, $X_i \in \mathcal{X}(M)$. The vertical differentiation d_{J_4} is determined by:

$$d_{J_4}=\left[i_{J_4}, d\right]=i_{J_4}d-di_{J_4} \qquad \rightarrow \qquad (4.1.5)$$

Where d is the usual exterior derivation. We saw that the closed Clifford $K\ddot{a}hler$ form is the closed 2-form given by $\Phi_L^{J_4}=-dd_{J_4}L$ such that

$$d_{J_4}=\frac{\partial}{\partial x_{4n+i}}dx_i-\frac{\partial}{\partial x_{2n+i}}dx_{n+i}+\frac{\partial}{\partial x_{n+i}}dx_{2n+i}-\frac{\partial}{\partial x_{7n+i}}dx_{3n+i}-$$
$$\frac{\partial}{\partial x_i}dx_{4n+i}+\frac{\partial}{\partial x_{6n+i}}dx_{5n+i}-\frac{\partial}{\partial x_{5n+i}}dx_{6n+i}+\frac{\partial}{\partial x_{3n+i}}dx_{7n+i}$$

Determined by operator:

$$d_{J_4}:\mathcal{F}(M)\rightarrow \Lambda^1 M \qquad \rightarrow \qquad (4.1.6)$$

Then

$$\Phi_L^{J_4}=-\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_j\wedge dx_i+\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j\wedge dx_{2n+i}$$

$$+\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j \wedge dx_{5n+i}$$

$$+\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} \wedge dx_i$$

$$+ \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} +$$

$$\frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} \wedge dx_{3n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} \wedge dx_{5n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{7n+i}$$

$$- \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+j}} dx_{2n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i}$$

$$- \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} dx_{2n+j} \wedge dx_{3n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} dx_{2n+j} \wedge dx_{5n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{2n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{7n+i}$$

$$- \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+i} \wedge dx_{n+i}$$

$$- \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} dx_{3n+j} \wedge dx_{3n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} dx_{3n+j} \wedge dx_{5n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{3n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{7n+i}$$

$$- \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{4n+i}} dx_{4n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} dx_{4n+j} \wedge dx_{n+i}$$

$$- \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} dx_{4n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} dx_{4n+j} \wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}dx_{4n+j}\wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j}\wedge dx_{5n+i}$$

$$+\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j}\wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j}\wedge dx_{7n+i}$$

$$-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j}\wedge dx_i + \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}\wedge dx_{n+i}$$

$$-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}\wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j}\wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{5n+j}\wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}\wedge dx_{5n+i}$$

$$+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j}\wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+j}}dx_{5n+j}\wedge dx_{7n+i}$$

$$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}\wedge dx_i + \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}\wedge dx_{n+i}$$

$$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}\wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}\wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{6n+j}\wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}\wedge dx_{5n+i}$$

$$+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+j}}dx_{6n+j}\wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}\wedge dx_{7n+i}$$

$$-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+i}\wedge dx_i + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}\wedge dx_{n+i}$$

$$-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}\wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}\wedge dx_{3n+i}$$

$$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{7n+j}\wedge dx_{4n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}\wedge dx_{5n+i}$$

$$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}\wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}\wedge dx_{7n+i}.$$

Let ξ be the second order differential equation by determined Eq(1) and given by Eq(4.1.1) and

$$i_\xi \Phi_L^{J_4} = -X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} \delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{n+i} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j +$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} \delta_i^j dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{4n+i} -$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} \delta_i^j dx_{5n+i} + X^{5n+i} \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j +$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} \delta_i^j dx_{6n+i} - X^{6n+i} \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_{7n+i} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{2n+i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} \delta_{n+i}^{n+j} dx_{3n+i} - X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{4n+i} - X^{4n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} \delta_{n+i}^{n+j} dx_{5n+i} + X^{5n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} \delta_{n+i}^{n+j} dx_{6n+i} - X^{6n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_{7n+i} + X^{7n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} -$$

$$X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} \delta_{2n+i}^{2n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{2n+j} +$$

$$X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta^{2n+j}_{2n+i}dx_{2n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta^{2n+j}_{2n+i}dx_{3n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\delta^{2n+j}_{2n+i}dx_{4n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}\delta^{2n+j}_{2n+i}dx_{5n+i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta^{2n+j}_{2n+i}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta^{2n+j}_{2n+i}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta^{3n+j}_{3n+i}dx_{i} + X^{i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta^{3n+j}_{3n+i}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta^{3n+j}_{3n+i}dx_{2n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta^{3n+j}_{3n+i}dx_{3n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}\delta^{3n+j}_{3n+i}dx_{4n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta^{3n+j}_{3n+i}dx_{5n+i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta^{3n+j}_{3n+i}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta^{3n+j}_{3n+i}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta^{4n+j}_{4n+i}dx_{i} + X^{i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta^{4n+j}_{4n+i}dx_{n+i} \quad - \quad X^{n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}\delta^{4n+j}_{4n+i}dx_{2n+i} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta^{4n+j}_{4n+i}dx_{3n+i} \quad - \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}\delta^{4n+j}_{4n+i}dx_{4n+i} \quad - \quad X^{4n+i}\frac{\partial^2}{\partial x_{4n+j}\partial x_{i}}dx_{4n+j} \quad -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta^{4n+j}_{4n+i}dx_{5n+i} \quad + \quad X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}\delta^{4n+j}_{4n+i}dx_{6n+i} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta^{4n+j}_{4n+i}dx_{7n+i} \quad + \quad X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta^{5n+j}_{5n+i}dx_{i} \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta^{5n+j}_{5n+i}dx_{n+i} \quad - \quad X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta^{5n+j}_{5n+i}dx_{2n+i} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta^{5n+j}_{5n+i}dx_{3n+i} - \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}\delta^{5n+j}_{5n+i}dx_{4n+i} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j} \quad -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta^{5n+j}_{5n+i}dx_{5n+i} \quad + \quad X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta^{5n+j}_{5n+i}dx_{6n+i} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta^{5n+j}_{5n+i}dx_{7n+i} \quad + \quad X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_i \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} \quad - \quad X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} - \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}\delta_{6n+i}^{6n+j}dx_{4n+i} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j} \quad -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i} + \quad X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i} - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i} \quad + \quad X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_i \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j} \quad +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{n+i} \quad - \quad X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i} \quad - \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}\delta_{7n+i}^{7n+j}dx_{4n+i} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j} \quad -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{5n+i} + \quad X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i} - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i} \quad + \quad X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}$$

Since the closed Clifford $Kähler$ form $\Phi_L^{J_4}$ on (M,V) is the symplectic structure, it holds

$$E_L^{J_4} = V_{J_4}(L) - L = X^i\frac{\partial L}{\partial x_{4n+i}} - X^{n+i}\frac{\partial L}{\partial x_{2n+i}} + X^{2n+i}\frac{\partial L}{\partial x_{n+i}} - X^{3n+i}\frac{\partial L}{\partial x_{7n+i}} - X^{4n+i}\frac{\partial L}{\partial x_i} + X^{5n+i}\frac{\partial}{\partial x_{6n+i}} - X^{6n+i}\frac{\partial}{\partial x_{5n+i}} + X^{7n+i}\frac{\partial}{\partial x_{3n+i}} - L$$

And thus

$$dE_L^{J_4} = X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_j - X^{n+i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j + X^{2n+i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_j - X^{4n+i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j + X^{5n+i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_j - X^{6n+i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_j$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j}$$

$$+X^i\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}$$

$$-X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} + X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{2n+j}$$

$$-X^{6n+j}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} + X^i\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j}$$

$$- X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} \quad -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j} \quad +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j} \quad -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j} \quad -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}dx_{4n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j} \quad +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j} \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j} \quad -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j} \quad -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j} \quad +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j} \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j} \quad -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j} \quad -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} \quad +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j} \quad + \quad X^{i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j} \quad -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j} \quad + \quad X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j} \quad -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j} \quad - \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j} \quad +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j} - \frac{\partial L}{\partial x_j}dx_j - \frac{\partial L}{\partial x_{n+j}}dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} -$$

$$\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} - \frac{\partial L}{\partial x_{4n+j}}dx_{4n+j} - \frac{\partial L}{\partial x_{5n+j}}dx_{5n+j} - \frac{\partial L}{\partial x_{6n+j}}dx_{6n+j} - \frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}$$

