Sudan University of Sciences and Technology
College of Graduate Studies

Mellin Transform
And
Isolated Singularities

تحويل ميلين والنقاط الشاذة المعزولة

A thesis submitted in partial fulfillment of the requirement for the M.Sc. Degree

In

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Contents

بسم الله الرحمن الرحيم

قال تعالي :

لِيَعْلَمَ أَنْ قَدْ أَبْلَغُوا رِسَالَاتِ رَبِّهِمْ وَأَحَاطَ بِمَا لَدَيْهِمْ وَأَحْصَى كُلَّ شَيْءٍ عَدَدًا ﴿٢٨﴾

صدق الله العظيم.

سورة الجن الأية (٢٨)

Dedication

This research paper is lovingly dedicated to our respective parents who have been our constant source of inspiration;
To my family who supports me in everything…….
To my friends who had helped me and supports me,
And most of all to the Almighty God who gives me strength and good health while doing this……..

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My thanks and my gratitude and deep appreciation to:

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Abstract

Mellin Transform is method for the exact calculation of one-dimensional definite integrals, and illustrates the application. The different types of singularity of a complex function f(z) are discussed and the definition of a residue at a pole is given. The residue theorem is used to evaluate contour integrals where the only singularities of f(z) inside the contour are poles.

Every singularity of a holomorphic function is isolated, but isolation of singularities is not alone sufficient to guarantee a function is holomorphic. Many important tools of complex analysis and the residue theorem require that all relevant singularities of the function be isolated.

الــخلاصة

تحويل ميلين هو طريقة لحساب تكاملات محددة ذات بعد واحد ويوضح تنفيذ التطبيقات ، وغالباً ما يهتم بالتكاملات المغلقة التي يصعب التوصل الي طرق حلها بالطرق العادية أو عن طريق جداول التكاملات المعتادة ومع ذلك توجد أساليب أخري بسيطة التطبيق ولكن تتطلب حسابات شاقة ، ولكن هذا التحويل يعتبر الأمثل وأكثر فائدة من الطرق الأخري .

وتم تعريف أنواع مختلفة من التفرد للدالة المركبة $f(z)$ نظراً لتعريف بقايا القطب ، ويتم إستخدام نظرية المتبقي لحساب التكاملات المغلقة وحيث كل النقاط الشاذة للدالة المركبة $f(z)$ داخل المسار المغلق تمثل أقطاب .

كل نقاط التفرد للدالة المركبة التامة هي معزولة علي الرغم من ذلك لا يمكن الجزم بأن الدالة المركبة تامة .

العديد من الأدوات المهمة للتحليل المركب ونظرية المتبقيات ذات صلة بالتفرد التي تجعل الدالة معزولة .

Introduction

In this research we have a singular point z_0 is called an isolated and isolated singular point of an analytic function f(z).

Isolated singular points include poles, removable singularities , essential singularities and branch points. Mellin was developer of the integral transform:

$$M\{ f(x)\}(s) := \phi(s) = \int_0^\infty x^{s-1} f(x)dx$$

Known as the Mellin transform.

We will study some examples , then we proceed to look at the correspondence between the asymptotic expansion of a function and singularities of the transformed function.

We use the Mellin transform in asymptotic analysis for estimating asymptotically harmonic sums.

And also the Mellin Transform is an integral transform , which is closely connected .

And also is extremely useful for certain applications including solving Laplace's equation in poler coordinates , as well as for estimating integrals.

Chapter (1)

Residue Theorem and Examples

Section(1.1) : Theorem(Residue Theorem)

Let ˡR ⊆ ¢ be an open region and w = f(z) be a complex holomorphic function in R − {z_1, z_2, …,z_n, …,z_L} where $1 \le n \le L$ and each point of the finite exceptional set {z_1, z_2,…,z_L} is an isolated of f(z) . Then ,for any C simple, closed and piecewise continuously differentiable contour such that no singularity of f(z) is on C and C encloses the isolated singularities z_1, z_2,…,z_n but no other singularity , then the following integral equality holds :

$$\oint_{C^+} f(z)\, dz = 2\pi i \sum_{k=1}^{n} \underset{z = z_k}{Res f(z)} \qquad (1.1)$$

Definition 1

Let w = f(z) be a complex function of $z \in$ ¢ holomorphic in ¢ $- \overline{D(0,R)}$= {z∈C $\big|$ $|z|>R$} we call residue at infinity of w = f(z) the quantity

$$\underset{Z=\infty}{\text{Res } f(z)} = -\underset{Z=0}{\text{Res}} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Theorem 1

Let f(z) be holomorphic function ¢ −{a_1,a_2,…,a_n} where $n \ge 0$ integer , with singularities a_1, a_2,…,a_n .

Then,

$$\underset{z=a_1}{\text{Res}}\ f(z) + \underset{z=a_2}{\text{Res}}\ f(z) + \dots + \underset{z=a_n}{\text{Res}}\ f(z) + \underset{z=a_\infty}{\text{Res}}\ f(z) = 0$$

Proof:

We consider any R>0 for which $|a_i|<R$,for all $i=1,2,\dots,n$ and $|a_i|<\frac{1}{R}$ for all $a_i\neq 0$.

We have

$$\underset{z=\infty}{Res}\ f(z) = \underset{z=0}{-Res}\ f(z) = -\frac{1}{2\pi i}\int_{C^+(0,\frac{1}{R})}\frac{1}{z^2}f\left(\frac{1}{z}\right)dz \tag{1.2}$$

We use the change of variable $w=\frac{1}{z}$, and we find

$$\text{Res}\ f(z) = -\frac{1}{2\pi i}\int_{C^+(0,1)} f(w)dw = -\left[\underset{z=a_1}{\text{Res}}\ f(z) + \underset{z=a_2}{\text{Res}}\ f(z) + \dots + \underset{z=a_n}{\text{Res}}\ f(z)\right],$$

And the result follows.

Examples :

Example 1

Find the integral

$$\oint_{C^+}\frac{5z-3}{z(z-2)}\,dz$$

If

a) C = C(0,1)

b) C = C(0,3)
c) C = C(4,1)

First, we compute the residues of function f(z) = $\frac{5z-3}{z(z-2)}$ at isolated singularities z_0 = 0 and z_0 = 2.

At all other point of ¢, this function is defined and holomorphic.

$$\underset{z=2}{\text{Res}}\ f(z) = [zf(z)]\Big|_{z=0} = \frac{5z-3}{z-2}\Big|_{z=0} = \frac{-3}{-2} = \frac{3}{2}$$

$$\underset{z=2}{\text{Res}}\ f(z) = \frac{5z-3}{z}\Big|_{z=2} = \frac{7}{2}$$

So, by the residue theorem, we have:

(a) Since z =0 is only singularity in side C = C(0,1), then

$$\oint_{C^+} \frac{5z-3}{z(z-2)}\, dz = 2\pi i.\frac{3}{2} = 3\pi i$$

(b) Now both singularities are in side C = C(0,3), and so

$$\oint_{C^+(0,3)} \frac{5z-3}{z(z-2)}\, dz = 2\pi i(\frac{3}{2} + \frac{7}{2} = 10\pi i$$

(c) Finally, in side C = C(4,1) there are no singularities of f(z).

There

$$\oint_{C^{+}(4,1)} \frac{5z-3}{z(z-2)}\,dz = 0\,.$$

Example 2

Evaluate the integral

$$\int_{C^{+}} \frac{dz}{e^{z}-1} \text{ if } \mathrm{C} = \mathrm{C}(0,3\pi).$$

Then ,by using the parameterization of C = C(0,3π) given by $Z = 3\pi e^{i\theta}$ with 0≤θ≤2π .

