Chapter(1)

General Introduction

Introduction:

Definition (1.1)

the Laplace Transformation

Given suitable function F(t) the Laplace transform, written f(s) is Defined by:

$$f(s) = \int_0^\infty F(t)\, e^{-st}\, dt$$

Theorem (1.1) [Linearity]

If $f_1(t)$ and $f_2(t)$ are two function whose

Laplace transform exists then:

$L\,[af_1(t) + bf_2(t)] = a[F_1(t)] + b\,L\,[F_2(t)] = af_1(s) + bf_2(s)$

Example (1.1)

Find the Laplace transform of the function F(t) = t

Solution:

$$L[F(t)] = f(s) = \int_0^\infty F(t)\, e^{-st}\, dt = \int_0^\infty t\, e^{-st} dt$$

0

Using integration by parts we get

u = t , du = dt

$v = e^{-st}$, $v = -\frac{1}{s}\, e^{-st}$

$$\int_0^\infty t\, e^{-st} dt = \left[-\frac{t}{s}\, e^{-st}\right]_{t=0}^{\infty} + \int_0^\infty \frac{1}{s}\, e^{-st} dt = \left[0 - 0 - \frac{1}{S^2}\, e^{-st}\right]_{t=0}^{\infty} = -\frac{1}{S^2}\, e^{-\infty} + \frac{1}{S^2}\, e^0 = \frac{1}{S^2}$$

Example(1.2):

Find the Laplace transform of the function F(t) = sin(t)

Solution:

$$L[F(t)] = f(s) = \int_0^\infty F(t)\, e^{-st}\, dt = \int_0^\infty \sin(t)\, e^{-st}\, dt$$

Using integration by parts we find

$u = \sin t$, $\frac{du}{dt} = \cos t$

$dv = e^{-st}\, dt$, $v = \frac{-1}{S}\, e^{-st}$

$$\int_0^\infty \sin t\, e^{-st}\, dt = [\sin t\, (-\frac{1}{S}\, e^{-st}]_{t=0}^{\infty} + \frac{1}{S} \int_0^\infty \cos t\, e^{-st}\, dt$$

$$= \frac{1}{S} \int_0^\infty \cos t\, e^{-st}\, dt$$

Using integration by parts again we have

$\frac{1}{S} \int_0^\infty \cos(t)\, e^{-st}\, dt$, $u = \cos t$ $\frac{du}{dt} = -\sin t$, $\int dv = \int e^{-st}\, dt$

$\mathbf{v = -\frac{1}{S}\, e^{-st}}$

$$\frac{1}{S} \int_0^\infty \cos t\, e^{-st}\, dt = \frac{1}{S}\left[-\frac{1}{S} \cos t\, e^{-st} \right]_{t=0}^{\infty} - \frac{1}{S^2} \int_0^\infty \sin\ t\, e^{-st}\, dt$$

$$I \int_0^\infty e^{-st} \sin t\, dt + \frac{1}{S^2} \int_0^\infty e^{-st} \sin t\, dt = \frac{1}{S^2}$$

$$[1 + \frac{1}{S^2}]\int_0^\infty \sin(t)\, e^{-st} v dt = \frac{1}{S^2} \quad , \left[\frac{S^2+1}{S^2}\right]\int_0^\infty \sin(t)\, e^{-st}\, dt = \frac{1}{S^2}$$

Divided both side by $\left[\frac{S^2+1}{S^2}\right]$ we get:

$$\int_0^\infty \sin t\, e^{-st}\, dt = \frac{1}{S^2+1}$$

Example find the Laplace transform of the function

$f(t) = e^{-at}$

solution $f(s) = \frac{1}{S - a}$

table(1.1)

in the following table, the Laplace transform for some fuction:

The Function	**Laplace Transform**
1	$\frac{1}{s}$
t	$\frac{1}{S^2}$
t^2	$\frac{2}{S^3}$
t^n	$\frac{n!}{S^{n+1}}$
Sin at	$\frac{a}{S^2 + a^2}$
Cos at	$\frac{s}{S^2 + a^2}$
e^{at}	$\frac{1}{S - a}$

$t e^{at}$	$\frac{1}{(S-a)^2}$
$t^n e^{at}$	$\frac{N!}{(S-a)^{(n+1)}}$
Sin h(at)	$\frac{a}{(S^2-a^2)}$
Coshat	$\frac{s}{(S^2-a^2)}$
e^{bt} Sin (at)	$\frac{a}{(S-a)^2+a^2}$
$e^{bt}\cos(at)$	$\frac{(S-b)}{(S-b)^2+a^2}$
$\frac{1}{a}\ e^{at}-1$	$\frac{1}{S(s-a)}$
$\delta(t)$	1
$\delta'(t)$	S
$\delta''(t)$	S^2

(1. 2) The convolution

Definition (1.2) The convolution of two given function f(t) and g(t) is written

f * g and is defined by the integral

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$L[f(t) * g(t)] = \int_0^{\infty}\int_0^{t} e^{-st} f(\tau)\, g(t - \tau) d\tau\, dt$$

Example(1.3)

Find the value of cost * sin t

Solution: -

Using the definition of convolution we have

$$\text{cost} * \text{sint} = \int_0^{t} \cos\tau \sin(t - \tau)\, dt$$

$$\text{Sin A.cos B} = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\int_0^{t} \frac{1}{2} (\sin t + \sin(t - 2\tau)\, d\tau$$

$$\frac{1}{2} [T\sin(t)]_{T=0}^{t} + \frac{1}{2} [\frac{1}{2} \cos(t - 2T)]_{T=0}^{t}$$

$$\frac{1}{2} t \sin t + \frac{1}{4} [\cos(-t) - \cos t]$$

$$\text{Cos t} * \text{sin t} = \frac{1}{2} t \sin(t)$$

Example(1.4):

Find the value of sin t * t^2

The solution is

$t^2 + 2\cos t - 2$

Inverse Laplace Transform

Example(1.5) **Find The Laplace inverse:**

$$L^{-1}\left[\frac{S}{S^2+1}\right] = L^{-1}\left[\frac{S}{S^2+1} - \frac{1}{S^2+1}\right]$$

$$L^{-1}\left[\frac{S}{S^2+1}\right] = \cos(t)$$

$$L^{-1}\left[\frac{1}{S^2+1}\right] = \sin(t)$$

$$L^{-1}\left[\frac{S}{S^2+1}\right]\left[\frac{1}{S^2+1}\right] = \cos t * \sin t = \frac{1}{2}\sin(t)$$

Example(1,6) :

Find $L^{-1}\left[\frac{1}{S^3}\ \frac{1}{(S^2+1)}\right]$

Solution :

$$L^{-1}\left[\frac{1}{S^3}\right] = \frac{1}{2}t^2 \ / \ L^{-1}\left[\frac{1}{S^2+1}\right] = \sin t$$

$$L^{-1}\ \frac{1}{S^3(S^2+1)} = \frac{1}{2}\ t^2 * \sin(t) = \frac{1}{2}\ (t^2 + 2\cos t - 2\ ]$$

$$L^{-1}\left[\frac{1}{S^3}\ \frac{1}{(S^2+1)}\right] = \frac{1}{2}\ (t^2 + 2\cos t - 2)$$

(1.3) first order different equation :

In this section, we solve some ordinary differential equation Using the Laplace transformation .

