Chapter 1

Additive Maps on Standard Operator Algebras Preserving InvertibilitiesorZero Divisors

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on infinite dimensional complex Banach spaces $X$and , respectively , and let Φ be an additive surjection from $\mathcal{A}$ onto $\mathcal{B}$. We prove that if Φ is unital and preserves any one of (left, right) invertibility, (left, right) zero divisors (left, right) topological divisors of zero , quasi – affinity , injectivity , surjectivity ,range density , lower –bounded ness and left (right) maximal ideals in both directions, then it has one of the following forms : isomorphism , conjugate isomorphism , anti- isomorphism .

Assume (P) is a property, we say that a map preserves property (P) if, for every T in the domain of Φ, T possesses $(P) \Rightarrow \Phi(\mathrm{T})$ possesses (P);Φ preserves (P) in both directions if T possesses $(P) \Rightarrow \Phi(\mathrm{T})$ possesses (P). Over the past decades, there hasbeen a considerable interest in the study of linear maps on operator algebras that preserve certain properties of operators. Many results having been obtained by now reveal the relation between linear structure and the algebraic structure of operator algebras, and help us to understand the operator algebras better. .

Some have been devoted to characterizing linear maps on operator algebras preserving some properties concerning the invertibility, kernel and range of operators. Let Xand Y be two complex Banach spaces, and $\mathcal{B}(\mathrm{X},\mathrm{Y})(\mathcal{B}(\mathrm{X}) if\ \mathrm{X}=\mathrm{Y})$ be the Banach space of all bounded linear operators from Xinto Y.Sour our proved that a unital linear bijective map preserving invertibility from$\mathcal{B}(\mathrm{X})$onto $\mathcal{B}(\mathrm{Y})$is either an isomorphism or an anti-isomorphism. Aupetit

showed that every unital linear surjection preserving invertibility in both directions between von Neumann algebras is a Jordan isomorphism. We improved the above result of Aupetit by omitting the assumption “in both directions”, and proved that every unital linear surjection preserving invertibility between von Neumann algebras is a Jordan homomorphism. It was shown in that every surjective unital linear map on $\mathcal{B}(\mathrm{X})$preserving injectivity of operators in both directions isan automorphism and every surjective unital linear map on $\mathcal{B}(\mathrm{H})$ preserving surjectivity of operators in both directions is an automorphism, where His a complex Hilbert space. We discussed the linear surjective maps compressing various parts of the spectrum containing the boundary of the spectrum on C^*-algebras $\mathcal{A}$of real rank zero and showed that such linear maps are Jordan homomorphisms. If A is a standard operator sub algebra of $\mathcal{B}(\mathrm{X})$, we also obtained the descriptions of unital linear surjective maps preserving the left invertibility, the right invertibility,the lower-bounded ness or the surjectivity of operators on A, the last result particularly generalizes the result concerning the surjectivity preservers mentioned above by omitting “in both directions” and considering the maps on general standard operator algebras on Banach spaces. Recall that a standard operator algebra on a Banach space Xis a closed subalgebra in $\mathcal{B}$(X)which contains the identity Iand the ideal of all finite rank operators. We show that every unital linear surjection from $\mathcal{B}$(X)onto $\mathcal{B}$(Y)which preserves the quasi-affinity of operators iseither an isomorphism or an anti-isomorphism, and every unital linear surjection from $\mathcal{B}$(X)onto $\mathcal{B}$(Y)which preserves the range density of operators is in fact an isomorphism.

Another interesting problem is to characterize the linear maps which preserve zero products. Semrl showed that every unital surjective linear map on $\mathcal{B}(X)$which preserves zero products in both directions is an automorphism. It was proved that every surjective linear map between standard operator algebras on Banach space is an isomorphism multiplied by a scalar if it preserves zero products in both directions. Similar results were obtained for bounded unital linear maps onnest algebras with atomic nests.A more general (and more difficult) situation would be to consider an algebra only as a ring, and to assume the maps being additive only. In this direction, only a few results concerning the preserver problem have been obtained .Hou and Gaoshowed that every surjective additive map on $\mathcal{B}(H)$preserving zero products in both directions is an automorphism or a conjugate automorphism multiplied by a scalar. The purpose is to discuss the additive maps between standard operator algebras on complex Banach spaces which preserve various properties in both directions concerning the invertibility, kernel and range of operators, zero divisors, by one method for all.

We first fix some notations. We always assume that X and Y are infinite dimensional complex Banach spaces. For $x \in X$ and $f \in X^*$, rank one operator $y \to \langle y, f\rangle_x$ is denoted by $\otimes\ f$, here, $\langle y, f\rangle$ denotes the value of $f\ at\ y$. As usual, $\mathbb{C}$ and $\mathbb{N}$ stand for complex plane and the set of natural numbers, respectively. Let M be a linear subspace of X, the dimension of M is denoted by $dim\ M$.

For$T \in \mathcal{B}(X), \sigma(T), \sigma_p(T), rng(T)\ and\ ker(T)$ denote the spectrum, point spectrum,range and kernel of T, respectively. $rank(T)$ denotes the rank of T which is the dimension of $rng(T)$. A map $\varphi: X \to Y$ is called conjugate linear if it is additive and$\varphi(\lambda x) = \bar{\lambda}\varphi(x)$ holds for all scalars $\lambda \in \mathbb{C}$ and vectors $x \in X$; more generally, $\varphi$is called τ-quasi-linear if it is additive and $\varphi(\lambda x) = \tau(\lambda)\varphi(x)$ holds for all scalars$\lambda \in \mathbb{C}$ and vectors $x \in X$, where τ is a ring automorphism of $\mathbb{C}$.

Let $\mathcal{A}$ be a Banach algebra. Recall that an element $T \in \mathcal{A}$ is called a left (resp.,right) zero divisor if there exists a non zero element $S \in \mathcal{A}$ such that $T S = 0$ (resp.,$ST = 0$). A zero divisor is an element of $\mathcal{A}$ which is both a left and a right zero divisor. We call T a left (resp., right) topological divisor of zero if there exists asequence $\{S_n\}_{n=1}^{\infty} \subset \mathcal{A}$ satisfying $\|S_n\| = 1$ such that $TS_n \to 0$(resp., $S_nT \to 0$).A topological divisor of zero is an element which is both a left and a right topological divisor of zero. We denote by $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}_l^{\mathcal{A}}$, $\mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}}$,$\mathcal{Z}_r^{\mathcal{A}}$,$\mathcal{TZ}^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}}$ and $\mathcal{TZ}_r^{\mathcal{A}}$ the subsets of all non invertible elements, left non invertible elements, right non invertible elements, zero divisors, left zero divisors, right zero divisors, topological divisors of zero, left topological divisors of zero and right topological divisors of zero, respectively, in $\mathcal{A}$. An element in $\mathcal{A}$ is called semi-invertible if it is either left invertible or right invertible. The notions of semi-zero divisor, semi-topological divisors of zero and semi-maximal ideals may be defined similarly.

Assume that $\mathcal{A}$ is a standard operator algebra on a complex Banach space X. Let $\Omega^{\mathcal{A}}$ and $\Theta^{\mathcal{A}}$ be any one of the subsets

$\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}_l^{\mathcal{A}}$, $\mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}}$, $\mathcal{Z}_r^{\mathcal{A}}$, $\mathcal{TZ}^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}}$,$\mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{S}_l^{\mathcal{A}} \cap \mathcal{S}_r^{\mathcal{A}}$,$\mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$ and $\mathcal{TZ}_l^{\mathcal{A}} \cup \mathcal{TZ}_r^{\mathcal{A}}$. We say that $\Theta^{\mathcal{A}}$ is the dual of $\Omega^{\mathcal{A}}$, denoted by $(\Omega^{\mathcal{A}})$, if, replacing $\mathcal{A}$ by $\mathcal{B}(H)$ with H an infinite dimensional complex Hilbert space, we have $T \in \Omega^{\mathcal{B}}(H) \Leftrightarrow T^* \in \Theta^{\mathcal{B}}(H)$holds true for every operator $T \in \mathcal{B}(H)$. It is clear, by the above definition, that $(\Omega^{\mathcal{A}})'' = ((\Omega^{\mathcal{A}})')' = \Omega^{\mathcal{A}}, (\mathcal{S}_l^{\mathcal{A}})' = \mathcal{S}_r^{\mathcal{A}}, (\mathcal{Z}^{\mathcal{A}})' = \mathcal{Z}^{\mathcal{A}}$ and $(\mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}})' = \mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$, etc.For $T \in \mathcal{A}, \sigma^{\mathcal{A}}(T)$ stands for the spectrum of T relative to $\mathcal{A}$. Let $\mathcal{F}_n(X)$ denotethe set of all operators in $\mathcal{B}(X)$ with rank not greater than n and $\mathbb{C}\mathbb{I} + \mathcal{F}_n(X) = \{\alpha I + F \mid \alpha \in \mathbb{C}$ and $F \in \mathcal{F}_n(X)\}$. The following lemma is useful in the sequel. It characterizes the rank one operators in terms of the subsets listed above and the operators in $\mathbb{C}\mathbb{I} + \mathcal{F}_2(X)$.

Lemma (1.1)[1]:-

Let $\mathcal{A}$ be a standard operator algebra on a complex Banach space X and $\Omega^{\mathcal{A}}$ denote any one of the subsets $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}_l^{\mathcal{A}}$, $\mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}}$, $\mathcal{Z}_r^{\mathcal{A}}$, $\mathcal{TZ}^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}}$, $\mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}} \cup \mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{S}_l^{\mathcal{A}} \cap \mathcal{S}_r^{\mathcal{A}}$ and $\mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$ of $\mathcal{A}$. Then, for an operator $A \in \mathcal{A}$, the following conditions are equivalent.

(a) Ahas rank one.

(b) For every $T \in \mathcal{A}$ and every scalar $c \neq 1$, if $T + A$ and $T + cA \in \Omega^{\mathcal{A}}$, then

$$T \in \Omega^{\mathcal{A}}.$$

(b') For every $T \in \mathbb{C}\mathbb{I} + \mathcal{F}_2$ and every scalar $c \neq 1$, if $T + A$ and $T + cA \in \Omega^{\mathcal{A}}$, then $T \in \Omega^{\mathcal{A}}$.

(c) For every $T \in \mathcal{A}$, if $T + A$ and $T + 2A$ $\Omega^{\mathcal{A}}$then $T \in \Omega^{\mathcal{A}}$.

(c') For every $T \in \mathbb{C}\mathbb{I} + \mathcal{F}_2$, if $T + A$ and T + $2A \in \Omega^{\mathcal{A}}$, then $T \in \Omega^{\mathcal{A}}$.

Proof:-

$(b) \Rightarrow (b') \Rightarrow (c')and(b) \Rightarrow (c) \Rightarrow (c')$ are obvious. For the sake of simplicity, we omit the superscript “A” of A in the proof of this lemma. $(a) \Rightarrow (b)$. Write $A = x \otimes f$. Assume, on the contrary, that there exists an element$T \in \mathcal{A}$, a non zero scalar $c \neq 1$ such that $T + A$ and $T + cA \in \Omega$ but $T \notin \Omega$, we will deduce a contradiction.

If $\Omega = \mathcal{S}_l^{\mathcal{A}}$, then T is left invertible and there is an element $S \in \mathcal{A}$ such that $ST = I$. Since $I + SA$ and $I + cSA$ are in $\mathcal{S}_l^{\mathcal{A}}, \{-1, -c^{-1}\} \subseteq \sigma^{\mathcal{A}}(SA) = \sigma(SA)$, this contradicts the fact that the spectrum of any rank one operator can not includestwo non zero points.

The cases $\Omega = \mathcal{S}_r^{\mathcal{A}}$, or $\mathcal{S}^{\mathcal{A}}$ are dealt with similarly, and then, it is clear that $(a) \Rightarrow (b)$ holds true for $\mathcal{S}_l^{\mathcal{A}} \cap \mathcal{S}_r^{\mathcal{A}}$.
If $\otimes = \mathcal{TZ}_l^{\mathcal{A}} \cup \mathcal{TZ}_r^{\mathcal{A}}$, then $T \notin \Omega$ implies T is a bijection and hence invertible as an operator with the inverse T^{-1} which may not belong to $\mathcal{A}$. However, $T^{-1}A \in \mathcal{A}$ is a rank one operator, and the above argument is also valid for this case.If $\Omega = \mathcal{TZ}_l^{\mathcal{A}}$, then T is lower bounded and hence, is injective and has closed range. $T + A$and$T + cA \in \Omega = \mathcal{TZ}_l^{\mathcal{A}}$ imply that there exist unit vector sequences$\{x_n\}$ and $\{u_n\}$ such that $\|(T + A)x_n\| \to 0$ and $\|(T + cA)u_n\| \to 0 \; as \; n \to \infty$. Since, as bounded subsets in $\mathbb{C}$, both $\{\langle x_n, f\rangle\}$ and $\{\langle u_n, f\rangle\}$ have convergent subsequences, without loss of generality, we may assume that$\langle x_n, f\rangle \to a$ and $\langle u_n, f\rangle \to b$ as$n \to \infty$. It follows that $T x_n \to -ax$ and $Tu_n \to -cbx$ as n $\to \infty$. Obviously, both $a \; and \; b$are non zero because $T \notin \mathcal{TZ}_l^{\mathcal{A}}$. Therefore one sees that $x \in rng(T)$. Pick $u \in X$ so that $Tu = x$, then $T(I + u \otimes f) = T + x \otimes f \in \mathcal{TZ}_l^{\mathcal{A}}$ implies that $I + u \otimes f$ is not invertible. Similarly, $I + cu \otimes f$ is not invertible. Thus the spectrum of rank one operator $u \otimes f$contains two distinguished nonzero points, a contradiction.

It is easy to verify from the case $\Omega = \mathcal{TZ}_l^{\mathcal{A}}$ that $(a) \Rightarrow (b)$ is still true for cases that$\Omega = \mathcal{TZ}_l^{\mathcal{A}}$or $TZ = \mathcal{TZ}_l^{\mathcal{A}} \cap \mathcal{TZ}_r^{\mathcal{A}}$.
Let $\Omega = \mathcal{Z}_l^{\mathcal{A}}$. Then T is injective and, as $c \neq 1$, there exist linearly independent vectors $u \; and \; v$ such that $T u = -\langle u, f\rangle x$and $Tv = -c\langle v, f\rangle x$. However, the injectivity of T implies that $u \; and \; v$ are linearly dependent, a contradiction. The cases $\Omega = \mathcal{Z}_l^{\mathcal{A}}$or $TZ = \mathcal{Z}_l^{\mathcal{A}} \cap \mathcal{Z}_r^{\mathcal{A}}$can be treated analogously.

By now the only case remained is $\Omega = \mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$, and it is obvious, by applying what have been proved above, that we need only to consider the case that $T + A \in \mathcal{Z}_l^{\mathcal{A}} \backslash \mathcal{Z}_r^{\mathcal{A}}$ and $T + cA \in \mathcal{Z}_l^{\mathcal{A}} \backslash \mathcal{Z}_r^{\mathcal{A}}$, while $T \notin \mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$. It follows from $T + A \in \mathcal{Z}_l^{\mathcal{A}} \backslash \mathcal{Z}_r^{\mathcal{A}}$ that there exists a vector u such that $Tu = x$. Since $T(I + cu \otimes f) = TcA \in \mathcal{Z}_r^{\mathcal{A}} \backslash \mathcal{Z}_l^{\mathcal{A}}$, $(I + cu \otimes f) \notin \mathcal{Z}_l^{\mathcal{A}}$ and hence is invertible. However, this implies $T + cA \notin \mathcal{Z}_r^{\mathcal{A}}$, a

contradiction.

$(c) \Rightarrow (a)$. Let $\otimes = \mathcal{Z}^{\mathcal{A}}$. Assume that rank$A > 1$, we will prove that condition (c) is not satisfied. Firstly assume that there exists a functional $f \in X^*$ such that $f, A^*f, (A^*)^2 f$ are linearly independent. Take vectors $x_0, x_1, x_2 \in X$ such that $x_i, (A^*)^j f = \delta_{ij}$ (the Kronecker symbol) for $i, j = 0, 1, 2$. Let

$$T = x_0 \otimes (3\|A\|f - A^*f) + x_1 \otimes (3\|A\|A^*f - 2(A^*)^2 f) + x_2 \otimes (3\|A\|f - A^*f) - 3\|A\|I.$$

Then $T \in \mathbb{C}I + \mathcal{F}_2(X)$, $(T^* + A^*)f = 0$and $(T^* + 2A^*)A^*f = 0$. Thus $T + A$and $T + 2A \in \mathcal{Z}_r^{\mathcal{A}}$. Furthermore,$(A - 3\|A\|I)[(A - 3\|A\|I)^{-1}(T + 3\|A\|I) + I] + A \in \mathcal{Z}_r^{\mathcal{A}}$ and since $(A - 3\|A\|I)^{-1}(T + 3\|A\|I)$ is of rank-2, we see that$(A - 3\|A\|I)^{-1}(T + 3\|A\|I) + I \in \mathcal{Z}_l^{\mathcal{A}}$. This implies that $T + A \in \mathcal{Z}_l^{\mathcal{A}}$. Similarly,$T + 2A \in \mathcal{Z}_l^{\mathcal{A}}$.Note that,$\mathcal{Z}^{\mathcal{A}} = \mathcal{Z}_l^{\mathcal{A}} \cap \mathcal{Z}_r^{\mathcal{A}}$. So we get $T + A$ and $T + 2A$ are in $\mathcal{Z}^{\mathcal{A}}$. However, it is easily checked that T is invertible in $\mathcal{A}$ and hence, can not be in$\mathcal{Z}^{\mathcal{A}}$.

Next assume that for any $g \in X^*$, the functionals $g, A^*g, (A^*)^2 g$ are linearly dependent. Then, A^* and consequently A, is an algebraic operator of degree not greater than two . That is, there exists a polynomial $p(t)$ of degree notgreater than two such that $p(A) = 0$. If the degree of $p(t)$is 1, then $A = aI$ for somescalar $a \neq 0$. Pick a vector $y \in Y$ so that $y, g = a$ and let $T = y \otimes g - 2aI$. Then$T$ is invertible, but $T + A$and $T + 2A \in \mathcal{Z}^{\mathcal{A}}$. So, from now on, we always assumethat A is not a scalar multiple of the identity and the degree of $p(t)$ is 2. In this casethere exist scalars $\alpha\ and\ \beta$such that $p(t) = (t - \alpha)(t - \beta)$.

Case (i). $\alpha \neq 0$and $\beta \neq \alpha$. If $\beta = 0$, then, since$rank\ A \geq 2, dim\ ker(A - \alpha I)\ 2$; if $\beta \neq 0$, then A is invertible and, at least one of the subspaces$ker(A - \alpha I)$ and $ker(A - \beta I)$ has dimension greater than 1 since X is of infinite dimension. So, without loss of generality, we may assume that $dim\ ker(A - \alpha I)\ 2$. Thus, there exist closed sub spaces $V_1, V_2\ and\ V_3\ of\ X$ with dim $V_2 = 1$ such that X has the space decomposition $X = V_1 + V_2 + V_3$ and A has the corresponded matrix representation

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha I_2 & 0 \\ 0 & 0 & \beta I_3 \end{pmatrix}.$$

whereI_2 and I_3 are the identities on V_2 and V_3, respectively. Let

$$T = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha I_2 & 0 \\ 0 & 0 & -\beta I_3 \end{pmatrix}.$$

then$T \in \mathbb{C}I + \mathcal{F}_1(X)$ is invertible but both $T + A$ and $T + 2A$ are zero divisors of $\mathcal{A}$.

Case (ii). $\alpha = \beta \neq 0$. Since $rankA > 1$, there exist linearly independent functional $f_1, f_2 \in X^*$ such that $A^* f_1 = \alpha f_1$ and $A^* f_1 = f_1 + \alpha f_2$. Take vectors $x_i \in X$ $(i = 1, 2)$ such that $\langle x_i, f_j \rangle = \delta_{ij}(i, j = 1, 2)$. If $\alpha \neq \pm 1$, let $T = (x_1 - x_2) \otimes f_1 + (\alpha^2 x_1 + x_2) \otimes f_2 - (\alpha + 1)I$. Then $T \in \mathbb{C}I + \mathcal{F}_2(X)$ and T is invertible since $\sigma^{\mathcal{A}}(T) = \{-\alpha - 1, -\alpha \pm i\alpha\}$. Because $(T^* + A^*)f_2 = 0$and $(T^* + 2A^*) \times (f_1 - \alpha f_2) = 0$, we have $T + A$ and $T + 2A \in \mathcal{Z}_r^{\mathcal{A}}$. Note that both $A - (\alpha + 1)I$and $2A - (\alpha + 1)I$ are invertible, this implies that we also have $T + A$ and $T + 2A \in \mathcal{Z}_l^{\mathcal{A}}$ and hence $T + A$and $T + 2A \in \mathcal{Z}^{\mathcal{A}}$. If $\alpha = 1$, let $T = (x_1 - 2x_2) \otimes f_1 + (-x_1 + x_2) \otimes f_2 - 3I$, then $T \in \mathbb{C}I + \mathcal{F}_2(X)$ and $\sigma(T) = \{-3, -2 \pm \sqrt{2}\}$.It is easy to check that $(T^* + A^*)(f_1 - f_2) = 0$ and$(T^* + 2A^*)f_2 = 0$, so we have $T + A$ and $T + 2A \in \mathcal{Z}_r^{\mathcal{A}}$. Since $3I - A$ and $3I - 2A$ are invertible, similar to the above argument, we also have that $T + A$and $T + 2A \in \mathcal{Z}_l^{\mathcal{A}}$. Thus both $T + A$and $T + 2A$ are zero divisors while T is not. If $\alpha = -1$, let $T = (-x_1 - 2x_2) \otimes f_1 + (-x_1 - x_2) \otimes f_2 + 3I$, then $T \in \mathbb{C}I + \mathcal{F}_2(X)$ and $\sigma(T) = \{3, 2 \pm \sqrt{2}\}$. As $(T^* + A^*)(f_1 + f_2) = 0$ and $(T^* + 2A^*)f_2 = 0$, thus $T + A$ and $T + 2A$ are zero divisor, but T is not.

Case (iii). $\alpha = \beta = 0$. Since $rank\, A > 1$, there exist functional, $g \in X^*$ such that $f, A^* f, g, A^* g$ are linearly independent. Let$f_1 = f, f_2 = A^* f, f_3 = g$ and $f_4 = A^* g$.Take vectors $x_i \in X$ $(i = 1, 2, 3, 4)$ such that $\langle x_i, f_j \rangle = \delta_{ij}(i, j = 1, 2, 3, 4)$. Put $T = x_2 \otimes f_1 + 2x_4 \otimes f_3 - \sqrt{2I}$. It is easily checked that $T \in \mathbb{C}I + \mathcal{F}_2(X)$is invertible and$(T^* + A^*)\left(A^* g + \sqrt{2g}\right) = 0$ as well as$(T^* + 2A^*)\left(\sqrt{2A^*} f + f\right) = 0$. The invertibility of $A - \sqrt{2I}$ and $2A - \sqrt{2I}$ also imply that there exist nonzero vectors $u\, and\, v$ such that $(T + A)u = 0$and $(T + 2A)v = 0$. Thus we get $T + A$ and $T + 2A \in \mathcal{Z}^{\mathcal{A}}$. This finishes the proof of $(c') \Rightarrow (a)$ for the case that

$\Omega = \mathcal{Z}^{\mathcal{A}}$.

Since every choice of has $\mathcal{Z}^{\mathcal{A}}$ as a subset and since every element in is not invertible in $\mathcal{A}$, we see from the arguments for the case $\Omega = \mathcal{Z}^{\mathcal{A}}$ above that $(c') \Rightarrow (a)$ also holds true for every choice of $\Omega^{\mathcal{A}}$, which completes the proof.

Lemma (1.2)[1]:-

Let $\mathcal{A}$ be a standard operator algebra on a complex Banach space $X$and $A, B \in \mathcal{A}$. If $A + R \in \Omega \Rightarrow B + R \in \Omega_1$ for every operator $R \in \mathbb{C}I + \mathcal{F}_1(X)$, then $A = B$. Here Ω and Ω_1 denote any one of the subsets $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}_l^{\mathcal{A}}$, $\mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}}$, $\mathcal{Z}_r^{\mathcal{A}}$, $\mathcal{TZ}^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}}$, $\mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}} \cup \mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{S}_l^{\mathcal{A}} \cap \mathcal{S}_r^{\mathcal{A}}$ and $\mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$of $\mathcal{A}$, respectively.

Proof:-

LetΩ and $\Omega_1$be any one of the 12 subsets in the lemma, respectively. We first note that $\mathcal{Z}^{\mathcal{A}} \subseteq \Omega \cap \Omega_1$. For any nonzero vector $x \in X$, denote $Ax = y$. Fix a scalar λ such that $|\lambda| > max\{\|A\|, \|B\|\}$ and $y \neq \lambda x$. Let $M = \{f \in X^* | \langle x, f \rangle = 1\}$.If $f \in M$, then $\lambda \in \sigma_p(A - (y - \lambda x) \otimes f)$. Since $|\lambda| > \|A\|$, one sees that λ also belongs to $\sigma_c(A - (y - \lambda x) \otimes f)$. Thus$A - \lambda - (y - \lambda x) \otimes f \in \mathcal{Z}^{\mathcal{A}} \subseteq \Omega$. It follows from the hypothesis that $B - \lambda - (y - \lambda x) \otimes f \in \Omega_1$ and consequently, $\lambda \in \sigma^{\mathcal{A}}(B - (y - \lambda x) \otimes f)$. Now $|\lambda| > \|B\|$ implies $\lambda \in \sigma_p(B - (y\,\lambda x) \otimes f)$. So there exists a nonzero vector u_f such that $(B - (y - \lambda x) \otimes f)u_f = \lambda u_f$. Note that $u_f = \langle u_f, f \rangle (B - \lambda)^{-1}(y - \lambda x)$. Let $u = (B - \lambda)^{-1}(y - \lambda x)$, then $(B - (y - \lambda x) \otimes f)u = \lambda u$ holds for every $f \in M$. If x and u are linearly independent, then there exists some $f \in M$ such that $\langle u, f \rangle = 0$, which leads to $(B - \lambda)u = 0$ and $u = 0$. This contradiction shows that$(B - (y - \lambda x) \otimes f)x = \lambda x$ and hence,$Bx = y$.From the arbitrariness of x it follows that $B = A$.

The following theorem is the basic result in this paper. It says that if a unital additive map from a standard operator algebra onto another one preserves any one ofthe mentioned twelve subsets, then it has one of the following forms: isomorphism, conjugate isomorphism, anti-isomorphism and conjugate anti-isomorphism.

Theorem (1.3)[1]:-

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on complex Banach spaces $X$and Y, respectively, and let $\Phi: \mathcal{A} \to \mathcal{B}$ be a unital surjective additive map. Let $\Omega^{\mathcal{R}}$ be any one of the subsets $\mathcal{S}^{\mathcal{A}}$, $\mathcal{S}_l^{\mathcal{A}}$, $\mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}}$, $\mathcal{Z}_r^{\mathcal{A}}$, $\mathcal{TZ}^{\mathcal{A}}$,$\mathcal{TZ}_l^{\mathcal{A}}$, $\mathcal{TZ}_r^{\mathcal{A}}$,$\mathcal{S}_l^{\mathcal{A}} \cap \mathcal{S}_r^{\mathcal{A}}$, $\mathcal{Z}_l^{\mathcal{A}} \cup \mathcal{Z}_r^{\mathcal{A}}$ and $\mathcal{TZ}_l^{\mathcal{A}} \cup \mathcal{TZ}_r^{\mathcal{A}}$ of $\mathcal{R}$ with $\mathcal{R} = A$ or $\mathcal{B}$.If $\Phi(T) \in \Omega^{\mathcal{B}} \Leftrightarrow T \in \Omega^{\mathcal{A}}$, then either there exists an invertible bounded linear orconjugate linear operator $A : X \to Y$ such that $\Phi(T) = AT\,A^{-1}$ for all $T \in \mathcal{A}$, orthere exists an invertible bounded linear or conjugate linear operator $A : X^* \to Y$such that $\Phi(T) = AT^*A^{-1}$ for all $T \in \mathcal{A}$. The last case can not occur if any one of X and Y is not reflexive, or if $\mathcal{A}$ contains a element S such that $S \in \Omega^{\mathcal{A}}$ but$S^* \notin A^*$,where $\mathcal{A}_* = \{T^* \mid T \in \mathcal{A}\}$.

Proof:-

Assume that $\Phi(T) \in B \Leftrightarrow T \in A$.

Claim 1.$\Phi$is injective

We first assert that, if $S \in \mathcal{A}$ such that $T + S \in \Omega^{\mathcal{A}} \Rightarrow T \in \Omega^{\mathcal{A}}$ for every $T \in \mathcal{A}$, then $S = 0$. This is an immediate consequence of Lemma (1.2). Now, the injectivity of Φ follows from this assertion. Indeed, if $\Phi(S) = 0$, then $T + S \in \Omega^{\mathcal{A}} \Rightarrow \Phi(T) \in \Omega^{\mathcal{B}} \Rightarrow T \in \Omega^{\mathcal{A}}$ for all $T \in \mathcal{A}$ and hence $S = 0$.

Claim 2.Φ preserves rank-one operators in both directions.