In side C = C(0,3π) ,the function $f(z) = \frac{dz}{e^{z}-1}$ has three

singularities , namely -2π ,o and 2πi

We find that each of the three residues :

$$\operatorname*{Res}_{z=0} f(z) = \left.\frac{z}{e^{z}-1}\right|_{z=0} = \left.\frac{1}{e^{z}}\right|_{z=0} = \frac{1}{1} = 1$$

$$\operatorname*{Res}_{z=2\pi i} f(z) = \left.\frac{z-2\pi i}{e^{z}-1}\right|_{z=2\pi i} = \left.\frac{1}{e^{z}}\right|_{z=2\pi i} = \frac{1}{1} = 1$$

$$\operatorname*{Res}_{z=-2\pi i} f(z) = \left.\frac{z+2\pi i}{e^{z}-1}\right|_{z=-2\pi i} = \left.\frac{1}{e^{z}}\right|_{z=-2\pi i} = \frac{1}{1} = 1$$

$$\int_{C^+} \frac{dz}{e^z - 1} = 2\pi i(1+1+1) = 6\pi i .$$

Section(1.2) Contour Integration and Improper Real Integrals

We are going to use integrations of complex functions along appropriately chosen contours to evaluate improper real integrals .This method is very powerful ,for it computes very difficult integrals and at same time proves their existence . Choosing the correct contour(s) and then applying the residue theorem .we are going to analyze the most important cases of such integral techniques which are sufficient for the needs of an undergraduate student .

Example 1 The integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \arctan(\infty) - \arctan(-\infty) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi$$

Has been computed elementarily .For easy practice ,we will use contour integration to establish this result.

We consider the complex function :

$$f(z) = \frac{1}{1+z^2} \quad \text{in } ₵$$

The denominator has two simple roots ,the $+i$ and $-i$ which are isolated singularities ,poles of order one .

So,

$$\underset{Z=\ i}{Res}\ f(z) = [(z-i)\ f(z)]\Big|_{z=i} = \left[\frac{z-i}{(z-i)(z+i)}\right]\Bigg|_{Z=i} = \frac{1}{2i}$$

We consider any R>1 and the contour C = $[-R, R]$+ S_R^+ consisting of two parts:

1-the straight segment of the x −axis$[-R, R]$ from $-R$ to R and,

2-the positively oriented upper half of C(0,R) , denoted by S_R^+ . see figure (1.1)

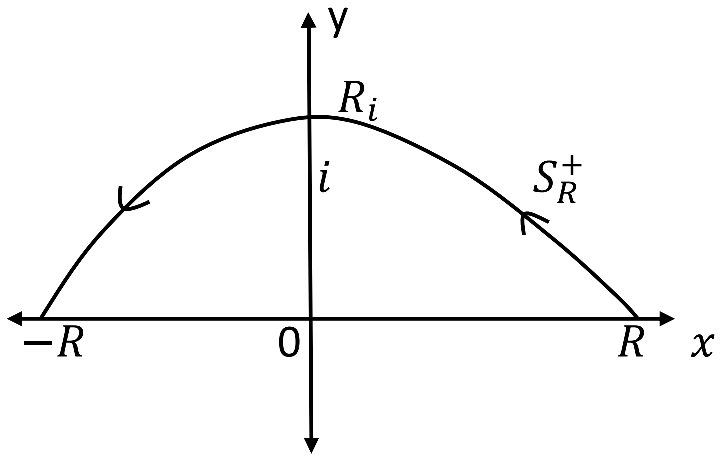


Figure (1.1)

These parts are respectively parameterized by:

(1) $\{z = x + i_0\,| - R \leq x \leq R\}$ and
(2) $\{z = Re^{i\theta}|0 \leq \theta \leq \pi\}$.

we have chosen the contour C in this way , so that at least one of the singularities ,namely the Z=$+i$, is enclosed in it . Then we apply the residue theorem, to find

$$\int_{C^+} \frac{dz}{1+e^z} = 2\pi i \cdot \frac{1}{2i} = \pi \qquad (1.3)$$

Remark : Since $f(x) = \frac{1}{1+x^2}$ is an even function in R we also get:

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} \ ,$$

Or

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2 \int_{-\infty}^{0} \frac{dx}{1+x^2} = 2 \int_{0}^{\infty} \frac{dx}{1+x^2}$$

Similarly we find

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx = 2 \int_{-\infty}^{0} \frac{x^2}{1+x^4} dx = 2 \int_{0}^{\infty} \frac{x^2}{1+x^4} dx = \frac{\pi}{\sqrt{2}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = 2 \int_{-\infty}^{0} \frac{dx}{1+x^6} = 2 \int_{0}^{\infty} \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^n} = \frac{\pi}{n \sin\left(\frac{(j+1)\pi}{L}\right)} \ ,$$

Where j=0,1,2,… and L > j+1 integer

Result:

For all integers L and j such that L > j+1 and j=0,1,2,… we have:

$$\int_0^\infty \frac{x^j}{1+x^L}\,dx = \frac{\pi}{L\sin\left(\frac{(j+1)\pi}{L}\right)} \quad (1.4)$$

Examples 2

(1) (a) where L=0,1,2,... integer , $0 < \alpha + 1 < L$ then

$$\int_0^\infty \frac{x^\alpha}{1+x^L} = \frac{\pi}{L\sin\left(\frac{(\alpha+1)\pi}{L}\right)}$$

(b) For 2n > 2m+1 and m=0,1,2,... then

$$\int_{-\infty}^\infty \frac{x^{2m}}{1+x^{2n}} = \frac{2\pi}{2n\sin\left(\frac{(2m+1)\pi}{2n}\right)} = \frac{\pi}{n\sin\left(\frac{(2m+1)\pi}{2n}\right)}$$

(2) For example $\alpha = \frac{1}{2}$, Find

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,dx$$

Solution:

$\alpha = \frac{1}{2}$, L = 2

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,dx = \frac{\pi}{2\sin\left(\frac{(\frac{1}{2}+1)\pi}{2}\right)} = \frac{\pi}{2\sin(135)} = \frac{\pi}{2.\frac{1}{\sqrt{2}}}$$

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,dx = \frac{\sqrt{2}}{2}.\pi$$

(3) in this example ,we have obtained the following :

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\ dx \ = \ \frac{\pi}{\sin[(\alpha+1)\pi]}$$

Notice that the only root of the denominator z = -1 .

We take as the branch cut the closed half line of none -negative real

semi - axis $\{z = x + i_0 \,|x \geq 0\}$.

We have chosen the positive continuous argument

0< arg(z)< 2π ,we write z = -1 = $e^{i\pi}$ then :

$$\underset{Z=\ -1}{Res}\ f(z) = \left[\frac{(z+1)z^{\alpha}}{(z+1)}\right]\Bigg|_{z=-1} = (-1)^{\alpha} = \binom{i\pi}{e}^{\alpha} = e^{i\alpha\pi}$$

From 0<r < 1 < R < ∞ and as appropriate contour

C = $[r, R]$+ A_R^+ + $[R, r]$ + A_R^- ,where $A_R^+(\theta)$ = R$e^{i\theta}$,

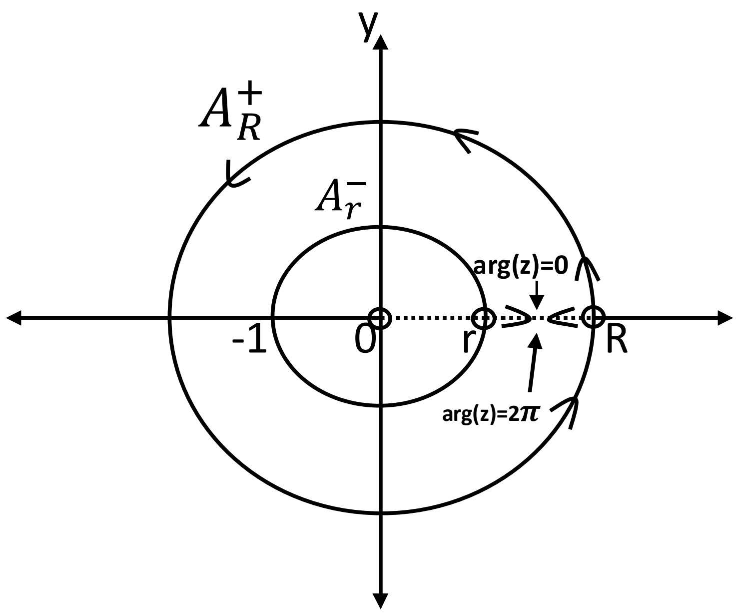
0< θ <2π and $A_r^-(\theta)$ = r$e^{i\theta}$, 0< θ <2π .