Definition(1.7):

The first order different equation of y = f(x) is

$$L\left[\frac{dx}{dt}\right] = S\,\tilde{x}(s) - x(0)$$

Example (1.8)

Solve The homogeneous order first differential

Equation:

$$\frac{dx}{dt} \cdot 3\,x = 0 \ , \ x(0) = x(0) = 1$$

Solution:

Taking Laplace Transform we have

$$L\left[\frac{dx}{dt}\right] + 3L\,[(x)] = 0$$

$$s\tilde{x}\,(s) - x(0) + 3\tilde{x}(s) = 0$$

$$(s + 3)\,\tilde{x}\ (S) = 1$$

$$\tilde{X}(s) = \frac{1}{S + 3}$$

Taking Laplace Inverse

$$L^{-1}\,[\,\tilde{x}\,(s)] = L^{-1}\left[\frac{1}{S+3}\right] , \ x\,(t) = e^{-3t}$$

Example (1.9):

Solve the homogeneous first order differential equation:

$$\frac{dx}{dt} + 3x = 0 \ , \ x(1) = 1$$

Solution:

Taking Laplace Transform we get:

$$L\left[\frac{dx}{dt}\right] + 3L\,[(x)] = 0$$

$$s\tilde{x}(s) - x(0) + 3\,\tilde{x}(s) = 0$$

$$(s + 3)\,\tilde{x}(s) = x(0)$$

$$\tilde{x}(s) = \frac{X\,(0)}{(s + 3)}$$

Taking Laplace Inverse we have :

$L^{-1}[\tilde{x}(s)] = L^{-1}\left|\ \frac{X(0)}{S+3}\ \right]$

$x(t) = x(0)(e^{-3t}\ ,\quad x(1) = 1$

Where t = 1 we have

$x(1) = x(0)\, e^{-3} = 1$

$x(0) = \frac{1}{e^{-3}} = e^{3}$

Then

$X(t) = e^{3}\, e^{-3t}$

$X(t) = e^{3(1-t)}$

Example (1.10)

Solve the non Homogeneous first order differential equation:

$\frac{dx}{dt} + 3x = \cos(3x)$

$X(0) = 0$

Solution:

Taking Laplace transform we have:

$L\ [(\frac{dx}{dt})] + 3L[(x)] = L\,[\cos(3t)]$

$s\,\tilde{x}(s) - x(0) + 3\tilde{x}(s) = \frac{s}{s^2+3^2}$

$\tilde{x}(s) = (\frac{s}{s^2+9})(\frac{1}{s+3})$

Taking Laplace Inverse we get:

$L^{-1}[\tilde{x}(s)] = L^{-1}[\frac{s}{s^2+9}][\frac{1}{s+3}]$

$x(t) = e^{-3t} * \cos(3t) = \int_0^t e^{-3(t-\tau)} \cos(3\tau)\, d\tau\ ,$

$x(t) = e^{-3t}\int^t e^{3t} \cos(3t)\, d\,\tau$

$u = e^{3t}\quad \frac{du}{dt} = 3e^{3t}$

$dv = \cos(t)$, $v = \frac{1}{3}\sin(3\tau)$

$$I = [\tfrac{1}{3} e^{3\tau} \sin(3T)]_{T=0}^{t} - \int_0^t e^{3T} \sin(3T)dt \quad , \quad u = e^{3t} \Rightarrow du = 3e^{3t}$$

$dv = \sin 3\tau \, d\tau$, $v = -\frac{1}{3}\cos 3\tau$

$$I = [\tfrac{1}{3} e^{3t} \sin(3t) - [-\tfrac{1}{3} e^{3t} \cos 3t]_{T=0}^{t} - \int \cos 3t \, e^{3t} dt$$

$$2I = \tfrac{1}{3} e^{3t} \sin(3t) + \tfrac{1}{3} e^{3t} \cos 3(t) - \tfrac{1}{3}$$

$$I = \frac{1}{6} e^{3t} \cos 3t - \frac{1}{6}$$

$$x(t) = I e^{-3t} = \frac{1}{6} \sin(3t) \ \frac{1}{6} \cos(3t) - \frac{1}{6} e^{-3t}$$

$$x(t) = \frac{1}{6} [\sin(3t) + \cos(3t) - e^{-3t}]$$

1.4 Second order differential equation:

$$L \left[\frac{d^2 x}{dt^2}\right] = s^2 \tilde{x}(s) - sx(0) - x'(0)$$

Example(1.11):

Use Laplace transform to solve the non- homogeneous second order differential equation:

$x(0) = 0$, $x'(0) = 0$

$$\frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 9x = \sin(t)$$

Solution:

Taking Laplace transform we have

$$[S^2 \tilde{x}(s) - Sx(0) - x'(0)] + 6[Sx(s) - x(0)] + 9\tilde{x}(s) = \frac{1}{S^2 + 1}$$

$$\tilde{x}(s) = \left(\frac{1}{S^2 + 1}\right)\left(\frac{1}{(S + 3)^2}\right)$$

we use the rational to have

$$\left(\frac{1}{S^2 + 1}\right)\left(\frac{1}{(S + 3)^2}\right) = \frac{A}{(S + 3)} + \frac{B}{(S + 3)^2} + \frac{CS + D}{(S^2 + 1)}$$

$1 = A(S + 3)(S + 3)^2 + B(S^2 + 1) + (CS + D)(S + 3)^2$

Take S = -3 we have

$1 = 10B$, $B = \frac{1}{10}$

then we have the following equation

$1 = A(S^3 + 2S^2 + S + 3) + B(S^2 + 1)$

$+ (CS^3 + 6CS^2 + 9CS + 6CS + 9D)$

We get the following system :

$C = - A$ (1)

$3A + B + 6C + D = 0$ (2)

$A + 9C + 6C = 0$ (3)

$1 = 3A + B + 9D$ (4)

