Chapter 1
Schur Multiplier Norms for Loewner Matrices

The Dounds immediadately lead to upper bounds on the ratio of Schatten q-norms of commutators $\|[A, f(B)]\|_q/\|[A, B]\|_q$. We also consider operator monotone Functions, for which sharper bounds are obtained.

The main impetus behind the work presented is to find good upper bounds on the ratio

$$\|[A, f(B)]\|_q/\|A, B\|_q \tag{1}$$

A basic property of any Schur multiplier norm is its self-duality. If $|||\cdot|||_D$ is the dual norm of $|||\cdot|||$, then $|||S_L||| = |||S_L|||_D$. In particular, $|||S_L|||_q = |||S_L|||_{q'}$, where $1/q = 1 - 1/q$. This can be proven easily using a standard duality argument. The importance of Schur multiplier norms for the problem considered follows from the following proposition:

Proposition $(1.1)[1]$: Let A be any matrix, and let B be Hermitian with eigenvalues b_i. Let L be the Loewner matrix of f at B :

$$L_{ij} := \begin{cases} \dfrac{f(b_i) - f(b_j)}{b_i - b_j}, & b_i \neq b_j, \\ f'(b_i), & b_i = b_j. \end{cases} \tag{2}$$

Then

$$|||[A, f(\mathrm{B})]||| \leq |||S_L||| \, |||[\mathrm{A}, \mathrm{B}]|||.$$

Proof. Working in the eigenbasis, the commutators can be expressed in terms of the Schur product as follows :

$$[A, B] = A \circ (b_i - b_j)^n{}_{i,j=1}, \qquad [A, f(B)] = A \circ \left(f(b_i) - f(b_j)\right)^n_{i,j=1}$$

Consider now the Loewner matrix L of the proposition. It is easy to see that this can be expressed in terms of L as

$$[A, f(B)] = [A, B] \circ L = S_L([A, B])$$

Hence, the norms of both commutators are related by

$$|||[\mathrm{A}, \mathrm{f(b)}]||| \leq |||S_L||| \; |||[\mathrm{A}, \mathrm{B}]|||$$

For the Schatten 2-norm (Frobenius norm), the induced Schur multiplier norm is easily calculated:

$$\begin{aligned} \|S_L\|_2 &= \max_A \frac{\|L \circ A\|_2}{\|A\|_2} \\ &= \max_A \left(\frac{\sum_{i,j} |L_{ij}|^2 |A_{ij}|^2}{\sum_{i,j} |A_{ij}|^2} \right)^{1/2} \\ &= \max_{i,j} |L_{ij}| \, . \end{aligned} \tag{3}$$

Computing Schur multiplier norms for other norms than the 2-norm is in general very difficult, and the fact that all entries of L are in a certain range by no means implies that $\|S_L\|$ should be in that range. Indeed, when L is upper triangular with all entries above the diagonal equal to 1, and all others 0, its Schur multiplier norm is $O(\log n)$.

Using complex interpolation, bounds for general Schatten q-norms can be derived from bounds for the 1-norm and the 2-norm. Indeed, by a direct application,for any $1 \le q \le 2$ we have

$$\|S_L\|_q \le \|S_L\|_1^{2-q} \|S_L\|_1^{q-1}. \quad (4)$$

The first and easiest class of functions treated here are the functions that are operator monotone on a given interval I.

Theorem$(\mathbf{1.2})[\mathbf{1}]$: Let f be an operator monotone function on the interval I. Let B be an $n \times n$ Hermitian matrix with spectrum in I. Let L be the Loewner matrix of f at B. Then, for all Schatten q-norms,

$$\|S_L\|_q \le f'(\lambda_{min}(B)) \quad (5)$$

Note that here f' is always non-negative over I.

Proof: If f is operator monotone, then its Loewner matrix L is a positive semidefinite matrix. By a theorem of Schur, S_L is then a completely positive map and $\|S_L\|$ (and hence $\|S_L\|_1$) is equal to $max_i\, L_{ii}$. In the present case, this number is equal to $\max_i f'(b_i)$. By the concavity of operator monotone functions, this maximum is equal to $f\ (min_i\, b_i)$.

For the Schatten 2-norm, we already found that $\|S_L\|_2 = max_{i,j} |L_{ij}|$. Again, in the present case $max_{i,j} |L_{ij}| = f'(min_i\, b_i)$, which proves the inequality for the Frobenius norm.

Finally, using the complex interpolation, these two results imply that holds for all Schatten norms. Indeed, for any $1 \le q \le 2$,

$$\|S_L\|_{q'} = \|S_L\|_q \le \|S_L\|_1^{2-q} \|S_L\|_2^{q-1} \le f'\left(\min_i b_i\right).$$

This immediately yields:

Corollary $(\mathbf{1.3})[\mathbf{1}]$: Let f be an operator monotone function on the interval I. Let B be an $n \times n$ Hermitian matrix with spectrum in I. Then, for any $q \ge 1$,

$$\|[A, f(B)]\|_q \le f'(\lambda_{min}(B))\|[A,B]\|_q$$

This corollary can be seen as a special case of a result by Kittaneh and Kosaki , which they dubbed the commutator version of the van Hemmen–Ando inequality:

Theorem $(\mathbf{1.4})[\mathbf{1}]$: (Kittaneh–Kosaki). *If A and B* are positive operators on a Hilbert space $\mathcal{H}$ such that $A \ge a \ge 0$ and $B \ge b \ge 0$, then for any operator monotone function f *on* $(0,\infty)$ and any operator X on $\mathcal{H}$,

$$\|f(A)X - Xf(B)\|_q \le C(a,b)\|AX - XB\|_q$$

where $1 \le q \le \infty$ and

$$C(a,b) := \begin{cases} \dfrac{f(a)-f(b)}{a-b} & a \ne b \\ f'^{(a)}, & a = b \end{cases}$$

Their proof proceeds along completiely different lines, and relies on an integral representation of operator monotone functions on $(0, \infty)$.

we obtain an intermediary result needed, which may be of independent interest. The numerical radius is defined as

$$w(A) = \sup_x \frac{|\langle Ax|x\rangle|}{\|x\|^2}$$

This is a norm, and its dual norm is

$$\|Y\|_{w^*} = \sup_x \frac{|TrY^*X|}{w(X)} = \sup_x\{|TrY^*X|: w(x) \le 1\},$$

Which we will call the w^* norm here. The unit ball of the w^* norm is the -absolute convex hull of the matrices of the form xx^* with $x \in \mathbb{C}^n$ and $\|x\| = 1$; i.e. it is the set of matrices $\sum_i \lambda_i\, x_i x_i^*$ for which $\sum_i |\lambda_i| \le 1$ and $\|x_i\| = 1$. This includes but is not limited to the normal matrices with trace norm not exceeding 1.

In general, the numerical radius never exceeds the spectral norm, $w(X) \le \|X\|$. Likewise, the w^* norm is bounded below by the trace norm. Indeed,

$$\|Y\|_{w^*} = \sup_x \frac{|TrY^*X|}{w(X)} \ge \sup_x \frac{|TrY^*X|}{\|X\|} = \|Y\|_1$$

For normal matrices X, the numerical radius is equal to the spectral norm: $w(Y) = \|Y\|$. Here we show that for normal matrices the w^* norm is equal to the trace norm.

Theorem(1.5)[1]: If Y is normal, then $\|Y\|_{w^*} = \|Y\|_1$

Proof: By a theorem of Ando, a matrix X has numerical radius at most one if and only if there exist contractions W and Z, where Z is Hermitian, such that

$$X = (\mathbb{I} + Z)^{1/2} W (\mathbb{I} - Z)^{1/2}$$

The definition of the w^* norm can therefore be rewritten as

$$\|Y\|_{w^*} = \sup_X\{|TrY^*|: w(X) \le 1\}$$

$$= \sup_{W,Z}\left\{\left|Tr\left(Y^* (\mathbb{I} + Z)^{1/2} W (\mathbb{I} - Z)^{1/2}\right)\right|: Z = Z^*, \|Z\| \le 1, \|W\| \le 1\right\}$$

$$= \sup_Z\left\{\sup_W\left\{\left|TrW\left((\mathbb{I} - Z)^{1/2} Y^* (\mathbb{I} + Z)^{1/2}\right)\right|: \|W\| \le 1\right\}: Z = Z^*, \|Z\| \le 1,\right\}$$

$$= \sup_Z\left\{\left\| (\mathbb{I} - Z)^{1/2} Y^* (\mathbb{I} + Z)^{1/2}\right\|_1: Z = Z^*, \|Z\| \le 1\right\}.$$

Since Y is normal , it has a unitary spectral decomposition $Y = \sum_{j=1}^n \lambda_j u_j u_j^*$, with $\{u_j\}_{j=1}^n$ an orthonormal basis of $\mathbb{C}^n$. Hence ,

$$\left\| (\mathbb{I} - Z)^{1/2} Y^* (\mathbb{I} + Z)^{1/2}\right\|_1 \le \sum_j |\lambda_j| \left\| (\mathbb{I} + Z)^{1/2} u_j u_j^* (\mathbb{I} + Z)^{1/2}\right\|_1$$

Noting that for any Hermitian contraction Z

$$\left\| (\mathbb{I} - Z)^{1/2} u_j u_j^* (\mathbb{I} + Z)^{1/2}\right\|_1 = \langle (\mathbb{I} - Z)^{1/2} u_j | u_j\rangle \le \left\|(\mathbb{I} - Z^2)^{1/2}\right\| \le 1.$$

We find

$$\left\| (\mathbb{I} - Z)^{1/2} Y^* (\mathbb{I} + Z)^{1/2}\right\|_1 \le \sum_j |\lambda_j| = \|Y\|_1$$

And therefore

$$\|Y\|_{w^*} \le \|Y\|_1.$$

Theorem(1.6)[1]: Let B be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^n$ sorted in non-decreasing order, $b_1 \le b_2 \le \cdots \le b_n$. let f be a function that is concave or convex on the interval $[b_1, b_n]$. let L be the loewner matrix of f at B then

$$\|S_L\|_2 \le \max(|f'(b_1)|, |f'(b_n)|). \tag{6}$$

Proof: the upper bound is given by $\max_{i,j} |L_{ij}|$. For concave f , the properties (R) of L imply that $\max_{i,j}|L_{ij}| = \max(|L_{11}|, |L_{nn}|)$. since $L_{ii} = f'(b_i)$. For convex f, simply replace f by $-f$ and note that both sides of the inequality are invariant under this sign change.
For the Schur multiplier trace norm (operator norm) we start with a technical proposition about certain standardised monotonously increasing concave functions, as the general case follows easily from this case.
Proposition(1.7)[1]: Let B be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order , $b_1 \le b_2 \le \cdots \le b_n$. Let g be a function that is concave on the interval $[b_1, b_n]$, and for which $g'(b_1) = 1$ and $g'(b_n) = 0$. Let K be the Loewner matrix of g at B. Then

$$|S_K|_1 = |S_k| \le 1 + \emptyset^{-1} \sum_{j=1}^{n} \left(1 - g'(b_j)\right), \qquad (7)$$

where $\emptyset$ is the Golden Ratio,$\emptyset = (1+\sqrt{5})/2 \approx 1.618$.
Note that the interpolation relation (4) can again be used to obtain bounds for general Schatten . norms .
Proof: The matrix K satisfies conditions (R) , and $k_{11} = 1$ and $k_{nn} = 0$. From this I will derive an upper bound on $\|S_k\|$ in terms of the diagonal elements $k_j = k_{jj}$.

The Schur multiplier norm of K can be characterised as

$$|S_K|_1 = |S_k| = \max_{x\in\mathbb{C}^n}\{\|k \circ (xx^*)\|_1 : \|x\| = 1\} .$$

We can find an upper bound on the trace norm of any matrix A by partitioning A as the block matrix

$$A = \begin{pmatrix} B & b \\ b^T & a \end{pmatrix},$$

where B is the upper left $(n-1) \times (n-1)$ submatrix of A, $a = A_{nn}$ and b is the $(n-1)$-dimensional vector consisting of the first $(n-1)$ entries of the last column of A . By a result of Bhatia and Kittaneh, the trace norm of A can be bounded above by the sum of the trace norms of the four blocks, i.e.

$$\|A\|_1 = \|B\|_1 + 2\|b\| + |a| .$$

when we apply this to the matrix $k \circ (xx^*)$, we have $a = k_{nn}|x_n|^2 = 0, b_i = \overline{x}_n x_i k_{in}$ and $B_{ij} = k_{ij} x_i \overline{x}_j$, for $i, j = 1, \cdots, n-1$.
since the vector x is normalised , the norm of the subvector of its first $n-1$ entries is equal to $\sqrt{1-|x_n|^2}$ .Introducing the $(n-1)$-dimensional normalised vector y with $y_i = x_i/\sqrt{1-|x_n|^2}$ for $i, j = 1, \ldots, n-1$, and partitioning K conformally with A as

$$k = \begin{pmatrix} Z & u \\ u^T & 0 \end{pmatrix}$$

we get $b = \overline{x}_n\sqrt{1-|x_n|^2}(y \circ u)$ and $B = (1-|x_n|^2)(Z \circ (yy^*))$. Hence

$$\|k \circ (xx^*)\|_1 \le (1-|x_n|^2)\|Z \circ (yy^*)\|_1 + 2|x_n|\sqrt{1-|x_n|^2}\|y \circ u\|$$

As the maximisation over x reduces maximisation over $|x_n|$ and over y, we obtain

$$\|S_k\| \leq \max_{0\leq x\leq 1}(1-x^2)\|S_Z\| + 2x\sqrt{1-x^2}\max_y\{\|y \circ u\| : \|y\| \leq 1\}.$$

The maximisation $\max_y \{\|y \circ u\| : \|y\| \leq 1\}$ yields $\max u_i, u_j$, which because of (R) is equal to k_{1n} and therefore bounded above by 1. Furthermore , substituting $a = \|S_Z\|$ and $x = \cos\theta$, the remaining max-imisation is

$$\max_{0\leq\theta\leq\pi/2} a\,(1-\text{con}2\theta)/2 + \sin 2\theta \ ,$$

which is the monotonously increasing function

$$v(a) := a/2 \ + \sqrt{1+(a/2)^2}$$

This gives our second relation :

$$\|S_k\| \leq v(\|S_Z\|). \tag{8}$$

Let us write Z in terms of a matrix K with upper left element 1 and lower right element 0 : $Z = k_{n-1}J + (1-k_{n-1})k'$. where J is the n x n matrix with $J_{ij} = 1$. Note that K' is a matrix that still obeys (R) but for which $k'_{n-1} = 0$ and $k'_1 = 1$, i.e. it has the same characteristics as the matrix K we started out with. The diagonal elements of K' in terms of those of K are given by

$$k'_j := \frac{k_j - k_{n-1}}{1-k_{n-1}}. \tag{9}$$

By convexity of the Schur multiplier norm and the fact that $\|S_J\| = 1$, we have

$$\|S_Z\| \leq k_{n-1} + (1-k_{n-1})\|S_{k'}\|,$$

so that, by (8)

$$\|S_Z\| \leq v(\,k_{n-1} + (1-k_{n-1})\|S_{k'}\|). \tag{10}$$

The two realtions (9) and (10) allow to find an easily computable upper bound on S_k via a recursion process. This process stops after n steps, as for a scalar $\|S_a\| = |a|$. In the recursion. We need in succession the elements $k_{n-1}, k'_{n-2}, k'_{n-3}, \cdots, k^{(m)}{}_{n-m}$ .which I'll abbreviate by a_m.for $m = 0, \dots, n-2,$.

Calculating it through, an explicit formula for the elements is $a_0 = k_{n-1}$

And , for $m = 1, \dots, n-2$,

$$a_m = k^{(m)}_{n-m-1} = \frac{k_{n-m-1} - k_{n-m}}{1-k_{n-m}}$$

The last element in this sequence is (since $k_1 = 1$)

$$a_{n-2} = \frac{k_1 - k_2}{1-k_2} = 1\,.$$

Then , denoting $\|S_{k^{(m)}}\| by\ s_m$, $s_m \leq v(a_m + (1-a_m)s_{m=1})$, $s_{n-2} = 1$.

Defining $t_m = s_m - 1$ and

$$b_m = 1 - a_m = \frac{1-k_{n-m-1}}{1-k_{n-m}},$$

We have

$$t_m \leq v(1 + b_m t_{m-1}) - 1, t_{n-2} = 0\,.$$

It is easily verified that $v(1+x) - 1 \leq 1/\emptyset + x$, where $\emptyset$ is the Golden Ratio . Thus

$$t_m \leq b_m t_{m+1} + 1/\emptyset\ , \quad t_{n-2} = 0\ ,$$

Whence

$$t_0 \leq \emptyset^{-1}(1 + b_0 + b_0 b_1 + \cdots + b_0 b_1 \dots b_{n-3})$$

It is immediately checked that
$b_0 b_1 \ldots b_j = 1 - k_{n-j-1}, for\ j = 0, \ldots, n-3\ and\ k_1 = 1, k_n = 0$,so that

$$t_0 \leq \emptyset^{-1} \sum_{j=1}^{n} (1 - k_j)$$

This finally yields $\|S_k\| \leq s_0 \leq 1 + \emptyset^{-1} \sum_{j=1}^{n} (1 - k_j)$ .As $k_{ii} = g'(b_i)$, the inequality of the proposition follows .

Corollary (1.8)[1]: Let B be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order, $b_1 \leq b_2 \leq \cdots \leq b_n$. Let h be a function that is concave on the interval $[b_1, b_n]$, and for which $h'(b_1) = 0$ and $h'(b_n) = -1$. Let K be the Loewner matrix of h at B.

Then.

$$\|S_k\| \leq 1 + \emptyset^{-1} \sum_{j=1}^{n} \left(1 + h'(b_j)\right). \tag{11}$$

Proof: This follows immediately from Proposition (1.7) with the matrix B replaced by $B' = b_1 + b_n - B$ and defining $h(x) = g(b_1 + b_n - x)$, so that $h'(b_j) = -g'(b_1 + b_n - b_j) = -g'(b'_j)$.

Theorem (1.9)[1]: Let B be a Hermitian $n \times n$ matrix with eigenvalues $(b_j)_{j=1}^{n}$ sorted in non-decreasing order, $b_1 \leq b_2 \leq \cdots \leq b_n$. Let f be a function that is concave on the interval $[b_1, b_n]$. Let L be the Loewner matrix of f at B.

Then

$$\|S_L\| \leq (\alpha - \beta) + \min\left(|\beta| + \emptyset^{-1} \sum_{j=1}^{n} \left(\alpha - f'(b_j)\right), |\alpha| + \emptyset^{-1} \sum_{j=1}^{n} \left(f'(b_j) - \beta\right) \right)$$

Where $\alpha = f'(b_1)$ and $\beta = f'(b_n)$. For any function that is convex on the interval $[b_1, b_n]$,

$$\|S_L\| \leq (\beta - \alpha) + \min\left(|\beta| + \emptyset^{-1} \sum_{j=1}^{n} \left(f'(b_j) - \alpha\right), |\alpha| + \emptyset^{-1} \sum_{j=1}^{n} \left(\beta - f'(b_j)\right) \right)$$

Proof: General concave functions f can be mapped to the standardised functions g and h . Note that

$$\alpha := f'(b_1) \geq f'(b_j) \geq f'(b_n) =: \beta .$$

First we write

$$F(x) = \beta x + (\alpha - \beta) g(x).$$

Then

$$(\alpha - \beta) g'(x) = f'(x) - \beta .$$

Letting L and K be the Loewner matrices of f and g, respectively, at B,

$$L = \beta J + (\alpha - \beta) k ,$$

where J is the matrix all of whose entries are 1. As $\|S_J\| = 1$.

$$\|S_L\| \le |\beta| + (\alpha-\beta)\|S_k\| \le |\beta| + (\alpha-\beta)\left(1 + \emptyset^{-1}\sum_{j=1}^{n}\left(1 - g'(b_j)\right)\right)$$

$$= |\beta| + (\alpha-\beta) + \emptyset^{-1}\sum_{j=1}^{n}((\alpha-\beta) - f'(b_j) - \beta)$$

$$= |\beta| + (\alpha-\beta) + \emptyset^{-1}\sum_{j=1}^{n}(\alpha - f'(b_j)) .$$

We can also write

$$f(x) = \alpha x + (\alpha-\beta)h(x)$$

and obtain in a similar way

$$\|S_L\| \le |\alpha| + (\alpha-\beta) + \emptyset^{-1}\sum_{j=1}^{n}(f'(b_j) - \beta)$$

Taking the minimum of both bounds yields the bound of the corollary.

For convex f we just replace f by –f and apply the result for concave functions. Since now $\alpha := f'(b_1) \le f'(b_j) \le f'(b_n) =: \beta$ an appropriate sign change has to be applied to the bound .

When the spectrum of B is not known, but it is known that $b_1 \le B \le b_n$, weaker bounds follow readily from this Theorem.

Corollary $(\mathbf{1.10})[\mathbf{1}]$: Let B be a Hermitian n x n matrix bounded as $b_1 \le B \le b_n$, Let f be a function that is either concave or convex on the interval $[b_1, b_n]$. Let L be the Loewner matrix of f at B. Then

$$\|S_L\| \le |\alpha-\beta|(1 + (n-1)\emptyset^{-1}) + \min(|\beta|, |\alpha|)$$

Where $\alpha = f'(b_1)$ and $\beta = f'(b_n)$.

Examples $(\mathbf{1.11})$ $[\mathbf{1}]$:

As a first application , we consider the function $f(x) = |x|$.

Theorem $(\mathbf{1.12})[\mathbf{1}]$: Let B be a Hermitian $n \times n$ matrix with r positive eigenvalues. Let L be the Loewner matrix of the function $f(x) = |x|$ at B. Then, for $1 \le r < n$.

$$\|S_L\| \le 3 + 2\emptyset^{-1}\min(r, n-r) .$$

If r is 0 or n , $\|S_L\|$ is 1 .

Proof: For $1 \le r < n$, $\alpha = f'(b_1) = 1, \beta = f'(b_n) = -1, f'(b_j) = 1$ for $n - r$ values of j , and $f'(b_j) = -1$. The bound follows by simple calculation.

Since the bounds only depend on the diagonal elements of the Loewner matrix. they are not expected to be sharp for specific functions. For the absolute value function, for example, it is known that in the $d = 2$ case the norm ratio lies between the values 1 and $\sqrt{2}$, whereas the theorem gives the bound $3 + 2/\emptyset$ for r=1.

For our second example, consider the following corollary of the main theorem. Let C be a Hermitian matrix with spectrum $c_1 \le c_2 \le \cdots \le c_n$. By putting $B = g(C)$ and $h = f \circ g$, we find :

Corollary $(\mathbf{1.13})[\mathbf{1}]$**:** For all $n \times n$ matrices A and for any monotonously increasing function g and any function h such that $f = h \circ g^{-1}$ is concave,

$$\frac{\|[A, h(C)]\|_1}{\|[A, g(C)]\|_1} \leq (\alpha - \beta) + \min\left(|\beta| + \emptyset^{-1} \sum_{j=1}^{n} (\alpha - f'(g(c_j))), |\alpha| + \emptyset^{-1} \sum_{j=1}^{n} (f'(g(c_j)) - \beta) \right),$$

Where $\alpha = f'(g(c_1))$ and $\beta = f'\big(g(c_n)\big)$.

Consider the functions $h(x) = \log x$ and $g(x) = \log(x) - \log(1 - x)$ .thus , $f(x) = x - \log(1 + e^x)$ which is monotonously increasing and concave , and $(f' \circ g)(x) = 1 - x$. The bound of the corollary then simplifies to

$$c_n - c_1 + \min\big(1 - c_n + \emptyset^{-1}(1 - nc_1), 1 - c_1 + \emptyset^{-1}(nc_n - 1)\big)$$

As $n_1 \geq 0$, this quantity is boded above by $1+\emptyset^{-1} = \emptyset$. we have therefore proven :

Corollary $(\mathbf{1.14})[\mathbf{1}]$**:** For any A and for any positive semidefinite C with Tr$C = 1$

$$\|[A, \log(C)]\|_1 \leq \emptyset \|[A, \log(C) - \log(\mathbb{I} - C]\|_1.$$

Theorem $(\mathbf{1.15})[\mathbf{5}]$**:** Let f be an operator convex function, then all Loewner matrices associated with f are conditionally negative definite one of the interesting relation, between operator monotone and convex function is that $f(t)$ is operator convex on $[0, \infty]$ if and only if $g(t) = f(t)$ it is operator monotone on $(0, \infty)$ this class an important role in analysis . The class of function $f(t) = tg(t)$ where g is operator convex seems equally in teresting in this context.

Theorem $(\mathbf{1.16})[\mathbf{6}]$**:** Let f be a c^1 function on $(0, \infty)$ and suppose $f'(t) > 0$, if for all $p_1, \dots, p_n$ the Loewner matrix $L_f(p_1, \dots, p_n)$ has exactly one positive eigenvalue , then the inversting function $g = f^{-1}$ is operator monotone.

Chapter 2

Complete Characterization of Hadamard Powers Preserving

Entrywise powers of symmetric matrices preserving positivity, monotonicity or convexity with respect to the Loewner ordering arise in various applications. Following Fitzgerald and Horn, it is well-known that there exits acritical exponent beyond which all entrywise powers preserve positive definiteness. Similar phenomena have also been shown by Hiai to accur for monotonicity and convexity.We extend the original problem by fully classifying the positive, monotone, or convex powers in amore general setting where additional rank constraints are imposed on the matrices. We also classify the entrywise powers that are super/sup. Additive with respect to the Loewner ordering. We extend all the previous characterizations to matrices with negative entries.

Section (2.1): Characterizing Entrywise Powers that are Loewner Positive

Definition $(\mathbf{2.1.1})[\mathbf{2}]$: Fix integers $n \geq 2$ and $1 \leq k \leq n$, and subsets $I \subset \mathbb{R}$. Let $\mathbb{P}_n^k k(I)$ denotethe subset of matrices in $\mathbb{P}_n(I)$ that have rank at most k. Define:

$$\mathcal{H}_{pos}(n,k) := \{\alpha \in \mathbb{R}\text{: the function } x^{\alpha} \text{ is positive on } \mathbb{P}_n^k([0,\infty))\},$$

$$\mathcal{H}_{pos}^{\phi}(n,k) := \{\alpha \in \mathbb{R}\text{: the function } \phi_{\alpha} \text{ is positive on } \mathbb{P}_n^k(\mathbb{R})\},$$

$$\mathcal{H}_{pos}(n,k) := \{\alpha \in R\text{: the function } \psi_{\alpha} \text{ is positive on } \mathbb{P}_{\mathrm{n}}(\mathbb{R})\}. \quad (1)$$

Similarly, let $\mathcal{H}_J(n,k), \mathcal{H}_J^{\phi}(n,k), \mathcal{H}_J^{\phi}(n,k)$ denote the entrywise powers preserving Loewner properties on $\mathbb{P}_n^k([0,\infty))$ or $\mathbb{P}_n^k(\mathbb{R})$, with $J \in$\{ positivity, monotonicity, convexity, super-additivity, sub-additivity\}.