Let $T \in \mathcal{A}$ with $rank\, T = 1$. For arbitrary $F \in \mathbb{C}I + \mathcal{F}_2(X) \subset \mathcal{B}$ there exists $S \in \mathcal{A}$ such that $\Phi(S) = F$. If both $F + \Phi(T)$ and $F + 2\Phi(T)$ are in $\Omega^{\mathcal{B}}$, then both S + T and S + 2T are in A. By Lemma (1.1)((a) $\Rightarrow$ (c)), we get $S \in \Omega^{\mathcal{A}}$ and hence $F \in \Omega^{\mathcal{B}}$. So, by Lemma (1.1)(($c'$) $\Rightarrow$ (a)) again, we have $rank\, \Phi(T) = 1$. Because $\Phi^{-1}(\Omega^{\mathcal{B}}) = \Omega^{\mathcal{A}}$, preserves rank-one operators in both directions.

Since Φ is additive, we see that the restriction of Φ to $\mathcal{F}(X)$ is a bijection between $\mathcal{F}(X)\, and\, \mathcal{F}(Y)$ and preserves rank-oneness in both directions. It follows that there exists a ring automorphism $\tau : \mathbb{C} \to \mathbb{C}$ and either

(i) there exist τ-quasi-linear bijective maps $A : X \to Y$ and $C : X^* \to Y^*$such that $(x \otimes f) = Ax \otimes Cf$ for all $x \in X$ and $f \in X^*$, or
(ii) there exist τ-quasi-linear bijective maps $A : X^* \to Y$ and $C : X \to Y^*$ such that $(x \otimes f) = Af \otimes Cx$ for all $x \in X$ and $f \in X^*$.
Note that the assumption that $\Phi(I) = I$ has not been used by far. Claim 3. If case (i) occurs, then $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$ for all $x \in X$and $f \in X^*$; if case (ii) occurs, then $\Phi(x \otimes f) = A(x \otimes f)^* A^{-1}$ for all $x \in X$ and $f \in X^*$. Assume that case (i) occurs, we first show that $\langle Ax, Cf\rangle = \tau(\langle x, f\rangle)$ for all$x \in X$ and $f \in X^*$. If$\langle x, f\rangle = 1$, then $I - x \otimes f \in \mathcal{Z}^{\mathcal{A}} \subseteq \Omega^{\mathcal{A}}$, and hence, $I - Ax \otimes Cf \in \Omega^{\mathcal{B}}$ as Φ is unital, which yields $\langle Ax, Cf\rangle = 1$. If $\langle x, f\rangle = \alpha \neq 0$, then$\tau(\alpha)^{-1}\langle Ax, Cf\rangle = \langle A(\alpha^{-1}x), Cf\rangle = \langle \alpha^{-1}x, f\rangle = 1$, so $\langle Ax, Cf\rangle = \tau(\alpha)$. Now assume that $\langle x, f\rangle = 0$, if $\langle Ax, Cf\rangle = \beta \neq 0$, then $I - A(\tau^{-1}(\beta^{-1})x) \otimes Cf \in \Omega^{\mathcal{B}}$,but this implies $I - \tau^{-1}(\beta^{-1})x \otimes f \in \Omega^{\mathcal{A}}$ and $\langle x, f\rangle = \tau^{-1}(\beta) \neq 0$, a contradiction. Thus, for any rank-1 operator $x \otimes f \in \mathcal{A}$and $y \in Y$, we have

$$\Phi(x \otimes f)y = (Ax \otimes Cf)y = \langle y, Cf\rangle Ax = \langle AA^{-1}Y, Cf\rangle Ax = \tau(\langle A^{-1}y, f\rangle)Ax$$
$$= A(x \otimes f)(A^{-1}y) = A(x \otimes f)A^{-1}y.$$

Therefore, $\Phi(x \otimes f) = A(x \otimes f)A^{-1}$.
If case (ii) occurs, then, similarly, we have $\langle Af, Cx\rangle = \tau(\langle x, f\rangle)$ for all $x \in X$ and $f \in X^*$, and consequently, the corresponding part of the claim is true.
Claim 4. $\tau(\lambda) = \lambda$for all $\lambda \in \mathbb{C}$ or $\tau(\lambda) = \bar{\lambda}$for all $\lambda \in \mathbb{C}$.
Since a nonzero continuous ring homomorphism of $\mathbb{C}$ must be either the identity or the complex conjugation , we need only to prove that τ is
continuous. Assume, on the contrary, that τ is not continuous, then $\tau$is not bounded on any neighborhood of 0.

Assume the case (i) occurs. Take a linear functional $g_1 \in Y^*$with $\|g_1\| \leq 1$,and then, pick a unit vector $u_1 \in X$ so that $\langle u_1, C^{-1}g_1\rangle \neq 0$. Since $\tau$is unbounded on$\{\lambda\langle u_1, C^{-1}g_1\rangle : |\lambda| < 2^{-1}\}$, $\tau(\lambda_1\langle u_1, C^{-1}g_1\rangle) > 1$ for some $\lambda_1$with $|\lambda_1| < 2^{-1}$. Let $x_1 = \lambda_1 u_1$. Then $\|x_1\| < 2^{-1}$and $|\tau(\langle x_1, C^{-1}g_1\rangle)| > 1$. Take $g_2 \in Y^*$ with$\|g_2\| \leq 1$ such that $C^{-1}g_2 \in \{x_1\}^{\perp}$. It is clear that $C^{-1}g_1$and $C^{-1}g_2$are linearly independent. Thus we can take a unit vector $u_2 \in X$ such that $\langle u_2, C^{-1}g_2\rangle \neq 0$ while$\langle u_2, C^{-1}g_1\rangle = 0$. By the unboundedness of τ on the set $\{\lambda\langle u_2, C^{-1}g_2\rangle : |\lambda| < 2^{-2}\}$, there exists λ_2 such that $|\tau(\lambda_2\langle u_2, C^{-1}g_2\rangle)| >2$. Let $x_2 = \lambda_2 u_2$. Then $\|x_2\| < 2^{-2}$and $|\tau(\langle u_2, C^{-1}g_1\rangle)| > 2$. Suppose that $x_1, x_2, \ldots, x_n$ and $g_1, g_2, \ldots, g_n$ are takenso that $0 < \|x_i\| < 2^{-i}$, $0 < \|g_i\| \leq 1, \langle u_i, C^{-1}g_k\rangle = 0$ whenever $i \neq k$, and$\tau(\langle u_i, C^{-1}g_k\rangle)| > i, i, k = 1, 2, \ldots, n$. Take g_{n+1} so that $C^{-1}g_{n+1} \in \{x_1, x_2, \ldots, x_n\}^{\perp}$ and $\|g_{n+1}\| \leq 1$. Then$C^{-1}g_{n+1} \notin V\{C^{-1}g_1, \ldots, C^{-1}g_n\}$, the linear spanof $\{C^{-1}g_k, k = 1, \ldots, n\}$. Pick u_{n+1} with $\|u_{n+1}\| = 1$ suchthat$\langle u_{n+1}, C^{-1}g_{n+1}\rangle \neq 0$ while $\langle u_{n+1}, C^{-1}g_i\rangle = 0$if $i = 1, 2, \ldots, n$. Since τ is unbounded on $\{\lambda\langle u_{n+1}, C^{-1}g_{n+1}\rangle : |\lambda| < 2^{-(n+1)}\}$, we get a λ_{n+1} with $|\lambda_{n+1}| < 2^{-(n+1)}$ such that $|\tau(\langle u_{n+1}, C^{-1}g_{n+1}\rangle)| > n + 1$, where $x_{n+1} = \lambda_{n+1}u_{n+1}$.Continuing this process, we get two sequences $\{x_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ which satisfy the conditions

(a) $\|x_n\| < 2^{-n}$and$\|g_n\| \leq 1$ for every n;

(b) $\langle x_n, C^{-1}g_k\rangle = 0$ whenever $n \neq k$;

(c) $|\tau(\langle x_n, C^{-1}g_n\rangle| > n$.

Note that $x = \sum_{n=1}^{\infty} x_n$ is a vector in X, so $Ax \in Y$. However, for any $n \in \mathbb{N}$, we Have

$$\|Ax\| \geq |\langle Ax, g_n\rangle| = |\tau(\langle x, C^{-1}g_n\rangle)| > n,$$

a contradiction. This shows that τ must be continuous.

Claim 5. $A$is a bounded linear or conjugate linear bijection.

This follows immediately from the fact $\langle Ax, Cf\rangle = \tau(\langle x, f\rangle)$ in case (i) (or,$\langle Af, Cx\rangle = \tau(\langle x, f\rangle)$ in case (ii)) and the Claim 4 as well as the Closed Graph Theorem.

Claim 6. $\Phi(T) = ATA^{-1}$for all $T \in \mathcal{A}$ if case (i) occurs, or $\Phi(T) = AT^*A^{-1}$for all $T \in \mathcal{A}$ if case (ii) occurs.

Suppose that the case (i) happens. Let $\Psi(T) = A^{-1}\Phi(T)A$ for every $T \in \mathcal{A}$.Then$\Psi: \mathcal{A} \to \mathcal{A}$ is a unital linear bijective map, $\Psi(\Omega^{\mathcal{A}}) = \Omega^{\mathcal{A}}$ and, by Claim 3$\Psi(x \otimes f) = x \otimes f$ for every rank-1 operator $x \otimes f \in \mathcal{A}$. Thus, for every $T \in \mathcal{A}$ and $\lambda \in \mathbb{C}, \Psi(T) + \lambda + x \otimes f \in \Omega^{\mathcal{A}}$ if and only if $T + \lambda + x \otimes f \in \Omega^{\mathcal{A}}$. It follows from Lemma (1.2) that we have $\Psi(T) = T$ and there fore, $\Phi(T) = AT A^{-1}$.

If the case (ii) occurs, similar to the proof of the case (i), we have that both $A : X^* \to Y$ and $C : X \to Y^*$ are invertible. Let $J : Y \to Y^{**}$ and $K : X \to X^{**}$ be then atural embeddings. Then, from $\langle Af, Cx\rangle = \tau(\langle x, f\rangle)$ with $\tau(\lambda) \equiv \lambda$ or $\tau(\lambda) \equiv \bar{\lambda}$ and the equation $\langle Wx, f\rangle = \overline{\langle x, W^*f\rangle}$ for conjugate linear operator W, we have$C^*JA = I_{X^*}$ and $A^*C = K$. So $J(Y) = Y^{**}$and $K(X) = X^{**}$. It follows that $X$and Y are reflexive.

Let $\Psi(T = A^*(T)^*(A^*)^{-1}$. Since X and Y are reflexive, $\Psi: \mathcal{A} \to \mathcal{A}$ is a unitallinear bijective map, and it is clear that $T \in \Omega^{\mathcal{A}}$ if and only if $\Psi(T) \in (\Omega^{\mathcal{A}})'$, where$(\Omega^{\mathcal{A}})'$ is the dual of $\Omega^{\mathcal{A}}$. By Claim 3, $\Psi(x \otimes f) = x \otimes f$ for every rank-1 element $x \otimes f \in \mathcal{A}$. Thus we have $T + R \in \Omega^{\mathcal{A}} \Leftrightarrow \Psi(T) + R \in (\Omega^{\mathcal{A}})'$ for every$R \in \mathbb{C}I + \mathcal{F}_1$. It follows from Lemma (1.2) that $\Psi(T) = T$ and hence ,$\Phi(T) = AT^*A^{-1}$.

If there exists an element $S \in \mathcal{A}$ such that $S \in \Omega^{\mathcal{A}}$but $S^* \notin \Omega^{\mathcal{A}_*}$, where $A_* = \{T^* | T \in \mathcal{A}\}$, then Φ cannot take the form $\Phi(\cdot) = A(\cdot)^*A^{-1}$ since $\Phi(S) = AS^* \times A^{-1} \notin \Omega^{\mathcal{B}}$. The proof is finished.

We remark that any form of isomorphism, conjugate isomorphism, anti-isomorphism and conjugate anti-automorphism that Φ takes in Theorem (1.3) may occur for every choice of $\Omega^{\mathcal{R}}$. This can be seen by assuming that both X and Y are reflexive but not separable, and by taking $\mathcal{A} = \mathbb{C}I + \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the ideal of compact operators or the norm closure of the ideal of finite-rank operators. We also remark that if $\Phi$is linear in Theorem (1.3), then $\Phi$is either an isomorphism or ananti-isomorphism.

Recall that an operator $T \in \mathcal{B}(X)$ is said to be quasi-affine if it is both injective and has a dense range; T is said to be lower-bounded if there exists a positive number $c > 0$ such that $\|T x\| \geq c\|x\|$ holds for all $x \in X$. Now we apply Theorem (1.3) to answer some preserver problems for additive maps on standard operator algebras in the following results. We point out that some of these preserver problems were not answered even for linear maps.

Theorem (1.4)[1]:-

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on complex Banach spaces $X$and Y, respectively, and let $\Phi: A \to B$ be a unital surjective additive map. Then the following are equivalent.

(a)Φpreserves the invertibility of elements in both directions.

(b)Φpreserves the semi-invertibility of elements in both directions.

(c)Φpreserves zero divisors in both directions.

(d)Φpreserves semi-zero divisors in both directions.

(e)Φpreserves topological divisors of zero in both directions.

(f)Φpreserves semi-topological divisors of zero in both directions.

(g)Φpreserves the quasi-affinity of operators in both directions.

(h)Φ preserves maximal semi-ideals in both directions.

(i) Either there exists an invertible bounded linear or conjugate linear operator $A: X \to Y$ such that $\Phi(T) = AT A^{-1}$ for all $T \in \mathcal{A}$, or there exists an invertible bounded linear or conjugate linear operator $A : X^* \to Y$ such that $\Phi(T) = AT^*A^{-1}$ for all $T \in \mathcal{A}$; the last case occurs only if X and Y are reflexive.

Proof:-

It is obvious that (i) implies each one of the conditions (a)–(h).

$(a) \Rightarrow (i), \ldots, (f) \Rightarrow (i)$ are immediate from Theorem (1.3) by taking $\Omega^{\mathcal{R}}$ the subsets $\mathcal{S}^{\mathcal{R}}$, $\mathcal{S}_l^{\mathcal{R}} \cap \mathcal{S}_r^{\mathcal{R}}, \mathcal{Z}^{\mathcal{R}}$, $\mathcal{Z}_l^{\mathcal{R}} \cup \mathcal{Z}_r^{\mathcal{R}}, \mathcal{T}\mathcal{Z}_r^{\mathcal{R}}$ *and* $\mathcal{T}\mathcal{Z}_l^{\mathcal{R}} \cup \mathcal{T}\mathcal{Z}_r^{\mathcal{R}}$ of $\mathcal{R}$, respectively, with $\mathcal{R} = \mathcal{A}$ *or* $\mathcal{B}$. As to $(g) \Rightarrow (i)$, we note that $T \in \mathcal{A}$ is quasi-affine as an operator in $\mathcal{B}(X)$ if and only if T is neither a left zero divisor nor a right zero divisor of $\mathcal{A}$ since $\mathcal{F}(X) \subset \mathcal{A}$, thus we have $(d) \Leftrightarrow (g)$. $(h) \Rightarrow (i)$ follows from $(a) \Rightarrow (i)$ because an element is neither in any maximal left ideals nor in any maximal right ideals if and only if it is invertible.

Theorem (1.5)[1]:-

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on complex Banach spaces $X$and Y, respectively, and let $\Phi: \mathcal{A} \to \mathcal{B}$ be a unital surjective additive map. Then any one of the following conditions (a)– (l) implies that (n) holds true.Moreover, if $\mathcal{A}$ contains a left invertible element which is not invertible, then the statements (a)– (m) are equivalent.

(a)Φ preserves the left invertibility of elements in both directions.

(b)Φ preserves the right invertibility of elements in both directions.

(c)Φ preserves left zero divisors in both directions.

(d)Φ preserves right zero divisors in both directions.

(e)Φ preserves left topological divisors of zero in both directions.

(f)Φ preserves right topological divisors of zero in both directions.

(g)Φ preserves the injectivity of operators in both directions.

(h)Φ preserves the range density of operators in both directions.

(i)Φ preserves the lower-boundedness of operators in both directions.

(j)Φ preserves the surjectivity of operators in both directions.

(k)Φ preserves maximal left ideals in both directions.

(l)Φ preserves maximal right ideals in both directions.

(m) There exists an invertible bounded linear or conjugate linear operator $A : X \to Y$ such that $\Phi(T) = AT\,A^{-1}$ for every $T \in \mathcal{A}$.

(n) Either there exists an invertible bounded linear or conjugate linear operator $A : X \to Y$ such that $\Phi(T) = AT\,A^{-1}$ holds for every $T \in \mathcal{A}$, or there exists an invertible bounded linear or conjugate linear operator $A : X^* \to Y$ such that $\Phi(T) = AT^*A^{-1}$ holds for every $T \in \mathcal{A}$. The last case occurs only if X and Y are reflexive.

Proof:-

$(a) \Rightarrow (n), \ldots, (f) \Rightarrow (n)$follow directly from Theorem (1.3). It is obvious that $(a) \Leftrightarrow (k)$ since an element fails to have a left inverse if and only if it is included in a maximal left ideal. Similarly, $(b) \Leftrightarrow (l)$. Since $\mathcal{A}$ and $\mathcal{B}$ are standard operator algebras, one checks easily that $(c) \Leftrightarrow (g), (d) \Leftrightarrow (h), (e) \Leftrightarrow (i)$ *and* $(f) \Leftrightarrow (j)$. Hence any one of (a)– (l) will imply (n).

Moreover, if $\mathcal{A}$ contains a left invertible element S which is not invertible, witha left inverse R, then S is also a right zero divisor, right topological divisor of zero but S^* is not; R is right invertible, left zero divisor, left topological divisor of zero but R^* is not. These ensure that Φ can not take the form $\Phi(T) = AT^*A^{-1}$ for every $T \in \mathcal{A}$. Therefore, each of the conditions (a)– (l) is equivalent to (m), completing the proof.

Corollary (1.6)[1]:-

Let $\Phi: \mathcal{B}(H) \to \mathcal{B}(K)$ be a unital surjective additive map, where H and K are infinite dimensional complex Hilbert spaces. Then the following statements are equivalent.

(a)Φ preserves the left invertibility of elements in both directions.

(b)Φ preserves the right invertibility of elements in both directions.

(c)Φ preserves the injectivity of operators in both directions.

(d)Φ preserves the range density of operators in both directions.

(e)Φ preserves the lower-boundedness of operators in both directions.

(f)Φ preserves the surjectivity of operators in both directions.

(g) There exists an invertible bounded linear or conjugate linear operator $A: H \to K$ such that $\Phi(T) = AT\,A^{-1}$ for every $T \in \mathcal{B}(H)$.

Proof:-

$(b) \Rightarrow (c)$. We use a method similar to that used . By Theorem (1.3), we have to verify that Φ can only take the form $\Phi(\cdot) = A(\cdot)A^{-1}$.To see this, assume, on the contrary, that Φ has the form $\Phi(T) = AT^*A^{-1}$ for every$T \in \mathcal{B}(X)$. Then, both X and $Y$are reflexive. There exists aseparable subspace W of Y and a linear projection P from Y onto W such that $\|P\| = 1$. Since W is a separable Banach space, according to
Ovsepian P elczynski' sresulton the existence of total bounded biorthogonal systems in separable Banach spaces, there is a vector sequence $\{y_n\} \subset W$ and a functional sequence$\{g_n\} \subset W^* = rng(P^*)$ such that (a)$g_m(y_n) = \delta_{mn}$ for $m, n = 1, 2, \ldots$; (b) the linearspan of $\{y_n\}$ is dense in W in the norm topology; (c) if $y \in W$ and $g_n(y) = 0$ forall $n \in \mathbb{N}$, then $y = 0$; (d) $sup_{\mathrm{n}}\|y_n\|\|g_n\| = M < \infty$.

Let $S = \sum_{n=1}^{\infty} 2^{-n}\, y_n \otimes g_n + I - P$. We claim that S is a bounded injective operator with dense range but not invertible on Y. Indeed, the boundedness of S follows from the condition (d) and $\|P\| = 1$, while the range density of S follows from thefact that $\{y_n\}_{n=1}^{\infty} \subset rng(S)$. Because$\sum_{n=1}^{\infty} 2^{-n}\, y_n \otimes g_n$ is compact, S is not invertible. From the surjectivity of Φ, we can find an operator $T \in \mathcal{B}(X)$ such that $\Phi(T) = S$. Itis clear that T has dense range. For any non zero functional $f \in X^*$,let $g = T^*f$ $(\neq 0)$. It is easily seen that $0 \in \sigma_p(T^* - g \otimes x)$ for arbitrary $x \in X$satisfying $\langle x, f\rangle = 1$. This implies that the range of $\Phi(T) - Ag \otimes h$ is not dense for arbitrary $h \in Y^*$satisfying $\langle Af, h\rangle = 1$. Hence for every $h \in Y^*$satisfying $\langle Af, h\rangle = 1$, there is anonzero functional $w \in Y^*$ such that $S^*w = \langle w, Ag\rangle h$. As $w \neq 0$ we have $S^*w \neq 0$and consequently, the range of S^* contains $span\{h \in Y^* |\, \langle Af, h\rangle = 1$for some $f \in X^*\}$. But,

$span\{h \in Y^* | \langle Af, h\rangle = 1\ for\ some\ \ f \in X^*\} = Y^*$ because $A : X^* \to Y$is invertible, which contradicts to the noninvertibility of S. So the second case cannot occur.
The proof of $(a) \Rightarrow (c)$ is similar.We have the following theorem .

Theorem (1.7)[1]:-

Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on complex Banach spaces X and Y, respectively, and let $\Phi: \mathcal{A} \to \mathcal{B}$ be a surjective additive map. Then Φ preserves zero products in both directions if and only if there exist a scalar cand an invertible bounded linear or conjugate linear operator $A : X \to Y$ such that$\Phi(T) = cATA^{-1}$ for every $T \in \mathcal{A}$.

Proof:-

We need only to prove the necessity. Assume that Φ preserves zero products in both directions. It is clear that Φ preserves left as well as right zero divisors in both directions. By the notice before the Claim 3 in the proof of Theorem (1.3), one sees that Φ is injective and preserves rank-oneness in both directions, and hence there exist τ -quasi-linear bijections A and C such that either the case (i) or the case (ii) listed there occurs. We claim that the case (ii) cannot happen. Assume, on the contrary, that the case (ii) occurs, then for every rank one operator $x \otimes f \in \mathcal{A}$ we have $\Phi(x \otimes f) = Af \otimes Cx$. Pick $u \in X$ and $f \in X^*$ so that $\langle u, f\rangle = 0$. Since A and C are surjective, there exist $x \in X$ and $h \in X^*$ such that $\langle Ah, Cx\rangle \neq 0$. However,$(x \otimes f)(u \otimes h) = 0$ implies $0 = \Phi(x \otimes f)\Phi(u \otimes h) = \langle Ah, Cx\rangle\, Af \otimes Cu \neq 0$, a contradiction. Thus, only the case (i) occurs, that is, $\Phi(x \otimes f) = Ax \otimes Cf$ holds for every rank one operator $x \otimes f \in \mathcal{A}$. Next we show that $\Phi(I) = cI$ for some nonzero scalar c. For any $x \otimes f \in \mathcal{A}$, if $\langle x, f\rangle = \alpha \neq 0$, since

$$(I - \alpha^{-1}x \otimes f)(x \otimes f) = (x \otimes f)(I - \alpha^{-1}x \otimes f) = 0,$$

andΦ is zero-product preserving, we have
$(\Phi(I) - \tau(\alpha)^{-1}Ax \otimes Cf)(Ax \otimes Cf) = \big((Ax \otimes Cf)\big)\big((\Phi(I) - \tau(\alpha)^{-1}Ax \otimes Cf)\big) = 0$,this yields that $\Phi(I)(Ax \otimes Cf) = (Ax \otimes Cf)\Phi(I)$; if $\langle x, f\rangle = 0$, pick a vector $u \in X$ so that $\langle u, f\rangle \neq 0$, then by what has just been proved we have $\Phi(I)((Ax + Au) \otimes Cf) = ((Ax + Au) \otimes Cf)\,\Phi(I)$ and $\Phi(I)(Au \otimes Cf) = (Au \otimes Cf)\,\Phi(I)$, these still imply $\Phi(I)(Ax \otimes Cf) = (Ax \otimes Cf)\,\Phi(I)$. Since both A and C are surjective, we see that $\Phi(I)$ commutes with every rank one operator and hence must be a multiple of the identity, that is, $\Phi(I) = cI$ for some scalar c. By the injectivity of Φ, $c \neq 0$. Now, $c^{-1}\,\Phi$ is a unital surjective additive map preserving left zero divisor in both directions, then applying Theorem (1.5), one completes the proof immediately.

Theorem (1.8) [5].

A finite dimensional Hopf algebra H is a symmetric Frobenius algebra if and only if H is unimodular and its antipode S satisfies $S_2$is an inner automorphism of H.

Chapter 2

Additive Maps onto Matrix Mpaces Compressing the Spectrum

We prove that given a unital C^*- algebra $\mathcal{A}$ and an additive and surjective map $T : \mathcal{A} \to \mathcal{M}_n$such that the spectrum of $T(x)$ is a subset of the spectrum of x for each $x \in \mathcal{A}$, then T is either an algebra morphism, or an algebra anti-morphism .

Let $\mathcal{A}$be a (complex) unitalBanach algebra, and denote its unit by 1. By $\sigma(a)$we shall denote the spectrum of the element $a \in \mathcal{A}$and $\rho(a)$will be its spectral radius. A well-known result in the theory of Banach algebras, the Gleason–Kahane–Żelazko theorem, states that if $f: \mathcal{A} \to \boldsymbol{C}$is $\boldsymbol{C}$-linear (that is, additive and homogeneous with respect to complex scalars) and $f(a) \in \sigma(a)$for every $a \in \mathcal{A}$, then $f$is multiplicative. Kowalski and Slodkowski generalized their result, by proving that if $f: \mathcal{A} \to C$with $f(0) = 0$satisfies

$$f(x) - f(y) \in \sigma(x - y)(x, y \in \mathcal{A}), \tag{1}$$

then$f$is automatically $\boldsymbol{C}$-linear, and therefore also multiplicative. (That $f$is $\boldsymbol{R}$-linear and the fact that $f(ia) = if(a)$for all $a \in \mathcal{A}$come automatically from the inclusions (1), which combine spectrum-preserving properties and additivity properties on the functional f.) In particular, if $f: \mathcal{A} \to C$ is additive and $f(x) \in \sigma(x)$for every $x \in \mathcal{A}$, then $f$is a character of $\mathcal{A}$.

The natural extension of the Gleason–Kahane–Żelazko theorem for the case when the range Coffis replaced by $\mathcal{M}_n$, the algebra of $n \times n$matrices over $\boldsymbol{C}$, was obtained by Aupetit .

Theorem (2.1) [2]:-

If $T: \mathcal{A} \to \mathcal{M}_n$is a surjective$\boldsymbol{C}$-linear map such that

$$\sigma\big(T(x)\big) \subseteq \sigma(x)(x \in \mathcal{A}), \tag{2}$$

then either

$$T(xy) = T(x)T(y)(x, y \in \mathcal{A}) \quad or \quad T(xy) = T(x)T(y)(x, y \in \mathcal{A}). \tag{3}$$

We state that if $T: \mathcal{A} \to \mathcal{M}_n$is linear, unital and onto, sending invertible elements from Ainto invertible elements of $\mathcal{M}_n$, then $T$is of the form (3). If (2)holds, then $x \in \mathcal{A}$invertible implies $0 \notin \sigma(x)$, thus by (2)we have $0 \notin \sigma(T(x))$, which means that the matrix $T(x)$is invertible. By Lemma(2.6) we also have that Tsends the unit element of $\mathcal{A}$into the unit element of $\mathcal{M}_n$. Thus, under the hypothesis of Theorem(2.1) we have that $T$is unital and invertibility-preserving. Under the hypothesis of Theorem(2.1) , the map

$T$is either an algebra morphism, or an algebra anti-morphism. We study the same type of problem as the one considered by Theorem(2.1), assuming only additivity instead of linearity over the complex field $\boldsymbol{C}$. Our first result states that if $\mathcal{A}$is supposed to be a C^*-algebra,then we arrive at the same conclusion by assuming only additivity on$T$.

Theorem (2.2)[2]:-

Let $\mathcal{A}$be a unitalC^*-algebra and suppose $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2) holds. Then $T$is of the form (3).
As a corollary, we obtain the following generalization for the case of additive maps defined on C^*-algebras which compress the spectrum.

Theorem (2.3)[2]:-

Let $\mathcal{A}$be a unitalC^*-algebra, and let $\mathcal{B}$be a complex, unitalBanach algebra having a separating family of irreducible finite-dimensional representations. Suppose $T: \mathcal{A} \to \mathcal{B}$is additive and onto such that (2)holds. Then $T$is a Jordan morphism, that is

$$T(x^2) = T(x)^2 (x \in \mathcal{A}).$$

For the general case of an arbitrary Banach algebra $\mathcal{A}$, we shall impose an extra surjectivity assumption on the map Tin order to obtain the same type of result.

Theorem (2.4)[2]:-

Let $\mathcal{A}$be a unitalBanach algebra and suppose $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2)holds. Suppose also that there exist $x_1, \ldots, x_{n^2} \in \mathcal{A}$such that

$$\{T(x_1) + T(ix_1)/i, \ldots, T(x_{n^2}) + T(ix_{n^2})/i\} \subseteq \mathcal{M}_n \tag{4}$$

are linearly independent over $\boldsymbol{C}$. Then $T$is of the form (2.3).