By the residue theorem,

we obtain

$$\int_{C^+} \frac{z^\alpha}{1+z} dz = \int_{[r,R]} \frac{z^\alpha}{1+z} dz + \int_{A_R^+} \frac{z^\alpha}{1+z}\ dz + \int_{[R,r]} \frac{z^\alpha}{1+z}\ dz\ + \int_{A_R^-} \frac{z^\alpha}{1+z}\ dz$$

$$\int_{C^+} \frac{z^\alpha}{1+z} dz = 2\pi i e^{i\alpha\pi}\ ,$$



Figure(1.2)

And we must take limits as R→ ∞ and as r→ 0^+

We get :

(1) Lim $\int_{A_R^+} \frac{z^\alpha}{1+z}\, dz = 0$

$R \to \infty$

(2) Lim $\int_{A_r^-} \frac{z^\alpha}{1+z}\, dz = 0$

$r \to 0^+$

Next we must compute the two partial integrals along the branch cut , that is over intervals [r,R] and [R,r] .

In the case of the (positive) segment [r,R] , as we travel along the contour the arc A_r^- indicates ,that we approach the branch cut from above ,and so the limit of the arg(z) is O . Hence along [r,R] we compute the real integral

$$\int_{r}^{R} \frac{(xe^{0i})^{\alpha}}{1+x}\, dx\,.$$

Taking limits ,we get

(3) $\text{Lim}_{\substack{r \to 0 \\ R \to \infty}} \int_r^R \frac{x^{\alpha}}{1+x}\, dx = \int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx$

But,in the case of (negative) segment [R,r] , we approach the branch cut from below , as arc A_R^+ indicates and so the limit of the arg(z) is 2π .Hence along [R,r] we must compute the integral

$$\int_{R}^{r} \frac{(xe^{2\pi i})^{\alpha}}{1+x}\, dx$$

(4) $\text{Lim}_{\substack{r \to 0 \\ R \to \infty}} \int_R^r \frac{(xe^{2\pi i})^{\alpha}}{1+x}\, dx = -e^{2\pi i^{\alpha}} \int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx$

By the four computed pieces (1),(2),(3) and (4) , we find

$$\int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx - e^{2\pi i^{\alpha}} \int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx = \left(1 - e^{2\pi i^{\alpha}}\right) \int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx = 2\pi i\, e^{i\alpha\pi} \quad (1.5)$$

Remark (1)

For $\alpha \geq 0$ or $\alpha \leq -1$

$$\int_0^{\infty} \frac{x^{\alpha}}{1+x}\, dx = \infty\,.$$

Example 3

(a) To find

$$\int_0^\infty \frac{x^\alpha}{b+x}\,dx\,,$$

We perform the u – substitution $x = bu \Rightarrow dx = bdu$,

$$\int_0^\infty \frac{x^\alpha}{1+x}\,dx = \int_0^\infty \frac{b^\alpha u^\alpha}{b+bu}\,bdu = \frac{b^{\alpha+1}}{b}\int_0^\infty \frac{u^\alpha}{1+u}\,du = b^\alpha \frac{\pi}{\sin[(\alpha+1)\pi]}.$$

(b) To find

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,du\,,$$

We perform the u – substitution

$u = x^{\sqrt{2}} \Leftrightarrow x = u^{\frac{1}{\sqrt{2}}}$ and $dx = \frac{1}{\sqrt{2}} u^{\frac{1}{\sqrt{2}}-1} du$ so ,

$$\int_0^\infty \frac{\sqrt{x}}{1+x^{\sqrt{2}}}\,dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{u^{\frac{3}{2\sqrt{2}}-1}}{1+u}\,du = \infty\,, \text{Since } \frac{3}{2\sqrt{2}} - 1 > 0\,.$$

Remark(2):

The integral evaluated in this example may be considered as integral of rational function of $\sin(\theta)$ and $\cos(\theta)$ which in a calculus course is treated with change of variables

$u = \tan\left(\frac{\theta}{2}\right) \Leftrightarrow \theta = 2arc\tan(u)$, called tangent of half – angle substitution.$d\theta = \frac{2}{1+u^2}\,du$, $\sin[2arc\tan(u)] = \frac{2u}{1+u^2}$,$\cos[2arc\tan(u)] = \frac{1-u^2}{1+u^2}$

These results change an (indefinite) integral of a rational function of $\sin(\theta)$ and $\cos(\theta)$ into an (indefinite) integral of a rational function of u , from this indefinite integral , we compute the definite .

For example , we find that

$$\int \frac{d\theta}{-1+\sin(\theta)} = \frac{2}{\tan\left(\frac{\theta}{2}\right)} + C \tag{1.6}$$

So,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{-1+\sin(\theta)} = \frac{2}{\tan\left(\frac{\theta}{2}\right)-1}\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{\tan\left(\frac{\pi}{4}\right)-1} - \frac{2}{\tan\left(-\frac{\pi}{2}\right)-1} = \frac{2}{0} - \frac{2}{-2}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta}{-1+\sin(\theta)} = \infty + 1 = \infty\,. \tag{1.7}$$

Chapter (2)

Mellin Transform

Section(2.1) *Basic concepts & Definitions*

Definition 1

The Mellin Transform of a real function $y = f(x)$ with $0 < x < \infty$ or $0 \leq x < \infty$ $[we\ consider\ f(x) = 0\ for\ x < 0]$ is defined by :

$$M\{ f(x)\}(s) := \phi(s) = \int_{0}^{\infty} x^{s-1} f(x) dx \qquad (2.1)$$

For all those s, s for which this integral exists .