Theorem $(\mathbf{2.1.2})[\mathbf{2}]$**:** (Main result). Fix an integer n ≥ 2. The sets of entrywise real powers that are Loewner positive, monotone, convex, and super/sub-additive, are as listed. We complete classification of the powers preserving various Loewner proper-ties, previous contributions in the literature are also included for completeness. Note that there are many cases which had not been considered previously and which we settle completely in this section. For sake of brevity, we will only briefly sketch proofs for the previously addressed cases (in order to mention how the rank constraint affects the problem). We instead focus our attention on the cases that remain open in the literature. Our original contributions in this section are:

J	$\mathcal{H}_J(n,k)$	$\mathcal{H}_J^{\phi}(n,k)$	$\mathcal{H}_J^{\Psi}(n,k)$
Positivity $K=1$ $2 \leq k \leq n$	$\mathbb{R}$ G–K–R $\mathbb{N} \cup [n-2,\infty)$ FitzGerald–Horn	$\mathbb{R}$ G–K–R $2\mathbb{N} \cup [n-2,\infty)$ FitzGerald–Horn,Hiai,Bhatia–Elsner,G–K–R	$\mathbb{R}$ G–K–R $(-1+2\mathbb{N}) \cup [n-2,\infty)$ FitzGerald–Horn,Hiai,G–K–R
Monotonicity $K=1$ $2 \leq k \leq n$	$[0,\infty)$ G–K–R $\mathbb{N} \cup [n-2,\infty)$ FitzGerald–Horn	$[0,\infty)$ G–K–R $2\mathbb{N} \cup [n-1,\infty)$ FitzGerald–Horn,Hiai, G–K–R	$[0,\infty)$ G–K–R $(-1+2\mathbb{N}) \cup [n-1,\infty)$ FitzGerald–Horn,Hiai,G–K–R
Convexity $K=1$ $2 \leq k \leq n$	$[1,\infty)$ G–K–R $\mathbb{N} \cup [n,\infty)$ Hiai, G–K–R	$[1,\infty)$ G–K–R $2\mathbb{N} \cup [n,\infty)$ Hiai, G–K–R	$[1,\infty)$ G–K–R $(-1+2\mathbb{N}) \cup [n,\infty)$ Hiai, G–K–R
Super-additivity $1 \leq k \leq n$	$\mathbb{N} \cup [n,\infty)$ G–K–R	$2\mathbb{N} \cup [n,\infty)$ G–K–R	$(-1+2\mathbb{N}) \cup [n,\infty)$ G–K–R
Sup-additivity $K=1$ $2 \leq k \leq n$	$[1,\infty) \cup \{1\} if n=2,$ $\{0,1\}$ if $n>2$ G–K–R $\{1\}$ G–K–R	ϕ G–K–R ϕ G–K–R	$\{0,1\}$ if $n=2$ $\{1\}$ if $n>2$ G–K–R $\{1\}$ G–K–R

(i) W ecomplete all of the previously unsolved cases involving powers preserving positivity, monotonicity, and convexity.
(ii) W eclassify all powers preserving super-additivity and sub-additivity. These properties have not been explored in the literature in the entrywise setting.
(iii) W ealso examine negative powers preserving Loewner properties, which were also previously unexplored.
(iv) Finally, we extend all of the above results – as well as those in the literature – by introducing rank constraints. Once again, we are able to obtain a complete classification of all real powers preserving the five aforementioned Loewner properties.

Similar to many settings in the literature, one can define Hadamard critical exponents for positivity, monotonicity, convexity, and super-additivity for $\mathbb{P}_n^k$ – these are the phase transition points akin. From Theorem (2.1.2), we immediately obtain the Hadamard critical exponents (CE) for the four Loewner properties for matrices with rank constraints:

Corollary $(\mathbf{2.1.3})[\mathbf{2}]$: Suppose $n \geq 2$ and $1 \leq k \leq n$. The Hadamard critical exponents for positivity, monotonicity, convexity, and super-additivity for $\mathbb{P}_n^k$ are $n-2, n-1, n, n$ respectively if $2 \leq k \leq n$, *and* $0, 0, 1, n$ respectively if $k = 1$. In particular, they are completely independent of the type of entrywise power used.

An interesting consequence of Corollary (2.1.3) is that if k ≥ 2, then the sets of fractional Hadamard powers $f_\alpha, \phi_\alpha, or\ \psi_\alpha$ that are Loewner positive, monotone, convex, or super-additive on $\mathbb{P}_n^k$ do not depend on k.

Thus, entrywise powers that preserve such properties on $\mathbb{P}_n^2$ automatically preserve them on all of $\mathbb{P}_n$. Corollary (2.1.3) also shows that the rank 1 caseis different from that of other k, in that three of the critical exponents do not depend on n if $k = 1$.This is not surprising for positivity because the functions $f_\alpha, \phi_\alpha, \psi_\alpha$ are all multiplicative. Furthermore, note that if $2 \leq k \leq n$, then entrywise maps are Loewner convex on $\mathbb{P}_n^k$ (i) if and only if they are Loewner super-additive. Finally, the structure of the $H_J(n,k)$-sets is different for J = sub-additivity, compared to the other Loewner properties.

We prove Theorem (2.1.2) by systematically studying entrywise powers that are (i) positive, (ii) monotone, (iii) convex, and (v) super/sub-additive with respect to the Loewner ordering. Thus in each of the next four sections, we gather previous results from the literature, and extend these in order to compute the sets $\mathcal{H}_J^I$I (n,k) f ormatrices with rank constraints. In doing so, as a special case we can complete the classification of powers $f_\alpha, \phi_\alpha, \psi_\alpha$ that are Loewner positive, monotone, or convex, for all matrices in $\mathbb{P}_n = \mathbb{P}_n^k$ (i.e., with no rank constraint). We then classify the entrywise real powers that are super/sub-additive and in the process demonstrate an interesting connection to Loewner convexity.

The study of Hadamard powers originates in the work of FitzGerald and Horn. We begin our analysis by stating one of their main results that characterizes the Hadamard powers preserving Loewner positivity.

Theorem $(\mathbf{2.1.4})[\mathbf{2}]$: Suppose A ∈ $\mathbb{P}_n$ ([0, ∞)) for some n ≥ 2. Then $A^{\circ\alpha} \in \mathbb{P}_n$ for all $\alpha \in \mathbb{N} \cup [n-2, \infty)$. *If* $\alpha \in (0, n-2)$ is not an integer, then there exists A ∈ $\mathbb{P}_n$ ((0, ∞)) such that $A^{\circ\alpha} \notin \mathbb{P}_n$. More precisely, Loewner positivity is not preserved for $A = ((1 + \epsilon ij))_{i.j=1,}^n$ for all sufficiently small $\epsilon > 0$ with $\alpha \in (0, n-2) \setminus \mathbb{N}$.

Thus, $\mathcal{H}_{pos}(n,n) = \mathbb{N} \cup [n-2,\infty)$ for all $2 \le n \in \mathbb{N}$. Additionally, Hiai showed that the same results as above hold for the critical exponent for the even and odd extensions ϕ_α and ψ_α:

Theorem (2.1.5)[2]:

If $n \ge 2$ and $\alpha \ge n-2$, them $\alpha \in \mathcal{H}^{\phi}_{pos}(n,n) \cap \mathcal{H}^{\psi}_{pos}(n,n)$.

Theorem (2.1.6)[2]: Suppose $n \ge 2$, and $r \in (0, n-2) \setminus 2\mathbb{N}$ is real. Then $r \notin \mathcal{H}^{\phi}_{pos}(n,n)$.

Theorem (2.1.7)[2]: Suppose $2 \le k \le n$ are integers with $n \ge 3$. Then,

$$\mathcal{H}_{pos}(n,k) = \mathbb{N} \cup [n-2,\infty), \mathcal{H}^{\phi}_{pos}(n,k) = 2\mathbb{N} \cup [n-2,\infty),$$
$$\mathcal{H}^{\psi}_{pos}(n,k) = (-1+2\mathbb{N}) \cup [n-2,\infty). \quad (2)$$

If instead $k = 1$ and/or $n = 2$, then

$$\mathcal{H}_{pos}(n,k) = \mathcal{H}^{\phi}_{pos}(n,k) = \mathcal{H}^{\psi}_{pos}(n,k) = (0,\infty). \quad (3)$$

Proof. First suppose $k = 1, n \ge 2$, and $A = uu^T \in \mathbb{P}^1_N$ for some $u \in \mathbb{R}^n$. Since the functions f_α, ψ_α, φ_α are multiplicative for all $\alpha \in \mathbb{R}$, we have $A^{\circ\alpha} = u^{\circ\alpha}(u^{\circ\alpha})^T \in \mathbb{P}^1_N$ for $u \in [0,\infty)^n$, and similarly for $\psi_\alpha[A], \emptyset_\alpha[A]$ for $u \in \mathbb{R}^n$. The result thus follows for $k = 1$.Furthermore, the result is obvious for $n = 2$ and all $\alpha \in \mathbb{R}$.

Now suppose that $2 \le k \le n$ and $n \ge 3$. We consider three cases corresponding to the three functions $f(x) = f_\alpha(x), \emptyset_\alpha(x)$, and $\psi_\alpha(x)$.

Case 1: $f(x) = f_\alpha(x)$. Consider the matrix

$$A := \begin{pmatrix} 1 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1 \end{pmatrix} \oplus 0_{(n-3)\,(n-3)} \in \mathbb{P}^2_n(0,\infty).$$

It is easily verified that $f_\alpha[A] \notin \mathbb{P}_n$ *for all* $\alpha \le 0$. Thus using Theorem (2.1.4) we have

$$\mathbb{N} \cup [n-2,\infty) = \mathcal{H}_{pos}(n,n) \subset \mathcal{H}_{pos}(n,k) \subset (0,\infty).$$

Now note that the counterexample $((1+\epsilon_{ij})) \in \mathbb{P}^2_n([0,\infty))$ provided in Theorem (2.1.4) is a rank 2 matrix and hence $\alpha \notin \mathcal{H}_{pos}(n,2)$ for any $\alpha \in (0, n-2) \setminus \mathbb{N}$. Thus $\mathcal{H}_{pos}(n,2) = \mathbb{N} \cup [n-2,\infty)$. Finally, since $\mathcal{H}_{pos}(n,k) \subset \mathcal{H}_{pos}(n,2)$, it follows that $\mathcal{H}_{pos}(n,k) = \mathbb{N} \cup [n-2,\infty)$.

Case 2: $f(x) = \phi_\alpha(x)$. Note that $2N \subset \mathcal{H}^{\phi}_{pos}(n,k)$ by the Schur product theorem. Using Theorem (2.1.5) and Case 1, it remains to show that no odd integer $\alpha \in (0, n-2)$ belongs to $\mathcal{H}^{\phi}_{pos}(n,k)$. To do so and for later use, first define the matrix A_r for $r \in \mathbb{N}$ as follows:

$$(A_r)_{ij} := (cos(i-j)\pi/r), \qquad 1 \le i,j \le r. \quad (4)$$

Note that $A_r\, \mathbb{P}^2_r$ since $A_r = uu^T + vv^T$, where $u := (cos(j\pi/r))^r_{j=1}$ and $v := (sin(j\pi/r))^r_{j=1}$. The matrix $\phi_\alpha[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3}$ for all $p \in (\alpha-1, \alpha+1)$. In particular, $\phi_p[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3}$. Since we are considering integer powers α such that $\alpha < n-2$, we have $\alpha + 3 \le n$, so

$$A_{\alpha+3} \oplus 0_{(n-\alpha-3)\times(n-\alpha-3)} \in \mathbb{P}^2_n, \qquad \phi_\alpha[A_{\alpha+3} \oplus 0_{(n-\alpha-3)\times(n-\alpha-3)}] \notin \mathbb{P}_n,$$

which proves that $\alpha \notin \mathcal{H}^{\phi}_{pos}(n,2)$ for any odd integer $\alpha \in (0, n-2)$, as desired.

Case 3: $f(x) = \psi_\alpha(x)$. Note that $-1+2\mathbb{N} \subset \mathcal{H}^{\psi}_{pos}(n,k)$ by the Schur product theorem. Using Theorem (2.1.5) and Case 1, it remains to show that no even integer $\alpha \in (0, n-2)$ belongs to $\mathcal{H}^{\psi}_{pos}(n,k)$. The approach in this part is to prove a result similar to the main, but for the function ψ_α. The proof is therefore omitted for brevity. For future reference we point out that the key step in the proof uses the following assertion:

$$\psi_p[A_{\alpha+3}] \notin \mathbb{P}_{\alpha+3} \; \forall p \in (\alpha-1, \alpha+1), \tag{5}$$

where $A_{\alpha+3}$ is defined in (4).

Our result on the full characterization of the even extensions of entrywise powers that preserve Loewner positivity, as given by Theorem (2.1.7), allows us to answer this question. By Theorem (2.1.7), the smallest $n \in \mathbb{N}$ such that $\varphi_p[A] \notin \mathbb{P}_n$ for at least one matrix $A \in \mathbb{P}_n$, $is\ n = \lfloor p \rfloor + 3$. Similarly, one can ask the analogous question for ψ: given $p \in (0,\infty) \setminus (-1+2\mathbb{N})$, find the smallest $n \in \mathbb{N}$ such that $\psi_p[A] \notin \mathbb{P}_n$ for at least one $A \in \mathbb{P}_n$. Once again by Theorem (2.1.7), the answer to this question is $n = \lfloor p \rfloor + 3$.

We now characterize the entrywise powers that are monotone with respect to the Loewner ordering. The following theorem by FitzGerald and Horn that is analogous but for monotonicity, answers the question for matrices with nonnegative entries. In what follows, we denote by $\mathbf{1}_{n\times n}$ *the* $n \times n$ matrix with all entries equal to 1.

Theorem (2.1.8)[2]: Suppose $0 < R \le \infty$, $I = (-R, R)$, and $f: I \to R$.

(i) For each $n \ge 3$, f is monotone on $\mathbb{P}_n(I)$ if and only if f is differentiable on I and f' is Loewner positive on $\mathbb{P}_n(I)$.

(ii) If $n \ge 1$ and $\alpha \ge n-1$, then $\alpha \in \mathcal{H}^{\varphi}_{\mathrm{mono}}(n,n) \cap \mathcal{H}^{\psi}_{\mathrm{mono}}(n,n)$. Theorem (2.1.9) is a powerful result, but cannot be applied directly to study entrywise functions preserving matrices in the more restricted set $\mathbb{P}^k_n$. We thus refine the first part of the theorem to also include rank constraints.

Proposition (2.1.9)[2]: Fix $0 < R \le \infty$, $I = (-R, R)$, and $2 \le k \le n$. Suppose $f: I \to R$ is differentiable. on I and Loewner monotone on $\mathbb{P}^k_n(I)$. If $A \in \mathbb{P}^k_n(I)$ is irreducible, .then $f'[A] \in \mathbb{P}_n$.

Proof: We first make the following observation (which in fact holds over any infinite field): Suppose $A_{n\times n}$ is a symmetric irreducible matrix. Then there exists a vector $\zeta \in Im\ A(the\ image\ of\ A)$ with no zero component.

To see why the observation is true, first suppose that all vectors in Im A have the ith component zero for some $1 \le i \le$ n– i.e., $e_i^T A v = 0$ for all vectors v.

Then the ith row (and hence column) of A is zero, which contradicts irreducibility.

Now fix vectors $w_i \in \mathrm{Im}\ A$ for all $1 \le i \le n$, such that the ith entry of w^i is nonzero. Let $W := [w_1|w_2| \cdots |w_n]$; then for all tuples $\boldsymbol{c} := ([c_1,\cdots,c_n]^T) \in \mathbb{R}^n$,

$$Wc = \sum_{i=1}^{n} c_i w_i \in Im\ A.$$

Consider the set S of all $\boldsymbol{c} \in \mathbb{R}^n$ such that W**c** has a zero entry. Then $S = \bigcup_{i=1}^n S_i$, where $\boldsymbol{c} \in S_i$ if $e_i^T W c = 0$. Note that S_i is a proper subspace of $\mathbb{R}^n$ since $e_i \notin S_i$ by assumption on w_i. Since $\mathbb{R}$ is an infinite field, S is a proper subset of $\mathbb{R}^n$, which proves the observation.

Now given an irreducible matrix $A \in \mathbb{P}_n^k(I)$, choose a vector $\zeta \in ImA$ as in the above observation. Let $A_\epsilon := A + \epsilon\zeta\zeta^T for\ \epsilon > 0;\ then\ A_\epsilon \in \mathbb{P}_n^k(I)$ since $\zeta \in ImA$. Therefore by monotonicity, $\frac{f[A_\epsilon]-f[A]}{\epsilon} \geq 0$. Letting $\epsilon \to 0^+$, it follows that $f'[A] \circ (\zeta\zeta^T) \geq 0$.Now let $\zeta^{\circ(-1)} := (\zeta^{-1}, \ldots, \zeta_n^{-1})^T$;then by the Schur Product Theorem, it follows that $f'[A] = f'[A] \circ (\zeta\zeta^T) \circ (\zeta^{\circ(-1)})(\zeta^{\circ(-1)}))^T) \geq 0$, which concludes the proof.

With the above results in hand, we now completely classify the powers preserving Loewner monotonicity, and also specify them when rank constraints are imposed.

Theorem $(\mathbf{2.1.10})[\mathbf{2}]$: Suppose $2 \leq k \leq n$ are integers. Then,

$$\mathcal{H}_{mono}(n,k) = \mathbb{N} \cup [n-1,\infty),\ \ \mathcal{H}^{\varphi}_{\mathrm{mono}}(\mathrm{n},\mathrm{k}) = 2\mathbb{N} \cup [n-1,\infty),$$
$$\mathcal{H}^{\psi}_{\mathrm{mono}}(\mathrm{n},\mathrm{k}) = (-1+2\mathbb{N}) \cup [n-1,\infty), \tag{6}$$

If instead $k = 1$, then

$$\mathcal{H}_{mono}(n,1) = \mathcal{H}^{\varphi}_{\mathrm{mono}}(\mathrm{n},1) = \mathcal{H}^{\psi}_{\mathrm{mono}}(\mathrm{n},1) = (0,\infty). \tag{7}$$

Proof: First suppose $k = 1 < n$ and $A = uu^T, B = vv^T \in \mathbb{P}_n^1$. If $A \geq B \geq 0$, then we claim that $v = cu$ for some $c \in [-1,1]$. To see the claim, assume to the contrary that u, v are linearly independent. We can then choose $w \in \mathbb{R}^n$ such that w is orthogonal to u but not to v. But then $w^T(A-B)w = -(w^Tv)^2 < 0$, which contradicts the assumption $A \geq B$. Thus u, v are linearly dependent. Since $A \geq B \geq 0$, it follows that $v = cu$ with $|c| \leq 1$. Now for all $\alpha \geq 0$ and all $A, B \in \mathbb{P}_n^1([0,\infty))$such that $A \geq B \geq 0$, we use the multiplicativity of f_α to compute:

$$f_\alpha[A] - f_\alpha[B] = f_\alpha[u]f_\alpha[u]^T - f_\alpha[cu]f_\alpha[cu]^T = (1-(c^2)^\alpha)f_\alpha[u]f_\alpha[u]^T \geq 0.$$

Thus f_α is monotone on $\mathbb{P}_n^1([0,\infty))$. Similar computations show the monotonicity of φ_α and $\varphi_\alpha$on $\mathbb{P}_n^1(\mathbb{R}) for\ all\ \alpha \geq 0$. The same computations also show that f_α, φ_α ,ψ_α are not monotone on $\mathbb{P}_n^1(I)$, for any $\alpha < 0$.

Now suppose $2 \leq k \leq n$. Note that if $A \geq B \geq 0$, then one inductively shows using the Schur product theorem that

$$\mathrm{A}^{\circ m} \geq \mathrm{B}^{\circ m}\ \forall m \leq N \Rightarrow \mathrm{A}^{\circ(N+1)} - \mathrm{B}^{\circ(N+1)}$$
$$= \sum_{m=0}^{N} \mathrm{A}^{\circ m} \circ (\mathrm{A}-\mathrm{B}) \circ \mathrm{B}^{\circ(N-m)} \geq 0. \tag{8}$$

Therefore every positive integer Hadamard power is monotone on $\mathbb{P}_n$. We now consider three cases corre-sponding to the three functions $f(x) = \mathrm{f}_\alpha(x), \varphi_\alpha(x)$, and $\psi_\alpha(x)$.

Case 1: $f(x) = \mathrm{f}_\alpha(x)$. First suppose $\alpha < 1$. By considering the matrices $B = \mathbf{1}_{2\times 2}\ and\ A = B + uu^T$ with $u = (1,-1)^T$, we immediately obtain that f_α is not monotone on $\mathbb{P}_2^2([0,\infty))$, and hence not on $\mathbb{P}_n^k([0,\infty))$ for all $\alpha < 1$. Now the above analysis imply that $\mathcal{H}_{mono}(n,k) \subset N \cup [n-1,\infty)$, since $((1+\epsilon_{ij})), \mathbf{1}_{n\times n} \in \mathbb{P}_n^2([0,\infty)) \subset \mathbb{P}_n^k([0,\infty))$ provide the necessary counterexample for $\alpha \in (0, n-1) \setminus \mathbb{N}$. Furthermore , $\mathbb{N} \cup [n-1,\infty) = \mathcal{H}_{mono}(n,k) \subset \mathcal{H}_{mono}(n,k)$, and thus $\mathcal{H}_{mono}(n,k) = \mathbb{N} \cup [n-1,\infty)$.

Case 2: $f(x) = \varphi_\alpha By\ E_q(8)\varphi_\alpha(x)\ \varphi_{2n}[A] = \mathrm{A}^{\circ 2n}$ preserves monotonicity on $\mathbb{P}_n$. From this observation, it follows that $2\mathbb{N} \cup [n-1,\infty) \subset \mathcal{H}^{\varphi}_{\mathrm{mono}}(\mathrm{n},\mathrm{n}) \subset \mathcal{H}^{\varphi}_{\mathrm{mono}}(\mathrm{n},\mathrm{k})$.

We now claim that

$$\mathcal{H}^{\varphi}_{\rm mono}(\rm n,k) \subset \mathcal{H}^{\varphi}_{\rm pos}(\rm n,k) \cap \mathcal{H}_{mono}(n,k) \subset \{n-2\} \cup 2\mathbb{N} \cup [n-1,\infty).$$

Indeed, the first inclusion above follows because every monotone function on $\mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$is simultaneously monotone on $\mathbb{P}^{\rm k}_{\rm n}([0,\infty))$and positive on $\mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$ by definition. The second inclusion above holds by Theorem(2.1.7)and Case 1.

It thus remains to consider if φ_{n-2} is monotone on $\mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$. We consider three sub-cases: if $n > 2$ is even, then $n-2 \in 2\mathbb{N} \cup [n-1,\infty)$. Hence $\mathcal{H}^{\varphi}_{\rm mono}(\rm n,k) = 2\mathbb{N} \cup [n-1,\infty)$ by the analysis mentioned above in Case 2. Next if $n = 3$, we produce a three-parameter family of matrices $A \geq \mathbf{1}_{3\times3} \geq 0$ in $\mathbb{P}_3(\mathbb{R})$such that $\varphi_1[A] \ngeq \varphi_1[\mathbf{1}_{3\times3}]$. Indeed, choose any $a > b > 0$ and $c \in (a^{-1}, b^{-1})$, and define

$$v := (a, b-c)^T, \qquad B := 1_{3\times3}, \qquad A := B + vv^T$$

Then both A, B are in $\mathbb{P}^2_3(\mathbb{R})$, and $\varphi_1[A], \varphi_1[B] \in \mathbb{P}_3(\mathbb{R})$. However,

$$\det(\varphi_1[A] - \varphi_1[B]) = det\begin{pmatrix} a^2 & ab & ac-2 \\ ab & b^2 & -bc \\ ac-2 & -bc & c^2 \end{pmatrix} = -4b^2(ac-1)^2 < 0.$$

Thus $\varphi_1$is not monotone on $\mathbb{P}^2_3(\mathbb{R})$,.

Finally, suppose $n > 3$ is odd and that $\varphi_{n-2}$is monotone on $\mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$. We then obtain a contradiction as follows: recall from $E_q(5)$ that the matrix A_n constructed in $E_q.(4)$ satisfies: $\psi_{n-3}[A_n] /\in \mathbb{P}_n$. (Here, $\alpha = n-3 = p$ is an even integer in $(0, n-2)$, since $n > 3$ is odd.) Moreover, $A_n \in \mathbb{P}^2_{\rm n}(\mathbb{R}) \subset \mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$is irreducible. Hence if φ_{n-2} is monotone on $\mathbb{P}^{\rm k}_{\rm n}(\mathbb{R})$, then by Proposition (2.1.9), $\psi_{n-3}[A_n] = \frac{1}{n-2}(\varphi_{n-2})'[A_n] \in \mathbb{P}_n$. This is a contradiction and so $\varphi_{n-2}$is not monotone for odd integers $n > 3$. This concludes the classification of the powers $\varphi_\alpha$that preserve Loewner monotonicity.

Case 3: $f(x) = \psi_\alpha(x)$.This case follows similarly to Case 2 and is therefore omitted.

Section (2.2): Characteizing Entrywise Powers that are Loewner Convex

We next characterize the entrywise powers that preserve Loewner convexity. Before proving the main result of this section, we need a few preliminary results. Recall that an $n \times n$ matrix A is said to be completely positive if $A = CC^T$ for some $n \times m$ matrix C with nonnegative entries. We denote the set of $n \times n$ completely positive matrices by CP_n.

Lemma (2.2.1)[2]: Suppose $I \subset \mathbb{R}$ is convex, $n \geq 2$, and $f: I \to R$ is continuously differentiable. Given two fixed matrices $A, B \in \mathbb{P}_n(I)$such that

$(i) A - B \in CP_n$;

$(ii) f[\lambda A + (1-\lambda)B] \leq \lambda f[A] + (1-\lambda) f[B]$ for all $0 \leq \lambda \leq 1$.

Then $f'[A] \geq f'[B]$.

Proof : Since $A - B \in C\mathbb{P}_n$, there exist vectors $v_1, \ldots, v_m \in [0,\infty)^n$ such that

$$A - B = v_1 v_1^T + \cdots + v_m v_m^T$$

For $1 \leq \rm k \leq m$, let $\rm A_k = B + v_{k+1} v_{k+1}^T + \cdots + v_m v_m^T$. Then $A =: A_0 \geq A_1 \geq \cdots \geq A_{m-1} \geq A_m := B$. The rest of the proof is the same as the first part of the proof.

Just as Proposition (2.1.9) was used in proving Theorem (2.1.10), we need the following preliminary result to classify the powers that preserve convexity.

Proposition $(2.2.2)[2]$: Fix $0 < R \leq \infty, I = (-R, R)$, and $2 \leq k \leq n$. Suppose $f: I \to \mathbb{R}$ is twice differentiable on I and Loewner convex on $\mathbb{P}_n^k$ (I). If $A \in \mathbb{P}_n^k$ (I) is irreducible, then $f''[A] \in \mathbb{P}_n$.

Proof: Given an irreducible matrix $A \in \mathbb{P}_n^k(I)$, choose a vector $\zeta \in ImA$ as in the observation at the beginning of the proof of Proposition (2.1.9). We now adapt the proof for the $k = n$ case, to the $2 \leq k < n$ case. Let $A_1 := A + \zeta\zeta^T$; then $A_1 \in \mathbb{P}_n^k(I)$ for $|\zeta|$ small enough since $\zeta \in ImA$. More generally, it easily follows that $\lambda A_1 + (1 - \lambda)A \in \mathbb{P}_n^k(I)$ for all $\lambda \in [0, 1]$. Since

$$f[\lambda A_1 + (1 - \lambda)A] \leq \lambda f[A_1] + (1 - \lambda)f[A] \quad \forall 0 \leq \lambda \leq 1$$

by convexity, it follows for $0 < \lambda < 1$ that

$$\frac{f[A + \lambda(A_1 - A)] - f(A)}{\lambda} \leq f[A_1] - f[A]$$

$$\frac{f[A_1 + (1 - \lambda)(A - A_1)] - f[A_1]}{1 - \lambda} \leq f[A] - f[A_1].$$

Letting $\lambda \to 0^+ or\ \lambda \to 1^-$, we obtain

$$(A_1 - A) \circ f'[A] \leq f[A_1] - f[A], \qquad (A - A_1) \circ f'[A_1] \leq f[A] - f[A_1]$$

Summing these two inequalities gives $(A_1 - A) \circ (f'[A_1] - f'[A]) \geq 0$. Note that $(A_1 - A)^{\circ -1} = (\zeta\zeta^T)^{\circ -1} \in \mathbb{P}_n^1$ and so $f'[A_1] - f'[A] \geq 0$.

Finally, given $\epsilon > 0$, define $A_\epsilon := A + \epsilon\zeta\zeta^T$. Then $A_\epsilon \in \mathbb{P}_n^k(I)$ and $f'[A_\epsilon] \geq f'[A]$ by the previous paragraph for $\sqrt{\epsilon}\zeta$. Therefore, for all $\epsilon > 0$, $\frac{f'[A_\epsilon] - f[A]}{\epsilon} \geq 0$. Letting $\epsilon \to 0^+$, it follows that $f''[A] \circ (\zeta\zeta^T) \geq 0$. Now let $\zeta^{\circ(-1)} := (\zeta_1^{-1}, \ldots, \zeta_n^{-1})^T$; then by the Schur Product Theorem,

$$0 \leq f''[A] \circ (\zeta\zeta^T) \circ \left(\zeta^{\circ(-1)}\left(\zeta^{\circ(-1)}\right)^T\right) = f''[A],$$

which concludes the proof.

Note that Lemma (2.2.1) and Proposition (2.2.2) generalize to the cone $\mathbb{P}_n$ of matrices with the Loewner ordering, the elementary results from real analysis that the first and second derivatives of a convex (twice) differentiable function are nondecreasing and nonnegative, respectively. These parallels have been explored by *Hiai* in detail for $I = (-R, R)$. We now state some assertions that concern Loewner convexity.

