Chapter 1
Locally Compact Groupoids

We show a first step toward extending the theory of Fourier-Stieltjes algebras from groups to groupoids. If G is locally compact (second countable) groupoid, we show that B(G), the linear span of the Borel positive definite functions on G, is a Banach algebra when represented as an algebra of completely bounded maps on a C*-algebra associated with G. This necessarily involves identifying equivalent elements of B(G). An example shows that the linear span of the continuous positive definite functions need not be complete.

Section (1.1): Background of Groupoids

As suggested by the title, this section connects two lines of earlier work, and we begin with an abbreviated history of each of these lines, in order of appearance. After the history, we will state our main results and outline the body of the section. We mention here that some basic definitions can be found and that we assume locally compact spaces are second countable. More background on groupoi-ds is available.The necessary background on Fourier- Stieltjes algebras can be obtained.

Introduced the notion of virtual group as a tool and context for several kinds of problems in analysis and geometry.Virtual groups are (equivalence classes of) groupoids having suitable measure theoretic structure and the property of ergodicity. Ergodicity makes agroupoid more group-like, but many results on groupoids do not require ergodicity. Among the structures which fit naturally into the study of groupoids are groups, group actions, equivalence relations (including foliations), ordinary spaces, and examples made from these by restricting to a part of the underlying space.

The original motivation for studying groupoids was provided by Mackey's theory of unitary representations of group extensions.The idea has been applied to that subject. In his original section, Mackey also showed the relevance of the idea for ergodic group actions in general , and a

number of applications have been made there.

Most uses of groupoids have been in the study of operator algebras, another approach to understanding and exploiting symmetry. Several pioneering section should be mentioned. Hahn proved the existence of Haar measures for measured groupoids, whether ergodic or not, and used this to make convolution algebras and study von Neumann algebras (is to define them as weakly closed *-algebras of bounded operators (on aHilbert space) containing the identity. In this definition the weak (operator) topology can be replaced by many other common topologies including the strong, ultrastrong or ultraweak operator topologies. The *-algebras of bounded operators that are closed in the norm topology are C*-algebras, so in particular any von Neumann algebra is a C*-algebra) [4] associated with measured groupoids. Feldman and Moore made a thorough analysis of ergodic equivalence relations that have countable equivalence classes, showing that the von Neumann algebras attached to them are exactly the factors that have Cartan subalgebras Connes introduced a variation on the approach of Mackey, particular by working without a chosen invariant measure class. This approach has some advantages for applications to foliations and to C*-algebras. Renault studied C*-algebras generated by convolution algebras on locally compact groupoids endowed with Haar systems, not using invariant measure classes. That measured groupoids may be assumed to have locally compact topologies. Thus the study of operator algebras associated with groupoid symmetry can always be confined to locally compact groupoids, whether one is interested in C*-algebras or von Neumann algebras.

Basically one can say that locally compact groupoids occur in situations Where there is symmetry that is made evident by the presence of an equivalence relation. Many of these are associated either with group actions or foliations. It can be surprising how group-like both group actions and

foliations can be. In particular, some of the section mentioned above have included information about the unitary representations of groupoids. However, there is no treatment of duality theory for groupoids. and we intend to make a beginning here.

Introduced Fourier and Fourier-Stieltjes algebras for non-commutative locally compact groups. Roughly, the Fourier−Stieltjes algebra of a locally compact group, G, denoted B(G), is the unitary representation theory of G equipped with some additional algebraic and geometric structure. More precisely, B(G) is the set of finite linear combinations of continuous positive definite functions on G equipped with a norm, which makes B(G) a commutative Banach algebra. The elements of B(G) are exactly the matrix entries of unitary representations of G. A primary source of intuition is the fact that when G is abelian, B(G) is the isometric, inverse Fourier −Stieltjes transform of $M^1(\hat{G})$,the convolution, Banach algebra of finite, regular Borel measures on $\hat{G}$,the dual group(of characters) of G. Thus B(G), as a Banach algebra, "is" $M^1(\hat{G})$.The fact that B(G) exists (as acommutative Banach algebra) when G is not abelian leads one to hope that a useful duality theory exists for non-abelian groups which is in spirit similar to the application rich Pontriagin−Van Kampen duality for abelian locally compact groups. That such a duality theory exists has been established by walter by proving

that

(i) B(G) is a complete invariant of G, i.e., $B(G_1)$ and $B(G_2)$ are isometrically isomorphic as Banach algebras, if and only if G_1 and G_2 are topologically isomorphic as locally compact groups, and

(ii) There is an explicit process for recovering G given its "dual object",

 B(G). Exactly how useful this theory will remains to be seen since all

but a few of the hoped important applications await rigorous proof.

For various reasons it turns out that it may be more fruitful to look at B(G) from a broader perspective than that afforded by the category of

locally compact groups. Namely, it is seen that there is a natural duality theory for a "large" collection of Banach algebras that extends in a precise way the Pontriagin duality for abelian groups as well as the above-mentioned duality for non-abelian groups. The theory of C*-algebras plays a large role both technically and intuitively in this duality theory.

In an effort to understand this new duality theory better, as well as to generate meaningful applications and examples of a concrete nature, in this section we have answered affirmatively the question: Does a locally compact groupoid G have a Fourier−Stieltjes algebra? For groupoids, than one candidate for the Fourier−Stieltjes algebra, and the details are more technical than for groups, but there is an affirmative answer.

The existence of a Fourier−Stieltjes algebra augurs well for future applications. In particular, one example suggests an interesting possibility: the algebra of continuous functions on X vanishing at infinity, $C_0(X)$, is the Fourier algebra of a locally compact space X. This opens up an entire "dual" approach to the currently exploding subject of non-commutative geometry, which at the moment is regarded more or less exclusively in terms of the associated C*-algebras (not the Fourier−Stieltjes algebras).

As for groups, the Fourier-Stieltjes algebra of a groupoid is the linear span of the positive definite functions and the algebra structure is given by pointwise operations. To provide the Banach space structure, we use C*-algebras attached to G, but we use them in a different way from Eymard, and also use C*-algebras associated with the equivalence relation that G induces on X.

To describe the various algebras, let us begin with the space $M_c(G)$ of compactly supported bounded Borel functions on G, and its subspace $C_c(G)$. Both are algebras under convolution, which is defined by using the Haar system, and have involutions. If R is the equivalence relation on X induced by G, defining $\theta(\gamma) = (r(\gamma), s(\gamma))$ gives a continuous homomorphism of G onto R using the relative product topology on R. The quotient

topology on R has some advantages: for example, if θ is one-to-one then θ is a homeomorphism. (G is said to be principal). Under the quotient topology R is σ-compact and we can provide it with a Borel measurable Haar system, which allows us to make a convolution*- algebra of the space $M_{0c}(R)$ of bounded Borel functions on R that are supported by the image of some compact set in G. we show how to make an algebra on G that contains a copy of the space M(X) of bounded Borel functions on X as well as $M_c(G)$, and this algebra is denoted by $M_c(G, X)$.The analog for R is denoted by $M_{0c}(R, X)$. Let $\overline{X}$ denote the one-point compactification of X.Then $C(\overline{X}) \subseteq M(X)$ so $+M_c(G,X)$ contains both $C_c(G)$ and $C(\overline{X})$.The span of these two subalgebras is denoted $C_c(G, \overline{X})$.

If ω is the universal representation of G, then ω carries each convolution algebra on G to an algebra of operators and thereby provides the contion algebra with a norm.The closures of the algebras of operators or the completions under the norms are useful in various ways, so we have notation for them: $C^*(G)$ is the completion of $C_c(G)$,$C^*(G,\overline{X})$ is the complication of $C_c(G,\overline{X})$, $M^*(G)$ is the completion of $M_c(G)$, and $M^*(G, X)$ is the completion of $M_c(G, X)$. Likewise for R we get $M^*(R)$ and $M^*(R, X)$ from $M_{0c}(R)$ and $M_{0c}(R, X)$. The algebra B(G) is isomorphic to a Banach algebra of completely bounded operators on $M^*(G)$, but the functions also correspond to completely bounded bimodule mappings from $C^*(G,\overline{X})$ to $M^*(R, X)$ as bimodules over $C(\overline{X})$.

we define a bounded Borel function p on a locally compact groupoid G with Haar System λ to be positive definite if

$$\iint f(\gamma_1)\bar{f}(\gamma_2)\, p(\gamma_2^{-1}\gamma_1)\, d\lambda^x(\gamma_1)\, d\lambda^x(\gamma_2) \geq 0 \qquad (1)$$

for every $f \in C_c(G)$. The set of these is denote P(G) and by definition the set B(G) is the linear span of P(G). In both sets two elements that agree except on a negligible set need to be identified, though we find it convenient to

indulge in the usual carelessness about maintaining the distinction. The primary result is

(i) B(G) is a Banach algebra. Results needed to prove this are:

(ii) Each p $\in$ P(G) can be represented in terms of a unitary representation of G and a cyclic "vector" for the representation.

(iii) Multiplication by a b $\in$ B(G) defines a completely bounded operator on M*(G) whose norm is at least the supremum norm of b.

(IV) The set of operators arising from elements of P(G) is closed in the space of completely bounded operators on M*(G).

In fact, B(G) is a Banach algebra of completely bounded operators on M*(G), and the elements of P(G) occur as completely positive operators.

In order to prove the completeness of B(G), we introduce an auxiliary groupoid. Let T_2 denote the transitive equivalence relation on the two point set {1, 2}, so that functions on T_2 are 2 × 2 matrices. Thus functions on G × T_2 can be regarded as 2 × 2 matrices of functions on G. Then each b $\in$ B(G) appears as a corner entry of a positive definite function on G × T_2 whose completely bounded norm is the same as that of b. Furthermore, such a corner entry is always in B(G). Combining these facts with the completeness of P(G × T_2) is what allows us to finish the proof of completeness of B(G).

The material can be outlined as follows which is devoted to Background material on three topics: locally compact groupoids, convolution algebras attached to them, and representations of groupoids and the algebras.

We give the definition of "positive definite function" and establish the connection between such functions and cyclic unitary representations of G.

We show that multiplication by a positive definite function is a completely positive operator on M*(G),using the main result. Also includes the proof that a positive definite function gives rise to a completely positive operator from C*(G,X) to M*(R,X).

All of these operators are bimodule maps over C($\overline{\text{X}}$), the algebra of continuous functions on the one-point compactification of the space of units of G. contains results about completely bounded bimodule maps.

Finally we are able to complete the proof that the linear combinations of positive definite functions constitute a Banach algebra, contains some counter examples.

The purpose is to give a source of some essential information about analysis on groupoids needed.

Much of our motivation comes from the fact that group actions give rise to groupoids, and that case was important in the development of the subject. However, we want to present a definition that has a different motivation, hoping to make the idea easier to grasp. Effros suggested this approach.

Start with two sets, X and G, and suppose that X is the set of vertices and G the set of edges of a directed graph. If the structure we are about to describe is present, we say that G is a groupoid on X. Suppose that we have a mapping taking values in G and defined on the set of pairs of edges for which the first edge starts from the vertex where the second edge terminates. For a groupoid of mappings, we want the operation to be composition and we want the right hand factor to be applied first. We want the operation to be associative and to have units and inverses.