This transform has a lot of applications in mathematics , engineering and computer science . The inverse transform is

$$M^{-1}\{\phi(s)\}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{s} \phi(s) ds \qquad (2.2)$$

For an appropriate constant C . we state that the Mellin transform exists if $x^k f(x)$ is absolutely integrable on $(0, \infty)$ for some $k > 0$ .Then ,the inverse transform also exists for $c > k$ .For example , the $\Gamma(p)$ is the Mellin transform of

$f(x) = e^{-x}$, with $x \in (0, \infty)$. Also,we have proved for all L =1,2,3,… integer and $\alpha \in R$ such that $L > \alpha + 1 > 0$

(or $L-1 > \alpha > -1$) we have

$$\int_0^{\infty} \frac{x^{\alpha}}{1+x^{L}}dx = \frac{\pi}{L\sin\left(\frac{(\alpha+1)\pi}{L}\right)}$$

Then ,under those condition , this integral can be viewed as

$$M\left\{\frac{1}{1+x^{L}}\right\}(\alpha+1) = \frac{\pi}{L\sin\left(\frac{(\alpha+1)\pi}{L}\right)}$$

Replacing $(\alpha+1)$ with S , We obtain the Mellin trans

$$M\left\{\frac{1}{1+x^{L}}\right\}(s) = \frac{\pi}{L\sin\left(\frac{(s)\pi}{L}\right)} .$$

Lemma: (Jordan's Lemma)

If $0 \le \theta_1 < \theta_2 \le \pi$ and $\mu > 0$,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt < \frac{\pi}{\mu} . \qquad (2.3)$$

Proof:

Since $e^{-\mu sin(t)} > 0$,

We obtain

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \le \int_0^{\pi} e^{-\mu\sin(t)}dt = \int_0^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt + \int_{\frac{\pi}{2}}^{\pi} e^{-\mu\sin(t)}dt .$$

Using $u = \pi - t$, we find

$$\int_{\frac{\pi}{2}}^{\pi} e^{-\mu\sin(t)}dt = \int_{0}^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt\,.$$

So ,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}dt \leq 2\int_{0}^{\frac{\pi}{2}} e^{-\mu\sin(t)}dt\ .$$

But , for $0 \leq t \leq \frac{\pi}{2}$, we have that $\frac{2t}{\pi} \leq \sin(t)$. This inequality is seen graphically , since $\sin(t)$ is a concave function in the interval $\left[0, \frac{\pi}{2}\right]$, There fore , $y = \sin(t)$ is greater than or equal to the straight segment function $y=\frac{2}{\pi}t$ in $\left[0, \frac{\pi}{2}\right]$.

So, $e^{-\mu\sin(t)} \leq e^{\frac{-2\mu t}{\pi}}$ for all $t \in \left[0, \frac{\pi}{2}\right]$.

Hence ,

$$\int_{\theta_1}^{\theta_2} e^{-\mu\sin(t)}\,dt \leq 2\int_{0}^{\frac{\pi}{2}} e^{\frac{-2\mu t}{\pi}}\,dt \leq 2\left(\frac{-\pi}{2\mu}\right)\left[e^{\frac{-2\mu t}{\pi}}\right]_{0}^{\frac{\pi}{2}} = \frac{-\pi}{\mu}(e^{-\mu} - 1)$$

$$= \frac{\pi}{\mu}(1 - e^{-\mu}) < \frac{\pi}{\mu}\,.$$

Remark

Jordan's Lemma also implies the following inequalities:

(1) If $\pi \leq \theta_1 < \theta_2 \leq 2\pi$ and $\lambda > 0$,

Then

$$\int_{\theta_1}^{\theta_2} e^{\lambda \sin(t)} dt < \frac{\pi}{\lambda}$$

(2) If $-\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$ and $\sigma > 0$, then

$$\int_{\theta_1}^{\theta_2} e^{-\sigma \cos(t)} dt < \frac{\pi}{\sigma} \ .$$

(3) If $\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \frac{3\pi}{2}$ and $\tau > 0$,then

$$\int_{\theta_1}^{\theta_2} e^{\tau \cos(t)} \, dt < \frac{\pi}{\tau} \ .$$

Section(2.2) *Examples*

Example (1)

New we can compute the **Fresnel** integrals by using contour integration $techniques$.

$$\int_0^\infty co\,s(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{2\pi}}{4} \tag{2.4}$$

We consider $f(z) = e^{-z^2}$, which holomorphic in C .

This function has no singularities in C.

Then , for any $R > 0$, we consider the contour

C = $[0,R]$+ A_R^+ + $[R\mathrm{e}^{\frac{i\pi}{4}},0]$, where A_R^+ is positive oriented are parameterized by $z = R\mathrm{e}^{i\theta}$ with $0 \leq \theta \leq \frac{\pi}{4}$

Then ,

$$\oint_{c^+} \mathrm{e}^{-z^2}\mathrm{dz} = \int_{[0,R]} \mathrm{e}^{-z^2}\mathrm{dz} + \int_{A_R^+} \mathrm{e}^{-z^2}\mathrm{dz} + \int_{[R\mathrm{e}^{\frac{\pi i}{4}},0]} \mathrm{e}^{-z^2}\mathrm{dz} = 0$$

New ,

$$\int_{[0,R]} \mathrm{e}^{-z^2}\mathrm{dz} = \int_0^R \mathrm{e}^{-x^2}\mathrm{d}x \text{ , and ,}$$

$$\lim_{R\to\infty}\int_0^R \mathrm{e}^{-x^2}\mathrm{d}x = \int_0^R \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2} \tag{2.5}$$

Next , since $\mathrm{e}^{\frac{i\pi}{2}} = i$ and along $[R\mathrm{e}^{\frac{i\pi}{4}},0]$ we have $z = xR\mathrm{e}^{\frac{i\pi}{4}}$ with $R > x > 0$, we get

$$\int_{[R\mathrm{e}^{\frac{i\pi}{4}},0]} \mathrm{e}^{-z^2}\mathrm{dz} = \int_R^0 \mathrm{e}^{-x^2\mathrm{e}^{\frac{i\pi}{2}}}\mathrm{e}^{\frac{i\pi}{4}}\mathrm{d}x = -\int_0^R \left(\mathrm{e}^{\frac{i\pi}{4}}\right).\mathrm{e}^{-x^2\mathrm{e}^{\frac{i\pi}{2}}}\mathrm{d}x = -\mathrm{e}^{\frac{i\pi}{4}}\int_0^R \mathrm{e}^{-x^2\mathrm{e}^{\frac{i\pi}{2}}}\mathrm{d}x$$

$$= \mathrm{e}^{\frac{-i\pi}{4}}\int_0^R \mathrm{e}^{-x^2 i}\mathrm{d}x \text{ .}$$

Lastly ,on A_R^+ we get

$$\int_{A_R^+} \mathrm{e}^{-z^2}\mathrm{dz} = iR\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\mathrm{e}^{2ix}}\ \mathrm{e}^{ix}\mathrm{d}x\ .$$

We observe that

$$\mathrm{e}^{-R^2\mathrm{e}^{2ix}} = e^{-R^2[\cos(2x)+i sin(2x)]} = e^{-R^2\cos(2x)} .\, e^{-iR^2\sin(2x)}\ .$$

Therefore , $\left|\mathrm{e}^{-R^2\mathrm{e}^{2ix}}\ \mathrm{e}^{ix}\right| = e^{-R^2\cos(2x)}$

So , we have

$$\left|\int_{A_R^+} \mathrm{e}^{-z^2}\mathrm{dz}\right| \le \int_{A_R^+} \left|\mathrm{e}^{-z^2}\right||\mathrm{dz}| \le \int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\cos(2\phi)}\,\mathrm{Rd}\phi = R\int_0^{\frac{\pi}{4}} \mathrm{e}^{-R^2\sin(2\mathrm{u})}\,\mathrm{d}u$$

$$= \frac{R}{2}\int_0^{\frac{\pi}{2}} \mathrm{e}^{-R^2\sin(v)}\,\mathrm{d}v < \frac{R}{2}\,.\frac{\pi}{R^2} = \frac{\pi}{2R}\ ,(v = 2u)$$

(by Jordan's Lemma)

Since $\lim_{R\to\infty}\frac{\pi}{2R} = 0$, we get $\lim_{R\to\infty}\int_{A_R^+} \mathrm{e}^{-z^2}\mathrm{d}z = 0$

So, as $R \to \infty$, we finally get

$$\frac{\sqrt{\pi}}{2} - \mathrm{e}^{\frac{i\pi}{4}}\int_0^{\infty} \mathrm{e}^{-ix^2}\mathrm{d}x = 0$$