By means of Eq(1), we calculate the following expressions

$$-X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}\delta_i^j dx_i + X^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_{n+i} - X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{2n+i} +$$

$$X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}\delta_i^j dx_{3n+i} + X^i\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j dx_{4n+i} - X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}\delta_i^j dx_{5n+i} +$$

$$X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}\delta_i^j dx_{6n+i} - X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{7n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}\delta_{n+i}^{n+j}dx_i$$

$$+ \quad X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{2n+i}$$

$$+ \quad X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}\delta_{n+i}^{n+j}dx_{3n+i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{4n+i}$$

$$- \quad X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}\delta_{n+i}^{n+j}dx_{5n+i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}\delta_{n+i}^{n+j}dx_{6n+i}$$

$$- \quad X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{7n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}\delta_{2n+i}^{2n+j}dx_i$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{2n+i}$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{4n+i}$$

$$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+j}}\delta_{2n+i}^{2n+j}dx_{5n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta_{2n+i}^{2n+j}dx_{6n+i}$$

$$- X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{7n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta_{3n+i}^{3n+j}dx_i$$

$$+ X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i}$$

$$+ X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} \; + \; X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{4n+i}$$

$$-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta_{3n+i}^{3n+j}dx_{5n+i} \; + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta_{3n+i}^{3n+j}dx_{6n+i}$$

$$- X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{7n+i} \; - \; X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_i$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{n+i} \; -X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}\delta_{4n+i}^{4n+j}dx_{2n+i}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{3n+i} + \; X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}\delta_{4n+i}^{4n+j}dx_{4n+i}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+i}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{5n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+i}\partial x_{5n+i}}\delta_{4n+i}^{4n+j}dx_{6n+i}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i} \; - \; X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_i$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{n+i} - \; X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i} + \; X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}\delta_{5n+i}^{5n+j}dx_{4n+i}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{5n+i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{6n+i}$$

$$- X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i} \; - \; X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_i$$

$$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} \; - \; X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}\delta_{6n+i}^{6n+j}dx_{4n+i}$$

$$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i}$$

$$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_i$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}\delta_{7n+i}^{7n+j}dx_{4n+i}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{5n+i}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i}$$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i}+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+$$

$$\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}+$$

$$\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+i}=0$$

If a curve determined by $\alpha: R\rightarrow M$ is taken to be an integral curve of ξ, then we found equation as follows:

$$-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_j-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_j-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_j-$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_j-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_j-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_j$$

$$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_j-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_j+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_{n+j}$$

$$+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{n+j}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{n+j}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_{2n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{2n+j}$$

$$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{2n+j}-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{2n+j}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{2n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{2n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{2n+j}$$

$$+X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}dx_{3n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{3n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{3n+j}$$

$$+ X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{3n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{3n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{3n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{3n+j} + X^{i}\frac{\partial^2 L}{\partial x_j \partial x_i}dx_{4n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{4n+j}$$

$$+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{4n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{4n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}dx_{4n+j}$$

$$+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{4n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{4n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{4n+j}$$

$$-X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}dx_{5n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{5n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{5n+j}$$

$$- X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{5n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{5n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{5n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{5n+j} + X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}dx_{6n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{6n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{6n+j} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$- X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_{7n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{7n+j}$$

$$- X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{7n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{7n+j}$$

$$- X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} dx_{7n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{3n+i}} dx_{7n+j}$$

$$- X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{3n+i}} dx_{7n+j} - X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{3n+i}} dx_{7n+j} + \frac{\partial L}{\partial x_j} dx_j$$

$$+ \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} + \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j}$$

$$+ \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j} + \frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0 \quad \rightarrow \quad (4.1.7)$$

Or

$$-[X^{i} \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{4n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{4n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{4n+i}}] dx_j + \frac{\partial L}{\partial x_j} dx_j$$

$$+[X^{i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{2n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{2n+i}}] dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j}$$

$$-[X^{i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{n+i}}] dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j}$$

$$+[\frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{7n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{7n+i}} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}]\,dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}$$

$$+[X^{i}\frac{\partial^2 L}{\partial x_j\partial x_i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}]\,dx_{4n+j} + \frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}$$

$$-[X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}]\,dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}$$

$$+[X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}]\,dx_{6n+j} + \frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}$$

$$-[X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}]\,dx_{7n+j} + \frac{\partial L}{\partial x_{7n+j}}dx_{7n+j} = 0$$

In this equation can be concise manner

$$-\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{4n+i}}dx_j + \frac{\partial L}{\partial x_j}dx_j + \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}}dx_{n+j} +$$

$$\frac{\partial L}{\partial x_{n+j}}dx_{n+j} - \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{n+i}}dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} +$$

$\sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{7n+i}} dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} + \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{i}} dx_{4n+j}$

$+ \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j} - \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{6n+i}} dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j} +$

$\sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{5n+i}} dx_{6n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j} -$

$\sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{3n+i}} dx_{7n+j} + \frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0 \qquad \rightarrow \qquad (4.1.8)$

Then we find the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{4n+i}} = 0 \ , \ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{2n+i}} = 0 \ , \ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) + \frac{\partial L}{\partial x_{n+i}} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) - \frac{\partial L}{\partial x_{7n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right) - \frac{\partial L}{\partial x_{i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right) + \frac{\partial L}{\partial x_{6n+i}} = 0 \ ,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right) - \frac{\partial L}{\partial x_{5n+i}} = 0 \ , \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right) + \frac{\partial L}{\partial x_{3n+i}} = 0. \qquad \rightarrow \qquad (4.1.9)$$

Such that the equations expressed in Eq(4.1.9) are named Euler-Lagrange equations structured on Clifford $K\ddot{a}hler$ manifold (M, V) by means of $\Phi_L^{J_4}$ and in the case, the triple $(M, \Phi_L^{J_4}, \xi)$ is said to be a mechanical system on Clifford $K\ddot{a}hler$ manifold (M, V).

Fifth, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_5}$ on Clifford $K\ddot{a}hler$ manifold (M, V).

Let J_5 be another local basis component on the Clifford $K\ddot{a}hler$ manifold(M, V). Let ξ take as in Eq(4.1.1). in the case, the vector field defined by

$$V_{J_5} = J_5(\xi) = X^i \frac{\partial}{\partial x_{5n+i}} - X^{n+i} \frac{\partial}{\partial x_{3n+i}} - X^{2n+i} \frac{\partial}{\partial x_{7n+i}} + X^{3n+i} \frac{\partial}{\partial x_{n+i}} + X^{4n+i} \frac{\partial}{\partial x_{6n+i}}$$

$$- X^{5n+i} \frac{\partial}{\partial x_i} - X^{6n+i} \frac{\partial}{\partial x_{4n+i}} + X^{7n+i} \frac{\partial}{\partial x_{2n+i}} \qquad \rightarrow \qquad (4.1.10)$$

Is Liouville vector field on Clifford $K\ddot{a}hler$ manifold (M, V).

The function given by $E_L^{J_5} = V_{J_5}(L) - L$ is energy function.

Then the operator i_{J_5} induced by J_5 and defined by

$$i_{J_5}\omega(X_1, X_2, \dots, X_r) = \sum_{i=1}^{r} \omega(X_1, \dots, J_5X_i, \dots, X_r) \quad \rightarrow \quad (4.1.11)$$

Is vertical derivation, where $\omega \in \Lambda^r M$, $X_i \in \mathcal{X}(M)$.

The vertical differentiation d_{J_5} is determined by

$$d_{J_5} = [i_{J_5}, d] = i_{J_5}d - di_{J_5} \quad \rightarrow \quad (4.1.12)$$

Taking into consideration J_5 , the closed Clifford $Kähler$ form is the closed 2-form given by $\Phi_L^{J_5} = -dd_{J_5}L$ such that

$$d_{J_5} = \frac{\partial}{\partial x_{5n+i}}dx_i - \frac{\partial}{\partial x_{3n+i}}dx_{n+i} - \frac{\partial}{\partial x_{7n+i}}dx_{2n+i} + \frac{\partial}{\partial x_{n+i}}dx_{3n+i} + \frac{\partial}{\partial x_{6n+i}}dx_{4n+i}$$
$$- \frac{\partial}{\partial x_i}dx_{5n+i} - \frac{\partial}{\partial x_{4n+i}}dx_{6n+i} + \frac{\partial}{\partial x_{2n+i}}dx_{7n+i} \quad \rightarrow \quad (4.1.13)$$

And given by operator

$$d_{J_5} : \mathcal{F}(M) \rightarrow \Lambda^1 M \quad \rightarrow \quad (4.1.14)$$