Theorem $(2.2.3)[2]$: *Suppose* $0 < R \leq \infty, I = (-R, R), and\ f: I \to \mathbb{R}$.

(i)For each $n \geq 2$, f is convex on $\mathbb{P}_n(I)$ if and only if f is differentiable on I and f' is monotone on $\mathbb{P}_n(I)$.

(ii)*If* $n \geq 1$ and $\alpha \geq n$, then $\alpha \in \mathcal{H}_{\text{conv}}^{\varphi}(\text{n}, \text{n}) \cap \mathcal{H}_{\text{conv}}^{\psi}(\text{n}, \text{n})$.

With the above results in hand, we now extend them in order to completely classify the powers preserving Loewner convexity, and also specify them when rank constraints are imposed.

Theorem (2.2.4) [2]: Suppose $2 \leq k \leq n$ are integers. Then,

$$\mathcal{H}_{conv}(n, k) = \mathbb{N} \cup [n, \infty), \quad \mathcal{H}_{\text{conv}}^{\varphi}(n, k) = 2\mathbb{N} \cup [n, \infty), \ \mathcal{H}_{\text{conv}}^{\psi}(n, k) = (-1 + 2\mathbb{N}) \cup [n, \infty). \tag{9}$$

If instead $k = 1$, then

$$\mathcal{H}_{conv}(n, 1) = \mathcal{H}_{\text{conv}}^{\varphi}(n, 1) = \mathcal{H}_{\text{conv}}^{\psi}(n, 1) = [1, \infty). \tag{10}$$

Proof: Suppose that $k = 1 < n \, and \, A = uu^T, B = vv^T \in \mathbb{P}_n^1$. If $A \geq B \geq 0$, then by the proof of Theorem(2.1.10), $v = cu$ for some $c \in [-1, 1]$. Now for any $\alpha > 0$ and $\lambda \in [0, 1]$,

$$\lambda f_\alpha[A] + (1 - \lambda) f_\alpha[B] - f_\alpha[\lambda A + (1 - \lambda) B]$$
$$= \left(\lambda + (1 - \lambda) c^{2\alpha} - \left(\lambda + c^2(1 - \lambda)\right)^\alpha\right) f_\alpha[A].$$

So f_α is convex on $\mathbb{P}_n^1(\mathbb{R})$ if and only if (using $b = c^2$)

$$\lambda + (1 - \lambda) b^\alpha \geq (\lambda + \left(b(1 - \lambda)\right)^\alpha, \forall \lambda, b \in [0, 1].$$

This condition is equivalent to the function $x \mapsto x^\alpha$ being convex on $[0, 1]$ and hence on $[0, \infty)$– in other words, if and only if $\alpha \geq 1$. A similar argument can be applied to analyze $\varphi_\alpha, \psi_\alpha$. If on the other hand $\alpha < 1$, then set $A := 1_{2\times 2} \oplus 0_{(n-2)\times(n-2)} \in \mathbb{P}_n^1(I)$ and $B := 0_{n\times n}$, and compute:

$$\frac{1}{2} f[A] + \frac{1}{2} f[B] - f\left[\frac{1}{2}(A + B)\right] = \frac{1}{2} f[A] - f\left[\frac{1}{2} A\right] = (2^{-1} - 2^{-\alpha}) f[A], \quad (11)$$

which is clearly not in $\mathbb{P}_n(\mathbb{R}) if \; \alpha < 1$. It follows that none of the functions $f = f_\alpha, \varphi_\alpha, \psi_\alpha$ is convex on $\mathbb{P}_n^k(I) for \; \alpha < 1, n \geq 2,$ and $1 \leq k \leq n$.

We now assume that $2 \leq k \leq n$, and show that $\mathbb{N} \cup [n, \infty) \subset \mathcal{H}_{conv}(n, k)$. We first assert that for any differentiable function $f: [0, \infty) \to \mathbb{R}$ such that $f'(x)$is monotone on $\mathbb{P}_n([0, \infty))$, then f is convex on $\mathbb{P}_n([0, \infty))$. This assertion parallels one implication in Theorem(2.1.10) for $I = [0, \infty)$ instead of $I = (-R, R)$. As the proof is similar to the proof of, it is omitted.

Next, letting $f(x) = x^\alpha \; for \; \alpha \in [n, \infty) \cup \mathbb{N}$, it follows immediately from Theorem(2.2.3) that f is convex on $\mathbb{P}_n([0, \infty))$.. Thus $\mathbb{N} \cup [n, \infty) \subset \mathcal{H}_{conv}(n, n) \subset \mathcal{H}_{conv}(n, k)$. Now note that for any $\alpha \geq 1, \varphi'_\alpha(x) = \alpha \psi_{\alpha-1}(x)$ and $\psi'_\alpha(x) = \alpha \varphi_{\alpha-1}(x)$. Thus using Theorem (2.2.3), it follows that $2\mathbb{N} \cup [n, \infty) \subset \mathcal{H}_{\mathrm{conv}}^{\varphi}(\mathrm{n, k})$ and $(-1 + 2\mathbb{N}) \cup [n, \infty) \subset \mathcal{H}_{\mathrm{conv}}^{\psi}(\mathrm{n, k})$.

Note that $\mathcal{H}_{conv}(n, k) \subset [1, \infty)$, and similarly for $\mathcal{H}_{\mathrm{conv}}^{\varphi}(\mathrm{n, k})$ and $\mathcal{H}_{\mathrm{conv}}^{\psi}(\mathrm{n, k})$. Thus to show the reverse inclusions, i.e., that $\mathcal{H}_{conv}(n, k) \subset \mathbb{N} \cup [n, \infty)$(and analogously for $\varphi_\alpha, \psi_\alpha$), we consider three cases corresponding to the three functions $f = f_\alpha, \varphi_\alpha, \psi_\alpha$.

Case 1: $f(x) = f\alpha(x). Let \; \alpha \in \mathcal{H}_{conv}(n, k)$ and consider the matrices $A = A_\epsilon = ((1 + \epsilon_{ij}))_{i,j=1}^n$ and $B = \mathbf{1}_{nXn}$ for $\epsilon > 0$. Since $A - B \in C\mathbb{P}_n$, by Lemma(2.2.1) for $I = [1, 1 + \epsilon n^2]$, we have $f'_\alpha[A] \geq f'_\alpha[B]$. It follows that $\alpha - 1 \geq n - 1$ or $\alpha \in \mathbb{N}$. Therefore $\mathcal{H}_{conv}(n, k) \subset \mathbb{N} \cup [n, \infty)$.

Case 2: $f(x) = \varphi_\alpha(x). Given \; \alpha \in \mathcal{H}_{\mathrm{conv}}^{\varphi}(\mathrm{n, k}) for \; k \geq 2$, first note by Case 1 that

$$\mathcal{H}_{\mathrm{conv}}^{\varphi}(\mathrm{n, k}) \subset \mathcal{H}_{\mathrm{conv}}^{\varphi}(\mathrm{n}, 2) \subset \mathcal{H}_{conv}(\mathrm{n}, 2) = \mathbb{N} \cup [n, \infty). \quad (12)$$

Thus it suffices to show that there is no odd integer in $S := (0, n) \cap \mathcal{H}_{\mathrm{conv}}^{\varphi}(n, 2)$. First note that for every odd integer $\alpha \in S$, the function is convex on $\mathbb{P}_{\alpha+1}^2(\mathbb{R})$. There are now two cases: first if $\alpha > 1$, then define $A_{\alpha+1} \in \mathbb{P}_{\alpha+1}^2(\mathbb{R})$. Now $A_{\alpha+1}$ is irreducible since $\alpha \geq 3$. Applying Proposition (2.2.2) to $A_{\alpha+1}$, we obtain $\varphi''_\alpha[A_{\alpha+1}] \geq 0$. Now if $2 \leq \alpha < n$, then this contradicts Case 2 of the proof of Theorem (2.1.7) since α is an odd integer. Therefore $\alpha \notin \mathcal{H}_{\mathrm{conv}}^{\varphi}(n, 2)$for all odd integers $\alpha \in (1, n)$. The second case is when $\alpha = 1$.

Recall that if $\alpha = 1$ and $\varphi_1: R \to R$ is convex on $\mathbb{P}_{\alpha+1}^2(\mathbb{R}) = \mathbb{P}_2(\mathbb{R})$, then $\varphi_1(x) = |x|$ would be differentiable on R, which is false. We conclude that $\mathcal{H}_{\mathrm{conv}}^{\varphi}(n, 2) \subset 2\mathbb{N} \cup [n, \infty)$.

Case 3: $f(x) = \psi_\alpha(x)$.We now prove that $\mathcal{H}^{\psi}_{\text{conv}}(n,2) \subset (-1 + 2\mathbb{N}) \cup [n,\infty)$. Once again it suffices to show that no even integer $\alpha \in (0,n)$ lies in $\mathcal{H}^{\psi}_{\text{conv}}(n,2)$. First assume that $\alpha > 2$. Then an argument similar to that for φ_α above (together with the analogous example in Theorem (2.1.7) for ψ_α) shows that $\alpha \notin \mathcal{H}^{\psi}_{\text{conv}}(n,2)$. Finally, if $\alpha = 2$, we provide a three-parameter family of counterexamples to show that $\psi_2$is not convex on $\mathbb{P}_3^2(\mathbb{R})$and hence not convex on $\mathbb{P}_n^2(\mathbb{R})$by adding blocks of zeros). To do so, choose $0 < b < a < \infty$ and $c \in (a^{-1}, \min(b^{-1}, 2a^{-1}))$, and define:

$$v := (a,b,-c)^T,\ \ B := \mathbf{1}_{3\times 3},\qquad A := B + vv^T$$

Clearly, $A, B \in \mathbb{P}_3^2(\mathbb{R}) and\ A \geq B$. Moreover,

$$C := \frac{1}{2}(\psi_2[A] + \psi_2[B]) - \psi_2[(A+B)/2]$$

$$= \frac{1}{4}\begin{pmatrix} a^4 & a^2b^2 & (3ac-2)(2-ac) \\ a^2b^2 & b^4 & b^2c^2 \\ (3ac-2)(2-ac) & b^2c^2 & c^4 \end{pmatrix}.$$

Now verify that $detC = -4^{-3}[2b(ac-1)]^4 < 0$. Thus $A \geq B \geq 0$ provide a family of counterexamples to the convexity $of\ \psi_2 on\ \mathbb{P}_3^2(\mathbb{R}) \left(with\ \lambda = \frac{1}{2}\right)$.

Powers that are Loewner super/sub-additive have been studied for matrix functions in parallel settings, where functions of matrices are evaluated through the Hermitian functional calculus instead of entrywise). We now characterize the powers that are Loewner super/sub-additive when applied entrywise.

Theorem (2.2.5) [2]: Suppose $1 \leq k \leq n$ are integers with $n \geq 2$. Then,

(i) $\mathcal{H}_{super}(n,k) = \mathbb{N} \cup [n,\infty), \mathcal{H}^{\varphi}_{\text{super}}(n,k) = 2\mathbb{N} \cup [n,\infty), \mathcal{H}^{\psi}_{\text{super}}(n,k) = (-1 + 2\mathbb{N}) \cup [n,\infty)$.

(ii) $\mathcal{H}_{sub}b(n,k) = \begin{cases} \{1\}, & if\ 2 \leq k \leq n, \\ \{0,1\}, & if\ k = 1, n > 2 \\ n > 2, (-\infty, 0] \cup \{1\}, & if\ (n,k) = (2,1). \end{cases}$

(iii) $\mathcal{H}^{\varphi}_{\text{sub}}(n,k) = \emptyset$ for $all\ 1 \leq k \leq n$.

(iv) $\mathcal{H}^{\psi}_{\text{sub}} = \{0,1\} if\ (n,k) = (2,1)$, and $\{1\}$otherwise.

Before we prove the result, note that it yields a hitherto unknown connection between super-additivity and convexity with respect to the Loewner ordering.

Proof: Super-additivity . Fix an integer $1 \leq k \leq n$. First apply the definition of super-additivity $to\ A = B = 1_{nXn} \in \mathbb{P}_n^k([0,\infty))$to conclude that if $\alpha \in R$and one of $f_\alpha, \varphi_\alpha, \psi_\alpha$ is Loewner super-additive, then $\alpha \geq 1$. We now consider three cases corresponding to the three functions $f(x) = f_\alpha(x), \varphi_\alpha(x)$, and $\psi_\alpha(x)$ for $\alpha \geq 1$.

Case 1: $f(x) = f_\alpha(x)$.That fαis super-additive on $\mathbb{P}_n([0,\infty))$for $\alpha \in N$follows by applying the binomial theorem. Now, suppose $\alpha \in (1,\infty) \setminus \mathbb{N}$. We adapt the argument to our situation. First assume that $\alpha \geq n$; then for $A, B \in \mathbb{P}_n([0,\infty))$

$$f_\alpha[A+B] = f_\alpha[A] + \alpha \int_0^1 B \circ f_{\alpha-1}[t(A+B) + (1-t)A]dt\,.$$

Note that $t(A+B)+(1-t)A = A+tB \geq tB$ for all $0 \leq t \leq 1$. Since $\alpha - 1 \geq n - 1$, it follows $f_{\alpha-1}[t(A+B)+(1-t)A] \geq t^{\alpha-1} f_{\alpha-1}[B]$. Therefore,

$$f_\alpha[A+B] \geq f_\alpha[A] + \alpha f_\alpha[B] \int_0^1 t^{\alpha-1} dt = f_\alpha[A] + f_\alpha[B]$$

This shows that $f_\alpha$is super-additive on $\mathbb{P}_n([0,\infty))$, and hence on $\mathbb{P}_n^k([0,\infty))$ if $\alpha \geq n$. The last remaining case is when $\alpha \in (1,n) \setminus \mathbb{N}$. Define $g_\alpha(x) := (1+x)^\alpha$. Given$\epsilon >$ $0 and\ v \in (0,1)^n$, apply Taylor's theorem entrywise to $g_\alpha[\epsilon v v^T]$to obtain:

$$g_\alpha[\epsilon v v^T] = 1_{n\times n} + \sum_{i=1}^{\lfloor \alpha \rfloor} \epsilon^i \binom{\alpha}{i} f_i[v] f_i[v]^T + O\left(\epsilon^{1+\lfloor \alpha \rfloor}\right) C, \qquad (13)$$

where $C = C(v)$is an $n \times n$ matrix that is independent of ϵ applied to $F(x) = \sum_{i=1}^{\lfloor \alpha \rfloor} \epsilon^i \binom{\alpha}{i} x^i - \epsilon^\alpha x^\alpha$ and $m = 1 + \lfloor \alpha \rfloor \leq n$, there exist $u \in (0,1)^n$and $x_\alpha \in \mathbb{R}^n$ such that $x_\alpha^T F[uu^T] x_\alpha - \epsilon^\alpha$. It follows that

$$x_\alpha^T (f_\alpha[1_{n\times n} + \epsilon u u^T] - 1_{n\times n} - \epsilon^\alpha f_\alpha[uu^T]) x_\alpha = O\left(\epsilon^{1+\lfloor \alpha \rfloor}\right) x_\alpha^T C x_\alpha - \epsilon^\alpha$$

and the last expression is negative for sufficiently small $\epsilon = \epsilon_0 > 0$. Hence $f_\alpha[1_{nXn} + \epsilon_0 uu^T] \ngeq f_\alpha[1_{nXn}] + f_\alpha[\epsilon_0 uu^T]$. This shows that f_α is not super-additive on $\mathbb{P}_n^1([0,\infty))$and hence on $\mathbb{P}_n^k([0,\infty))$, for $\alpha \in (1,n) \setminus \mathbb{N}$.

Case 2: $f(x) = \varphi_\alpha(x)$.Clearly, the assertion holds if $\alpha \in 2N$ and $1 \leq k \leq n$, since in that case $\varphi_\alpha \equiv x^\alpha. Next\ if\ \alpha \geq n \geq 2$, then as in Case 1, for $A, B \in \mathbb{P}_n(\mathbb{R})$,

$$\varphi_\alpha[\mathrm{A}+\mathrm{B}] = \varphi_\alpha[\mathrm{A}] + \alpha \int_0^1 B \circ \psi_{\alpha-1}[\mathrm{t}(\mathrm{A}+\mathrm{B}) + (1-\mathrm{t})\mathrm{A}]\mathrm{dt}\,.$$

Since $\alpha - 1 \geq n-1$, by Theorem (2.1.10) the function $\psi_{\alpha-1}$is monotone on $\mathbb{P}_n(\mathbb{R})$. Thus,

$$\varphi_\alpha[\mathrm{A}+\mathrm{B}] \geq \varphi_\alpha[\mathrm{A}] + \alpha B \circ \psi_{\alpha-1}[B] \int_0^1 t^{\alpha-1} dt = \varphi_\alpha[A] + \varphi_\alpha[B]$$

It follows that $\varphi_\alpha$is super-additive on $\mathbb{P}_n^k(\mathbb{R})$for $\alpha \in 2\mathbb{N} \cup [n,\infty)$. Next note by Case1 that $\varphi_\alpha$is not super-additive on $\mathbb{P}_n^k(\mathbb{R})$for $\alpha \in (1,n) \setminus \mathbb{N}$. It thus remains to prove that φ_α is not super-additive on $\mathbb{P}_n^k(R) for\ \alpha \in (-1+2\mathbb{N}) \cap [1,n)$. Note that for all $u, v \in \mathbb{R}_n, if\ \varphi\alpha$is super-additive, then

$$\varphi_\alpha[uu^T + vv^T] \geq \varphi_\alpha[uu^T] + \varphi_\alpha[vv^T] = \varphi_\alpha[u]\varphi_\alpha[v]^T + \varphi_\alpha[v]\varphi_\alpha[v]^T \in \mathbb{P}_n(\mathbb{R})$$

Thus, if $\varphi_\alpha$is super-additive, then it is also positive on $\mathbb{P}_n^2(\mathbb{R})$. We conclude by Theorem (2.1.7) that $\varphi_\alpha$is not super-additive for $\alpha \in (-1+2\mathbb{N}) \cap [1, n-2)$.

The only two powers left to consider are $\alpha = n-2$ with $n$odd, and $\alpha = n-1$ with n even. In other words, n is of the form $n = 2l$ or $n = 2l+1$ with $l \geq 1$. Thus , $\alpha = 2l - 1 \geq 1$ in both cases. We claim that $\varphi_{2l-1}$is not super-additive on $\mathbb{P}_n^1(\mathbb{R})$. To show the claim, first observe that if $v \in (-1,1)^n$, then $1 + v_i v_j > 0$ for all i, j, and so by the binomial theorem,

$$\varphi_{2l-1}[1_{n\times n} + vv^T] - 1_{n\times n} - \varphi_{2l-1}[vv^T] = \sum_{i=1}^{2l-1} \binom{2l-1}{i} g_i[vv^T] - \varphi_{2l-1}[vv^T]$$

where $g_i(x) = x^i for\ x \in \mathbb{R}$ and $i = 1,\ldots,2l-1$. By Corollary (2.2.8) applied to the functions $g_1, g_2, \ldots, g_{2l-1} = \psi_{2l-1}, \varphi_{2l-1} and\ m = 2l \le n$, there exist $u \in (-1,1)n$ and $x \in \mathbb{R}^n$ such that

$$x^T(\varphi_{2l-1}[1_{n\times n} + uu^T] - 1_{n\times n} - \varphi_{2l-1}[uu^T])x = -1$$

This shows that φ_{2l-1} is not super-additive on $\mathbb{P}_n^1(\mathbb{R})$, hence not on $\mathbb{P}_n^k(\mathbb{R})$.

Case 3: $f(x) = \psi_\alpha(x)$. The proof is similar to that of Case 2 and is thus omitted.

(i) Sub-additivity for f_α. First note that if $\alpha \in \mathbb{R}$, applying the definition of sub-additivity to $A = B = 1_{n\times n} \in \mathbb{P}_n^1([0,\infty))$ shows that f_α is not Loewner sub-additive for $\alpha > 1$. Clearly f_1 is sub-additive on $\mathbb{P}_n(I)$, so it remains to study $f_\alpha\ for\ \alpha < 1$. Now suppose $2 \le k \le n\ and\ \alpha < 1$. By Theorem (2.1.7), there exists $A \in \mathbb{P}_n^2(I)$ such that $f_\alpha[A] \notin \mathbb{P}_n$. Setting $B = A$, we obtain:

$f_\alpha[A] + f_\alpha[B] - f_\alpha[A+B] = (2 - 2^\alpha)f[A] \notin \mathbb{P}_n$. It follows that f_α is not sub-additive on $\mathbb{P}_n^k(I)$ for $\alpha < 1$. This settles the assertion for $2 \le k \le n$.

The last case is if $k = 1$ and $\alpha < 1$. For ease of exposition, the analysis in this case is divided into several sub-cases:

Sub-case 1: $\alpha \in (0,1)$. Given $0 < \epsilon < 1 and\ v \in (0,1)^n$, apply Taylor's theorem entrywise to $g_\alpha[\epsilon\, vv^T]$, where $g_\alpha(x) = (1+x)^\alpha$ as above. We obtain:

$$\mathrm{f}_\alpha[1_{\mathrm{n\times n}} + \epsilon vv^{\mathrm{T}}] - 1_{\mathrm{n\times n}} - \mathrm{f}_\alpha[\epsilon vv^{\mathrm{T}}] = \epsilon\alpha vv^{\mathrm{T}} - \epsilon^\alpha f_\alpha[vv^{\mathrm{T}}] + O(\epsilon^2)C\ .$$

where $C = C(v)$ is an $n \times n$ matrix that is independent of ϵ. By Corollary (2.2.8) with $F(x) = \epsilon\alpha x - \epsilon^\alpha x^\alpha$ and $m = 2 \le n$, there exist $u \in (0,1)^n$ and $x_\alpha \in \mathbb{R}_n$ such that

$$x_\alpha^T(f_\alpha[1_{n\times n} + \epsilon uu^T] - 1_{n\times n} - f_\alpha[uu^T])x_\alpha = \epsilon^\alpha + O(\epsilon^2)x_\alpha^T C x_\alpha$$

which is positive for $\epsilon > 0$ small enough. Therefore f_α is not sub-additive $on\ \mathbb{P}_n^1([0,\infty))$ for $\alpha \in (0,1)$.

Sub-case 2: $\alpha = 0$. To see why f_0 is indeed sub-additive on $\mathbb{P}_n^1([0,\infty))$, given a subset $S \subset \{1,\ldots,n\}$ we define 1_S to be the matrix with (i,j) entry $1\ if\ i,j \in S$ and 0 otherwise. Now given $A = (a_{ij}) \in \mathbb{P}_n^1([0,\infty))$, define $S(A) := \{i: a_{ii} \neq 0\}$. Then $f_0[A]1_{S(A)} = f_0$, $f_0[B] = 1_{S(B)} for\ A, B \in \mathbb{P}_n^1([0,\infty))$, and hence by inclusion–exclusion, $f_0[A] + f_0[B] - f_0[A+B] = 1_{S(A)\cap S(B)} \in \mathbb{P}_n^1([0,\infty))$. Thus f_0 is sub-additive on $\mathbb{P}_n^1([0,\infty))$ as claimed.

Sub-case 3: $\alpha < 0, n \ge 3$. The fact that f_α is not subadditive on $\mathbb{P}_n^1([0,\infty))$ for $\alpha < 0$ follows by an argument similar to Sub-case 1.

The argument is omitted for the sake of brevity.

Sub-case 4: $\alpha < 0, (n,k) = (2,1)$. The bulk of the work in classifying the sub-additive entrywise powers f_α lies in the remaining case of $\mathbb{P}_2^1$ with $\alpha < 0$. We first show that f_α is sub-additive on $\mathbb{P}_2^1([0,\infty))$ for all $\alpha < 0$. Setting $A := (a,b)^T(a,b)$ and $B := (c,d)^T(c,d)$, the problem translates to showing that

$$\left(f_\alpha(a^2) + f_\alpha(c^2) - f_\alpha(a^2+c^2)\right).f_\alpha(b^2) + f_\alpha(d^2) - f_\alpha(b^2+d^2)$$
$$\ge \left(f_\alpha(ab) + f_\alpha(cd) - f_\alpha(ab+cd)\right)^2.$$

Note that if any of $a,b,c,d = 0$ then the inequality is clear. Thus we may assume $a,b,c,d > 0$. Now define

$$f(x,y) := x^\alpha + y^\alpha - (x+y)^\alpha, \qquad g(x,y) := log f(e^x, e^y)$$

Then proving the above inequality is equivalent to showing that $(g(x_1,y_1) + g(x_2,y_2))/2 \ge g((x_1+x_2)/2, (y_1+y_2))/2)$ i.e., that g is midpoint-convex on $\mathbb{R}^2$. Since g is smooth, it suffices to show that its Hessian det $H_g(x,y)$ is positive semidefinite at all points in $\mathbb{R}^2$. Astraightforward but longwinded computation demonstrates that det $H_g(x,y) = 0$ for all $x, y \in \mathbb{R}^2$ Thus it suffices to show that g_{xx} is nonnegative on $\mathbb{R}$ (and the result for g_{yy} follows by symmetry). We now compute, setting $E := e^x + e^y$ for notational convenience:

$$f(e^x, e^y) g_{xx}(x,y)$$
$$= (e^{\alpha x} + e^{\alpha y} - E^{\alpha})(\alpha^2 e^{\alpha x} - \alpha e^x E^{\alpha-1} - \alpha(\alpha-1)e^{2x}E^{\alpha-2})$$
$$- \alpha^2 (e^x E^{\alpha-1} - e^{\alpha x})^2$$
$$= \alpha^2 e^{\alpha(x+y)} E^{\alpha-2}\left(E^{2-\alpha} - \left(e^{(2-\alpha)x} + e^{(2-\alpha)y}\right)\right)$$
$$+ \alpha e^{x+y} E^{\alpha-2}\left(E^{\alpha} - (e^{\alpha x} + e^{\alpha y})\right)$$

Note that the difference in the first term is nonnegative because $x^{2-\alpha}$ is super-additive, while the difference in the second term is nonpositive because x^{α} is sub-additive. Thus both terms are nonnegative, which concludes the proof for f_α.

(ii) Sub-additivity for φ_α. By considering the matrices $A = uu^T, B = vv^T$ with $u = (1,1)^T$ and $v = (1,-1)^T$, it immediately follows that $\varphi_\alpha$is not Loewner sub-additive on $\mathbb{P}_2^1$ (R), and hence not sub-additive on $\mathbb{P}_n^k(\mathbb{R})$for all (n,k).

(iii) Sub-additivity for ψ_α. First note that $\psi_1(x) = x$ is sub-additive on $\mathbb{P}_n^k(\mathbb{R})$for all (n,k). It is also not difficult to show that ψ_0 is sub-additive on $\mathbb{P}_2^1(\mathbb{R})$. Using part (a), it remains to prove that ψ_0 is not sub-additive on $\mathbb{P}_n^1(R)$for $n > 2$, and $\psi_\alpha$is not sub-additive on $\mathbb{P}_2^1(R)$for $\alpha < 0$.

To see why ψ_0 is not sub-additive on $\mathbb{P}_n^1(\mathbb{R})$for $n \geq 3$, use the following three-parameter family of rank one counterexamples:

$$(A(a,b,c) := (-a,c,c)^T(-a,c,c), B(a,b,c) := (c,-b,c)^T(c,-b,c)),$$
$$0 < a < b < c.$$

It remains to show that $\psi_\alpha$is not sub-additive on $\mathbb{P}_2^1(\mathbb{R})$for any $\alpha < 0. Let\ A := (1,-1)^T(1,-1)$and $B := (1,1/2)^T(1,1/2)$. Then,

$$\psi_\alpha[A] + \psi_\alpha[B] - \psi_\alpha[A+B] = \begin{pmatrix} 2-2^{\alpha} & -1+2^{1-\alpha} \\ -1+2^{1-\alpha} & 1+(1/4)^{\alpha}-(5/4)^{\alpha} \end{pmatrix} =: C_\alpha.$$

We claim that $\det C_\alpha < 0$ for all $\alpha < 0$, which shows that ψ_α is not sub-additive on $\mathbb{P}_2^1(\mathbb{R})$ and completes the proof. To see why the claim holds, compute for $\alpha < 0$:

$$4^{\alpha}(2-2^{\alpha})\det C_\alpha = 4^{\alpha} + 2^{\alpha} - 1 - 5^{\alpha}$$

Note that the function $f_\alpha(x) = x^{\alpha}$ is convex on $(0,\infty)$for $\alpha < 0$, so Jensen's inequality yields:

$$2^{\alpha} = f_\alpha\left(\frac{3}{4}.1 + \frac{1}{4}.5\right) < \frac{3}{4} + \frac{1}{4}.5^{\alpha}, \qquad 4^{\alpha} = f_\alpha\left(\frac{1}{4}.1 + \frac{3}{4}.5\right) < \frac{1}{4} + \frac{3}{4}.5^{\alpha}$$

Adding the two inequalities shows that det $C_\alpha < 0$ for $\alpha < 0$, and the proof is complete.