To describe this in more detail, we use two functions r and s from G onto X, such that each $\gamma \in$ G is an edge from $s(\gamma)$ to $r(\gamma)$. Then for γ and γ' in G, the element $\gamma\gamma'$ of G is defined iff $s(\gamma) = r(\gamma')$. We write $G^{(2)} = \{(\gamma, \gamma') \in G \times G : s(\gamma) = r(\gamma')\}$. We also assume there is given a mapping $x \mapsto i_x$ of X into G and an involution $\gamma \mapsto \gamma^{-1}$ on G. Then we require the following properties:

(i) (associativity) If $s(\gamma_1) = r(\gamma_2)$, then $s(\gamma_1 \gamma_2) = s(\gamma_2)$, and $r(\gamma_1 \gamma_2) = r(\gamma_1)$. If, also, $s(\gamma_2) = r(\gamma_3)$, then $(\gamma_1 \gamma_2) \gamma_3 = \gamma_1(\gamma_2 \gamma_3)$.

(ii) (units) If $x \in$ X, then $r(i_x) = s(i_x) = x$. If $\gamma \in$ G, then $\gamma\, i_{s\,(\gamma)} = i_{r\,(\gamma)}\, \gamma = \gamma$.

(iii) (inverses) $r(\gamma^{-1}) = s(\gamma)$, $s(\gamma^{-1}) = r(\gamma)$, $\gamma\, \gamma^{-1} = i_{r\,(\gamma)}$, and $\gamma^{-1}\, \gamma = i_{s\,(\gamma)}$.

Examples (1.1.1) [1]: (i) Suppose a group H acts on a set X (on the left). Set g= H × X, identify X with {e} × X, and define r(h, x) = hx, s(h,x) = x . Then we can define (h_1, x_1) $(h_2 , x_2) = (h_1 h_2 , x_2)$ if $x_1 = h_2 x_2$, $i_x = (e,x)$ and $(h, x)^{-1} = (h^{-1}, hx)$, to make a groupoid. (Right actions work better for left Haar measures as we see below, and then we have s(x, h) = xh,$r(x,$h) = x).

(ii) To make a groupoid from an equivalence relation R on a set X, identify X with the diagonal in X × X, define r(x, y) = x, s(x, y) = y, (x, y)(y, z) = (x, z) and $(x, y)^{-1} = $ (y,x).

(iii) Let X be the set of open sets in $\mathbb{R}^n$, and let G be the set of diffeomorphisms between elements of X. For γ ∈ G, let s(γ) be the domain of the mapping and let r(γ) be its range. Let the product be function composition and let the inverse be the inverse of functions.

Every groupoid determines a natural equivalence relation on its set of units, namel x ~ y iff there is a γ: x→y. The equivalence class of x is denoted [x] and is called its orbit. As a subset of X × X, this equivalence relation is R={(r(γ), s(γ)) : γ ∈ G}. The function θ = (r, s) mapping G to R is a groupoid homomorphism and G is called principal iff θ is one-one, i.e G is isomorphic to an equivalence relation. If G arises from a group action G is principal iff the action is free (the only element of the group that has any fixed points is the identiy).

If G is a groupoid on X, and Y⊆X is non-empty, we call r^{-1}(Y) ∩ s^{-1}(y) the restriction of G to Y, and write G│Y for it. In terms of graphs, G│Y is the set of all edges in G that connect points of Y. G│Y is a subgroupoid of G, and a groupoid on Y. For each x ∈ X, G│{x} is a group called the stabilizer of x or the isotropy of x.

If A and B are subsets of a groupoid G, we define the product AB of the two sets to be { γ γ' : γ ∈ A, γ' ∈ B, r(γ') = s(γ)}. If A has a single element

y_0, we write $\gamma_0 B$ for AB. Thus $YGY = G \,|\, Y$ and $xGx = G \,|\, \{x\}$ if $Y \subseteq X$ and $x \in X$. We also use the sets $r^{-1}(x) = xG$ and $s^{-1}(x) = Gx$ when $x \in X$.

A groupoid G is a Borel groupoid if G has a Borel structure, X is a Borel set when regarded as a subset of G, and r, s, $()^{-1}$ and multiplication are Borel functions. We will consider only Borel groupoids which are at least analytic, and then $X = \{\gamma : r(\gamma) = \gamma\}$ is Borel if r is Borel. A groupoid G is topological if it has a topology such that X is closed, and r, s, $()^{-1}$ and multiplications are continuous, while r and s are open. Again these proper-ties are not independent. It is necessary for r to be open in order to prove that AB is open whenever A and B are open.

We write M(G) for the space of bounded Borel measurable functions on G, whenever G is a Borel groupoid. If G has a topology in which it is compact (a countable union of compact sets), we write $M_c(G)$ for the subspace of M(G) of functions having compact support .

If G is an analytic Borel groupoid, we say a measure μ on G is quasisymmetric if it has the same null sets as its image $(\mu)^{-1}$ under $()^{-1}$. Thus μ and $(\mu)^{-1}$ are in the same measure class, and the measure class $[\mu]$ (set of measures with the same null sets as μ) is invariant under $()^{-1}$. For measures on G, this global symmetry is just the same as if G were a group. We give the definitions for groupoids that extend the notions of invariance and quasi-invariance of measures under translation on a group or under other actions of the group.

Because translation on the left by a groupoid element γ makes sense only on $s(\gamma)G$, and similarly for right translation, the notions of invariance and quasi-invariance are more complicated for groupoids than for groups .

Following Connes we say that the kernel is a function v assigning a σ-finite (positive) measure v^x on G to each $x \in X$, so that these two statements are true:

(i) $v^x(G \setminus xG)$ is always 0. One may say that v^x concentrated on xG.

(ii) If $f \in M(G)$, and $f \geq 0$, the function $v(f) : X \to [x, \infty]$ defined by

$v(f)(x) = v^x(f) = \int f \, dv^x$ is Borel.

Given an element $\gamma \in G$, the mapping $\gamma' \mapsto \gamma \gamma'$ is a Borel isomorphism of $s(\gamma)G$ onto $r(\gamma)G$ and thus maps $v^{s(\gamma)}$ to a measure $\gamma v^{s(\gamma)}$ on $r(\gamma)G$, for every kernel v. A kernel v is called left invariant provided $v^{r(\gamma)} = \gamma v^{s(\gamma)}$ for all $\gamma \in G$. It is called (left) quasiinvariant if $v^{r(\gamma)}$ and $\gamma v^{s(\gamma)}$ are equivalent for all $\gamma \in G$.

A left invariant kernel, λ, on a Borel groupoid G is called a Borel Haar system. Then defining λ_x to be the image of λ^x under inversion produces a right Borel Haar system. A Borel Haar system λ on a locally compact groupoid is called a Haar system if supp(λ^x) is always xG and $\lambda(f) \in C_c(X)$ for each $f \in C_c(G)$. In particular, each λ^x is a Radon measure. For discussi - ons of Haar systems .

When λ is a Haar system, it can be convenient to have a left quasi kernel λ_1 consisting of probability measures equivalent to the measures λ^x. It is not difficult to show that there is a continuous, strictly positive, function f on G such that for every $x \in X$, $\int f \, d\lambda^x = 1$. We choose one such f and write λ_1^x for the measure $f\lambda^x$. We also write μ_1^x for the probability measure $s(\lambda_1^x)$ on X; these measures also depend on x in a BoreI way.

If λ is a Borel Haar system on a Borel groupoid G and μ is probability measure on X, we can form a measure

$$v = \int \lambda^x \, d\mu(x) : \int f dv = \int \int f(\gamma) d\lambda^x (\gamma) d\mu(x) \qquad (2)$$

We often write λ^μ for this measure v. Suppose that $G = X \times H$, where X is a right H-space, and give G the groupoid structure that comes from the group action. Let λ be a left Haar measure on H. For each $x \in X$, let ε^x be the point mass at x, and define $\lambda^x = \varepsilon^x \times \lambda$, to get a Borel left Haar system. If μ is a σ-finite measure on X for this gropoid , then $v = \lambda^\mu = \mu \times \lambda$ and the class $[v]$ is symmetric iff μ is quasi-invariant under the group action, i.e., for every Borel set $E \square X$ and every group element h, $\mu(E) = 0$ iff $\mu(Eh) = 0$. The fact that if μ is quasi-invariant under almost all elements of the group ,

then it is quasi-invariant. Hence, on a general Borel groupoid with Borel Haar system λ, a σ-finite measure μ on X is called quasi-invariant iff λ^μ is quasisymmetric. In that case, a result of Peter Hahn, combined with shows that there is a Borel homomorphism Δ_μ, of G to the multiplicative positive real numbers such that

$$\Delta_\mu = \frac{d\lambda^\mu}{d(\lambda^\mu)^{-1}} \tag{3}$$

This homomorphism is called the modular function by analogy with locally compact groups. If μ is quasi-invariant, and Y is a μ-conull Borel set in X, the restriction G │ Y is called in essential.

We often refer to the set of all quasi-invariant σ-finite measures on X, and will denote that set by Q. We say a Borel set N □ X is Q-null provided μ(N)= 0 for every μ ε Q. It follows from the existence, and uniqueness upto equivalence, of a quasi-invariant σ-finite measure on each orbit.That N is Q-null iff λ^x(GN) is always 0.The measures μ_1^x introduced above are in this class, and any measure in Q equivalent to such a measure is called transitive because it is concentrated on a single orbit. For a Borel set N□G, we say N is λ^Q-null iff λ^μ(N) = 0 whenever μ ε Q. A function f on X is Q-essentially bounded iff the restriction of f to the complement of some Q-null set is bounded, and then $\|f\|_\infty$ is defined to be the smallest element of {B: $|f| \le$ B μ-almost everywhere for every μ ε Q}.The space of Q -essentially bounded functions on X will be denoted by $L^\infty(Q)$. A similar definition is used for the space $L^\infty(\lambda^Q)$ of λ^Q-essentially bounded functions on G, except that the measures $\mu^\lambda$are used.

Examples (1.1.2) [1]: (i) If G = X × H, where X and H are locally compact and H is a group, let ε^xdenote the unit point mass at x for x ε X and let λ be a left Haar measure on H. Then $\lambda^x = \varepsilon^x \times \lambda$ defines a Haar system for G.

(ii) If E is an analytic equivalence relation on X and each equivalence class is countable, we can let λ^x be counting measure on $\{x\} \times [x]$ to get a left invariant system of measures.

(iii) Here is an example of a locally compact groupoid that has a Borel Haar system but no Haar system. Let G = [0, 1/2] × {0} ∪ [1/2, 1] × Z/2. This is a field of groups. To get a Borel Haar system, we can make each λ^x a multiple of the Haar measure on {0} or $\mathbb{Z}/2$. Then $\lambda^{1/2}(\{1/2,0\}) = \lambda^{1/2}(\{1/2,1\}) > 0$ and if we let f be the characteristic function of [1/2,1] × {1} then the function λ(f) has a jump at 1/2. We could easily change to another locally compact topology on this G and get a Haar system. In general, it may be necessary to change the topology on G and pass to an in essential restriction in order to get a Haar system.

We use several convolution algebras, and will introduce them here. There are two basic convolutions, a convolution of functions that can be defined in the presence of a Borel Haar system, and a convolution of kernels that does not depend on any such system. If the groupoid is locally compact and the Haar system is continuous, then C_c(G) is an algebra under the convolution of functions. We will see that convolution of functions can be subsumed under convolution of kernels by replacing each function by the kernel obtained by multiplying the Haar system by the Function.

First, let G be a Borel groupoid with a Borel Haar system λ. If f, g are non-negative Borel functions on G, then $\int f(\gamma_1)\, g(\gamma_2)\, d\lambda^{r(\gamma_1)}(\gamma_2)$ is a Borel function of γ_1 , so by taking linear combinations and monotone limits we see that whenever F is a non-negative Borel function on G × G the integral $\int F(\gamma_1, \gamma_2)\, d\lambda^{r(\gamma_1)}(\gamma_2)$ depends on γ_1 in a Borel manner. Then for non negative f, g ∈ M(G), we can let $F(\gamma_1, \gamma_2) = f(\gamma_1)\, g(\gamma_1^{-1}\gamma_2)$ when r(γ_2) = r(γ_1) and F(γ_1, γ_2) = 0 otherwise, and see that $\int f(\gamma_1)\, g(\gamma_1^{-1}\gamma_2)\, d\lambda^{r(\gamma_2)}(\gamma_1)$ is a Borel function of γ_2. Denote this function by $f * $ g, provided that it is always finite valued. Then $f * $ g ∈ M(G). The function

$f * g$ is called the convolution of f and g. Convolution can be extended to more general function using linearity .