Then

$$\Phi_L^{J_5} = -\frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}dx_j \wedge dx_{2n+i}$$
$$- \frac{\partial^2 L}{\partial x_j \partial x_{n+i}}dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}dx_j \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_j \partial x_i}dx_j \wedge dx_{5n+i}$$
$$+ \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}}dx_j \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}}dx_j \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}}dx_{n+j} \wedge dx_i$$
$$+ \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}}dx_{n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}}dx_{n+j} \wedge dx_{2n+i} -$$
$$\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}}dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}}dx_{n+j} \wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{n+j}\partial x_i} dx_{n+j} \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}} dx_{n+j} \wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}} dx_{n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}} dx_{2n+j} \wedge dx_i$$

$$+ \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}} dx_{2n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}} dx_{2n+j} \wedge dx_{2n+i}$$

$$- \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}} dx_{2n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}} dx_{2n+j} \wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{2n+j}\partial x_i} dx_{2n+j} \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}} dx_{2n+j} \wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}} dx_{2n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}} dx_{3n+j} \wedge dx_i$$

$$+ \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}} dx_{3n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}} dx_{3n+j} \wedge dx_{2n+i}$$

$$- \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}} dx_{3n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}} dx_{3n+j} \wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{3n+j}\partial x_i} dx_{3n+j} \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}} dx_{3n+j} \wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}} dx_{3n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}} dx_{4n+j} \wedge dx_i$$

$$+ \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}} dx_{4n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}} dx_{4n+j} \wedge dx_{2n+i}$$

$$- \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}} dx_{4n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}} dx_{4n+j} \wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{4n+j}\partial x_i} dx_{4n+j} \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}} dx_{4n+j} \wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}} dx_{4n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}} dx_{5n+j} \wedge dx_i$$

$$+ \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}} dx_{5n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}} dx_{5n+j} \wedge dx_{2n+i}$$

$$-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}\wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}\wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j}\wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j}\wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}\wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}\wedge dx_{i}$$

$$+ \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}\wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}\wedge dx_{2n+i}$$

$$- \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}\wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}\wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j}\wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}\wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}\wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}\wedge dx_{i}$$

$$+ \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}\wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}\wedge dx_{2n+i}$$

$$- \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}\wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}\wedge dx_{4n+i}$$

$$+ \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j}\wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}\wedge dx_{6n+i}$$

$$- \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}\wedge dx_{7n+i}.$$

Let ξ be the second order differential equation by determined Eq(1) and given by Eq(4.1.1) and

$$i_\xi \Phi_L^{J_5} = -X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}\delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}\delta_i^j dx_{n+i} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}\delta_i^j dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}dx_j -$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}}\delta_i^j dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}}dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}\delta_i^j dx_{4n+i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \; +$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} \delta_i^j dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{7n+i} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \; +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} \delta_{n+i}^{n+j} dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} \; -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} \delta_{n+i}^{n+j} dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} \delta_{n+i}^{n+j} dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} \; +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_i} \delta_{n+i}^{n+j} dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} \delta_{n+i}^{n+j} dx_{6n+i}$$

$$- X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} \delta_{n+i}^{n+j} dx_{7n+i} \; +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} \delta_{2n+i}^{2n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{2n+j}$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} \delta_{2n+i}^{2n+j} dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j}$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} \delta_{2n+i}^{2n+j} dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} dx_{2n+j}$$

$$- X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} \delta_{2n+i}^{2n+j} dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j}$$

$$- X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} \delta_{2n+i}^{2n+j} dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} dx_{2n+j}$$

$$+ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} \delta_{2n+i}^{2n+j} dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j}$$

$$+ \quad X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} \delta_{2n+i}^{2n+j} dx_{6n+i} \quad - \quad X^{6n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{2n+j}$$

$$- \quad X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} \delta_{2n+i}^{2n+j} dx_{7n+i} \quad + \quad X^{7n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j}$$

$$- \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} \delta_{3n+i}^{3n+j} dx_{i} \quad + \quad X^{i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{3n+j} \quad +$$

$$X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} \delta_{3n+i}^{3n+j} dx_{n+i} \quad - \quad X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \quad +$$

$$X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} \delta_{3n+i}^{3n+j} dx_{2n+i} \quad - \quad X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} dx_{3n+j}$$

$$- \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} \delta_{3n+i}^{3n+j} dx_{3n+i} \quad + \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j}$$

$$- \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} \delta_{3n+i}^{3n+j} dx_{4n+i} + X^{4n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} dx_{3n+j}$$

$$+ \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{i}} \delta_{3n+i}^{3n+j} dx_{5n+i} \quad - \quad X^{5n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{i}} dx_{3n+j}$$

$$+ \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} \delta_{3n+i}^{3n+j} dx_{6n+i} - X^{6n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{3n+j}$$

$$- X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} \delta_{3n+i}^{3n+j} dx_{7n+i} + X^{7n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j}$$

$$- \quad X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{5n+i}} \delta_{4n+i}^{4n+j} dx_{i} \quad + \quad X^{i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{5n+i}} dx_{4n+j}$$

$$+ \quad X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} \delta_{4n+i}^{4n+j} dx_{n+i} \quad - \quad X^{n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} dx_{4n+j}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} \delta_{4n+i}^{4n+j} dx_{2n+i} - \quad X^{2n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} dx_{4n+j}$$

$$- X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} \delta_{4n+i}^{4n+j} dx_{3n+i} + \quad X^{3n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} dx_{4n+j}$$

$$- X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} \delta_{4n+i}^{4n+j} dx_{4n+i} + X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} dx_{4n+j}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}\delta_{4n+i}^{4n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}dx_{4n+j}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j}$$

$$- X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$- X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_i + X^{i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j}$$

$$+ X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}$$

$$+ X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j}$$

$$- X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}$$

$$- X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}$$

$$+ X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}\delta_{5n+i}^{5n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{5n+j}$$

$$+ X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j}$$

$$- X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}$$

$$- X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_i + X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}$$

$$- X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}$$

$$- X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}\delta_{6n+i}^{6n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j}$$

$$+ X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}$$

$$- X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}$$

$$- X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{i} + X^{i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}$$

$$- X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}$$

$$- X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}\delta_{7n+i}^{7n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j}$$

$$+ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}$$

$$- X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}$$

Since the closed Clifford $K\ddot{a}hler$ form $\Phi_L^{J_5}$ $on\ M$ is the symplectic structure.

$$E_L^{J_5} = V_{J_5}(L) - L = X^{i}\frac{\partial L}{\partial x_{5n+i}} - X^{n+i}\frac{\partial L}{\partial x_{3n+i}} - X^{2n+i}\frac{\partial L}{\partial x_{7n+i}} + X^{3n+i}\frac{\partial L}{\partial x_{n+i}} +$$

$$X^{4n+i}\frac{\partial L}{\partial x_{6n+i}} - X^{5n+i}\frac{\partial L}{\partial x_{i}} - X^{6n+i}\frac{\partial L}{\partial x_{4n+i}} + X^{7n+i}\frac{\partial L}{\partial x_{2n+i}} - L$$

And thus

$$dE_L^{J_5} = X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j +$$

$$X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^{4n+i} \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j - X^{5n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j -$$

$$X^{6n+i} \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j + X^{7n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j}$$

$$+X^{4n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j}$$

$$+X^{7n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + X^i \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{2n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j}$$

$$-X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + X^{4n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} dx_{2n+j} -$$

$$X^{5n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{2n+j} + X^{7n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j}$$

$$+X^i \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{3n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} dx_{3n+j}$$

$$+X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} + X^{4n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} dx_{3n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j}$$

$$-X^{6n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{3n+j} + X^{7n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} + X^i \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{5n+i}} dx_{4n+j}$$

$$-X^{n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} dx_{4n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} dx_{4n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} dx_{4n+j} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} dx_{4n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_i} dx_{4n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{4n+i}} dx_{4n+j}$$

$$+X^{7n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} dx_{4n+j} + X^i \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{5n+i}} dx_{5n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{3n+i}} dx_{5n+j}$$

$$- X^{2n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{7n+i}} dx_{5n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{n+i}} dx_{5n+j} + X^{4n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{6n+i}} dx_{5n+j} -$$

$$X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_i} dx_{5n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} dx_{5n+j} + X^{7n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} dx_{5n+j}$$

$+X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}$

$+X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j}$

$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}+X^{i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}$

$-X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}+$

$X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}$

$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}-\frac{\partial L}{\partial x_j}dx_j-\frac{\partial L}{\partial x_{n+j}}dx_{n+j}-\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}-$

$\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}-\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}-\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}-\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}-\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}$

By means of Eq(1), we calculate the following expressions

$-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}\delta_i^j dx_i+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{n+i}+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}\delta_i^j dx_{2n+i}-$

$X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{3n+i}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}\delta_i^j dx_{4n+i}+X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{i}}\delta_i^j dx_{5n+i}+$

$X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}\delta_i^j dx_{6n+i}-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_{7n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}\delta_{n+i}^{n+j}dx_i+$

$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}\delta_{n+i}^{n+j}dx_{2n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+i}$