$A = \frac{3}{50}$,$b = \frac{5}{50}$ $C = \frac{3}{50}$, $D = \frac{4}{50}$

now we have

$$\tilde{X}(s) = \left[\frac{3}{50}\left(\frac{1}{S + 3}\right) + \frac{5}{50} \frac{1}{(S + 3)^2} - \frac{3}{50}\left(\frac{1}{S^3 + 1}\right) \right.$$

$$\frac{4}{50}\left[\frac{1}{S^2 + 1}\right]$$

then we taking Laplace inverse we have ,the following:

$$L^{-1}[X(s)] = L^{-1}\left[\frac{3}{50}\left(\frac{1}{S + 3}\right) + \frac{5}{50}\left(\frac{1}{(S + 3)^2}\right) \right.$$

$- \frac{3}{50} \left(\frac{S}{S^2 + 1} \right) + \frac{3}{50} \left(\frac{1}{S^2 + 1} \right)$

The solution is,

$X(t) = \frac{3}{50} e^{-3t} + \frac{5}{50} t e^{-3t} - \frac{3}{50} \cos(t) + \frac{4}{50} \sin t$

(1.12)Example:

Solve the second order differential equation by

Laplace Transform:

$\frac{d^2x}{dt^2} + y = 0$, with $x(0) = 1$, $x^-(0) = 0$

Solution :

Taking Laplace Transform we have:

$S^2 \tilde{x}(s) - sx(0) - \bar{x}(0) + \tilde{x}(s) = 0$

$(S^2 + 1) \tilde{x}(s) = s$, $\tilde{x}(s) = \frac{s}{S^2 + 1}$

Taking Laplace inverse we have

$x(t) = \cos t$

(1.13) Example:

Solve the equation

$\frac{d^2x}{dt^2} + y = t$, $x(0) = 1$, $\bar{x}^-(0) = 0$

Solution: taking Laplace we have

$S^2 \tilde{x}(s) - sx(0) - x^-(0) + \tilde{x}(s) = \frac{1}{S^2}$

$(S^2 + 1)x(s) = \frac{1}{S^2} + S$, $x(s) = \frac{1}{S^2(S^2 + 1)} + \frac{S}{(S^2 + 1)}$

Taking Laplace Inverse we have

$x(t) = L^{-1} [(\frac{1}{S^2} (\frac{1}{S^2 + 1}) + L^{-1}(\frac{S}{S^2 + 1})$

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$$x(t) = \cos t + \int_0^t (t - \tau) \sin(\tau)\, d\tau$$

$u = t - \tau$, $\frac{du}{dt} = -1$

$dv = \sin \tau\, d\tau$, $v = -\cos \tau$

$$\int_0^t (t - \tau) \sin \tau\, d\tau = [(\tau + t) \cos\ t]_{\tau=0}^{t} - \int_0^t \cos \tau\, d\tau$$

$u(t) = \cos(t) - \sin(t) + t$

(1.14) Example:

Using Laplace transform to solve second order differential

Equation:

$$\frac{d^2y}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t} \qquad , t \geq 0$$

$x(0) = 1$, $x = 0$ at $t = 0$

Solution:

Taking Laplace transform we get:

$$[S^2 \tilde{x}(s) - Sx(0) - \acute{x}(0)] + 5\,[S\tilde{x}(s) - x(0)] + 6\,\tilde{x}(s) = \frac{2}{S+1}$$

$$(s^2 + 5S + 6)x(s) = \frac{2}{(S+1)} + (S + 5)$$

$$\tilde{x}(s) = \frac{2}{(S+1)(S+2)(S+3)} + \frac{(S+5)}{(S+2)(S+3)}$$

then we use the rational fraction to get:

$$\frac{2}{(S+1)(S+2)(S+3)} = \frac{A}{(S+1)} + \frac{B}{(S+2)} + \frac{C}{(S+3)}$$

$2 = A(S + 2)(S + 3) + B(S + 1)(S + 3) + C (S + 1)(S + 2)$

Let $S = -1$ we get $A = 1$

Let S = -2 we get B = -2

Let S = -3 we get C = 1 then

$$\frac{2}{(S + 1)\,(S + 2)(S + 3)} = \frac{B}{(S + 2)} + \frac{C}{(S + 3)}$$

S + 5 = A(S + 3) + B(S + 2)

Let S = -3 , B = - 2

Let S = -2 then A = 3

$$\frac{S + 5}{(S + 2)(S + 3)} = \frac{3}{(S + 2)} - \frac{2}{(S + 3)}$$

then we have the following equation:

$$\tilde{x}(s) = \left(\frac{1}{(S + 1)}\right) - 2\left(\frac{1}{(S + 2)}\right) + \frac{1}{(S + 3)} + \frac{3}{(S + 2)} - \frac{2}{(S + 3)}$$

then taking Laplace transform inverse we get the solution

$X(t) = e^{-t} + e^{-2t} - e^{-3t}$

Chapter 2

Introduction

Double Laplace Transform for solving the Heat Equation

The heat equation ,known as fundamental equation in mathematical, physics ,applied mathematical and engineering .It is also known that there are two types of the heat equation ,the homogeneous equation that have constant coefficients with many classical solutions such as the separation of variable ,[1],the method of characteristics[2,3],the single Laplace Transform [4] ,the non –homogeneous equation with constant Coefficients solve by means of the double Laplace transform [8] and operation calculus[5,6].

In this chapter we use double Laplace transform to solve the a homogeneous and non Homogeneous heat equation. Some example will be to solve demonstrate of the double Laplace transform method for solving the heat equation.

First of all ,we recall the following definition given by Kilicman and Gadain [10] .

Definition (2,1)

The double Laplace transform is defined by

$$L_x L_t [f(x,t) , (p,s)] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)\, dt\, dx, \text{ where } x,t > 0 \text{ , and } p,s \text{ complex numbers} \quad (2.1)$$

Double Laplace transform for the first order partial derivative with respect to t is defined by :

$$L_x\, L_t \left[\frac{\partial f(x,t)}{\partial t} ,(p,s) \right] = s\, F(p,s) - F(p,0) \qquad (2.2)$$

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by defined is t to respect with derivative:

$$L_x\, L_t \left[\frac{\partial f(x,t)}{\partial x} ,(p,s) \right] = p\, F(p,s) - F(0,s) \qquad (2.3)$$

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by defined t to respect with:

$$L_x\, L_t \left[\frac{\partial^2 F(x,t)}{\partial^2 x^2} , (p,s) \right] = p^2\, F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x} \qquad (4.4)$$

$$L_x\, L_t \quad \frac{\partial^2 F(x,t)}{\partial^2 x^2} , (p,s) \quad = p^2\, F(p,s) - pF(0,s) - \qquad (4.4)$$

Example (2.1):

Use the double Laplace transform to solve the Homogeneous Heat equation:

$u_t = u_{xx}$

With boundary condition $u(0,t) = 0$, $u_x(0,t) = e^{-t}$

And initial condition $u(x,0) = \sin x$

Solution:

Taking double Laplace transform and single Laplace transform we have

$s\, u(p,s) - u(p,0) = p^2 u(p,s) - pu(0,s) - u_x(0,s)$,

$(p^2 - s)u(p,s)$

$= pu(0,s) + u_x(0,s) - u(p,0)$,

$$(p^2 - s)u(p,s) = \frac{1}{(s+1)} - \frac{1}{(p^2+1)} = \frac{P^2 + 1 - s - 1}{(s+1)(p^2+1)}$$

$$(p^2 - s)u(p,s) = \frac{(p^2 - s)}{(s+1)(p^2+1)}$$

Divided both said by $(p^2 - s)$ we get

$$u(p,s) = \frac{1}{(s+1)} \quad \frac{1}{(p^2+1)}$$

Taking double Laplace transform inverse we have

$u(x,t) = e^{-t} \sin x$

Example (2.2):

Use double Laplace transform to solve homogenous heat equation, with homogenous boundary conditions

$u_t = u_{xx}$, $x > 0$, $t > 0$

Boundary conditions $u(0,t) = e^{-t}$, $u_x(0,t) = 1$,initial condition $u(x,0) = x + \cos t$

Taking double Laplace transform and single Laplace we have

$s\,u(p,s) - u(p,0) = p^2 u(p,s) - pu(0,s) - u_x(0,s)$

$(p^2 - s)u(p,s) = pu(0,s) + u_x(0,s) - u(p,0)$

$$\frac{p}{(s+1)} + \frac{1}{s} - \frac{1}{P^2} - \frac{p}{(P^2+1)}$$

$$(p^2 - s)u(p,s) = \left(\frac{p}{(s+1)} - \frac{p}{(p^2+1)}\right) + \left(\frac{1}{s} - \frac{1}{s}\right)$$

$$= \frac{P^3 + p - sp - p}{(s+1)(P^2+1)} + \frac{(P^2 - s)}{S\,P^2}$$

$$(p^2 - s)u(p,s) = \frac{p(P^2 - s)}{(S+1)(p^2+1)} + \frac{(P - s)}{S\,P^2}$$

Divided both side by $(p^2 - s)$ we have

$$u(p,s) = \left(\frac{1}{S+1}\right)\left(\frac{p}{P^2+1}\right) + \left(\frac{1}{P^2}\right)\left(\frac{1}{S}\right)$$

Taking double Laplace transform inverse we get

$u(x,t) = e^{-t}\cos x + x$

Example(2.3):

Use the double Laplace transform to solve the following heat problem

$u_t = u_{xx} - u$

boundary condition $u(0,t) = 0$, $u =(0,t) = e^{-2t}$

initial condition $U(x,0) = \sin(x)$

Solution:

Taking double Laplace transform and single Laplace transform we have:

$$su(p,s) - u(p,0) = p^2u(p,s) - pu(0,s) - u_x(0,s) - u(p,s)$$

$$(p^2 - s - 1)u(p,s) = pu(0,s) + u_x(0,s) - u(p,0)$$

$$= \frac{1}{S+2} - \frac{1}{P^2+1} = \frac{P^2 + 1 - s - 2}{(p^2+1)(S+2)}$$

$$(p^2 - s - 1)u(p,s) = \frac{P^2 - s - 1}{(p^2+1)(S+2)}$$

Divided both said by $(p^2 - s - 1)$ we get:

$$u(p,s) = \left(\frac{1}{S+2}\right)\left(\frac{1}{P^2+1}\right.$$

Taking double Laplace transform inverse we get:

$$u(x,t) = e^{-2t}\sin(x)$$

Example(2.4):

Use the double Laplace transform to solve the in homogenous heat equation:

$u_t = u_{xx} - 2 \quad , x > 0 , t > 0$

boundary condition $u(0,t) = e^{-t} \quad , t \geq 0$

initial condition $u(x,0) = x^2 + \cos x$, $u_x(0,t) = 0$

Solution:

Taking double Laplace transform we have:

$$s\,u(p,s) - u(p,0) = p^2\,u(p,s) - p\,u(0,s) - u_x(0,s) - \frac{2}{ps}$$

$$(p^2 - s)u(p,s) = p\,u(0,s) + u_x(0,s) + \frac{2}{Ps} - u(p,0)$$

$$= \frac{p}{s+1} + \frac{2}{ps} - \frac{2}{P^3} - \frac{p}{p^2+1}$$

$$\left[\left(\frac{p}{s+1}\right) - \left(\frac{p}{p^2+1}\right)\right] + \left[\left(\frac{2}{ps}\right) - \left(\frac{2}{P3}\right)\right]$$

$$\frac{P^2 + p - ps - p}{(P^2+1)(s+1)} + \frac{2P^2 - 2s}{P^3 s} = \frac{P(P^2 - s)}{(P^2+1)(s+1)} + \frac{2(P^2 - s)}{P^3 s}$$

Divided both side by $(p^2 - s)$ we have:

$$u(p,s) = \left(\frac{p}{P^2+1}\right)\left(\frac{1}{s+1}\right) + \left(\frac{2}{P^3}\right)\left(\frac{1}{s}\right)$$

Taking double Laplace transform inverse have:

$$u(x,t) = e^{-t}\cos x + x^2$$

3.2 Double Laplace Transform for solving a convection problems

Example (2.5):

Find the solution of a convection problems :

$u_t + uu_x = 0$

initial condition $u(x,0) = -x$

Solution:

Taking Laplace transform we get

$su(p,s) - u(p,0) + L[uu_x] = 0$

$$su(p,s) = -\frac{1}{P^2} - L[uu_x]$$

$$\frac{-1}{sP^2} \qquad \frac{1}{s}$$

$u(p,s) = \frac{}{} - - \frac{}{} L[uu_x]$

Taking the inverse of the Laplace transform we have

$u(x,t) = - x - L^{-1} [\frac{1}{S} L(uu_x))]$

We decompose the solution as an infinite sum given below

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ and $uu_x = \sum_{n=0}^{\infty} H_n(u)$

where $H_n(u)$ are He's polynomial there represent the non- liner

Terms:

$H_0(u) = u_0u_{0x}$,$H_1(u) = u_0u_{1x} + u_1u_{0x}$

$H_2(u) = u_0u_{2x} + u_1u_{1x} + u_2u_{0x}$

Then , $\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0p^0 + u_1 p^1 + u_2p^2 + \ldots\ldots\ldots =$

$- x - L^{-1} [\frac{1}{S} L(H_0(u) p^0 + H_1(u) p^1 + H_2(u)p^2 + \ldots\ldots.$