We conclude by discussing the following questions that naturally arise from the above analysis:

(i) Is it possible to find matrices of rank exactly k(for some $1 \leq k \leq n$) for which a non-integer power less than $n-2$ is not Loewner positive? Similar questions can also be asked for monotonicity, convexity, and super/sub-additivity.

(ii)Can we compute the Hadamard critical exponents for convex combinations of the two-sided power functions $\varphi_\alpha, \psi_\alpha$?

Corollary $(\mathbf{2.2.6})[\mathbf{2}]$**:** $Fix\ \alpha > 0$ and integers $2 \le k \le n. A$ fractional power function $f = f_\alpha, \varphi_\alpha, \psi_\alpha$ is Loewner convex on $\mathbb{P}_n^k(I)$ if and only if f is Loewner super-additive. Here $I = [0, \infty) if\ f = f\alpha$ and $I = R$ otherwise.

In order to prove Theorem (2.2.5), we extend classical results about generalized Vandermonde determinants to the odd and even extensions of the power functions.

Proposition (2.2.7) [2]: Let $0 < R \le \infty$. Then,

(i)the functions $\{f_\alpha : \alpha \in R\} \cup \{f \equiv 1\}$are linearly independent on $I = [0, R)$;

(ii)the functions $\{\varphi_\alpha, \psi_\alpha : \alpha \in R\} \cup \{f \equiv 1\}$are linearly independent on $I = (-R, R)$.

Proof: Fix $\alpha_1 < \cdots < \alpha_n\ and\ define\ \alpha := (\alpha_1, \ldots, \alpha_n)$. We first show that the set of functions $\{x^{\alpha_i} : i = 1, \ldots, n\} \cup \{f \equiv 1\}$ is linearly independent on $[0, R)$. Indeed,fix $x := (0, x_1, \ldots, x_n) \in R^n$ for any $0 < x_1 < \cdots < x_n < R$;

$$x^{\circ \alpha_j} := \left(0, x_1^{\alpha_j}, \cdots, x_n^{\alpha_j}\right), \qquad j = 1, \cdots, n,$$

and $(1, 1, \ldots, 1)$ are linearly independent.

We next show that the set of functions $\{\varphi_{\alpha_i}, \psi_{\alpha_i} : i = 1, \ldots, n\} \cup \{f \equiv 1\}$ is linearly independent on $(-R, R)$. Indeed, fix $\boldsymbol{x}' := (x_1, \ldots, x_n)$ with $x_i \in (0, R)$ as above; then by the above analysis,

$$\psi(x', \alpha) := \begin{pmatrix} \left(\varphi_{\alpha_i}(x_j)\right)_{i,j=1}^n & \left(\varphi_{\alpha_i}(-x_j)\right)_{i,j=1}^n \\ \left(\psi_{\alpha_i}(x_j)\right)_{i,j=1}^n & \left(\psi_{\alpha_i}(-x_j)\right)_{i,j=1}^n \end{pmatrix}. \tag{14}$$

Is a nonsingular matrix, since it is of the form $\begin{pmatrix} M & M \\ M & -M \end{pmatrix}$ for a nonsingular matrix M. But then $\begin{pmatrix} \Psi(x', \alpha) & 0_{2n\times 1} \\ 1_{1\times 2n} & 1 \end{pmatrix}$ is also nonsingular, whence the points $\pm x_1, \ldots, \pm x_n$, 0provide the required nonsingular matrix. This proves the second assertion.

Proposition (2.2.7) has the following consequence that is repeatedly used in proving Theorem (2.2.5).

Corollary $(\mathbf{2.2.8})[\mathbf{2}]$**:** Let $0 < R \le \infty\ and\ I = (-R, R) or\ I = [0, R)$. Fix integers $1 \le m \le n$ and scalars $c_1, \ldots, c_m$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_m$. Suppose $\{g_1, \ldots, g_m\} \subset \{\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_m}, \psi_{\alpha_1}, \ldots, \psi_{\alpha_m}\}$, and define

$F(x) := \sum_{i=1}^m c_i g_i(x)$. Then there exist vectors $u \in \left(I \cap (-1, 1)\right)^n and\ v_i \in \mathbb{R}^n$ that do not depend on c_i, such that $v_i^T F[uu^T] v_I = c_i\ \ for\ all\ i = 1, \ldots, m.$

Proof: Suppose first $I = (-R, R)$. Choose scalars $\alpha_n > \alpha_{n-1} > \cdots > \alpha_{m+1} > \alpha_m$. By Proposition (2.2.7), the functions $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_n}, \psi_{\alpha_1}, \ldots, \psi_{\alpha_n}$ are linearly independent. Thus, as in the proof of Proposition (2.2.7), for all pairwise distinct $x_1, \ldots, x_n \in (0, 1) \cap I$, the matrix $\Psi(x, \alpha)$as in Theorem(2.2.5) is nonsingular, where $x := (x_1, \ldots, x_n)$ and $\alpha := (\alpha_1, \ldots, \alpha_n)$. Now consider the submatrix $C_{m\times 2n}$ of $\Psi(x, \alpha)$ whose rows correspond to the functions gifor $1 \le i \le m$. Since Chas full rank, choose elements $u_1, u_2, \ldots, u_n$ from among the $\pm x_i$ such that the matrix $(g_i(u_j))_{ij}^m = 1$ is nonsingular. Now set $u := (u_1, \ldots, u_n)^T$; then the vectors $g_i[u]$ are linearly independent. Choose v_i to be orthogonal to $g_j[u] for\ j \ne i,$ and such that $v_i^T g_i[u] = 1$. It follows that $v_i^T F[uu^T] v_i = c_i$ for all $i.$ The proof is similar for $I = [0, R)$.

We now classify the entrywise powers that are Loewner super/sub-additive.

Proposition $(2.2.9)[2]$: Let $0 < R \leq \infty, I = [0, R) or (-R, R)$, and f: I → ℝ be continuous. Suppose $1 \leq l < k \leq n$ are integers, and $A, B \geq 0$ are matrices in $\mathbb{P}_n(I)$such that rank $A = l$ and one of the following properties is satisfied:

(i)$f[A] \notin \mathbb{P}_n$;

(ii)$A \geq B \geq 0$ and $f[A] \ngeq f[B]$;

(iii)$A \geq B \geq 0$ and $f[\lambda A + (1-\lambda)B] \nleq \lambda f[A] + (1-\lambda)f[B]$ for some $0 < \lambda < 1$;

(v)$f[A + B] \ngeq f[A] + f[B]$;

(vi)$f[A + B] \nleq f[A] + f[B]$.

Then there exist $A', B' \geq 0$ such that rank $A' = k$ and the same property holds when A, B are replaced by A', B' respectively.

Note that the special cases of $l = 1$, 2 answer question (1) above.

Proof: We show the result for property (i) monotonicity; the analogous results for (ii) positivity, (iii)convexity, (v)super-additivity, and (vi)sub-additivity are shown similarly. Suppose $A \geq B \geq 0$ and rank $A = l$, but $f[A] \ngeq f[B]$. Then there exists a nonzero vector $v \in \mathbb{R}_n$ such that $v^T f[A]v < v^T f[B]v$. Now write $A = \sum_{i=1}^{l} \lambda_i u_i u_i^T$ where $\lambda_i = 0$ and uiare the nonzero eigenvalues and eigenvectors respectively. Extend the uito an orthonormal set $\{u_1, \ldots, u_k\}$, and define $C := \sum_{i=l+1}^{k} u_i u_i^T$. Clearly, $A + \epsilon C \geq B + \epsilon C \geq 0$ and $A + \epsilon C, B + \epsilon C \in \mathbb{P}_n(I)$for small $\epsilon > 0. Since$

$$0 > v^T f[A]v - v^T f[B]v = \lim_{\epsilon \to 0^+} v^T (f[A + \epsilon C] - f[B + \epsilon C])v$$

and f is continuous, there exists small $\epsilon_0 > 0$ such that $f[A + \epsilon_0 C] \ngeq f[B + \epsilon_0 C]$. Now setting $A' := A + \epsilon_0 C, B' := B + \epsilon_0 C$ completes the proof, since $A' \in \mathbb{P}_n^k(I)$.

Definition $(2.2.10)[7]$: let $I \subset R$ be an interval with interior I^0 . A function $f \in C(I)$ is said to be a bsolutely monotonic on I if it is in $C^\infty(I^0)$ and $f^{(k)}(X) \geq 0$ for every $X \in I^0$ and every $k \geq 0$.

Chapter 3
Some Applications of Variation of Loewner Chains in Several Complex Variables

We show atopological property of the class S^0 of mappings with parametric representation on the Euclidean unit bull $\mathbb{B}^n$, which for $n \geq 2$ immediately implies the density of the automorphisms of $\mathbb{C}^n$ that, restricted to $\mathbb{B}^n$, have parametric representation, and second, to show that every normalized univalent mapping on $\mathbb{B}^n$ whose image is Runge and which is $\mathbb{C}^1$ up to the boundary embeds in to a normalized Loewner chain with range $\mathbb{C}^n$.

Definition $(\mathbf{3.1})[\mathbf{3}]$: A family $(f_t)_{t\geq 0}$ of mappings is called a normalized subordination chain if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in $\mathcal{H}_0$ and, for every $0 \leq s \leq t$, there exists $\varphi_{s,t} : \mathbb{B}^n \to \mathbb{B}^n$ holomorphic such that $f_s = f_t \circ \varphi_{s,t}$. A normalized subordination chain $(f_t)_{t\geq 0}$ is called a normalized Loewner chain if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in S, and if, in addition, $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family, then $(f_t)_{t\geq 0}$is called a normal Loewner chain.
For every normalized subordination chain $(f_t)_{t\geq 0}$we denote

$$R(f_t) := \bigcup_{t\geq 0} f_t \ (\mathbb{B}^n).$$

This set is called the (Loewner) range of $(f_t)_{t\geq 0}$.

We say that a mapping $f \in S$ embeds into a normalized Loewner chain $(f_t)_{t\geq 0}$ if $f_0 = f$
Let

$$S^0 := \{ f \in S |\, f \text{ embeds into a normal Loewner chain } (f_t)_{t\geq 0}\},$$

$$S^1 := \{ f \in S | f \text{ embeds into a normalized Loewner chain } (f_t)_{t\geq 0} \text{with } R(f_t) = \mathbb{C}^n\}$$

and

$$S_R := \{ f \in S |\, f\ (\mathbb{B}^n)\ is\ Runge\}.$$

For the definition and basic properties of the Runge domains, one can consult .
The class S^0 is known as the class of mappings with parametric representation on $\mathbb{B}^n$ and it is a compact set in $\mathcal{H}_0$ with respect to the compact-open topology.
Let

$$M := \{h \in \mathcal{H}_0(\mathbb{B}^n) | \Re\langle h(z), z\rangle \geq 0\ for\ all\ z \in \mathbb{B}^n\}.$$

Various applications of this family in the theory of univalent mappings on $\mathbb{B}^n$, and the references therein.
Definition $(\mathbf{3.2})[\mathbf{3}]$: A mapping $G: \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is called a Herglotz vector field if $G(z,\cdot)$is measurable, for every $z \in \mathbb{B}^n$, and $G(\cdot,t) \in M$, for a.e. $t \in [0,\infty)$.
For every Herglotz vector field G, we can associate the following Loewner–Kufarev PDE:

$$\frac{\partial \mathrm{f_t}}{\partial \mathrm{t}}(z) = d(f_t)zG(z,t), \tag{1}$$

for a.e. $t \geq 0$ and for every $z \in \mathbb{B}^n$.

Definition $(3.3)[3]$: A family $(f_t)_{t\geq 0}$ of mappings is called a normalized solution to the Loewner–Kufarev PDE associated to a Herglotz vector field G if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in $\mathcal{H}_0$, the mapping $t \to f_t(z)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$ and it satisfies with G.
A normalized solution $(f_t)_{t\geq 0}$ to a Loewner–Kufarev PDE is called a normalized univalent solution if $\{e^{-t}f_t\}_{t\geq 0}$ is a family in S, and if, in addition, $\{e^{-t}f_t\}_{t\geq 0}$ is a normal family, then $(f_t)_{t\geq 0}$ is called a canonical solution.
Theorem $(3.4)[3]$:
(i) Every normalized subordination chain $(f_t)_{t\geq 0}$ is a normalized solution to a Loewner–Kufarev PDE.
(ii) Every Loewner–Kufarev PDE has a unique canonical solution. Moreover, this solution is a normal Loewner chain with range $\mathbb{C}^n$.
(iii) Let $(f_t)_{t\geq 0}$ be a normalized solution to the Loewner–Kufarev PDE(1) associated to a Herglotz vector field G. Then $(f_t)_{t\geq 0}$ is a normalized subordination chain and there exists a holomorphic mapping $\Phi\colon \mathbb{C}^n \to R(f_t)$, with $\Phi(0) = 0$ and $d\Phi_0 = I$, such that $f_t = \Phi \circ g_t$, for every $t \geq 0$, where $(g_t)_{t\geq 0}$is the canonical solution of (1) with G. In particular, $(f_t)_{t\geq 0}$ is a normalized Loewner chain if and only if Φ is biholomorphic (so, in this case, the range $R(f_t)$ is a Fatou–Bieberbach domain).
Conversely, for every holomorphic mapping $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$, with $\Phi(0) = 0$ and $d\Phi_0 = I$, and every canonical solution $(g_t)_{t\geq 0}$ to a Loewner–Kufarev PDE (1) associated to a Herglotz vector field G, $(\Phi \circ g_t)_{t\geq 0}$ is a normalized subordination chain that satisfies (1) with G.
By Theorem (4), we can deduce that in the case n = 1):

(i) every normalized solution to a Loewner–Kufarev PDE is a normalized subordination chain and vice versa;
(ii) every normalized univalent solution to a Loewner–Kufarev PDE is a normalized Loewner chain and vice versa;
(iii) every canonical solution to a Loewner–Kufarev PDE is a Normal Loewner chain and vice versa.

We have the following characterizations:

(i) $(f_t)_{t\geq 0}$is normalized subordination chain if and only if there exist an entire holomorphic mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^n$, with $\Phi(0) = 0$ and $d\Phi_0 = I$, and a normal Loewner chain $(g_t)_{t\geq 0}$ such that $f_t = \Phi \circ g_t$, for every $t \geq 0$;
(ii) $(f_t)_{t\geq 0}$is normalized Loewner chain if and only if there exist an entire univalent mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^n$, with $\Phi(0) = 0$ and $d\Phi_0 = I$, and a normal Loewner chain $(g_t)_{t\geq 0}$ such that $f_t = \Phi \circ g_t$, for every $t \geq 0$.
Moreover, if $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is an entire holomorphic mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^n$, with $\Phi(0) = 0$ and $d\Phi_\circ = I$, and $(g_t)_{t\geq 0}$ a normal Loewner chain, then $(\Phi \circ g_t)_{t\geq 0}$ is a normalized Loewner chain if and only if Φ is univalent and, considering also Theorem (4),
(iii) $(\Phi \circ g_t)_{t\geq 0}$ is a normal Loewner chain if and only if Φ = I.

$$Aut_0\,(\mathbb{C}^n) := \{\Phi\ \Phi \text{ is an automorphism of } \mathbb{C}^n\ with\ \Phi(0) = 0, d\Phi_0 = I\},$$
$$A := \{\varphi \mid \varphi = \Phi\mid_{\mathbb{B}^n}, \text{where } \Phi \in Aut_0(\mathbb{C}^n)\}$$

and

$$A^0 := A \cap S^0 .$$

We mention that examples of mappings in A^0 can be found. If $n = 1$, then we have $A^0 = A = \{I\}$ and $S^0 = S^1 = S = S_R$. The last equality is a consequence of the classical result that a domain in $\mathbb{C}$ is Runge if and only if it is simply connected.
If $n \geq 2$, then we have $S^0 \subsetneq S^1$.
From Theorem (4) we can deduce that

$$S^1 = \{ f \in S | f = \Phi \ g, \text{where } g \in S^0 \ and \ \Phi \in Aut\ _0\mathbb{C}^n\}. \qquad (2)$$

As mentioned in the introduction , it is proved that for $n \geq 2$

$$S^1 \subset S_R \subsetneq S.$$

By Andersén–Lempert theorem, it was proved that for $n \geq 2$:

$$S_R = \bar{A},$$

where the closure is with respect to the compact-open topology on $\mathcal{H}_0$.
Now, we can easy see that for $n \geq 2$

$$A \subsetneq S^1 \subset S_R = \bar{A} \subsetneq S. \qquad (3)$$

We note that for $n \geq 2$ Schleißinger proved that

$$S^0 \subsetneq \bar{A}, \qquad (4)$$

and then considered the following : is $\overline{A^0} = S^0$?
Our first result is that (roughly speaking) S^0 is "absorbing" in $\mathcal{H}_0$ in the following sense: if a sequence in $\mathcal{H}_0$ converges, uniformly on compacta of $\mathbb{B}^n$, to a mapping in S^0, then there exists a subsequence which rescaled in a natural prescribed way is in S^0 and still converges to the same mapping. As a consequence of this result we give a simple proof of the fact that $\overline{A^0} = S^0$ for $n \geq 2$. The same "absorbing" property holds also for S^1.
The authors have been interested in the following question: is $S^1 = S_R$? Our second result gives a partial answer to this question, namely we prove that every mapping in S_R which is of class C^1 up to the boundary is in S^1.
Our proofs heavily rely on the following definition and result obtained .
Definition (3.5)[3]: A normalized Loewner chain $(f_t)_{t\geq 0}$ is geräumig in $[0, T)$, for some $T > 0$, if there exists $a, b > 0$ and $c \in (0, 1]$ such that :
(i) for all $t \in [0,T)$ and all $z \in \mathbb{B}^n$, $\mu(d(f_t)_z) := min_{\|v\|=1}\|d(f\ ) v\| \geq a$,
(ii)for $a.e. t \in [0,T)$ and all $z \in \mathbb{B}^n$, $\left\|\frac{\partial f_t}{\partial t}(z)\right\| \leq b$,
(iii) for $a.e. t \in [0,T)$ andall $z \in \mathbb{B}^n$, $\Re\langle (d(f_t)^{-1} \frac{\partial f_t}{\partial t}(z), z \rangle \geq c\,\|z\|^2$.
We have $\mu(A) = \frac{1}{\|A^{-1}\|}$, for every invertible linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$.
Theorem (3.6)[3]: Assume that $(f_t)_{t\geq 0}$ is a normalized Loewner chain, respectively a normal Loewner chain. If $(f_t)_{t\geq 0}$ is geräumig in $[0, T)$, for some $T > 0$, then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, setting

$$\alpha(t) := \begin{cases} \varepsilon(1 - \frac{t}{T}), & t \in [0,T) \\ 0, & t \in [T,\infty), \end{cases}$$

The family $(f_t + \alpha(t)h)_{t\geq 0}$ is a normalized Loewner chain, respectively a normal Loewner chain, for every $h : \mathbb{B}^n \to \mathbb{C}^n$ holomorphic with $h(0) = 0$, $dh_0 = 0$, $\sup_{z\in\mathbb{B}^n} \|h(z)\| \leq 1$ and $\sup_{z\in\mathbb{B}^n}\|dh_z\| \leq 1$
For any $g \in \mathcal{H}_0$ and $T^r \in (0, 1)$ we denote by g_r the mapping which satisfies: $g_r(z) = \frac{1}{r}g(rz), for\ all\ z \in \mathbb{B}^n$.

If the mapping has some index, e.g. g_α then we denote by $g_{r,\alpha}$ the corresponding rescaled mapping.
We can deduce the next lemma.
Lemma (3.7)[3]: Let $n \in \mathbb{N}^*$, $g \in S^0$ and $(g_t)_{t\geq 0}$ be a normal Loewner chain into which g embeds. Then

(i) for every $r \in (0,1)$, $\left(g_{r,t}\right)_{t\geq 0}$ is a normal Loewner chain which is geräumig in $[0,T)$, for any $T > 0$; in particular, $g_r \in S^0$;
(ii) for every $r \in (0,1)$ and for every univalent mapping $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ with $\varphi(0) = 0$ and $d\varphi_0 = I$, $\left(\varphi \circ g_{r,t}\right)_{t\geq 0}$ is a normalized Loewner chain which is geräumig in $[0,T)$, for any $T > 0$; in particular, $\varphi \circ g_r \in S^1$, for $\varphi \in \mathrm{Aut}\,(\mathbb{C}^n)$.

Proof: Fix arbitrary $r \in (0,1)$ and $T > 0$.
(i) Assume that condition (i) of Definition (3.5) does not hold. Then there exists a sequence $(t_k)_{k\in\mathbb{N}}$ in $[0,T)$ which converges to some $t \in [0,T]$ and a sequence $(z_k)_{k\in\mathbb{N}}$ in $\mathbb{B}^n$ which converges to some $z \in \overline{\mathbb{B}}^n$ such that $\left\|d(g_{rt_k})_{z_k}\right\| = \left\|d(g_{t_k})rz_k\right\| \to 0, when\ k \to \infty$. implies that $g_{t_k} \to g_t$, when $k \to \infty$, uniformly on compacta of $\mathbb{B}^n$. So $\|d(g_t)_{rzk}\| \to \|d(g_t)_{rz}\|$, when $k \to \infty$. But then $\|d(g_t)_{rz}\| = 0$, which is a contradiction. So there exists $a > 0$ such that, for all $t \in [0,T)$ and all $z \in \mathbb{B}^n$, $\mu(d(g_{r,t})) \geq a$.
Condition (ii) follows, using the fact that the class M is compact and combining Cauchy's integral formula with the following inequality:

$$\|g_t(z)\| \leq e^t \frac{\|z\|}{1-\|z\|}, for\ all\ t \geq 0, z \in \mathbb{B}^n. \tag{5}$$

So there exists $b > 0$ such that, for a.e. $t \in [0,T)$ and all $z \in \mathbb{B}^n$, $\left\|\frac{\partial g_{r,t}}{\partial t}(z)\right\| \leq b$.
Condition (iii) of Definition (3.5) follows immediately. So there exists $c \in (0,1]$ such that, for a.e. $t \in [0,T)$ and all $z \in \mathbb{B}^n$,

$$\Re\left\langle \left(d(g_{r,t})\right)^{-1} \frac{\partial g_{r,t}}{\partial t}(z), z \right\rangle \geq c\,\|z\|^2.$$

Since $(g_t)_{t\geq 0}$ is a normal Loewner chain, $\left(g_{r,t}\right)_{t\geq 0}$ is a normal Loewner chain which is geräumig in $[0,T)$.
(ii) We consider $a, b > 0$ and $c \in (0,1]$ obtained previously.
Since $sup_{t\in[0,T),z\in\mathbb{B}^n} \left\|g_{r,t}(z)\right\| < \infty$, we have

$$\lambda_1 := \sup_{w\in g_{r,t}(\mathbb{B}^n),t\in[0,T)} \|(d\phi_w)^{-1}\| \in (0,\infty)$$

and

$$\lambda_2 := \sup_{w\in g_{r,t}(\mathbb{B}^n),t\in[0,T)} \|(d\phi_w)^{-1}\| \in (0,\infty)$$

So

$$\mu\left(d(\varphi \circ g_{r,t})z\right) \geq \frac{1}{\left\|(d\phi g_{r,t}(z))^{-1}\right\| \left\|(d(g_{r,t})z)^{-1}\right\|} \geq \frac{\alpha}{\lambda_1},$$

for all $t \in [0, T)$ and all $z \in \mathbb{B}^n$,

$$\left\|\frac{\partial(\phi\ o\ g_{r,t})}{\partial t}(z)\right\| \leq \|\ d\phi g_{r,t}(z)\| \left\|\frac{\partial g_{r,t}}{\partial t}(z)\right\| \leq \lambda_1 b,$$

for a.e. $t \in [0, T)$ and all $z \in \mathbb{B}^n$, and

$$R\left\langle\left(d(\phi\ o\ g_{r,t})(z)\right)^{-1}\frac{\partial(\phi \circ g_{r,t})}{\partial t}(z), z\right\rangle = R\left\langle dg_{r,t}(z)^{-1}\frac{dg_{r,t}}{\partial t}(z)\right\rangle \geq c\|z\|^2,$$

for a.e. $t \in [0, T)$ and all $z \in \mathbb{B}^n$.

Since $(g_{r,t})_{t\geq 0}$ is a normal Loewner chain and the three conditions of Definition (3.5) are satisfied,$(\phi \circ g_{r,t})_{t\geq 0}$ is a normalized Loewner chain which is geräumig in $[0, T)$.

Proposition (3.8)[3]: Let $n \in \mathbb{N}^*$, $g \in S^0$ and $(g_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{H}_0$ which converges, uniformly on compacta of $\mathbb{B}^n$, to g. Then for every sequence $(r_k)_{k\in\mathbb{N}}$ in (0, 1) convergent to 1 there exists a subsequence of indexes $(j_k)_{k\in\mathbb{N}}$ such that $g_{\mathrm{rk,jk}} \in S^0$, for all $k \in \mathbb{N}$, and $(g_{\mathrm{rk,jk}})_{k\in\mathbb{N}}$ converges, uniformly on compacta of $\mathbb{B}^n$, to g.

The same property holds for S^1.

Proof: Fix arbitrary $r \in (0, 1)$.

Let $(g_t)_{t\geq 0}$ be a normal Loewner chain into which g embeds.

By Lemma (3.7) (i)$g \in S^0$ and $(g_{r,t})_{t\geq 0}$ is a normal Loewner chain which is geräumig in $[0, T)$, for some $T > 0$.

We observe that $g_{r,j} \to g_r$, when $j \to \infty$, uniformly on compacta of $\frac{1}{r}\mathbb{B}^n$.

Let $\varepsilon > 0$bearbitrary. By the previous observation, we deduce that there exists $j_\varepsilon \in N$ sufficiently large such that

$$\sup_{z\in\mathbb{B}^n}\|g_{r,j}(z) - g_r(z)\| \leq \varepsilon \text{ and} \sup_{z\in\mathbb{B}^n}\left\|d(g_{r,j\varepsilon})_z - d(g_r)_z\right\| \leq \varepsilon$$

Let $h_{r,j_\varepsilon} := g_{r,j_\varepsilon} - g_r$. Note that h_{r,j_ε} is a holomorphic mapping on $\mathbb{B}^n$ with $h_{r,j_\varepsilon}(0) = 0$ and $d(h_{r,j_\varepsilon}) = 0$.

For a sufficiently small $\varepsilon > 0$, by Theorem(3.6), we have that $g_r + h_{r,j_\varepsilon} \in S^0$, hence $g_{r,j\varepsilon} \in S^0$.

Now, let $(r_k)_{k\in\mathbb{N}}$ be a sequence in (0, 1), convergent to 1.

For every $k \in \mathbb{N}$, by the previous argument, there exists $j_k \in \mathbb{N}$ such that $g_{\mathrm{rk,jk}} \in S^0$.

Next, we show that $g_{rk,jk} \to g$, when $k \to \infty$, uniformly on compacta of $\mathbb{B}^n$.

Let $K := \mathbb{B}^n(0,R)$, with arbitrary $R \in (0, 1)$.

Since $g_{rk,jk} \to g$, when $k \to \infty$, uniformly on compacta of $\mathbb{B}^n$, we have that $sup_{\zeta\in K}\|g_{\mathrm{rk,jk}}(\zeta) - g_{\mathrm{rk}}(\zeta)\| \to 0$, when $k \to \infty$.

So it is sufficient to prove that $sup_{\zeta\in K}\|g(r_k\zeta) - g(\zeta)\| \to 0$, when $k \to \infty$.

By the mean value theorem for vector-valued functions we have

$$\|g(r_k\zeta)) - g(\zeta)\| \leq (1 - r_k)\,\zeta \sup_{z\in K}\|dg\|,$$

for all $\zeta \in K$, so the proof is finished.

To prove that the same property holds for S^1, one can redo the same proof, using this time Lemma (3.7) (ii) and Theorem (3.6)for normalized Loewner chains.

Schleißinger show that $S^0 \subset \bar{A}$, then suggested that we may have $S^0 = \overline{A^0}$ for $n \geq 2$. In the following, we prove that, indeed, this is a fact.

Corollary $(\mathbf{3.9})$**[3]:** If $n \in N$ such that $n \geq 2$, then $S^0 = \overline{A^0}$.