We do not know whetherthe assumption that the matrices in (4)span$\mathcal{M}_n$over the complex field may be removed from the statement of Theorem(2.4). We believe that this hypothesis can be eliminated, being a consequence of the fact that $T$is surjective and that (2)holds, but we have not been able to prove it. An important part of the proof of Theorem(2.4) can be carried out without the surjectivity hypothesis given by (4)being assumed, using only the surjectivity of the map T. Throughout this part, $\mathcal{A}$will denote an arbitrary unitalBanach algebra. The first result shows that, as in the C-linear case , under the hypothesis of Theorem(2.2)we have that the continuity of the map $T$is automatic.

Theorem (2.5) [2]:- Let Tbe an additive map from $\mathcal{A}$onto $\mathcal{M}_n$such that

$$\rho\big(T(a)\big) \leq \rho(a) (a \in \mathcal{A}). \tag{5}$$

Then $T$is continuous, and therefore also $\boldsymbol{R}$-linear.

Proof:-

Since $T$is supposed to beadditive, it is sufficient to prove the continuity at $0 \in \mathcal{A}$. Suppose that $a_k \to 0$in $\mathcal{A}$and let us prove first that $(T(a_k))_k \subseteq \mathcal{M}_n$is bounded. Using the surjectivity of T, it is sufficient to prove that given any $x \in \mathcal{A}$then $(tr(T(a_k)T(x)))_k \subseteq C$is bounded, where $tr(\cdot)$denotes the usual trace on$\mathcal{M}_n$. By (5), for each $k$we have that

$$\rho((T(a_k + x))^2) = \left(\rho(T(a_k + x))\right)^2 \leq (\rho(a_k + x))^2 \leq \|a_k + x\|^2 \leq (\|a_k\| + \|x\|)^2,$$

which implies

$$\left|tr\left(T(a_k)^2 + 2tr\big(T(a_k)T(x)\big) + tr(T(x)^2)\right)\right| \leq n(\|a_k\|^2 + 2\|a_k\|\|x\| + \|x\|^2).$$

Since $a_k \to 0$and $\rho(T(a_k)) \leq \rho(a_k) \leq \|a_k\|$for each k, this gives $\rho(T(a_k)) \to 0$and therefore $tr(T(a_k)^2) \to 0$. Thus

$$2 \lim_{k\to\infty} sup\, |tr(T(a_k)T(x))| \leq n\|x\|^2 + |tr(T(x))^2|,$$

and therefore $(tr(T(a_k)T(x)))_k$is bounded, as desired.
Since $\mathcal{M}_n$is finite dimensional, without loss of generality we may suppose that $T(a_k) \to w \in \mathcal{M}_n$, and let us prove that $w = 0$. We shall use the fact that the spectral radius on a general Banach algebra is upper semicontinuous and the fact that on $\mathcal{M}_n$the spectral radius is continuous .Given any $a \in \mathcal{A}$and $m \in N$, by (5)we have $\rho(T(ma_k + a)) \leq \rho(ma_k + a)$. Using that $T$is additive, this gives $\rho(mT(a_k) + T(a)) \leq \rho(ma_k + a)$. Therefore

$$\lim_{k\to\infty} sup\, \rho(mT(a_k) + T(a)) \leq \lim_{k\to\infty} sup\, \rho(ma_k + a).$$

Since the spectral radius is continuous on $\mathcal{M}_n$, that $T(a_k) \to w$in $\mathcal{M}_n$gives

$$\lim_{k\to\infty} sup\, \rho\big(mT(a_k) + T(a)\big) = \lim_{k\to\infty} \rho\big(mT(a_k) + T(a)\big) = \rho\big(mw + T(a)\big).$$

Since the spectral radius is upper semi continuous on $\mathcal{A}$, that $a_k \to 0$ in $\mathcal{A}$ gives

$$\lim_{k\to\infty} sup\, (ma_k + a) \leq \rho(a).$$

Hence given any $a \in \mathcal{A}$we have that $\rho(mw + T(a)) \leq \rho(a)$for all $m \in N$. Since $T$is supposed to besurjective, we deduce that given any $b \in \mathcal{M}_n$we can find $\mathcal{M}_b \geq 0$such that

$$\rho(mw+b)\leq \mathcal{M}_b(m\in N). \qquad (6)$$

Taking $b=0$ in (6) we get $\rho(w)=0$. If $w\in\mathcal{M}_n$ were not zero, we may write it as

$$w=y^{-1}\begin{bmatrix}0 & 1 & 0\dots & 0\\ 0 & 0 & *\dots & 0\\ \vdots & \vdots & \ddots\vdots & \vdots\\ 0 & 0 & \cdots 0 & *\\ 0 & 0 & \cdots 0 & 0\end{bmatrix}y$$

For some invertible $y\in\mathcal{M}_n$. For

$$b=y^{-1}\begin{bmatrix}0 & 0 & \cdots 0\\ 1 & * & \dots 0\\ \vdots & \vdots & \ddots\vdots\\ 0 & 0 & \cdots 0\end{bmatrix}y\in\mathcal{M}_n$$

we have that λ^2-m divides the characteristic polynomial of $mw+b$. Hence $\rho(mw+b)\geq\sqrt{m}$ for all $m\in N$, contradicting (6).

The following lemma and Theorem(2.5) show that an additive surjective map $T:\mathcal{A}\to\mathcal{M}_n$ satisfying (2) is automatically unital.

Lemma (2.6) [2]:-

Let $T:\mathcal{A}\to\mathcal{M}_n$ be additive and onto such that (2) holds. Then $T(\lambda 1)=\lambda I_n$ for every $\lambda\in C$, where I_n is the unit matrix of $\mathcal{M}_n$.

Proof:-

By Theorem(2.5) we have that T is continuous, and therefore also $\boldsymbol{R}$-linear. Since T is onto, by the open mapping theorem for surjective $\boldsymbol{R}$-linear maps we find $N>0$ such that $y\in\mathcal{M}_n$ implies the existence of $x\in\mathcal{A}$ such that $T(x)=y$ and $\|x\|\leq N\|y\|$. Let $\lambda\in C$ and denote $u=T(\lambda 1)\in\mathcal{M}_n$. Then given any $y\in\mathcal{M}_n$, we have

$$\sigma\big(\lambda I_n-(u+y)\big)=\lambda-\sigma(u+y)=\lambda-\sigma\big(T(\lambda 1+x)\big)\subseteq\lambda-\sigma(\lambda 1+x)$$
$$=\lambda-\big(\lambda+\sigma(x)\big)=-\sigma(x),$$

Where $x\in\mathcal{A}$ was such that $T(x)=y$ and $\|x\|\leq N\|y\|$. Thus

$$\rho\big(\lambda I_n-(u+y)\big)\leq\rho(x)\leq\|x\|\leq N\|y\|,$$

That is

$$\rho\big((\lambda I_n-u)-y\big)\leq N\|y\|, \qquad (y\in\mathcal{M}_n).$$

The Zemánek characterization of the radical implies that $\lambda I_n - u$belongs to the radical of $\mathcal{M}_n$. That is, $u = \lambda I_n$, since $\mathcal{M}_n$is semisimple.

Suppose now that $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2)holds. By Theorem(2.5), we have that $T$is $\boldsymbol{R}$-linear. Following an idea , given any $r \in \boldsymbol{R}$we have

$$\sigma\left(e^{ir}T\left(e^{-ir}x\right)\right) \subseteq e^{ir}\sigma\left(e^{-ir}x\right) = \sigma(x)$$

For every $x \in \mathcal{A}$. From the $\boldsymbol{R}$-linearity of Twe also have

$$\begin{aligned} e^{ir}T\left(e^{-ir}x\right) &= (\cos r + i \sin r)(\cos r\,.\,T(x) - \sin r\,.\,T(ir)) \\ &= T(x)(cos^2 r + i \sin r\,.\cos r) - T(ix)(\cos r\,.\sin r + i\, sin^2 r) \\ &= T(x) + T(ix)/2 + e^{2ir}(T(x) - T(ix)/i)/2. \end{aligned}$$

Thus

$$\sigma(R(x) + \xi S(x)) \subseteq \sigma(x)(x \in \mathcal{A};\ \xi \in C, |\xi| = 1), \tag{7}$$

where we have denoted

$$R(x) = \frac{T(x) + T(ix)/i}{2} \qquad (x \in \mathcal{A})$$

And

$$S(x) = \frac{T(x) - T(ix)/i}{2}(x \in \mathcal{A}).$$

Since $T$is $\boldsymbol{R}$-linear, one can easily check that $R\ and\ S$are both $\boldsymbol{R}$-linear transformations from $\mathcal{A}$into $\mathcal{M}_n$. More than that, $R(ix) = iR(x)$for every $x \in \mathcal{A}$, and therefore $R$is $\boldsymbol{C}$-linear. Also, $S(ix) = -iS(x)$for every $x \in \mathcal{A}$, and therefore $S$is conjugate-linear. Thus

$$T(x) = R(x) + S(x)(x \in \mathcal{A}),$$

where$R$is $\boldsymbol{C}$-linear and Sis $\overline{\boldsymbol{C}}$-linear. Observe also that by Lemma (2.6) we have $R(1) = I_n$and $S(2.1) = 0 \in \mathcal{M}_n$.
The inclusions (7)imply the following spectral inequalities for the maps $R$and$S$.

Theorem (2.7) [2]:-

Suppose $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2)holds. Then $R(e^a) \in \mathcal{M}_n$is invertible for each $a \in \mathcal{A}$, and

$$\rho\left(S(xe^a)\left(R(e^a)\right)^{-1}\right) \leq \rho(x)(a, x \in \mathcal{A}). \tag{8}$$

Proof:-

Consider $x \in \mathcal{A}$ with $\rho\,(x) < 1$and an arbitrary $a\ \in \mathcal{A}$. Let $\xi\ \in C$with $|\xi| = 1$. Then for each $r \in \boldsymbol{R}$we have that $(\bar{\xi} - x)e^{ra} \in \mathcal{A}$is invertible, and (2)gives

$$\sigma(T((\bar{\xi} - x)e^{ra}))\ \subseteq \boldsymbol{C}\backslash\{0\}. \text{ Then}$$

$$\sigma\big(\bar{\xi}R(e^{ra})\big) - T(xe^{ra}) + \xi S(e^{ra}) \subseteq C\{0\},$$

And therefore

$$\sigma\big(R(e^{ra})\big) - \xi T(xe^{ra}) + \xi^2 S(e^{ra}) \subseteq C\backslash\{0\}, \qquad (a \in \mathcal{A}, \rho(x) < 1, |\xi| = 1, r \in R). \tag{9}$$

This leads us to consider the family of analytic multi valued functions $(K_r)_{r\in[0,1]}$given by

$$K_r(\lambda) = \sigma\big(R(e^{ra}) - \lambda T(xe^{ra}) + \lambda^2 S(e^{ra})\big)(\lambda \in C).$$

Since by Theorem(2.5)we have that T, Rand $S$are continuous and since the spectrum function is continuous on matrices, for each $\lambda\ \in C$we have that the function $r \to K_r(\lambda)$is continuous with respect to r. We apply now the multivalued form of Rouché's Theorem given to see that

$$\big(K_0(D)\backslash K_1(D)\big)\bigcup\big(K_1(D)\backslash K_0(D)\big) \subseteq \bigcup\{K_r(\xi)\colon r \in [0,1],\ \ |\xi| = 1\}.$$

(By $\boldsymbol{D}$we have denoted the open unit disk in $\boldsymbol{C}$.) Now (9)implies that $0 \notin K_r(\xi)$for $r \in [0,1]$and $|\xi| = 1$, and therefore $(K_1(D)\backslash K_0(D)) \subseteq C\backslash\{0\}$. That $R(1) = I_n$and $S(1) = 0$imply $K_0(\lambda)\ = \sigma(I_n - \lambda T(x))$. But $\sigma(T(x))\ \subseteq \sigma(x)\ \subseteq D$, and therefore $K_0(\lambda) \subseteq C\backslash\{0\}$for all $\lambda \in D$. That $K_1(D)\backslash K_1(D)$does not contain $0\ \in C$implies then $K_1(D)\ \subseteq C\backslash\{0\}$, and therefore

$$\sigma\big(R(e^{ra}) - \lambda T(xe^{a}) + \lambda^2 S(e^{a})\big) \subseteq \mathrm{C}\backslash\{0\} \quad (\mathrm{a} \in \mathcal{A}, \rho(x) < 1, |\lambda| < 1). \tag{10}$$

Taking $\lambda\ = 0$in (10), we see that $R(e^{a})$is an invertible matrix. Denoting $s = T(xe^{a})(R(e^{a}))^{-1} \in \mathcal{M}_n$and $p = S(e^{a})(R(e^{a}))^{-1} \in \mathcal{M}_n$, we infer from (10)that $det(\mu^2 I_n - \mu s\ + p) \neq 0$for $|\mu| > 1$. Let us observe now that $\mu\ \to det(\mu^2 I_n - \mu s\ + p)$is just the characteristic polynomial of

$$\begin{bmatrix} 0 & I_n \\ -\rho & s \end{bmatrix} \in \mathcal{M}_{2n},$$

And therefore

$$\rho\left(\begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & T(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1 \qquad (a \in \mathcal{A}, \rho(x) < 1).$$

For $x \to \eta x$ with $|\eta| = 1$ we get

$$\rho\left(\begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & 0 \end{bmatrix} + \eta \begin{bmatrix} 0 & 0 \\ 0 & R(xe^a)R(e^a)^{-1} \end{bmatrix} + \bar{\eta} \begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1,$$

And therefore

$$\rho\left(\eta \begin{bmatrix} 0 & I_n \\ -S(e^a)R(e^a)^{-1} & 0 \end{bmatrix} + \eta^2 \begin{bmatrix} 0 & 0 \\ 0 & R(xe^a)R(e^a)^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1$$

for all $|\eta| = 1$. Using Vesentini's theorem and the maximum principle for subharmonic functions we infer that

$$\rho\left(\begin{bmatrix} 0 & 0 \\ 0 & S(xe^a)R(e^a)^{-1} \end{bmatrix}\right) \leq 1.$$

Therefore $\rho(S(xe^a)R(e^a)^{-1}) \leq 1$ for all $a \in \mathcal{A}$ and for all $x \in$ with $\rho(x) < 1$. Using the fact that S is conjugate-homogeneous, we obtain (8).

Let us remark that, under the hypothesis of Theorem(2.7), we have $det S(x) = 0$ for every $x \in \mathcal{A}$. Indeed, taking $x = 1$ in (8) we see that $\rho(S(e^a)(R(e^a))^{-1}) \leq 1$ for every $a \in \mathcal{A}$. There fore $\rho(S(e^{\lambda a})(R(e^{\lambda a}))^{-1}) \leq 1$ for every $a \in \mathcal{A}$ and $\lambda \in C$. This implies that the analytic function $\lambda \to \overline{det\, S(e^{\lambda a})} det(R(e^{\lambda a}))^{-1}$ is bounded on C. Classical Liouville's theorem implies that it is therefore constant. Since $S(1) = 0$, then $det S(e^{\lambda a}) = 0$ for $\lambda = 0$, and therefore $\overline{det S(e^{\lambda a})} det\left(R\left(e^{\lambda a}\right)\right)^{-1} = 0$ for every $\lambda \in C$. Thus $det S(e^{\lambda a}) = 0$ on C, and in particular $det S(e^a) = 0$. Now if $x \in \mathcal{A}$ is arbitrary, the holomorphic functional calculus shows that $\xi 1 + x \in \mathcal{A}$ is an exponential for $|\xi| > \rho(x)$. Then $det(S(x)) = det(S(\xi 1 + x)) = 0$, where ξ was chosen such that $|\xi| > \rho(x)$.

Let us observe that until now in this part, the only surjectivity assumption that was used in the proofs is the one we have on the map T. By Theorem(2.5) we have that R is continuous, and by Lemma(2.6) we have that R is unital. By Theorem(2.7), the map R sends exponentials from $\mathcal{A}$ into invertible matrices. Then the proof shows that given any complex polynomial p, we have

$$tr(R(p(x)y)) = tr\big(p(R(x))R(y)\big) \qquad (x, y \in \mathcal{A})$$

In particular, tr(R(xy)) =tr(R(x)R(y)) for every x and y, and

$$tr(((R(x)^2 - R(x^2))R(y)) = 0 (x, y \in A) \qquad . \quad (11)$$

If we further suppose (4)to be true, then $R$is also surjective and (11)implies that $R(x)^2 = R(x^2)$for every $x \in \mathcal{A}$. Thus $R$is a Jordan morphism and therefore, since $\mathcal{M}_n$is prime, of the form (3). We shall use this property in the proofs of both Theorem(2.2)and Theorem(2.4).

Theorem (2.2) [2]:-

Let $\mathcal{A}$be a unitalC^*-algebra and suppose $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2)holds. Then $T$is of the form (3).
As a corollary, we obtain the following generalization for the case of additive maps defined on C^*-algebras which compress the spectrum.

Proof:-

By Theorem(2.7), we have that (8)holds. Let $a \in \mathcal{A}$be a self-adjoint element. Then for every $r \in R$we have that $e^{ira} \in \mathcal{A}$is a unitary element. In particular, $||e^{ira}|| = \rho(e^{ira}) = 1$. For an arbitrary $y \in \mathcal{A}$and $\lambda = \alpha + i\beta \in C$, where $\alpha, \beta \in R$, by taking $x = ye - 2i\beta a \in \mathcal{A}$in (8)we see that

$$\rho\left(S\left(ye^{\overline{\lambda}a}\right)\left(R\left(e^{\lambda a}\right)\right)^{-1}\right) = \rho\left(S\left(ye^{-2i\beta a}e^{(\alpha+i\beta)a}\right)\left(R\left(e^{(\alpha+i\beta)a}\right)\right)^{-1}\right) \leq \rho\left(ye^{-2i\beta a}\right)$$
$$\leq \left\|ye^{-2i\beta a}\right\| \leq \|y\|\left\|e^{-2i\beta a}\right\| = \|y\|.$$

The continuity of $S$and R, together with the facts that $S$is conjugate-linear and $R$is $\boldsymbol{C}$-linear imply that $\lambda \to S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1}$is analytic from $\boldsymbol{C}$into $\mathcal{M}_n$. Then Liouville's Spectral Theorem implies that $\lambda \to S(ye^{\overline{\lambda}a})(R(e^{\lambda a}))^{-1}$is constant on $\boldsymbol{C}$. In particular, for every λwe have

$$\sigma\left(S\left(ye^{\overline{\lambda}a}\right)\left(R\left(e^{\lambda a}\right)\right)^{-1}\right) = \sigma\left(S\left(y1\right)\left(R(1)\right)^{-1}\right) = \sigma\left(S(y)\right),$$

the last equality being true since by Lemma (2.6) we have that $R(1)$ is the $n \times n$ identity matrix. Thus

$$\sigma\big((S(y) + \lambda S(ya) + \lambda^2(ya^2)/2 + \cdots)(I_n - \lambda R(a) + \cdots)\big)$$
$$= \sigma(S(y))(\lambda \in C). \qquad (12)$$

Taking $y = 1$ in (12), since by Lemma (2.6) we have that $S(1)$ is the $n \times n$ zero matrix we obtain that

$$\rho\big((\lambda S(a) + \lambda^2 S(a^2)/2 + \cdots)(I_n - \lambda R(a) + \cdots)\big) = 0 \qquad (\lambda \in C).$$

Dividing the last equality by $\lambda \neq 0$ and letting $\lambda \to 0$ we see that $\rho\ (S\ (a)) = 0$. This holds for any arbitraryself-adjoint element $a \in \mathcal{A}$; if $x \in \mathcal{A}$ is now arbitrary, with $x =$

$a + ib$ where $a, b \in \mathcal{A}$ are self-adjoint elements,then $\rho(S(a + rb)) = 0$ for every $r \in R$, the element $a + rb \in \mathcal{A}$ being self-adjoint. Thus $\rho(S(a) + rS(b)) = 0$for every $r \in R$, and for the analytic function $\lambda \to S(a) + \lambda S(b)$ this implies that $\rho(S(a) + \lambda S(b)) = 0$for every $\lambda \in C$. Taking $\lambda = -i$ we infer that $\rho(S(x)) = 0$, equality which holds for every $x \in \mathcal{A}$. Now if$y, z \in \mathcal{A}$ are arbitrary elements, we have $\rho(S(y) + \lambda S(z)) = \rho(S(y + \lambda z)) = 0$ for every $\lambda \in C$. In particular$tr((S(y) + \lambda S(z))2) = 0$ for every $\lambda \in C$, and therefore

$$tr\big(S(y)S(z)\big) = 0 \qquad (y, z \in \mathcal{A}). \tag{13}$$

Equation (12) implies that given any $y \in \mathcal{A}$ and any self-adjoint element $a \in \mathcal{A}$ we have

$$tr\big((S(y) + \lambda S(ya) + \lambda^2 (ya^2)/2 + \cdots)(I_n - \lambda R(a) + \cdots)\big) = tr\big(S(y)\big)(\lambda \in C).$$

Computing the coefficient of λ, we see that $tr(S(ya)) = tr(S(y)R(a))$. That $\rho(S(ya)) = 0$ gives $tr(S(ya)) = 0$,and therefore $tr(S(y)R(a)) = 0$. By (13) we also have $tr(S(y)S(a)) = 0$. Now if $x = a + ib$ is arbitrary, where$a, b \in \mathcal{A}$ are self-adjoint elements, then

$$\begin{aligned} tr\big(S(y)T(x)\big) &= tr\big(S(y)R(x)\big) + tr\big(S(y)S(x)\big) \\ &= tr(S(y)R(a) + itr(S(y)R(b) - itr\big(S(y)S(b)\big) = 0. \end{aligned}$$

Thus $tr(S(y)T(x)) = 0$ for every $x, y \in \mathcal{A}$. The surjectivity of T implies that S is identically zero. Thus$T = R$. In particular R is surjective, and then (11) implies that R is a Jordan morphism and therefore ofthe form (3). Thus, the same is true for $T = R$too. As a corollary, we obtain the characterization of additive, surjective, spectrum compressing maps into Banach algebras having a separating family of irreducible finite-dimensional representations.

Theorem (2.3) [2]:-

Let $\mathcal{A}$be a unitalC^*-algebra, and let $\mathcal{B}$be a complex, unitalBanach algebra having a separating family of irreducible finite-dimensional representations. Suppose $T: \mathcal{A} \to \mathcal{B}$is additive and onto such that (2)holds. Then $T$is a Jordan morphism, that is

$$T(x^2) = T(x)^2 (x \in \mathcal{A}).$$

For the general case of an arbitrary Banach algebra $\mathcal{A}$, we shall impose an extra surjectivity assumption on the map Tin order to obtain the same type of result.

Proof:-

Let π be a finite-dimensional irreducible representation of B. Using the Jacobsondensity theorem, we have that $\pi: B \to \mathcal{M}_n$ is surjective, for some $n \geq 1$. Define $T_\pi : \mathcal{A} \to \mathcal{M}_n$byputting $T_\pi = \pi \circ T$. Then T_π is additive and onto, and

$$\sigma(T_\pi(x)) = \sigma\left(\pi(T(x))\right) \subseteq \sigma(x)(x \in \mathcal{A}).$$

We use then Theorem (2.2) to see that T_π is a Jordan morphism. Thus

$$\pi(T(x^2) - T(x^2)) = 0 \qquad (x \in \mathcal{A}),$$

and using now the fact that $\mathcal{B}$ has a separating family of irreducible finite-dimensional representations weconclude that $T(x^2) = T(x^2)$ for all $x \in \mathcal{A}$.

Theorem (2.4) [2]:-

Let $\mathcal{A}$be a unitalBanach algebra and suppose $T: \mathcal{A} \to \mathcal{M}_n$is a surjective additive map such that (2)holds. Suppose also that there exist $x_1, \ldots, x_{n^2} \in \mathcal{A}$such that

$$\{T(x_1) + T(ix_1)/i, \ldots, T(x_{n^2}) + T(ix_{n^2})/i\} \subseteq \mathcal{M}_n \tag{4}$$

are linearly independent over $\boldsymbol{C}$. Then $T$is of the form (3).

We do not know whetherthe assumption that the matrices in (4)span$\mathcal{M}_n$over the complex field may be removed from the statement of Theorem(2.4). We believe that this hypothesis can be eliminated, being a consequence of the fact that $T$is surjective and that (2)holds, but we have not been able to prove it. An important part of the proof of Theorem(2.4)can be carried out without the surjectivity hypothesis given by (4)being assumed, using only the surjectivity of the map T. Throughout this part, $\mathcal{A}$will denote an arbitrary unitalBanach algebra. The first result shows that, as in the C-linear case , under the hypothesis of Theorem(2.2) we have that the continuity of the map $T$is automatic.

Proof:-

We have seen that for $R(x) = (T(x + T(ix)/i)/2$and $S(x) = (T(x) - T(ix)/i)/2$, the map $R: \mathcal{A} \to \mathcal{M}_n$is C-linear, while $S: \mathcal{A} \to \mathcal{M}_n$ is conjugate-linear. Also, the hypothesis (4) implies that R is also onto,and therefore the final remark in previous part implies that R is either an algebra morphism, or an algebraanti-morphism. Let us suppose, for example, that R is a morphism.

Consider an arbitrary $y \in \mathcal{A}$ with $\rho(y) < 1$ and an arbitrary $\xi \in C$with $|\xi| = 1$. The holomorphicfunctional calculus shows that $\xi 1 - y \in \mathcal{A}$ is an exponential, and then (8) implies that $\rho(S(x(\xi 1 - y))(R(\xi 1 - y))^{-1}) \leq \rho(x)$ for all x. That is, $\rho((\bar{\xi} S(x) - S(xy))(I_n - \bar{\xi} R(y)^{-1}) \leq \rho(x)$. Since R is an algebra morphism,then$\rho(R(y)) \leq$

$\rho(y) < 1$, and we then have $(I_n - \bar{\xi}R(y))^{-1} = I_n + \bar{\xi}R(y) + \bar{\xi}^2 R(y)^2 + \cdots$. The sub harmonic function

$$\mu \mapsto \rho\big(\mu S(x) - S(xy)(I_n + \mu R(y) + \mu^2 R(y)^2 + \cdots)\big)$$

is well-defined on a neighborhood of the closed unit disk and is bounded by $\rho(x)$ for $|\mu| = 1$. Using themaximum principle we see that, for $\rho(y) < 1$ and $x \in \mathcal{A}$ we have

$$\rho\big((\mu S(x) - S(xy))(I_n + \mu R(x) + \mu^2 R(y)^2 + \cdots)\big) \le \rho(x)(|\mu| \le 1) \qquad (14)$$

For $\mu = 0$ in (14) we get $\rho(S(xy)) \le \rho(x)\ for\ \rho(y) < 1$, and using once more the conjugate-homogeneity of$S$ we infer that

$$\rho\big(S(xy)\big) \le \rho(x)\rho(y)(x, y \in \mathcal{A}), \qquad (15)$$

Taking the trace of the analytic function in the left hand side of the inequality from (14) and computing the coefficients of μ and μ^2, the Cauchy inequalities imply the existence of $c_1 > 0$ and $c_2 > 0$ such that

$$|tr(S(x) - tr(S(xy)R(y))| \le c_1\rho(x)(x \in \mathcal{A}, \rho(y) < 1) \qquad (16)$$

And

$$|tr(S(x)R(y) - tr(S(xy)R(y)^2)| \le c_2\rho(x)(x \in \mathcal{A}, \rho(y) < 1). \qquad (17)$$

Taking $y = 1$ in (15) we get $|tr(S(x))| \le n\rho(x)$, and then from (16) we infer the existence of $c_3 > 0$ suchthat $|tr(S(xy)R(y))| \le c_3\rho(x)$ for all $x, y \in \mathcal{A}$, with $\rho(y) < 1$. Since S is conjugate-homogeneous and R ishomogeneous, this gives

$$|tr(S(xy)R(y)| \le c_3\rho(x)\rho(y)^2(x, y \in \mathcal{A}). \qquad (18)$$

Using the homogeneity of S and R in (17), we have

$$|tr(S(x)R(y))\rho(y)^2 - tr(S(xy)R(y)^2)| \le c_3\rho(x)\rho(y)^3(x, y \in \mathcal{A}). \qquad (19)$$

(For arbitrary $x, y \in \mathcal{A}$and $\varepsilon > 0$, applying (17) to x and $y/(\rho(y) + \varepsilon)$ we see that

$$|tr(S(x)R(y))(\rho(y) + \varepsilon)^2 - tr(S(xy)R(y)^2)| \le c_2\rho(x)(\rho(y) + \varepsilon)^3, (x, y \in \mathcal{A}).$$

and then we let $\varepsilon \to 0$.) If $R(y)^2 = 0$, then (19) gives $|tr(S(x)R(y))| \le c_2\rho(x)\rho(y)$ for all $x \in \mathcal{A}$. If$R(y)^2 = R(y)$, then using (18) in (19) we obtain that for all $x \in \mathcal{A}$ we have

$$|tr(S(x)R(y))\rho(y)^2| \le c_3\rho(x)\rho(y)^2 + c_2\rho(x)\rho(y)^3.$$

Thus, there exist $c_4, c_5 \geq 0$ such that for $y \in \mathcal{A}$ if we have either $R(y)^2 = 0$or $R\ (y)^2 = R(y)$, then

$$|tr(S(x)R(y))| \leq \rho(x)\big(c_4 + c_5\rho(x)\big)(x \in \mathcal{A}). \qquad (20)$$

Let now $x, u \in \mathcal{A}$ be arbitrary and $y \in \mathcal{A}$ such that either $R(y)^2 = 0$, or $R(y)^2 = R(y)$. Since R is amorphism, for each invertible element $w \in \mathcal{A}$ we have that $R(y)^2 = 0$ implies $(R(w^{-1}yw))^2 = R(w^{-1}y^2w) = R(w^{-1})R(y)^2R(w) = 0$, and analogously $R(y)^2 = R(y)$ implies $R(w^{-1}yw)^2 = R(w^{-1}yw)$. By (20), the entire function

$$\lambda \longmapsto tr(S(x)R(e^{-\lambda u}ye^{\lambda u}))$$

is then bounded on C, and therefore, by classical Liouville's theorem, is constant. The coefficient of $\lambda$for its Taylor series is therefore zero, and using one more the fact that R is a morphism we infer that$tr\left(S(x)R(u)R(y)\right) - tr(S(x)R(y)R(u)) = 0$. Thus

$$tr\Big(\big(S(x)R(u) - R(u)S(x)\big)R(y)\Big) = 0,$$

for all $y \in \mathcal{A}$ such that either $R(y)^2 = 0$, or $R(y)^2 = R(y)$. Since R is surjective, by taking y such that$R(y)$ has 1 on the (j,k) entry and zeroes everywhere else, we obtain that $S(x)R(u) - R(u)S(x) = 0$forall $x, u \in \mathcal{A}$. We use once more the surjectivity of R to infer that $S(A) \subseteq CI_n$. Since $detS(x)$ is always
zero (see the remark following the proof of Theorem (2.7)), we obtain that S itself is identically zero. Therefore$T = R$, and the theorem is proved.