Comparing the coefficient of like power of p we get

P^0: $u_0(x,t) = -x$

P^1: $u_1(x,t) = -L^{-1} [\frac{1}{S} L H_0(u)] = -L^{-1}[\frac{1}{S} L(x)]$

$= -L^{-1}[\frac{1}{S} (\frac{1}{S^2})] = - xt$

P^2: $u_2(x,t) = -L^{-1} [\frac{1}{S} L H_1(u)] = -L^{-1} \frac{1}{S} L(u_0u_{1x} + u_1u_{0x})]$

$= - L^{-1}[\frac{1}{S} L (2xt) = - L^{-1}[\frac{1}{S} (\frac{2}{p^2S^2})$

$= - L^{-1} [(\frac{1}{p^2})(\frac{2}{S^3})] = -xt^2$

P^3: $u_3(x,t) = -L^{-1} [\frac{1}{S} L H_2(u)]$

$= -L^{-1}[\frac{1}{S} L(u_0u_{2x} + u_1u_{1x}) + u_2u_{0x}]$

$= - L^{-1}[\ \frac{1}{S}\ L\ \ (-x)(-t^2) + (-xt)(-t) + (-xt^2)(-1)$

$- L^{-1}[\ \frac{1}{S}\ L(3xt^2)]$

$= - L^{-1}\ [(\ \frac{1}{p^2}\ )(\ \frac{6}{S^4}\ )] = -xt^3$

then the solution is given by the summation

$$\sum_{N=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots\ldots\ldots\ldots$$

$= -x - xt - xt^2 - xt^3 - \ldots\ldots\ldots\ldots..$

$= [\ -x(1 + t + t^2 + t^3 + \ldots\ldots\ldots\ldots..$

The sum of sequence is $(1 + t + t^2 + t^3 + \ldots\ ) = \frac{1}{1 - t}$

$u(x,t) = \ (\ \frac{x}{(t - 1}\ )$

Example (2.6)

Solve the non-homogeneous convection equation:

$u_t + uu_x = x + xt^2$

the initial condition ,$u(x,0) = 0$

Solution:

We apply the double Laplace transform we have:

$$su(p,s) - u(p,0) + L(uu_x) = \frac{1}{sp^2} + \frac{2}{P^2s^3}$$

$$u(p,s) =(\ \frac{1}{p^2}\ )(\ \frac{1}{s^2}\ )\ +\ (\ \frac{1}{p^2}\ )(\ \frac{2}{s4}\ ) - \frac{1}{s} L(uu_x)$$

Applying inverse Laplace transform we get:

$$u(x,t) = xt + \tfrac{1}{3}\, xt^3 - L^{-1}\ [\ \frac{1}{s}\ L(uu_x)$$

We decompose the solution on an infinite sum given below

$$u(x,t) = \sum_{N=0}^{\infty} u_n(x,t)\ = \sum_{N=0}^{\infty} p^n\, u_n(x,t) = p^0u_0 + p^1u_1 + p^2u_2 + \ldots\ldots..$$

$$= (xt + \tfrac{1}{3} xt^3)p^0 - L^{-1}[\frac{1}{S} (L\, H_n(u))p]$$

$H_0(u) = u_0u_{0\,x}$, $H_1(u) = u_0u_{1\,x} + u_1u_{0\,x}$

$H_2(u) = u_0u_{2\,x} + u_1u_{1\,x} + u_2u_{0\,x}$

Comparing the coefficient of like power of p we have

P^0: $u_0(x,t) = xt + \tfrac{1}{3} xt^3$

P^1: $u_1(x,t) = - L^{-1}[\frac{1}{S} H_0(u)] = - L^{-1}[\frac{1}{S} (L(u_0u_{0\,x}))]$

$- L^{-1}[$ [redacted] $(L(xt + \tfrac{1}{3} xt^3)(t + \tfrac{1}{3} t^3))]$

$= - L^{-}$ [redacted] $\frac{1}{S} L(xt^2 + \tfrac{2}{3} xt^4 + \frac{1}{9} xt^6)]$

$$= - L^{-1}[\frac{1}{S} (\frac{1}{P^2} \frac{1}{S^3} + (\frac{1}{P^3})(\frac{16}{S^5}) + \frac{1}{9} (\frac{720}{P^2 S^7})]$$

$$= - L^{-1}[\frac{1}{P^2 S^3} + \frac{16}{P^2 S^6} + \frac{720}{9P^2 S^8}]$$

$$u_1(x,t) = - \tfrac{1}{3} xt^3 - \frac{2}{15} xt^5 - \frac{1}{63} xt^7$$

$$P^2 : u_2(x,t) = - L^{-1} [\frac{1}{S} L(H_1(u)] = - L^{-1}[\frac{1}{S} L(u_0u_{1x} + u_1u_{0\,x}]$$

$$L^{-1}[\frac{1}{S} L(- \tfrac{1}{3} xt^4 - \frac{2}{15} xt^6 - \frac{1}{63} xt^8$$

$$- \frac{1}{9} Xt^6 - \frac{2}{45} xt^8 - \frac{1}{189} xt^{10} - \frac{1}{63} xt^4 - \frac{1}{9} xt^6$$

$$- \frac{2}{15} Xt^6 - \frac{2}{45} xt^8 - \frac{1}{63} xt^8 - \frac{1}{189} xt^{10})]$$

$$= - L^{-1}\left[\frac{1}{S} L\left(- \frac{2}{3} Xt^4 - \frac{22}{45} xt^6 - \frac{38}{315} xt^8\right.\right.$$

$$\left.\left. - \frac{2}{189} xt^{10}\right)\right]$$

$$= - L^{-1}\left[\frac{1}{s}\left(- \frac{16}{P^2 S^5} - \frac{352}{P^2 S^7} - \frac{4864}{p^2 S^9} - \frac{36400}{p^2 S^{11}}\right)\right]$$

$$== - L^{-1}\left[\left(- \frac{16}{P^2 S^6} - \frac{352}{P^2 S^8} - \frac{4864}{p^2 S^{10}} - \frac{36400}{p^2 S^{12}}\right)\right]$$

Taking inverse double Laplace transform we have

$$u_2(x,t) = \frac{2}{15} xt^5 + \frac{22}{315} xt^7 + \frac{38}{2835} xt^9 + \frac{2}{2079} xt^{11}$$

$$u(x,t) = \sum_{N=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots\ldots$$

$u(x,t)= xt$

Example (2.7):

Solve the a non linear partial differential equation:

$u_{xx} - u_x u_{yy} = - x + u$

I.C $u(0,y) = \sin y$, $U_x(0,y) = 1$

Solution:

Taking double Laplace transform we have:

$$P^2 u(p,s) – p\, u(0,s) - u_x(0,s) = - \frac{1}{P^2} \cdot L(u + u_x u_{yy})$$

$$u(p,s) = \frac{1}{P^2} - \frac{1}{P^4} + \frac{1}{P^2} L(u + u_x u_{yy})$$

Taking inverse double Laplace transform we get

1 2: 1

6 P^2

$u(x,y) = x + \sin y - \frac{}{} x^3 + L^{-1}[\frac{}{} L(U + U_x U_{yy})]$

Let $u(x,y) = \sum_{n=0}^{\infty} u(x,y) = u_0(x,y) + u_1(x,y) + u_2(x,y) + u_3(x,y) + \ldots.$

$$\sum_{N=0}^{\infty} u_n(x,y)\, p^n = x + \sin y - \frac{1}{6} x^3 + L^{-1}[\frac{1}{P^2} (L(u_n + u_{n\,x} u_{n\,y\,y})]$$

$u_0(x,y) = x + \sin y$

$u_1(x,y) = - \frac{X^3}{6} + L^{-1}[\frac{1}{P^2} L(u_0 + u_{0\,x} u_{0\,y\,y})]$

$= \frac{X^3}{6} + L^{-1} [(\frac{1}{P^2} (x + \sin y - \sin y) = - \frac{X^3}{6} + \frac{X^3}{6} = 0$

$u_n(x,t) = 0$, for all $n \geq 1$

the total solution of the above problem is given by:

$u(x,t) = \sum_{N=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots\ldots\ldots\ldots$

$u(x,t) = x + \sin y$

Chapter (3):

Double Laplace Transform for solving the wave equation

(3.1) Introduction:

In this chapter we use double Laplace transform to solve the homogeneous and non homogeneous wave equation

(3.2) Some examples In this section we

use double Laplace transform to solve the Homogeneous and non-homogenous

wave equation:

Example (3.1):

Consider the Homogeneous wave equation find the solution by Double Laplace transforms to solve it

$u_{xx} = u_{tt}$, $x > 0$, $t > 0$

B.C $u(0,t) = \sin(t)$, $u_x(0,t) = 1$

I.C $u(x,0) = x$, $u_t(x,0) = \cos(x)$

Solution:

Taking double Laplace transform and single Laplace transform

We have:

$s^2 u(p,s) – su(p,0) – u_t(p,0) = p^2 u(p,s) – pu(0,s) – u_x(0,s)$

$(p^2 – s^2) u(p,s) = pu(0,s) + u_x(0,s) – su(p,0) – u_t(p,0)$

$$= \frac{p}{s^2+1} + \frac{1}{s} - \frac{s}{P^2} - \frac{p}{P^2+1}$$

$$(p^2 - s^2)u(p,s) = \left(\frac{p}{s^2+1} - \frac{p}{P^2+1} \right) + \left(- \frac{s}{P^2} + \frac{1}{s} \right)$$

$$= \frac{P^3 + p - ps^2 - p}{(p^2+1)(s^2+1)} + \frac{P^2 - s^2}{Sp^2}$$

$$(P^2 - S^2)u(p,s) = \frac{P(p^2 - s^2)}{(p^2+1)(s^2+1)} + \frac{P^2 - s^2}{Sp^2}$$

$$u(p,s) = \left(\frac{p}{p^2+1} \right)\left(\frac{1}{s^2+1} \right) + \left(\frac{1}{s} \right)\left(\frac{1}{P^2} \right)$$

Taking double Laplace transform inverse we have

$u(x,t) = \cos(x) \sin(t) + x$

Example (3.2):

Using double Laplace transform to solve the homogeneous

wave equation

$u_{tt} = u_{xx}$, $x > 0$, $t > 0$

B.C : $u(0,t) = 0$, $u_x(0,t) = \cos(t)$

I.C : $u(x,0) = \sin(x)$, $u_t(x,0) = 0$

Solution:

Taking double Laplace transform and single Laplace transform

we have :

$$p^2u(p,s) – pu(0,s) – u_x(0,s) = s^2u(p,s) – su(p,0) – u_t(p,0)$$

$$(p^2-s^2)u(p,s) = pu(0,s) + u_x(0,s) –su(p,0)-u_t(p,0)$$

$$= \frac{s}{p^2 + 1} - \frac{s}{s^2 + 1} = \frac{Sp^2 + s – s^3 - s}{(p^2 + 1)(s^2 + 1)}$$

$$(p^2 - s^2)u(p,s) = \frac{S(p^2 – s^2)}{(p^2 + 1)(s^2 + 1)}$$

Divided both side by $(p^2 - s^2)$ we get

$$u(p,s) = \left[\frac{s}{s^2 + 1}\right]\left[\frac{s}{p^2 + 1}\right]$$

Taking double Laplace transform inverse we have

$$u(x,t) = \sin(x)\cos(t)$$

Example(3.3):

Using double Laplace transform to solve the homogeneous wave equation :

$u_{tt}=u_{xx}$,B.C $u(0,t) = 1+ \sin(t)$,$u_x(o,t) = 0$

I.C$u(x,0) =1$,$u_t(x,0)=\cos(x)$

Solution:

Taking double Laplace transform and single Laplace transform

We have:

$$p^2u(p,s) – pu(0,s) – u_x(0,s) = su(p,s) – su(p,0) – u_t(p,0)$$

$$(p^2 – s^{2-}) u(p,s) = pu(0,s) + u_x(0,s) – su(p,0) – uS_t(p,0)$$

$$\frac{1}{s} \qquad \frac{1}{S^2 + 1}$$

$$(p^2 - s^{2-})\, U(p,s) = p\left(\frac{}{} + \frac{}{} \right) + 0 -$$

$$S\left[\frac{1}{p} \right] \cdot \frac{p}{p^2 + 1}$$

$$(p^2 - s^2)u(p,s) = \frac{p}{s} + \frac{p}{s^2 + 1} - \frac{s}{p} - \frac{p}{p^2 + 1}$$

$$= \left(\frac{p}{S} - \frac{s}{p} \right) + \left(\frac{p}{S^2 + 1} - \frac{p}{p^2 + 1} \right)$$

$$= \left(\frac{P^2 - s^2}{Sp} \right) + \frac{P^3 + p - ps^2 - p}{(p^2 + 1)(S^2 + 1)}$$

$$(p^2 - s^2)U(p,s) = \frac{P^2 - s^2}{ps} + \frac{p(p^2 - s^2)}{(p^2 + 1)(s^2 + 1)}$$

Divided both side by $(p^2 - s^2)$ we have:

$$u(p,s) = \left(\frac{1}{p} \right)\left(\frac{1}{s} \right) + \left(\frac{p}{P^2 + 1} \right)\left(\frac{1}{s^2 + 1} \right)$$

Taking double Laplace transform inverse we have:

$$u(x,t) = 1 + \cos(x)\sin(t)$$

Example (3.4)