Proof: Let $g \in S^0$.

By (4) we have that there exists a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in A such that $\varphi_j \to g$, when $j \to \infty$, uniformly on compacta of $\mathbb{B}^n$.

Let $(r_k)_{k\in\mathbb{N}}$ be a sequence in $(0, 1)$ convergent to 1. By proposition (3.8)

There exists a subsequence of indexes $(j_k)_{k\in\mathbb{N}}$ such that $\varphi_{rk,jk} \in S^0$, for all $k\in\mathbb{N}$, and $(\varphi_{rk,jk})_{k\in\mathbb{N}}$ converges, uniformly on compacta of $\mathbb{B}^n$, to g.

We observe that $(\varphi_{rk,jk})_{k\in\mathbb{N}}$ is also a sequence in A. Hence $g \in \overline{A^0}$.

Denote

$$C^1\left(\overline{\mathbb{B}^n}\right) := \{f \in C^1(\mathbb{B}^n) \mid f \text{ and } df \text{ extend continuously to } \mathbb{B}^n\}.$$

The following result is an improvement of a recent result due to Arosio, Bracci and Wold.

Theorem $(\mathbf{3.10})$**[3]**: $S_R \cap C^1(\overline{\mathbb{B}}^n) \subset S^1$, for any $n \in \mathbb{N}^*$.

Proof: If $n = 1$, then we have $S^1 = S_R$, as mentioned in the introduction.

Let $n \geq 2$.

Let $f \in S_R \cap C^1(\overline{\mathbb{B}}^n)$.

Let $\varepsilon > 0$ be arbitrary.

Let $(r_j)_{j\in\mathbb{N}}$ be a sequence in $(0, 1)$ convergent to 1.

In the following we denote also by f the continuous extension of f to $\overline{\mathbb{B}}^n$.

Since f is continuous on $\overline{\mathbb{B}}^n$, there exists a sequence $(z_j)_{j\in\mathbb{N}}$ in $\partial\mathbb{B}^n$ such that

$$\left\|f_{rj}(z_j) - f(z_j)\right\| = \max_{z\in\overline{\mathbb{B}}^n} \left\|f_{rj}(z) - f(z)\right\|.$$

Up to a subsequence, we can assume $(z_j)_{j\in\mathbb{N}}$ converges to a point $z_0 \in \partial\mathbb{B}^n$. Since f is continuous on $\overline{\mathbb{B}}^n$,

it follows that $f_{rj}(z_j) \to f(z_0)$, when $j \to \infty$, hence $sup_{z\in\mathbb{B}^n}\, sup_{z\in\mathbb{B}^n} \left\|f_{rj}(z) - f(z)\right\| \to 0, when\ j \to \infty$.

Since the same argument is valid for df, we can choose $j\in\mathbb{N}$ sufficiently large such that

$$\sup_{z\in\mathbb{B}^n} \left\|f_{rj}(z) - f(z)\right\| \leq \frac{\varepsilon}{2}. \tag{6}$$

and

$$\sup_{z\in\mathbb{B}^n} \left\|d\left(f_{rj}\right)_{(z)} - df_z\right\| \leq \frac{\varepsilon}{2}. \tag{7}$$

For simplicity we denote $r := r_j$.

By (3) we deduce that there exists a sequence $(f_i)_{i\in\mathbb{N}}$in S^1 such that $f_i \to f$, uniformly on compacta of $\mathbb{B}^n$, when $i \to \infty$. Since $\left(f_{r,i}\right)_{i\in\mathbb{N}}$converges, uniformly on compacta of $\frac{1}{r}\mathbb{B}^n$, to f, we can choose $i \in \mathbb{N}$, sufficiently large such that

$$\sup_{z\in\mathbb{B}^n} \left\|f_{r,i}(z) - f_r(z)\right\| \leq \frac{\varepsilon}{2} \tag{8}$$

and

$$\sup_{z\in\mathbb{B}^n} \left\|d\left(f_{r,i}\right)_{(z)} - d(f_r)_z\right\| \leq \frac{\varepsilon}{2} \tag{9}$$

Combining the inequalities (6), (9) we have

$$\sup_{z\in\mathbb{B}^n} \left\|f_{(z)} - f_{r,i}(z)\right\| \leq \varepsilon \quad and \quad \sup_{z\in\mathbb{B}^n} \left\|df_z - d\left(f_{r,i}\right)_z\right\| \leq \varepsilon \tag{10}$$

Since f $\in S^1$, by (2) we deduce that there exist $\varphi_i \in$ Aut $(\mathbb{C}^n)$ and $g_i \in S^0$ such that $f_i = \varphi_i \circ g_i$.
Note that

$$f_{r,i}(z) = \frac{1}{r}\varphi_i\left(r\frac{1}{r}g_i(rz)\right),$$

for all $z \in \mathbb{B}^n$.
For simplicity we denote $\phi := \phi_{r,i}$ and $g := g_i$. We have $\phi \in \mathrm{Aut}_0(\mathbb{C}^n)$, $g_r \in S^0$ and $f_{r,i} = \phi \circ g_r$. By (10) we have

$$\sup_{z\in\mathbb{B}^n} \left\|f_{(z)} - (\phi \circ g_r)(z)\right\| \leq \varepsilon \quad and \quad \sup_{z\in\mathbb{B}^n} \|df_z - d(\phi \circ g_r)_z\| \leq \varepsilon \qquad (11).$$

Let $(g_t)_{t\geq 0}$ be a normal Loewner chain into which g embeds.
By Lemma (3.7) (ii) we have that $\left(\phi \circ g_{r,t}\right)_{t\geq 0}$ is a normalized Loewner chain which is geräumig in $[0, T)$, for some time $T > 0$.
Let $h_r := f - \phi \circ g_r$. Note that h is a holomorphic mapping on $\mathbb{B}^n$, with $h_r(0) = 0$ and $d(h_r)_0 = 0$.
For a sufficiently small $\varepsilon > 0$, by Theorem (3.6), we have that $(f_t)_{t\geq 0}$ with $f_t \coloneqq \phi \circ g_{r,t} + \alpha(t)h_r$, for all $t \geq 0$, is a normalized Loewner chain, where

$$\alpha(t) := \begin{cases} 1 - \dfrac{t}{T} & t \in [0, T), \\ 0 & \in [T, \infty) \end{cases}$$

By Theorem (3.4)(i), $(f_t)_t \geq 0$ is a normalized univalent solution of a Loewner–Kufarev PDE:

$$\frac{\partial f_t}{\partial f}(z) = d(f_t)_z G(z, t),$$

for a.e. $t \geq 0$ andfor every $z \in \mathbb{B}^n$, where G is a Herglotz vector field. By Theorem (3.4)(iii) there exists a normalized holomorphic mapping $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$f_t = \varphi \circ k_t, \quad \text{for all } t \geq 0,$$

where $(k_t)_t \geq 0$ is the canonical solution of the above Loewner–Kufarev PDE.

Since: $f_t = \varphi \circ g_{r,t} = \varphi \circ k_t$, for all $t \geq T$ (because $\alpha(t) = 0$,for all $t \geq T$), $g_{r,t}, k_t \in S$, for all $t \geq T$, φ is a normalized automorphism of $\mathbb{C}^n$ and $R(g_t) = R(k_t) = \mathbb{C}^n$, we deduce that φ is also a normalized automorphism of $\mathbb{C}^n$.
We observe that $f = f_0 = \varphi \circ k_0$ and $k \in S^0$. So, in view of (2), we deduce that $f \in S^1$.
Theorem (3.11)[8]: Let h(z,t) be Herglotz vector field of order ∞ on B^q such that $h(z,t) = A_z + O(|z|^2)$ with

$$2\min\{\mathrm{Re}\langle A_z, z\rangle : |z| = 1\} > \max\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{sp}(A)\}$$

hermition where $\langle\cdot,\cdot\rangle$ is the her mition product on c^q then the boeuner PDF $\frac{df_t(z)}{dt} = Df_t(z)h(z,t)$, admits alocally Lipshitz univalent solution $(f_t : B^q \to c^q)$. The range $U_t \geq 0$ of (B^q) of any such solution is biholomorphic to c^q.

Chapter 4

Nevanlinna Representations in Several Variabls

We find Four different representation formulae and we show that every function in the Loewner class has one of the four representations, corresponding precisely to four different growth conditions at infinity.

Section (4.1) Nevanlinna's Representations

Theorem (4.1.1)[4]: (Nevanlinna's Representation). Let h be a function defined on Π. There exists a finite positive measure μ on ℝ such that

$$h(z) = \int \frac{d\mu}{t - z} \qquad (1)$$

if and only if $h \in p$ and

$$\liminf_{y\to\infty} y\,|h(iy)| < \infty. \qquad (2)$$

A closely related theorem, also referred to in the literature as Nevanlinna's Represen- tation, provides an integral representation for a general element of p

Theorem (4.1.2)[4]: A function $h{:}\,\Pi \to C$ belongs to the Pick class p if and only if there exist $a \in \mathbb{R}, b \geq 0$ and a finite positive Borel measure μ on R such that

$$h(z) = a + bz + \int \frac{1 + tz}{t - z} d\mu\,(t) \qquad (3)$$

for all $z \in \Pi$. Moreover, for any $h \in p$, the numbers $a \in R, b \geq 0$ and the measure $\mu \geq 0$ in the representation (3) are uniquely determined.

What are the several-variable analogs of Nevanlinna's theorems? In this paper we shall propose four types of Nevanlinna representation for various subclasses of the n-variable Pick class P_n, where P_n is defined to be the set of analytic functions h on the polyhalf-plane Π_n such that $\operatorname{Im} h \geq 0$. In addition, we shall present necessary and sufficient conditions for a function defined on Πn to possess a representation of a given type in terms of asymptotic growth conditions at ∞.

The integral representation (1) of those functions in the Pick class that satisfy condition (2) can be written in the form

$$h(z) = \langle (A - z)^{-1}\, 1, 1\rangle_{L^2(\mu)}\,,$$

where A is the operation of multiplication by the independent variable on $L^2(\mu)$ and 1 is the constant function 1. We propose that an appropriate n-variable analog of the Cauchy transform is the formula

$$h(z_1, \ldots, z_n) = \langle (A - z_1 Y_1 - \cdots - z_n Y_n)^{-1}\, v, v\rangle_{\mathcal{H}} \text{ for } z_1, \ldots, z_n \in \Pi, \qquad (4)$$

where $\mathcal{H}$ is a Hilbert space, A is a densely defined self-adjoint operator on $\mathcal{H}$, $Y_1, \ldots, Y_n$ are positive contractions on $\mathcal{H}$ summing to 1 and v is a vector in $\mathcal{H}$.

Theorem $(4.1.3)$[4]: A function h defined on Π belongs to P if and only if the function A defined

$$on\ \Pi\ \times\ \Pi\ by$$

$$A(z,\mathcal{W}) = \frac{\mathrm{h(z)}\ -\ \overline{\mathrm{h}(\mathcal{W})}}{\mathrm{z}\ -\ \overline{\mathcal{W}}}$$

is positive semidefinite, that is, for all $n \geq 1$, $z_1,\ldots,z_n \in \Pi$, $c_1,\ldots,c_n \in C$,

$$\sum A\,(z_j,z_i)\bar{c}_i c_j\ \geq\ 0.$$

The following theorem, leads to a generalization of Theorem (4.1.3) to two variables. The Schur class of the polydisc, denoted by S_n, is the set of analytic functions on the polydisc $\mathbb{D}^n$ that are bounded by 1 inmodulus.

Theorem$(4.1.4)$[4]: A function φ defined on $\mathbb{D}^2$ belongs to S_2 if and only if there exist positive semidefinite functions A and A on $\mathbb{D}^2 \times \mathbb{D}^2$ such that

$$1 - \overline{\varphi(\mu)}\varphi(\lambda) = (1 - \bar{\mu}_1\lambda_1)A_1(\lambda,\mu) + (1 - \bar{\mu}_2\lambda_2)A_2(\lambda,\mu). \qquad (5)$$

By way of the transformations

$$z = i\frac{1+\lambda}{1-\lambda}\quad \lambda = \frac{z\ -\ i}{z+\ i}, \qquad (6)$$

and

$$h(z) = i\frac{1+\varphi(\lambda)}{1-\varphi(\lambda)},\quad \varphi(\lambda) = \frac{h(z)-i}{h(z)+i} \qquad (7)$$

there is a one-to-one correspondence between functions in the Schur and Pick classes. Under these transformations, Theorem (4.1.4) becomes the following generalization of Pick's theorem to two variables.

Theorem $(4.1.5)$[4]: A function h defined on Π^2 belongs to P_2if and only if there exist positive semidefinite functions A and A on $\Pi^2 \times \Pi^2$ such that

$$h(z) - \overline{h(w)} = (z_1 - \overline{w_1})A_1(z,w) + (z_2\ -\ w_2)A_2(z,w).$$

In the light of Theorems (4.1.2) and (4.1.5) we define the Loewner class $\mathcal{L}_n$ to be the set of analytic functions h on Π^n with the property that there exist n positive semidefinite functions $A_1,\ldots,A_n$ on Π^n such that

$$h(z)\ -\ \overline{h(w)} = \sum_{j-1}^{n}(z_j - \overline{w_j})A_j(z,w) \qquad (8)$$

for all $z,w \in \Pi^n$. The Loewner class $\mathcal{L}_n$ played a key, which gave a generalization to several variables of Loewner's characterization of the one-variable operator-monotone functions. As the following theorem makes clear, $\mathcal{L}_n$ also has a fundamental role to play in the understanding of Nevanlinna representations in several variables.

Theorem $(\mathbf{4.1.6})[\mathbf{4}]$: A function h defined on Π^n has a representation of the form (8) if and only if h $\in \mathcal{L}_n$ and

$$\liminf_{y\to\infty} y\,|h(iy,\ldots,iy)| < \infty. \tag{9}$$

In the cases when $n = 1$ and $n = 2$, Theorems (4.1.2) and (4.1.5) assert that $L_n = P_n$, and so for $n = 1$, Theorem (4.1.6) is Nevanlinna's classical Theorem (4.1.1), and when $n = 2$, Theorem (4.1.6) is a straightforward generalization of that result to two variables. When there are more than two variables, it is known that the Loewner class is a proper subset of the Pick class, $L_n \neq P_n$. Nevertheless, Nevanlinna's result survives as a theorem about the representation of elements of $\mathcal{L}_n$. Other than the work in very little is known about the representation of functions in P_n for three or more variables.

For a function h on Π^n, we call the formula (3) a Nevanlinna representation of type 1. Thus, Theorem (4.1.6) can be rephrased as the assertion that h has a Nevanlinna representation of type 1 if and only if $h \in \mathcal{L}_n$ and h satisfies condition (9). Somewhat more complicated representation formulae are needed to generalize Theorem (4.1.2). We identify three further representation formulae, of increasing generality, and show that every function in $\mathcal{L}_n$ has a representation of one or more of the four types.

For a function h defined on Π^n, we refer to a formula

$$h(z_1,\ldots,z_n) = a + \langle (A - z_1Y_1 - \cdots - z_nY_n)^{-1}v, v\rangle_{\mathcal{H}} \quad \text{for } z_1,\ldots,z_n \in \Pi, \tag{10}$$

where a is a constant, $\mathcal{H}$ is a Hilbert space, A is a densely defined self-adjoint operator on $\mathcal{H}$, $Y_1,\ldots,Y_n$ are positive contractions on $\mathcal{H}$ summing to 1 and v is a vector in $\mathcal{H}$, as a Nevanlinna representation of type 2.

Theorem $(\mathbf{4.1.7})[\mathbf{4}]$: A function h defined on Π^n has a Nevanlinna representation of type 2 if and only if $h \in \mathcal{L}_n$ and

$$\liminf_{y\to\infty} y \operatorname{Im} h(iy,\ldots,iy) < \infty. \tag{11}$$

A Nevanlinna representation of type 3 of a function h defined on Π^n is of the form

$$h(z) = a + \langle (1 - iA)\big(A - z_y\big)(1 + z_y A)(1 - iA)^{-1} v, v\rangle \text{ for all } z \in \Pi^n$$

for some real a, some self-adjoint operator A and some vector v, where $Y_1,\ldots,Y_n$ are operators as in equation (4.1.5) above and $z_Y = z_1Y_1 + \cdots + z_nY_n$.

Theorem $(\mathbf{4.1.8})[\mathbf{4}]$**:** A function h defined on Π^n has a Nevan Linna representation of type 3 if and only if $h \in \mathcal{L}_n$ and

$$\liminf_{y\to\infty} \frac{1}{y} \operatorname{Im} h(iy,\ldots,iy) = 0.$$

Finally, Nevanlinna representations of type 4aregiven by the formula

$$h(z) = a + \langle M(z)v, v\rangle, \tag{12}$$

where a $\in \mathbb{R}$ and M(z) i san operator of the form

$$\begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - zp \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \left(zp \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix}^{-1}, \tag{13}$$

acting on an orthogonal direct sum of Hilbert spaces N⊕M. In equation (12), v is a vector in N⊕M. In equation (13), A is a densely-defined self-adjoint operator acting on M and zP is the operator acting on N⊕M via the formula

$$zP = \sum z_i P_i$$

where $P_1, \ldots, P_n$ are pairwise orthogonal projections acting on N⊕M that sum to 1.

Theorem (4.1.9) [4]: Let h be a function defined on Π^n. Then h has a Nevanlinna representation of type 4 if and only if h ∈ $\mathcal{L}_n$.

A weaker, "generic" version of Theorem (4.1.9), where it was used to show that elements in Ln are locally operator-monotone.

It turns out that for 1 ≤ k ≤ 4, if h is a function on Π^n and h has a Nevanlinna representation of type k, then for k ≤ j ≤ 4, h also has a Nevanlinna representation of type j. Thus, it is natural to define the type of a function in Ln to be the smallest k such that h has a Nevanlinna representation of type k.

For h ∈ $\mathcal{L}_n$ the type of h can be characterized in function-theoretic terms through the use of a geometric idea due to Carathéodory. A carapoint for a function φ in the Schur class S_n is a point τ ∈ $\mathbb{T}$ such that

$$\liminf_{\lambda \to \tau} \frac{1-|\varphi(\lambda)|}{1-\|\lambda\|_\infty} < \infty.$$

where

$$\|\lambda\|_\infty = \max_{1 \le i \le n} |\lambda_i|$$

Carathéodory introduced this notion in one variable, along the way to refining earlier. The following was Carathéodory's main result; the notation $\lambda \overset{nt}{\to} \tau$ means that λ tends nontangentially to τ.

Theorem (4.1.10)[4]: Let φ ∈S_1, τ ∈ $\mathbb{T}$. If τ is a carapoint for φ, then φ is nontangentially differentiable at τ, that is, there exist values φ (τ) and φ' (τ) such that

$$\lim_{\lambda \overset{nt}{\to} \tau} \frac{\varphi(\lambda) - \varphi(\tau) - \varphi'(\tau)(\lambda - \tau)}{\lambda - \tau} = 0$$

In particular, if τ is a carapoint for φ then there exists a unique point φ (τ) ∈ T such that $\varphi(\lambda) \to \varphi(\tau)$ as $\lambda \overset{nt}{\to} \tau$.

In several variables, carapoints have been studied. The strong conclusion of nontangential differentiability is lost in several variables; however, at a carapoint τ, there still exists a unimodular nontangential limit φ (τ).

As the point χ = (1,…,1) is transformed to the point ∞ = (∞,…,∞) by Theorem (4.1.10), it is natural to say that a function h ∈ $\mathcal{L}_n$ has a carapoint at ∞ if the associated Schur function

φ, given by the transformation in equation (7), has a carapoint at χ, and in that case to define $h(\infty)$ by

$$h(\infty) = i\frac{1+\ \varphi\,(\chi)}{1\ -\ \varphi\,(\chi)} \tag{14}$$

The connection between carapoints and function types is given in the following Theorem.

Theorem (4.1.11)[4]: For a function $h \in \mathcal{L}_n$,

(i) h is of type 1 if and only if ∞ is a carapoint of h and $h(\infty) = 0$;
(ii) h is of type 2 if and only if ∞ is a carapoint of h and $h(\infty) \in \mathbb{R} \setminus \{0\}$;
(iii) h is of type 3 if and only if ∞ is not a carapoint of h;
(v) h is of type 4 if and only if ∞ is a carapoint of h and $h(\infty) = \infty$.

As is clear from the formulae used to define the various Nevanlinna representations, Nevanlinna representations are generalizations of the resolvent of a self-adjoint operator. These structured resolvents, are analytic operator-valued functions on the polyhalf-plane Π^n with non-negative imaginary part, fully analogous to the familiar resolvent operator. There are also struc-tured resolvent identities for them.

In modern texts Nevanlinna's representation is derived from the Herglotz Representa-tion with the aid of the Cayley transform . We introduce the n-variable strong Herglotz class and then prove Theorem (4.1.9) by applying the Cayley transform.

We derive the Nevanlinna representations of types 3, 2, and 1, we show how they arise naturally from the underlying Hilbert space geometry we give function-theoretic conditions for a function $h \in \mathcal{L}_n$ to possess a representation of a given type.

We introduce the notion of carapoints for functions in the Pick class and we establish the criteria for the type of a function using the language of carapoints.

We give the growth estimates for functions in $\mathcal{L}_n$ that flow from our analysis of structured resolvents, and we present resolvent identities for structured resolvents.

Results related to ours from a system-theoretic perspective have been obtained in recent works of J.A.Ball and D. Where Krein space methods are applied to similar problems.

The resolvent operator $(A - z)^{-1}$ 1 of a densely defined self-adjoint operator A on a Hilbert space plays a prominent role in spectral theory. It has the following properties.

(i) It is an analytic bounded operator-valued function of z in the upper half-plane Π;
(ii) it satisfies the growth estimate $\|(A\ -\ z)^{-1}\| \ \le\ 1/\operatorname{Im} z$ for $z \in \Pi$;
(iii) $(A - z)^{-1}$ has non-negative imaginary part for all $z \in \Pi$;
(v) it satisfies the "resolvent identity".

Here we are interested in several-variable analogs of the resolvent. These will again be operator-valued analytic functions with non-negative imaginary part, but now on the polyhalf-plane Π^n. Because of the additional complexities in several variables we encounter three different types of resolvent; all of them have the four listed properties, with very slight modifications, and therefore deserve the name structured resolvent.

For any Hilbert space $\mathcal{H}$, a positive decomposition of $\mathcal{H}$ will mean an n-tuple $Y = (Y_1, \dots, Y_n)$ of positive contractions on H that sum to the identity operator. For any $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and any n-tuple $T = (T_1, \dots, T_n)$ of bounded operators we denote by z_T the operator $\sum_j z_j T_j$. Here each T_j is a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, for some Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, so that zT is also a bounded operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Definition(4.1.12)[4]: Let A be a closed densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and let Y be a positive decomposition of $\mathcal{H}$. The structured resolvent of A of type 2 corresponding to Y is the operator-valued function

$$z-(A-z_Y)^{-1}:\Pi^n \to \mathcal{L}(\mathcal{H}).$$

The following observation is essentially.

Proposition (4.1.13)[4]: For A and Y as in Definition (4.1.12) the structured resolvent $(A-z\ )^{-1}$ is well defined on Π^n and satisfies, for all $z\in\Pi^n$,

$$\|(A-z_Y)^{-1}\| \le \frac{1}{\min_j \operatorname{Im} z_j}. \tag{15}$$

Moreover

$$\begin{aligned} Im\,((A-z_Y)^{-1}) &= (A-z_Y^*)^{-1}(Im\, z_Y)(A-z_Y)^{-1} \\ &= (A-z_Y)^{-1}(Im\, z_Y)(A-z_Y^*)^{-1} \ge 0. \end{aligned} \tag{16}$$

The range of the bounded operator $(A-z_Y)^{-1}$ is of course $\mathcal{D}$(A), the domain of A.

Proof. For any vector ξ in the domain of A,

$$\begin{aligned} \|(A-z_Y)\xi\|\,\|\xi\| &\ge |\,(\langle (A-z_Y)\xi,\xi\,|\rangle \\ &\ge |\mathrm{Im}\,(A-z_Y)\xi,\xi\,| \\ &= (\langle \mathrm{Im}\, z_Y)\xi,\xi\rangle \\ &= \sum_j (\mathrm{Im}\, z_j)\,\langle Y_j\xi,\xi\rangle \\ &\ge (\min_j \mathrm{Im}\ z_j)\,\langle \sum_j Y_j\xi,\xi\rangle \\ &= (\min_j \mathrm{Im}\ z_j)\,\|\xi\|^2. \end{aligned}$$

Thus $A-z_Y$ has lower bound minj $\mathrm{Im}\, z_j > 0$, and so has a bounded left inverse. Asimilar argument with z replaced by $\overline{z}$ shows that $(A-z_Y)^*$ also has a bounded left inverse, and so $A-z_Y$ has a bounded inverse and the inequality (15) holds.
The identities (16) are easy.
Resolvents of type 2 are the simplest several-variable analogues of the familiar one-variable resolvent but they are not sufficient for the analysis of the several-variable Pick class. To this end we introduce two further generalizations. Let us first recall some basic facts about closed unbounded operators.
Lemma (4.1.14)[4] : Let T be a closed densely defined operator on a Hilbert space $\mathcal{H}$, with domain $\mathcal{D}$(T). The operator $1+T^*T$ is a bijection from $D(T^*T)$ to H, and the operators

$$B \overset{\text{def}}{=} (1+T^*T)^{-1},\ C \overset{\text{def}}{=} T(1+T^*T)^{-1}$$

are everywhere defined and contractive on $\mathcal{H}$. Moreover B is self-adjoint and positive, and $\operatorname{ran} C \subset D(T^*)$.

Proof: Although the final statement about ran C is not explicitly stated. We must show that for all $v \in \mathcal{H}$ there exists $y \in \mathcal{H}$ such that, for all $h \in \mathcal{H}$,

$$\langle Th, Cv \rangle = \langle h, y \rangle .$$

It is straightforward to check that this relation holds for $y = v - Bv$, and so $\operatorname{ran} C \subset D(T^*)$.

Definition (4.1.15)[4]: Let A be a closed densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and let Y be a positive decomposition of $\mathcal{H}$. The structured resolvent of A of type 3 corresponding to Y is the operator-valued function $M : \Pi^n \to \mathcal{L}(\mathcal{H})$ given by

$$M(z) = (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1} . \tag{17}$$

We denote the ℓ_1 norm on $\mathbb{C}^n$ by $\|\cdot\|_1$. Note that $\|z_Y\| \le \|z\|_1$ for all $z \in \mathbb{C}^n$ and all positive decompositions Y .

Proposition (4.1.16)[4]: For A and Y as in Definition (4.1.15) the structured resolvent M(z) of type 3 given by equation (17) is well defined as a bounded operator on $\mathcal{H}$ for all $z \in \Pi^n$ and satisfies

$$\|M(z)\| \le (1 + 2\|z\|_1)\left(1 + \frac{1 + \|z\|_1}{\min_j \operatorname{Im} z_j}\right) . \tag{18}$$

Proof. Since

$$1 + z_Y A = 1 - iz_Y + iz_Y(1 - iA): D(A) \to \mathcal{H}$$

and $(1 - iA)^{-1}$ is a contraction on all of $\mathcal{H}$, with range $\mathcal{D}(A)$, the operator $(1+z_Y A)(1 - iA)^{-1}$ 1 is well defined as an operator on $\mathcal{H}$ and

$$\begin{aligned}\|(1 + z_Y A)(1 - iA)\|^{-1} &= \|(1 - iz_Y)(1 - iA)^{-1} + iz_Y\| \\ &\le \|1 - iz_Y\| + \|z_Y\| \\ &\le 1 + 2\|z_Y\| \\ &\le 1 + 2\|z\|_1 .\end{aligned} \tag{19}$$

Similarly $(1 - iA)(A - z\ )^{-1}$ is well defined on $\mathcal{H}$, and since

$$i(A - z_Y) = -(1 - iA) + (1 - iz_Y): \mathcal{D}(A) \to \mathcal{H}$$

we have

$$i = -(1 - iA)(A - z_Y)^{-1} + (1 - iz_Y)(A - z_Y)^{-1} : \mathcal{H} \to \mathcal{H}.$$

Thus, by virtue of the bound (15),

$$\begin{aligned}\|(1 - iA)(A - z_Y)^{-1}\| &= \|i - (1 - iz_Y)(A - z_Y)^{-1}\| \\ &\le 1 + \|1 - iz_Y\|\,\|(A - z_Y)^{-1}\| \\ &\le 1 + \frac{1 + \|z\|_1}{\min_j \operatorname{Im} z_j} .\end{aligned} \tag{20}$$

On combining the estimates (20) and (19) we obtain the bound (18).