Then ,

$$\int_{0}^{\infty} [\cos(x^2) - i sin(x^2)]\mathrm{d}x = \frac{\sqrt{\pi}}{2}.\mathrm{e}^{\frac{-i\pi}{4}} = \frac{\sqrt{\pi}}{2}\left[\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = \frac{\sqrt{2\pi}}{2}[1-i]\,.$$

Now , we separate the real and imaginary parts of this equality and obtain the final result

$$\int_{0}^{\infty} \cos(x^2)\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4} = \int_{0}^{\infty} \sin(x^2)\,\mathrm{d}x \qquad (2.6)$$

Example (2)

In this example, we show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\cos(4x)\,\mathrm{d}x = \mathrm{e}^{-4}\sqrt{\pi}\ . \qquad (2.7)$$

For any $R > 0$, we consider the rectangle contour in figure 2.1

$c^{+} = [-R, R] + [R, R + 2i] + [R + 2i, -R + 2i] + [-R + 2i, -R]$

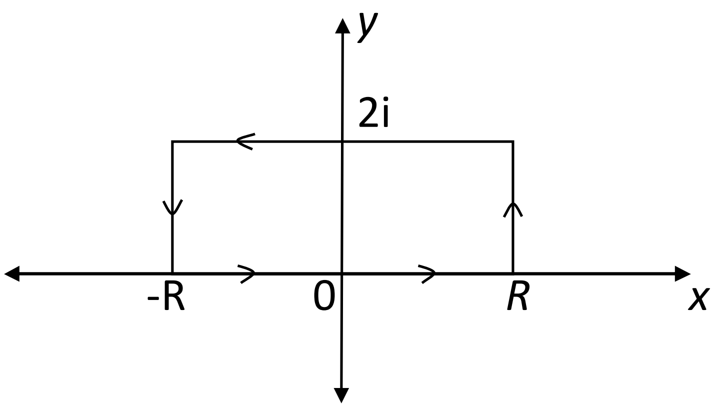


Figure (2.1) for example 2

A gain, as the previous example , we have

$$\int_{C^+} e^{-z^2} dz = 0$$

And so

$$\int_{-R}^{R} e^{-x^2} dx + \int_{0}^{2} e^{-(R+iy)^2} i dy + \int_{R}^{-R} e^{-(x+2i)^2} dx + \int_{2}^{0} e^{-(R+iy)^2} i dy = 0$$

Then, we have

$$\left| \int_{0}^{2} e^{-(R+iy)^2} i dy \right| \le \int_{0}^{2} \left| e^{-R^2} e^{-2\mathrm{Riy}} e^{y^2} \right| dy$$

$$= \int_{0}^{2} e^{-R^2} e^{y^2} dy \le 2e^{-R^2} e^{4} \to 0 , \qquad \text{as R} \to \infty$$

Similarly ,

$$\left| \int_{0}^{2} e^{-(R+iy)^2} i dy \right| \le 2e^{-R^2} e^{4} \to 0 , \mathrm{R} \to \infty .$$

Hence , taking limit as R→ ∞ , we get

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-(x+2i)^2} dx .$$

Thus ,

$$\mathrm{Re}\left[\int_{-\infty}^{\infty} e^{-(x+2i)^2} dx \right] = \mathrm{Re}\left[\int_{-\infty}^{\infty} e^{-x^2} dx \right] = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} .$$

Developing this , we get

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \cos(4x)\, \mathrm{d}x = \mathrm{e}^{-4}\sqrt{\pi}\ .$$

Remark 1

Since $f(x)= \mathrm{e}^{-x^2}\cos(4x)$ is even , we get

$$\int_{-\infty}^{0} \mathrm{e}^{-x^2} \cos(4x)\, \mathrm{d}x = \int_{0}^{\infty} \mathrm{e}^{-x^2} \cos(4x)\, \mathrm{d}x = \frac{\mathrm{e}^{-4}\sqrt{\pi}}{2}$$

Remark 2

The equality of the imaginary parts gives

$$\int_{\infty-}^{\infty} \mathrm{e}^{-x^2} \sin(4x)\, \mathrm{d}x = 0$$

A fact already known, since the function $f(x)= \mathrm{e}^{-x^2}\sin(4x)$ is odd and the integral exists .

Remark 3

For any $R > 0$, if we integrate e^{-z^2} over the contour $C = [0, R] + [R, R + 2i] + [R + 2i, 2i] + [2i, 0]$ (=rectangle) .

Chapter (3)

Infinite Isolated Singularities and Integrals

Section (3.1)

Suppose we need to evaluate ,

$$\int_{-\infty}^{\infty} f(x)dx$$

By means of residues, where $f(z)$ is holomorphic in the upper closed halt plane $\mathrm{Im}(\mathrm{z}) \geq 0$, or in the lower closed halt plane $\mathrm{Im}(\mathrm{z}) \leq 0$ except at a set of (countably) infinite isolated singularities $A = \{z_n \,|\, n \in N\}$.

Let us assume that we work in the upper closed half plane $[\,\mathrm{Im}(\mathrm{z}) \geq 0]$

Then ,by the residue theorem, we get

$$\oint_{C_L^+} f(z)dz = 2\pi i \sum_{z=z_k} Resf(z) \; , \qquad (3.1)$$

And so

$$\int_{-R_L}^{R_L} f(x)dx = 2\pi i \sum_{k=1}^{k_L} \frac{Resf(z)}{z = z_k} - \int_{p_L} f(z)dz$$

Taking limits as $L \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{n=1}^{\infty} \underset{Z = z_n}{Resf(z)} \tag{3.2}$$

More generally ,we have computed the principal value of the integral, if this principal value exist, since the limit is taken over the symmetrical intervals $[R, R] \to (-\infty, \infty)$ as $L \to \infty$. I.e ,

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{n=1}^{\infty} \underset{Z = z_n}{Resf(z)} \ . \tag{3.3}$$

Example 1

We easily observe that the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \int_{-\infty}^{\infty} \frac{\text{sech}\,(x)}{1+x^2} dx \quad \text{exist} .$$

We will try to evaluate it as a convergent infinite series.

The corresponding complex function $f(z) = \frac{1}{(1+z^2)\cosh(z)}$ has singularities at the solutions of $(1+z^2)\cosh(z) = 0$. These are $z = \pm i$ and $z = \frac{2k+1}{2}\pi i$ with z ∈ ℤ .

No singularity lies on the $x - axis$ and there are infinitely many isolated singularities in either of half planes.

We choose to work in the upper closed half plane in which the singularities are :

$$i\ , \frac{1}{2}\pi i, \frac{3}{2}\pi i\ , \frac{5}{2}\pi i\ , \ldots$$

We have observed that the singularities are simple poles . Now, we must calculate their residues.

$$\operatorname*{Res}_{z=i} f(z) = \frac{z-i}{(z-i)(z+i)\cosh(z)}\Bigg|_{z=i} = \frac{1}{2i\cosh(i)} = \frac{1}{2i\cos(1)}$$

For any $n \geq 1$ integer we have

$$\operatorname*{Res}_{z=\frac{2n-1}{2}\pi i} f(z) = \frac{z-\frac{(2n-1)}{2}\pi i}{(z^2+1)\cosh(z)}\Bigg|_{z=\frac{(2n-1)}{2}\pi i} = \frac{1}{(z^2+1)\sinh(z)}\Bigg|_{z=\frac{(2n-1)}{2}\pi i}$$

$$= \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]\sinh\left(\frac{(2n-1)}{2}\pi i\right)} = \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right]i\sin\left(\frac{(2n-1)}{2}\pi\right)}$$

$$= \frac{1}{\left[1-\left(\frac{2n-1}{2}\right)^2\pi^2\right](-1)^{n-1}i} = \frac{(-1)^n}{\left[\left(\frac{2n-1}{2}\right)^2\pi^2-1\right]i}$$