Using double Laplace transform to solve the non- homogenous wave equation:

$$u_{tt} = u_{xx} - 3u \qquad , x \geq 0 \quad , \quad t \geq 0$$

B.C $U(0,t) = \sin(2t)$, $u_t(0,t) = 0$

I.C $U(x,0) = 0$, $U_t(x,0) = 2\cos(x)$

Solution:

By taking double Laplace transform and single Laplace transform we have:

$pu(p,s) – pu(0,s) – u_x(0,s) – 3u(p,s)=s^2u(p,s) – su(p,0) – u_t(p,0) +$

$(p^2 – s^2 – 1) = pu(0,s) + u_x(0,s) – su(p,0) – u_t(p,0)$

$$= \left[p \quad \frac{2}{s^2+4} \quad - \quad \frac{2p}{p^2+1} \right]$$

$$(p^2 – s^2 – 3)u(p,s) = \left[\frac{2p}{s^2+4} \quad - \quad \frac{2p}{p^2+1} \right] \quad = \quad \frac{2p^3 \sout{+ 2p - 2ps^2 – 8p}}{(p^2+1)(s^2+2^2)}$$

$$\frac{2p(p^2 – s^2 – 3)}{(p^2+1)(s^2+2^2)} \quad \frac{p}{^2)} \quad – s^2 – 3)u(p,s)$$

Divided both side by $(p^2 – s^2 – 3)$ we have:

$$u(p,s) = \left[\frac{p}{p^2+1} \right] \left[\frac{2}{s^2+2^2} \right]$$

Taking double Laplace inverse transform we have:

$u(x,t) = \cos(x)\sin(2t)$

Example (3.5):

Using double Laplace transform to solve the homogeneous wave equation :

$u_{tt} = u_{xx} -2 \ , x \geq 0 \ , t \geq 0$

B.C $U(0,t) = 0 , U_x(0,t) = \sin(t)$

I.C $u(x,0)= x^2 , u_t(x,0) = \sin(x)s$

Solution:

Taking double Laplace transform and single Laplace transform

we have :

$$s^2u(p,s) – su(p,0) – u_t(p,0) = p^2u(p,s) – pu(0,s) – u_x(0,s) - \frac{2}{ps}$$

$\frac{2}{ps}$

$(P^2-s^2)\, u(p,s) = pu(0,s) + u_x(0,s) + \quad \text{——} - su(p,0) - u_t(p,0)$

$$= \frac{1}{s^2+1} + \frac{2}{Ps} - \frac{2s}{P^3} - \frac{1}{p^2+1} : \frac{1}{s^2+1} - \frac{1}{p^2+1} + \frac{2}{Ps} - \frac{2s}{P^3}$$

$$= (p^2 - s^2)u(p,s) = \frac{p^2 + 1 - s^2 - 1}{(p^2+1)(s^2+2^2)} + \frac{2p^2 - 2s^2}{P^3 s}$$

$$= \frac{p^2 - s^2}{(p^2+1)(s^2+1^2)} \quad \frac{2(p^2 - s^2)}{p^3 s}$$

Divided both said by $(p^2 - s^2)$ we have:

$$U(p,s) = \left(\frac{1}{p^2+1}\right)\left(\frac{1}{S^2+1}\right)\left(\frac{2}{P^3}\right)\frac{1}{s}\ )$$

Taking double Laplace inverse we have :

$U(x,t) = x^2 + \sin(x)\sin(t)$

Example (3.6) :

Using double Laplace transform to solve anon homogeneous wave equation:

$u_{tt} = u_{xx} + \sin x$,

B.c $u(0,t) = 0$, $u_x(0,t) = 1 + \sin(t)$

I.c $u(x,0) = \sin(x)$, $u_t(x,0) = \sin(x)$

Solution:

Taking double Laplace and single Laplace transform we have :

$$s^2u(p,s) - su(p,0) - u_t(p,0) = p^2u(p,s) - pu(0,s) - u_x(0,s)$$

$$(p^2 - s^2)u(p,s) = pu(0,s) + u_x(0,s) - su(p,0) - u_t(p,0) + \frac{1}{p^2+1}$$

$$(p^2 - s^2)\, u(p,s) = u_x(0,s) - \frac{1}{p^2+1} - su(p,0) - u_t(p,0)$$

$$= \frac{1}{s} + \frac{1}{s^2+1} - \frac{1}{p^2+1} - \frac{s}{P^2+1} - \frac{1}{P^2+1}$$

$$= \frac{1}{s} + \frac{(-s-1)}{p^2+1} + \frac{1}{s^2+1} - \frac{1}{P^2+1}$$

$$= \frac{(p^2+1-s^2-s)}{S(p^2+1)} + \frac{(p^2+1-s^2-1)}{(p^2+1)(s^2+1)}$$

$$= \frac{(p^2-s^2)}{S(p^2+1)} + \frac{(1-s)}{S(p^2+1} \qquad (p^2-s^2) \qquad \frac{(p^2-s^2)}{+1)}$$

Divided both side byddd

$$= U(p,s) = \frac{1}{S(p^2+1)} + \frac{1}{(p^2+1)(s^2+1)} +$$

$$\left[\frac{1}{S(p^2+1)(s^2-p^2)} - \frac{1}{(p^2+1)(s^2-p^2)}\right]$$

Taking double Laplace inverse we have:

$$L^{-1}\, U(p,s) = L^{-1}\left[\frac{1}{S(p^2+1)} + \frac{1}{(p^2+1)(s^2+1)}\right] +$$

$$\left[L^{-1}\ \frac{1}{S(p^2+1)(s^2-p^2)} - L^{-1}\ \frac{1}{(p^2+1)(s^2-p^2)}\right]$$

$U(x,t) = \sin(x) + \sin(x)\sin(t)$

Chapter (4)

Double Laplace ,Transform

(4.1) Introduction:

In this chapter we use Double Laplace transform to solve the Telegraphic equation and partial integodifferential equation, we follow the method that was proposes by Kilicman and El tayeb[7].Where they extended one-dimensional convolution theorem to two-dimensional case[9].