The following alternative formula for the structured resolvent of type 3, valid on the dense subspace D(A) of $\mathcal{H}$, allows us to show that Im M(z) ≥ 0.

Proposition (**4.1.17**)[**4**]: For A and Y as in Definition (4.1.15) and z∈Π^n

$$M(z)|D(A) = (1- iA)\{(A - z_Y)^{-1} - A(1+ A^2)^{-1}\}(1+ iA) \quad (21)$$
$$= (1- iA)(A - z_Y)^{-1}(1+ iA) - A : \mathcal{D}(A) \to \mathcal{H}. \quad (22)$$

Moreover, for every v ∈ $\mathcal{D}$ (A),

$$\text{Im}\langle M(z)v, v\rangle = \langle (1 - iA)(A - z_Y^*)^{-1}(Im\, z_Y)(A - z_Y)^{-1}(1+ iA)v, v\rangle \geq 0. \quad (23)$$

Proof: By Lemma (4.1.14) the operator $A(1 + A^2)^{-1}$ is contractive on $\mathcal{H}$ and has range contained in $\mathcal{D}$ (A). On $\mathcal{D}(A^2)$ wehave the identity

$$1 + z_Y A = 1 + A^2 - (A - z_Y)A.$$

Since $(1 + A^2)^{-1}$ maps $\mathcal{H}$ into $\mathcal{D}(A^2)$ we have

$$(1+ z_Y A)(1+ A^2)^{-1} = 1 - (A - z_Y)A(1+ A^2)^{-1} : \mathcal{H} \to \mathcal{H},$$

and therefore

$$(A - z_Y)^{-1}(1+ z_Y A)(1+ A^2)^{-1} = (A - z_Y)^{-1} - A(1+ A^2)^{-1} : \mathcal{H} \to \mathcal{D}(A). \quad (24)$$

Clearly

$$(1+ A^2)^{-1}(1+ iA) = (1- iA)^{-1} \text{ on } \mathcal{D}(A)$$

and so, on multiplying equation (24) fore-and-aftby 1±iA, we deduce that, as operators from $\mathcal{D}$(A) to $\mathcal{H}$,

$$M(z)|\mathcal{D}(A) = (1- iA)(A - z_Y)^{-1}(1+ z_Y A)(1 - iA)^{-1}$$
$$= (1- iA)(A - z_Y)^{-1}(1+ z_Y A)(1+ A^2)^{-1}(1+ iA)$$
$$= (1- iA)\{(A - z_Y)^{-1} - A(1+ A^2)^{-1}\}(1+ iA).$$

This establishes equation (21).
The expression (22) follows from equation (21) since

$$(1 - iA)A(1+ A^2)^{-1}(1+ iA) = A \text{ on } \mathcal{D}(A).$$

By equation (22) we have, for any z ∈ Π^n and v ∈ $\mathcal{D}$ (A),

$$Im\langle M(z)v, v\rangle = Im\langle (1 - iA)(A - z_Y)^{-1}(1+ iA)v, v\rangle - Im\langle Av, v\rangle$$
$$= Im\langle (A - z_Y)^{-1}(1+ iA)v, (1+ iA)v\rangle$$

and hence, by equation (16),

$$Im\, M\langle (z)v, v\rangle = \langle (A - z_Y^*)^{-1}(Im\, z_Y)(A - z_Y)^{-1}(1+ iA)v, (1+ iA)v\rangle,$$

and so equation (23) holds.

Corollary (4.1.18)[4]: For A and Y as in Definition (4.1.15) the structured resolvent M(z) given by equation (17) satisfies Im M(z) ≥ 0 for all $z \in \Pi^n$.

For,by Propositions (4.1.16) and (4.1.17), M(z) is a bounded operator on $\mathcal{H}$, and Im $\langle M(z)v, v\rangle \geq 0$ for $v \in \mathcal{D}(A)$. The conclusion follows by the density of $\mathcal{D}$(A) and continuity.

In the case of bounded A there is yet another expression for the structured resolvent of type 3.

Proposition (4.1.19)[4]: If A is a bounded self-adjoint operator on H and Y is a positive de-composition of $\mathcal{H}$ then, for z ∈ Π^n,

$$M(z) = (1 + iA)^{-1} (1 + Az_Y)(A - z_Y)^{-1} (1 + iA). \qquad (25)$$

Proof. Since A is bounded it is defined on all of $\mathcal{H}$. We have

$$1 + Az_Y = 1 + A^2 - A(A - z_Y)$$

and hence

$$(1 + Az_Y)(A - z_Y) = (1 + A^2)(A - z_Y)^{-1} - A.$$

Thus

$$(1 + iA)^{-1} (1 + Az_Y)(A - z_Y)^{-1} (1 + iA) = (1 - iA)(A - z_Y)^{-1} (1 + iA) - A = \mathrm{M(z)}$$

by equation (22).

Here are two examples of structured resolvents of type 3, one on $\mathbb{C}^2$ and one on aninfinite-dimensional space.

Example (4.1.20) [4]: Let

$$\mathcal{H} = \mathbb{C}^2, A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, Y_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, Y_2 = 1 - Y_1, Y = (Y_1, Y_2).$$

Then

$$M(z) = (1 - iA)(A - z_Y)^{-1} (1 + z_Y A)(1 - iA)^{-1}$$
$$= \frac{1}{1 - z_1 z_2} \begin{bmatrix} (1 + z_1)(1 + z_2) & -i(z_1 - z_2) \\ i(z_1 - z_2) & -(1 - z_1)(1 - z_2) \end{bmatrix}$$

Example (4.1.21) [4]: Let $\mathcal{H}$ = $L^2(\mathbb{R})$, let A be the operation of multiplication by the indepen-dent variable t and let Y = P,Q where P, Q are the orthogonal projection operators onto the subspaces of even and odd functions respectively in L^2. Thus

$$Pf(t) = \frac{1}{2}\{f(t) + f(-t)\}, Qf(t) = \frac{1}{2}\{f(t) - f(-t)\}.$$

Let Y'= (Q, P). Note that

$$PA = A\,Q, Q\,A = AP$$

and hence

$$z_{Y'} A = Az_Y, z_Y z_{Y'} = z_1 z_2 = z_{Y'} z_Y.$$

It follows that z_Y and $z_{Y'}$ commute with A^2, and it may be checked that

$$(A - z_Y)^{-1} = (A^2 - z_1 z_2)^{-1} (z_{Y'} + A) = (z_{Y'} + A)(A^2 - z_1 z_2)^{-1}$$

and hence

$$(A - z_Y)^{-1} (1 + z_Y A) = (A^2 - z_1 z_2)^{-1} ((1 + A^2) z_Y + (1 + z_1 z_2)A).$$

A straightforward calculation now shows that the structured resolvent M(z) of A corresponding to Y is given by

$$(M(z)f)(t)=\frac{\left(\frac{1}{2}(z_1\ +\ z_2\ )(1+\ t^2)+(1+z_1z_2\ )t\right)f(t)+\frac{1}{2}(z_2-\ z_1\ )(1\ -\ it)^2f(-t)}{t^2-z_1z_2}$$

for all z ∈ Π^2, f ∈ $L^2(\mathbb{R})$ and t ∈ ℝ. In particular, we note for future use that if f is an even function,

$$M(z)f)(t)=\frac{t(1+z_1z_2)(1-\mathrm{it})(\mathrm{it}z_1\ +\ z_2)}{t^2-z_1z_2}f(t) \tag{26}$$

The third and last form of structured resolvent that we consider has a 2 × 2 matricial form. As will become clear, this extra complication is needed for the description of the most general type of function in the several-variable Loewner class.

By an orthogonal decomposition of a Hilbert space $\mathcal{H}$ we shall mean an n-tuple $P=(P_1,\ldots,P_n)$ oforthogonal projection operators with pairwise orthogonal ranges such that $\sum_{j=1}^{n}P_j$ is the identity operator.

Proposition (4.1.22) [4]: Let $\mathcal{H}$ be the orthogonal direct sum of Hilbert spaces N, M, let A be a densely defined self-adjoint operator on M with domain $\mathcal{D}$(A) and let P be an orthogonal decomposition of H. For every z ∈ Π^n the operator on H given with respect to the decomposition N⊕M by the matricial formula

$$m(z)=\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z_p\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{1}$$

$$\times\left(z_p\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}^{-1} \tag{27}$$

is a bounded operator defined on all of H, and

$$\|M(z)\|\ \leq(1+\ \sqrt{10}\ \|z\|_1\ )\left(1+\frac{1+\sqrt{2}\ \|z\|_1}{\min_j \mathrm{Im}\, z_j}\right). \tag{28}$$

Proof. Let $z\in\Pi^{\mathrm{n}}$. Let the projection P_j have operator matrix

$$p_j=\begin{bmatrix}X_j & B_j\\ B_j & Y_j\end{bmatrix} \tag{29}$$

with respect to the decomposition H = N⊕M . Then

$$X\ =(X_1,\ldots,X_n)\ =(Y_1,\ldots,Y_n)$$

are positive decompositions of N, M respectively, and

$$B\ =(B_1,\ldots,B_n),\qquad B^*=(B_1^*,\ldots,B_n^*)$$

are n-tuples of contractions summing to 0, from M to N and from N to M respectively. Since the B_j are contractions we have

$$\|z_B\|\ \leq\|\ z\|_1.$$

For any z ∈ $\mathbb{C}^n$,

$$z_P=\begin{bmatrix}z_X & z_B\\ z_{B^*} & z_Y\end{bmatrix} \tag{30}$$

Consider the third and fourth factors in the product on the right hand side; the product of these two factors is well defined as an operator on $\mathcal{H}$ since $(1-iA)^{-1}$ maps M to $\mathcal{D}$(A). It is even a bounded operator, since, by virtue,

$$\left(z_p\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}^{-1}=\begin{bmatrix}iz_x & z_B A(1-iA)^{-1}\\ iz_{B^*} & (1+z_Y A)(1-iA)^{-1}\end{bmatrix} \tag{31}$$

Since

$$\|(A(1-iA))^{-1}\| = \|i(1-(1-iA)^{-1})\| \le 2$$

we can immediately see that the operator (31) is bounded. We can get an estimate of the norm of the operator matrix (31) if we replace each of the four operator entries by an upper bound for its norm. We find that

$$\left\|\left(z_p\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}^{-1}\right\| \le \left\|\begin{bmatrix}\|z\|_1 & 2\|z\|_1\\ \|z\|_1 & 1+2\|z\|_1\end{bmatrix}\right\|$$
$$\le 1+\|z\|_1\left\|\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}\right\| = 1+\sqrt{10}\|z\|_1 \tag{32}$$

Now consider the second factor in the definition (4.1.23) of M(z). We find that

$$\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z_p\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^1=\begin{bmatrix}1 & -z_B\\ 0 & A-z_Y\end{bmatrix}^{-1}\begin{bmatrix}1 & z_B(A-z_Y)^{-1}\\ 0 & (A-z_Y)^{-1}\end{bmatrix} \tag{33}$$

which maps H into N $\oplus$ $\mathcal{D}$(A). Hence the product of the first two factors in the product on the right hand side of equation (27) is

$$\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z_p\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^1\begin{bmatrix}-i & iz_B(A-z_Y)^{-1}\\ 0 & (1-iA)(A-z_Y)^{-1}\end{bmatrix} \tag{34}$$

Since

$$\|(1-iA)(A-z_Y)^{-1}\| = \|(1-iz_Y)(A-z_Y)^{-1}-i\| \le 1+\|1-iz_Y\|\|A-z_Y\|$$
$$\le +\frac{1+\|z\|_1}{\min_j \mathrm{Im}\, z_j}$$

we deduce from equation (34) that

$$\left\|\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-zp\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)^{-1}\right\| \le \left\|\begin{bmatrix}1 & \|z\|_1\|(A-z_Y)^{-1}\|\\ 0 & 1+(1+\|z\|_1)(A-z_Y)^{-1}\end{bmatrix}\right\|$$
$$\le 1+\left\|\begin{bmatrix}0 & \|z\|_1\\ 0 & 1+\|z\|_1\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & \|(A-z_Y)^{-1}\|\end{bmatrix}\right\|$$
$$\le 1+\frac{1+\sqrt{2}\|z\|_1}{\min_j \mathrm{Im}\, z_j} \tag{35}$$

$\min_j Im\, z_j$

On combining the estimates (35) and (34) we obtain the bound (28) for $\|M(z)\|$.

Definition (4.1.23) [4]: Let $\mathcal{H}$ be the orthogonal direct sum of Hilbert spaces N, M, let A be a densely defined self-adjoint operator on M with domain $\mathcal{D}$(A) and let P be an orthogonal decomposition of $\mathcal{H}$. The structured resolvent of A of type 4 corresponding to P is the operator-valued function M : $\Pi^n \to \mathcal{L}(\mathcal{H})$ g ivenby equation (27).

We shall also refer to M(z) as the matricial resolvent of A with respect to P. The important property that Im M(z) ≥ 0 is not at once apparent from the formula (27); as with structured resolvents of type 3, there are alternative formulae from which this nproperty is more easily shown. Once again the alternatives suffer the minor drawback that they give M(z) o nlyon a dense subspace of $\mathcal{H}$.

Proposition (4.1.24) [4]: With the notation of Definition (4.1.23), as operators on N ⊕ D (A),

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & A(1+A^2)^{-1} \end{bmatrix} z_p + \begin{bmatrix} 0 & 0 \\ 0 & (1+A^2)^{-1} \end{bmatrix}\right) \times \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_p\right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} \tag{36}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_p + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_p\right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \tag{37}$$

$$= \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\left(z_p \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \tag{38}$$

for all $z \in \Pi^n$. Moreover, for all $z, w \in \Pi^n$,

$$M(z) - M(w)^* = \begin{bmatrix} -i & 0 \\ 0 & 1-iA \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - w_p^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1} \times (z_p - w_p^*)\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_p\right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1+iA \end{bmatrix} \tag{39}$$

on N ⊕ D (A).

Proof. By Lemma (4.1.14) the operators $(1 + A^2)^{-1}$ and

$$C \overset{\text{def}}{=} Im(1 - iA)^{-1} = A(1 + A^2)^{-1}$$

are self-adjoint contractions defined on all of M. Furthermore,

$$ran(1 + A^2)^{-1} = \mathcal{D}(A^2), \qquad ran\, C \subset \mathcal{D}(A).$$

We claim that, as operators on N ⊕ $\mathcal{D}$ (A),

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)^{-1}\left(z_p \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} z_p + \begin{bmatrix} 0 & 0 \\ 0 & (1 + A^2)^{-1} \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & (1 + A^2)^{-1} \end{bmatrix} z_p\right)^{-1} \tag{40}$$

We have

$$
\begin{aligned}
&\left(z_p\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}z_p\right)\\
&\quad=\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}+z_p\begin{bmatrix}1 & 0\\ 0 & AC\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}z_p-z_p\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}z_p\\
&\quad=\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}+z_p\left(\begin{bmatrix}1 & 0\\ 0 & AC\end{bmatrix}-1\right)+\left(1-\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}\right)z_p\\
&\quad-z_p\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}z_p\\
&\quad=\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}-z_p\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & AC\end{bmatrix}z_p-z_p\begin{bmatrix}0 & 0\\ 0 & C\end{bmatrix}z_p\\
&\quad=\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-z_p\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}z_p+\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}\right)
\end{aligned}
$$

This is an identity between operators on H, in both cases a composition $\mathcal{H}\to \mathrm{N}\oplus\mathcal{D}(\mathrm{A})\to\mathcal{H}$, and moreover the first factor on the left hand side and the second factor on the right hand side are invertible, from $\mathrm{N}\oplus\mathcal{D}(\mathrm{A})$ to $\mathcal{H}$ and from $\mathcal{H}$ to $\mathrm{N}\oplus\mathcal{D}(\mathrm{A})$ respectively. We may pre-and post-multiply appropriately to obtain equation (40), but note that the equation is then only valid as an identity between operators on $\mathrm{N}\oplus\mathrm{D}(\mathrm{A})$.

On combining equations (27) and (40) we deduce that

$$
\begin{aligned}
M(z)=&\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}z_p+\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}\right)\\
&\times\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}z_p\right)^{-1}\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}^{-1}
\end{aligned}
$$

Since

$$
\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1+A^2\end{bmatrix}^{-1}\begin{bmatrix}i & 0\\ 0 & 1+iA\end{bmatrix}
$$

and

$$
\begin{bmatrix}1 & 0\\ 0 & 1+A^2\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}z_p\right)=\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}z_p
$$

we deduce further that

$$
\begin{aligned}
M(z)=&\begin{bmatrix}-i & 0\\ 0 & 1-iA\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & C\end{bmatrix}z_p+\begin{bmatrix}0 & 0\\ 0 & (1+A^2)^{-1}\end{bmatrix}\right)\\
&\times\left(\begin{bmatrix}1 & 0\\ 0 & A\end{bmatrix}-\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}z_p\right)^{-1}\begin{bmatrix}i & 0\\ 0 & 1+iA\end{bmatrix}
\end{aligned}
\tag{41}
$$

which proves equation (36). It is straightforward to verify that

$$\left(\begin{bmatrix}1 & 0\\0 & C\end{bmatrix} z_p + \begin{bmatrix}0 & 0\\0 & (1\,+\,A^2)^{-1}\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} z_p\right)^{-1}$$

$$= \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} z_p + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} z_p\right)^{-1} \tag{42}$$

$$- \begin{bmatrix}0 & 0\\0 & A(1\,+\,A^2)^{-1}\end{bmatrix} \tag{43}$$

Clearly

$$\begin{bmatrix}-i & 0\\0 & 1 - iA\end{bmatrix}\begin{bmatrix}0 & 0\\0 & A(1\,+\,A^2)^{-1}\end{bmatrix}\begin{bmatrix}i & 0\\0 & 1 + iA\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & A\end{bmatrix}$$

and so on suitably pre- and post-multiplying equation (42), we obtain equation (37). To prove equation (38), check first that

$$\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - z_p\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} z_p + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$$

$$= \left(z_p\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} z_p\right)$$

as operators on N $\oplus$ $\mathcal{D}$(A). It follows that

$$\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} z_p + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} z_p\right)^{-1} =$$

$$\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - z_p\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)^{-1}\left(z_p\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$$

as operators from H to N $\oplus$ $\mathcal{D}$ (A). On combining this relation with equation (37) we derive the expression (38) for M(z)|N $\oplus$ $\mathcal{D}$ (A).
We now derive the identity (39). Let

$$D = \begin{bmatrix}i & 0\\0 & 1 + iA\end{bmatrix}$$

and consider z,w $\in \Pi^n$. By equation (36)

$$M(z) = D^{\,*}W(z)D \tag{44}$$

on N $\oplus$ $\mathcal{D}$(A), where

$$W(z) = R(z)S(z)^{-1} - \begin{bmatrix}0 & 0\\0 & A(1\,+\,A^2)^{-1}\end{bmatrix} \tag{45}$$

and

$$R(z)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} z_p + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, S(z) = \begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} z_p$$

We have seen that S(z) is invertible for any z $\in \Pi^n$, so that W(z) is a bounded operator on $\mathcal{H}$. Clearly

$$M(z) \;-\; M(w)^* \;=\; D^*\,(R(z)S(z)^* \;-\; S(w)^{*-1}\,R(w)^*)\,D$$
$$=\; D^*\,S(w)^{*-1}\,(S(w)^*\,R(z) \;-\; R(w)^*\,S(z))\,S(z)^{-1}\,D.$$

Here

$$S(w)^* R(z) - R(w)^* S(z)$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z_p + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - w_p^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \left(w_p^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_p \right)$$
$$= z_p - w_p^*$$

Hence

$$M(z) - M(w)^* = D^* S(w)^{*-1} (z_p - w_p^*) S(z)^{-1} D,$$

which is equation (39).

The next result shows that the matricial resolvent belongs not just to the operator Pick class, but to the smaller operator Loewner class.

Proposition (4.1.25) [4]: With the notation of Definition (4.1.23), there exists an analytic operator valued function $F : \Pi^n \to \mathcal{L}(\mathrm{H})$ such that for all $z, w \in \Pi^n$,

$$M(z) - M(w)^* = F(w)^* (z - \overline{w})_P F(z) \qquad (46)$$

on $\mathcal{H}$.

Proof. The identity (39) shows that such a relation holds on $\mathrm{N} \oplus \mathcal{D}(\mathrm{A})$; we must extend it to all of $\mathcal{H}$. Write P_j as an operator matrix with respect to the decomposition $\mathcal{H} = \mathrm{N} \oplus \mathrm{M}$, as in equation (30). Then z has the matricial expression (29). For $z \in \Pi^n$

$$F^{\#}(z) = \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} z_p \right)^{-1} \begin{bmatrix} i & 0 \\ 0 & 1 + iA \end{bmatrix}$$

Then $F^{\#}(z)$ is an operator from $\mathrm{N} \oplus \mathcal{D}(\mathrm{A})$ to $\mathcal{H}$, and we find that

$$F^{\#}(z) = \begin{bmatrix} 1 & 0 \\ -z_{B^*} & A - z_Y \end{bmatrix}^{-1} \begin{bmatrix} i & 0 \\ 0 & 1 + iA \end{bmatrix}$$
$$= \begin{bmatrix} i & 0 \\ i(A - z_Y)^{-1} z_{B^*} & i + (A - z_Y)^{-1}(1 + iz_Y) \end{bmatrix} : N \oplus \mathcal{D}(A) \to \mathcal{H}.$$

Let

$$F(z) = \begin{bmatrix} i & 0 \\ i(A - z_Y)^{-1} z_{B^*} & i + (A - z_Y)^{-1}(1 + iz_Y) \end{bmatrix} : N \oplus M \to \mathcal{H}. \qquad (47)$$

Since

$$(A - z_Y)^{-1} (1 + iA) = i + (A - z_Y)^{-1} (1 + iz_Y)$$

on $\mathrm{N} \oplus \mathcal{D}(\mathrm{A})$ andthe right hand side of the last equation is a bounded operator on all of $\mathcal{H}$, it is clear that, for every $z \in \Pi^n$, F(z) isa continuous extension to $\mathcal{H}$ of F (z) and is a bounded operator. Furthermore F is analytic on Π^n.

By Proposition (4.1.25), equation (39), the relation (46) holds on the dense subspace $\mathrm{N} \oplus \mathcal{D}(\mathrm{A})$ of $\mathcal{H}$ for every $z, w \in \Pi^n$. Since the operators on both sides of equation (46) are continuous on $\mathcal{H}$, the equation holds throughout $\mathcal{H}$.

Corollary (4.1.26)[4]: A matricial resolvent has a non-negative imaginary part at every point of Π^n.

Proof: In the notation, on choosing $w = z$ in and dividing by 2i we obtain the relation

$$\text{Im } M(z) = F(z)^* \, (Im\, z_P\,) F(z)$$

on $\mathcal{H}$. We have

$$\text{Im } z_P \; = \sum_j (Im\, z_j) P_j \; \geq \; 0 \; ,$$

and so Im M(z) ≥ 0 on $\mathcal{H}$ for all z ∈ Π^n.

Here is a concrete example of a matricial resolvent.

Example (4.1.27)[4]: The function

$$M(z) = \frac{1}{z_1 + z_2} \begin{bmatrix} 2z_1 z_2 & i(z_1 - z_2) \\ -i(z_1 - z_2) & -2 \end{bmatrix} \tag{48}$$

is the matricial resolvent corresponding to

$$\mathcal{H} \; = \; \mathbb{C}^2 \, , N \; = \; M \; = \; \mathbb{C}, A \; = 0 \; on \; \mathbb{C}, \qquad p_1 \; = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, p_2 = 1 - p_1.$$

We derive a multivariable analog of the most general form of Nevanlinna representation for functions in the one-variable Pick class Theorem (4.1.2). We start with a multivariable Herglotz. We shall say that an analytic operator-valued function F on $\mathbb{D}^n$ is a Herglotz function if Re F(λ) ≥ 0 for all λ ∈ $\mathbb{D}^n$. For present purposes we need the following modification of the notion.

Definition (4.1.28)[4]: An analytic function F : $\mathbb{D}^n \to \mathcal{L}(K)$, where K is a Hilbert space, is a strong Herglotz function if, for every commuting n-tuple T = ($T_1, \ldots, T_n$) of operators on a Hilbert space and for 0 ≤ r < 1, Re F(rT) ≥ 0.

These functions were called F_n-Herglotz functions. The class of strong Herglotz functions has also been called the Herglotz–Agler class. It is clear that every strong Herglotz function is a Herglotz function, and in the cases n = 1 and 2 the converse is also true.

Theorem (4.1.29)[4]: Let K be a Hilbert space and let $F: \mathcal{D}^2 \to \mathcal{L}(K)$ be a strong Herglotz function such that F(0) = 1. There exist a Hilbert space $\mathcal{H}$, an orthogonal decomposition P of $\mathcal{H}$, an isometric linear operator V : K ⟶ $\mathcal{H}$ and a unitary operator U on $\mathcal{H}$ such that, for all λ ∈ $\mathcal{D}^n$,

$$F(\lambda) = V^* \frac{1 + U\lambda P}{1 - U\lambda P} \tag{49}$$

Conversely, every function $F: \mathcal{D}^n \to \mathcal{L}\,(K)$ expressible in the form (49) for some $\mathcal{H}$, P, V and U with the stated properties is a strong Herglotz function and satisfies F(0) = 1. Note that $\lambda_P = \sum_j \lambda_j P_j$ has operator norm at most $\lambda < 1$ for $\lambda \in \mathcal{D}^n$, and hence equation (49) does define F as an analytic operator-valued function on $\mathcal{D}^n$.

On specializing to scalar-valued functions in the n-variable Herglotz class we obtain the following consequence.

Corollary (4.1.30)[4]: Let f be a scalar-valued strong Herglotz function on $\mathcal{D}^n$. There exists a Hilbert space $\mathcal{H}$, a unitaryoperator $\mathcal{L}$ on $\mathcal{H}$, an orthogonal decomposition P of $\mathcal{H}$, a real number a and a vector v $\in \mathcal{H}$ such that, for all $\lambda \in \mathcal{D}^n$,

$$f(\lambda) = -ia + \langle (L - \lambda_p)^{-1} (L + \lambda_p) v, v \rangle . \qquad (50)$$

Conversely, for any $\mathcal{H}$, L, P, a and v with the properties described, equation (50) defines f as an n-variable strong Herglotz function.

Again, the right hand side of equation (50) is an analytic function of $\lambda \in \mathcal{D}^n$ since

$$(L - \lambda_p)^{-1} = L^{-1} (1 - \lambda_p L^{-1})^{-1}$$

is a bounded operator and is analytic in λ.

Definition (4.1.31)[4]: A Nevanlinna representation of type 4 of a function h : $\Pi^n \to C$ consists of an orthogonally decomposed Hilbert space $\mathcal{H} = N \oplus M$, aself-adjoint densely defined operator A on M, an orthogonal decomposition P of $\mathcal{H}$, a realnumber a and a vector v $\in \mathcal{H}$

such that

$$h(z) = a + \langle M(z) v, v \rangle \qquad (51)$$

for all $z \in \Pi^n$, where M(z) isthe structured resolvent of A of type 4 corresponding to P.

We wish to convert Corollary (4.1.30) to a representation theorem for suitable analytic functions on Π^n. The fact that the corollary only applies to strong Herglotz functions

results in representation theorems for a subclass of the Pick class P_n. Recall from the introduction:

Definition (4.1.32)[4]: The Loewner class L_n is the set of analytic functions h on Π^n with the property that there exist n positive semi-definite functions $A_n, \dots, A_n$ on $\Pi^n \times \Pi^n$, analytic in the first argument, such that

$$h(z) - \overline{h(w)} = \sum_{j-1}^{n} (z_j - \overline{w_j}) A_j(z, w)$$

for all z,w $\in \Pi^n$.

A function h on Π^n belongs to $\mathcal{L}$ if and only if it corresponds under conjugation by the Cayley transform to a function in the Schur–Agler class of the polydisc.
Another characterization: $h \in \mathcal{L}_n$ if and only if, for every commuting n-tuple T of bounded operators with strictly positive imaginary parts, h(T) has positive imaginary part.
We can now prove Theorem (4.1.9) from the introduction: a function h defined on Π^n has a Nevanlinna representation of type 4 if and only if $h \in \mathcal{L}_n$.