$-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}\delta_{n+i}^{n+j}dx_{4n+i}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}\delta_{n+i}^{n+j}dx_{5n+i}+$

$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}\delta_{n+i}^{n+j}dx_{6n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_{7n+i}-$

$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta_{2n+i}^{2n+j}dx_i+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i}+$

$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j}dx_{2n+i}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i}-$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}\delta_{2n+i}^{2n+j}dx_{4n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\delta_{2n+i}^{2n+j}dx_{5n+i} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}\delta_{2n+i}^{2n+j}dx_{6n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_{7n+i}$$

$$-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta_{3n+i}^{3n+j}dx_{i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta_{3n+i}^{3n+j}dx_{4n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}\delta_{3n+i}^{3n+j}dx_{5n+i} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta_{3n+i}^{3n+j}dx_{6n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{7n+i}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}\delta_{4n+i}^{4n+j}dx_{i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{n+i}$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{2n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}\delta_{4n+i}^{4n+j}dx_{3n+i} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}\delta_{4n+i}^{4n+j}dx_{5n+i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_{6n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i}$$

$$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{n+i} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{4n+i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}\delta_{5n+i}^{5n+j}dx_{5n+i} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_{6n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i}$$

$$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{4n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}\delta_{6n+i}^{6n+j}dx_{5n+i} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i}$$

$$- X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{n+i} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{4n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}\delta_{7n+i}^{7n+j}dx_{5n+i} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i} +$$

$$\frac{\partial L}{\partial x_j}dx_j + \frac{\partial L}{\partial x_{n+j}}dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} + \frac{\partial L}{\partial x_{4n+j}}dx_{4n+j} +$$

$$\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j} + \frac{\partial L}{\partial x_{6n+j}}dx_{6n+j} + \frac{\partial L}{\partial x_{7n+j}}dx_{7n+j} = 0$$

If a curve determined by $\alpha: R \rightarrow M$ is taken to be an integral curve of ξ, then we found equation as follows:

$$-X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{5n+i}}dx_j - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_j - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_j -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_j - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_j - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_j -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_j - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_j + X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{3n+i}}dx_{n+j} +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{n+j} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{n+j} + X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{7n+i}}dx_{2n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{2n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{2n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{2n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{2n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{2n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{2n+j} +$$

$$-X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{n+i}}dx_{3n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{3n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{3n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{3n+j}$$

$$-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{3n+j} - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{3n+j} - X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{6n+i}}dx_{4n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{4n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{4n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{4n+j}$$

$$-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{4n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{4n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{4n+j} + X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{i}}dx_{5n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}dx_{5n+j} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}dx_{5n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{5n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}dx_{5n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{5n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{5n+j} +$$

$$X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{4n+i}}dx_{6n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{6n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{6n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{6n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{6n+j} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{6n+j} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{6n+j} - X^{i}\frac{\partial^2 L}{\partial x_{j}\partial x_{2n+i}}dx_{7n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{7n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{7n+j} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{7n+j}$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{7n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{7n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{7n+j} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+$$

$$\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+$$

$$\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0 \qquad \rightarrow \qquad (4.1.15)$$

Or

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}+$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}]\,dx_j+\frac{\partial L}{\partial x_j}dx_j$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}+$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}]\,dx_{n+j}+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}$$

$$+[X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}+$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}]\,dx_{2n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-[X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}+$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}]\,dx_{3n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}$$

$$-[X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{6n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{6n+i}}] \, dx_{4n+j} + \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j}$$

$$+[X^i \frac{\partial^2 L}{\partial x_j \partial x_i} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} + X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_i}$$

$$+X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_i} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_i} + X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_i}] dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j}$$

$$+[X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{4n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{4n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{4n+i}}] \, dx_{6n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j}$$

$$-[X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{2n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{2n+i}}] \, dx_{7n+j} + \frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0$$

In this equation can be concise manner

$$-\sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{5n+i}} dx_j + \frac{\partial L}{\partial x_j} dx_j + \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{3n+i}} dx_{n+j} +$$

$$\frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{7n+i}} dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} -$$

$$\sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{n+i}} dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} - \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_{6n+i}} dx_{4n+j}$$

$$+ \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j} + \sum_{a=0}^{7} X^{an+i} \frac{\partial^2 L}{\partial x_{an+j} \partial x_i} dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j} +$$

$$\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{4n+i}}dx_{6n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}-\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}}dx_{7n+j}$$

$$\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0 \quad \rightarrow \quad (4.1.16)$$

Then we find the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{5n+i}}=0\ ,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0\ ,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0\ ,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0\ ,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{6n+i}}=0\ ,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_i}=0\ ,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0\ ,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0 \quad \rightarrow \quad (4.1.17)$$

Such that the equations expressed in Eq(4.1.17) are named Euler-Lagrange equations structured on Clifford $K\ddot{a}hler$ manifold (M,V) by means of $\Phi_L^{J_5}$ in the case, the triple $(M,\Phi_L^{J_5},\xi)$ is called a mechanical system on Clifford $K\ddot{a}hler$ manifold (M,V).

Sixth, we present Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_6}$ on Clifford $K\ddot{a}hler$ manifold (M,V).

Let J_6 be a local basis on Clifford $K\ddot{a}hler$ manifold (M,V).

Let semispray ξ give as in Eq(4.1.1). So, Liouville vector field on Clifford $K\ddot{a}hler$ manifold (M,V) is the vector field defined by

$$V_{J_6}=J_6(\xi)=X^i\frac{\partial}{\partial x_{6n+i}}-X^{n+i}\frac{\partial}{\partial x_{7n+i}}-X^{2n+i}\frac{\partial}{\partial x_{3n+i}}+X^{3n+i}\frac{\partial}{\partial x_{2n+i}}+$$

$$X^{4n+i}\frac{\partial}{\partial x_{5n+i}}-X^{5n+i}\frac{\partial}{\partial x_{4n+i}}-X^{6n+i}\frac{\partial}{\partial x_i}+X^{7n+i}\frac{\partial}{\partial x_{n+i}} \rightarrow (4.1.18)$$

The function given by $E_L^{J_6}=V_{J_6}(L)-L$ is energy function.

The function i_{J_6} induced by J_6 and given by

$$i_{J_6}\omega(X_1,X_2,\dots,X_r)=\sum_{i=1}^{r}\omega(X_1,\dots,J_6X_i,\dots,X_r) \quad \rightarrow \quad (4.1.19)$$

Is said to be vertical derivation, where $\omega \in \Lambda^r M$, $X_i \in \mathcal{X}(M)$. The vertical differentiation d_{J_6} is determined by

$$d_{J_6} = \left[i_{J_6}, d\right] = i_{J_6}d - di_{J_6} \qquad \rightarrow \qquad (4.1.20)$$

We say the closed $K\ddot{a}hler$ form is the closed 2-form given by $\Phi_L^{J_6} = -dd_{J_6}L$ such that

$$d_{J_6} = \frac{\partial}{\partial x_{6n+i}}dx_i - \frac{\partial}{\partial x_{7n+i}}dx_{n+i} - \frac{\partial}{\partial x_{3n+i}}dx_{2n+i} + \frac{\partial}{\partial x_{2n+i}}dx_{3n+i} + \frac{\partial}{\partial x_{5n+i}}dx_{4n+i} - \frac{\partial}{\partial x_{4n+i}}dx_{5n+i} - \frac{\partial}{\partial x_i}dx_{6n+i} + \frac{\partial}{\partial x_{n+i}}dx_{7n+i}$$

And

$$d_{J_6} : \mathcal{F}(M) \rightarrow \Lambda^1 M \qquad \rightarrow \qquad (4.1.21)$$