So ,finally

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)\cosh(x)} = 2\pi i\left\{\frac{1}{2i\cos(1)} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2\pi^2-1\right]i}\right.$$

$$= \frac{\pi}{\cos(1)} 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2\pi^2-1\right]},$$

And we see that this series converges absolutely.

since $f(x) = \frac{1}{(1+x^2)\cosh(x)}$ is an even function, we also get

$$\int_{-\infty}^{0} \frac{dx}{(1+x^2)\cosh(x)} =$$

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2 \pi^2 - 1\right]} \tag{3.4}$$

Section (3.2) **Isolated Singularities on Coordinate Axis and Cauchy Principal Value**

We begin by proving again result

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2 \pi^2 - 1\right]} \tag{3.5}$$

We have already seen the closed upper half plane the function

$f(z) = \frac{1}{(1+z^2)\cosh(z)}$ Has simple poles at $i, \frac{1}{2}\pi i, \frac{3}{2}\pi i, \frac{5}{2}\pi i, \ldots$

With corresponding residues:

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{2i\cos(1)}$$

And for any $k \geq 1$ integer

$$\operatorname*{Res}_{z=\frac{(2k-1)}{2}\pi i} f(z) = \frac{(-1)^k}{\left[\left(\frac{(2k-1)}{2}\right)^2 \pi^2 - 1\right] i}$$

And we obtain the result

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2 \pi^2 - 1\right]}$$

Remarks:

1. Here, we first prove that

$$\int_0^\infty \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{2\cos(1)} + \pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(n-\frac{1}{2}\right)^2\pi^2-1} \ ,$$

Then:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\cosh(x)} = \frac{\pi}{\cos(1)} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{\left[\left(n-\frac{1}{2}\right)^2\pi^2-1\right]} \qquad (3.6)$$

2. If there were isolated singularities w's of f(z) in the interior of the contour C_k^+'S , then we should add the

$$2\pi i \sum_{w} \underset{Z=w}{Res} f(z)$$

in the second side of the equality .

3. Since by construction of contour C_k^+'S we avoid a symmetrical interval $(P-\delta_i, P+\delta_i)$ around each pole P of $f(z)$ located in upper closed half plane , by letting $\delta \to 0$ we obtain :

$$\text{P.V}\int_0^\infty f(iy)\, d(iy) = \text{P.V}\int_0^\infty \frac{d(iy)}{[1+(iy)^2]\cosh(iy)}$$

$$= \text{P.V}\int_0^\infty \frac{dy}{(1-y^2)\cos(y)} = 0 \ ,$$

And so

$$\text{P.V} \int_0^\infty \frac{dy}{(1-y^2)\cos(y)} = 0 \,.$$

4. The integrand function has infinitely many isolated simple poles on the negative imaginary half axis . Then in the same way , we get

$$\text{P.V} \int_{-\infty}^\infty \frac{dy}{(1-y^2)\cos(y)} = 0 \,, \tag{3.7}$$

Section (3.3) Cauchy Principle Value

If a function $f(x)$ has fintely many isolated singularities (x_i) with $i = 1,2, \ldots$,n∈N, on the x-axis , we Pic small $\varepsilon > 0$ and big $R > 0$ and then

$$\text{P.V.} \int_{-\infty}^\infty f(x)dx =$$

$$\lim_{\substack{\varepsilon\to 0^+ \\ R\to\infty}} \left[\int_{-R}^{x_1-\varepsilon} f(x)dx + \sum_{i=1}^{n} \int_{x_i-\varepsilon}^{x_i+\varepsilon} f(x)dx + \int_{x_n+\varepsilon}^{R} f(x)dx \right]. \tag{3.8}$$

Now , we state:

Theorem 1

Suppose a complex function $f(z)$ with allowed to have finitely many simple (isolated) poles on the x axis .

Then we have :

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx$$

$$= 2\pi i \sum [residues\ of\ f(z)\ \ in\ the\ upperhafe\ plane\ ]$$
$$+ \pi i \sum [residues\ of\ f(z)\ \ on\ the\ real\ axis\ ]$$

Example (1)

Find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4 - 1} dx \ .$$

The complex function $f(z) = \frac{1}{z^4 - 1}$

$$f(z) = \frac{1}{(z^2-1)(z^2+1)} = \frac{1}{(z-1)(z+1)(z-i)(z+i)}$$

Has two simple poles on the x – axis , the number $z = 1$ and $z = -1$, one simple pole , $z = i$ in the upper half plane and satisfies the condition of theorem 1 . The corresponding residues are :

$$\underset{z=1}{\text{Res}}\ f(z) = \left. \frac{z-1}{(z-1)(z+1)(z^2+1)} \right|_{z=1} = \frac{1}{2.(1+1)} = \frac{1}{4} \ ,$$

$$\underset{z=-1}{\text{Res}}\ f(z) = \left. \frac{z+1}{(z-1)(z+1)(z^2+1)} \right|_{z=1} = \frac{1}{-2.(1+1)} = -\frac{1}{4} \ ,$$

$$\underset{z=i}{\text{Res}}\, f(z) = \left.\frac{z-i}{(z^2-1)(z-i)(z+i)}\right|_{z=i} = \frac{1}{-2.(2i)} = -\frac{1}{4i}\ ,$$

Therefore, by theorem 1 we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4-1} dx = 2\pi i \left(-\frac{1}{4i}\right) + \pi i \left(\frac{1}{4} - \frac{1}{4}\right) = -\frac{\pi}{2}.$$

Example (2)

We continue with Example . There we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)\sinh(x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\csc(x)dx}{1+x^2} = 0\ ,$$

The singularities of $f(z) = \frac{1}{(1+z^2)\sinh(z)}$ are roots of the denominator $g(z) = (1+z^2)\sinh(z)$ namely : $z = \pm i$ and $z = k\pi i$ with $k \in \mathbb{Z}$. These roots of $g(z)$ are simple , and so they are simple poles of $f(z)$.

For $k = 0$, we get the pole $z = 0$, which is on the x – axis and

$$\underset{z=0}{\text{Res}}\, f(z) = \left.\frac{z}{(1+z^2)\sinh(z)}\right|_{z=0} = 1\ .$$

$$\underset{z=i}{\text{Res}}\, f(z) = \left.\frac{z-i}{(z-i)(z+i)\sinh(z)}\right|_{z=i} = \left.\frac{1}{(z+i)sin(iz)}\right|_{z=i} = \frac{1}{2i(sin(-1))}$$

$= \frac{1}{2sin(1)} = -\frac{1}{2}$ csch(1)

At $z = k\pi i$, with $k \in \mathbb{N}$, we find

$$\underset{z=k\pi i}{\text{Res } f(z)} = \frac{z-k\pi i}{(1+z^2)\sinh(z)}\Bigg|_{z = k\pi i} = \frac{1}{(1-k^2\pi^2)(-1)^k} = \frac{(-1)^{k-1}}{\pi^2\left[k^2-\left(\frac{1}{\pi}\right)^2\right]} .$$

Therefore by Theorem 1 , and since the above principal value is zero , it must be

$$\pi i \,.\, 1+2\pi i \,.\, \left[-\frac{1}{2}\csc(1)\right]+2\pi i \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi^2\left[k^2-\left(\frac{1}{\pi}\right)^2\right]}$$

$$= \pi i\left\{1 - \csc(1) + \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\left[k^2-\left(\frac{1}{\pi}\right)^2\right]}\right\} = 0,$$

And the obvious fact that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(a^2 + x^2)\sinh(x)} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\text{csch}(x)}{(a^2 + x^2)}\, dx = 0 .$$

Example (3)

$$\text{Find} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2 - 1}$$

The complex function $f(z) = \frac{1}{z^2-1}$,

$z^2 - 1 = 0 \Rightarrow z = \pm 1$

We have two simple poles on real axis .