Define the space $I_r(G,\lambda)$ to be { $f \in M(G) : \lambda (|f|)$ is bounded}, and give it a norm by letting $\|f\|_{i,r}$ be the sup norm of the Borel function $\lambda (|f|)$.We can define an involution on M(G) by letting $f^b(\gamma) = \bar{f}(\gamma^{-1})$ for $f \in M(G)$, $\gamma \in G$. If we set $I(G,\lambda) = I_r(G,\lambda) \cap (I_r(G,\lambda))^b$, then we can define $\|f\|_I$ to be the maximum of $\|f\|_{I,r}$ and $\|f^b\|_{I,r}$ for $f \in I(G, \lambda)$, obtaining a normed algebra on which the involution is an isometry.

If G is locally compact and λ is a Haar system, then $C_c(G)$ is a $*$-sub-algebra of $I(G, \lambda)$. In the inductive limit topology, $C_c(G)$ is a topological algebra.

The second kind of convolution can be introduced after the objects are defined: A complex kernel is a function v assigning a complex measure v^x on G so that

(i) v^x is always concentrated on xG .

(ii) if $f \in M(G)$, the function $v(f)$ taking $x \in X$ to $v^x(f)$ is Borel.

We define K(G) to be the space of bounded complex kernels on G, i.e those for which the total variation of v^x is a bounded function of x.

If $\gamma \in G$ and $v \in K(G)$ we can map $v^{s(\gamma)}$ to a measure on r(γ)G, via left translation by $\gamma v^{s(\gamma)}$. If $v_1, v_2 \in K(G)$ we can define the convolution $v = v_1 * v_2$ by $v^x = \int \gamma v_2^{s(\gamma)} dv_1^x(\gamma)$, as we do in defining Haar systems. Denote this measure by $\gamma v^{s(\gamma)}$.If $v_1, v_2 \in K(G)$ we can define the convolution $v = v_1 * v_2$ by $v^x = \int \gamma v_2^{s(\gamma)} dv_1^x(\gamma)$. We can also define an action of K(G) on $I_r(G,\lambda)$ as follows If $v \in K(G)$, $f \in I_r(G, \lambda)$ and $\gamma' \in G$ set

$$L(v)f(\gamma') = \int f(\gamma^{-1} \gamma')dv^{r(\gamma')}(\gamma) \qquad (4)$$

It is not difficult to verify that L(v) is a bounded operator whose norm is at most the essential supremum of the total variation norms of the signed measures v^x If v_1 and v_2 are in K(G) and $f \in I_r(G, \lambda)$, then we can calculate

$$\left(L(v_1)(L(v_2)f)\right)(\gamma) = \int (L(v_2)f(\gamma_1^{-1}\gamma)dv_1^{r(\gamma)}(\gamma_1)$$

$$= \int \int f(\gamma_2^{-1}\ \gamma_1^{-1}\gamma)dv_2^{s(\gamma 1)}(\gamma_2)dv_1^{r(\gamma)}(\gamma_1)$$

$$= \int \int f(\gamma_2^{-1}\gamma)dv_2^{s(\gamma 1)}(\gamma_1^{-1}\gamma_2)dv_1^{r(\gamma)}(\gamma_1)\quad (5)$$

$$= \int f(\gamma_2^{-1}\gamma)d(v_1 * v_2)^{r(\gamma)}(\gamma_2)$$

showing that L takes convolution to composition of operators. Since L is faithful, K(G) is an algebra under convolution. If f, $g \in I_r(G,\lambda)$ it is not difficult to verify that $f\lambda \in K(G)$ and $L(f\lambda)\ g = f * g$:

$$L(f\lambda)g(\gamma) = \int g(\gamma_1^{-1}\gamma)f(\gamma_1)d\lambda^{r(\gamma)}(\gamma_1) \qquad (6)$$

Since L is faithful and convolution is associative, it follows that

$f\lambda * g\lambda = (f*g)\lambda$. Thus $I_r(G, \lambda)\ \lambda = \{\ f\lambda\colon f\ I_r(G, \lambda)\}$ is a subalgebra of K(G) isomorphic to $I_r(G, \lambda)$. If G is locally compact and has a Haar system λ, the calculations just made also show that $C_c(G)\ \lambda$ is a subalgebra of K(G) isomorphic to $C_c(G)$.

Next we want to enlarge $C_c(G)\ \lambda$ to a subalgebra of K(G) that contains a copy of $C_c(X)$. We denote the one-point compactification of X by $\overline{X}$ .The mapping $f \rightarrow f\ |X$ takes $C(\overline{X})$ one-one onto the algebra of continuous function on X that have a limit at infinity. We identify $C(\overline{X})$ with that sub-algebra of C(X) but continue to write $C(\overline{X})$. Notice that there is also a sub-algebra of K(G) isomorphic to $C(\overline{X})$, obtained as follows. First define ε to be the kernel that assigns the point mass at x to each $x \in X$, which we denote by ε^x as above. Next notice that K(G) is closed under multiplication by any bounded Borel function on G, so if h ∈ M(X) and $v \in K(G)$, we can define hv to be $(h \circ r)v$, and $vh = (h \circ s)v$. (These agree with the naturally defined left and right multiplication of M(X) on $I_r(G, \lambda)$ when the latter is regarded as a space of kernels). Then M(X)ε is a subalgebra of K(G) isomorphic to M(X), and that algebra includes $C(\overline{X})\varepsilon$,which is isomorphic to $C(\overline{X})$.

If we write $C_c(G,\bar{X})$ for the sum of $C(\bar{X})\varepsilon$ and $C_c(G) \lambda$ as subspaces of $K(G)$, it can be seen that $C_c(G, \bar{X})$ is a subalgebra. Also the involution on $C_c(G)$ extends in a natural way to $C_c(G, \bar{X})$. We need the algebra $C_c(G,\bar{X})$ because it generates a C*-algebra that contains $C(\bar{X})$ as a subalgebra enabling us to apply a result on completely bounded bimodule mappings.

On the other hand, the algebra $C_c(G)$ has an approximate. In order to state the existence theorem, we need to introduce some of their terminology. They call a set L in G r-relatively compact if KL is relatively compact for every compact set $K \subseteq X$. There exists a decreasing sequence $U_1, U_2 , \ldots$ of open r-relatively compact sets whose intersection is X. There also exists an increasing sequence of compact sets in X, $K_1 , K_2 , \ldots$ whose interiors exhaust X. These come from the second countability of G, and they allow us to make a sequence that is an approximate unit (instead of a more general net). We call a function f in $C_c(G)$ symmetric if $f^b=f$.

Theorem (1.1.3) [1]: There is a sequence e_1, e_2, ... of symmetric function in $C_c^+(G)$ such that for

each n we have

(i) $\text{supp}(e_n) \subseteq U_n$, and

(ii) $\int e(\gamma)d\lambda^x(\gamma) \geq 1-n^{-1}$ for $x \in K_n$, and ≤ 1 for all $x \in X$.

Such a sequence is a two-sided approximate unit for $C_c(G)$ in its inductive limit topology, i.e., for uniform convergence on compact sets.

A (unitary) representation of a locally compact groupoid G is given by a Hilbert G-bundle K over X, the unit space of G; this means we have two functions that have some properties:

(i) a Hilbert space $K(x)$ for each x. We form $\Gamma_K =\{(x, v): x \in X, v \in K(x)\}$, called the graph of K, and require that Γ_K have astandard Borel structure such that the projection onto X is Borel and there is a countable set of Borel sections of Γ_K such that for each x the set of their values at x is dense in $K(x)$.

(ii) a Borel homomorphism π of G into the unitary groupoid of the bundle

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K, i.e., for each γ, $\pi\,(\gamma):K(s(\gamma))\rightarrow K(r(\gamma))$ is unitary, and π is a Borel function.

This can also be said as follows: (K, π) is a Borel function on G taking values in the category of Hilbert spaces.

Given a representation π of G, and a measure $\mu \in Q$, we can obtain from them a $*$ -representation of $M_c(G)$. Before describing the representation, we need another item of above and define $v_0 = \Delta^{-1/2}v$, obtaining a symmetric measure. Next we make a Hilbert space, $L^2(\mu;K)$, of square integrable sections of K. For $f \in M_c(G)$ we define $\pi^\mu(f)$ on $L^2(\mu;K)$ by setting

$$(\pi^\mu(f)\xi|\eta) = \int f(\gamma)\big(\pi(\gamma)\xi \circ s(\gamma)|\eta \circ r(\gamma)\big)dv_0\,(\gamma) \tag{7}$$

For $\xi, \eta \in L^2(\mu;K)$. Then π^μ is a $*$ -representation of $M_c(G)$ with $\pi^\mu\big\|(f)\big\| \leq \big\|f\big\|_{I,\mu}$ so its restriction to $C_c(G)$ has the same property. We denote the restriction by the same symbol, depending on context to distinguish the two. Later we will also use another method of integrating a unitary representation of G, one that is due to Hahn and does not use the symmetrized measure.

It can be convenient to choose μ to be finite, say a probability measure so we need to know that $\mu' \sim \mu$ implies $\pi^{\mu'}$ is unitarily equivalent to π^μ. To prove this implication, take ρ to be a positive Borel function whose square is the Radon- Nikodym derivative of μ' with respect to μ. Then

$$p^2 \circ r = \frac{d\lambda^{\mu'}}{d\lambda^\mu} \tag{8}$$

and

$$p^2 \circ s = \frac{d(\lambda^{\mu'})^{-1}}{d(\lambda^\mu)^{-1}} \tag{9}$$

So

$$(p^2 \circ r)\Delta_\mu = (p^2 \circ s)\,\Delta_{\mu'} \tag{10}$$

Hence we can define $V: L^2(\mu',K)\rightarrow L^2(\mu, K)$ by $V\xi = \rho\xi$ to get the necessary unitary equivalence. To see that it is indeed an intertwining operator, compute to see that the inner products are equal:

$$(\pi^\mu(f)\, V\xi \mid V\eta) = (\pi^{\mu'}(f)\, \xi \mid \eta).$$

It is natural to ask whether every continuous representation of $C_c(G)$ can be obtained by integrating a unitary representation of G, as is true for groups. An affirmative answer to this question was provided by an ingenious argument due to Renault, and it follows that every representation of $M_c(G)$ bounded by $\parallel \parallel_I$ can be obtained by integrating a unitary representation of G. Another discussion of this result Renault's theorem is:

Theorem (1.1.4) [1]: Let G be a locally compact groupoid that has a Haar system, and let H_0 be a dense subspace of a (separable) Hilbert space H. suppose that L is representation of $C_c(G)$ by operators on H_0 such that

(i) L is non-degenerate;

(ii) L is continuous in the sense that for every pair of vectors ξ, $\eta \in H_0$, the linear functional $L_{\xi,\eta}$ defined by $L_{\xi,\eta}(f) =(L(f)\, \xi \mid \eta)$ is continuous relative to the inductive limit topology on $C_c(G)$;

(iii) L preserves the involution, i.e.,$(\xi \mid L(f^b)\eta) = (L(f)\xi \mid \eta)$ for $\xi,\eta \in H_0$ and $f \in C_c(G)$.