Corollary (2.8) [6].

Let A be a semi-simple commutative Banach algebras with the unite A and B acommutative Banach algebra with the unite B, respectively. Suppose that T is a multiplicative map from A onto B and preserves the spectrum. Then B is semi-simple and T is an isomorphism from A onto B, in particular, there exists a homeomorphism Φ from $M_B$onto $M_A$such that the equation

(Tf)(y) = f_Φ(y)_(y ∈M_A)

holds for every f ∈A..

Chapter 3

ACriterion for Integrability of Matrix Coefficients with Respect to Asymmetric Space

The group case reduces to Casselman's square-integrability criterion. As a consequence we assert that certain families of symmetric spaces are strongly tempered in the sense of Sakellaridis and Venkatesh. For some other families our result implies that matrix coefficients of all irreducible, discrete series representations are $\boldsymbol{G^{\theta}}$- integrable.

Section (3.1) Preliminaries on the Symmetric Subgroup:-

Let Fbe a $p-$adic field. Let$G$be the group of $F-$points of a reductive $F-$group, θ aninvolution on $G$and $H = G^{\theta}$the subgroup of θ-fixed points. In this work we providea criterion for H-integrability of matrix coefficients of admissible representations of Gin terms of their exponents along $\theta-$stable parabolic subgroups of G. In the group case$(G = H \times H, \theta(x,y) = (y,x))$ our result reduces to Casselman's square-integrability criterion.

For a smooth representation $\pi$of$G$, let Hom $H(\pi, C)$ be the space of $H-$invariant linear forms on π. As apparent, for example, from the general treatment , this space plays an essential role in the harmonic analysis of the space G/H. See also for the study of $H-$invariant linear forms on induced representations in the context of a $p-$adic symmetric space and in the more general setting of a spherical variety.

Furthermore, the understanding of $H-$invariant linear forms in the local setting has applications to the study of period integrals of automorphic forms. A conjecture of Ichino-Ikeda treats a different setting in which the pair (G, H) is of the Gross-Prasad type. Itclaims, roughly speaking, that under appropriate assumptions, the Hermitian form on an irreducible, tempered, automorphic representation of G associated to the absolute value squared of the $H-$period integral factorizes as a product of local $H-$integrals of the associated matrix coefficients. The conjectural frame work suggests a generalization of this phenomenon, which will include the symmetric case. (For an explicit factorization of a somewhat different nature .)

Integrability of matrix coefficients provides an explicit construction of the local components of period integrals of automorphic forms. Factorizable period integrals, in turn , are intimately related with special values of L-functions and with Langlands functoriality conjectures.

The above global conjectures suggest to study the following purely local questions. Let $A_G$be the maximal split torus in the centre of $G$and $A_G^{+}$the connected component of its intersection with$H$. Let πbe a smooth representation of$G$and $\tilde{v}$a smooth linear form in its contragredient$\tilde{\pi}$.

(i) Is the linear form

$$\ell_{\tilde{v}}, \pi(v) := \int_{H/A_G^+} \tilde{v}(\pi(h)v)dh$$

well defined on $\pi$by an absolutely convergent integral? (When this is the case $\ell_{\tilde{v},H} \in$Hom$H(\pi, C)$.)

(ii)Is it non-zero?

The answer we provide for the first question is a relative analogue of Casselman's criterion.

We recall that, essentially, that criterion says that an admissible representation $\pi$of $G$is square-integrable if and only if all its exponents are positive. The two main ingredients in its proof are:

(a) The Cartan decomposition of G, which allows to test convergence of a G-integral by convergence of a series summed over a positive cone in the lattice associated with a maximal split torus in G.

(b) Casselman's pairing, which is a tool to study the asymptotics of matrix coefficients in a positive enough cone in terms of its Jacquet modules along parabolic subgroups and eventually, in terms of the exponents of the representation.

Similarly, testing $H-$integrability, can be put in terms of convergence of a series summed over a positive cone in a maximal split torus in$H$. In order to apply the asymptotics of matrix coefficients of representations of G one has to relate positivity of the cone in H to positivity of relevant cones in$G$. We achieve this by further studying a root system, introduced by Helminck-Wang, associated to a symmetric space G/H. It is a root system containing that of Hthat we refer to as the descendent root system.

A key ingredient in our proof is the relation, obtained in Corollary (3.1.8), between the twonotions of positivity.

In what follows. Let P_1 be a minimal $\theta-$stable parabolicsubgroup of $G$and P_0 a minimal parabolic subgroup of Gcontained in P_1. There exists a maximal split torus A_0 of Gin P_0 that is $\theta-$stable. Let$a_0^* = X^*(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$ where $X^*(A_0)$ is the lattice of $F-$characters ofA_0. Then θacts as an involution on a_0^* and gives rise to a decomposition

$$a_0^* = (a_0^*)_\theta^+ \oplus (a_0^*)_\theta^-$$

where $(a_0^*)_\theta^\pm$is the $\pm 1-$eigenspace of θ.

Let$P$be a parabolic subgroup of$G$containing P_0 (a standard parabolic subgroup) witha standard Levi decomposition $P = M \ltimes U$and let $A_M$be the maximal split torus in the centre of$M$. Then a_0 admits a decomposition

$$a_0^* = a_M^* \oplus (a_0^M)^*$$

where $a_M^* = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. Assume that P(and therefore also$M$)is$\theta -$stable. Then $\theta -$acts ona_M^*as an involution and decomposes it into the $\bullet \pm 1 -$eigenspaces

$$a_M^* = (a_M^*)_\theta^+ \oplus (a_M^*)_\theta^- .$$

Let

$$\lambda \longmapsto (\lambda_M)_\theta^+ : a_0^* \longrightarrow (a_M)_\theta^+$$

be the projection to the first component with respect to the decomposition

$$a_0^* = (a_0^*)_\theta^+ \oplus (a_0^*)_\theta^- \oplus (a_0^M)^* .$$

Let A_M^+be the connected component of A_M^θ .Then$(a_M^*)_\theta^+ \simeq X(A_M^+) \otimes_{\mathbb{Z}} \mathbb{R}$and in particular$(a_0^*)_\theta^+ \simeq X(A_0^+) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $\sum^G$ be the root system of$G$with respect to A_0 and let $\triangle$ be the set of simple rootsdetermined by P_0. Let $\triangle^{G/H}$ (M)be the set of non-zero restrictions to $A_M^+$of the elements of $\triangle$. We say that $\lambda \in a_0^*$ is $M -$relatively positive if $(\lambda_M)_\theta^+$is a linear combination of the elements of $\triangle^{G/H}$ (M) with positive coefficients.

There are two other root systems relevant to our main result. The root system $\sum^H$ of $H$with respect to A_0^+ and the descendent root system$\sum^{G/H}$ which is the set of roots of A_0^+ in Lie(G). Let $W^H$and $W^{G/H}$be the associated Weyl groups. By definition,$\sum^H \subseteq \sum^G$ and this induces the imbedding $W^H \subseteq W^{G/H}$. In Corollary (3.1.8) we define a particular set of representatives $[W^{G/H}/W^H]$for the coset space $W^{G/H}/W^H$.

Let $\rho_0^G \in a_0^*$ be the usual half sum of positive roots in $\sum^G$(summed with multiplicities).Note that similarly, $\rho_0^H \in (a_0^*)_\theta^+$and that $W^{G/H}$H acts on $(a_0^*)_\theta^+$.Our main result takes the following form.

Theorem (3.1.1)[3]:-

Let π be an admissible *representation of G.* The matrix coefficient of $\pi$is$H -$ integrable if and only if for every $\theta -$ stable, standard parabolic subgroup $P = M \ltimes U$of$G$, *any exponent*χofπ*along*P*and any* $w \in [W^{G/H}/W^H]$*we have that*$\rho_0^G - 2w\rho_0^H + Re(\chi)$is $M -$ *relatively positive.*

For the definition of exponents of admissible representations see latter. For thedefinition of $Re(\chi) \in a_M^*$for a character $\chi$of $A_M$see (1).

Following Sakellaridis-Venkatesh, we say that $G/H$is strongly tempered (resp. strongly discrete) if matrix coefficients of irreducible, tempered (resp. discrete series) representations of $G$are all H-integrable.

Pairs of the Gross-Prasad type are strongly tempered in the special orthogonal case andin the unitary case. As a consequence of the general criterion obtained in this work, we provide examples of symmetric spaces that are strongly tempered or at least strongly discrete. We recapitulate the results here.

Corollary(3.1.2) [3] :-

Let E/F be a quadratic extension and $J \in GL_n(F)$ a symmetric matrix.

$$In\ the\ following\ cases\ G/H\ is\ strongly\ tempered:$$

G	H
$GL_n(E)$	$O_J(F)$
$\mathrm{U}_{J,E/F}(F)$	$O_J(F)$
$\mathrm{Sp}_{2n}(F)$	$\mathrm{U}_{J,E/F}(F)$
$\mathrm{GL}_2(F)$	$\mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$

Here$O_J$is the orthogonal group associated to$J$and$\mathrm{U}_{J,E/F}$the unitary group associated to $J$and E/F.

In the following cases$G/H$is strongly discrete:

G	H
$G'(E)$	$G'(F)$
$\mathrm{GL}_{2n}(F)$	$GL_n(E)$
$\mathrm{GL}_{2n}(F)$	$\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$
$\mathrm{GL}_{2n+1}(F)$	$\mathrm{GL}_n(F) \times \mathrm{GL}_{n+1}(F)$

HereG' is any reductive group defined over$F$.

For real symmetric spaces it is shown that weak positivity of $\rho_0^G - 2\rho_0^H$isequivalent to$L^2(G/H)$ being tempered. It will be interesting to study the relation between temperedness of $L^2(G/H)$ and the above properties, strongly tempered/discrete, in the p −adic case.

When $G$is split over$F$, Sakellaridis and Venkatesh show that ifG/Hisstrongly tempered then all H −invariant linear forms of an irreducible, square- integrablerepresentation $\pi$of Gemerge as H −integrals of matrix coefficients, i.e.,

$$\mathrm{Hom}_H(\pi, \mathbb{C}) = \{\ell_{\tilde{v},H}: \tilde{v} \in \tilde{\pi}\}.$$

We apply this result in end latter to some examples of symmetric spaces that are strongly tempered by our criterion. This expands on some similar recently obtained results. Pairs of Gross-Prasad type are strongly tempered and of multiplicity one. For those cases, the non-vanishing of H −integrals of matrix coefficients was obtained. For irreducible cuspidal representations it is shown for all symmetric spaces that all H −invariant linear forms emerge as H −integrals of matrix coefficients.

We recall in end of this section some basic facts about symmetric spaces. In particular we recall the definition of the descendent root system associated to a symmetric space $G/H$by Helminck and Wang and prove some relations with the root systems of $G$and of Hthat are relevant to the rest of this work. We prove the main result, a criterion for $H-$integrability of matrix coefficients. We provide examples of strongly tempered/discrete symmetric spaces based on our main result. We apply results of Sakellaridis and Venkatesh to provide examples where $H-$invariant linear forms emerge as integrals of matrix coefficients.

Let Fbe a$p-$adic field. In general, if $\boldsymbol{X}$ is an algebraic variety defined over F(an$F-$Variety) we write $X = \boldsymbol{X}(F)$ for its $F-$points.

Let G be an algebraic $F-$group and A_G the maximal $F-$split torus in the centre of G.We denote by$X^*(G)$ the group of F-rational characters of G. Let $a_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$ andlet $a_G = Hom_{\mathbb{R}}(a_G^*, \mathbb{R})$ be its dual vector space with the natural pairing $\langle .,. \rangle = \langle .,. \rangle_G$. Wehave $a_G^* = a_{A_G}^*$.

To $\lambda \otimes a \in a_G^*$we associate the character g $g \mapsto |\lambda(g)|^a$of$G$. This extends to a bijection from $a_G^*$to the group of positive continuous characters of$G$. We denote by $Re(\chi) \in a_G^*$the pre-image of a positive character$\chi: G \to R_{>0}$. If $\chi: G \to \mathbb{C}^*$ is any continuous homomorphism then we set

$$Re(\chi) = Re(|\chi|). \quad (1)$$

Let $X_*(G)$ be the set of one parameter subgroups of G(i.e., $F-$homomorphisms $\mathbb{G}_m \to \boldsymbol{G}$).For an $F-$torus$\boldsymbol{T}$, $X_*(T)$ is a free abelian group of finite rank. The natural pairing of$X_*(T)$ with $X^*(G)$ allows us to identify $a_T$with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$

Let $\delta_G$be the modulus function of G^1.

From now on assume that $\boldsymbol{G}$ is a connected reductive group. Let $\boldsymbol{A}_G$be the maximal$F-$split torus in the centre of$\boldsymbol{G}$.

Let $\boldsymbol{P}_0 = \boldsymbol{M}_0 \ltimes \boldsymbol{U}_0$ be a minimal parabolic $F-$subgroup of $\boldsymbol{G}$ with Levi component $\boldsymbol{M}_0$and unipotent radical $\boldsymbol{U}_0$. Set $\boldsymbol{A}_0 = A_{M_0}, \boldsymbol{a}_0 = \boldsymbol{a}_{M_0}$and $\boldsymbol{a}_0^* = \boldsymbol{a}_{M_0}^*$.Then $\boldsymbol{A}_0$ is a maximal$F-$split torus in$\boldsymbol{G}$.

A parabolic $F-$subgroup P of G is called semi-standard if it contains $\boldsymbol{A}_0$, and standardif it contains $\boldsymbol{P}_0$ If $\boldsymbol{P}$ is semi-standard, it admits a unique Levi subgroup $\boldsymbol{M}$ containing$\boldsymbol{A}_0$. We will say that $\boldsymbol{M}$ is a semi-standard Levi subgroup of$\boldsymbol{G}$. When we write that$\boldsymbol{P} = \boldsymbol{M} \ltimes \boldsymbol{U}$ is a semi-standard parabolic $F-$subgroup of$\boldsymbol{G}$, we will mean that $\boldsymbol{M}$ is the unique semi-standard Levi subgroup of $\boldsymbol{P}$ and $\boldsymbol{U}$ is the unipotent radical of$\boldsymbol{P}$.

$$\boldsymbol{a}_0 = \boldsymbol{a}_M \otimes a_0^M.$$

More generally, if $\boldsymbol{Q} = \boldsymbol{L} \ltimes \boldsymbol{V}$ is another semi-standard $F-$parabolic subgroup of $\boldsymbol{G}$ containing$\boldsymbol{P}$ then $a_L$is a subspace of $a_M$and there is a canonical decomposition

$$\boldsymbol{a}_M = \boldsymbol{a}_L \otimes a_M^L.$$

Similar decompositions apply to the dual spaces. For $\lambda \in \boldsymbol{a}_0^*$ we denote by $\lambda_M$its projection to $\boldsymbol{a}_M^*$ and by λ_M^Lits projection to $(\boldsymbol{a}_M^L)^*$.
Let T be an$F$ −split torus in$\boldsymbol{G}$. If $0 \neq \mathcal{V} \in Lie(G)$ and $0 \neq \alpha \in X^*(T)$ are such that$Ad(t)v = \alpha(t)v, t \in \boldsymbol{T}$ then we say that $\alpha$is a root of $G$with respect to $T$and$v$is a root vector with rootα. Let $R(T, G)$ be the set of all roots of $G$with respect to$T$.

Let $\sum = \sum^G R(A_0, G)$ It is a subset of $X^*(A_0)$ that spans $(\boldsymbol{a}_0^G)^*$and forms a root system. $Let \sum^{>0} = \sum^{G,>0} = R(A_0, P_0)$be the set of positive roots and Δ=$\Delta^G$the basis of simple roots with respect to P_0. Let W^Gdenote the Weyl group of $\sum^G$ .For a standard parabolic subgroup $\boldsymbol{P} = \boldsymbol{M} \ltimes \boldsymbol{U}$ of $\boldsymbol{G}$ let Δ^M=$\Delta \cap \sum^M$ be the set of simple roots of M with respect to $M \cap P_0$. Furthermore, let

$$\Delta_M = \left\{\alpha|_{A_M} : \alpha \in \Delta^M\right\} \backslash\{0\}.$$

For $\lambda \in X_*(G)$ we associate a parabolic F −subgroup $\boldsymbol{P}(\lambda) = \boldsymbol{P_G}(\lambda)$. It is defined as the set of points $x \in \boldsymbol{G}$ so that the map $a \mapsto \lambda(\mathrm{a})x\lambda(\mathrm{a})^{-1}: \mathbb{G}_m \to \boldsymbol{G}$extends to an$F$ −rational map $\mathbb{G}_a \to G$. (Here we view the multiplicative group $\mathbb{G}_m$as a subvariety of the additive group $\mathbb{G}_a$.) It naturally comes with a Levi decomposition$\boldsymbol{P}(\lambda) = \boldsymbol{M}(\lambda) \ltimes \boldsymbol{U}(\lambda)$ where the Levi component$\boldsymbol{M}(\lambda)$ is the centralizer of the image of $\lambda$and the unipotent radical consists of the elements $x$where the above extended map sends 0 to the identity in $\boldsymbol{G}$. The group $\boldsymbol{P}(-\lambda)$ is the parabolic subgroup of $\boldsymbol{G}$ opposite to $\boldsymbol{P}(\lambda)$ so that $\boldsymbol{P}(\lambda) \cap \boldsymbol{P}(-\lambda) = \boldsymbol{M}(\lambda)$. Every parabolic F −subgroup of $\boldsymbol{G}$ is of the form $\boldsymbol{P}(\lambda)$ for some $\in X_*(G)$.
Furthermore, every semi-standard parabolic F −subgroup of G is of the form $P(\lambda)$ where $\lambda \in X_*(A_0)$.(In fact, semi-standard parabolic F −subgroups of G are in bijection with facets of $a_0$with respect to root hyperplanes associated to$\sum$.)

For a subset $I \subseteq \Delta$let $\lambda_I \in X_*(A_0)$be such that $\langle \alpha, \lambda_I \rangle = 0.$ for all $\alpha \in I$and $\langle \alpha, \lambda_I \rangle > 0$ for all $\alpha \in \Delta/I$.Then$\boldsymbol{P_I} = \boldsymbol{P}(\lambda_I)$ is a standard parabolic$F$ −subgroup of G. In fact, $\boldsymbol{P_I}$is independent of a choice of $\lambda_I$as above and $I \mapsto \boldsymbol{P_I}$is an order preserving bijection between subsets of $\Delta$and standard parabolic F −subgroups of$G$. We denote by $\boldsymbol{P_I} = \boldsymbol{M_I} \ltimes \boldsymbol{U_I}$ the associated Levi decomposition and let $\boldsymbol{A_I} = \boldsymbol{A_{M_I}}$. Then $A_I$is the connected component of $\cap_{\alpha \in I} \ker\alpha \subseteq A_0$and Δ^{MI}. Note that $P_\emptyset = P_0$ and $P_\Delta = G$.
Let T be an$F$ −split torus. For a subset$S \subseteq X^*(T)$let

$$a_T^{s,>0} = \{x \in a_T : \langle \alpha, x \rangle > 0, \alpha \in S\}$$

$a_T^{s,\geq 0}$be its closure and

$$X_*(T)^{s,\geq 0} = X_*(T) \cap a_T^{s,\geq 0}.$$

Also let

$$C(T, S) = \left\{\sum_{\alpha \in S} a_\alpha \alpha : a_\alpha \in \mathbb{R}_{>0} \alpha \in S\right\}$$

and let $\bar{C}(T, S)$ be its closure.
Fix a uniformizer ϖ of Fonce and for all. Then,$X_*(T)$ can be embedded in $T$by

$x \mapsto x(\bar{w})$. We denote the image of this embedding by C_T. Then $T/C_T$is compact.
Let$C_T^{s,\geq 0}$be the image of $X_*(T)^{s,\geq 0}$in C_T..
Let $P = M \ltimes U$ be a standard parabolic F −subgroup of G. For $\epsilon > 0$ let

$$C_{AM}^{s,>0}(\epsilon) = \{a \in C_{AM}: |\alpha(a)|_F < \epsilon, \alpha \in \Delta_M\}.$$

Note that if $\epsilon \leq 1$ then$C_{AM}^{>0}(\epsilon) \subseteq C_{AM}^{\Delta_M,\geq 0}$,
Let

$$C_0^{\geq 0}(\epsilon) = C_{A_0}^{\Delta^G,\geq 0}$$

and fix a maximal compact subgroup $K = K^G$ of G'adapt`eáA_0' in the terminology. By our choice of$K$there exists a finite set F_0 in M_0 such that

$$G = \coprod_{0 \in C_0^{\geq 0}} \coprod_{f \in F_0} KfcK.$$

Fix a Haar measure on $G$and let vol(X) denote the measure of a subset $X$of G.

Lemma (3.1.3) [3]:-

There exists a basis$\mathcal{J}$of neighbourhoods of the identity in G consisting of open normal subgroups of$K$such that

$$\text{vol}(K_0 fcK_0) = \delta_{P_0}^{-1}(fc)\,\text{vol}(k_0)\ for\ all\ K_0 \in \mathcal{J}$$

Let θbe an involution on G defined over$F$and

$$\mathbf{H} = \boldsymbol{G}^\theta = \{g \in \boldsymbol{G}: \theta(g) = g\}.$$

We further denote by $\theta$the differential of its action on$G$. It is an involution on Lie$(G)$and

$$\text{Lie }(H) = \text{Lie }(G)^\theta \tag{2}$$

Let $\mathbf{H}^\circ$be the connected component of the identity in$\mathbf{H}$. It is a connected reductive F −group and $\mathbf{H}^\circ$is of finite index in $\mathbf{H}$.
For a θ −stable F −torus $\mathbf{T}$ in G let $\mathbf{T}^+$ (resp. $\mathbf{T}^-$) be the maximal subtorus of $\mathbf{T}^\theta$(resp. $\{t \in T: \theta(t) = t^{-1}\}$). Then $\mathbf{T} = \mathbf{T}^+\mathbf{T}^-$ In particular, an element of $X^*(T)$is determined byits restrictions to $\mathbf{T}^+$ and $\mathbf{T}^-$
Note that θinduces an involution on the set$X_*(G)$ that we further denote by θ, its fixedpoints are precisely the elements of $X_*(H)$.

Lemma (3.1.4) [3]:-

The collection of parabolic$F$ −subgroups of$\mathbf{H}^\circ$is the set of groups of the form$\mathbf{P} \cap \mathbf{H}^\circ$where$\mathbf{P}$is a$\theta$ −stable parabolic$F$ −subgroup of$\mathbf{G}$.

Proof:-

A parabolic F −subgroup of $\mathbf{H}^\circ$is of the form $P_{\mathbf{H}^\circ}(\lambda)$, where $\lambda \in X_*(H^\circ) \subseteq X_*(G)$.It follows by definition that $\mathbf{P}_{\mathbf{H}^\circ}(\lambda) = \mathbf{P}_\mathbf{G}(\lambda) \cap \mathbf{H}^\circ$.Note further that $\theta(\lambda) = \lambda$andtherefore $\theta(\mathbf{P}_\mathbf{G}(\lambda)) = \mathbf{P}_\mathbf{G}(\theta(\lambda)) = \mathbf{P}_\mathbf{G}(\lambda)$, i.e.,$\mathbf{P}_\mathbf{G}(\lambda)$ is a θ −stable parabolic F-subgroup of$\boldsymbol{G}$.

Conversely, suppose that P is a $\theta-$stable parabolic $F-$subgroup of G. There exists a maximal $\theta-$stable torus A of G contained inside P. Now, there exists $\lambda \in X_*(A^+)$such that $P = \mathbf{P_G}(\lambda)$.Since$A^+ \subseteq \mathbf{H}^\circ$, the $F-$subgroup $\boldsymbol{P} \cap \mathbf{H}^\circ = \mathbf{P}_{\mathbf{H}^\circ}(\lambda)$ of $\mathbf{H}^\circ$is parabolic.

Fix a minimal parabolic $F-$subgroup $\boldsymbol{P_0^H}$ of $\mathbf{H}^\circ$. Let $\boldsymbol{P_1}$ be minimal amongst the$\theta-$stable parabolic $F-$subgroups$\boldsymbol{P}$of $\boldsymbol{G}$ such that $\boldsymbol{P} \cap \mathbf{H}^\circ = \boldsymbol{P_0^H}$. It follows from Lemma (3.1.4) that $\boldsymbol{P_1}$ is in fact a minimal $\theta-$stable parabolic $F-$subgroup of $\boldsymbol{G}$.

We may choose the minimal parabolic $F-$subgroup $\mathbf{P_0}$of $\boldsymbol{G}$ to be contained in $\mathbf{P_1}$. We may and do further choose $\mathbf{A_0}$ to be $\theta-$stable. Thus θacts on $X_*(\mathrm{A}_0)$, $X^*(\mathrm{A}_0)$, a_0 and $\boldsymbol{a_0^*}$.

Note that if $\alpha \in \sum^G$ has root vector $v \in \mathrm{Lie}(G)$ then

$$Ad\big(\theta(a)\big)\theta(v) = \theta(Ad(a)v) = \alpha(a)\theta(v),\ \ a \in \boldsymbol{A_0},$$

i.e., $\theta(v)$ is a root vector for $\theta(\alpha)$ and therefore θacts on $\sum^G$ and maps the root space of $\alpha$to that of$\theta(\alpha)$.

If $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ is a semi-standard $\theta-$stable parabolic $F-$subgroup of $\mathbf{G}$then$\mathbf{U}$ and $\mathbf{M}$are $\theta-$stable by the uniqueness of the semi-standard Levi decomposition. Thus, AM is also $\theta-$stable.

$\boldsymbol{A_0^+}$is a maximal $F-$split torus of H and the standard Levidecomposition $\mathbf{P}_1 = \mathbf{M}_1 \ltimes \mathbf{U}_1$ is such that $\mathbf{M}_1$ is the centralizer of $\boldsymbol{A_0^+}$ in $\mathbf{G}$.

Since θacts as an involution on a_0 it decomposes it into a direct sum of the $\pm 1-$eigenspaces which we denote by $(a_0)_\theta^\pm$. Similarly.

$$a_0^* = (a_0^*)_\theta^+ \otimes (a_0^*)_\theta^- .$$

The inclusion $X_*(\mathrm{A}_0^+) \subseteq X_*(\mathrm{A}_0)$ induces the identification

$$X_*(\mathrm{A}_0^+) \otimes_{\mathbb{Z}} \mathbb{R} \simeq (a_0)_\theta^+.$$

It is straightforward that the pairing $\langle .,. \rangle_G$is θinvariant and therefore $(a_0^*)_\theta^\pm$is the dualof $(a_0)_\theta^\pm$.Thus,$\langle .,. \rangle_G$restricted to $(a_0^*)_\theta^+ \times (a_0^*)_\theta^-$is the natural pairing $\langle .,. \rangle_H$defined with respect to A_0^+.

Let $\mathbf{P} = \mathbf{M} \ltimes \mathbf{U}$ be a standard, $\theta-$stable parabolic $F-$subgroup of G. Then θacts asan involution on $\mathrm{a_M}$and we obtain a decomposition $\mathrm{a_M} = (\mathrm{a_M})_\theta^+ \otimes (\mathrm{a_M})_\theta^-$to the • $\pm 1-$eigenspaces . A similar decomposition holds for the dual space and $(\mathrm{a_M})_\theta^\pm$is the dual of $(a_\mathrm{M})_\theta^\pm$. We have $(a_\mathrm{M})_\theta^+ = a_{\mathrm{A_M^+}}$and similarly for the dual space. We denote by $\lambda_\theta^\pm$theprojection of $\lambda \in a_\mathrm{M}^*$to $(a_\mathrm{M}^*)_\theta^\pm$.