The corresponding residues :

$$\underset{z=1}{\text{Res}}\, f(z) = \left.\frac{z-1}{(z-1)(z+1)}\right|_{z=1} = \frac{1}{2} \ ,$$

$$\underset{z=-1}{\text{Res}}\, f(z) = \left.\frac{z+1}{(z-1)(z+1)}\right|_{z=-1} = -\frac{1}{2} \ ,$$

Therefore , by Residue Theorem , we find

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i.\left[\frac{1}{2} - \frac{1}{2}\right] = 0 \,.$$

Chapter (4)

Infinite Isolated Singularities and Series

We consider a holomrophic function $f(z)$ in $₵ - A$, where A $\subset$ ₵ is a countable set of isolated singularities of $f(z)$.

Suppose we can find simple closed contours (circles , squares parallelograms , etc .) C_L with interior D_L ,such that:

$$\text{(a) } ₵ = \bigcup_{L=1}^{\infty} D_L \text{ , } and$$

$$(b) \lim_{L=\infty} \oint_{C_L} f(z)dz = 0 \text{ .} \qquad (4.1)$$

Then

$$(c) \sum_{w \in A} \begin{matrix} \text{Res f(z)} \\ \text{z} = w \end{matrix} = 0 \text{ .} \qquad (4.2)$$

This allows us to evaluate certain infinite series , as we do in the following :

Example 1

For any a ∈ ₵ ,such that $a \neq ni$ with $n \in \mathbb{Z}$ (i.e., a_i is not an integer) ,prove

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth(\pi a) \text{ ,}$$

From which we immediately obtain the result :

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) - \frac{1}{2a^2} = \frac{1}{2a}\left[\pi \coth(\pi a) - \frac{1}{a}\right]$$

We consider the function

$f(z)$= $\frac{1}{z^2+a^2}$ cot(πz)

Which has simple poles at $z = \pm ai$ and $z = n$,for all $n \in \mathbb{Z}$.
In this case, are the(positively oriented) squares with vertices $\pm(n + \frac{1}{2}) \pm (n + \frac{1}{2})i$, for n = 1,2,3,… (see figure 4.1)

The residues are :

$\underset{z=ai}{\text{Res}}\, f(z) = \frac{1}{2ai} \cot(\pi ai) = \frac{-1}{2a} \coth(\pi a)$,

$\underset{z=-ai}{\text{Res}}\, f(z) = \frac{1}{-2ai} \cot(-\pi ai) = \frac{-1}{2a} \coth(\pi a)$,

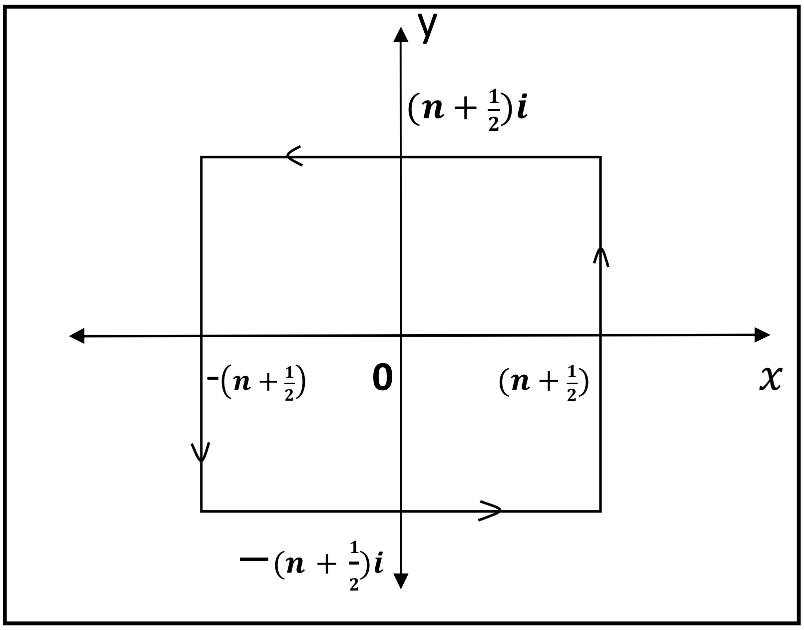


Figure 4.1 Contour for Example 1

And $\forall n \in \mathbb{Z}$, we have

$$\operatorname*{Res}_{z=n} f(z) = \left.\frac{(z-n)\cos(\pi z)}{(z^2+a^2)sin(\pi z)}\right|_{z=n} = \left.\frac{(-1)^n}{n^2+a^2}.\frac{1}{\pi\cos(\pi z)}\right|_{z=n} = \frac{1}{\pi(n^2+a^2)}$$

Now ,

$$\lim_{L\to\infty} \oint_{C_n^+} f(z)dz = 0$$

Obviously ,

$$\oint_{C_n^+} f(z)dz = I_1(\mathrm{n}) + I_2(\mathrm{n}) + I_3(\mathrm{n}) + I_4(\mathrm{n}) \qquad (4.3)$$

With

$$I_1(\mathrm{n}) = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{1}{\left[x-(n+\frac{1}{2})i\right]^2 + a^2} cot\left\{\pi\left[x-(n+\tfrac{1}{2})i\right]\right\} dx ,$$

$$I_2(n) = \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{1}{\left[(n+\frac{1}{2})+yi\right]^2 + a^2} cot\left\{\pi\left[(n+\tfrac{1}{2})+yi\right]\right\} dy ,$$

$$I_3(\mathrm{n}) = \int_{(n+\frac{1}{2})}^{-(n+\frac{1}{2})} \frac{1}{\left[x+(n+\frac{1}{2})i\right]^2+a^2} cot\left\{\pi\left[x+(n+\frac{1}{2})i\right]\right\} dx \text{ , } I_4(n)$$

$$= \int_{(n+\frac{1}{2})}^{-(n+\frac{1}{2})} \frac{1}{\left[-(n+\frac{1}{2})+yi\right]^2+a^2} cot\left\{\pi\left[-(n+\frac{1}{2})+yi\right]\right\} dy \text{ ,}$$

We will show that for j = 1,2,3,4

$$\lim_{n\to\infty} I_j(n) = 0 \qquad (4.4)$$

We will do this for j = 1 and 2 , similarly with j = 3 and 4 ,for n large.

$$\left|cot\pi\left[\pi(x-\left(n+\tfrac{1}{2}\right)i)\right]\right| < 2 \text{ .}$$

Hence ,for n large

$$|I_1(\mathrm{n})| \leq \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{2}{x^2+(n+\frac{1}{2})^2-a^2} dx =$$

$$\left[\frac{2}{\sqrt{(n+\frac{1}{2})^2-a^2}} arctan\left[\frac{x}{\sqrt{(n+\frac{1}{2})^2-a^2}}\right]\right]_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})}$$

$$=\frac{2}{\sqrt{(n+\frac{1}{2})^2-a^2}} \cdot \left\{arctan\left[\frac{n+\frac{1}{2}}{\sqrt{(n+\frac{1}{2})^2-a^2}}\right] - arctan\left[-\frac{n+\frac{1}{2}}{\sqrt{(n+\frac{1}{2})^2-a^2}}\right]\right\}$$

$\to \frac{2}{\infty}[\arctan(1) - \arctan(-1)] = \frac{2}{\infty}\left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = \frac{\pi}{\infty}$ = 0 ,as $n \to \infty$.