Then the operators L(f) are bounded. The representation of $C_c(G)$ on H obtained from L is equivalent to one obtained by integrating a unitary representation of G using a probability measure $\mu \in Q$. In particular, L is continuous relative to $\parallel \parallel_I$.

Renault defined a norm on $C_c(G)$ by $\parallel f \parallel$ =sup $\{ \parallel L(f) \parallel$: L is a bounded representation of $C_c(G)\}$. Theorem (1.1.5) shows that we could get the same norm by using the representations π^μ in place of the L's. The completion of $C_c(G)$ with respect to the norm just defined is a C*-algebra denoted C*(G). Every positive linear functional of norm one on a C*-algebra gives rise to a representation of the algebra and a cyclic vector in the Hilbert space of the representation. The direct sum of all these cyclic representations is called the universal representation of the C*-algebra. We will denote this representation by ω. For C*(G), we know that every one of the cyclic representations is of the form π^μ, so ω can also be regarded as a

representation of $M_c(G)$. We will write $M^*(G)$ for the operator norm closure of $\omega(M_c(G))$. Since ω is an isomorphism on $C^*(G)$, we can regard $C^*(G)$ as a subalgebra of $M^*(G)$. We will also refer to ω as the universal representation of G itself.

In proving that L can be obtained by integration, Renault shows that there is a representation of $C_c(X)$, say ϕ associated with L such that for $f \in C_c(G)$ and $h \in C_c(X)$ we have

$$L\big((h \circ r)f\big) = \phi(h)L(f) \tag{11}$$

and

$$L\big(f(h \circ s)\big) = L(f)\phi(h) \tag{12}$$

Then ϕ extends in the obvious way to a unital representation of $C(\overline{X})$ and can be used to extend L to a representation of $C_c(G, \overline{X})$:

$$L(f\lambda + g\varepsilon) = L(f) + \phi(g) \tag{13}$$

We can verify, easily, that this defines a unital representation of

$C_c(G, \overline{X})$. We extend ω to $C(G, \overline{X})$ in this way, and also to $M_c(G, \overline{X})$. Then we define $C^*(G, \overline{X})$ to be the operator norm closure of $\omega(C_c(G, \overline{X}))$ and

$M^*(G, X)$ to be the closure of $\omega(M_c(G,X))$.

For some computations we need another norm. Let $\mu \in \mathcal{Q}$, let $f \in M_c(G)$ and define

$$\|f\|_{II,\mu} = \sup\left\{ \int \ |f(\gamma)g \circ r(\gamma)h \circ s(\gamma)| \Delta_\mu(\gamma)^{-\frac{1}{2}} d\lambda^\mu(\gamma)\right\} \tag{14}$$

the supremum being taken over unit vectors g, h $\in L^2(\mu)$. Then define $\|f\|_{II}$ to be sup $\{\|f\|_{II,\mu}: \mu \in \mathcal{Q}\}$. Three facts about this norm should be mentioned. The first is that if π is a unitary representation of G, then $\|\pi^\mu (f)\| \le \|f\|_{II,\mu}$. Thus $\|\omega(f)\| \le \|f\|_{II}$, because $\|\omega(f)\| = \sup \{\|\pi^\mu(f)\|: \pi$ is a unitary representation.and $\mu \in \mathcal{Q}\}$. Next, if π is the one dimensional trivial representation and $f \ge 0$ then $\|\pi^\mu(f)\| = \|f\|_{II,\mu}$. It follows that if

$0 \le f \in M_c(G)$ then

$$\|\omega(f)\| = \| f \|_{II.} \tag{15}$$

A third fact is this : if $b \in L^\infty(\lambda^Q)$ and $f \in M_c(G)$ then for any $\mu \in Q$ we have

$$\| bf \|_{II,\mu} \leq \| b \|_\infty \| f \|_{II,\mu} \tag{16}$$

So

$$\| bf \|_{II} \leq \| b \|_\infty \| f \|_{II} \tag{17}$$

Lemma (1.1.5) [1]: if $0 \leq f \in M_c(G)$ and $b \in L^\infty(\lambda^Q)$, then $\| \omega(bf) \| \leq \| b \|_\infty \| \omega(f) \|$.

Proof:

Using the three properties of $\| \ \|_{II,\mu}$ mentioned just above, we have

$$\| \omega(bf) \| \leq sup \left\{ \| bf \|_{II,\mu} : \mu \in Q \right\}$$
$$\leq sup \left\{ \| b \|_\infty \| f \|_{II,\mu} : \mu \in Q \right\} \tag{18}$$
$$= \| b \|_\infty \| \omega(f) \|.$$

Section (1.2): Measure and Positive Definite Functions

A basic lemma is needed for our construction of positive definite functions from completely positive maps. After proving that lemma, we also need to prepare some detailed information about Haar systems on locally compact groupoids and how they relate to Borel Haar systems on the associated equivalence relations. Most of that information .

As preparation for the proof of the lemma in question, we recall a basic fact about measures and function spaces. Suppose that (X, B) is a set with algebra and that A is asubalgebra of B that generates B as algebra. Let μ be any finite measure defined on B. The measure of the symmetric difference between two sets is the same as the distance between their characteristic functions in $L^1(\mu)$, and hence provides a (pseudo) metric on B. The closure of A in B is σ- algebra that contains A and hence is B. For us, it is important that the fact of density is independent of μ. This implies similar properties for the set S(A), our notation for the set of linear combinations of characteristic functions of sets in A using coefficients from Q[i], which is Q with $\sqrt{-1}$ adjoined. By looking first at simple functions, it is easy to show that S(A) is always dense in $L^1(\mu)$. In the same way, we see that for any $f \in L^1(\mu)$,

$$\|f\|_1 = \sup\{ \left| \int f\varphi d\mu \right| : \varphi \in S(A) and \left| \varphi \right| \leq 1 \} \qquad (19)$$

which is a supremum indexed by a family independent of μ. When A can be taken to be countable, as is the case when X is a standard Borel space, these facts are particularly useful.

A similar situation arises if X is locally compact. In that case, there is a countable dense subset S(X) of $C_c(X)$ that is an algebra over Q[i], and any such S(X) is dense in $L^1(\mu)$ for every finite measure μ on X .

The next lemma is a generalization of the fact that for two measure spaces, functions on the product and functions from one measure space to the functions on the other can be identified. In our setting, the measure on

the image space must be allowed to vary.

Lemma (1.2.1) [1]: Let X and Y be standard Borel spaces and let $x \mapsto v^x$ be aBorel function from X to finite Borel measures on Y. Suppose that f is a function on X selecting an element $f(x)$ of $L^1(v^x)$ for each $x \in X$ so that the function $x \mapsto f(x) v^x \& x$ is Borel, taking values in the space of complex valued Borel measures. Then there is a Borel function F on X × Y such that for every $x \in X$ the function F(x , .) is integrable relative to v^x and in the class $f(x)$. The function F can be chosen so that if $f(x) \in L^\infty(v^x)$. then F(x, .) is bounded by $\|f(x)\|_\infty$. It is possible to choose F meeting those conditions and so that if $v^x = v^{x'}$ and $f(x) = f(x')$ then F(x , .) = F(x', .) (every where on y).

Proof:

For the proof we must have a way, that does not depend on x directly, to choose representatives of classes approximating $f(x)$. For this we choose first a countable algebra, A, of Borel sets in Y that generates the σ- algebra of Borel sets, so we can use the facts mentioned before the statement of the lemma. List S(A) as a sequence, s_1 , s_2 , ... For convenience, let us write $x \sim x'$ to mean that $v^x = v^{x'}$ and $f(x) = f(x')$, and say that such points are equivalent .

Now we are ready to describe the basic step which will be used repeatedly in the proof. If ε > 0 and $x \in X$ define j(x, ε) to be the least element of $\{i : \|f(x) - s_i\|_{l^1(v^x)} < \varepsilon\}$. It is clear that j(. , ε) takes the same valueat equivalent points of X, and we will show that j(.,ε) is Borel function. This will follow if we can show that for each bounded Borel function h on Y, $\{x : \|f(x) - h\|_{l^1(v^x)} < \varepsilon\}$ is a Borel set. We can get that from the fact that norms can be computed as, suprema, because for each φ ∈ S(A), $\int f(x) - h)$φ dv^x is a Borel function of x and hence so is its absolute value.

If we define $g(x) = s_{j(x, \varepsilon)}$ (as an element of $L^1(v^x)$) and $G(x, y) = s_{j(x, \varepsilon)}$ (y), then $g(x) = g(x')$ and $G(x,.) = G(x',.)$ (everywhere on Y) when ever $x \sim x'$. Also, both these functions are Borel.

Apply this process first to f with $\varepsilon = 2^{-1}$ to obtain G_1 and g_1. Then apply it to $f - g_1$ with $\varepsilon = 2^{-2}$ to obtain G_2 and g_2, etc. For each n the value of the function $f - (g_1 + \ldots + g_n)$ at a point x is an element of $L^1(v^x)$ having norm $< 2^{-n}$. Thus for $n \geq 2, \|g_n(x)\|_1 < 3(2^{-n})$. It follows that for each x the sum $\sum_{n \geq 1} |G_n(x, y)|$ is finite for almost all y. Inductively, we see that $G_n(x,.) = G_n(x',.)$ if $x \sim x'$. The set $N = \{(x,y) \in X \times Y: \sum_{n \geq 1} |G_n(x, y)| = \infty\}$ is a Borel set in $X \times Y$ and the slices of N over x and x' are the same set if $x \sim x'$. Now change each G_n to be 0 on N. Then the sum is always finite and we still have $G_n(x,.) = G_n(x', .)$ if $x \sim x'$.

Define $F(x,y) \sum_{n \geq 1} G_n(x,y)$ .Then F is Borel and satisfies the first and last conclusions of the theorem. Thus the slice of the Borel set $\{(x, y) : |F(x, y)| > \|f(x)\|_\infty\}$ over every point of X is of measure 0 and the slices of this set are the same over equivalent points of X. Change F to be on that set, and all the desired conditions are satisfied .

Now we are going to present some results on the fine structure of the Haar system, as developed by Renault. Renault decomposes the Haar system λ over a Borel Haar system α on R, by studying the action of G on a special group bundle, and we summarize the results here. Recall that the isotropy group bundle of G, denoted by G', is defined to be $\{\gamma \in G : r(\gamma) = s(\gamma)\} = \cup \{xGx: x \in X\}$. This closed in G and hence locally compact, so the space of closed subsets of G' is a compact space in the Fell topology. Let $\sum^{(0)}$ be the space of closed subgroups of the fibers in G', which is a closed subset of the space of closed subsets. Then the set $\sum = \{(H, \gamma) \in \sum^{(0)} \times G' | \gamma \in H\}$ is called the canonical group bundle of $\sum^{(0)}$. G acts on $\sum$ and on $\sum^{(0)}$ by conjugation: if $(H_1 , \gamma_1) \in \sum, \gamma \in G$, and $s(\gamma_1) = r(\gamma)$, then

$$(H_1, \gamma_1)\gamma = (\gamma^{-1} H_1 \gamma, \gamma^{-1} \gamma_1 \gamma) \tag{20}$$

while if H $\in \sum^{(0)}$, say H $\subseteq xGx$, and r(γ) = x, then H. $\gamma = \gamma^{-1}H\gamma$. We want to make a Borel choice of Haar measures on the groups xGx. One way to do this is to choose a continuous function F_0 on G that is non-negative, 1 at each $x \in$ X and has compact support on each xG. Then for each $x \in$ X choose a left Haar measure β^x on xGx so the integral of F_0 with respect to β^x is1. Likewise, choose a function F on $\sum$ that is non-negative, 1 at each point (H, e), and has support that intersects every {H} $\times$ H in a compact set, and make a similar choice of Haar system on $\sum$, $\beta^{\mathbf{H}}$.