Every $\theta-$stable, semi-standard parabolic $F-$subgroup of G is of the form $\mathbf{P_G}(\lambda)$ for some $\lambda \in X_*(\mathrm{A}_0^+)$.In particular, there exists $\lambda_1 \in X_*(\mathrm{A}_0^+)$ such that $\mathbf{P_1} = \mathbf{P_G}(\lambda_1)$.

Let$\sum^H = R(\mathrm{A}_0^+, H)$be the root system of $H, \sum^{H,>0} = R(\mathrm{A}_0^+, \mathrm{P}_0^\mathrm{H})$ the subset of positiveroots and $\Delta^H$the basis of simple roots with respect to $\mathrm{P}_0^\mathrm{H}$and$W^H$the Weyl group of $\sum^H$.

Let $\sum^{G/H} = R(A_0^+, G)$be the set of roots of A_0^+in Lie(G). Clearly $\sum^{H} \subseteq \sum^{G/H}$ .It follows that, unlessempty, $\sum^{G/H}$ is a root system with Weyl group$W^{G/H} = N_G(A_0^+)/G_G(A_0^+)$(Recall that $G_G(A_0^+) = M_1$). In particular, $W^H \subseteq W^{G/H}$. Furthermore, if $\sum^{G/H}$ is empty then $H/A_G^+$is compact. This case will be of little interest to us and we assume in what follows that $H/A_G^+$is isotropic. We call$\sum^{G/H}$ the descendent root system.

Since the root space decomposition of Lie(G) with respect to A_0 automatically provides a decomposition of Lie(G) into A_0^+ -eigenspaces we have

$$\sum^{G/H} = \{\alpha|_{A_0^+} : \alpha \in \sum^{G}\}\backslash\{0\}. \tag{3}$$

Lemma (3.1.5) [3]:-

Let $\alpha \in \sum^G$ be such that $\alpha|_{A_0^+} \in \sum^G$ .Then$\alpha \in \sum^{G,>0}$ if and only if $\alpha|_{A_0^+} \in \sum^{H,>0}$.

Proof:-

Recall that $\lambda_1 \in X_*(A_0^+)$ is such that $P_1 = P_{\mathbf{G}}(\lambda_1)$ and $P_0^H = \mathbf{P}_{\mathbf{H}^\circ}(\lambda_1)$. Thus,
$\alpha|_{A_0^+} \in \sum^{G,>0}$ if and only if ,$\langle \alpha|_{A_0^+} , \lambda_1\rangle_H > 0$. Our embedding of $X^*(A_0^+)$ in $(a_0^*)_\theta^+\theta$ identifies $\alpha|_{A_0^+}$with $\frac{1}{2}\big(\alpha + \theta(\alpha)\big)$. Since $\theta(\lambda_1) = \lambda_1$it follows that

$$\langle \alpha , \lambda_1\rangle_G = \langle \tfrac{1}{2}\big(\alpha + \theta(\alpha)\big) , \lambda_1\rangle_G = \langle \alpha|_{A_0^+} , \lambda_1\rangle_H \quad .$$

Since $U_1 \subseteq U_0$ it follows immediately that if $\alpha|_{A_0^+} \in \sum^{H,>0}$ then $\alpha \in \sum^{G,>0}$ Conversely, if $\alpha \in \sum^{G,>0}$ then $\langle \alpha , \lambda_1\rangle_G \geq 0$. If $\langle \alpha , \lambda_1\rangle_G = 0$ then $\alpha \in R(M_1, A_0)$.But since A_0^+ contained in the centre of M_1 this contradicts the fact that $\alpha|_{A_0^+}$ 0is non-trivial. It follows that $\langle \alpha , \lambda_1\rangle_G > 0$ and therefore that $\alpha|_{A_0^+} \in \sum^{H,>0}$.

$$\theta(x)|_{A_0^+} = x|_{A_0^+} \text{and} \theta(x)|_{A_0^-} = -x|_{A_0^-} \text{ for all } x = X^*(A_0) \tag{4}$$

It follows that

$$x + \theta(x) = 0 \text{ if and only if } x|_{A_0^+} = 0 \tag{5}$$

Let

$$\Delta^G\,[\theta = -1] = \{\alpha \in \Delta^G ; \theta(\alpha) = -\alpha\}\underline{\underline{\delta}}\{\alpha \in \Delta^G : \alpha|_{A_0^+} = 0\}.$$

LetX_0 be the subgroup of $X^*(A_0^+)$ generated by $\Delta^G\,[\theta = -1]$. Also set

$$\Delta^G\,[\theta \neq -1] = \Delta^G / \Delta^G\,[\theta = -1]$$

Lemma (3.1.6) [3]:-

For every $\alpha \in \Delta^G\,[\theta \neq -1]$there exist $\beta \in \Delta^G\,[\theta = -1]$and $x \in X_0$ such that $\theta(\alpha) = \beta + x$.

Proof:-

It follows from the definitions that X_0 is $\theta -$stable. Thus, the action of θon $X^*(A_0)$ induces an action (that we still denote by θ) as an involution on $\Gamma := X^*(A_0)/X_0$.

Let $\alpha \in \Delta^G\,[\theta \neq -1]$. If $\theta(\alpha) = \alpha$then $\beta = \alpha, x = 0$ and we are done. Assumethat $\theta(\alpha) \neq \alpha$. Let $v \in$ Lie(G) be a root vector for α. Then $\theta(v)$ is a root vector for$\theta(\alpha)$ and by our assumption $v$and $\theta(v)$ are linearly independent. It follows from (4) that$v +$

$\theta(v) \in \mathrm{Lie}(G)^{\theta} = \mathrm{Lie}(H)$ is a root vector for the root $\alpha|_{A_0^+} \in \sum^{H}$. By Lemma (3.1.5)$\alpha|_{A_0^+} \in \sum^{H,>0}$ and $\theta(\alpha) \in \sum^{G,>0}$

Let $x \mapsto \bar{x}$ be the projection of $X^*(A_0)$ to Γ and let $\Delta^G [\theta \neq -1] = \{\alpha_1, \dots, \alpha_t\}$. Clearly, $\{\bar{\alpha}_1, \dots, \bar{\alpha}_t\}$are $\mathbb{Z}$ −linearly independent in Γ. Since $\theta(\alpha_i) \in \sum^{G,>0}$ for all i, it follows that there exists $M = (n_{i,j}) \in M_t(\mathbb{Z})$, a matrix of non-negative integers, such that

$$\overline{\theta(\alpha_\iota)} = \sum_{j=1}^{t} n_{i,j}\, \bar{\alpha}_i.$$

Since $\theta$is an involution we get that $M^2 = I_t$is the identity matrix. It is now straightforward that$M$is a permutation matrix. The lemma follows

$$\Delta^{G/H} = \{\alpha|_{A_0^+} : \alpha \in \Delta^G [\theta \neq -1]\} = \{\alpha|_{A_0^+} : \alpha \in \Delta^G\} \backslash \{0\} \subseteq X^*(A_0^+)$$

Proposition (3.1.7) [3]:-

The set$\Delta^{G/H}$is a basis of simple roots for the descendentroot system$\sum^{G/H}$.

Proof:-

Let $\beta = \alpha|_{A_0^+} \in \sum^{G/H}$ with $\alpha \in \sum^G$ (see (3)). Then either αor $-\alpha$is a

Linear combination with positive integer coefficients of elements of Δ. Restricting to A_0^+ we getthat, respectively, βor $-\beta$is a linear combination with positive integer coefficients of elements of $\Delta^{G/H}$ .To prove the proposition we therefore only need to show that $\Delta^{G/H}$is linearly independent. Set $\Delta^{G/H} = \{\beta_1, \dots, \beta_t\}$and fix $\alpha_1, \dots, \alpha_t \in \Delta^G [\theta \neq -1]$ so that $\beta_i = \alpha_i|_{A_0^+}$, $i = 1, \dots, t$. Let $\alpha_i' \in \Delta^G [\theta \neq -1]$ be given by Lemma (3.1.6) so that$\theta(\alpha_i) - \alpha_i' \in X_0$. After rearrangement we may assume that there exist $k, 0 \leq k \leq t$such that $\alpha_i' = \alpha_i$if and only if $i \leq k$. Note that $\{\alpha_i : i = 1, \dots, t\} \cup \{\alpha_i' : k < i \leq t\}$is a subset of exactly $2t - k$ elements in $\Delta^G [\theta \neq -1]$.

Suppose that $x_1\beta_1, \dots, x_t\beta_t = 0, x_1, \dots, x_t \in \mathbb{R}$and let $\gamma = x_1\alpha_1, \dots, x_t\alpha_t$. Then $\gamma|_{A_0^+} = 0$and by (5) $\gamma + \theta(\gamma) = 0$. Therefore

$$\sum_{i=1}^{k} 2x_i\alpha_i + \sum_{i=k+1}^{t} x_i(\alpha_i + \alpha_i') \in X_0.$$

From the linear independence of Δ^Git follows that $x_i = 0$ for all i.The proposition follows.

Note that our identifications give an action of the Weyl group $W^{G/H}$on the vector space $(a_0)_\theta^+$and on its dual $(a_0^*)_\theta^+$.

Corollary (3.1.8) [3]:-

We have

(a)$\Delta^G \subseteq \bar{C}(A_0^+, \Delta^{G/H})$;

(b) $[(a_0)_\theta^+]^{\Delta^{G/H,\geq 0}} \subseteq [(a_0)_\theta^+]^{\Delta^{H,\geq 0}}$ *and hence* $X_*(A_0^+)^{\Delta^{G/H,\geq 0}} \subseteq X_*(A_0^+)^{\Delta^{H,\geq 0}}$;

(c) The set

$$[W^{G/H}/W^{H}] := \{w \in W^{G/H}: w^{-1}\,[(a_0)^+_\theta]^{\Delta^{G/H},\geq 0} \subseteq\ [(a_0)^+_\theta]^{\Delta^{H},\geq 0}\}$$

$$forms\ a\ complete\ set\ of\ representatives\ for W^{G/H}/W^{H} and$$

$$X_*(A_0^+)^{\Delta^{H,\geq 0}} = \cup_{w\in [W^{G/H}/W^{H}]} w^{-1} X_*(A_0^+)^{\Delta^{G/H},\geq 0};$$

(d)*For every* $w \in\ [W^{G/H}/W^{H}], w(\sum^{H,\geq 0}) \subseteq \bar{C}(A_0^+,\Delta^{G/H})$.

Proof:-

Since $\sum^{H} \subseteq \sum^{G/H}$ it follows from (c) and Lemma (3.1.5) that every element of $\sum^{H,\geq 0}$ is a restriction to $A_0^+$of an element of$\sum^{G,\geq 0}$. In particular, if $\beta = \alpha|_{A_0^+} \in \Delta^{H}$with $\alpha \in \sum^{G,\geq 0} \subseteq \bar{C}(A_0,\Delta^{G})$then writing αas a positive linear combination of elements of Δ^{G} and restricting to A_0^+ shows that $\beta \in \bar{C}(A_0^+,\Delta^{G/H})$. This shows part (a).

Part (b) is straightforward from part (a).

Recall that $\sum^{H} \subseteq \sum^{G/H}$ are root systems in $(a_0^*)^+_\theta$. For$\lambda \in (a_0^*)^+_\theta$Let

$$H_\lambda = \{x \in (a_0)^+_\theta: \langle \lambda, x\rangle = 0\}.$$

We have the Weyl chamber decomposition in the dual space

$$(a_0)^+_\theta \backslash (\cup_{\alpha\in\sum^{H}} H_\alpha) = \coprod_{w\in W^{H}} w\,[(a_0)^+_\theta]^{\Delta^{H},\geq 0}$$

with respect to the root system $\sum^{H}$. The union is of connected components. By Proposition (3.1.7) we similarly have a decomposition

$$(a_0)^+_\theta \backslash (\cup_{\alpha\in\sum^{G/H}} H_\alpha) = \coprod_{w\in W^{G/H}} w\,[(a_0)^+_\theta]^{G/H,\geq 0}$$

with respect to the root system $\sum^{G/H}$.

Since $\cup_{\alpha\in\sum^{H}} H_\alpha \subseteq \cup_{\alpha\in\sum^{G/H}} H_\alpha$,any connected component of $(a_0)^+_\theta \backslash (\cup_{\alpha\in\sum^{H}} H_\alpha)$is contained in a connected component of $(a_0)^+_\theta \backslash (\cup_{\alpha\in\sum^{G/H}} H_\alpha)$. In particular, taking closures we have

$$[(a_0)^+_\theta]^{\Delta^{H},\geq 0} = \cup_{w\in [W^{G/H}/W^{H}]} w^{-1}\,[(a_0)^+_\theta]^{\Delta^{G/H},\geq 0}$$

And part (c) follows.

Finally, for all$w \in\ [W^{G/H}/W^{H}], \alpha \in \sum^{H,\geq 0}$ and $\lambda \in\ [(a_0)^+_\theta]^{\Delta^{G/H},\geq 0}$we have

$$\langle w(\alpha), \lambda\rangle = \langle \alpha, w^{-1}(\lambda)\rangle \geq 0$$

Note, that $[(a_0)^+_\theta]^{\Delta^{G/H},\geq 0}$and $\bar{C}(A_0^+,\Delta^{G/H})$are both closed convex cones in Euclidean spaces, in the sense that they are closed under linear combinations with positive coefficients. Hence, by duality of convex cones, we have

$$\bar{C}(A_0^+,\Delta^{G/H}) = \{x \in (a_0^*)^+_\theta : \langle \alpha, \lambda\rangle \geq 0, \forall \lambda \in\ [(a_0)^+_\theta]^{\Delta^{G/H},\geq 0}\}$$

The corollary follows.

Lemma (3.1.9) [3]:-

(a)*The dual lattices* $X^* = X^*(A_0^+/\,A_G^+)$ *and* $X_* = X_*(A_0^+)/X_*(\,A_G^+)$ *are of rank* $|\Delta^{G/H}|$

$(b) There\ exists\ a\ set \{y_\alpha : \alpha \in \Delta^{G/H}\} \subseteq X_*(A_0^+) such\ that \langle \alpha, y_\alpha \rangle > 0\ \ and\ \langle \alpha, y_\beta \rangle = 0\ for\ all\ \ \alpha \neq \beta\ \ in\ \ \Delta^{G/H}$

$(c) For\ such\ a\ set \{y_\alpha : \alpha \in \Delta^{G/H}\}, let\ Y\ be\ the\ subgroup\ of X_* generatedby\ the\ images\ of\ the y_\alpha's\ and Y^{\geq 0} be\ the\ subset\ of Y givenby\ images\ of\ \ elementsof\ the\ form$

$$\sum_{\alpha \in \Delta^{G/H}} n_\alpha y_\alpha\ with \mathbb{Z}_{\geq 0}$$

$Then\ Y\ isof\ finite\ index\ in X_* and\ there\ exists\ acomplete\ set\ of\ representatives\ E\ for X_*/Y so\ that\ we\ have\ the\ disjoint\ union$

$$X_*(A_0^+)^{\Delta^{G/H},\geq 0} / X_*(A_G^+) = \coprod_{e \in E} e + Y^{\geq 0}$$

Proof:-

By definition we have

$$\bigcap_{\beta \in \Delta^{G/H}} \ker \beta \subseteq \bigcap_{\alpha \in \Delta^G} ker\ \alpha$$

Hence, since A_G is the connected component of $\bigcap_{\alpha \in \Delta^G} ker \alpha$, we also have that A_G^+ is the connected component of $\bigcap_{\beta \in \Delta^{G/H}} \ker \beta$.

It follows that $\Delta^{G/H}$ embeds into X_* and its image is a basis of the $\mathbb{Q}-$vector space $X^* \otimes_{\mathbb{Z}} \mathbb{Q}$. In particular part (a) follows. For each element of the dual basis (of $X^* \otimes_{\mathbb{Z}} \mathbb{Q}$.) there is a positive integer that multiplies it into X_*. Choosing representatives mod $X_*(A_G^+)$ we obtain a set $\{y_\alpha : \alpha \in \Delta^{G/H}\}\}$ as in (b). As its image in X_* is a basis of $X^* \otimes_{\mathbb{Z}} \mathbb{Q}$ it follows that Y is of finite index in X_*.

Let E' be a complete set of representatives for X_*/Y and let $c_\alpha = \langle \alpha, y_\alpha \rangle > 0, \alpha \in \Delta^{G/H}$. For $e' \in E'$ let $m_{e',\alpha} \in \mathbb{Z}$ be minimal such that $\langle \alpha, e' \rangle + m_{e',\alpha} c_\alpha > 0$ and let $e = e' + \sum_{\alpha \in \Delta^{G/H}} m_{e',\alpha} y_\alpha$. Then $E = \{e : e' \in E'\}$ is still a complete set of representatives for X_*/Y. Note that

$$\langle \alpha, e \rangle = \langle \alpha, e' \rangle + m_{e',\alpha} c_\alpha \geq 0$$

Hence $E \subseteq X_*(A_0^+)^{\Delta^{G/H},\geq 0}$ and $\langle \alpha, e \rangle = \min_{x \in X_*(A_0^+)^{\Delta^{G/H},\geq 0} \cap (e+Y)} \langle \alpha, x \rangle$ for all $\alpha \in \Delta^{G/H}$.

It follows that

$$X_*(A_0^+)^{\Delta^{G/H},\geq 0} \cap (e + Y) = e + Y^{\geq 0}$$

And part (3) follows. _

Let $\boldsymbol{P} : \Delta^G [\theta \neq -1] \to \Delta^{G/H}$ be the surjective map defined by restriction to A_0^+.

Lemma (3.1.10) [3]:-

Let $I \subseteq \Delta^G$ .Then $\boldsymbol{P}_I$ is $\theta-$stable if and only if there exists $J \subseteq \Delta^{G/H}$ such that $I = \Delta^G [\theta \neq -1] \cup P^{-1}(J)$. In particula, $\boldsymbol{P}_I = \boldsymbol{P}_{\Delta^G[\theta=-1]}$.

Proof:-

Assume that $\boldsymbol{P}_I$is θ −stable. Recall that we may take $\boldsymbol{\lambda}_I \in X_*(\mathrm{A}_0^+) \subseteq (a_0)_\theta^+$so that $P_I = P_G(\lambda_I)$. By definition $\Delta^G [\theta = -1] \subseteq (a_0^*)_\theta^-$and therefore, $\langle \alpha, y_I \rangle_G = 0$ for all $\alpha \in \Delta^G [\theta = -1]$. As argued in the proof of Lemma (3.1.6), for $\alpha\ \Delta^G [\theta \neq -1]$ we have $\langle \alpha, y_I \rangle_G = \langle P(\alpha), y_I \rangle_H$. It follows that

$$I = \{\alpha \in \Delta^G\ \langle \alpha, y_I \rangle_G = 0\} = \Delta^G [\theta = -1] \cup P^{-1}(J),$$

Where$J = \left\{\alpha \in \Delta^{G/H}\ \langle \beta, \lambda_I \rangle_H = 0\right\}$

Conversely, let J $\subseteq \Delta^{G/H}$and $I = \Delta^G [\theta = -1] \cup P^{-1}(J)$. It follows from Proposition (3.1.7) and Lemma (3.1.9)(1) that there exists $\mu \in X_*(\mathrm{A}_0^+)$ such that $\langle \beta, \mu \rangle_H = 0$ if $\beta \in J$and $\langle \beta, \mu \rangle_H > 0$ if $\beta \in \Delta^{G/H} \setminus J$. Arguing as above we get that $I = \{\alpha \in \Delta^G\ \langle \alpha, \mu \rangle_G = 0\}$.

Therefore $P_I = P_G(\mu)$. As in Lemma (3.1.4) it follows that $P_I$is θ −stable.For a standard, θ −stable parabolic F −subgroup $P = M \ltimes U$ of G let

$$\Delta^{G/H} (M) = \{\beta|_{\mathrm{A}_\mathrm{M}^+} : \beta \in \Delta^{G/H}\} \setminus \{0\} = \{\alpha|_{\mathrm{A}_\mathrm{M}^+} : \alpha \in \Delta^G\} \setminus \{0\}.$$

Let $J \subseteq \Delta^{G/H}$and $I = \Delta^G [\theta = -1] \cup P^{-1}(J)$ be such that $P = P_I$.

Lemma (3.1.11) [3]:-

Restriction to A_M^+defines a bijection between $\Delta^{G/H} \setminus J$and $\Delta^{G/H} (M)$. Furthermore ,$\Delta^{G/H} (M)$is linearly independent.

Proof:-

Recall that

$$I = \Delta^M = \{\alpha \in \Delta^G : \alpha|_{\mathrm{A}_\mathrm{M}} = 0\}.$$

Therefore

$$\Delta^{G/H} (M) = \{\beta|_{\mathrm{A}_\mathrm{M}^+} : \beta \in \Delta^{G/H} \setminus J\} \setminus \{0\}.$$

Let $\Delta^{G/H} \setminus J = \{\beta_1, \ldots, \beta_t\}$. To conclude the lemma it is enough to show that for $x_1, \ldots, x_t \in \mathbb{R}$we have, if $x_1\beta_1, \ldots, x_t\beta_t$is trivial on A_M^+then $x_i = 0$ for all $i = 1, \ldots, t$.

If $\beta \in \Delta^{G/H} \setminus J$ then $\beta = \alpha|_{\mathrm{A}_0^+}$for some $\alpha \in \Delta^G [\theta \neq -1] \setminus I$. Assume that $\sum_{i=1}^t x_i \beta_i|_{\mathrm{A}_\mathrm{M}^+} = 0$. Let $\alpha_i \in \Delta^G [\theta \neq -1] \setminus I$ be such that $\alpha_i|_{\mathrm{A}_0^+} = \beta_i$ and let $\gamma = \sum_{i=1}^t a_i \alpha_i$. Then $\gamma|_{\mathrm{A}_\mathrm{M}^+} = 0$ and therefore by a standard argument that we already applied we have $(\gamma + \theta(\gamma))|_{\mathrm{A}_\mathrm{M}} = 0$.

Therefore, $\gamma + \theta(\gamma)$ is a linear combination of elements of $I = \Delta^M$. On the other hand, let $\alpha'_i \in \Delta^G [\theta \neq -1]$ be given by Lemma (3.1.6) so that $\theta(\alpha_i) - \alpha'_i \in \mathrm{x}_0$. Since $\alpha_i, \theta(\alpha_i)$ and α'_icoincide on A_0^+ , it follows that $\alpha'_i$is not trivial on $\mathrm{A}_\mathrm{M}$and therefore $\alpha'_i \in \Delta^G \setminus I$. Since $\Delta^G [\theta = -1] \subseteq I$ every element of X0 is a linear combination of elements of I. It follows that $\sum_{i=1}^t x_i(\alpha_i + \alpha_i')$is in the span of I. Arguing as in the proof of Proposition (3.1.7), by the linear independence of Δ^Git follows that $x_i = 0$ for all $i$and the lemma follows.

We call $C\left(\mathrm{A}_{\mathrm{M}}^{+},\Delta^{G/H}(M)\right)$ the cone of relatively positive elements in $(a_0^*)_\theta^+$. Recall that

$$a_0^* = (a_0^M)^* \otimes (a_M^*)_\theta^+ \otimes (a_M^*)_\theta^-.$$

Definition (3.1.12) [3]:-

$An\ element \lambda \in a_0^* is\ called\ M$ −relatively positive(resp. weakly positive) if its projection $(\lambda_M)_\theta^+ to (a_M)_\theta^+ is\ in C\left(\mathrm{A}_{\mathrm{M}}^{+},\Delta^{G/H}(M)\right)$

$$\left(resp. \bar{C}\left(\mathrm{A}_{\mathrm{M}}^{+},\Delta^{G/H}(M)\right)\right).$$

Corollary (3.1.13) [3]:-

With the above notation we have

$$\Delta^{G/H}(M) = \{\alpha|_{\mathrm{A}_{\mathrm{M}}} : \alpha \in \Delta_M\} \backslash \{0\}.$$

$Thus, any \lambda \in C(A_M, \Delta_M)$ is$M$ −r$elatively\ positive\ and\ any \lambda \in \bar{C}(A_M, \Delta_M) is M - relatively\ weakly\ positive.$

Proof:-

It follows from Lemma (3.1.11) that every element of $\Delta^{G/H}(M)$ is of the form $\beta|_{\mathrm{A}_{\mathrm{M}}^{+}}$for some $\beta \in \Delta^{G/H} \setminus J$. Let $\alpha \in \Delta^G$be such that $\alpha|_{\mathrm{A}_0^+} = \beta$. Then $\alpha \notin I$,and therefore $\alpha|_{\mathrm{A}_{\mathrm{M}}} \neq 0$ i.e., $\gamma := \alpha|_{\mathrm{A}_{\mathrm{M}}} \in \Delta_M$is such that $\gamma|_{\mathrm{A}_{\mathrm{M}}^{+}} = \beta|_{\mathrm{A}_{\mathrm{M}}^{+}}$. Conversely, if $\beta \in \Delta_M$is such that $\beta|_{\mathrm{A}_{\mathrm{M}}^{+}} \neq 0$ then $\beta = \alpha|_{\mathrm{A}_{\mathrm{M}}}$ for some $\alpha \in \Delta^G$ $[\theta \neq -1]$. Thus, $\gamma := \alpha|_{\mathrm{A}_0^+} \in \Delta^{G/H}$is such that $\beta|_{\mathrm{A}_{\mathrm{M}}^{+}} = \gamma|_{\mathrm{A}_{\mathrm{M}}^{+}}$and therefore $\beta|_{\mathrm{A}_{\mathrm{M}}^{+}} \in \Delta^{G/H}(M)$. The rest of the corollary is now straightforward.

Section (3.2) H– integrability and non – vanishing:-

We apply Lemma (3.1.3) to $H^\circ$with respect to the minimal parabolic Subgroup P_0^H and the maximal F −split torus A_0^+. Write $P_0^H = M_0^H \ltimes U_0^H$ where $M_0^H$is the centralizer in $H^\circ$of A_0^+ and therefore $M_0^H \subseteq M_1^\theta$. Let

$$C_0^{H,\geq 0} = C_{\mathrm{A}_0^+}^{\Delta^H,\geq 0}.$$

Choose a finite subset F_0^H of M_0^H in such a way that

$$H^\circ = \coprod_{f \in F_0^H} \coprod_{c \in C_0^{H,\geq 0}} K^{H^\circ} f c K^{H^\circ}$$

Holds. We further insure that F_0^H is such that Lemma (3.1.3) holds for $H^\circ$with I^Has a basisof open normal subgroups of K^{H°.

For a subset $X$of $C_{\mathrm{A}_0^+}$let $[X]$ be its image under the projection to $C_{\mathrm{A}_0^+}/C_{\mathrm{A}_{\mathrm{G}}^+}$.

Let $C^\infty(\mathrm{A}_{\mathrm{G}}^+ \backslash G)$be the space of functions $\phi: G \to \mathbb{C}$ such that $\phi(ag) = \phi(g), g \in G, a \in \mathrm{A}_{\mathrm{G}}^+$and there exists an open subgroup K_0 of Gsuch that $\phi$is bi$-K_0$ −invariant.

Proposition (3.2.1) [3]:-

Let $\phi \in C^{\infty}(\mathrm{A}_{\mathrm{G}}^{+}\backslash G)$. Then the following conditions are equivalent:

$$(a)\int_{\mathrm{A}_{\mathrm{G}}^{+}\backslash H}|\phi(h)|dh<\infty;$$

$$(b)\sum\nolimits_{s\in[C_0^{H,\geq 0}]}\delta_{P_0^H}^{-1}(s)|\phi(h_1sh_2)|<\infty forall h_1,h_2\in H.$$

Proof:-

Since$C_{\mathrm{A}_{\mathrm{G}}^{+}}$is cocompact in $\mathrm{A}_{\mathrm{G}}^{+}$, condition (1) holds if and only if$\int_{C_{\mathrm{A}_{\mathrm{G}}^{+}}\backslash H}|\phi(h)|dh<\infty$. Let D be a (finite) set of representatives for $H/H^{\circ}$and let$K_0\in I^H$ IH be such that $\phi(d.)$

$$H=\coprod_{d\in D}\coprod_{f\in F_0^H}\coprod_{c\in C_0^{H,\geq 0}}dK^{H^{\circ}}fcK^{H^{\circ}}$$

and therefore

$$\int_{C_{\mathrm{A}_{\mathrm{G}}^{+}}\backslash H}|\phi(h)|dh\leq\sum_{d\in D}\sum_{f\in F_0^H}\sum_{e_1,e_2\in E}\sum_{s\in[C_0^{H,\geq 0}]}\int_{K_0e_1fse_2K_0}|\phi(dh)|dh=$$

$$\sum_{d\in D}\sum_{f\in F_0^H}\sum_{e_1,e_2\in E}\sum_{s\in[C_0^{H,\geq 0}]}|\phi(de_1fse_2)|\mathrm{vol}(K_0e_1fse_2K_0).$$

Note further that

$$\mathrm{vol}(K_0e_1fse_2K_0)=\mathrm{vol}(e_1K_0fsK_0e_2)=\mathrm{vol}(K_0fsK_0)=\delta_{P_0^H}^{-1}(fs)\,\mathrm{vol}(K_0)$$

Where the identities follow respectively by the normality of K_0 in $K^{H^{\circ}}$, the invariance ofthe Haar measure on $H$and Lemma (3.1.3) Thus,

$$\int_{C_{\mathrm{A}_{\mathrm{G}}^{+}}\backslash H}|\phi(h)|dh\leq\mathrm{vol}(K_0)\sum_{d\in D}\sum_{f\in F_0^H}\delta_{P_0^H}^{-1}(f)\sum_{e_1,e_2\in E}\sum_{s\in[C_0^{H,\geq 0}]}\delta_{P_0^H}^{-1}(s)\,|\phi(e_1se_2)|.$$

Since the sums over d,f,e_1,e_2 are finite clearly (2) implies (1). Similarly, if

$$X=\bigcup\nolimits_{s\in[C_0^{H,\geq 0}]}K_0sK_0$$

Then

$$\mathrm{vol}(K_0)\sum_{s\in[C_0^{H,\geq 0}]}|\phi(h_1sh_2)|=\int_{C_{\mathrm{A}_{\mathrm{G}}^{+}}\backslash h_1X\,h_2}|\phi(h)|dh\leq\int_{C_{\mathrm{A}_{\mathrm{G}}^{+}}\backslash H}|\phi(h)|dh$$

and therefore (1) implies (2).