Similarly, for n large, we have

$$|I_2(\mathrm{n})| \leq \int_{-(n+\frac{1}{2})}^{(n+\frac{1}{2})} \frac{1}{y^2+(n+\frac{1}{2})^2-a^2}\,dy \to 0\,, \text{as } n \to \infty$$

And

$I_3(\mathrm{n}) \to \infty$, as $n \to \infty$ and $I_4(\mathrm{n}) \to 0$, as $n \to \infty$

Therefore,

$$\lim_{n\to\infty} \oint_{C_n} f(z)dz = 0$$

Then ,

$$\sum_{w\in A} \begin{matrix}\text{Res f(z)}\\ \text{z} = w\end{matrix} = 0$$

And so

$$2\,.\frac{-1}{2a}\,.\coth(\pi a)\sum_{\text{n}=-\infty}^{\infty}\frac{1}{\pi(n^2+a^2)} = 0\,.$$

Hence, we obtain the following :

Result :

$$\forall a \in ₵\,, \sum_{\text{n}=-\infty}^{\infty}\frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)\,.$$

Corollary 1

Notice that this final sum can be rewritten as

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi a \cosh(\pi a) - \sinh(\pi a)}{2a^2 \sinh(\pi a)}$$

Corollary 2

By letting a be real in the above corollary , we find Euler's sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

Example 2

By means of series summation ,we will evaluate the integral

$$\int_{0}^{\infty} \frac{\sin(ax)}{e^{bx} + c} dx ,$$

Where a , b and c are appropriate complex constants.

In the process, we will discover the conditions that a , b and c must satisfy.

First , we rewrite the sin(ax) by its exponential form , and then we expand the integrand $\frac{\sin(ax)}{e^{bx}+c}$ as an infinite series by means of the geometric series, as follows:

$$\frac{\sin(ax)}{e^{bx}+c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)e^{-bx} \frac{1}{1+ce^{-bx}} . \qquad (4.5)$$

To apply the geometric series expansion to the last fraction ,we need mini mum condition$\left|ce^{-bx}\right| < 1$for all $x \in (0,\infty)$ Therefore, we must stipulate the conditions :

1) $Re(b) > 0$
2) $|c| \le 1$

Then, we have

$$\frac{\sin(ax)}{e^{bx}+c} = \frac{1}{2i}\left(e^{iax} - e^{-iax}\right)e^{-bx}\ \sum_{n=0}^{\infty}(-ce^{-bx})^n$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}(-c)^n\left(e^{[ia-b(n+1)]x} - e^{-[ia+b(n+1)]x}\right). \qquad (4.6)$$

3) $|Im(a)| < (0+1)Re(b)$ = Re(b) .

We observe that the function $f(x) = \frac{\sin(ax)}{e^{bx}-1}$ is continuous for $x > 0$ and approaches the value $\frac{a}{b}$, as $x \to 0$. so, we can use as dominating function the following function

$$g(x) = \begin{cases} B\,,\ \ if\ \ 0 \le x \le 1 \\ \frac{e^{|Im(a)|x}}{e^{Re(b)x}-1}\ \ ,\ \ if\ 1 < x < \infty \end{cases} \qquad (4.7)$$

For

$B = \max_{0\le x\le 1}|f(x)|$ is finite .

Now , we integrate the above series, in (4.5) , term by term and use the fact for any real numbers $u, v, \alpha\ and\ \beta$ the formula

$$\int_{u}^{v} e^{(\alpha+i\beta)x} \; dx = \frac{e^{v(\alpha+i\beta)} - e^{u(\alpha+i\beta)}}{\alpha + i\beta} \tag{4.8}$$

If a , b and c are complex constants , such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| \le 1$, then

$$\int_{0}^{\infty} \frac{\sin(ax)}{e^{bx} + c} dx = a \sum_{n=0}^{\infty} (-c)^n \frac{1}{a^2 + b^2(n+1)^2} = \frac{a}{b^2} \sum_{n=0}^{\infty} \frac{(-c)^{n-1}}{n^2 + \left(\frac{a}{b}\right)^2} \tag{4.9}$$

Example 3

If a , b and c are complex constants, such that $Re(b) > 0$, $|Im(a)| < Re(b)$ and $|c| < 1$, then

$$\int_{0}^{\infty} \frac{\cos(ax)}{e^{bx} + 1} dx = \frac{1}{b} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}\, n}{n^2 + \left(\frac{a}{b}\right)^2} \quad , \tag{4.10}$$

Notice that here we have $|c| < 1$, in general.

If $|c| = 1$, we need to check the formula for the individual c
For instance , with c =1 we get

$$\int_{0}^{\infty} \frac{\cos(ax)}{e^{bx} + 1} dx = \sum_{n=1}^{\infty} \frac{(-c)^{n-1}\, n}{n^2 + 1} \quad , \quad etc$$

Also for a =0 and b such that $Re(b) > 0$, we find

$$\int_{0}^{\infty} \frac{1}{e^{bx} + 1} \; dx \overset{x = \ln(u)}{=} \int_{1}^{\infty} \frac{1}{u(u^b + 1)} \; du = \frac{1}{b} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{\ln(2)}{b}$$

Which can also be verified by the substitution

$v = u^b$. But if $c = -1$,

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + \left(\frac{a}{b}\right)^2}$$

In this case , besides the infinite interval of integration, the integral becomes also improper at $x = 0$ .If we take the derivative of equation (4.10) with respect to a , we find : If a , b and c are complex constants, such that $Re(b) > 0$,

$|Im(a)| < Re(b)$ and $|c| < 1$, then

$$\int_0^{\infty} \frac{x\sin(ax)}{e^{bx} + c}\, dx = \frac{2a}{b^3} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}\ n}{\left[n^2 + \left(\frac{a}{b}\right)^2\right]^2} \ .$$

If now we divide this equation by a and take the limit as $a \to 0$, We find : Under the conditions $Re(b) > 0$,and $|c| < 1$,we have

$$\int_0^{\infty} \frac{x^2}{e^{bx} + c}\, dx = \frac{2}{b^3} \sum_{n=1}^{\infty} \frac{(-c)^{n-1}}{n^3} \ ,$$

And so

$$\int_0^{\infty} \frac{x^2}{e^{bx} + 1}\, dx = \frac{2}{b^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \ ,$$

And

$$\int_0^{\infty} \frac{x^2}{e^{bx} - 1}\, dx = \frac{2}{b^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \ .$$

Putting $x = \ln u$ in the last two equalities, with $b > 0$ real,
We get

$$\int_1^\infty \frac{ln^2(u)}{u(u^b+1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3} \quad .$$

And

$$\int_1^\infty \frac{ln^2(u)}{u(u^b-1)}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3} \quad .$$

If now we let $u = \frac{1}{x}$ in the above two integrals , with

$b > 0$ real ,we obtain

$$\int_0^1 \frac{x^{b-1}\ \ ln^2(x)}{1+x^6}\,dx = \frac{2}{b^3}\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^3} \quad .$$

And

$$\int_0^1 \frac{x^{b-1}\ \ ln^2(x)}{1-x^6}\,du = \frac{2}{b^3}\sum_{n=1}^\infty \frac{1}{n^3} \quad .$$

If in equation (4.11) we replace a with ia and use the identity $\cos(iz)$ = $\cosh(z)$, we find : If a , b and c are complex constants , such that $Re(b) > 0$, $|Re(a)| < Re(b)$ and

$|c| < 1$, then

$$\int_0^\infty \frac{\cosh\,(ax)}{e^{bx}+1}\,dx = \frac{1}{b}\sum_{n=1}^\infty \frac{(-1)^{n-1}\ n}{n^2-\left(\frac{a}{b}\right)^2} \, .$$

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