Example(4.1):

Use the double Laplace transform to solve the homogeneous Telegraph equation given by:

$$u_{xx} - u_{tt} - u_t - u = 0 \qquad (46)$$

With boundary conditions(B.C)

$u(0,t) = e^{-t}$, $u_x(0,t) = e^{-t}$

And initial conditions (4.7)

$u(x,0) = e^x$, $u_t(x,0) = -e^x$

Solution:

Taking double Laplace transform of the equation(4.6) and single Laplace transform for the equation(4.7) we obtain

$[p^2\, u(p,s) - p\, u(0,s,) - u_x\,(0,s)] - [s^2\, u(p,s) - s\, u(p,0) - u_t(p,0)]$

$-[s\, u(p,s) - u(p,0)] - u(p,s) = 0$

$(p^2 - s^2 - s - 1)\, u(p,s) = p\, u(0,s) + u_x(0,s) - s\, u(p,0) - u_t(p,0) +$ $u(p,0)$

$$u(p,s) = \frac{p}{s+1} + \frac{1}{s+1} - \frac{s}{p-1} \cdot \frac{1}{p-1} + \frac{1}{p-1}$$

$$= \frac{P+1}{s+1} - \frac{s}{p-1} = \frac{p^2 - 1 - s^2 - s}{(s+1)(p-1)}$$

$$u(p,s) = \frac{1}{(s+1)(p-1)}$$

Taking double Laplace inverse we have

$$L^{-1}\,[u(p,s)] = L^{-1}\left[\frac{1}{s+1}\right]\left[\frac{1}{s+1}\right]$$

$u(x,t) = e^{x-t}$

Example(4.2):

Used double Laplace transform to solve the homogeneous telegraph equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t^2} + 2\,\frac{\partial u}{\partial t} + u$$

(i.C) $u(x,0) = e^x$, $u_t\,(x,0) = -\,2e^x$

(B.C) $u(0,t) = e^{-2t}$, $u_x(0,t) = e^{-2t}$

Solution:

Taking double Laplace transform and single Laplace transform

we have

$[p^2 u(p,s) – p\, u(0,s) – u_x(0,s)] = [s^2 u(p,s) – s\, u(p,0)- u_t (p,0)$

$+ 2[s\, u(p,s) – u(p,0)] + u(p,s)$

$(p^2 – s^2 – 2s – 1) u(p,s) = p\, u(0,s) + u_x(0,s) – s\, u(p,0) – u_t(p,0)$

$- 2u(p,0)$

$$\frac{p}{s+2} + \frac{1}{s+2} - \frac{s}{p-1} + \frac{2}{p-1} - \frac{2}{p-1}$$

$$= \frac{p}{s+2} + \frac{1}{s+2} - \frac{s}{p-1} = \frac{P+1}{s+2} - \frac{s}{p-1} = \frac{p^2 - 1 - s^2 - 2s}{(s+2)(p-1)}$$

$$u(p,s) = \left[\frac{1}{s+2}\right]\left[\frac{1}{P-1}\right]$$

Taking double Laplace transform inverse we have

$u (x,t) = e^{x + t}$

Example (4.3):

Use double Laplace transform to solve the non homogenous

Telegraph equation defined by:

$u_{xx} - u_{tt} - u_t – u = - 2e^{x + t}$

With boundary condition

$u(0,t) = e^t \quad , \; u_x(0,t) = e^t$

And initial condition

$u(x,0) = e^x \quad , u_t(0,t) = e^x$

Solution:

Use double Laplace Transform and single Laplace transform we have

$[p^2u(p,s) - pu(0,s) - u_x(0,s)] - [s^2u(p,s) - su(p,0) - u_t(p,0)]$

$- [su(p,s) - u(p,0)] - [u(p,s)] = \dfrac{-2}{(p-1)(s+1)}$

$(p^2 - s^2 - s - 1)u(p,s) = pu(0,s) + u_x(0,s) - su(p,0) - u_t(p,0)$

$- u(p,0) = \dfrac{-2}{(p-1)(s+1)}$

$(p^2 - s^2 - s - 1)u(p,s) = \dfrac{p}{(s-1)} + \dfrac{1}{(s-1)} - \dfrac{s}{(p-1)} - \dfrac{s}{(p-1)} - \dfrac{1}{(p-1)}$

$= \dfrac{(p+1)}{(s-1)} + \dfrac{(-s-2)}{(p-1)} - \dfrac{2}{(p-1)(s-1)} = \dfrac{(p^2 - 1 - s^2 - 2s + s + 2 - 2}{(p-1)(s-1)}$

$(p^2 - s^2 - s - 1)u(p,s) = \dfrac{(p^2 - s^2 - s - 1)}{(p-1)(s-1)}$

$u(p,s) = \left(\dfrac{1}{(p-1)}\right)\left(\dfrac{1}{(s-1)}\right)$

Taking double Laplace transform inverse we obtain:

$u(x,t) = e^{x+t}$

Example(4.4):

Solve the partial integral differential equation by using double Laplace transform:

$$u_{tt} - u_{xx} + u + \int_0^x\int_0^t e^{x-\alpha+t-\beta}\, u(\alpha,\beta)\,d\alpha\, d\beta = e^{x+t} + xt\, e^{x+t}$$

With boundary condition

$u(0,t) = e^t \quad , \quad u_t(0,t) = e^t$

IC = is $u(x,0) = e^x$, $u_t(x,0) = e^x$

Solution:

Taking double Laplace transform ,we have

$$[s^2u(p,s) - su(p,0) - u_t(p,0)] - [p^2 u(p,s) - pu(0,s) - u_x(0,s)]$$

$$+ u(p,s) + \frac{1}{(p-1)(s-1)} u(p,s) = \frac{1}{(p-1)(s-1)} + \frac{1}{(p-1)^2(s-1)^2}$$

$$[s^2 + p^2 + \frac{1}{(p-1)(s-1)}] u(p,s) = \frac{1}{(p-1)(s-1)} + \frac{1}{(p-1)^2(s-1)^2}$$

$$+ su(p,0) + u_t(p,0) - pu(0,s) - u_x(0,s)$$

$$[s^2 + p^2 + \frac{1}{(p-1)(s-1)}] u(ps) = \frac{1}{(p-1)(s-1)} + \frac{1}{(p-1)^2(s-1)^2}$$

$$+ \frac{s}{(p-1)} + \frac{1}{(p-1)} - \frac{p}{(s-1)} - \frac{1}{(s-1)} = \frac{(S-1)}{(p-1)} + \frac{(-p-1)}{(s-1)}$$

$$u(p,s) \left[\frac{(s^2 - p^2 + 1)(p-1)(s-1)}{(p-1)(s-1)} + 1\right] =$$

$$\frac{(p-1)(s-1) + 1 + (s+1)(p+1)(s-1)^2 - (p+1)(p-1)^2(s-1)}{(p-1)(s-1)}$$

$$\frac{[(ps - p - s + 1)(s^2 - 1 + 1 - p^2 + 1] + 1}{(p-1)(s-1)}$$

$$u(p,s)[s^2 - p^2 + 1)(p-1)(s-1) + 1] = \frac{(s^2 - p^2 + 1)(p-1)(s-1) + 1}{(p-1)(s-1)}$$

$$= u(p,s) = \frac{1}{(p-1)(s-1)}$$

Taking double Laplace transform inverse we have

$u(x,t) = e^{x + t}$