Let (π,V) be an admissible, smooth (complex valued) representation of G.For a parabolic subgroup$P=\ M\ltimes U$of G, let $(r_p(\pi),r_p(V))$ denote the normalized

Jacquet module of $\pi$with respect to$P$. It is an admissible representation of M. We say that a character $\chi$of $A_M$is an exponent of πalong P, if it is an A_M–eigenvalue on $r_p(V)$, i.e., there exists $0 \neq v \in r_p(V)$ such that $r_p(\pi)(a)v = \chi(a)v, a \in A_M$. See for a more detailed discussion of this definition. If $\pi$is of finite length then so is $r_p(\pi)$. In this case, the exponents are the restrictions to$A_M$of the central characters of the irreducible components in a decomposition series for $r_p(\pi)$.

Let$\varepsilon_P(\pi)$denote the set of all exponents of πalong P.

Let πbe an admissible representation of $G$and let $\tilde{\pi}$be its contragredient. For $v \in \pi$and $\tilde{v} \in \tilde{\pi}$the function

$$C_{v,\tilde{v}}(g) = \tilde{v}(\pi(g)v), \qquad g \in G$$

Is called a matrix coefficient of π.Let$M(\pi)$be the space of all matrix coefficients of π.

Casselman developed a tool to study the a symptotics of matrix coefficients of πin terms of matrix coefficients of Jacquet modules of π. We recall the results relevant to us.

Let$P = M \ltimes U$be a standard parabolic subgroup of $G$and let $P-$be the opposite parabolic. Casselman defined an$M-$invariant pairing on $r_p(\pi) \times r_{p-}(\pi)$that identifies $r_{p-}(\pi)$as the contragredient of $r_p(\pi)$.Let$v_p$denote the projection of $v \in \pi$to $r_p(\pi)$. It follows that for $v \in V$and $\tilde{v} \in \tilde{\pi}$we have$C_{v_p,\tilde{v}_{p-}} \in M(r_p(\pi))$. Moreover, there exists $\epsilon > 0$ such that

$$C_{v,\tilde{v}}(a) = \delta_P^{1/2} C_{v_p,\tilde{v}_{p-}}(a), \qquad a \in C_{A_M}^{>0}(\epsilon) \tag{6}$$

Let

$$\rho_0^G = Re\left(\delta_{P_0}^{1/2}\right) \in (a_0^G)^*$$

And $\rho_M^G = (\rho_0^G)_M \in (a_M^G)^*$its projection with respect to a standard Levi subgroup $M$of G.

Note that if $P = M \ltimes U$is a standard, $\theta-$stable parabolic subgroup of Gthen$(\rho_M^G)_\theta^+ = Re\left(\delta_P^{1/2}\Big|_{A_M^+}\right)$.

Proposition (3.2.2) [3]:-

Letπbe an admissible representationof$G$so thatA_G^+acts onπ as a unitary character and let ω be a character ofA_0^+ / A_G^+ G. *The* followingare equivalent.

(a)*For every*$c \in M(\pi)$*we have*

$$\sum_{S \in [C_{A_0^+}^{\Delta G/H, \geq 0}]} |c(s)\omega(s)| < \infty;$$

(b)*For every standard,* $\theta-$ *stable parabolic* $F-$ *subgroup* $P = M \ltimes U$ *of* G*and for every* $\chi \in \varepsilon_P(\pi)$ *we have* $Re(\chi) + Re(\varepsilon) + \rho_0^G$is M$-$ *relatively positive.*

Proof:-

Let$\{y_\alpha: \alpha \in \Delta^{G/H}\}$be as in Lemma (3.1.9)(2). In the notation of the lemma let$t_\alpha = y_\alpha(\overline{\omega}), \varepsilon = \{e(\overline{\omega}): e \in E\}$and

$$S = \{y(\overline{\omega}): y \in Y^{\geq 0}\} = \left\{\prod_{\alpha \in \Delta^{G/H}} t_\alpha^{n_\alpha} : n_\alpha \in \mathbb{Z}_{\geq 0} \text{for all} \alpha \in \Delta^{G/H}\right\}.$$

It follows from Lemma (3.1.9)(3) that we have the disjoint union

$$\left[C_{\mathrm{A}_0^+}^{\Delta^{G/H},\geq 0}\right] = \coprod_{\epsilon \in \varepsilon} \epsilon S.$$

For a subset $J \subseteq \Delta^{G/H}$and a positive integer Nlet

$$S_J(N)_0 = \left\{\prod_{\alpha \in \Delta^{G/H} \setminus J} t_\alpha^{n_\alpha} : N < n_\alpha\right\}, \quad S_J(N)_1 = \left\{\prod_{\alpha \in J} t_\alpha^{n_\alpha} : 0 \leq n_\alpha \leq N\right\}$$

$$S_J(N) = S_J(N)_0 S_J(N)_1 \subseteq S.$$

Note that $S_J(N)_1$ is a finite set. Clearly, for any fixed Nwe have the disjoint union

$$S = \coprod_{J \subseteq \Delta^{G/H}} S_J(N).$$

And therefore

$$\sum_{s \in \left[C_{\mathrm{A}_0^+}^{\Delta^{G/H},\geq 0}\right]} |c(s)\omega(s)| = \sum_{\epsilon \in \varepsilon} \sum_{J \subseteq \Delta^{G/H}} \sum_{s \in S_J(N)} |c(\epsilon s)\omega(\epsilon s)|$$

$$= \sum_{\epsilon \in \varepsilon} \sum_{J \subseteq \Delta^{G/H}} \sum_{s \in S_J(N)_1} |\omega(\epsilon s)| \sum_{s \in S_J(N)_0} |c(\epsilon t s)\omega(s)|.$$

Since $c(\epsilon t) \in M(\pi)$ and the first three summations on the right hand side are over a finite set, we see that condition (1) is equivalent to the condition:

For every $c \in M(\pi)$ and $J \subseteq \Delta^{G/H}$there exists$N > 0$ such that we have

$$\sum_{s \in S_J(N)_0} |c(s)\omega(s)| < \infty. \tag{7}$$

For $J \subseteq \Delta^{G/H}$let $I = \Delta^G\ [\theta = -1] \cup P^{-1}(J)$ and $P = M \ltimes U = P_I$. Let $S_M$be the lattice generated by $\{t_\alpha: \alpha \in \Delta^{G/H} \setminus J\}$. We further formulate the condition:

$$\sum_{s \in S_J(N)_0} \delta_P^{1/2}(s)|Q(s)\chi(s)\omega(s)| < \infty for all N > 0, J \subseteq \Delta^{G/H}, \tag{8}$$

$\chi \in \varepsilon_P(\pi)$and polynomials Qon $S_M$with complex coefficients.

Clearly (8) holds if and only if for all $J \subseteq \Delta^{G/H}, \chi \in \varepsilon_P(\pi)$ and $\alpha \in \Delta^{G/H} \setminus J$we have $\delta_P^{1/2}(t_\alpha)|\chi\omega(t_\alpha)| < 1$. Note that $S_J(N)_0$is contained in $\mathrm{A}_0^+\mathrm{M}$ and that $\delta_{P_0}|_{A_M} = \delta_P|_{A_M}$.

By Lemma (3.1.11) we get that (2) is equivalent to (8). It is therefore enough to show that conditions (7) and (8) are equivalent.

Assume that condition (8) holds. Fix $c \in M(\pi)$ and $J \subseteq \Delta^{G/H}$ (so that $I = \Delta^G[\theta = -1] \cup P^{-1}(J) and P = M \ltimes U = P_I$). Let $\tilde{c} \in M\left(r_p(\pi)\right)$ be the matrix coefficient associated by the Casselman pairing and $\epsilon > 0$ be given by (6) so that

$$c(a) = \delta_P^{1/2}(a)\tilde{c}(a), \qquad a \in c_{A_M}^{>0}(\epsilon)$$

An element of Δ_M is of the form $\alpha|_{A_M}$ for some $\alpha \in \Delta^G \setminus I$. Hence $\in \Delta^{G/H} \setminus J$. It therefore follows from the definition of the sets $S_J(N)_0$ that there exists N large enough so that $S_J(N)_0 \subseteq c_{A_M}^{>0}(\epsilon)$. To show that condition (7) holds it is therefore enough to show that

$$\sum_{s \in S_J(N)_0} \delta_P^{1/2}(s)|\tilde{c}(s)\omega(s)| < \infty.$$

A standard argument shows that there exist polynomials $Q_\chi, \chi \in \varepsilon_P(\pi)$ on S_M, only finitely many of which are non-zero, so that

$$\tilde{c}(s) = \sum_{\chi \in \varepsilon_P(\pi)} Q_\chi(s)\, \chi(\in s), \;\; s \in S_M.$$

Hence (7) follows immediately from (8).

Conversely, assume that (8) does not hold. Let $J \subseteq \Delta^{G/H}, \alpha \in \Delta^{G/H} \setminus J$, and, in the above notation, $\chi \in \varepsilon_P(\pi)$ be such that $\delta_P^{1/2}(t_\alpha)|\chi\omega(t_\alpha)| \geq 1$. Then $\sum_{s \in S_J(N)_0} \delta_P^{1/2}(s)|\chi(s)\omega(s)| = \infty$ for all $N > 0$. Set $c = C_{v,\tilde{v}}$ where $v \in \pi$ is such that vP is an eigenvector of AM with eigen value χ (this realizes χ as an exponent of π along P) and $\tilde{v} \in \tilde{\pi}$ is such that $\langle v_P, \tilde{v}_P \rangle = 1$. Then, $\tilde{c}|_{\mathrm{A_M}} = \chi$ and the above argument applying the Casselman pairing shows that for N large enough

$$\sum_{s \in S_J(N)_0} |c(s)\omega(s)| = \sum_{s \in S_J(N)_0} \delta_P^{1/2}|\tilde{c}(s)\omega(s)| = \infty.$$

Thus, condition (7) fails to hold. (Indeed, $S_J(N_2)_0 \subseteq S_J(N_1)_0$ for $N_1 < N_2$ and therefore, if condition (7) holds then it is satisfied with N arbitrarily large.)

Definition (3.2.3) [3]:-

We say that a smooth representation π of G/A_G^+ is $H-$integrable if for any $c \in M(\pi)$ we have

$$\int_{H/A_G^+} |c(h)|\, dh < \infty.$$

Let $\rho_0^H = Re\left(\delta_{P_0^H}^{1/2}\right)$ and recall that the set $[W^{G/H}/W^H]$ was defined in Corollary (3.1.7)(3). We can now formulate our main result.

Theorem (3.2.4) [3]:-

Let π be an admissible representation of G/A_G^+. Then $\pi$is H −integrable if and only if for any θ −stable, standard parabolic subgroup $P = M \ltimes U$of $G$and any $\chi \in \varepsilon_P(\pi)$,the element $Re(\chi) + \rho_0^H - 2\omega(\rho_0^H)$ is M −relatively positive for all $w \in [W^{G/H}/W^H]$.

Proof:-

Let $N^{G/H}$be a subset of $N_G(A_0^+)$ consisting of a choice of a representative $n$forevery element $w \in [W^{G/H}/W^H]$. Since every (left or right) translation by $G$of an element of $M(\pi)$ is again in $M(\pi)$ it follows from Proposition (3.2.1) (in its notation) that $\pi$is H −integrable if and only if

$$\sum_{s \in C_0^{H,\geq 0}} \delta_{P_0^H}^{-1}(s)|c(s)| < \infty \text{ for all } c \in M(\pi) \tag{9}$$

By Corollary (3.1.7) we have

$$\left[C_0^{H,\geq 0}\right] = \cup_{n \in N^{G/H}} n^{-1}\left[C_{A_0^+}^{\Delta^{G/H},\geq 0}\right] n$$

And therefore,

$$\sum_{s \in C_0^{H,\geq 0}} \delta_{P_0^H}^{-1}(s)|c(s)| < \infty$$

If and only if

$$\sum_{s \in C_{A_0^+}^{\Delta^{G/H},\geq 0}} \delta_{P_0^H}^{-1}(n^{-1}sn)|c(n^{-1}sn)| < \infty$$

for all $n \in N^{G/H}$. Note that $c(n^{-1}.n) \in M(\pi)$ and that $Re\left(\delta_{P_0^H} c(n^{-1}.n)\right) = 2\omega(\rho_0^H)$, when $n$represents $w \in [W^{G/H}/W^H]$. It now follows from Proposition (3.2.2) (applied with$\omega = \delta_{P_0^H}^{-1} n^{-1}.n|_{A_0^+}$ that (9) is equivalent to the condition in the statement of the theorem.

Theorem (3.2.4) points on the significance of the exponents

$$\rho_{G/H}^{w} := (\rho_0^G)_\theta^+ - 2w(\rho_0^H) = \left(\rho_{M_1}^G\right)_\theta^+ - 2w(\rho_0^H) \in \left(a_{M_1}^*\right)_\theta^+ = (a_0^*)_\theta^+$$

for $w \in [W^{G/H}/W^H]$. We will now present means to compute these exponents using the action of θon the various root data involved.

For $\alpha \in \sum^{G/H}$ letL_α^H(resp.L_α^H) be the weight space of αin Lie(G) (resp. Lie(H)).

Thus $L_\alpha^H = 0$if $\alpha \notin \sum^H$. Set

$$M_\alpha^G = \dim L_\alpha^G, \quad M_\alpha^H = \dim L_\alpha^H.$$

Since A_0^+ is θ −fixed, its adjoint action on Lie(G) commutes with the θ −action. Thus, each L_α^G is aθ −invariant subspace of Lie(G).

Lemma (3.2.5) [3]:-

Let$\alpha \in \sum^{\mathrm{G/H}}$ and set

$$m_{\theta,\alpha} = Tr\left(\theta|_{L_\alpha^G}\right).$$

$(a) we\ have\ \ m_{\theta,\alpha} = 2M_\alpha^H - M_\alpha^G.$

$(b) if\ \theta(\beta) \neq \beta for\ every\ \beta \in \sum^{\mathrm{G}}\ such\ that \beta|_{A_\alpha^+} = \alpha\ then\ m_{\theta,\alpha} = 0$

Proof:-

The linear involution θon L_α^Gdecomposes the space into a sum of the eigenspacesrelated to the eigenvalues 1 and-1. The $1-$eigenspace is precisely $L_\alpha^G \cap \mathrm{Lie}(G)^\theta = L_\alpha^H$. Thus, $m_{\theta,\alpha} = 1. M_\alpha^H + (-1).(M_\alpha^G - L_\alpha^H)$.

Suppose that $\alpha$is as in the assumption of (2). Then there is an even number of elements of $\sum^{\mathrm{G}}$ whose restriction to$A_0^+$is $\alpha$and we can enumerate them as $\{\beta_1, \dots, \beta_k, \gamma_1, \dots, \gamma_k\}$ with $\theta(\beta_i) = \gamma_i$. Thus, L_α^Gadmits a decomposition $L_\alpha^G = V_1 \oplus V_2$with $\theta(V_1) = V_2$(indeed take $V_1$to be the direct sum of root eigenspaces in $\mathrm{Lie}(G)$ with respect to $\{\beta_1, \dots, \beta_k\}$and similarly V_2 with respect to $\{\gamma_1, \dots, \gamma_k\}$). Evidently, this implies that $\theta|_{L_\alpha^G}$ is of zero trace.

Let $\sum^{\mathrm{G/H,>0}} := \sum^{\mathrm{G/H}} \cap \bar{C}(\mathrm{A}_0^+, \Delta^{\mathrm{G/H}})$be the set of positive roots in $\sum^{\mathrm{G/H}}$. These are the non-zero restrictions to A+0 of roots in $\sum^{\mathrm{G,>0}}$.

Proposition (3.2.6) [3]:-

For every $w \in [W^{G/H}/W^H]$. we have

$$\rho_{G/H}^w = -\frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{G/H,>0}}.} m_{\theta,w^{-1}}(\alpha)^\alpha.$$

Proof. Recall that $\delta_{P_0}(a) = |det(Ad(a)|\mathrm{Lie}(P_0))|F, a \in A_0$. Applied to Hthis gives

$$\rho_0^H = \frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{H,>0}}} M_\alpha^H\, \alpha.$$

Applied to $G$and composed with the projection of ρ_0^H to $(a_0^*)_\theta^+$we have

$$(\rho_0^H)_\theta^+ = \frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{G/H,>0}}} M_\alpha^G\, \alpha.$$

Now, let $w \in [W^{G/H}/W^H]$ be given. Then,

$$w(\rho_0^H) = \frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{H,>0}}} M_\alpha^H\, w(\alpha) = \frac{1}{2} \sum_{\alpha \in w(\sum^{\mathrm{H,>0}})} M_{w^{-1}(\alpha)}^H\, \alpha = \frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{G/H,>0}}} M_{w^{-1}(\alpha)}^H\, \alpha. \quad (10)$$

The last equality is obtained as follows. By Corollary (3.1.7)(4) we have $w(\sum^{\mathrm{H,>0}}) \subseteq \sum^{\mathrm{G/H,>0}}$.

The equality will therefore follow if we show that $M_\beta^H = 0$ (i.e., that $\beta \notin \sum^{\mathrm{H}}$) for $\beta \in w^{-1}\left(\sum^{\mathrm{G/H,>0}}\right)\setminus \sum^{\mathrm{H,>0}}$ .Assume by contradiction that$-\beta \in \sum^{\mathrm{H,>0}}$. As above, by Corollary(3.1.7)(4) we have $-w(\beta) \in \sum^{\mathrm{G/H,>0}}$, i.e., both $\pm w(\beta) \in \sum^{\mathrm{G/H,>0}}$ which is a contradiction.

Finally, there exists $n \in N_G(A_0^+)$ (a representative of w^{-1}) such that $Ad(n)(L_\alpha^G) = L_{w^{-1}(\alpha)}^G$for all $\alpha \in \sum^{\mathrm{G/H,>0}}$. Hence,$M_\alpha^G = M_{w^{-1}(\alpha)}^G$ and we can write

$$(\rho_0^H)_\theta^+ = \frac{1}{2} \sum_{\alpha \in \sum^{\mathrm{G/H,>0}}} M_{w^{-1}(\alpha)}^G \, \alpha. \qquad (11)$$

The statement now follows from (10), (11) and Lemma (3.2.5).We examine our criterion for H-integrability of matrix coefficients oncertain symmetric spaces. Sakellaridis and Venkatesh defined the notion of

a strongly tempered spherical variety. We recall the definition and make an analogous definition for square-integrable representations.

Definition (3.2.7) [3]:-

We say that G/H is strongly tempered (resp. strongly discrete) if every irreducible tempered (resp. square-integrable) smooth representation π of G is H-integrable.

We provide examples of families of symmetric spaces for which the above properties hold. In order to be able to apply Theorem (3.2.4) to this problem, we first need to recall Casselman's criterion for square integrability and a similar criterion for temperdness.

Theorem (3.2.8) [3]:-

Let π be an admissible representation of G for which the centre of Gacts by a unitary character. Then π is square-integrable (resp. tempered) if and only if$Re(\chi) \in C(A_M, \Delta_M)$ (resp. $Re(\chi) \in \bar{C}(A_M, \Delta_M)$), for any standard parabolic F-subgroup$P = M \ltimes U$ of G and any $\chi \in \varepsilon_P(\pi)$.

It is straightforward from the definitions that an M_1-relatively (weakly) positive element of $(a_0^*)_\theta^+$ is also M-relatively (weakly) positive for every standard θ-stable Levi subgroup M.The following is therefore a straight for ward consequence of Corollary (3.1.13) and Theorems (3.2.4) and (3.2.8)

Corollary(3.2.9) [3]:-

If the relative test characters $\rho_{G/H}^w$are M_1-relatively positive (resp. weakly positive) for all $w \in [W^{G/H}/W^H]$, then G/H is strongly tempered (resp. strongly discrete).

Let E/F be a quadratic field extension. Let H be a connected, reductive F-group and $G = Res_{E/F}(H_E)$ be the restriction of scalars from E to F of the group H considered as an E-group. Thus, $G \simeq H(E)$. The Galois involution of E/F defines an involution on G that we denote by θ. We identify$H$ with G^θ and call G/H a Galois symmetric space.

Since H is defined over F, so are the Lie algebra $Lie(H)$ and the adjoint action on it.Hence, we have

$$Lie(G) \simeq Lie(H)(E) = Lie(H) \otimes_F E$$

and the action of $h \in H$ is given as $Ad(h)(v \otimes e) = Ad(h)v \otimes e, v \in Lie(H), e \in E$.It follows, that any eigenvalue of $Ad(A_0^+)$ on $Lie(G)$ is also an eigenvalue on $Lie(H)$ and therefore $\sum^{G/H} = \sum^H$. In particular, $W^{G/H} = W^H$.

Since standard parabolic subgroups of H are in bijection with subsets of Δ^H, θ-stable ,standard parabolic subgroups of G are in bijection with subsets of $\Delta^{G/H}$and $\Delta^H = \Delta^{G/H}$ the map $P \mapsto P^\theta$ is a bijection between θ-stable, standard parabolic Fsubgroupsof G and standard parabolic F-subgroups of H with inverse $Q \longmapsto Res_{E/F}\ (Q_E)$.In particular, we have $P_1^\theta = P_0^H$.We have the following

Lemma (3.2.10) [3]:-

Let P be a θ-stable, stanadrd parabolic F-subgroup of G. Then $\delta_P^{1/2}|_{P\theta} = \delta_{P\theta}$.

It follows that $(\rho_{M_1}^G)_\theta^+ = 2\rho_0^H$ and hence $\rho_{G/H}^e = 0$ where e is the identity in $W^{G/H}$. Hence, the following is immediate from Corollary(3.2.9).

Corollary (3.2.11) [3]:-

Every Galois symmetric space G/H is strongly discrete.We can also state the precise criterion inferred from an application of Theorem (3.2.4) to the Galois case.

Theorem (3.2.12) [3]:-

Let G/H be a Galois symmetric space and let πbe an admissible representation of G/A_G^+. Then π is H-integrable if and only if for any θ-stable parabolic subgroup$P = M \ltimes U$ of G and any $\chi \in \varepsilon_P(\pi)$, the element $Re(\chi)$ is M-relatively positive.

Assume now in addition that $A_0 = A_0^+$. Then by (3) $\sum^G = \sum^{G/H} = \sum^H$ and in particular $\Delta^G = \Delta^H$. Thus, standard parabolic subgroups of G are all θ-stable and in particular $P_0 = P_1$. In paricular, for any standard parabolic subgroup $P = M \ltimes U$ of G we have $AM = A_M^+$and $\Delta_M = \Delta^{G/H}(M)$. The following is therefore immediate from Theorems (3.2.12) and (3.2.8).

Corollary (3.2.13) [3]:-

Assume that G/H is a Galois symmetric space and $A_0 = A_0^+$. Let π be an admissible representation of G/A_G. Then π is H-integrable if and only if π is square-integrable.

Let $G = GL_n$. Every symmetric matrix $J \in GL_n$ defines an F-involution $\theta(g) = J^t g^{-1} J^{-1}$ on G. Denote the associated orthogonal group by $O_J = G^\theta = H$.

After G-conjugation if necessary, we may assume without loss of generality that J is of the form

$$\begin{pmatrix} & & w_r \\ & J_0 & \\ w_r & & \end{pmatrix}$$

where$J_0 \in GL_{n-2r}$ defines an anisotropic quadratric form (r is the Witt index of J) and $w_r \in GL_r$ is the permutation matrix $(w_r)_{i,j} = \delta_{i,r+1-j}$. We may and do further assume that J_0 is diagonal.

We choose the torus of diagonal matrices in G to be the θ-stable maximal F-split torus A_0. We Write $\epsilon_i \in a_0^*$ for the character of A_0 that takes a diagonal matrix to its i-th entry and identify$a_0^* \simeq \mathbb{R}^n$ by identifying $\{\epsilon_1, \ldots, \epsilon_n\}$ with the standard basis of $\mathbb{R}^n$. Note that

$$A_0^+ = \{diag\ (a_1, \ldots, a_r, 1, \ldots, \ldots, a_r^{-1}, \ldots, a_r^{-1}): a_i \in F^*,\ \ i = 1, \ldots, r\}.$$

We write

$$\eta_i = \epsilon_i|_{A_0^+} \in (a_0)_\theta^+.$$

Let P_0 be the Borel subgroup of upper triangular matrices in G. For a decomposition $n_1 + \ldots + n_k = n$ let $P_{(n_1+\ldots+n_k)} = M_{(n_1+\ldots+n_k)} \ltimes U_{(n_1+\ldots+n_k)}$ be the associated standard parabolic subgroups of $G$with its standard Levi decomposition, where the Levi subgroup $M_{(n_1+\ldots+n_k)}$is isomorphic to $L_{n_1} \times \ldots \times GL_{n_k}$.

Then $P_1 = P_{(1,\ldots,1,2n-r,1,\ldots,1)} = M_1 \ltimes U_1$ is a standard, minimal θ-stable parabolic F-subgroup of G. The intersection $P_0^H = P_1 \cap H^\circ$ is a minimal parabolic F-subgroup of H°.The root system

$$\textstyle\sum^{G/H} = \begin{cases} \{\pm(\eta_i \pm \eta_j): 1 \le i \ne j \le r\} \cup \{\pm\eta_i, \pm 2\eta_i: i = 1, \ldots, r\} & 2r < n \\ \{\pm(\eta_i \pm \eta_j): 1 \le i \ne j \le r\} \cup \{\pm 2\eta_i: i = 1, \ldots, r\} & 2r = n \end{cases} \qquad (12)$$

Is of type BC_r when $2r < n$ and of type $C_r$when $2r = n$. We have

$$\Delta^{\frac{G}{H}} = \begin{cases} \{\eta_i - \eta_{i+1}\}_{i=1}^{r-1} \cup \{\eta_r\} & 2r < n \\ \{\eta_i - \eta_{i+1}\}_{i=1}^{r-1} \cup \{2\eta_r\} & 2r = n. \end{cases} \qquad (13)$$

We write $E_{i,j} \subseteq Lie(G) = gl_n(F)$ for the one-dimensional subspace of matrices vanishing outside the (i,j)-th entry. These are the weight spaces for the roots in $\sum^G$. For integers$a \le b$ let $[a,b] = \{a, a+1, \ldots, b\}$ be the corresponding interval of integers. Note thatthe action of θ on $gl_n(F)$ (given by $\theta(X) = -J^t XJ^{-1}$)satisfies $\theta(E_{i,j}) = E_{n+1-j,n+1-i}$whenever $i,j \in [1,r] \cup [n+1-r,n]$ and $\theta(Ei,j) = E_{j,n+1-i}$ for $1 \le i \le r$ and $r < j \le n-r$.It easily follows that for $\alpha \in \sum^{G/H} \setminus \{2\eta_1, \ldots, 2\eta_r\}$ and every $\beta \in \sum^G$ such that $\beta|_{A_0^+} = \alpha$ we have $\theta(\beta) \ne \beta$. Thus, by Lemma (3.2.5)(2), $m_{\theta,\alpha} = 0$. Further more, θ acts by -1 on$L_{2\eta_i}^G = E_{i,n+1-i}$ and there fore $m_{\theta,2\eta_i} = -1$.

In case $n = 2r$ (H is an F-split orthogonal group), the root system $\sum^H$ is of type D_R, $\Delta^H = \{\eta_i - \eta_{i+1}\}_{i=1}^{r-1} \cup \{\eta_{r-1} + \eta_r\}$ and WH is an index 2 subgroup of $W^{G/H}$. It is easyto check that $[W^{G/H}/W^H] = \{e, \epsilon\}$, where ϵ is the simple reflection associated with the root $2\eta_r \in \Delta^{G/H}$ and e is the identity. It is straight for ward that $m_{\theta,\epsilon^{-1}(\alpha)} = m_{\theta,\alpha}$ for all $\alpha \in \sum^{G/H}$. It there fore follows from Proposition (3.2.6)that $\rho_{G/H}^\epsilon = \rho_{G/H}^e$.

Otherwise, when $2r < \sum^H$ is of type B_r, $\Delta^H = \{\eta_i - \eta_{i+1}\}_{i=1}^{r-1} \cup \{\eta_r\}$ and $W^H = W^{G/H}$.

In all cases, combining this with Proposition (3.2.6) the relative test characters are given by

$$\rho_{G/H}^{w} = \sum_{i=1}^{r} \eta_i = \sum_{j=1}^{r-1} j\,.\left(\eta_j - \eta_{j+1}\right) + r.\eta_r, \qquad w \in [W^{\frac{G}{H}}/W^H]. \tag{14}$$

This is M_1-relatively positive by the second equality. Thus, from Corollary (3.2.9) we deducethe following.