Then

$$\Phi_L^{J_6} = -\frac{\partial^2 L}{\partial x_j \partial x_{6n+i}}dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}}dx_j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_j \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}}dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}}dx_j \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}}dx_j \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_j \partial x_i}dx_j \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_j \partial x_{n+i}}dx_j \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}}dx_{n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}}dx_{n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}}dx_{n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}}dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}}dx_{n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}}dx_{n+j} \wedge dx_{5n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i}dx_{n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}}dx_{n+j} \wedge dx_{7n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}}dx_{2n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}}dx_{2n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}}dx_{2n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}}dx_{2n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}}dx_{2n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}}dx_{2n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i} dx_{2n+j} \wedge dx_{6n+i} \quad - \quad \frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}} dx_{2n+j} \wedge dx_{7n+i} -$$

$$\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}} dx_{3n+j} \wedge dx_i \quad + \quad \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}} dx_{3n+j} \wedge dx_{n+i} \; +$$

$$\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}} dx_{3n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}} dx_{3n+j} \wedge dx_{3n+i} -$$

$$\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}} dx_{3n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}} dx_{3n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i} dx_{3n+j} \wedge dx_{6n+i} \; - \; \frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}} dx_{3n+j} \wedge dx_{7n+i} -$$

$$\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}} dx_{4n+j} \wedge dx_i \quad + \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}} dx_{4n+j} \wedge dx_{n+i} \; +$$

$$\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}} dx_{4n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}} dx_{4n+j} \wedge dx_{3n+i} -$$

$$\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}} dx_{4n+j} \wedge dx_{4n+i} \; + \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}} dx_{4n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i} dx_{4n+j} \wedge dx_{6n+i} \; - \; \frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}} dx_{4n+j} \wedge dx_{7n+i} \; -$$

$$\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}} dx_{5n+j} \wedge dx_i \quad + \; \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}} dx_{5n+j} \wedge dx_{n+i} \; +$$

$$\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}} dx_{5n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}} dx_{5n+j} \wedge dx_{3n+i} -$$

$$\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}} dx_{5n+j} \wedge dx_{4n+i} + \; \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}} dx_{5n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i} dx_{5n+j} \wedge dx_{6n+i} \; - \; \frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}} dx_{5n+j} \wedge dx_{7n+i} -$$

$$\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}} dx_{6n+j} \wedge dx_i \quad + \; \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}} dx_{6n+j} \wedge dx_{n+i} \; +$$

$$\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}} dx_{6n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}} dx_{6n+j} \wedge dx_{3n+i} -$$

$$\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}} dx_{6n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}} dx_{6n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}} dx_{6n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}} dx_{6n+j} \wedge dx_{7n+i} -$$

$$\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}} dx_{7n+j} \wedge dx_{i} + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}} dx_{7n+j} \wedge dx_{n+i} +$$

$$\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}} dx_{7n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}} dx_{7n+j} \wedge dx_{3n+i} -$$

$$\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}} dx_{7n+j} \wedge dx_{4n+i} + \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}} dx_{7n+j} \wedge dx_{5n+i} +$$

$$\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}} dx_{7n+j} \wedge dx_{6n+i} - \frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}} dx_{7n+j} \wedge dx_{7n+i}$$

Let ξ be the second order differential equation by determined Eq(1) and given by Eq(4.1.1) and

$$i_\xi \Phi_L^{J_6} = -X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} \delta_i^j dx_i + X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} \delta_i^j dx_{n+i} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} \delta_i^j dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j -$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} \delta_i^j dx_{3n+i} + X^{3n+i} \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} \delta_i^j dx_{4n+i} +$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j + X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} \delta_i^j dx_{5n+i} - X^{5n+i} \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j +$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_i} \delta_i^j dx_{6n+i} - X^{6n+i} \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} \delta_i^j dx_{7n+i} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} \delta_{n+i}^{n+j} dx_i + X^i \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} \delta_{n+i}^{n+j} dx_{n+i} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} +$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} \delta_{n+i}^{n+j} dx_{2n+i} - X^{2n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_{3n+i} \quad + \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}\delta_{n+i}^{n+j}dx_{4n+i} \quad + \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{n+j} \; +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}\delta_{n+i}^{n+j}dx_{5n+i} \; - \; X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{n+j} \; +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}\delta_{n+i}^{n+j}dx_{6n+i} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}dx_{n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{7n+i} \; + \; X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}\delta_{2n+i}^{2n+j}dx_{i} \; + \; X^{i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{2n+j} \; +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} \; - \; X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} \; +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{2n+i} \; - \; X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} \quad + \quad X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta_{2n+i}^{2n+j}dx_{4n+i} \quad + \quad X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} \; +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}\delta_{2n+i}^{2n+j}dx_{5n+i} \; - \; X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j} \; +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\delta_{2n+i}^{2n+j}dx_{6n+i} \quad - \quad X^{6n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}dx_{2n+j} \; -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{7n+i} \; + \; X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta_{3n+i}^{3n+j}dx_{i} \; + \; X^{i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j} \; +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j}dx_{n+i} \; - \; X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} \; +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta_{3n+i}^{3n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta_{3n+i}^{3n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}\delta_{3n+i}^{3n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{i} + X^{i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{3n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}\delta_{4n+i}^{4n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_{5n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}\delta_{4n+i}^{4n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}dx_{4n+j} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i} + X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{i} + X^{i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{n+i} \;-\; X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j} \;+$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i} \;-\; X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j} \;-$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i} \;+\; X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j} \;-$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{4n+i} \;+\; X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j} \;+$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_{5n+i} \;-\; X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j} \;+$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}\delta_{5n+i}^{5n+j}dx_{6n+i} \;-\; X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j} \;-$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i} \;+\; X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j} \;-$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{i} \;+\; X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j} \;+$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} \;-\; X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j} \;+$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i} \;-\; X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j} \;-$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} \;+\; X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j} \;-$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{4n+i} \;+\; X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} \;+$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i} \;-\; X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j} \;+$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}\delta_{6n+i}^{6n+j}dx_{6n+i} \;-\; X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j} \;-$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i} \;+\; X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j} \;-$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta^{7n+j}_{7n+i}dx_i \ + \ X^i\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j} \ +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta^{7n+j}_{7n+i}dx_{n+i} \ - \ X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j} \ +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta^{7n+j}_{7n+i}dx_{2n+i} \ - \ X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j} \ -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta^{7n+j}_{7n+i}dx_{3n+i} \ + \ X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j} \ -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta^{7n+j}_{7n+i}dx_{4n+i} \ + \ X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j} \ +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta^{7n+j}_{7n+i}dx_{5n+i} \ - \ X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j} \ +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}\delta^{7n+j}_{7n+i}dx_{6n+i} \ - \ X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{7n+j} \ -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta^{7n+j}_{7n+i}dx_{7n+i} \ + \ X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}$$

Since the closed Clifford $K\ddot{a}hler$ form $\Phi_L^{J_6}$ $on\ M$ is the symplectic structure.

$$E_L^{J_6} = V_{J_6}(L) - L = X^i\frac{\partial L}{\partial x_{6n+i}} - X^{n+i}\frac{\partial L}{\partial x_{7n+i}} - X^{2n+i}\frac{\partial L}{\partial x_{3n+i}} + X^{3n+i}\frac{\partial L}{\partial x_{2n+i}}$$

$$+X^{4n+i}\frac{\partial L}{\partial x_{5n+i}} - X^{5n+i}\frac{\partial L}{\partial x_{4n+i}} - X^{6n+i}\frac{\partial L}{\partial x_i} + X^{7n+i}\frac{\partial L}{\partial x_{n+i}} - L \quad \rightarrow \quad (4.1.22)$$

And thus

$$dE_L^{J_6} = X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_j - X^{n+i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_j - X^{2n+i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j + X^{4n+i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_j - X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_j -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j + X^{7n+i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j + X^i\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}dx_{n+j} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} + X^{i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{2n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}dx_{2n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}$$

$$X^{i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} + X^{i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}dx_{4n+j}$$

$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j} + X^{i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}$$

$$X^{i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j} - X^{n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j} + X^{i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j} - X^{2n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}$$

$$+X^{4n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j} - X^{5n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j} - X^{6n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j}$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j} - \frac{\partial L}{\partial x_j}dx_j - \frac{\partial L}{\partial x_{n+j}}dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} -$$

$$\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} - \frac{\partial L}{\partial x_{4n+j}}dx_{4n+j} - \frac{\partial L}{\partial x_{5n+j}}dx_{5n+j} - \frac{\partial L}{\partial x_{6n+j}}dx_{6n+j} - \frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}.$$

By means of Eq(1), we calculate the following expressions.