Form the groupoid $\sum^{(0)} * G$ = {(H, γ) : s(H) = r(γ)} arising from the action of G on $\sum^{(0)}$. Then the essential uniqueness of Haar measures guarantees the existence of a 1-cocycle, δ, on $\sum^{(0)} * G$ so that for every (H, γ) $\in \sum^{(0)} * G$ we have

$$\gamma^{-1}\beta^H\gamma = \delta(H, \gamma)^{-1}\beta^{\gamma-1H\gamma} \qquad (21)$$

Renault proves that δ is continuous. The cohomology class of δ is determined by G, and Renault calls it the isotropy modulus function of G.

To shorten some formulas , we write G(x) for xGx. Renault defines $\delta(\gamma)$ = δ(G(r(γ)), γ) to get a 1-cocycle, also called δ, on G such that for every $x \in$ X, $\delta | xGx$ is the modular function for B^x . The preimage in $\sum^{(0)} * G$ of a compact set in G is compact, so δ and δ^{-1}= 1/δ are bounded on compact sets in G. Renault defines $\beta_y^x = \gamma\beta^y$ if $\gamma \in xGy$. If γ' is another element of xGy, then $\gamma^{-1}\gamma' \in$ yGy, and since β^y is a left Haar measure on yGy, it follows that β_y^x is independent of the choice of γ. With this apparatus in place, it is possible to describe a decomposition of the Haar system λ for G over the equivalence relation R={(r(γ),s(γ) $\gamma \in$ G}.This R is the image of G under the homomorphism θ (=(r,s)),so it is aσ- compact groupoid. Furthermore, there is unique Borel Haar system α for R with the property that for every $x \in$ X we have

$$\lambda^x = \int \beta_y^z \, d\alpha^x(z, y). \qquad (22)$$

Now suppose that $\mu \in Q$ so that we can form α^μ and λ^μ, getting quasisym metric measures. If $\underline{\Delta}=d\alpha^\mu/d(\alpha^\mu)^{-1})$ then $\delta\underline{\Delta} \circ \theta$ will serve as $d\lambda^\mu/d(\lambda^\mu)^{-1}$i.e., we can always take $\Delta_\mu=\delta\Delta_\mu \circ \theta$. We shall see that sometimes $\underline{\Delta}_\mu = 1$ so $\Delta\mu =\delta$.

For each x, the measure α^x is concentrated on $\{x\} \times \{x\}$ so there is a measure μ^x on $[x]$ such that $\alpha^x = \varepsilon^x \times \mu^x$, where ε^x is the unit point mass at $x \in X \subseteq G$. Since α is a Haar system, we have $\mu^x, = \mu^y$ if $x \sim y$, and the function $x \to \mu^x$, is Borel. If we take μ' to be the measure μ^z for some $z \in X$, then μ' is quasi-invariant. We give a different proof.
First of all,
$$\alpha^{\mu'} = \int \alpha^X d\mu'(x) = \mu^Z \times \mu^Z, \tag{23}$$
so $\alpha^{\mu'}$ is symmetric. Hence $\underline{\Delta}_{\mu'}=1$. Next we consider $\lambda^{\mu'} =$
$\int \int \beta_y^x d\mu^Z(x)d\mu^Z(y).$ Since
Gz is locally compact, there is a Borel function c: [z] $\to$Gz such that for every $x \in$ [z] we have c(x) $\in x$Gz. The value of c(z) can be taken to be z. We can use c to define a Borel isomorphism ψ: G $|$ [z]$\to$[z] $\times$ G(z) $\times$ [z] by
$$\psi\,(\gamma) = r(\gamma),c\big(r(\gamma)\big)^{-1}\gamma c\big(s(\gamma)\big), s(\gamma)) \tag{24}$$
By the uniqueness of Haar measure, as above, we see that ψ always carries β_y^x to a positive multiple of $\varepsilon^x \times \beta^z \times \varepsilon^y$, and that multiple is a Borel function of the pair $(x,\,y)$. Hence carry $\lambda^{\mu'}$to a measure equivalent to $\mu^z \times \beta^z \times \mu^z$. It follows $\lambda^{\mu'} is$ quasisymmetric, as needed .

Since λ is a Haar system, we know that if K is a compact set in G then the function $x\mapsto\lambda^x$(K) is bounded. We will use the formula for λ^xin terms of α^x to prove that $x\mapsto\alpha^x(\theta(K))$ is also bounded, and also that μ^x is finite on compact sets for the quotient topology on $[x]$. Let F be the function used above to make a choice of Haar measures β^μ. If S is the support of F, then $\beta^y\,(s) \geq 1$ for every y $\in$ X. To prove the boundedness statement above, let K be a compact subset of G and set K_1=K(s(K) S). Because both factors are

compact, so is K_1, so $x \mapsto \lambda^x(K_1)$ is bounded. For $(x, y) \in \theta(K)$, choose $\gamma \in K$ such that $\theta(\gamma) = (x, y)$. Then $\gamma S \subseteq K_1$, so $\beta_y^x(K_1) \geq 1$.

Hence

$$\lambda^x(k_1) = \int \beta_y^x(k_1) d\alpha^x(x, y)$$

$$\geq \int_{s(xK)} \beta_y^x(k_1) d\alpha^x(x, y) \tag{25}$$

$$\geq \alpha^x(\theta(K)$$

For the second assertion, suppose that C is a compact set in $[x]$ for the quotient topology. Since xG is locally compact and s is continuous and open from xG to $[x]$, there is a compact set K contained in xG whose image contains C. Then $\theta(K) \subseteq xR$, so the boundedness result just prove that $\mu^x(s(K)) = \alpha^x(\theta(K))$ is finite. Hence μ^x is a σ-finite.

It is also true that finiteness of μ^x on compact sets forces λ^x to be finite on compact sets, by an argument.

Define $M_{0c}(R)$ to be the space of bounded Borel functions on R that vanish off sets of the form $\theta(K)$, where K is a compact subset of G. Now we know that $M_{0c}(R) \subseteq I(R, \lambda)$, and it is not difficult to show that $M_{0c}(R)$ is a *-subalgebra of $I(G, \lambda)$. The definition of this algebra is admittedly somewhat unusual, but the algebra will serve a useful purpose in proving the main step along one way to prove the completeness of the Fourier-Stieltjes algebra of G. The point is that R is a kind of shadow of G, and we need a convoliotion algebra on it that is a shadow of the same kind.

We will characterize the functions on a locally compact groupoid that are diagonal matrix entries of unitary representations as the functions that are what we call positive definite. For this to be meaningful, we need a good definition of "positive definite." This is more complicated than for locally compact groups because unitary representations of locally compact groupoids can be Borel functions without being continuous. Thus we make our definition using integrals, and must even identify two functions that

agree λ^Q-almost everywhere, as defined. we will need to construct a positive definite function from a parametrized family of functions, each of which is positive definite on a transitive subroupoid. Thus we prove the representation theorem in that broader context. For a locally compact groupoid that has a Haar system, the notion of positive definite function can be defined in the least restrictive way as follows:

Definition (1.2.2) [1]: Let G be a locally compact groupoid and let λ be a left Haar system on G. Then a bounded Borel function p on G is called positive definite if for each $x \in X$ and each f in $C_c(G)$ we have

$$\int \int f(\gamma_1)\bar{f}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \geq 0 \qquad (26)$$

The set of all such p's will be denoted by p(G). We say that two elements of P(G) are equivalent iff they agree λ^Q-almost every where.

Lemma (1.2.3) [1]: Let π be a unitary representation of G on a Hilbert bund and let ξ be a bounded Borel section of H. Define $p(\gamma) = (\pi(\gamma) \xi \circ s(\gamma) | \xi \circ r(\gamma))$ for $\gamma \in G$. Then $p \in P(G)$.

Proof:

Fix $x \in X$ and $f \in C_c(G)$. Then for $\eta \in H(x)$

$$\left| \int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma) | \eta d\lambda^x(\gamma) \right| \leq \|f\|_1 \|\xi\|_\infty \|\eta\| \qquad (27)$$

so there is an element $\zeta(x) \in H(x)$ such that for all $\eta \in H(x)$ we have

$$\int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma)|\eta)d\lambda^x(\gamma) = (\zeta(x)|\eta). \qquad (28)$$

Indeed, this defines a Borel section, ζ, of H.The Borel character of ζ follows from the fact that $(\pi(\gamma) \xi_1 \circ s(\gamma) | \eta_1 \circ r(\gamma))$ is a Borel whenever ξ_1 and η_1 are Borel sections of H. For this section ζ the integral involved in the condition (P) is equal to $(\zeta(x) | \zeta(x))$, which is certainly non-negative.

Lemma (1.2.4) [1]: If $p \in P(G)$, the formula

$$(f | g)_x = \int \int f(\gamma_1)\bar{g}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \qquad (29)$$

defines a semi-inner product on $L^1(\lambda^x)$. Let H(x) denote the Hilbert space completion of the resulting inner product space. Then H is a Hilbert bundle

over X. For $\gamma_1 \in G$, define $\pi(\gamma_1)$ from $L^1(\lambda^{s(\gamma_1)})$ to $L^1(\lambda^{r(\gamma_1)})$ by $(\pi(\gamma_1) f)(\gamma) = f(\gamma_1^{-1}, \gamma)$. Then π determines a unitary representation, also denoted by π, on the bundle H.

Proof:

The form $(f \mid g)_x$ is clearly linear in f and conjugate linear in g. Since the vector space is complex the Hermitian symmetry follows from positive definiteness.

Let $N(x) = \{ f \in L^1(\lambda^x) : (f \mid f)_x = 0\}$ and set $F(x) = L^1(\lambda^x)/ N(x)$, the corresponding inner-product space. Write $H(x)$ for the completion of $F(x)$. Let $\mid \ \mid_x$ be the norm (or semi-norm) arising from $(\mid)_x$. For f, $g \in L^1(\lambda^x)$,

$\mid (f \mid g)_x \mid \leq \|p\|_\infty \|f\|_1 \|g\|_1$ so $\mid f \mid_x \leq \|p\|_\infty^{1/2} \|f\|$. it follow that the image of $C_c(xG)$, which is the image of $C_c(G)$, is dense in $H(x)$.

Now we want to make a Borel structure on the graph of H, denoted by $\Gamma = \Gamma_H = \{(x, \xi) : x \in X, \xi \in H(x)\}$. The process used is fairly standard. First, if $f \in C_c(G)$ and $x \in X$, define $\sigma(f)(x)$ to be the element of $H(x)$ represented by $f \mid xG$. This defines a section $\sigma(f)$ of the graph of H. We want all $\sigma(f)$'s to be Borel sections, and that tells us how to define the Borel structure. For $f \in C_c(G)$ define ψ_f on Γ by $\psi_f (x, \xi) = (\sigma(f)(x) \mid \xi)_x$. Then give Γ the smallest Borel structure relative to which the projection to X is Borel along with all the functions ψ_f ($f \in C_c(G)$). It follows from the fact that p is Borel and bounded that each section $\sigma(g)$ for $g \in C_c(G)$ is indeed a Borel section. Since G is second countable, there is a countable set dense in $C_c(G)$. For any countable dense set of f's, the ψ_f's would determine the same Borel structure as $\{\psi_f : f \in C_c(G)\}$, so the latter is standard: Apply the Gram-Schmidt process in a pointwise manner to a dense sequence of sections of the form $\sigma(f)$ to get a sequence g_1 , g_2 , . . . of Borel functions such that

(i) $g_n \mid xG$ is always in $L^1(\lambda^x)$

(ii) if $f \in C_c(G)$ and $n \geq 1$, then $x \rightarrow (\sigma(f) \mid \sigma(g_n))_x$ is aBorel function.