Corollary (3.2.14) [3]:-

The symmetric space $GL_n\,/O_J$ is strongly tempered for every symmetric matrix $J\ \in\ GL_n$.

We provide another family of strongly tempered symmetric spaces. The computation of relative test characters in the case at hand reduces to that of the previous. We there fore maintain all the notation defined and use different letters todenote the symmetric space we now consider.

Recall that $E\ =\ F[\tau\,]/F$ is a quadratic extension with Galois involution σ. We consider the following embedding of O_J as a the group of fixed points of an involution on the unitary group associated with $J$and E/F.

Let $G' = Res_{E/F}(G_E)$ and consider σ as the Galois involution on G'.Note that the involution θ on $G = (G')^{\sigma}$ extends to an involution on G' by the same formula $\theta(g) = J\,{}^{t}g^{-1}J^{-1}, g\ \in\ G'$ and that σ and θ commute. Let $\theta' = \theta\sigma = \sigma\theta$ and$U = U_{J,E/F}\ = (G')^{\theta'}$ be the associated unitary group.

Note that σ restricts to an involution on $U$and $U^{\sigma} = O_J = H$. We consider now the symmetric space U/H.

From this explicit construction it is easy to see that there exists a σ-stable, maximal$F$-split torus $A_0^U$of U such that A_0^+ is the maximal F-split torus in $(A_0^U)^{\sigma}$. Furthermore,

$$P_1^U = Res_{E/F}\left(P_{1,\dots,1,2n-r,1,\dots,1}\right) \cap U$$

is a minimal σ-stable parabolic F-subgroup of U such that $P_1^U \cap H^{\circ} = P_1^H$.

We consider $Lie(U)$ as the θ'-fixed subspace of $Lie(G') \simeq gl_n(E) = gl_n(F) + \tau.gl_n(F)$.Thus,

$$Lie(U) = \{X + \tau Y : X, Y \in gl_n(F), X = -J^tXJ^{-1}, Y = J^tYJ^{-1}\}.$$

By studying the adjoint action of A_0^+ on $Lie(U)$ we observe that $\sum^{U/H} = \sum^{G/H}$ (whereon both sides we view elements as characters on A_0^+) and $\Delta^{U/H}{=}\Delta^{G/H}$. Hence $\sum^{U/H>0} = \sum^{G/H>0}$. Furthermore, for every $\alpha \in \sum^{U/H>0}\backslash\{2\eta_1, \dots, 2\eta_r\}$ there is a subspace $V_\alpha \subseteq gl_n(F)$ (explicated bellow) so that $L_\alpha^U = L_\alpha^{U,+} \oplus L_\alpha^{U,-}$ where $L_\alpha^{U,+} = \{v + \theta'(v) : v \in V_\alpha\}\ and\ L_\alpha^{U,-} = \{v + \theta'(v) : v \in \tau V_\alpha\}$ For all such α we have $dim L_\alpha^{U,+} = dim\, L_\alpha^{U,-}$ and clearly σacts by $\bullet \pm 1$ on$L_\alpha^{U,\pm}$ respectively.Therefore$m_{\sigma,\alpha} =$

$0 = m_{\theta,\alpha}$. Also $L_{2\eta_i} = \tau E_{i,n+1-i}$ is one dimensional and clearly $m_{\sigma,2\eta_i} = -1 = m_{\theta,2\eta_i}$ $for\ i = 1,\ldots,r$.

It follows that $m_{\sigma,\alpha} = m_{\theta,\alpha}$ for all $\alpha \in \sum^{U/H} = \sum^{G/H}$. This allows us to argue verbatimas in Corollary (3.2.14) to deduce the following.

Corollary (3.2.15) [3]:-

Let E/F be a quadratic extension and $J \in GL_n$ a symmetric matrix.Then the symmetric space $U_{J,E/F}\,/O_J$ is strongly tempered.

For the sake of completeness, we provide here the above mentioned spaces V_α that complete the reduction of our computation to that of the previous. For $1 \le i < j \le r$ we have

$$V_{\eta_i-\eta_j} = E_{i,j} \quad and \quad V_{\eta_i+\eta_j} = E_{i,n+1-j}$$

whereas if $2r < n$ for $i = 1,\ldots,r$ we have

$$V_{\eta_i} = \bigoplus_{j=r+1}^{n-r} E_{i,j}.$$

Let $G = GL_{2n}$and $\nu = \tau^2 \in F$. Define the involution $\theta(g) = tgt^{-1}$ on G where

$$t = diag\left(\begin{pmatrix}0 & v^{-1}\\1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & v^{-1}\\1 & 0\end{pmatrix}\right).$$

Note that $H = G^\theta \simeq GL_n(E)$. We can choose A_0 to be the diagonal torus in G. It isθ-stable and

$$A_0^+ = \{diag\,(a_1, a_1, a_2, a_2, \ldots, a_n, a_n) : a_i \in F^*, i = 1,\ldots,n\}.$$

We can take $P_1 = P_{(2,\ldots,2)}$ to be the minimal θ-stable parabolic subgroup of G so that $P_0^H = P_1 \cap H$ is a minimal parabolic subgroup of $H = H^\circ$.

We then have$\sum^{G/H} = \sum^H$and $W^{G/H} = W^H$. For each $\alpha \in \sum^{G/H}$ there are four roots in$\sum^G$such that $\beta|_{A_0^+} = \alpha$. The involution θ does not fix any of the four. Thus, by Lemma (3.2.5)(2), $m_{\theta,\alpha} = 0$ for all $\alpha \in \sum^{G/H,>0}$. In particular, the relative test character $\rho_{G/H}^e = 0$.From Corollary (3.2.9) we have the following.

Corollary (3.2.16) [3]:-

The symmetric space $GL_{2n}(F)/GL_n(E)$ is strongly discrete .To describe an explicit realization of the symmetric space that we consider next it is convenient to maintain the notation of the previous. For a symmetric matrix $J \in GL_n$, wecan embed the corresponding unitary group $U_{J,E/F}$ in Sp_{2n} as follows. To J= (a_{ij}) weassociate the anti-symmetric matrix $A_J \in GL_{2n}$whose whose (i,j)-th 2×2 block is givenby

$$\begin{pmatrix}0 & a_{i,j}\\a_{i,j} & 0\end{pmatrix}.$$

Let $\sigma(g) = A_J{}^t g^{-1} A_J^{-1}$ be the involution on G so that $G^\sigma = Sp_{AJ} \simeq Sp_{2n}$. Note thatthe involutions σ and θ commute, hence θ restricts to an involution on $Sp_{AJ}$and $Sp_{AJ}^\theta \simeq$

$U_{J,E/F}$. The group $U_{w_n,E/F}$ is quasi-split over F. It is well known that if n is odd thenevery unitary group is $GL_n(E)$-conjugate to $U_{w_n,E/F}$. If n is even then there are two conjugacy classes of non-isomorphic unitary groups determined by the norm class of the discriminant. We consider the two cases as follows.
Let $G_1' = Sp_{A_{w_n}} \simeq Sp_{2n}\ and\ U_1 = (G_1')^{\theta} \simeq U_{w_n,E/F}$. If n is even let $\delta \in F^*$ be suchthat δ detwn–2 $det\ w_n$ is not a norm from $E\ to\ F$ and let

$$J_2 = \begin{pmatrix} & & w_{n/2-1} \\ & d & \\ w_{n/2-1} & & \end{pmatrix}$$

where$d = diag(1,\delta)$. Set $G_2' = Sp_{A_{J_2}} \simeq Sp_{2n}\ and\ U_2 = (G_2')^{\theta} \simeq U_{J,E/F}$ the non-quasi splitunitary group.
In order to unify notation for the two cases at hand we set $J = w_n (resp. J = J_2)$ and$G' = G_1' (resp. G' = G_2')$ and let $U = (G')^{\theta}$ be the corresponding unitary group. We canchoose the minimal θ-stable parabolic subgroup P_1' of G' to be

$$P_1' = \begin{cases} P_{(2^{(n)})} \cap G' & J = w_n \\ P_{(2^{(n/2-1)},4.2^{(n/2-1)})} \cap G' & J = J_2 \end{cases}$$

where $(2^{(a)}) = \overbrace{(2,\dots,2)}^{a}$. It contains a θ-stable maximal F-split torus $A_0'$of G', such that$(A_0')^+$ is the maximal F-split torus of U such that

$$(A_0')^+ = \{diag\ (a_1, a_1, \dots, a_r, a_r, I_{2n-4r}, a_r^{-1}, a_r^{-1}, \dots, a_1^{-1}, a_1^{-1}): \quad a_i \in F^*, \\ i = 1, \dots, r\},$$

where$r = \lfloor n/2 \rfloor$ in the quasi-split case, and $r = n/2 - 1$ in the non-quasi-split case. Forour computation we recall that

$$Lie(G'') = \{X \in gl_n(F): -A_{J^t} X A_J^{-1} = X\}.$$

The root system $\sum^{G'/U}$ is of the same type as in the example of subsection 3.2.6. Namely ,$\sum^{G'/U}$ is of type BC_r when $2r < n$ and of type $C_r$when $2r = n$. We may therefore denotethe roots as in (12) where η_i is the character of $(A_0')^+$ that satisfies

$$\eta_i\big(diag\ (a_1, a_1, \dots, a_r, a_r, I_{2n-4r}, a_r^{-1}, a_r^{-1}, \dots, a_1^{-1}, a_1^{-1})\big) = a_i, \ \ i = 1, \dots, r.$$

The simple roots $\Delta^{G'/U}$ are then given by (13).We now have$\sum^U = \sum^{G'/U}$ and therefore $W^U = W^{G'/U}$ in all cases.
It is now a straightforward verification that for any $\alpha \in \sum^{G'/U} \setminus \{2\eta_1, \dots, 2\eta_r\}$ there are four roots β in $\sum^{G'}$ such that $\beta|_{(A_0')^+} = \alpha$ and the involution θ fixes none of them. It follows from Lemma (3.2.5)(2) that $m_{\theta,\alpha} = 0$ for all $\alpha \in \sum^{G'/U} \setminus \{2\eta_1, \dots, 2\eta_r\}$.
For $k = 1, \dots, r$ the root space $L_{2\eta_k}^{G'}$ consists of matrices $X \in Lie(G')$ such that the(i,j)-th 2×2 block of X is zero unless $i = k = n + 1 - j$ in which case it is of the form$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$for some $a, b, c \in F$. We denote such an element by $X_{a,b,c}$. Then

$$\theta(X_{a,b,c}) = X_{-a,v^{-1}c,vb}$$

and there fore $m_{\theta,2\eta_k} = -1$. It now follows from Proposition (3.2.6) that the relative test character is given by

$$\rho^{e}_{\frac{G'}{H}} = \sum_{i=1}^{r} \eta_i$$

which is M_1'-relatively positive (M_1' is the Levi subgroup of P_1' containing A_0') by the secondequality in (14). Thus, from Corollary (3.2.9) we deduce the following.

Corollary (3.2.17) [3]:-

For every symmetric matrix $J \in GL_n(F)$ the symmetric space $Sp_{A_J}/U_{J,E/F}$is strongly tempered. The symmetric spaces $GL_{2n}(F)/GL_n(F \times GL_v(F)$ and $GL_{2n+1}(F)/\ GLn(F)\ \times GL_{n+1}(F)$ are strongly discrete:-

Let $G\ =\ GL_{n_1+n_2}$ and $\theta(g) = tgt^{-1}, g \in G$ where$t = diag(I_{n_1}, -I_{n_2})$. Then, $H = G^{\theta} \simeq GL_{n_1} \times\ GL_{n_2}$.

Let $P_1\ =\ P_0$ be the standard Borel subgroup of upper triangular matrices and A_0 the diagonal torus in G. Then $A_0 = A_0^+$ and there fore $\sum^G = \sum^{G/H}$ is of type $A_{n_1+n_2-1}$.For $1 \le i \ne j \le n_1 + n_2$ let $\alpha_{i,j} \in \sum^G$ be the root corresponding to the weight space$E_{i,j}$ defined. Then, $\sum^{G/H} = \sum^G =\ \{\beta_1, \ldots, \beta_{n_1+n_2-1}\}$. where $\beta_i = \alpha_{i,i+1}$, for $1 \le i \le n_1 + n_2 - 1$. We identify $W^G = W^{G/H}$ with the group $S_{n_1+n_2}$ of permutations on$\{1, \ldots, n_1 + n_2\}$ so that $w(\alpha_{i,j}) = \alpha_{w(i),w(j)}$ for all $w\ \in W^G$. The set $[W^{G/H}/W^H]$ consists ofall permutations that satisfy $w(i) < w(j)$ for all $1 \le i < j \le n_1$and $n_1 + 1 \le i < j \le n_1 + n_2$.

Lemma (3.2.18) [3]:-

If either $n_2 = n_1$ or $n_2 = n_1 + 1$ then $\rho^w_{G/H}$ is M_1-relatively weakly positivefor every $w \in [W^{G/H}\ /W^H]$. If $n_1 = n_2 = 1$ then $\rho^w_{G/H}$ is M_1-relatively positive for every $w \in [W^{G/H}\ /W^H]$.

Proof:-

For every $w \in [W^{G/H}\ /W^H]$, we write

$$\rho^w_{G/H} = a^w_1\beta_1 + \cdots + a^w_{n_1+n_2-1}\beta_{n_1+n_2-1}$$

with half-integers aw i . Then $\rho^w_{G/H}$ is M_1-relatively weakly positive if and only if $a^w_k \ge 0$forall $1 \le k \le n_1 + n_2 - 1$. It is M_1-relatively positive when the inequalities are strict. Note that for $1 \le i \ne j \le n_1 + n_2$ we have

$$a_{\theta,\alpha_{i,j}} = \begin{cases} 1 & i,j > n_1 \ \ or\ \ i,j \le n_1 \\ -1 & otherwise \end{cases}$$

and that $\alpha_{i,j}\ =\ \beta_i +\ \beta_{i+1} + \cdot\cdot\cdot + \beta_{j-1}$ for all $i\ <\ j$. Set

$$d(w,k) = \# \left\{(i,j):\ \ 1 \le i \le k < j \le n_1 + n_2, m_{\theta,\alpha_{w^{-1}(i),w^{-1}(j)}} = 1\right\}.$$

By Proposition (3.2.6) we have

$$a_k^w = -\frac{1}{2}\left[d(w,k) - \# \left\{(i,j):\ \ 1 \le i \le k < j \le n_1+n_2, m_{\theta,\alpha_{w^{-1}(i),w^{-1}(j)}} = -1\right\}\right]$$
$$= -\frac{1}{2}[d(w,k) - (k(n_1+n_2-k) - d(w,k))]$$
$$= \frac{k(n_1+n_2-k)}{2} - d(w,k).$$

Note that translating by w^{-1}we get that

$$d(w,k) = \# \left\{(i,j): \quad \begin{matrix} 1 \le w(i) \le k < w(j) \le n_1+n_2. \\ either\ 1 \le i < j \le n_1 \ \ or\ \ n_1+1 \le i < j \le n_1+n_2 \end{matrix}\right\}.$$

Let

$$e_w = \begin{cases} \max\{1 \le i \le n_1 : w(i) \le k\} & w(1) \le k \\ 0 & k < w(1). \end{cases}$$

Note that $e_w \le\ k$,

$$k - e_w = \begin{cases} \max\{1 \le i \le n_2 : w(n_1+i) \le k\} & w(n_1+1) \le k \\ 0 & k < w(n_1+1) \end{cases}$$

and $\{w(i): 1 \le i \le e_w\} \cup \{w(n_1+i): 1 \le i\ \le k - e_w\} = \{1,\dots,k\}$. It follows that

$$d(w,k) = e_w(n_1 - e_w) + (k - e_w)\big(n_2 - (k - e_w)\big).$$

Thus, in order to have $a_k^w \ge\ 0$ we need to show that

$$\frac{k\big(k-(n_1+n_2)\big)}{2} \le e_w(e_w - n_1) + (k - e_w)\big((k - e_w) - n_2\big). \qquad (15)$$

Consider first the case $n_1 = n_2$ and let $\phi(t) = t(t\ -\ n_1)$, $t\ \in \mathbb{R}$. It is a convex real function and therefore

$$2\phi(k/2) \le \phi(e_w) + \phi(k - e_w)$$

(this is precisely the inequality (15)) and equality holds if and only if $e_w = k - e_w$. This shows that $a_k^w \ge\ 0$ in this case. If in addition $n_1 =\ 1$ then $k\ =\ 1$ and $e_w \ne k - e_w$. Thus inthis case $a_1^w > 0$ and $\rho_{G/H}^w$ is M_1-relatively positive.

Assume now that $n_2 = n_1 + 1$. If $e_w = k - e_w$ then (15) is always an equality.

Assumenow that $e_w \ne k - e_w$ and let $\psi(t)\ =\ t^2 - t\left(\frac{t-e_w}{k-2e_w} + n_1\right), t\ \in \mathbb{R}$. Again, it is a real function with non-negative second derivative and therefore

$$2\psi(k/2) \le \psi(e_w) + \psi(k - e_w)$$

which is precisely the inequality (15). The lemma follows. The following is now immediate from Lemma (3.2.18) and Corollary (3.2.9).

Corollary (3.2.19) [3]:-

The symmetric spaces $GL_{2n}/GL_n \times GL_n$ and $GL_{2n+1}/GL_n \times GL_{n+1}$ are strongly discrete. The symmetric space $GL_2/GL_1 \times GL_1$ is strongly tempered. For an H-integrable representation π of G and a vector $\tilde{v}$ in $\tilde{\pi}$ let $\ell_{\tilde{v},\mathrm{H}}$ be the linear formonπ defined by

$$\ell_{\tilde{v},\mathrm{H}}(v) = \int_{H/A_G^+} c_{v,\tilde{v}}\,(h)\,dh.$$

We write $L_H^\pi = \{\ell_{\tilde{v},\mathrm{H}}: \tilde{v} \in \tilde{\pi}\} \subseteq Hom_H(\pi,\mathbb{C})$ for the subspace of H-invariant linear formson π emerging as integrals of matrix coefficients.

Let $X = G/H$ be the G-symmetric space associated with θ. $X$is calledstrongly tempered if G/H_z is strongly tempered (in the sense of Definition (3.2.7)), for every$z \in X$ where H_z is the stabilizer of z in G. The statement of assumes that X is strongly tempered, but the proof considers a single G-orbit at a time.It therefore implies the following.

Theorem (3.2.20) [3]:-(Sakellaridis-Venkatesh)

Assume that G is F-split and that G/H is stronglytempered. If π is an irreducible, square-integrable representation of G then

$$L_H^\pi = Hom_H(\pi,\mathbb{C}).$$

If π is a representation of G parabolically induced from an irreducible, square-integrable representation of a Levi subgroup of G then we have the implication

$$Hom_H(\pi,\mathbb{C}) \neq 0 \quad \Rightarrow \quad L_H^\pi \neq 0.$$

The following is therefore an immediate consequence of Theorem (3.2.20) and Corollaries (3.2.16), (3.2.17) and (3.2.19).

Corollary (3.2.21) [3]:-

For the following symmetric spaces G/H and for every irreducible square-integrable representation π of G we have

$$L_H^\pi = Hom_H(\pi,\mathbb{C}).$$

$(a) GL_n/O_J$for a symmetric matrix $J \in GL_n$.

$(b) Sp_{2n}\,/U_{J,E/F}$for a symmetric matrix $J \in GL_n$.

$(c) GL_2/GL_1 \times GL_1$.

When $G = GL_n$, it follows from Zelevinsky's classification that representations of G parabolically induced from irreducible square-integrable are precisely the irreducible temperedrepresentations of G. We therefore also have the following.

Corollary (3.2.22) [3]:-

In cases 1 and 3 of Corollary (3.2.21), for every irreducible tempered representation π of G we have

$$Hom_H(\pi,\mathbb{C}) \neq 0 \quad \Rightarrow \quad L_H^\pi \neq 0.$$

Lemma(3.2.22) [7]

Let _ be an irreducible representation of a reductive p-adic group and let P = MN be a parabolic subgroup of G. Suppose that M is a direct product of two reductive subgroups M_1andM_2. Let T_1be an irreducible representation of $\mathrm{M}_1$and let T_2be a representation of

M_2. Suppose $\pi \hookrightarrow \mathrm{Ind}_P^Q(T_1 \otimes T_2)$:Then there exists an irreducible representation T_2' such that $\pi \hookrightarrow \mathrm{Ind}_P^Q(T_1 \otimes T_2')$

Chapter 4

Quasi – Compact Endomorphisms and Primary Ideals in Commutative UnitalBanach Algebras

Among other things , our results lead to the observation that when B is strongly regular , every Riesz endomorphism of B is quasi-nilpotent on an invariant maximal ideal . Some of the implications of our work for various other types of function algebra are explored

Section(4.1) Spectral Projection and Primary Ideals

Let X be a complex Banach space. The essential spectrum $\sigma_e(T)$ of abounded operator $T : X \longrightarrow X$ is the set of all complex numbers λ for which thedifference $\lambda - T$ is not a Fredholm operator. $T$is quasi-compact if

$$\sigma_e(T) \subseteq \{\lambda: |\lambda| < 1\},$$

And Riesz if $\sigma_e(T) \subseteq \{0\}$.

Let Bbe a commutative unitalBanach algebra. An endomorphism of B is a bounded linear operator $T: B \longrightarrow B$which is multiplicative and preserves the multiplicative identity $1 \in B$. Operators of this type have received a great deal of attention , and their properties are well understood in certain cases. As the following theorem of Feinstein and Kamowitz makes clear, quasi-compact endomorphisms are rather special.

Theorem (4.1.1) [4]:-

Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B. Let Tbe a quasi-compact endomorphism of B. Then:

(i) $\sigma(T) \subseteq \{\lambda : |\lambda| < 1\} \cup \{1\}$;
(ii) the eigenvalue 1has(algebraic) multiplicity 1and eigenspace $\mathbb{C} \cdot 1$;
(iii) there is a character $x_0 \in \Phi_B$such that the sequence $(T^n)_{n=1}^{\infty}$converges in norm to the $rank$ 1projection $b \mapsto x_0(b) \cdot 1$. We have the following result.

Theorem (4.1.2) [4]:-

Let B be a semi-prime commutative unital Banach algebra with connected character space Φ_B. Let Tbe a quasi-compact endomorphism of B, and suppose that $T^* x_0 = x_0$. Then there is a family $J$of T-invariant closed primary ideals of finite codimension in Band hull $\{x_0\}$, for which

$$\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\} = \bigcup_{I \in J} \sigma\left(\frac{T}{I}\right). \quad (1)$$

The maximal ideal $M(x_0) = \{b \in B: x_0(b) = 0\}$always belongs to J.
Here

$$r_e(T) = \inf\{r > 0: |\lambda| \leq r \ \ for\ all\ \ \lambda \in \sigma_e(T)\}$$

is the essential spectral radius of T, and (for each $I \in J$) $T/I$is the endomorphism of B/I which satisfies

$$(T/I)(b + I) = Tb + I \quad (b \in B)$$

for every $b \in B$.

An ideal in a commutative Banach algebra is said to be primary if it is simultaneously modular and contained in only one maximal ideal. We will later exploit the fact that, for many algebras of differentiable functions, these ideals are often of a very particular form. For example, if $\Omega$is a convexbounded domain in $\mathbb{R}^d$,every closed primary ideal in $C^k(\bar{\Omega})$contains an ideal of the form

$$\{u \in B: (D^\alpha u)(u) = 0 \ \ for\ all\ \ |\alpha| \leq k\}$$

for some $x \in \bar{\Omega}$. We will see later that this implies that every Riesz endomorphism $T$of $C^k(\bar{\Omega})$satisfies $\sigma(T) = \{0,1\}$. This conclusion will, in fact, be shown to apply throughout a large class of Shilov-regular function algebras on the closed unit ball in $\mathbb{R}^d$. The situation for algebras of infinitely differentiable functions is much more interesting. In this direction, Theorem(4.1.2) can be used to reproduce a large proportion of the results in the literature concerning the spectra of Riesz endomorphisms on algebras of holomorphic functions on domains in $\mathbb{C}$. Among these is the following (now well-known) Theorem of Kamowitz. Below, $\mathbb{D}$is the open unit disk, and $z: \bar{\mathbb{D}} \to \mathbb{C}$is the associated inclusion map.

Theorem (4.1.3) [4]:-

Let Tbe a Riesz endomorphism of the disk algebra $A(\bar{\mathbb{D}})$, and suppose that the function $\phi = Tz$fixes a point $p = \phi(p)$in the open unit disk. Then the spectrum of $T$is given by

$$\sigma(T) = \{0,1\} \cup \left\{\phi'(p)^k : k \in \mathbb{N}\right\}.$$

Asimilar conclusion applies for a large number of other algebras of holomorphic functions on $\bar{\mathbb{D}}$. Theorem(4.1.2) will allow us to prove a significant generalisation of Theorem(4.1.3) , one which subsumes a large number of existing results in this area. In

particular, we will show that the same conclusion applies in every unital Banach algebra obtained by completing the algebra of polynomials in $z$with respect to a norm for which the multiplication operators $f \mapsto (z-p)f$are bounded below (for $p \in \mathbb{D}$).
The machinery described is then used to prove a version of Theorem(4.1.2) . Recovery of this central result is followed by a brief examination of some of the implications of our work for determining the spectra of Riesz endomorphisms of algebras possessing particular primary ideal structures. We devoted to examining some of the implications of our results for Riesz endomorphisms in two rather different classes of function algebra: one class contains only regular algebras, and the other consists solely of algebras of functions which are holomorphic in the unit disk.

Given any linear map $T: X \longrightarrow Y$between vector spaces $X$and Y, we will henceforth write

$$\ker(T) = T^{-1}(0) = \{x \in X: Tx = 0\} \ and \ Im\,(T) = \{Tx: x \in X\}$$

for the kernel and range of T.
Let Xbe a non-zero complex Banach space. The symbol $L(X)$denotes the unitalBanach algebra of all bounded linear operators on X. Given any $T \in L(X)$, we set

$$\rho(T) = \{\lambda \in \mathbb{C}: \lambda - T \ is\ invertible\ L(X)\}, \quad \sigma(T) = \mathbb{C}\backslash\rho(T),$$

$$\rho_e(T) = \{\lambda \in \mathbb{C}: \lambda - T\ is\ a\ Fredholm\ operator\}, \quad \sigma_e(T) = \mathbb{C}\backslash\rho_e(T).$$

These are referred to as the resolvent, spectrum, essential resolvent and essential spectrum of Trespectively.
Suppose that $X$is infinite dimensional and let $K(X)$be the closed, proper ideal in $L(X)$consisting of all $T \in L(X)$which are compact. It follows from Atkinson's theorem that, for each $T \in L(X), \sigma_e(T)$is precisely the spectrum of $T + K(X)$in the Calkin algebra $L(X)/K(X)$. As such, $\sigma_e(T)$is a non-empty, compact subset of $\mathbb{C}$with $\sigma_e(T) \subseteq \sigma(T)$. An important connection between the essential and actual spectra of an operator is provided by the so-called punctured neighbourhood theorem, which we now quickly recall.

Theorem (4.1.4) [4]:-

Let Xbe a complex Banach space, let $T \in L(X)$and let Ube a component of $\rho_e(T)$. Then either $U \subseteq \sigma(T)$or $U \cap \sigma(T)$is at most countable and each $\lambda \in U \cap \sigma(T)$is isolated in $\sigma(T)$.
Now let σbe a non-empty compact subset of $\mathbb{C}$. We write $\mathcal{O}(\sigma)$for the algebra of germs of holomorphic functions over σ, equipped with the inductive compact open topology . We use the symbol $z_\sigma$for the germ over $\sigma$of the complex co-ordinate functional $z$on $\mathbb{C}$. Given disjoint compact subsets $\sigma$and $\sigma'$of $\mathbb{C}$, we denote by $1_{\sigma,\sigma\prime}$, the germ over $\sigma \cup \sigma'$obtained in the following manner. Let U, U'be disjoint open neighbourhoods of $\sigma$and

σ'respectively. Now let $h$be the holomorphic function on $U \cup U'$which is 1on $U$and 0on U' and let $1_{\sigma,\sigma'}$be the germ of hover $\sigma \cup \sigma'$. It is clear that $1_{\sigma,\sigma'}$is an idempotent in $\mathcal{O}(\sigma \cup \sigma')$which does not depend on the particular choice of $U$and U'.

For an element $b \in B$of a unitalBanach algebra B, the main single-variable holomorphic functional calculus theorem asserts that there is a unique continuous unital algebra homomorphism $\Theta_b: \mathcal{O}(\sigma(b)) \to B$satisfying the condition $\Theta_b(z_{\sigma(b)}) = b$. It is standard that elements of $Im(\Theta_b)$commute with b. If $\sigma \subseteq \sigma(b)$and both $\sigma$and its complement $\sigma' = \sigma(b)\backslash\sigma$are compact, we define $P_b(\sigma) = \Theta_b(1_{\sigma,\sigma'})$, and refer to this as the spectral projection of aover σ. Each $1_{\sigma,\sigma'}$is an idempotent in $\mathcal{O}(\sigma(b))$and $\Theta_b$is multiplicative, so each spectral projection $P_b(\sigma)$is an idempotent in Bwhich commutes with b.The next lemma summarises some pertinent properties of these idempotents when $B = L(X)$for a Banach space X.