$$-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}\delta_i^j dx_i + X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}\delta_i^j dx_{n+i} + X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{2n+i} -$$

$$X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_{3n+i} - X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}\delta_i^j dx_{4n+i} + X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}\delta_i^j dx_{5n+i} +$$

$$X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{i}}\delta_i^j dx_{6n+i} - X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{7n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}\delta_{n+i}^{n+j} dx_{i} +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}\delta_{n+i}^{n+j} dx_{n+i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j} dx_{2n+i} - X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j} dx_{3n+i} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}\delta_{n+i}^{n+j} dx_{4n+i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}\delta_{n+i}^{n+j} dx_{5n+i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}\delta_{n+i}^{n+j} dx_{6n+i} -$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j} dx_{7n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}\delta_{2n+i}^{2n+j} dx_{i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j} dx_{n+i} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j} dx_{2n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j} dx_{3n+i} -$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta_{2n+i}^{2n+j} dx_{4n+i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}\delta_{2n+i}^{2n+j} dx_{5n+i} +$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\delta_{2n+i}^{2n+j} dx_{6n+i} - X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j} dx_{7n+i} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta_{3n+i}^{3n+j} dx_{i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j} dx_{n+i} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta_{3n+i}^{3n+j}dx_{4n+i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta_{3n+i}^{3n+j}dx_{5n+i} +$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}\delta_{3n+i}^{3n+j}dx_{6n+i} - X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{7n+i} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{n+i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{2n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{3n+i} -$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}\delta_{4n+i}^{4n+j}dx_{4n+i} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_{5n+i} +$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}\delta_{4n+i}^{4n+j}dx_{6n+i} - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{n+i} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i} -$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{4n+i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_{5n+i} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}\delta_{5n+i}^{5n+j}dx_{6n+i} - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{n+i} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i} -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{4n+i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i} +$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}\delta_{6n+i}^{6n+j}dx_{6n+i} - X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i} -$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta^{7n+j}_{7n+i}dx_i \;+\; X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta^{7n+j}_{7n+i}dx_{n+i} \;+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta^{7n+j}_{7n+i}dx_{2n+i} \;-\; X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta^{7n+j}_{7n+i}dx_{3n+i} \,-$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta^{7n+j}_{7n+i}dx_{4n+i} \;+\; X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta^{7n+j}_{7n+i}dx_{5n+i} \;+$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}\delta^{6n+j}_{6n+i}dx_{6n+i} - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta^{7n+j}_{7n+i}dx_{7n+i} + \frac{\partial L}{\partial x_j}dx_j$$

$$\frac{\partial L}{\partial x_{n+j}}dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} + \frac{\partial L}{\partial x_{4n+j}}dx_{4n+j} + \frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}$$

$$+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j} + \frac{\partial L}{\partial x_{7n+j}}dx_{7n+j} = 0$$

If a curve determined by $\alpha: R \to M$ is taken to be an integral curve of ξ, then we found equation as follows:

$$-X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_j \;-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_j \;-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_j -$$

$$X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_j - X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_j - X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_j -$$

$$X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_j - X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_j + X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_{n+j} +$$

$$X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{n+j} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{n+j}$$

$$X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{n+j} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{n+j}$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{n+j} + X^{i}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{2n+j} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{2n+j}$$

$$X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{2n+j} + X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{2n+j} +$$

$$X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{2n+j} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{2n+j} + X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{2n+j} -$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{3n+j} - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{3n+j} -$$

$$X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} - X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} dx_{3n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} dx_{3n+j} -$$

$$X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{2n+i}} dx_{3n+j} - X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{2n+i}} dx_{3n+j} - X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_{4n+j}$$

$$-X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{4n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{4n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{4n+j} -$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{5n+i}} dx_{4n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{5n+i}} dx_{4n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{5n+i}} dx_{4n+j} -$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{5n+i}} dx_{4n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_{5n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{5n+j} +$$

$$X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{5n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{5n+j} + X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{4n+i}} dx_{5n+j} +$$

$$X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} dx_{5n+j} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{4n+i}} dx_{5n+j} + X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{4n+i}} dx_{5n+j} +$$

$$X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{6n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{6n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{6n+j} +$$

$$X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{6n+j} + X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_i} dx_{6n+j} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_i} dx_{6n+j}$$

$$+X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_i} dx_{6n+j} + X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_i} dx_{6n+j} - X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{7n+j} -$$

$$X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{7n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{7n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{7n+j} -$$

$$X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{n+i}} dx_{7n+j} - X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{n+i}} dx_{7n+j} - X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{n+i}} dx_{7n+j}$$

$$-X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{n+i}} dx_{7n+j} + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} +$$

$$\frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} + \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j} +$$

$$\frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0 \qquad \rightarrow \quad (4.1.23)$$

Or

$$-[X^i \frac{\partial^2 L}{\partial x_j \partial x_{6n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{6n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{6n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{6n+i}}]\, dx_j + \frac{\partial L}{\partial x_j} dx_j$$

$$+[X^i \frac{\partial^2 L}{\partial x_j \partial x_{7n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{7n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{7n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{7n+i}}]\, dx_{n+j} + \frac{\partial L}{\partial x_{n+j}} dx_{n+j}$$

$$+[X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{3n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{3n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{3n+i}}]\, dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j}$$

$$-[X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{2n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{2n+i}}]\, dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j}$$

$$-[X^i \frac{\partial^2 L}{\partial x_j \partial x_{5n+i}} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}}$$

$$+ X^{4n+i} \frac{\partial^2 L}{\partial x_{4n+j} \partial x_{5n+i}} + X^{5n+i} \frac{\partial^2 L}{\partial x_{5n+j} \partial x_{5n+i}} + X^{6n+i} \frac{\partial^2 L}{\partial x_{6n+j} \partial x_{5n+i}} +$$

$$X^{7n+i} \frac{\partial^2 L}{\partial x_{7n+j} \partial x_{5n+i}}]\, dx_{4n+j} + \frac{\partial L}{\partial x_{4n+j}} dx_{4n+j}$$

$$+[X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{4n+i}} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{4n+i}} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}]\, dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j}$$

$$+[X^{i}\frac{\partial^2 L}{\partial x_j \partial x_i} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}]\, dx_{6n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j}$$

$$-[X^{i}\frac{\partial^2 L}{\partial x_j \partial x_{n+i}} + X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}} + X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}} + X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}$$

$$+ X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{n+i}} + X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}} + X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}} +$$

$$X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}]\, dx_{7n+j} + \frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0$$

In this equation can be concise manner.

$$-\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{6n+i}} dx_j + \frac{\partial L}{\partial x_j} dx_j + \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{7n+i}} dx_{n+j}$$

$$+\frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{3n+i}} dx_{2n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j}$$

$$-\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}} dx_{3n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} - \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{5n+i}} dx_{4n+j} +$$

$$\frac{\partial L}{\partial x_{4n+j}} dx_{4n+j} + \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{4n+i}} dx_{5n+j} + \frac{\partial L}{\partial x_{5n+j}} dx_{5n+j} +$$

$$\sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_i} dx_{6n+j} + \frac{\partial L}{\partial x_{6n+j}} dx_{6n+j} - \sum_{a=0}^{7} X^{an+i}\frac{\partial^2 L}{\partial x_{an+j}\partial x_{n+i}} dx_{7n+j}$$

$$+\frac{\partial L}{\partial x_{7n+j}} dx_{7n+j} = 0.$$

Then we find the equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{6n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0,$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_i}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0 \qquad \rightarrow \qquad (4.1.24)$$

Thus equations obtained in Eq(4.1.24) infer Euler-Lagrange equations structured by means of $\Phi_L^{J_6}$ on Clifford $K\ddot{a}hler$ manifold (M,V) and so, the triple $(M,\Phi_L^{J_6},\xi)$ is called a mechanical system on Clifford $K\ddot{a}hler$ manifold (M,V).

4-2 Conclusion:

Form above, Lagrangian formalisms has intrinsically been described taking into account a canonical local basis $\{J_i\}$, $i=\overline{1,6}\ \ of\ V$ on Clifford $K\ddot{a}hler$ manifold (M,V).

The paths of semispray ξ on Clifford $K\ddot{a}hler$ manifold are the solutions Euler-Lagrange equations raised in (4.1.9),(4.1.17) and (4.1.24) and also obtained by a canonical local basis $\{J_i\}$, $i=\overline{1,6}$ of vector bundle V on Clifford $K\ddot{a}hler$ manifold (M,V). One may be shown that these equations are very important to explain the rotational spatial mechanics problems.

Chapter Five

Hamiltonian Dynamical Systems on Clifford $K\ddot{a}hler$ Manifolds

5-1 Hamilton Mechanics:

In this section, we obtain Hamilton equations and Hamilton mechanical system for quantum and classical mechanics by means of a canonical local basis $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*, J_6^*\}$ $of\ V$ on Clifford $K\ddot{a}hler$ manifold (M, V). We saw that the Hamilton equations using basis $\{J_1^*, J_2^*, J_3^*\}$ $of\ V\ on\ (R^8, V)$. In this study, it is seen that they are the same as the equations obtained by operators J_1^*, J_2^*, J_3^* $of\ V$ on Clifford $K\ddot{a}hler$ manifold (M, V). If we redetermine them, they are respectively:

Fist:

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dx_{2n+i}}{dt} = -\frac{\partial H}{\partial x_{4n+i}}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial H}{\partial x_{5n+i}},$$

$$\frac{dx_{4n+i}}{dt} = \frac{\partial H}{\partial x_{2n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{6n+i}}{dt} = -\frac{\partial H}{\partial x_{7n+i}}, \quad \frac{dx_{7n+i}}{dt} = \frac{\partial H}{\partial x_{6n+i}}.$$

Second:

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{2n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{4n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dx_{3n+i}}{dt} = -\frac{\partial H}{\partial x_{6n+i}},$$

$$\frac{dx_{4n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}, \quad \frac{dx_{5n+i}}{dt} = \frac{\partial H}{\partial x_{7n+i}}, \quad \frac{dx_{6n+i}}{dt} = \frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{7n+i}}{dt} = -\frac{\partial H}{\partial x_{5n+i}}.$$

Third:

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{3n+i}}, \quad \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{5n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_{6n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_i},$$

$$\frac{dx_{4n+i}}{dt} = -\frac{\partial H}{\partial x_{7n+i}}, \quad \frac{dx_{5n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}}, \quad \frac{dx_{6n+i}}{dt} = -\frac{\partial H}{\partial x_{2n+i}}, \quad \frac{dx_{7n+i}}{dt} = \frac{\partial H}{\partial x_{4n+i}}.$$

Fourth, let (M, V) be a Clifford $K\ddot{a}hler$ manifold. Suppose that a component of almost Clifford structure V^*, a Liouville form and a 1-form on Clifford $K\ddot{a}hler$ manifold (M, V) are given by J_4^*, $\lambda_{J_4^*}$ $and\ \omega J_4^*$, respectively.