(iii) for each x the non-zero elements of $\{\sigma(g_n)(x): n \geq 1\})$ Form an orthonormal basis of H(x).

Then it is easy to show that Γ is isomorphic to the disjoint union of a sequence of product bundles $X_n \times K_n$, where $\{X_\infty, X_1 , X_2, ...\}$ is a Borel partition of X and each K_n is a Hilbert space of dimension n. Thus Γ is standard because each $X_n \times K_n$ is standard.

If $f \in L^1(\lambda^x)$ and $\gamma_1 : x \rightarrow y$ is in G, define $\pi(\gamma_1) f$ by $(\pi(\gamma_1)(f))(\gamma) = f(\gamma_1^{-1}\gamma)$ for $\gamma \in yG$. Since λ is left invariant, $\pi(\gamma_1)f \in L^1(\lambda^y)$. Notice that $\pi(\gamma_1^{-1})$ is the inverse of $\pi(\gamma_1)$. If g is another element of $L^1(\lambda^x)$, then

$$(\pi(\gamma_1)f \mid \pi(\gamma_1 g)_y = \int \int f(\gamma_1^{-1}\gamma_2)\bar{g}(\gamma_1^{-1}\gamma_3)p(\gamma_3^{-1}\gamma_2)d\lambda^y(\gamma_2)d\lambda^y(\gamma_3)$$

$$= \int \int f(\gamma_2)\bar{g}(\gamma_3)p(\gamma_3^{-1}\gamma_2)d\lambda^x(\gamma_2)d\lambda^x(\gamma_3) \qquad (30)$$

$$=(f \mid g)_x$$

Hence $\pi(\gamma_1)$ extends to a unitary operator from H(x) to H(γ), for which we use the same notation.

To work with the bundle and with the representation we need to restrict to subsets spaces where the various operation are defined. There are two fibered products, $\Gamma \times' \Gamma = \{(x,\xi, x,\xi') : x \in X, \xi, \xi' \in H(x)\}$, a subset of $\Gamma \times \Gamma$, and $G \times' \Gamma \subseteq G \times \Gamma$, defined to be $\{(\gamma, x, \xi) : s(\gamma) = x, \xi \in H(x)\}$. Let us show that $(\gamma,x,\xi)\mapsto(r(\gamma), \pi(\gamma)\xi)$ is Borel from $G \times' \Gamma$ to Γ. The composition of this map with the projection to X is clearly Borel.

Let $f \in C_c(G)$ and compose the map with ψ_f. The value of the composition at (γ, x, ξ) is $\psi_f (r(\gamma), \pi(\gamma) \xi) = (\pi(\gamma))^{-1}((\sigma(f)(r(\gamma)))|\xi)_x$. This is the value of another composition.

$$G \times' \Gamma \rightarrow \Gamma \times' \Gamma \rightarrow \mathbb{C},$$

where the first function takes (γ, x, ξ) to $(s(\gamma), \pi(\gamma^{-1})(\sigma(f)(r(\gamma)))$; (x, ξ) and the second is the inner product function. The first function is Borel if each component is, so let us see that the first component is a Borel function of γ.

composition of it with projection is s and hence Borel. If g $\in C_c$(G),

$$\psi_g\big(s(\gamma),\pi(\gamma^{-1})\big(\sigma(f)(r(\gamma))\big)\big) =$$

$$\int\int \bar{f}(\gamma\gamma_2)g(\gamma_1)p(\gamma_2^{-1}\gamma_1)\,d\lambda^{s(\gamma)}(\gamma_1)\,d\lambda^{s(\gamma)}(\gamma_2)$$

$$= \int_{G\times G\times G} \mathsf{fd}(\varepsilon^\gamma \times \lambda^{s(\gamma)} \times \lambda^{s(\gamma)}), \qquad (31)$$

Where F is the first function that is 0 at $(\gamma_0 , \gamma_1 , \gamma_2)$ unless $s(\gamma_0) = r(\gamma_1) = r(\gamma_2)$ and then its value is $\bar{f}(\gamma_0\gamma_1)$ $g(\gamma_2)$ $p(\gamma_2^{-1}\gamma_1)$. A fairly standard argument then shows that $\gamma\mapsto\psi_g(s(\gamma)),\pi(\gamma^{-1})(\sigma(f)(r(\gamma)))$ is Borel, as desired.

To show that the inner product is Borel on $\Gamma \times \Gamma'$, we use the functions g_n used to show that the bundle is standard. Indeed,

$$((x,\xi)\,|\,(x,\eta))_x = \sum_{n\geq 1} \psi_{gn}(x,\xi)\bar{\psi}_{gn}(x,\eta) \qquad (32)$$

which is a Borel function. It follows that $(\gamma, x, \xi)\mapsto\psi_f(r(\gamma), \pi(\gamma)\xi)$ is Borel, as needed .

This completes the construction of a unitary representation from a positive definite function. From now on, subscripts will be used on inner products and norms associated with such bundles only when necessary to make clear which space is involved. Our next task is to find a (cyclic) section ξ_p such that $p(\gamma) = (\pi(\gamma)\ \xi_p(s(\gamma))\ |\ \xi_p(r(\gamma)))$ for $\lambda^{\mathcal{Q}}$-almost every $\gamma \in G$.

The argument can be outlined as follows. Given a $\mu \in \mathcal{Q}$ we let $H(\mu)$ denote the Hilbert space of square integrable sections of H, which is some times written $L^2(\mu, H)$. There is no loss of generality in assuming that μ is a probability measure, since changing to an equivalent measure produces an equivalent representation. The representation π of G can be integrated to give a representation of $C_c(G)$ on $H(\mu)$, denoted by π_μ, using the formulati-on of Hahn, rather than that of Renault. The definition is given below. If u_1, u_2, ... is a symmetric approximate unit for $C_c(G)$, the sequence of sections $\sigma(u_1)$, $\sigma(u_2)$, . . . has a subsequence that converges weakly to a section ξ_μ such that for $f \in C_c(G)$ we have $\pi_\mu(f)\,\xi_\mu = \sigma(f)$, and the matrix entry made

from π and ξ_μ agrees with p a.e. relative to get a section p not depending on μ, we observe that if we had such a ξ_p, then for $f \in C_c(G)$ we would get $\int fpd\, \lambda^x = (\sigma(f)|\, \xi_p)_x$. Thus we consider the set D(p) of those $x \in X$ for which $\int fpd\, \lambda^x$, as a linear function of f in $C_c(G)$ "extends" to a bounded linear functional on H(x). We need to know that

D(p) is conull for every $\mu \in Q$, and this follows from the existence of ξ_μ .we let $\xi_p(x)$ be the vector representing that linear functionl, and verify that is the section we wanted.

Before giving details, we introduce the space $L^{1,2}(\lambda,\ \mu)$, consisting of those Borel functions f for which

$$\|f\|^2_{1,2} = \int \big(\int |f(\gamma)|\, d\lambda^x\, (\gamma)\big)^2\, d\mu(x) > \infty \tag{33}$$

Now, begin by taking H(μ) as defined above, and observe that for $f \in L^{1,2}(\lambda,\ \mu)$, the section $\sigma(f)$ is in H(μ), and $\|\sigma(f)\| \leq \|p\|^{1/2}_\infty \|f\|_{1,2}$ so that $f_n \to f$ in $L^{1,2}(\lambda,\ \mu)$ implies $\sigma(f_n) \to \sigma(f)$ in H(μ). To make the proof work, we must integrate the representation π to get π_μ having the property that for f, g $\in C_c(G)$, $\pi_\mu(f)(\sigma(g)) = \sigma(f * g)$. This can be done if we use the method, which applies to I(G, λ),which is a subspace of $L^{1,2}(\lambda,\ \mu)$, because μ is a probability measure.

For $f \in I(G,\ \lambda)$ we define $\pi_\mu(f)$ by saying that for sections ξ, η in H(μ) we have

$$(\pi_\mu(f)\xi\,|\,\gamma) = \int f(\gamma)\Big(\pi(\gamma)\xi(s(\gamma))\Big|\eta(r(\gamma))\Big)\, d\lambda^\mu(\gamma) \tag{34}$$

The integral defines a bounded sesquilinear form, so the formula defines a bounded operator $\pi_\mu(f)$. It is proved that π_μ is a bounded representation of I(G, λ). If $f \in I(G,\ \lambda)$ and $\xi \in$ H(μ), then $\pi_\mu(f)\, \xi$ is represented by a section whose value at almost every x is $\int f(\gamma)\ \pi(\gamma)\ \xi(s(\gamma))\ d\lambda^x(\gamma)$, where the integral is defined weakly. If g $\in$ I(G, λ), we have

$$(\pi_\mu(f)\sigma(g)\,|\,\eta = \int f(\gamma)\Big(\pi(\gamma)\Big(\sigma(g)(s(\gamma))\Big)\Big|\eta(r(\gamma))\Big)\, d\lambda^x\, (\gamma)$$

$$= \int \int f(\gamma) \left(\pi(\gamma) \left(\sigma(g)(s(\gamma)) \right) \Big| \eta(x) \right) d\lambda^x (\gamma) d\mu(x) \qquad (35)$$

$$= \int (\sigma(f * g)(r(\gamma)) \big| \eta(r(\gamma)) d\lambda^\mu (\gamma),$$

because $\pi(\gamma)(\sigma(g)(s(\gamma)))$ is represented by a function on $r(\gamma)$ G whose value at a point γ_1 is $g(\gamma^{-1}\gamma_1)$.

Lemma (1.2.5) [1]: Let G be a locally compact groupoid with a Haar system λ. Suppose that $\mu \in \mathcal{Q}$ is a probability measure. If p is a positive definite function on G and (H, π) is constructed from p as in the proof of Lemma (1.2.4), then there is a section $\xi_\mu \in H(\mu)$ such that

(i) $\left| \xi_\mu(x) \right|_x^2 \leq \| p \|_\infty$ for $x \in X$

(ii) $\pi_\mu(f)\xi_\mu = \sigma(f)$ for $f \in C_c(G)$

(iii) $p(\gamma) = (\pi(\gamma) \, \xi_\mu(s(\gamma)) \big| \, \xi_\mu(r(\gamma)))$ a.e $d\lambda^\mu(\gamma)$

Proof:

Let u_1, u_2, . . . be a symmetric approximate unit for G. Then $\left| \sigma(u_i)(x) \right| \leq \| p \|_\infty^{1/2}$ for each x and i, so $\| \sigma(u_i) \| \leq \| p \|_\infty^{1/2}$ for each i. Thus $\sigma(u_1), \sigma(u_2)$, . . . has subsequence converging weakly to avector $\xi_\mu \in H(\mu)$. We may suppose that subsequence is $\sigma(u_1)$, $\sigma(u_2)$, ... If, for every Borel set E in X, P(E) is the projection of H(μ) onto the subspace determined by sections that vanish off E, then $P(E)\sigma(u_n)$ converges weakly to P(E) which has norm at most $(\| p \|_\infty \mu(E))^{1/2}$. It it follows that $\left| \xi_\mu(x) \right|_x^2$ for a.e. x, and we can change ξ_μ to make it true for all x. For $f \in C_c(G)$, $f * u_i \rightarrow f$ uniformly and all these functions vanish off a fixed compact set. Thus $f * u_i \rightarrow f$ in $L^{1,2}$, and $\sigma(f * u_i) \rightarrow \sigma(f)$ in H(μ). Hence $\pi_\mu(f) \, \sigma(u_i)$ converges to $\sigma(f)$. we also know that $\pi_\mu(f)$ is bounded operator, so $\pi_\mu(f)\sigma(u_i)$ converges weakly to $\pi_\mu(f) \, \xi_\mu$. Hence $\pi_\mu(f) \, \xi_\mu = \sigma(f)$, as elements of H($\mu$).