Theorem (4.1.33)[4]: Let h be a function defined on Π^n. then h has a nevanlinna represent tation of type 4 if and only if $h \in \mathcal{L}_n$

Proof: Let $h \in \mathcal{L}_n$. Define an n-variable Herglotz function $f : \mathbb{D}^n \to \mathbb{C}$ by

$$f(\lambda) = -ih(z) \tag{52}$$

where

$$z_j = i\frac{1+\lambda_j}{1-\lambda_j} \quad for\, j = 1, \dots, n. \tag{53}$$

When $\lambda \in \mathbb{D}^n$ the point z belongs to Π^n, and so $f(\lambda)$ is well defined, and since $\text{Im}\, h(z) \geq 0$ we have $\text{Re}\, f(\lambda) \geq 0$, so that f is indeed a Herglotz function. In fact f is even a strong Herglotz function: since h $\in \mathcal{L}_n$, the function $\varphi \in S_n$ corresponding to h lies in the Schur–Agler class of the polydisc, and so $f = (1+\varphi)/(1-\varphi)$ i s a strong Herglotz function.
By Corollary (4.1.30) there exist a real number a, a Hilbert space $\mathcal{H}$, a vector $v \in \mathcal{H}$, a unitary operator L on $\mathcal{H}$ and an orthogonal decomposition P on $\mathcal{H}$ such that, for all $z \in \Pi^n$,

$$\begin{aligned} h(z) = if(\lambda) &= a + \langle i(L-\lambda)^{-1}(L+\lambda)v, v\rangle \\ &= a + \langle i[L-(z-i)(z+i)^{-1}]^{-1}[L+(z-i)(z+i)^{-1}]v, v\rangle . \end{aligned}$$

Here and in the rest of this section z, λ are identified with the operators z_P , λ_P on $\mathcal{H}$, and in consequence the relation

$$\lambda = \frac{z-i}{z+i}$$

is meaningful and valid.
For $z \in \Pi^n$ let

$$M(z) = i\,(L-\lambda)^{-1}(L+\lambda) = i\left(L - \frac{z-i}{z+i}\right)^{-1}\left(L + \frac{z-i}{z+i}\right). \tag{54}$$

Since L is unitary on $\mathcal{H}$ and $\lambda \in \mathcal{D}^n$, the operator M(z) i sbounded on $\mathcal{H}$ for every $z \in \Pi^n$, by

$$h(z) = a + \langle M(z)v, v\rangle \tag{55}$$

for all $z \in \Pi^2$. Theorem (4.1.33) will follow provided we can show that M(z) is given by equation (27) for a suitable self-adjoint operator A.
Observe that

$$\begin{aligned} M(z) &= i((z+i)L-(z-i))^{-1}((z+i)L+(z-i)) \\ &= i\,(z(L-1)+i(L+1))^{-1}(z(L+1)+i(L-1)) . \end{aligned} \tag{56}$$

We wish to take out a factor $1-L$ from both factors in equation (56), but this may be impossible since $1-L$ can have a nonzero kernel. Accordingly we decompose $\mathcal{H}$ into $N \oplus M$ where $N = \ker(1-L)$, $M = N^{\perp}$. With respect to this decomposition we can write L as an operator matrix

$$L = \begin{bmatrix} 1 & 0 \\ 0 & L_0 \end{bmatrix},$$

where L_0 is unitary and $\ker(1-L_0) = \{0\}$. Substituting into equation (56) we have

$$M(z) = i\left(z\begin{bmatrix}0 & 0\\0 & L_0-1\end{bmatrix} + i\begin{bmatrix}2 & 0\\0 & L_0+1\end{bmatrix}\right)^{-1}\left(z\begin{bmatrix}2 & 0\\0 & L_0+1\end{bmatrix} + i\begin{bmatrix}0 & 0\\0 & L_0-1\end{bmatrix}z\right)$$
$$= \left(-z\begin{bmatrix}0 & 0\\0 & 1-L_0\end{bmatrix} + \begin{bmatrix}2_i & 0\\0 & i(1+L_0)\end{bmatrix}\right)^{-1}\left(z\begin{bmatrix}2i & 0\\0 & i(1+L_0)\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1-L_0\end{bmatrix}\right) \qquad (57)$$

Formally we may now write

$$M(z) = \begin{bmatrix}-\frac{1}{2}i & 0\\0 & (1-L_0)^{-1}\end{bmatrix}\left(-z\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & i\frac{1+L_0}{1-L_0}\end{bmatrix}\right)^{-1} \times \left(z\begin{bmatrix}1 & 0\\0 & i\frac{1+L_0}{1-L_0}\end{bmatrix} + \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}2i & 0\\0 & i(1-L_0)\end{bmatrix} \qquad (58)$$

but whereas equation (57) is a relation between bounded operators defined on all of $\mathcal{H}$, equation (58) involves unbounded, partially defined operators and we must verify that the product of operators on the right hand side is meaningful.
Let

$$A = i\,\frac{1+L_0}{1-L_0}.$$

Since L_0 is unitary on M and $\ker(1-L_0) = \{0\}$, the operator A is self-adjoint and densely defined on M. The domain $\mathcal{D}(A)$ of A is the dense subspace $\operatorname{ran}(1-L_0)$ of M. It follows from the definition of A that

$$(1-L_0)^{-1} = \frac{1}{2}(1 - iA), \qquad (59)$$

which is an equation between bijective operators from $\mathcal{D}(A)$ to M. Likewise

$$1+L_0 = -2iA(1 - iA)^{-1} : M \to \mathcal{D}(A) \qquad (60)$$

are bounded operators.
Let us continue the calculation from the first factor on the right hand side of equation (57). Since $\ker(1-L_0) = \{0\}$, the right hand side of the relation

$$-z\begin{bmatrix}0 & 0\\0 & 1-L_0\end{bmatrix} + \begin{bmatrix}2i & 0\\0 & i(1+L_0)\end{bmatrix} = \left(-z\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & A\end{bmatrix}\right)\begin{bmatrix}2i & 0\\0 & 1-L_0\end{bmatrix}$$

comprises a bijective map from $\mathcal{H}$ to $N \oplus \mathcal{D}(A)$ followed by a bijection from $N \oplus \mathcal{D}(A)$ to $\mathcal{H}$ (recall the equation (33)). We may therefore take inverses in the equation to obtain

$$\left(-z\begin{bmatrix}0 & 0\\0 & 1-L_0\end{bmatrix} + \begin{bmatrix}2i & 0\\0 & i(1+L_0)\end{bmatrix}\right)^{-1} = \begin{bmatrix}-\frac{1}{2}i & 0\\0 & (1-L_0)^{-1}\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - z\begin{bmatrix}1 & 0\\0 & A\end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix}-\frac{1}{2}i & 0\\0 & \frac{1}{2}(1-iA)^{-1}\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & A\end{bmatrix} - z\begin{bmatrix}0 & 0\\0 & A\end{bmatrix}\right)^{-1} \qquad (61)$$

as operators on $N \oplus \mathcal{D}(A)$.

Similar reasoning applies to the equation

$$\begin{bmatrix} 2i & 0 \\ 0 & i(1+L_0) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1-L_0 \end{bmatrix} \tag{62}$$

$$= \left(z\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 2i & 0 \\ 0 & 1-L_0 \end{bmatrix} = \left(z\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}(1-iA)^{-1} \end{bmatrix}^{-1};$$

it is valid as an equation between operators on $\mathcal{H}$. The right hand side comprises an operator from $\mathcal{H}$ to N $\oplus$ $\mathcal{D}$(A) followed by an operator from N $\oplus$ $\mathcal{D}$ (A) t o $\mathcal{H}$, and so both sides of the equation denote an operator on $\mathcal{H}$.
On combining equations (57), (61) and (62) we obtain

$$M(z) = \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}(1-iA) \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$$\times \left(z\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}(1-iA) \end{bmatrix}$$

Pre-multiply this equation by 2 and post-multiply by 1 to deduce that M(z) i sindeed the structured resolvent of A of type 4 corresponding to P, as defined in equation (27).
Thus the formula (55) is a Nevanlinna representation of h of type 4.
Conversely, let $h \in \mathcal{L}_n$ have a type 4 representation (51). By Proposition (4.1.25) there exists an analytic operator-valued function F : $\Pi^n \to \mathcal{L}(\mathcal{H})$ s uchthat, for all z,w ∈ Π^n,

$$M(z) - M(w)^* = F(w)^* (z - \bar{w})_p F(z) \tag{63}$$

on H. Hence

$$\begin{aligned} h(z) - \overline{h(w)} &= \langle (M(z) - M(w)^*)v, v \rangle \\ &= \langle F(w)^* (z - \bar{w})_p F(z)v, v \rangle \\ &= \sum_{j=1}^{n} (z_j - \bar{w}_j) A_j(z,w) \end{aligned}$$

for all z,w ∈ Π^n, where

$$A_j(z,w) = \langle P_j F(z)v, F(w)v \rangle .$$

The A are clearly positive semidefinite on Π^n, and hence h belongs to the Loewner class $\mathcal{L}_n$.

Nevanlinna representations of type 4 have the virtue of being general for functions in $\mathcal{L}_n$, but they are undeniably cumbersome. In this section we shall show that there are three simpler representation formulae, corresponding to increasingly stringent growth conditions on h ∈ $\mathcal{L}_n$.
In Nevanlinna's one-variable representation formula of Theorem (4.1.2),

$$h(z) = a + bz + \int \frac{1 + t_z}{t - z} \, d\mu(t), \qquad (64)$$

it may be the case for a particular h ∈ P that the bz term is absent. The analogous situation in two variables is that the space N in a type 4 representation may be zero. Equivalently, in the corresponding Herglotz representation, the unitary operator L does not have 1 as an eigenvalue. This suggests the following notion.

Definition (4.1.34)[4]: A Nevanlinna representation of type 3 of a function h on Π^n consists of a Hilbert space $\mathcal{H}$, aself-adjoint densely defined operator A on $\mathcal{H}$, apositive decomposition Y of $\mathcal{H}$, a realnumber a and a vector v ∈ $\mathcal{H}$ such that, for all z ∈ Π^n,

$$h(z) = a + \langle (1 - iA)(A - z_Y)^{-1} (1 + z_Y A)(1 - iA)^{-1} v, v \rangle . \qquad (65)$$

Thus h has a type 3 representation if $h(z) = a + \langle M(z)v, v \rangle$ where M(z) is the structured resolvent of A of type 3 corresponding to Y .

The authors derived a somewhat simpler representation which can also be regarded as an analog of the case b = 0 of Nevanlinna's one-variable formula (64).

Definition (4.1.35)[4]: A Nevanlinna representation of type 2 of a function h on Π^n consists of a Hilbert space $\mathcal{H}$, aself-adjoint densely defined operator A on $\mathcal{H}$, apositive decomposition Y of $\mathcal{H}$, a realnumber a and a vector α ∈ $\mathcal{H}$ such that, for all z ∈ Π^n

$$h(z) = a + \langle (A - z_Y)^{-1} \alpha, \alpha \rangle. \qquad (66)$$

This means of course that, for all z ∈ Π^n,

$$h(z) = a + \langle M(z)\alpha, \alpha \rangle$$

where M(z) is the structured resolvent of A of type 2 corresponding to Y.

We wish to understand the relationship between type 3 and type 2 representations.

Proposition (4.1.36)[4]: If $h \in P_n$ has a type 2 representation then h has a type 3 representation.

Conversely, if $h \in P_n$ has a type 3 representation as in equation (65) with the additional property that $v \in \mathcal{D}(A)$ then h has a type 2 representation.

Proof. Suppose that $h \in P_n$ has the type 2 representation

$$h(z) = a_0 + \langle (A - z_Y)\, \alpha, \alpha \rangle$$

for some $a_0 \in \mathbb{R}$, positive decomposition Y and α ∈ $\mathcal{H}$. We must show that h has a representation of the form (65) for some a ∈ ℝ and v ∈ $\mathcal{H}$. By Proposition (4.1.17): it suffices to find a ∈ ℝ and v ∈ $\mathcal{D}$(A) such that

$$h(z) = a + \langle (1 - iA)[(A - z_Y)^{-1} - A(1 + A^2)^{-1}] (1 + iA)v, v \rangle$$

for all z ∈ Π^n

To this end, let $C = A(1 + A2)^{-1}$ and let

$$a = a_0 + \langle C\alpha, \alpha \rangle . \qquad (67)$$

Since 1 + iA is invertible on $\mathcal{H}$ and ran $(1 + iA)^{-1} \subset \mathcal{D}$(A) we may define

$$v = (1 + iA)^{-1} \alpha \in \mathcal{D}(A). \qquad (68)$$

Then

$$h(z) = a_0 + \langle (A - z_Y)^{-1}\alpha, \alpha\rangle = a - \langle\langle C\alpha, \alpha\rangle + (A - z_Y)^{-1}\alpha, \alpha\rangle$$
$$= a + \langle \{(A - z_Y)^{-1} - C\}(1 + iA)v, (1 + iA)v\rangle$$
$$= a + \langle (1 - iA)\{(A - z_Y)^{-1} - C\}(1 + iA)v, v\rangle$$

as required. Thus h has a type 3 representation.
Conversely, let h have a type 3 representation (65) such that v $\in \mathcal{D}$(A), that is

$$h(z) = a + \langle M(z)v, v\rangle$$

where a $\in \mathbb{R}$ and M is the structured resolvent of A of type 3 corresponding to Y. Since v $\in \mathcal{D}$(A) w emay define the vector $\alpha \overset{\text{def}}{=} (1 + iA)v \in \mathcal{H}$, and furthermore, by Proposition (4.1.17),

$$h(z) = a + \langle (1 - iA)\{(A - z_Y)^{-1} - C\}(1 + iA)v, v\rangle$$
$$= a + \langle \{(A - z_Y)^{-1} - C\}\alpha, \alpha\rangle = a - \langle C\alpha, \alpha\rangle + \langle (A - z_Y)^{-1}\alpha, \alpha\rangle$$
$$= a_0 + \langle (A - z_Y)^{-1}\alpha, \alpha\rangle,$$

where $a_0 \in \mathbb{R}$ is given by equation (67). Thus h has a representation of type 2.

A special case of a type 2 representation occurs when the constant term a in equation (66) is 0. In one variable, this corresponds to Nevanlinna's characterization of the Cauchy transforms of positive finite measures on R. Accordingly we define a type 1 representation of $h \in \mathcal{L}_n$ to be the special case of a type 2 representation of h in which $a = 0$ in equation (66).

Definition (4.1.37)[4]: An analytic function h on Π^n has a Nevanlinna representation of type 1 if there exist a Hilbert space $\mathcal{H}$, a densely defined self-adjoint operator A on $\mathcal{H}$, a positive decomposition Y of $\mathcal{H}$ and a vector $\alpha \in$ H such that, for all z $\in \Pi^n$,

$$h(z) = \langle (A - z_Y)^{-1}\alpha, \alpha\rangle. \qquad (69)$$

A representation of type 1 is obviously a representation of type 2. The following proposition is an immediate corollary of Proposition (4.1.36).

Proposition (4.1.38)[4]: A function h $\in \mathcal{L}_n$ has a type 1 representation if and only if h has a type 3 representation as in equation (65) with the additional properties that v $\in \mathcal{D}$(A) and

$$a - \langle A(1 + A^2)^{-1}\alpha, \alpha\rangle = 0.$$

For consistency with our earlier terminology for structured resolvents and represen-tations we should have to define a structured resolvent of type 1 to be the same as a structured resolvent of type 2. We refrain from making such a confusing definition.
We conclude by giving examples of the four types of Nevanlinna repre-sentation in two variables.

Example (4.1.39)[4]: The formula

(i) $$h(z) = -\frac{1}{z_1+z_2} = \langle (0 - z_Y)^{-1} v, v\rangle_C,$$

where Y $= \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v = 1/\sqrt{2}$, exhibits a representation of type 1, with A $= 0$.

(ii) Likewise

$$h(z) = 1 - \frac{1}{z_1 + z_2} = 1 + \langle (0 - z_Y)^{-1} v, v \rangle_C$$

is a representation of type 2.

(iii) Let

$$h(z) = \begin{cases} \frac{1}{1 + z_1 z_2}\left(z_1 - z_2 + \frac{i z_2 (1 + z_1^2)}{\sqrt{z_1 z_2}}\right) if\ z_1 z_2 \neq -1 \\ \frac{1}{2}(z_1 + z_2) \qquad\qquad if\ z_1 z_2 = -1 \end{cases} \tag{70}$$

where we take the branch of the square root that is analytic in $\mathbb{C} \setminus [0, \infty)$ w ith range Π. We claim that $h \in P_2$ and that h has the type 3 representation

$$h(z) = \langle M(z) v, v \rangle_{L^2(\mathbb{R})} , \tag{71}$$

where M(z) i sthe structured resolvent of type 3 given in Example (4.1.21) and $v(t) = 1/\sqrt{\pi(1 + t^2)}$. To see this, let h be temporarily defined by equation (71). Since v is an even function in $L^2(R)$, equation (26) tells us that

$$h(z) = \int_{-\infty}^{\infty} \frac{t(1 + z_1 z_2) + (1 - it)(itz_1 + z_2)}{\pi(t^2 - z_1 z_2)(1 + t^2)} dt.$$

Since the denominator is an even function of t, the integrals of all the odd powers of t in the numerator vanish, and we have, provided $z_1 z_2 \neq -1$,

$$h(z) = \frac{2}{\pi} \int_0^{\infty} \frac{z_2 + t^2 z_1}{(t^2 - z_1 z_2)(1 + t^2)} dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{z_2 (1 + z_1^2)}{1 - z_1 z_2} \frac{1}{t^2 - z_1 z_2} + \frac{z_1 - z_2}{1 + z_1 z_2} \frac{1}{1 + t^2} dt$$

Now, for $w \in \Pi$,

$$\int_0^{\infty} \frac{dt}{t^2 - w^2} = \frac{i\pi}{2w}$$

and so we find that h is indeed given by equation (70) in the case that $z_1 z_2 \neq -1$. When $z_1 z_2 = -1$ we have

$$h(z) \frac{2}{\pi} \int_0^{\infty} \frac{z_2 + z_1 t^2}{(1 + t^2)^2} dt = \frac{1}{2}(z_1 z_2)$$

Thus equation (71) is a type 3 representation of the function h given by equation (70). This function is constant and equal to i on the diagonal $z_1 = z_2$.

(v) The function

$$h(z)=\frac{z_1z_2}{z_1+z_2}=-\left(-\frac{1}{z_1}-\frac{1}{z_2}\right)^{-1}$$

clearly belongs to P_2. It has the representation of type 4

$$h(z)=\langle M(z)v,v\rangle_{\mathbb{C}^2}$$

where M(z) i sthe matricial resolvent given in Example (4.1.27) and

$$v=\frac{1}{\sqrt{2}}\binom{1}{0}$$

We claim that each of the above representations is of the simplest available type for the function in question; for example, the function h in part (v) does not have a Nevan- Linna representation of type 3. To prove this claim.

Section (4.2): Asymptotic Behavior and Types of Representations

We shall give function-theoretic conditions for a function in $\mathcal{L}_n$ to have a representation of a given type. These conditions will be in terms of the asymptotic behavior of the function at ∞.

Every function in $\mathcal{L}_n$ has a type 4 representation, by Theorem(4.1.9). Let us characterize the functions that possess a type 3 representation. We denote by χ the vector (1,...,1) of ones in $\mathbb{C}^n$. The following statement contains Theorem(4.1.8).

Theorem (4.2.1)[4]. The following three conditions are equivalent for a function $h\in\mathcal{L}_n$.

(i) The function h has a Nevanlinna representation of type 3;

(ii) $$\lim_{n\to\infty}\frac{1}{s}\,Im\,h(is\chi)=0; \qquad (72)$$

(iii) $$\lim_{n\to\infty}\frac{1}{s}\,Im\,h(is\chi)=0; \qquad (73)$$

Proof. (i)⇒(iii) Suppose that h has a Nevanlinna representation of type 3:

$$h(z)=a+\langle(1-iA)(A-z_Y)^{-1}(1+z_YA)(1-iA)^{-1}v,v\rangle \qquad (74)$$

for suitable $a\in\mathbb{R}$, $\mathcal{H}$, A, Y and v $\in\mathcal{H}$. Since

$$(is\chi)_Y=\sum_i is\,Y_j=is$$

we have

$$h(is\chi)=a+\langle(1-iA)(A-is)^{-1}(1+isA)(1-iA)^{-1}v,v\rangle\,.$$

Let v be the scalar spectral measure for A corresponding to the vector v $\in\mathcal{H}$. By the Spectral Theorem

$$h(is\chi) = a + \int (1 - it)(t - is)^{-1} (1 + ist)(1 - it)^{-1} \, d\nu(t)$$
$$= a + \int \frac{1 + ist}{\mathrm{t} - is} \, d\nu(t).$$

Since

$$\mathrm{Im} \frac{1 + \mathrm{ist}}{\mathrm{t} - \mathrm{is}} = \frac{\mathrm{s}(1 + \mathrm{t}^2)}{\mathrm{s}^2 + \mathrm{t}^2} \ ,$$

we have

$$\frac{1}{s} \mathrm{Im}\, h(is\chi) = \int \frac{1 + \mathrm{t}^2}{\mathrm{s}^2 - \mathrm{t}^2} \, d\nu(t).$$

The integrand decreases monotonically to 0 as $s \to \infty$ and so, by the Monotone Convergence Theorem, equation (73) holds.

(iii)⇒(ii) is trivial.

(ii)⇒(i) Now suppose that $h \in \mathcal{L}_n$ and

$$\liminf_{s \to \infty} \frac{1}{s} \, Im\, h(is\chi) = 0$$

By Theorem (4.1.9), h has a Nevanlinna representation of type 4: that is, there exist a, $\mathcal{H}$, $N \subset \mathcal{H}$, operators A, Y on $N^{\perp}$ and a vector v ∈ $\mathcal{H}$ with the properties described in Definition (4.1.34) such that

$$h(z) = a + \langle M(z)v, v\rangle$$

for all z ∈ Π^n, where

$$M(z) = \begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_P \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \times \left(z_P \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix}^{-1} \tag{75}$$

Thus, for s > 0, since once again $(is\chi)_P = is$,

$$M(is\chi) = \begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (A - is)^{-1} \end{bmatrix} \begin{bmatrix} is & 0 \\ 0 & 1 + isA \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & (1 - iA)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} is & 0 \\ 0 & (1 - iA)(A - is)^{-1}(1 + isA)(1 - iA)^{-1} \end{bmatrix}$$

Let the projections of v onto N, $N^{\perp}$ be v_1, v_2 respectively. Then

$$h(is\chi) = a + \langle M(is\chi)v, v\rangle$$
$$= a + is\, \|v_1\|^2 + \langle (1 - iA)(A - is)^{-1} (1 + isA)(1 - iA)^{-1} v_2, v_2\rangle$$

and therefore

$$\frac{1}{2} Im\, h(is\chi) = \|v_1\|^2 + \frac{1}{2} Im \langle (1 - iA)(A - is)^{-1} (1 + isA)(1 - iA)^{-1} \geq v_2, v_2\rangle$$
$$\geq \|v_1\|^2$$

Hence

$$0 = \liminf_{s\to\infty} \frac{1}{s} \ \operatorname{Im} h(is\chi) \geq \|v_1\|^2$$

It follows that $v_1 = 0$.

Let the compression of the projection P_j to $N^{\perp}$ be Y_j : then $Y = (Y_1, \ldots, Y_n)$ is a positive decomposition of $N^{\perp}$, and the compression of z_p to $N^{\perp}$ is z_Y. The $(2, 2)$ block $M_{22}(z)$ in M(z) i s

$$M_{22}(z) = (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1}.$$

Since $v_1 = 0$ i t follows that

$$h(z) = \langle a + M(z)v, v\rangle = a + \langle M_{22}(z)v_2, v_2\rangle$$
$$= a + \langle (1 - iA)(A - z_Y)^{-1}(1 + z_Y A)(1 - iA)^{-1} v_2, v_2\rangle,$$

which is the desired type 3 representation of h.

It is shown that condition (iii) in the above theorem is also a necessary and sufficient condition that −ih have a Π^n-impedance-conservative realization.

Type 2 representations were characterized by the following theorem in the case of two variables. The following result, which contains Theorem (4.1.7), shows that the result holds generally.

Theorem [4.2.2][4]: The following three conditions are equivalent for a function $h \in \mathcal{L}_n$.

(i) The function h has a Nevanlinna representation of type 2;

(ii) $$\liminf_{s\to\infty} \ s \operatorname{Im} h(is\chi) < \infty; \tag{76}$$

(iii) $$\lim_{n\to\infty} s \ \operatorname{Im} h(is\chi) < \infty. \tag{77}$$

Proof. (i)⇒(iii) Suppose that h has the type 2 representation $h(z) = a + \langle (A - z)^{-1} 1v, v\rangle$ for a suitable real a, self-adjoint A, positive decomposition Y and vector v. Let ν be the scalar spectral measure for A corresponding to the vector v. Then, for s > 0, A−(isχ)Y = A − is and so

$$s \operatorname{Im} h(is\chi) = s \operatorname{Im} \int \frac{d\nu(t)}{t - is} = \int \frac{s^2 d\nu(t)}{t^2 + s^2}$$

The integrand is positive and increases monotonically to 1 as $s \to \infty$. Hence, by the Dominated Convergence Theorem

$$\lim_{s\to\infty} s \ \ Im\, h(is\chi) = (\mathbb{R}) = \|v\|^2 < \infty.$$

Hence (ii)⇒(iii).

(iii)⇒(ii) is trivial.

(ii)⇒(i) Suppose (ii) holds. A fortiori,

$$\lim_{s\to\infty} \frac{1}{s} Im\, h(isx) = 0$$

by Theorem (4.2.1) h has a type 3 for suitable a ∈ ℝ, $\mathcal{H}$, A, Y and v ∈ $\mathcal{H}$. Let ν be the scalar spectral measure for A corresponding to the vector v. Then for s > 0

$$s\, Im\, h(is\chi) = s\, Im \int \frac{1 + ist}{t - is} d\nu(t) = \int \frac{s^2(1 + t^2)}{t^2 + s^2} d\nu(t)$$

As s $\rightarrow \infty$ the integrand increases monotonically to $1 + t^2$. Condition (ii) now implies that

$$\int (1 + t^2)\, dv(t) < \infty.$$

It follows that v $\in \mathcal{D}$(A). Hence, h has a representation of type 2.

We proved Theorem (4.2.2) for $n = 2$ usinga different approach from the present one

From this theorem the characterization of type1 representations follows just as in the one-variable case. We obtain a strengthening of Theorem (4.1.6).

Theorem (4.2.3)[4]: The following three conditions are equivalent for a function $h \in \mathcal{L}_n$.

(i) The function h has a Nevanlinna representation of type 1;
(ii) $\liminf_{s\to\infty} s\,|h(is\chi)| < \infty$;

(iii) $\lim_{s\to\infty} s\,|h(is\chi)| < \infty$; (78)

Proof: We follow Lax's treatment of the one-variable Nevanlinna theorem.
(i)⇒(iii) Suppose that h has a type 1 representation as in equation (69) for some $\mathcal{H}$, A, Y and v. Then

$$h(is\chi) = \langle (A - is)^{-1}\alpha, \alpha\rangle = \langle (A + is)(A^2 + s^2)^{-1}\alpha, \alpha\rangle,$$

and so

$$Re\, sh(is\chi) = \langle sA(A^2 + s^2)^{-1}\alpha, \alpha\rangle, Im\, sh(is\chi) = \langle s^2(A^2 + s^2)^{-1}\alpha, \alpha\rangle.$$

Let ν be the scalar spectral measure for A corresponding to the vector $\alpha \in \mathcal{H}$. Then

$$Re\, sh(is\chi) = \int \frac{st}{t^2 + s^2}\, dv(t), Im\, sh(is\chi) = \int \frac{s^2}{t^2 + s^2}\, dv(t).$$

The integrand in the first integral tends pointwise in t to 0 as $s \rightarrow \infty$, and by the inequality of the means it is no greater than 1/2 ; thus the integral tends to 0 as $s \rightarrow \infty$ by the Dominated Convergence Theorem. The integrand in the second integral increases monotonically to 1 as $s \rightarrow \infty$. Thus

$$Re\, sh(is\chi) \rightarrow 0, Im\, sh(is\chi) \rightarrow \|\alpha\|^2 \; as\; s \rightarrow \infty.$$

Hence the inequality (78) holds. Thus (i)⇒(iii).
(iii)⇒(ii) is trivial.
(ii)⇒(i) Suppose that

$$\liminf_{s\to\infty} s\,|h(is\chi)| < \infty;$$

$$\liminf_{s\to\infty} s\, Im\, h(is\chi) \le \liminf_{s\to\infty} s\,|h(is\chi)| < \infty,$$

h satisfies condition (76) of Theorem (4.2.2). Therefore h has a representation of type 2, say

$$h(z) = a + \langle (A - zY)^{-1}\alpha, \alpha\rangle.$$

It remains to show that $a = 0$. The inequality (78) implies that there exists a sequence (s_j) tending to ∞ such that $h(is_j\chi) \to 0$. But

$$Re\, h(is_j\chi) = a + \langle A(A^2 + s_n^2)^{-1}\alpha, \alpha\rangle \to a.$$

Hence $a = 0$ and h has a type 1 representation. This establishes (ii)⇒(i).