Putting

$$\omega J_4^* = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i}$$
$$+x_{4n+i} dx_{4n+i} + x_{5n+i} dx_{5n+i} + x_{6n+i} dx_{6n+i} + x_{7n+i} dx_{7n+i})$$

We have

$$\lambda_{J_4^*} = J_4^*(\omega J_4^*) = \frac{1}{2}(x_i dx_{4n+i} - x_{n+i} dx_{2n+i} + x_{2n+i} dx_{n+i} - x_{3n+i} dx_{7n+i}$$
$$-x_{4n+i} dx_i + x_{5n+i} dx_{6n+i} - x_{6n+i} dx_{5n+i} + x_{7n+i} dx_{3n+i})$$

It is known that if $\Phi_{J_4^*}$ is a closed $K\ddot{a}hler$ form on Clifford $K\ddot{a}hler$ manifold (M, V), then $\Phi_{J_4^*}$ is also a symplectic structure on Clifford $K\ddot{a}hler$ manifold(M, V).

Take into consideration that Hamilton vector field X associated with Hamilton energy H is given by

$$X = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}} + X^{4n+i} \frac{\partial}{\partial x_{4n+i}} +$$
$$X^{5n+i} \frac{\partial}{\partial x_{5n+i}} + X^{6n+i} \frac{\partial}{\partial x_{6n+i}} + X^{7n+i} \frac{\partial}{\partial x_{7n+i}}. \qquad \rightarrow \qquad (5.1.1)$$

Then

$$\Phi_{J_4^*} = -d\lambda_{J_4^*} = dx_{n+i} \wedge dx_{2n+i} + dx_{3n+i} \wedge dx_{7n+i} + dx_{4n+i} \wedge dx_i +$$
$$dx_{6n+i} \wedge dx_{5n+i} \qquad \rightarrow \qquad (5.1.2)$$

And

$$i_X \Phi_{J_4^*} = \Phi_{J_4^*}(X) = X^{n+i} dx_{2n+i} - X^{2n+i} dx_{n+i} + X^{3n+i} dx_{7n+i} - X^{7n+i} dx_{3n+i}$$
$$+X^{4n+i} dx_i - X^i dx_{4n+i} + X^{6n+i} dx_{5n+i} - X^{5n+i} dx_{6n+i} \rightarrow (5.1.3)$$

Furthermore, the differential of Hamilton energy is obtained as follows:

$$dH = \frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial x_{n+i}} dx_{n+i} + \frac{\partial H}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial H}{\partial x_{3n+i}} dx_{3n+i} + \frac{\partial H}{\partial x_{4n+i}} dx_{4n+i}$$
$$+ \frac{\partial H}{\partial x_{5n+i}} dx_{5n+i} + \frac{\partial H}{\partial x_{6n+i}} dx_{6n+i} + \frac{\partial H}{\partial x_{7n+i}} dx_{7n+i} \qquad \rightarrow \quad (5.1.4)$$

According to Eq(2) if equaled Eq(5.1.3) and Eq(5.1.4), the Hamilton vector field is calculated as follows:

$$X = -\frac{\partial H}{\partial x_{4n+i}}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_{n+i}} - \frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_{2n+i}} + \frac{\partial H}{\partial x_{7n+i}}\frac{\partial}{\partial x_{3n+i}} +$$

$$\frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{4n+i}} - \frac{\partial H}{\partial x_{6n+i}}\frac{\partial}{\partial x_{5n+i}} + \frac{\partial H}{\partial x_{5n+i}}\frac{\partial}{\partial x_{6n+i}} - \frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_{7n+i}} \quad \rightarrow \quad (5.1.5)$$

Assume that a curve

$$\alpha : R \rightarrow M \qquad \rightarrow \qquad (5.1.6)$$

Be an integral curve of the Hamilton vector field X, i.e. ,

$$X\big(\alpha(t)\big) = \dot{\alpha} \;\; , \;\; t \in R \qquad \rightarrow \qquad (5.1.7)$$

In the local coordinates, it is found that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}) \quad \rightarrow \qquad (5.1.8)$$

And

$$\dot{\alpha}(t) = \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt}\frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt}\frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt}\frac{\partial}{\partial x_{3n+i}} +$$

$$\frac{dx_{4n+i}}{dt}\frac{\partial}{\partial x_{4n+i}} + \frac{dx_{5n+i}}{dt}\frac{\partial}{\partial x_{5n+i}} + \frac{dx_{6n+i}}{dt}\frac{\partial}{\partial x_{6n+i}} + \frac{dx_{7n+i}}{dt}\frac{\partial}{\partial x_{7n+i}} \quad \rightarrow (5.1.9)$$

Thinking out Eq(5.1.7) if equaled Eq(5.1.5) and Eq(1.5.9), it follows:

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{4n+i}} \; , \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{2n+i}} \;\; , \frac{dx_{2n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}} \; , \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_{7n+i}} \, ,$$

$$\frac{dx_{4n+i}}{dt} = \frac{\partial H}{\partial x_i} \, , \frac{dx_{5n+i}}{dt} = -\frac{\partial H}{\partial x_{6n+i}} \;\; , \frac{dx_{6n+i}}{dt} = \frac{\partial H}{\partial x_{5n+i}} \; , \frac{dx_{7n+i}}{dt} = -\frac{\partial H}{\partial x_{3n+i}} \quad \rightarrow (5.1.10)$$

Hence, the equations obtained in Eq(5.1.10) are shown to be Hamilton equations with respect to component J_4^* of almost Clifford structure V^* on Clifford $K\ddot{a}hler$ manifold(M, V) , and then the triple $(M, \Phi_{J_4^*}\, , X)$ is said to be a Hamilton mechanical system on Clifford $K\ddot{a}hler$ manifold(M, V).

Fifth, let (M,V) be a Clifford $K\ddot{a}hler$ manifold. Assume that an element of almost Clifford structure V^*, a Liouville form and a 1-form on Clifford $K\ddot{a}hler$ manifold (M,V) are determined by J_5^*, $\lambda_{J_5^*}$ and ωJ_5^*, respectively.

Setting

$$\omega J_5^* = \frac{1}{2}(x_i dx_i + x_{n+i}dx_{n+i} + x_{2n+i}dx_{2n+i} + x_{3n+i}dx_{3n+i}$$

$$+x_{4n+i}dx_{4n+i} + x_{5n+i}dx_{5n+i} + x_{6n+i}dx_{6n+i} + x_{7n+i}dx_{7n+i})$$

We have

$$\lambda_{J_5^*} = J_5^*(\omega J_5^*) = \frac{1}{2}(x_i dx_{5n+i} - x_{n+i}dx_{3n+i} - x_{2n+i}dx_{7n+i} + x_{3n+i}dx_{n+i}$$

$$+x_{4n+i}dx_{6n+i} - x_{5n+i}dx_i - x_{6n+i}dx_{4n+i} + x_{7n+i}dx_{2n+i}).$$

Assume that X is a Hamilton vector field related to Hamilton energy H and given by Eq(5.1.1).