It follows from this that if f, g $\in C_c(G)$, then $(\sigma(f) \big| \sigma(g)) = (\pi_\mu(f) \, \xi_\mu) \big| \pi_\mu(g) \, \xi_\mu)$ and this can be written as

$$\int\int\int (\pi(\gamma_1)\xi_\mu(s(\gamma_1)) \big| \pi(\gamma_2)\xi_\mu(s(\gamma_2)) f(\gamma_1)\bar{g}(\gamma_2) d\lambda^x(\gamma_1) d\lambda^x(\gamma_2) d\mu(x) \quad (36)$$

which is equal to

$$\iiint (\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1)) \mid \xi_\mu(s(\gamma_2)))f(\gamma_1)\bar{g}(\gamma_2)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2)d\mu(x) \quad (37)$$

If $h \in C_c(X)$ we can replace f in this calculation by $hf = (h \circ r) f$. From this it follows that if f, $g \in C_c(G)$ then for μ-almost every x we have $(\sigma(f) \mid \sigma(g))_x$

$$= \iint (\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1)) \mid \xi_\mu(s(\gamma_2)))f(\gamma_1)\bar{g}(\gamma_2)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \quad (38)$$

By the definition of $(\mid)_x$, this shows that for μ-almost every x,

$$p(\gamma_2^{-1}\gamma_1) = \left(\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1))\big|\xi_\mu(s(\gamma_2))\right) \quad (39)$$

is true for $\lambda^x \times \lambda^x$-almost all pairs ($\gamma_1$, γ_2). For each such x, for λ^x-almost every γ_2, the formula (39) is true for λ^x-almost ever γ_1, i.e., $p(\gamma)=$

$(\pi(\gamma)\xi_\mu\left(s(\gamma)\big|\xi_\mu(r(\gamma))\right)$ for $\lambda^{s(\gamma_2)}$-almost all γ. Indeed, the set $\{s(\gamma_2):(39)$

holds for λ^x-almost every $\gamma_1\}$ is conull in $[x]$.

Theorem (1.2.6) [1]: Let G be a locally compact groupoid and let λ be a Haar system on G. If p is a positive definite function on G and (H, π) is the associated unitary G-bundle over X, then there is a bounded section ξ_p of H such that if $\mu \in \mathcal{Q}$, then

(i) $p(\gamma) = (\pi(\gamma) \xi_p(s(\gamma)) \mid \xi_p(r(\gamma)))$ a.e. $d\lambda^\mu(\gamma)$

(ii) if $f \in C_c(G)$, then $\pi_\mu(f) \xi_p = \sigma(f)$ in $H(\mu)$.

If p is continuous, then ξ_p can be chosen to be continuous and $p(\gamma) = (\pi(\gamma) \xi_p (s(\gamma)) \mid \xi_p(r(\gamma)))$ for all γ.

Proof:

Define $D = D(p) = \{x \in X : f\mapsto\int fp\, d\lambda^x = \lambda^x(fp)$ extends from $C_c(G)$ to give a bounded linear functional on $H(x)$ of norm at most $\|p\|_\infty^{1/2}\}$. For each $f \in C_c(G)$, $\lambda(fp)$ and $x\mapsto (f \mid f)_x$ are Borel functions, and boundedness can be tested on a countable dense set, so D is a Borel set. For $x \in D$, there is a unique vector $\xi_p(x) \in H(x)$ such that $(\sigma(f)(x) \mid \xi_p(x))_x = \lambda^x(fp)$ for $f \in C_c(G)$,

and if we let $\xi_p(x) = 0$ for $x \notin D$, ξ_p is a Borel section of H, bounded by $\|p\|_\infty^{1/2}$. We need to show that D is Q-conull, i.e., conull for each $\mu \in Q$.

Let $\mu \in Q$. Then there is a $\xi_\mu \in H(\mu)$ satisfying (i), (ii), (iii) of Lemma(1.2.5) and thus for each $f \in C_c(G)$ we have

$$(\sigma(f)(x) \mid \xi_\mu(x))_x = ((\pi_\mu(f)\xi_\mu(x) \mid \xi_\mu(x))_x = \lambda^x(fp) \qquad (40)$$

for μ-a.e. x. Since bounded linear functionals are determined by their values on a countable dense set, and since boundedness of a linear functional canbe tested on a countable dense set, there is a μ-conull set D_μ such that for $x \in D_\mu$ and $f \in C_c(G)$,

$$(\sigma(f)(x) \mid \xi_\mu(x))_x = \lambda^x(fp) \qquad (41)$$

Thus $D_\mu \subseteq D$, from which it follows that D is μ-conull and $\xi_p(x) = \xi_\mu(x)$ a.e. This fact and Lemma (1.2.4) combine to establish the truth of statement (i) and (ii) in the theorem. By the definitions of D and ξ_p , it follows that p is bounded by $\|p\|_\infty^{1/2}$.

To complete the proof, we show first that if p is continuous, then D =X. Again take a $\mu \in Q$ and the section ξ_μ . We have $\lambda^x(fp) = (\sigma(f)(x) \mid \xi_\mu(x))_x$ for μ-a.e. x, and for such x's,

$$\left| \lambda^x(fp) \right| \leq \left| \sigma(f)(x) \right|_x \|\xi_\mu(x)\|$$
$$\leq \left| \sigma(f)(x) \right|_x \|p\|_\infty^{1/2}. \qquad (42)$$

Since p is continuous, both $\lambda(fp)$ and $x \mapsto (\sigma(f), \sigma(f))_x$ are continuous, so this estimate holds on the support of μ. The supports of the μ's in Q fill X, So

$$\left| \lambda^x(fp) \right| \leq \left| \sigma(f)(x) \right| \|p\|_\infty^{1/2} \qquad (43)$$

for all f in $C_c(G)$ and all x. Thus D = X. Now $p(\gamma) = (\pi(\gamma)\,\xi_p(s(\gamma)) \mid \xi_p(r(\gamma)))$ a.e. $d\lambda^\mu(\gamma)$ for every μ, so it will end the proof if we can show that the second function is continuous. By a partition of unity argument, this wil follow if we can show that $(\pi(\gamma)\,\xi(s(\gamma)) \mid \xi(r(\gamma)))$ is a continuous function of γ for every continuous section ξ of compact support. We can reduce to

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considering $\xi = \sigma(f)$ for $f \in C_c(G)$, by using partitions of unity and uniform limits. Then we have

$$\left(\pi(\gamma)\xi(s(\gamma))\big|\xi(r(\gamma))\right) =$$

$$\iint f(\gamma^{-1}\gamma_1)\bar{f}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^{r(\gamma)}(\gamma_1)d\lambda^{r(\gamma)}(\gamma_2) \quad (44)$$

Continuity of this function of γ can be deduced by applying the following easy lemma and a variant of it using the second coordinate projection, because the integrands can be extended to functions satisfying the hypotheses of the lemma.

Lemma (1.2.7) [1]: Suppose G is a locally compact groupoid with a Haar system λ and let F be a continuous complex valued function on G $\times$ G. Let p_1 : G $\times$ G$\to$G be the first coordinate projection. Suppose that for every compact set C $\subseteq$ G the set $p_1^{-1}(C) \cap \sup(F)$ is compact.Then, $\int F(\gamma,\gamma_2) d\lambda^{r(\gamma)}(\gamma_2)$ is a continuous function of γ.

We have an existence theorem, but we should show that the results are essentially the same for any two equivalent elements of P(G).

Theorem (1.2.8) [1]: Suppose that p, q $\in$ P(G) and that p=q λ^ϱ-a.e. Then the associated representations (H$_p$, π_p) and (H$_q$, π_q) are the same, and the sections ξ_p and ξ_q agree Q-a.e.

Proof:

Let z $\in$ X and consider the inner products on $L^1(\lambda^z)$ defined using p and q. Denote them by $(\ |\ )_p$ and $(\ |\ )_q$ respectively. To prove they are the same, it will suffice to show that $p(\gamma_2^{-1}\gamma_1) = q(\gamma_2^{-1}\gamma_1)$ for $\lambda^z \times \lambda^z$-almost every pair $(\gamma_1$, $\gamma_2)$, because the inner products are defined by double integrals using these functions and measures.

Let μ_1^z be a quasiin variant probability measure equivalent to the measure μ^z that was associated with the orbit [z]. Let E be the set of $x \in X$ for which p = q a.e. relative to λ^x. Then $\mu^z(E) =1$, so $\{\gamma: s(\gamma) \in E\} = GE$ is λ^z-conull. If $\gamma \in zGE, \{\gamma_1 \in zG : p(\gamma^{-1}\gamma_1) = q(\gamma^{-1}\gamma_1)\}$ is conull relative to λ^z by

translation invariance of the Haar system. By the Fubini Theorem, we get the desired agreement a.e.

This shows that the Hilbert bundles H_p and H_q are identical, and since the formula for the representation is just left translation in each case, the representations are the same.

To show that the sections ξ_p and ξ_q agree Q-a.e., we resort to the definitions, namely, $\xi_p(x)$ and $\xi_q(x)$ are determined by the fact that for

$f \in C_c(G)$.

$$(\sigma (f) \mid \xi_p(x)) = \lambda^x(fp)$$

and

$$(\sigma (f) \mid \xi_q(x)) = \lambda^x(fq)$$

Let F be the set of $x \in X$ for which $\xi_p(x) = \xi_q(x)$. Since the two sections are Borel, F is a Borel set. We need to show that if $\mu \in Q$, then $\mu(X \backslash F) = 0$. We know that for each $f \in C_c(G)$ the two functions $\lambda(fp)$ and $\lambda(fq)$ agree almost every where relative to μ. Let C be a countable dense set in $C_c(G)$, and let N be a μ-null set such that $x \notin N$ and $f \in C$ imply $\lambda^x(fp) = \lambda^x(fq)$. Since p and q are bounded, this equality is preserved under limits in $C_c(G)$,so it holds for $x \notin N$ and all $f \in C_c(G)$. Thus F contains the complement of N, as desired.

Theorem (1.2.9) [1]: Sums and products of positive definite functions are positive definite.

Proof:

This is immediate from the existence of direct sums and tensor products (let us first consider aspecial case: let us say V,W are free vector spaces for the sets S,T respectively. That is, V= F(S), W=F(T). In this special case, the tensor product is defined as F(S) $\otimes$ F(T) = F(S×T). In most typical cases, any vector spase can be immediately understood as the free vector spase for some set, so this definition suffices. However, there is also an explicit wa of constructing the tensor product directly from V,W , without appeal to S,T.

In general, given two vector spaces V and W over a field K, the tensor product U of V and W,denoted as U =V⊗W is defined as the vector space whose elements and operations are constructed) **[5]** of representations, because of Theorem (1.2.6).

We consider the enlarging the space from which we construct that fibers of the Hilbert bundle H_p using the positive definite function p. It will be convenient to replace the algebra $C_c(G)$ by the larger algebra $C_c(G,\overline{X})$,an algebra of kernels introduced, and we will need to know that using the latter in our construction does not change the fibers in that bundle.

Definition (1.2.10) [1]: Let G be a locally compact groupoid and let λ be a left Haar system on G. Then a bounded Borel function p on G is called strictly positive definite if for each $x \in X$ and each v in $C_c(G, \overline{X})$ we have

$$\iint p(\gamma_2^{-1}\gamma_1)dv^x(\gamma_1)dv^{-x}(\gamma_2) \geq 0. \tag{45}$$

The set of all such p's will be denoted by P'(G). Two functions p, q ∈ P'(G) will be called equivalent iff they agree λ^Q-almost everywhere on G and their restrictions to X agree Q-almost everywhere.