Lemma (4.1.5) [4]:-

Let Xbe a non-zero complex Banach space and let $T \in L(X)$. Let $\sigma \subseteq \sigma(T)$and suppose that both $\sigma$and $\sigma(T)\backslash\sigma$are compact when considered as subsets of $\mathbb{C}$. Then $P_T(\sigma)$has the following properties:

(i) The subspaces $Im(P_T(\sigma))$and $Ker(PT(\sigma))$are T-invariant closed subspaces of Xsuch that $X = Im(P_T(\sigma)) \oplus Ker(P_T(\sigma))$. If σis a non-empty proper subset of $\sigma(T)$then the spectra of the restrictions of $T$to $Im(P_T(\sigma))$and $Ker(P_T(\sigma))$are σ and $\sigma(T)\backslash\sigma$respectively. The projection $P_T(\sigma)$is zero if and only if $\sigma$is empty and is the identity operator if and only if $\sigma = \sigma(T)$.

(ii) If there is a number $t > 0$such that $|\lambda| < t$for all $\lambda \in \sigma$and $|\lambda| > t$for all $\lambda \in \sigma(T)\backslash\sigma$then $Im(P_T(\sigma)) = \{x \in X: \|T^n x\|/t^n \to 0 \; as n \to \infty\}$.

(iii) If σconsists of a single isolated point $\lambda$of $\sigma(T)$and $\lambda - T$is Fredholm then $P_T(\sigma)$is a finite rank operator. In this case, there is a non-negative integer $k$for which $Im(P_T(\sigma)) = Ker((\lambda - T)^k)$.

It follows from (iii) that if $\sigma$is a finite set of isolated points of $\sigma(T)$belonging to $\rho_e(T)$then $P_T(\sigma)$is a finite rank operator; it is simply the sum of the finite rank operators $P_T(\{\lambda\}) for \; \lambda \in \sigma$(this is easy to see by writing $1_{\sigma,\sigma(T)\backslash\sigma}$as the sum of germs of the form $1_{\{\lambda\},\sigma(T)\backslash\{\lambda\}} \; for \; \lambda \in \sigma$). Assertion (ii) appears as an exercise , and relies on considerations approaching so-called local spectral theory. The explicit description of $Im(P_T(\sigma))$provided to us by (ii) in the circumstances described will be very useful later.

Let B be a commutative unital Banach algebra, and let $I \subseteq B$be an ideal. We write

$$\mathscr{k}(I) = \Phi_B \cap I^{\perp} = \{x \in \Phi_B : I \subseteq M(x)\},$$

and refer to this as the hull of I. Let $x \in \Phi_B$. Having agreed to use the symbol $\pi_B(x)$to denote the set of all closed primary ideals in B contained in the maximal ideal $M(x) = x^{-1}(0)$, we have

$$\pi_B(x) = \{I : I \text{ is a closed ideal in } B \text{ and } \{x\} = \mathscr{k}(I)\}.$$

Let Abe a second commutative unital Banach algebra and let $Hom(A, B)$be the set of all bounded linear operators $T: A \to B$which are multiplicative and which send the multiplicative identity in $A$to that in B. An operator of this type obviously induces a continuous map

$$T^{\dagger}: \Phi_B \to \Phi_A$$

such that $T^{\dagger}(x) = T * x$for every $x \in \Phi_B$(where $T^*: B^* \to A^*$is the usual Banach space adjoint of T). Indeed, it is through this map that homomorphisms of commutative Banach algebras are typically studied.

Let Tbe an endomorphism of B, and let $I \subseteq B$be a closed T-invariant ideal. Letting $Q_I: B \to B/I$be the quotient map, it is standard that $Q_I^{\dagger}$maps Φ_B/Ihomeo-morphically onto $\mathscr{k}(I)$. In fact, it does us no particular harm to identify $Q_I^{\dagger}$with the inclusion map of $\mathscr{k}(I)$into Φ_B. The fact that $I$is T-invariant obviously makes h(I)in-variant under $T^{\dagger}$, so $T^{\dagger}$restricts to give a map $\tau: \mathscr{k}(I) \to \mathscr{k}(I)$(given, of course, by $\tau(x) = T^*x$for each $x \in \mathscr{k}(I)$). Since the endomorphism $T/I$is defined by the property that$Q_I T = (T/I)Q_I$, it is clear that $\tau$is topologically conjugate to $(T/I)^{\dagger}$. It follows, in particular, that when $I$is a closed ideal for which $T/I$is invertible, T^*maps $\mathscr{k}(I)$ontoitself. This observation will turn out to be very important.

Given a closed T-invariant ideal $I$for which $T/I$is invertible in $L(B/I)$, the hull $\mathscr{k}(I)$now clearly belongs to the collection

$$\mathcal{C} = \{E \subseteq \Phi_B : T^*E = E\}.$$

The union $F(T^{\dagger}) = \bigcup_{E \in \mathcal{C}} E$is the so-called fixed setof $T^{\dagger}$, and is known to coincide with the intersection $\bigcap_{n=1}^{\infty} T^{*n}\Phi_B$. The following lemma is now a consequence of part(iii) of Theorem(4.1.1).

Lemma (4.1.6) [4]:-

Let Bbe a semi-prime commutative unital Banach algebra with connected character space Φ_B, and let Tbe a quasi-compact endomorphism of B. Suppose that $T^*x_0 = x_0$for some $x_0 \in \Phi_B$, and that $I \subseteq B$is a closed T-invariant ideal for which $T/I$is invertible. Then $I \in \pi_B(x_0)$.

Proof:-

By Theorem(4.1.1) , the sequence $(T^n)_{n=1}^{\infty}$converges in norm to a projection of the form $b \mapsto y_0(b)\ \cdot 1$for some $y_0 \in \Phi_B$. As Feinstein and Kamowitz note , this implies that $F(T^\dagger)\ = \{y_0\}$. The result now follows by noting that any fixed point of $T^\dagger$belongs to $F(T^\dagger)$; our insistence that $T^* x_0 = x_0$is simply to ensure that $x_0 = y_0$.

The following Lemma reveals the source of the ideals mentioned in Theorem(4.1.2) ; they will be the kernels of certain spectral projections.

Lemma (4.1.7) [4]:-

Let Bbe a semi-prime commutative unitalBanach algebra with connected character space, and let Tbe a quasi-compact endomorphism of B. Let

$$\sigma_r = \sigma(T)\ \cap\ \{\lambda :\ |\lambda|\ \geq\ r\}$$

for any $r_e(T) < r \leq 1$. Then $\sigma_r$is finite, and $I_r = Ker(P_T(\sigma_r))$is a closed T-invariant ideal of finite codimension in B.

Proof:-

It is clear that $\sigma_r$is a compact subset of $\mathbb{C}$. Applying Theorem(4.1.4) with Uequal to the unbounded component of $\rho_e(T)$, we observe that, in addition, every $\lambda\ \in \sigma_r$is isolated in $\sigma(T)$. As a compact subset of $\mathbb{C}$with no accumulation points, $\sigma_r$is finite. It is clear from Lemma(4.1.5) (and the remarks immediately following it) that $I_r = Ker(P_T(\sigma_r))$is a closed T-invariant subspace of finite codimension in B.
We have not yet established that $I_r$is an ideal. To do this, we first observe that since $\sigma(T)$ is compact and every point of $\sigma_r$is isolated in $\sigma(T)$, the complement $\sigma(T)\ \backslash\sigma_r$is also a compact subset of $\mathbb{C}$. This implies, in particular, that there is a $0 < t(r) < r$for such that

$$\sigma_r = \sigma(T) \cap \{\lambda: |\lambda| > t(r)\}$$

$$and \quad \sigma(T)\backslash\sigma_r = \sigma(T) \cap \{\lambda: |\lambda| > t(r)\}.$$

Invoking part (ii) of Lemma(4.1.5), we see that

$$I_r = \{b \in B: \|T^n b\|/t(r)^n \to 0\ \ as\ n \to \infty\} \qquad (2)$$

The fact that Iris an ideal now follows from Theorem1which ensures, of course, that $T$is power bounded.

It is clear from part (i) of Theorem(4.1.1)that

$$Im\left(P_T(\sigma_1)\right) = Im\left(P_T(\{1\})\right).$$

By part (ii) of the same theorem, this subspace has dimension 1. It follows that $I_1 = Ker(P_T(\sigma_1))$is a maximal ideal. It is already clear that the ideal $I_r$is contained in the maximal ideal $I_1$for each $r_e(T) < r \leq 1$. With Lemma(4.1.6) at our disposal, the following observation helps us show (among other things) that $I_1$is the onlymaximal ideal with this property.

Lemma (4.1.8) [4]:-

Let $B$and Tbe as in Lemma(4.1.7) and define $\sigma_r$and $I_r$as before for $r_e(T) < r \leq 1$. Then $\sigma(T/I_r) = \sigma_r$for all such$r$.

Proof:-

Let $r_e(T) < r \leq 1$, and let

$$i_r: Im\left(P_T(\sigma_r)\right) \to B \quad and \quad Q_r: B \to B/I_r$$

be the appropriate inclusion and quotient maps. Composing these in the obvious fashion, we obtain a Banach space isomorphism

$$U_r = Q_r i_r: Im\left(P_T(\sigma_r)\right) \to B/I_r$$

Letting $T_r$be the restriction of $T$to $Im(P_T(\sigma_r))$, we have the equations $i_r T_r = T i_r$and $Q_r T = (T/I) Q_r$.

Taken together, these lead to the intertwining relation $(T/I)U_r = U_r T_r$. Since this implies that $T_r$and T/Ihave the same spectrum, the result now follows from part (i) of Lemma(4.1.5).

We have now assembled everything we need in order to prove Theorem(4.1.2). However , before we do so, it is perhaps worth making explicit the respective rôles of Lemmas (4.1.6),(4.1. 7)and (4.1.8). Under the hypotheses of Theorem(4.1.2) , Lemma (4.1.7) provides us with a family of closed T-invariant ideals of finite codimension in B; these are simply the kernels of the spectral projections $P_T(\sigma_R)$for $r_e(T) < r \leq 1$. Lemma (4.1.8) does two things. Unsurprisingly, it tells us that

$$\sigma(T) \cap \{\lambda: |\lambda| > r_e(T)\} = \bigcup_{r_e(T)<r\leq 1} \sigma(T/I_r).$$

However, this is not its only rôle. At this stage in the proof, we do not yet know anything about the hulls $\hbar(I_r)$, except that they all contain x_0. It is here that Lemma(4.1.8)really comes to our rescue; the fact that $0 \notin \sigma_r$means that each of the endomorphisms$T/I_r$is

invertible, and it is this information which (via Lemma(4.1.6)) allows us to show that each $I_r$belongs to $\pi_B(x_0)$.
We now prove the following strong form of Theorem(4.1.2).

Theorem (4.1.9) [4]:-

Let Bbe a semi-prime commutative unitalBanach algebra with connected character space Φ_B, and let Tbe a quasi-compact endomorphism of B. Let $x_0 \in \Phi_B$satisfy $T^*x_0 = x_0$. Then there is a family $J$of closed T-invariant ideals with the following properties

(i) $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\} = \bigcup_{I\in J} \sigma(T/I)$;
(ii) $J \subseteq \pi_B(x_0)$;
(iii) the maximal ideal $M(x_0)$always belongs to J;
(iv) each $I \in J$is the kernel of a finite rank spectral projection associated with T;
(v) $J$is at-most-countable, and its elements form a chain, in the sense that if $I, I' \in J$then either $I \subseteq I'$ or $I' \subseteq I$;
(vi) for $r_e(T) < r \leq 1$, there is some $I \in J$for which $\sigma(T_{|I})$is contained in the disk $\{\lambda : |\lambda| < r\}$.

Proof:-

Invoking Lemma(4.1.7), and setting

$$J = \left\{Ker\left(P_T(\sigma_R)\right) : r_e(T) < r \leq 1\right\}, \tag{3}$$

we have immediately that $J$is a family of T-invariant closed ideals of finite codimension in B. Assertions (i)–(vi) are now proved as follows.

(i) Since $\sigma(T) \cap \{\lambda : |\lambda| > r_e(T)\}$is the union of the sets

$$r_e = \sigma(T) \cap \{\lambda: |\lambda| \geq r\}$$

for $r_e(T) < r \leq 1$, this part is clear from Lemma 8.
(ii) Making a second appeal to Lemma (4.1.8), we observe that for each ideal $I \in J$, the spectrum $\sigma(T/I)$consists solely of points of modulus strictly larger than 0. This means, in particular, that each of the operators $(T/I)I \in J$is invertible. Lemma(4.1.6) therefore implies that $J \subseteq \pi_B(x_0)$.
(iii) Showing that $I_1 = Ker(P_T(\sigma_1))$is a maximal ideal is achieved using the argument described in the comments before Lemma(4.1.8). Explicitly, a combination of Theorem(4.1.1) and Lemma(4.1.5) gives us

$$B = I\,m\left(P_r(\{1\})\right) \oplus I_1.$$

Using part (ii) of Theorem(4.1.1), this implies that I_1has codimension1 $in\ B$. Having already shown that $I_1$is an ideal, this gives us what we need.

(iv) This is immediate from definition (3).

(v) The fact that $J$is at-most-countable is a straightforward consequence of the punc-turedneighbourhood theorem (Theorem(4.1.4)). That Jis a chain in the sense indicated is easily proved using description (2)from the proof of Lemma(4.1.7).

(vi) Let $r_e(T) < r \leq 1$. Setting $I = Ker(P_T(\sigma_r))$, Lemma(4.1.5)tells us that

$$\sigma(T_{|I}) = \sigma(T) \backslash \sigma_t = \sigma(T) \cap \{\lambda: |\lambda| < r\}.$$

The proof is now complete.

Function algebra is a semi simple commutative unital Banach algebra, considered as an algebra of continuous functions on its character space. Let Bbe such an algebra. Then, as we will recall, $B$is said to be:

(a) Regular if for each closed subset $F \subseteq \Phi_B$and each point$x \in \Phi_B \backslash F$, there is some $f \in B$satisfying $f(x) = 1$and $f(F) \subseteq \{0\}$; and

(b) Strongly regular if$\pi_B(x) = \{M(x)\}$for every $x \in \Phi_B$.

When $B$is regular and $x \in \Phi_B$, we write $J(x)$for the closure of the ideal

$$J_0(x) = \{f \in B: f^{-1}(0)\ is\ a\ neighbourhood\ of\ x\}.$$

A celebrated result of Shilov ensures that when $B$is regular, $J(x)$is the intersection of all the closed ideals $I \in \pi_B(x)$. We exploit this to establish the following consequence of Theorem(4.1.9).

Theorem (4.1.10) [4]:-

Let Bbe a regular function algebra with connected character space Φ_B,. Let T be a Riesz endomorphism of B, and suppose that $T^* x_0 = x_0$. Then $J(x_0)$is a T-invariant closed ideal, and

$$\{0\} \cup \sigma(T) = \{0\} \cup (T/J(x_0)).$$

Proof:-

Showing that $J(x_0)$is T-invariant is straightforward, and can be achieved by using the fact that $Tf = f \circ T^\dagger$for each $f \in B$. Let $J \subseteq \pi_B(x_0)$be the family of ideals supplied by Theorem(4.1.9) , and fix any ε. Then, by part (vi), there is some $I \in J$such that

$$\sigma(T_{|I}) \subseteq \{\lambda: |\lambda| < \varepsilon\}.$$

As the restriction of a Riesz operator to a closed T-invariant subspace, $T_{|I}$is also a Riesz operator. This implies, in particular, that $\rho(T_{|I})$is connected, hence that $\sigma(T_{|N}) \subseteq \sigma(T_{|I})$for every $T_{|I}$-invariant closed subspace $N \subseteq I$. Using Shilov's result, we therefore see that

$$\sigma\left(T_{|J(x_0)}\right) \subseteq \{\lambda: |\lambda| < \varepsilon\}.$$

Since an identical conclusion is available for every $\varepsilon > 0$, $T_{|J(x_0)}$is quasi-nilpotent. The result now follows from the standard spectral inclusions $\sigma(T) \subseteq \sigma(T_{|N}) \cup \sigma(T_{|N})$and $\sigma(T_{|N}) \subseteq \sigma(T) \cup \sigma(T_{|N})$, which hold for any closed T-invariant subspace $N$of$B$. The proofs associated with the latest inclusions can be found .

The next result indicates, among other things, that a Riesz endomorphism of a strongly regular function algebra can never have a nontrivial spectrum.

Corollary (4.1.11) [4]:-

Let Bbe a regular function algebra with connected character space, and let Tbe a Riesz endomorphism of$B$. Let $x_0$be the element of $\Phi_B$for which $T^*x_0 = x_0$, and suppose that $J(x_0)$has finite codimension in B. Then $\sigma(T) = \{0, 1\}$.

Proof:-

By the previous result, the nonzero spectrum of Tcoincides with that of $T/J(x_0)$ (an operator on a finite dimensional space). This means that $\sigma(T)$is finite. Now choose any nonzero point $\lambda$of $\sigma(T)$. Since $T$is a Riesz operator, $\lambda$is necessarily an eigenvalue. It follows that $\lambda^n \in \sigma(T)$for every $n \in \mathbb{N}$, which leads to a contradiction unless $\lambda = 1$. Although the class of algebras to which Corollary (4.1.11) applies is very large, quite a number of standard regular function algebras lie entirely beyond its reach. Examples include the 'big' Lipschitz algebras $Lip(Y)$over compact metric spaces. However, even these algebras are not immune to the following theorem.

Theorem (4.1.12) [4]:- Let Bbe a semi-prime commutative unitalBanach algebra with connected character space Φ_B. Let Tbe a quasi-compact endomorphism of B, and let $x_0$be a character for which $T^*x_0 = x_0$. Suppose that

$$\sigma(T) \cap \{\lambda: |\lambda| > r_e(T)\}$$

contains a point other than 1. Then there is a bounded point derivation $d$at $x_0$such that $T^*d \neq 0$.

Before proceeding with the proof, we remind that a bounded point derivation(on a commutative unital Banach algebra B) is bounded linear functional $d \in B^*$such that $d(uv) = x(u)dv + x(v)du$holds for some $x \in \Phi_B$and all $u, v \in B$.
In this case, $d$is said to be a bounded point derivation at x. It is standard that Bsupports a bounded point derivation at some $x \in \Phi_B$if and only if $M(x) \neq \overline{M(x)^2}$.

Proof:-

Our hypothesis on $\sigma(T)$means that there is some $r > 0$for which the set $\sigma_r = \sigma(T) \cap \{\lambda : |\lambda| \geq r\}$contains at least two points. The ideal $I_r = Ker(P_T(\sigma_r))$therefore has codimension at least 2. As a non-trivial finite dimensional commutative unitalBanach algebra with exactly one maximal ideal, the quotient algebra B/I_rsupports at least one nonzero bounded point derivation; call this d_0. Letting $Q: B \to B/I_r$be the quotient map, set $d = Q^* d_0$; it is straightforward to verify that this is a bounded point derivation on Bat $x_0$with

$$T^* d = Q^* (T/I_r)^* d_0.$$

The result follows, since $Q^*$is injective and (by Lemma(4.1.8)), $(T/I)^*$is invertible.This result complements an existing result of Udo Klein who proved an analogous assertion for compact endomorphisms of uniform algebras.

Section(4.2) : Applications for Concrete Function Algebras :-

It goes without saying that the results have some serious consequences for Riesz endomorphisms of strongly regular algebras such as $C(X)$and $lip(Y)$(where $X$is any connected compact Hausdorff space and $Y$is any connected compact metric space); it is immediate from any of the three previous results that such operators all have spectra equal to $\{0, 1\}$. Our aim, at least, is to describe another large class of function algebras with this property. For simplicity, we work with function algebras defined on (the closure of) the open ball Ωin $\mathbb{R}^d$. However much of what we have to say applies equally to algebras on $\mathbb{T}^d$and other connected compact smooth manifolds.

We write $C^k(\bar{\Omega})$for the algebra of k-times continu-ously differentiable functions $u : \Omega \to \mathbb{C}$which, together with their partial derivatives $D^\alpha u$of orders $|\alpha| \leq k$, extend to be continuous on $\bar{\Omega}$. This is a regular function algebra under the norm

$$\|u\|_{k,\infty,\Omega} = \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \lim_{x \in \Omega} |(D^\alpha u)(x)|, \qquad (u \in C^k(\bar{\Omega})).$$

Setting $C^\infty(\bar{\Omega}) = \bigcap_{k \geq 1} C^k(\bar{\Omega})$, we now recall a famous result of Shilov; a particularly straightforward proof of this theorem can be found by Mirkil's treatise.

Theorem (4.2.1) [4]:-(Shilov, 1950)

Let Bbe a Banach space of functions on $\bar{\Omega}$(or any other compact smooth manifold), equipped with pointwise addition and a topology stronger than that of pointwise convergence on $\bar{\Omega}$. Then $C^{\infty}(\bar{\Omega}) \subset B$if and only if $C^{k}(\bar{\Omega}) \subseteq B$for some $k \in \mathbb{N}$.The following result is also due to Shilov, and can be found in, of Gelfand, Shilov and Raikov.

Theorem (4.2.2) [4]:-

Let $B'$and B''be regular function algebras with the same space $X$of maximal ideals; furthermore, let $B' \subseteq B''$with B'dense in B''. If $J' \subset B'$and $J'' \subset B''$are the minimal primary ideals corresponding to the same point $x_0 \in X$and the quotient algebra $B'/J'$is finite dimensional then $B''/J''$is also finite dimensional, and its dimension is no greater than that of B'/J'.

With these two results at our disposal, we can give the following application of Corollary (4.1.11). The regularity of the domain Ωmeans that the next result applies, in particular, when $B$is one of the Sobolev algebras

$$W^{s,p}(\Omega) = \{u \in L^{p}(\Omega) : D^{\alpha}u \in L^{p}(\Omega)\ for\ |\alpha| \leq s\}$$

with$1 \leq p, s < +\infty$and $sp > d$.

Theorem (4.2.3) [4]:-

Let Bbe Banach algebra of continuous functions on $\bar{\Omega}$for which $C^{\infty}(\bar{\Omega})$is dense in B. Let Tbe a Riesz endomorphism of B. Then $\sigma(T) = \{0,1\}$.

Proof:-

It is clear from Theorem(4.2.1)that, under our hypotheses, there is some $k \in \mathbb{N}$for which $C^{k}(\bar{\Omega})$is dense in B. Since the inclusion maps $C^{k}(\bar{\Omega}) \to B$and $B \to C(\bar{\Omega})$are both continuous, there are constants $C_1$and C_2such that

$$\|f\|_{0,\infty,\Omega} \leq C_1\|f\|_{B} \leq C_2\|f\|_{k,\infty,\Omega}$$

for every $f \in C^{k}(\bar{\Omega})$.

Since the spectral radius of any element of $C^{k}(\bar{\Omega})$is therefore the same with respect to Bas it is with respect to $C^{k}(\bar{\Omega})$, it follows that these two algebras have ‘the same’ space of maximal ideals. This latest assertion is, of course, to be understood in the sense that the restriction map $x \mapsto x_{|C^{k}(\bar{\Omega})}$is a homeomorphism of $\Phi_B$onto $\Phi_{C^{k}(\bar{\Omega})}$. Recalling that every minimal primary ideal in $C^{k}(\bar{\Omega})$is of the form

$$J' = \{u \in C^k(\bar{\Omega}) : (D^\alpha u)(x) = 0 \; for \; |\alpha| \le k\},$$

for some $x \in \bar{\Omega}$, an application of Theorem(4.2.2)(with $B' = C^k(\bar{\Omega})$and $B'' = B$) indicates that $B/J(x)$is finite dimensional for every $x \in \Phi_B$. The result is now an obvious consequence of Corollary (4.1.11).

Our next application demonstrates a little of what can be achieved when we appeal to Theorem(4.1.9) more directly. For the remainder of this essay, $B$is a unitalBanach algebra of functions which are continuous on the open unit disk and holomorphic on its interior. We will also assume that Bcontains the inclusion map $z: \bar{\mathbb{D}} \longrightarrow \mathbb{C}$. The only other restriction we will impose is that

$$(z-p)^k B = \{f \in B: f(p) = f'(p) = \cdot\cdot\cdot = f^{(k-1)}(p) = 0\} \qquad (4)$$

for each $p \in \mathbb{D}$(the opendisk) and each $k \in \mathbb{N}$. We note that this condition is automatically satisfied when the polynomials (in z) are dense in $B$and there is a $c_p > 0$for each $p \in \mathbb{D}$such that

$$\|f\| \le c_p \|(z - p)f\|$$

for each $f \in B$. The disk algebra $A(\bar{\mathbb{D}})$is easily seen to have this property.Our insistence that (4)holds is purely to give us access to the following theorem of Domar from 1982.

Theorem (4.2.4) [4]:-

Let Bbe a commutative unitalBanach algebra, let $f \in B$, and suppose that, for each $n \in \mathbb{N}$, the principal ideal $M_n = f^n B$ has codimension$n$. Then the $M_n$are theonly closed primary ideals of finite-codimension with $M_n \subseteq M_1$.

In light of our assumptions on B, this implies that the only closed primary ideals at p of finite codimension are of the form (4).

We can now prove the following theorem; given the looseness of our assumptions on B, it subsumes a large number of existing algebra-specific results .

Theorem (4.2.5) [4]:-

Let ϕbe a continuous self-map of $\bar{\mathbb{D}}$for which $f \circ \phi \in B$for every $f \in B$. Suppose that the operator defined by

$$Tf = f \circ \phi, \quad (f \in B)$$

isRiesz, and that ϕhas a fixed point pin $\mathbb{D}$, the open unit disk. Then

$$\sigma(T) = \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.$$

Proof:-

Combining Domar's theorem with assumption (4), we have

$$\pi_B(p) = \{M_k : k \in \mathbb{N}\}, \qquad (5)$$

where $M_k = \{f \in B: f(p) = f'(p) = \cdots = f^{(k-1)}(p) = 0\}$ for each $k \in \mathbb{N}$. It is clear that B cannot have any idempotents other than the functions $f = 0$ and $f = 1$ so, by the Shilov idempotent theorem, Φ_B is connected. Invoking Theorem(4.1.9) , there is a family $J \subseteq \pi_B(p)$ of closed, primary ideals of finite codimension in B such that

$$\sigma(T) \setminus \{0\} = \bigcup_{I \in J} \sigma(T/I).$$

We know, of course, that each $I \in J$ is equal to M_k for some $k \in \mathbb{N}$, and this makes the problem of determining $\sigma(T)$ particularly tractable. Fixing any $k \in \mathbb{N}$, let $\mathbb{C}[y]$ be the algebra of formal polynomials with coefficients in $\mathbb{C}$, and let j_p^k be the operator from B into $\mathbb{C}[y]/y^k\mathbb{C}[y]$ given by

$$j_p^k f = \sum_{l=0}^{k-1} \frac{1}{l!} f^{(l)}(p) Z^l, \qquad (f \in B)$$

where Z is the residue class of y in $\mathbb{C}[y]/y^k\mathbb{C}[y]$. It is easy to see that given any $I \in J$, there is a $k \in \mathbb{N}$ for which T/I is similar to an endomorphism $T_k: \mathbb{C}[y]/y^k\mathbb{C}[y] \to \mathbb{C}[y]/y^k\mathbb{C}[y]$, where

$$T_k\left(\sum_{l=0}^{k-1} a_l Z^l\right) = \sum_{l=0}^{k-1} a_l \left(j_p^k \phi - p\right)^l, \quad k \in \mathbb{N} \qquad (6)$$

Here, we adopt the convention that, in all cases, $(j_p^k \phi - p)^0$ is the identity element in $\mathbb{C}[y]/y^k\mathbb{C}[y]$. A routine calculation shows that the matrix of T_k (with respect to the basis $1, Z, Z^2, \ldots, Z^{k-1}$) is lower triangular, with $1, \phi'(p), \phi'(p)^2, \ldots, \phi'(p)^{k-1}$ along the diagonal. Thus, given any $I \in J$,

$$\sigma(T/I) = \{\phi'(p)^j : j = 0, 1, \ldots, k - 1\}$$

for some $k \in \mathbb{N}$. This is enough for us to be able to conclude that

$$\sigma(T) \subseteq \{0, 1\} \cup \{\phi'(p)^k : k \in \mathbb{N}\}.$$

That $\{0, 1\} \subseteq \sigma(T)$ is obvious, so it only remains to show that $\phi'(p)^k \in \sigma(T)$ for every $k \in \mathbb{N}$. If it were guaranteed that $J = \pi_B(p)$, this would already be clear. However, this is not what Theorem(4.1.9)tells us. To complete the proof, we consider the bounded linear functional on Bgiven by

$$\delta'_p(f) = f'(p)\,, \quad (f \in B).$$

The chain rule now gives $T^*\delta'_p = \phi'(p)\delta'_p$. As $T$is Riesz, this means that either $\phi'(p)$ =0, or $\phi'(p)$is an eigen value of T. The result now follows from the fact that the set of eigen values of $T$is closed under powers. We have said nothing about the situation when $p \in \Phi_B\backslash\mathbb{D}$. Here, the result is much more dependent on the algebra under consideration. However, progress can still be made with the help of Theorem (4.1.12); the situation is particularly straightforward when Bhas no non-zero bounded point derivations at pointsof $\Phi_B\backslash\mathbb{D}$.

Theorem (4.2.6) [8].

Let I be a secondary ideal of a commutative ring R**.** Then If Q is a weakly primary ideal (resp. weakly prime ideal) of R, then $\mathrm{I} \cap \mathrm{Q}$ is secondary.

List of Symbols

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