Taking into consideration

$$\Phi_{J_5^*} = -d\lambda_{J_5^*} = dx_{n+i}\wedge dx_{3n+i} + dx_{2n+i}\wedge dx_{7n+i} + dx_{5n+i}\wedge dx_i +$$

$$dx_{6n+i}\wedge dx_{4n+i} \qquad \rightarrow \qquad (5.1.11)$$

Then we find

$$i_X\Phi_{J_5^*} = \Phi_{J_5^*}(X) = X^{n+i}dx_{3n+i} - X^{3n+i}dx_{n+i} + X^{2n+i}dx_{7n+i} - X^{7n+i}dx_{2n+i}$$

$$+X^{5n+i}dx_i - X^i dx_{5n+i} + X^{6n+i}dx_{4n+i} - X^{4n+i}dx_{6n+i} \rightarrow (5.1.12)$$

According to Eq(2) if we equal Eq(5.1.4) and Eq(5.1.12) it follows

$$X = -\frac{\partial H}{\partial x_{5n+i}}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_{n+i}} - \frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_{3n+i}} + \frac{\partial H}{\partial x_{7n+i}}\frac{\partial}{\partial x_{2n+i}} -$$

$$\frac{\partial H}{\partial x_{6n+i}}\frac{\partial}{\partial x_{4n+i}} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{5n+i}} + \frac{\partial H}{\partial x_{4n+i}}\frac{\partial}{\partial x_{6n+i}} - \frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_{7n+i}} \quad \rightarrow \quad (5.1.13)$$

Taking Eq(5.1.7), Eqs.(5.1.9) and (5.1.13) are equal, we obtain equations

$$\frac{dx_i}{dt}=-\frac{\partial H}{\partial x_{5n+i}},\frac{dx_{n+i}}{dt}=\frac{\partial H}{\partial x_{3n+i}},\frac{dx_{2n+i}}{dt}=\frac{\partial H}{\partial x_{7n+i}},\frac{dx_{3n+i}}{dt}=-\frac{\partial H}{\partial x_{n+i}},$$

$$\frac{dx_{4n+i}}{dt}=-\frac{\partial H}{\partial x_{6n+i}},\frac{dx_{5n+i}}{dt}=\frac{\partial H}{\partial x_i},\frac{dx_{6n+i}}{dt}=\frac{\partial H}{\partial x_{4n+i}},\frac{dx_{7n+i}}{dt}=-\frac{\partial H}{\partial x_{2n+i}} \rightarrow (5.1.14)$$

In the end, the equations found in Eq(5.1.14) are seen to be Hamilton equations with respect to component J_5^* of almost Clifford structure V^* on Clifford $K\ddot{a}hler$ manifold(M,V) , and then the triple $(M,\Phi_{J_5^*},X)$ is named a Hamilton mechanical system on Clifford $K\ddot{a}hler$ manifold(M,V).

Sixth, let (M,V) be Clifford $K\ddot{a}hler$ manifold. By J_6^* , $\lambda_{J_6^*}$ and ωJ_6^*, we denote a component of almost Clifford structure V^*, a Liouville form and a 1-form on Clifford $K\ddot{a}hler$ manifold(M,V), respectively.

Let ωJ_6^* be determined by

$$\omega J_6^*=\frac{1}{2}(x_i dx_i+x_{n+i}dx_{n+i}+x_{2n+i}dx_{2n+i}+x_{3n+i}dx_{3n+i}$$
$$+x_{4n+i}dx_{4n+i}+x_{5n+i}dx_{5n+i}+x_{6n+i}dx_{6n+i}+x_{7n+i}dx_{7n+i})$$

Then it yields

$$\lambda_{J_6^*}=J_6^*(\omega J_6^*)=\frac{1}{2}(x_i dx_{6n+i}-x_{n+i}dx_{7n+i}-x_{2n+i}dx_{3n+i}+x_{3n+i}dx_{2n+i}$$
$$+x_{4n+i}dx_{5n+i}-x_{5n+i}dx_{4n+i}-x_{6n+i}dx_i+x_{7n+i}dx_{n+i}).$$

It is known that if $\Phi_{J_6^*}$ is a closed $K\ddot{a}hler$ form on Clifford $K\ddot{a}hler$ manifold(M,V) , then $\Phi_{J_6^*}$ is also a symplectic structure on Clifford $K\ddot{a}hler$ manifold(M,V).

Take X. It is Hamilton vector field connected with Hamilton energy H and given by Eq(5.1.1).

Considering

$$\Phi_{J_6^*}=-d\lambda_{J_6^*}=dx_{n+i}\wedge dx_{7n+i}+dx_{2n+i}\wedge dx_{3n+i}+dx_{5n+i}\wedge dx_{4n+i}+$$
$$dx_{6n+i}\wedge dx_i \qquad \rightarrow \qquad (5.1.15)$$

We calculate

$$i_X\Phi_{J_6^*} = \Phi_{J_6^*}(X) = X^{n+i}dx_{7n+i} - X^{7n+i}dx_{n+i} + X^{2n+i}dx_{3n+i} - X^{3n+i}dx_{2n+i}$$

$$+X^{5n+i}dx_{4n+i} - X^{4n+i}dx_{5n+i} + X^{6n+i}dx_i - X^i dx_{6n+i} \rightarrow (5.1.16)$$

According to Eq(2) , Eqs(5.1.4) and (5.1.16) are equaled, Hamilton vector field is found as follows:

$$X = -\frac{\partial H}{\partial x_{6n+i}}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_{7n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_{2n+i}} - \frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_{3n+i}} -$$

$$\frac{\partial H}{\partial x_{5n+i}}\frac{\partial}{\partial x_{4n+i}} + \frac{\partial H}{\partial x_{4n+i}}\frac{\partial}{\partial x_{5n+i}} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{6n+i}} - \frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_{7n+i}} \quad \rightarrow \quad (5.1.17)$$

Considering Eq(5.1.7), we equal Eq(5.1.9) and Eq(5.1.17), it holds

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{6n+i}}, \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{7n+i}}, \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_{3n+i}}, \frac{dx_{3n+i}}{dt} = -\frac{\partial H}{\partial x_{2n+i}},$$

$$\frac{dx_{4n+i}}{dt} = -\frac{\partial H}{\partial x_{5n+i}}, \frac{dx_{5n+i}}{dt} = \frac{\partial H}{\partial x_{4n+i}}, \frac{dx_{6n+i}}{dt} = \frac{\partial H}{\partial x_i}, \frac{dx_{7n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}} \rightarrow (5.1.18)$$

Finally, the equations calculated in Eq(5.1.18) are called to be Hamilton equations with respect to component J_6^* of almost Clifford structure V^* on Clifford $Kähler$ manifold(M, V) , and then the triple $(M, \Phi_{J_5^*}, X)$ is said to be a Hamilton mechanical system on Clifford $Kähler$ manifold(M, V).

5-2 Conclusion

Hamilton formalisms has intrinsically been described with taking into account the basis $\{J_1^*, J_2^*, J_3^*, J_4^*, J_5^*, J_6^*\}$ of almost Clifford structure V^* on Clifford $Kähler$ manifold (M, V).

Hamilton models arise to be a very important tool since they present a simple method to describe the model for dynamical systems. In solving problems in classical mechanics, the rotational mechanical system will then be easily usable model.

Since a new model for dynamic systems on subspaces and spaces is needed, equations (5.1.10), (5.1.14) and (5.1.18) are only considered to be a first step t o realize how Clifford geometry has been used in understanding modeling and solving problems in different physical fields.

For further research, the Hamilton vector fields and equations obtained here are advised to deal with in problems of quantum and classical mechanics of physics.

List of Symbols:-

No	Symbols	Meaning
1	ξ	Vector fields
2	Φ_L	Indicates the symplectic form
3	TM	Tangent bundle
4	M	Manifold
5	E	Euclidean space
6	$\mathcal{F}_p$	A derivation of the algebra
7	$Der(\mathcal{F}_p)$	Set of all derivation on $\mathcal{F}_p$
8	$T_p f$	Tangent map (at p)
9	$\Gamma(M, TM)$	Set of all smooth vector fields on M
10	T	Tensor
11	$T^r(V)$	Covariant r-tensor on V
12	$T_r(V)$	Contra-variant r-tensor on V
13	$T \otimes S$	Tensor product
14	$\oplus$	Direct sum
15	$SymT$	Symmetric tensor
16	$\Sigma^r(V)$	r-tensor on V
17	$AltT$	Alternating tensor of symmetric
18	$\wedge$	Exterior (wedge) product
19	δ	Kronecker delta

20	$\mathcal{F}(\mathbb{C})$	Algebra germs
21	$\mathcal{C}\ell_{0,3}$	Clifford algebra
22	$L: TM \to R$	A regular Lagrangian function
23	E_L	Is the energy associated to L
24	$H: T^*Q \to R$	A regular Hamilton function
25	$\mathcal{F}(M)$	The set of functions on M
26	$\mathcal{X}(M)$	The set of vector fields on M
27	$\Lambda^1(M)$	The set of 1-forms on M
28	$T(M)$	The tangent space
29	$T^*(M)$	The cotangent space
30	$d_{J_i}\ \ i = \overline{1,6}$	Vertical differentiation

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