How can we recognize from function-theoretic properties whether a given function in the n-variable Loewner class admits a Nevanlinna representation of a given type? In the preceding section it was shown that it depends on growth along a single ray through the origin. In this section we describe the notion of carapoints at infinity for a function in the Pick class, and in the next section we shall give succinct criteria for the four types in the language of carapoints.

Carapoints (though not with this nomenclature) were first introduced by Carathéodory for a function φ on the unit disc, as a hypothesis in the "Julia–Carathéodory Lemma". For any $\tau \in T$, a function φ in the Schur class satisfies the Carathéodory condition at τ if

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - |\lambda|} < \infty. \tag{79}$$

The notion has been generalized to other domains by many authors. Consider domains $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ and an analytic function φ from U to the closure of V.The function φ is said to satisfy Carathéodory's condition at $\tau \in \partial U$ if

$$\liminf_{\lambda \to \tau} \frac{\mathrm{dist}(\varphi(\lambda), \partial V)}{\mathrm{dist}(\lambda, \partial U)} < \infty. \tag{80}$$

Thus, for example, when $U = \Pi^n$, $V = \Pi$, a function $h \in P_n$ satisfies Carathéodory's condition at the point $x \in \mathbb{R}^n$ if

$$\liminf_{z \to x} \frac{\mathrm{Im}\, h(z)}{min_j \mathrm{Im}\, z_j} < \infty. \tag{81}$$

This definition works well for finite points in ∂U, but for our present purpose we need to consider points at infinity in the boundaries of Π^n and Π. We shall introduce a variant of Carathéodory's condition for the class P_n with the aid of the Cayley transform

$$z = i\frac{1+\lambda}{1-\lambda}, \quad \lambda = \frac{z-i}{z+i}, \tag{82}$$

which furnishes a conformal map between $\mathbb{D}$ and Π, and hence a biholomorphic map between $\mathbb{D}^n$ and Π^n by coordinatewise action. We obtain a one-to-one correspondence between $S_n \setminus \{1\}$ and P_n via the formulae

$$h(z) = i\frac{1+\varphi(\lambda)}{1-\varphi(\lambda)}, \quad \varphi(\lambda) = \frac{h(z)-i}{h(z)+i} \tag{83}$$

where 1 is the constant function equal to 1and λ, z are related by (82). For $\varphi \in S_n$ we define $\tau \in \mathbb{T}^n$ to be a carapoint of φ if

$$\liminf_{\lambda \to \tau} \frac{1 - |\varphi(\lambda)|}{1 - \|\lambda\|_\infty} < \infty. \tag{84}$$

We can now extend the notion of carapoints to points at infinity. The point $(\infty,\dots,\infty)$ in the boundary of Π^n corresponds to the point χ in the closed unit polydisc; as in the last section, χ denotes the point $(1,\dots,1) \in \mathbb{C}^n$.

Definition (4.2.4)[4]: Let h be a function in the Pick class P_n with associated function φ in the Schur class S_n given by equation (83). Let $\tau \in \mathbb{T}^n$, $x \in (\mathbb{R} \cup \infty)^n$ be related by

$$x_j = i\frac{1+\tau_j}{1-\tau_j} \; for\; j = 1,\dots,n. \tag{85}$$

We say that x is a carapoint for h if τ is a carapoint for φ. We say that h has a carapoint at ∞ if h has a carapoint at $(\infty,\dots,\infty)$, that is, if φ has a carapoint at χ.
Note that, for a point $x \in \mathbb{R}^n$, to say that x is a carapoint of h is not the same as saying that h satisfies the Carathéodory condition (81) at x. Consider the function $h(z) = -1/z$ in P_n . Clearly h does not satisfy Carathéodory's condition at $0 \in \mathbb{R}^n$.
However, the function φ in S_n corresponding to h is $\varphi(\lambda) = -\lambda_1$, which does have a carapoint at $-\chi$, the point in $\mathbb{T}^n$ corresponding to $0 \in \mathbb{R}^n$. Hence h has a carapoint at 0.
We shall be mainly concerned with carapoints at 0 and ∞. The following observation will help us identify them. For any $h \in P_n$ we define $h^b \in P$ by

$$h^b(z) = h\left(-\frac{1}{z_1},\dots,\frac{1}{z_n}\right) \; for\; z \in \Pi^n.$$

For $\varphi \in S_n$ we define

$$\varphi^b(\lambda) = \varphi(-\lambda).$$

If h and φ are corresponding functions, as in equations (83), then so are h^b and φ^b .

Proposition (4.2.5)[4]: The following conditions are equivalent for a function $h \in P_n$.

(i) ∞ is a carapoint for h;
(ii) 0 is a carapoint for h^b ;
(iii)

$$\liminf_{y\to 0^+}\frac{\operatorname{Im} h^b(iy\chi)}{y|h^b(iy\chi)+i|^2} < \infty;$$

(v)

$$\liminf_{y\to\infty}\frac{y\operatorname{Im} h(iy\chi)}{y|h(iy\chi)+i|^2} < \infty;$$

Proof. (i)⇔(ii) Since $-\chi \in \mathbb{T}^n$ corresponds under the Cayley transform to $0 \in \mathbb{R}^n$, we have

$$\infty\; is\; a\; carapoint\; of\; h \Leftrightarrow \chi\; is\; a\; carapoint\; of\; \varphi \Leftrightarrow -\chi\; is\; a\; carapoint\; of\; \varphi^b$$
$$\Leftrightarrow 0\; is\; acarapointofh^b .$$

(ii)⇔(iii) A consequence of the n-variable Julia–Carathéodory, is that $\tau \in \mathbb{T}^n$ is a carapoint of $\varphi \in S^n$ if and only if

$$\liminf_{r\to 1^-}\frac{1-|\varphi(r\tau)|}{1-r} < \infty.$$

It follows that

$$0 \text{ is a carapoint for } h^b \Leftrightarrow -\chi \text{ is a carapoint for } \varphi^b \Leftrightarrow \liminf_{r\to 1^-} \frac{1-|\varphi^b(-r\chi)|}{1-r} < \infty.$$

$$\Leftrightarrow \liminf_{r\to 1^-} \frac{1-|\varphi^b(-r,-r)|^2}{1-r^2} < \infty.$$

Let $i, y \in \Pi$ be the Cayley transform of $-r \in (-1,0)$, so that $y \to 0^+$ as $r \to 1^-$. In view of the identity

$$\frac{1-|\varphi(\lambda)|^2}{1-\|\lambda\|_\infty^2} = \left(\max_j \frac{\left|z_j+i\right|^2}{\operatorname{Im} z_j}\right)\frac{\operatorname{Im} h(z)}{|h(z)+i|^2} \tag{86}$$

we have

$$0 \text{ is a carapoint for} h^b \Leftrightarrow \liminf_{y\to 0+} \frac{|iy+i|^2}{y}\frac{\operatorname{Im} h^b(iy\chi)}{|h^b(iy\chi)+i|^2} < \infty$$

$$\Leftrightarrow \liminf_{y\to 0^+} \frac{\operatorname{Im} h^b(iy\chi)}{y|h^b(iy\chi)+i|^2} < \infty.$$

(iii)⇔(v) Replace y by 1/y.

Corollary $(4.2.6)[4]$: If $f \in P_n$ satisfies Carathéodory's condition

$$\liminf_{z\to x} \frac{Im\, f(z)}{Im\, z} < \infty \tag{87}$$

at $x \in \mathbb{R}^n$ then x is a carapoint for f. If

$$\liminf_{y\to\infty} \mathrm{y} \operatorname{Im} f(iy\chi) < \infty$$

then ∞ is a carapoint for f.

Proof. Let $h = f^b \in P_n$. Clearly $|h^b(z) + i| \geq 1$for all $z \in \Pi^n$. If the condition (87) holds for $x = 0$ then

$$\liminf_{z\to 0} \frac{\operatorname{Im} h^b(z)}{|h^b(z)+i|^2 \min_j \operatorname{Im} z_j} \leq \liminf_{z\to 0} \frac{\operatorname{Im} h^b(z)}{\min_j \operatorname{Im} z_j} < \infty.$$

and hence, by (ii)⇔(iii) of Proposition (4.2.5), 0 is a carapoint for $h^b = f$. The case of a general $x \in \mathbb{R}^n$ follows by translation.

If $h \in P_n$ has a carapoint at $x \in (\mathbb{R} \cup \infty)^n$ then it has a value at x in a natural sense.
If $\varphi \in S_n$ has a carapoint at $\tau \in \mathbb{T}^n$, there exists a unimodular constant $\varphi(\tau)$such that

$$\lim_{\lambda \overset{nt}{\to} \tau} \varphi(\lambda) = \varphi(\tau). \tag{88}$$

Here $\lambda \overset{nt}{\to} \tau$ means that λ tends nontangentially to τ in $\mathbb{R}^n$.

Definition $(4.2.7)[4]$: If $h \in P_n$ has a carapoint at $x \in (\mathbb{R} \cup \infty)^n$ then we define

$$h(x)\begin{cases} \infty & \text{if } \varphi(\tau) = 1 \\ i\dfrac{1+\varphi(\tau)}{1-\varphi(\tau)} & \text{if } \varphi(\tau) \neq 1 \end{cases}$$

where $\tau \in \mathbb{T}^n$ corresponds to x as in equation (85).

Thus $h(\infty) \in \mathbb{R} \cup \{\infty\}$when ∞ is a carapoint of h.

In the example $h(z) = -1/z_1$, since the value of $\varphi(-\lambda)$ at $-\chi$ is 1, we have $h(0) = \infty$. Although the value of $h(\infty)$ is defined in terms of the Schur class function φ, it can be expressed more directly in terms of h.

Proposition $(\mathbf{4.2.8})[\mathbf{4}]$: If ∞ is a carapoint of h then

$$h(\infty) = h^b\,(0) = \lim_{\substack{nt \\ z\to\infty}} h(z). \tag{89}$$

Here we say that $z \overset{nt}{\to} \infty$ if $z \to (\infty,\dots,\infty)$ in the set $\{z \in \Pi^n : (-1/z_1,\dots, -1/z_n) \in S\}$ for some set $S \subset \Pi^n$ that approaches 0 nontangentially, or equivalently, if $z \to (\infty,\dots,\infty)$ in a set on which $\|z\|_\infty \min_j Im\, z_j$ is bounded.

Proof: Clearly

$$h(\infty) = \infty \Leftrightarrow \varphi(\chi) = 1 \Leftrightarrow \varphi^b\,(-\chi) = 1 \Leftrightarrow h^b\,(0) = \infty.$$

Similarly, for $\xi \in \mathbb{R}$,

$$h(\infty) = \xi \Leftrightarrow \varphi(\chi) = \frac{\xi - i}{\xi + i} \Leftrightarrow \varphi^b\,(-\chi) = \frac{\xi - i}{\xi + i} \Leftrightarrow h^b\,(0) = \xi.$$

Thus, whether $h(\infty)$ is finite or infinite, $h(\infty) = h^b\,(0)$. follows from the.

We shall show that the type of a function $h \in \mathcal{L}_n$ is entirely determined by whether or not ∞ is a carapoint of h and by the value of $h(\infty)$. Let us make precise the notion of the type of a function in $\mathcal{L}_n$.

Definition $(\mathbf{4.2.9})[\mathbf{4}]$**:** A function $h \in \mathcal{L}_n$ is of type 1ifit has a Nevanlinna representation of type 1. For $n = 2$, 3 or 4 we say that h is of type n if h has a Nevanlinna representation of type n but has no representation of type $n - 1$.

Clearly every function in $\mathcal{L}_\mathrm{n}$ is of exactly one of the types 1 to 4. We shall now show Theorem (4.1.11).

Theorem$(\mathbf{4.2.10})[\mathbf{4}]$: A function h defiend on Π^n has a nevanlinna representation or type 2 if and only if $h \in \mathcal{L}_\mathrm{n}$ are $\lim_{y\to\infty} \inf y\, Im\, h(iy,\dots,iy) S_n$ for any function $h \in \mathcal{L}_n$.

(i) h is of type 1 if and only if ∞ is a carapoint of h and $h(\infty) = 0$;
(ii) h is of type 2 if and only if ∞ is a carapoint of h and $h(\infty) \in \mathbb{R}\backslash\{0\}$;
(iii) h is of type 3 if and only if ∞ is not a carapoint of h;
(v) h is of type 4 if and only if ∞ is a carapoint of h and $h(\infty) = \infty$.

Proof. Let $h \in \mathcal{L}_\mathrm{n}$ have a type 2 representation $h(z) = a + \langle (A - z)^{-1} v, v\rangle$ with $a \neq 0$. By Theorem (4.2.2),

$$\liminf_{y\to\infty} \mathrm{y}\, \mathrm{Im}\, h(iy\chi) < \infty$$

By Corollary (4.2.6) , ∞ is a carapoint for h. Furthermore, by Proposition(4.2.8).

$$h(\infty) = \lim_{y\to\infty} h(iy\chi) = a \in \mathbb{R}\backslash\{0\}.$$

Conversely, suppose that ∞ is a carapoint for h and $h(\infty) \in \mathbb{R} \setminus \{0\}$. By Proposition (4.2.5).

$$\lim_{y\to\infty} \frac{y \operatorname{Im} h(iy\chi)}{|h(iy\chi)+i|^2} < \infty$$

while by Proposition (4.2.8)

$$\lim_{y\to\infty} |h(iy\chi)+i|^2 = h(\infty)^2 + 1 \in (1, \infty).$$

On combining these two limits we find that

$$\liminf_{y\to\infty} y \operatorname{Im} h(iy\chi) < \infty,$$

and so, by Theorem (4.2.2), h has a representation of type 2. Since $h(\infty) = 0$ itis clear that h does not have a representation of type 1. Thus (ii) holds.

A trivial modification of the above argument proves that (i) is also true.

(v) Let h be of type 4. Then h has no type 3 representation, and so, by Theorem (4.2.1), there exists $\delta > 0$ and a sequence (s_n) o f positive numbers tending to ∞ such that

$$\frac{1}{s_n} Im\, h(is_n\chi) \geq \delta > 0.$$

Let $y_n = 1/s_n$; then $-1/(is_n) = iy_n$, and we have

$$y_n\, Im\, h^b\, (iy_n\chi) \geq \delta \; for\ all\ n \geq 1. \tag{90}$$

Since $\left|h^b\,(z)+i\right| > Im\, h^b(z)$ f or all z, we have

$$\liminf_{z\to 0} \frac{\operatorname{Im} h^b\,(\mathrm{z})}{|h^b(z)+i|^2 \min_j \operatorname{Im} z_j} \leq \liminf_{z\to 0} \frac{1}{\operatorname{Im} h^b\,(\mathrm{z}) \min_{\mathrm{j}} \operatorname{Im} \mathrm{z_j}} \leq \liminf_{n\to\infty} \frac{1}{\mathrm{y_n} \operatorname{Im} h^b\,(iy_n\chi)}$$
$$\leq 1/\delta.$$

Hence (0, 0) is a carapoint of h^b, and so ∞ is a carapoint of h. Since $y_n \to 0$ it follows from the inequality (4.89) that $\operatorname{Im} h^b(iy_n\chi) \to \infty$, hence that $h^b(0) = \infty$, and therefore that $h(\infty) = \infty$.

Conversely, suppose that ∞ is a carapoint of h and that $h(\infty) = \infty$. We shall show that

$$\liminf_{n\to\infty} \frac{1}{s} Im\, h(is_n\chi) \neq 0 \tag{91}$$

and it will follow from Theorem(4.2.1) that h does not have a representation of type 3, that is, h is of type 4.

Let $\varphi \in S_n$ correspond to h and let $r \in (0, 1)$ correspond to is $\in$ Π. Then

$$\frac{1}{s} Im\, h(is_n\chi) \frac{1-r}{1+r} \quad \frac{1-|\varphi(r\chi)|^2}{1-|\varphi(r\chi)|^2}$$
$$\frac{1-|\varphi(r\chi)|^2}{1-r^2} \quad \frac{(1-r)^2}{1-|\varphi(r\chi)|^2} \tag{92}$$

By hypothesis, χ is a carapoint for φ and $\varphi(\chi) = 1$. By definition of carapoint,

$$\liminf_{z\to x} \frac{1-|\varphi(\mathrm{z})|^2}{1-\|z\|_\infty^2} = \alpha < \infty \text{ for all s} > 0.$$

The n-variable Julia–Carathéodory Lemma now tells us that $\alpha > 0$ and

$$\frac{1-|\varphi(\mathrm{r}\chi)|^2}{|1-r|^2} \le \alpha \frac{1-|\varphi(\mathrm{r}\chi)|^2}{1-r^2} \quad for\ all\ r \in (0,1) \tag{93}$$

On combining equations (92) and (93) we obtain

$$\frac{1}{s} Im\ h(is\chi) \ge \frac{1}{\alpha} > 0\ for\ all\ s > 0.$$

Thus the relation (91) is true, and so, by , h is of type 4.
now follows easily. The function $h \in \mathcal{L}_n$ is of type 3 if and only if it is not of types 1, 2 or 4, hence if and only if it is not the case that ∞ is a carapoint for h and $h(\infty) \in \mathbb{R} \cup \{\infty\}$, hence if and only if ∞ is not a carapoint of h.

We now show that there are functions in the Pick class P_2 of all four types. We return to Example (4.1.39) and show that the functions in P_2 which we presented there are indeed of the stated types.

Example $(\mathbf{4.2.11})[\mathbf{4}]$: The function

$$h(z) - \frac{1}{z_1 + z_2} = \langle \left(0 - z_y\right)^{-1} v, v\rangle_{\mathbb{C}}$$

where $Y = \frac{1}{2}$ and $v = 1/\sqrt{2}$, is obviously of type 1. Let us nevertheless check that ∞ is a carapoint of h and $h(\infty) = 0,$ in accordance with . We have $h(iy, iy) = \frac{1}{2} i/y$ and hence

$$\liminf_{y \to 0+} y\ Im\ h(iy, iy) = \frac{1}{2}.$$

(i) Thus ∞ is a carapoint for h by Proposition (4.2.5). Moreover $h(iy, iy) \to 0\ as\ y \to \infty$, and therefore $h(\infty) = 0$.

(ii) It is immediate that the function $1 + h$, with h as in (i), is of type 2, and that ∞ is a carapoint of $1 + h$ with value 1.

(iii) We have seen that the function

$$h(z) = \begin{cases} \frac{1}{1 + z_1 z_2}\left(z_1 - z_2 + \frac{i z_2 (1 + z_1^2)}{\sqrt{z_1 z_2}}\right) & if\ z_1 z_2 \neq -1 \\ \frac{1}{2}(z_1 z_2) & if\ z_1 z_2 = -1 \end{cases} \tag{94}$$

has a representation of type 3. To show that h is indeed of type 3 we must prove that ∞ is not a carapoint of h.
For all $y > 0$ we have $h(iy, iy) = i$. Hence

$$\liminf_{y \to \infty} \frac{\mathrm{y\ Im}\ h(iy, iy)}{|h(iy, iy) + i|^2} = \liminf_{y \to \infty} \frac{y}{4} = \infty$$

By Proposition (4.2.5), ∞ is not a carapoint for h. Thus h is of type 3.

(v) The function

$$h(z) = \frac{z_1 z_2}{z_1 + z_2} = -1/\left(-\frac{1}{z_1} - \frac{1}{z_2}\right)$$

is clearly in P_2. We gave a type 4 representation of h in Example (4.1.39). We claim that ∞ is a carapoint of h. We have $h(iy, iy) = \frac{1}{2} iy$, and thus

$$\liminf_{y\to\infty} \frac{y \operatorname{Im} h(iy, iy)}{|h(iy, iy) + i|^2} = \liminf_{y\to\infty} \frac{\frac{1}{2} y^2}{\left|\frac{1}{2} iy + i\right|^2} = 2$$

Hence ∞ is a carapoint for h. Furthermore $h(iy, iy) = \frac{1}{2} iy \to \infty$ as $y \to \infty$, and so $h(\infty) = \infty$. Thus h is of type 4.

Another example of a function of type 4 is $h(z) = \sqrt{z_1 z_2}$

The Nevanlinna representation formulae give rise to growth estimates for functions in the n-variable Loewner class. It turns out that growth is mild, both at infinity and close to the real axis. Even though the type of a function is determined by its growth on the single ray $\{iy\chi : y > 0\}$, in turn the growth of the function on the entire polyhalf-plane is constrained by its type.

Consider first the one-variable case. If h is the Cauchy transform of a finite positive measure μ then

$$|h(z)| \le \int \frac{d\mu(t)}{|t - z|} \le \int \frac{d\mu(t)}{\operatorname{Im} z} = \frac{C}{\operatorname{Im} z}$$

for some $C > 0$ and for all $z \in \Pi$. For a general function h in the Pick class, by Nevanlinna's representation there exist $a \in \mathbb{R}, b \ge 0$ anda finite positive measure μ on $\mathbb{R}$ such that, for all $z \in \Pi$,

$$h(z)\ \alpha + bz + \int \frac{1 + tz}{t - z} d\mu(t)$$

$$\alpha + bz + \int \frac{1 + z^2}{t - z} + z\, d\mu(t)$$

and therefore

$$|h(z)| \le |a| + b|z| + \left(\frac{1 + |z|^2}{\operatorname{Im} z} + |z|\right) \mu(\mathbb{R}) \le C\left(1 + |z| + \frac{1 + |z|^2}{\operatorname{Im} z}\right)$$

for some $C > 0$.

Similar estimates hold for the Loewner class.

Proposition $(4.2.12)$[4]: For any function $h \in \mathcal{L}_n$ there exists a non-negative number C such that, for all $z \in \Pi^n$,

$$|h(z)| \leq C\left(1 + \|z\|_1 + \frac{1 + \|z\|_1^2}{\min_j \operatorname{Im} z_j}\right). \tag{95}$$

For any function $h \in \mathcal{L}_n$ of type 2 there exists a non-negative number C such that, for all $z \in \Pi^n$,

$$|h(z)| \leq C\left(1 + \frac{1}{\min_j \operatorname{Im} z_j}\right) \tag{96}$$

For any function $h \in \mathcal{L}_n$ of type 1 there exists a non-negative number C such that, for all $z \in \Pi^n$,

$$|h(z)| \leq \frac{C}{\min_j \operatorname{Im} z_j}. \tag{97}$$

Proof. Let $h \in \mathcal{L}_n$. Let N, M, A, P, and v be as in Theorem (4.1.9)

$$h(z) = a + \langle M(z)v, v\rangle$$

for all $z \in \Pi^n$, where M(z) is the matricial resolvent given. By Proposition (4.1.22) we have, for all $z \in \Pi^n$,

$$\|M(z)\| \leq (1 + \sqrt{10}\,\|z\|_1)\left(1 + \frac{1 + \sqrt{2}\,\|z\|_1}{\min_j \operatorname{Im} z_j}\right)$$
$$\leq 1 + \sqrt{10}\,\|z\|_1 + B\frac{1 + \|z\|_1 + \|z\|_1^2}{\min_j \operatorname{Im} z_j}$$

for a suitable choice of $B \geq 0$. Hence

$$|h(z)| \leq |a| + \|M(z)\|\,\|v\|^2 \leq |a| + \left(1 + \sqrt{10}\,\|z\|_1 + B\frac{1 + \|z\|_1 + \|z\|_1^2}{\min_j \operatorname{Im} z_j}\right)\|v\|^2$$

Since

$$1 + \|z\|_1 + \|z\|_1^2 \leq \frac{3}{2}(1 + \|z_1\|^2),$$

we have

$$|h(z)| \leq C\left(1 + \|z\|_1 + \frac{1 + \|z\|_1^2}{\min_j \operatorname{Im} z_j}\right)$$

for some choice of $C > 0$ and for all $z \in \Pi^n$. Thus the estimate (95) holds.

To conclude the paper we point out that there are structured analogs of the classical resolvent identity

$$(A - z)^{-1} - (A - w)^{-1} = (z - w)(A - z)^{-1}(A - w)^{-1}$$

for any z, w in the resolvent set of an operator A.

Proposition $(4.2.13)$[4]: Let A be a densely defined self-adjoint operator on a Hilbert space $\mathcal{H}$ and let Y be a positive decomposition of $\mathcal{H}$. For all $z, w \in \Pi^n$

$$(A - z_Y)^{-1} - (A - w_Y)^{-1} = (A - z_Y)^{-1}(z - w)_Y\ (A - w_Y)^{-1} \quad (98)$$

If M(z) is the structured resolvent of type 3 corresponding to A and Y then

$$M(z) - M(w)\ |\mathcal{D}(A) = (1 - iA)(A - z_Y)^{-1}(z - w)_Y (A - w_Y)^{-1}(1 + iA). \quad (99)$$

Proof. The first of these identities is immediate.

$$M(z) - M(w)\ |\mathcal{D}(A) = (1 - iA)\ (A - z_Y)^{-1} - (A - w_Y)^{-1}(1 + iA),$$

and the identity (99) follows from equation (98).

Proposition (4.2.14)[4]: Let $\mathcal{H}$ be the orthogonal direct sum of Hilbert spaces N, M, let A be a densely defined self-adjoint operator on M with domain $\mathcal{D}(A)$ and let P be an orthogonal decomposition of $\mathcal{H}$. For every $z, w \in \Pi^n$, as operators on $N \oplus \mathcal{D}(A)$,

$$M(z) - M(w) = \begin{bmatrix} -i & 0 \\ 0 & 1 - iA \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ o & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} (z - w)P$$

$$\times \left(\begin{bmatrix} 1 & 0 \\ o & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} w_p \begin{bmatrix} i & 0 \\ 0 & 1 + iA \end{bmatrix} \quad (100)$$

Proof. Let

$$D = \begin{bmatrix} i & 0 \\ 0 & 1 + iA \end{bmatrix} : N \oplus \mathcal{D}(A) \to \mathcal{H}.$$

By equations (37) and (38) we have

$$M(z) - M(w) | N \oplus \mathcal{D}(A)$$

$$= D^* \left\{ \left(\begin{bmatrix} 1 & 0 \\ o & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \left(z_p \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right.$$

$$\left. - \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_p + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w_p \right)^{-1} \right\} D$$

$$= D^* \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \left\{ \left(z_p \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right.$$

$$\left. - \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w_p \right) - \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - z_p \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w_p + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}$$

$$\times \left(\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} w_p \right)^{-1} D$$

List of Symbols

Symbol		Page
max	Maximum	1
min	Minimum	2
sup	Supremum	2
pos	Positive	9
$\oplus$	Direct sum	12
mono	Monotone	13
Im	Image	13
conv	Convex	16
det	Determinant	18
Aut	Automorphism	27
a.e	Almost every where	28
inf	Infimum	32
L^2	Hilbert space	32
Re	Real	49
ker	Kernel	49
dist	Distance	59

References

[1] Koenraad M. R. Audenaert, Schur multiplier norms for Loewner Matrices, Linear Algebra and its Applications 439(2013), 2598-2608.

[2] Dominique Guillot, Apoorva Khare, Bala Rajaratnam , Complete characterization of Hodamard powers preserving Loewner positivity ,monotonicity, and convexity, Journal of Mathematical Analysis and Applications 425(2015), 489-507.

[3] Mihai Iancu, some application of variation of Loewner chains in several complex variables, Journal of Mathematical Analysis and Applications 421(2015), 1469-1478.

[4] J. Agler[a], R. Tully-Doyle[b], N. J. Young[c,d], Nevanilinna representations in several variables, Journal of Functional Analysis 270(2016), 3000-3046.

[5] F. Hansen and G. K. Pedersen, Jensen's inequality for operators and Loewner's theorem Math.Ann, 258(1982), 229-241.

[6] T.Ando, K. Okubo, Induced norms of the Schur multiplier operator, Linear Algebra Appl. 147(1991) 181-199.

[7] Rejendra Bhatia, Ludwig Elsner. Positivity preserving Hadamard matrix Function. Positivity 11(4)(2007), 583-588.

[8] L.Arosio, Resonances in Loewner equations, Adv. Math 227(2011), 1413-1435.