We have P'(G) ⊆ P(G), and would like to know that the sets are equal. This is not true because a function p can satisfy condition (P) and be negative everywhere on X unless there is a μ ∈ $\mathcal{Q}$ such that $\lambda^\mu(X) > 0$. Actions by non-discrete groups give rise to groupoids for which $\mathcal{Q}$ contains no such μ . However, we have proved that every equivalence class in P(G) contains a diagonal matrix entry. Thus a kind of reverse of the containment would follow from the analog of Lemma (1.2.3) namely Lemma (1.2.12) below showing that diagonal matrix entries are in p'(G). This meaning of the reverse containment would be that every class in P(G) contains an element of P'(G), or that diagonal matrix entries are in P'(G).

If two diagonal matrix entries are equivalent in P(G), are they equivalent in P'(G)π? The affirmative answer is given in Lemma (1.2.14).

Lemma (1.2.11) [1]: Let π be a unitary representation of G on the Hilbert bundle K, and let ξ be a bounded Borel section of K. Define $p(\gamma) = (\pi(\gamma)\xi \circ s(\gamma) \mid \xi \circ r(\gamma))$ for $\gamma \in$ G.

Then $p \in P'(G)$.

Proof:

As in the proof of Lemma (1.2.3), for $f \in C_c(G)$, there is a section of the bundle such that for each $x \in X$ and every $\eta \in K(x)$ we have

$$\int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma) \mid \eta) d\varepsilon^x(\gamma)$$

$$= (\zeta(x) \mid \eta \tag{46}$$

If $g \in C(\overline{X})$, $x \in X$, and $\eta \in K(x)$, then

$$\int g(\gamma)(\pi(\gamma)\xi \circ s(\gamma) \mid \eta) d\varepsilon^x(\gamma) = g(x)(\xi(x) \mid \eta \tag{47}$$

If $v = \lambda f + g\varepsilon$, these show that the integral involved in the condition (p') is equal to$(\zeta(x) + g(x)\xi(x) \mid \zeta(x) + g(x)\xi(x))$, which is certainly non-negative.

Corollary (1.2.12) [1]: Every equivalence class in P(G) contains an element of p'(G).

Lemma (1.2.13) [1]: Let (H, π) be a representation of G, let u_1 , u_2, ... be a symmetric approximate ,as described in Theorem (1.1.4), and let ξ be a bounded Borel section of H. Suppose that $\mu \in Q$, and let π_μ be the integrated form of π as defined just before the statement of Lemma (1.2.4).Then $\pi_\mu(u_n) \to \xi$ as $n \to \infty$.

Proof:

By construction of the functions u_n, $\|u_n\|_I \leq 1$ for each n, so every $\pi_\mu(u_n)$ has norm at most 1. Hence it suffices to find a dense set of vectors satisfying the conclusion. Each vector of the form $\pi_\mu(g)\,\eta$ satisfies the conclusion, and hence vectors in the linear span of the set of such vectors do also. That linear span is dense.

Lemma (1.2.14) [1]: Suppose that π and π_1 are representations of G on bundles K and K_1 , and that ξ and ξ_1 are bounded Borel sections of K and

K_1 . If $\pi(\gamma)\xi \circ s(\gamma) \,|\, \xi \circ r(\gamma)) = (\pi_1(\gamma)\xi_1 \circ s(\gamma) \,|\, \xi_1 \circ r(\gamma))$ for almost every γ relative to λ^{ϱ}, then $(\xi(x)|\xi(x)) = (\xi_1(x)|\xi_1(x))$ for almost every $x \in X$ relative to Q.

Proof:

Since ξ and ξ_1 are Borel sections the set E of $x \in X$ for which $(\xi(x)|\xi\ (x))$ $=\xi_1(x)(\xi_1(x)$ is a Borel set. We need to prove that for $\mu \in Q$, $\mu(E)=1$. The hypothesis implies that for $f \in M_c(G)$,we have $\pi_\mu(f)\,\xi\,|\,\xi) = (\pi_{1,\mu}(f)\,\xi_1\,|\,\xi_1)$, these being inner products associated with the integrated forms using Hahn's method (c f. Lemma (1.2.5), and the paragraph before it). Let φ and φ_1 be the representations of $C(\overline{X})$ by multiplication on the sections of K and K_1 .Then it follows from the discussion following the statement of Theorem (1.1.4) that for h $\in C(\overline{X})$ and $f \in C_c(G)$.

$$(\varphi(h)\pi_\mu(f)\xi \,|\, \xi = (\varphi(h)\pi_{1,\mu}(f)\xi_1 \,|\, \xi_1 \tag{48}$$

Now for the $f \in C_c(G)$ take the terms of a symmetric approximate unit, to see that for all h $\in C(\overline{X})$,

$$(\varphi(h)\xi \,|\, \xi = (\varphi_1)(h)\xi_1 \,|\, \xi_1) \tag{49}$$

This means that for all h $\in C(\overline{X})$

$$\int h(x)\big(\xi(x)|\xi(x)\big)d\mu(x) = \int h(x)(\xi_1(x) \,|\, \xi_1\ (x)d\mu(x). \tag{50}$$

Thus E is indeed μ-conull.

After this, we will always take elements of p(G) or p'(G) to be diagonal matrix entries, and understand that they are determined a.e.on X as well as on G.

Chapter 2

Groupoids and Theoretical Measure

This chapter is a continuation of chapter 1. For groups, B(G) is isometric to the Banach space dual of C*(G). For groupoids, the best analog of that fact is to be found in a representation of B(G) as a Banach space of completely bounded maps from a C*- algebra associated with G to a C*-algebra associated with the equivalence relation induced by G.

Section (2.1): Complete Positivity

We introduce second and third ways to view elements of P(G), namely in terms of completely positive mappings.Theorem(2.1.1) is a first step toward getting Banach algebras of completely bounded maps on M*(G) and on C*(G) we obtained C*(G) by completing C_c(G), and defined ω to be the direct sum of the (cyclic) representations C*(G) that arise from normalized positive linear functionals on C*(G). Let H_ω be the Hilbert space of ω. By a theorem of Renault, stated, each representation of C_c(G) can be gotten by integrating a unitary representation of G. Thus ω │ C_c(G) is also a direct sum of certain repress of certain representations π^μ. The process of integration allows us to regard each π^μ, and hence ω, as a representation of either M_c(G) or C_c(G). We call ω the universal representation of G. We also defined M*(G) to be the operator norm closure of ω(M_c(G)), and notice that C*(G) is isomorphic to the norm closure of ω(C_c(G)). If G is a group, of course M*(G) = C*(G), but these two algebras can be different for groupoids.

Theorem (2.1.1) [1]: Let p be a positive definite function on G. Let ω be the universal representation of G, and define $T_p(\omega(f)) = \omega(pf)$ for $f \in$ M_c(G). ThenT_p extends to a completely positive map of M*(G) to M*(G) with completely

bounded norm equal to the Q-essential supremum of { $p(x) : x \in$ X}

Proof: (We modify a proof for groupoids). We remind the reader that this Q-essential supremum is the infimum of {B: if $\mu \in Q$, then $p \leq \mu$-a.e.}. Also, in working with ω we will use its construction as a direct sum.

We will need to find a formula for T_p, in order to prove that the mapping is completely positive. For this we begin with two vectors ξ, η in one summand of H_ω given by an integrated representation π^μ. This means that we begin with a measure $\mu \in Q$ and a Hilbert bundle K over X. The subspace of H_ω in question is $L^2(\mu;K)$, and the restriction of ω to this subspace is the integrated form of a representation, π, of G. We are using Renault's form here, as described: take $v = \int \lambda^x d\mu(x)$ and $v_0 = \Delta_\mu^{-1/2}v$. Then for $f \in M_c(G)$.

$$T_p\omega(f) \xi| \eta) = (\omega(pf \xi| \eta)$$
$$= \int P(\gamma)\ f(\gamma)(\pi(\gamma)\ \xi(s(\gamma)) \mid \eta(r(\gamma)))\ dv_0(\gamma)$$
$$= (\pi^\mu(Pf)\ \xi| \eta) \tag{1}$$

By Theorem (1.2.5) there are a Hilbert bundle K_p on X, a (unitary) Borel representation π_p of G on K_p and a bounded Borel section ξ_p of K_p such that $p(\gamma) = (\pi_p(\gamma)\ \xi_p \circ s(\gamma) \mid \xi_p \circ r(\gamma))$ for λ^μ-a.e. $\gamma \in G$. By Theorem (1.2.8), π_p is unique, and the section ξ_p is determined Q-a.e. Thus we can continue the calculation from above as follows:

$$= \int f(\gamma) \left(\pi_p(\gamma)\xi_p \circ s(\gamma) \middle| \xi_p \circ r(\gamma)\right)\left(\pi(\gamma)\xi \circ s(\gamma)| \eta \circ r(\gamma)\right)dv_0(\gamma)$$

$$= \int f(\gamma)((\pi_p \otimes \pi(\gamma))(\xi_p \otimes \xi) \circ s(\gamma) \mid (\xi_p \otimes \eta) \circ r(\gamma))dv_0(\gamma) \tag{2}$$

$$= \left((\pi_p \otimes \pi)(f)(\xi_p \otimes \xi) \middle| \xi_p \otimes \eta\right)$$

$$= ((\pi_p \otimes \omega)(f)(\xi_p \otimes \xi) \mid \xi_p \otimes \eta)$$

Here $\xi_p \otimes \xi$ and $\xi_p \otimes \eta$ are in $L^2(\mu; K_p \otimes K)$. In summary we have

$$\left(T_p\omega(f)\xi| \eta\right) = ((\xi_p \otimes \omega)(f)V_{p,\mu,K}\xi \mid V_{p,\mu,K}\eta, \tag{3}$$

where $V_{p, \mu, K}: L^2(\mu;K) \to L^2(\mu;K_p \otimes K)$ is defined by $V_{p, \mu, K}\xi = \xi_p \otimes \xi$. This is a bounded operator because the section ξ_p is bounded and the usual

techniques for multiplication operators apply. If we let V_p be the direct sum of the operators $V_{p, \mu, K}$ over all pairs we have $T_p\omega(f) = V_p^*(\pi_p \otimes \omega)(f) V_p$. A theorem of Stinespring shows that T_p is completely positive with completely bounded norm equal to $\|V_p\|^2$. But V_p is given by a tensor multiplication which behaves like a scalar multiplication operator, so

$$\|V_p\|^2 = \text{ess sup}\{\|\xi_p(x)\|^2 : x \in X\} = \text{ess sup}\{p(x) : x \in X\} \qquad (4)$$

The proof of Theorem(2.1.1) also proves this :

Theorem (2.1.2) [1]: Let p be a positive definite function on G, let $\mu \in Q$ and let π be a representation of G. Define $T'_p(\pi^\mu(f)) = \pi^\mu(pf)$ for $f \in M_c(G)$. Then T'_p extends to a completely positive map of the norm (to closure of $\pi^\mu(M_c(G))$ to itself, this being the quotient of the T_p defined in Theorem (2.1.1)The completely bounded norm of T_p as an operator on $cl(\pi^\mu(M_c(G)))$ is the μ-essential supremum of $\{p(x) : x \in X\}$.

Although the norm on the Fourier−Stieltjes algebra of a groupoid comes from its representation by completely bounded maps rather than as the Banach space dual of the C*-algebra as it does for groups, the latter fact has a remnant. Here we prove just one lemma regarding that remnant.

Lemma (2.1.3) [1]: Let p be a positive definite function on G, and let μ be aprobability measure in Q. Define $\psi_{p,\mu}(\omega f(\gamma)) = \int f(\gamma)p(\gamma)d\nu(\gamma)$ for $f \in C_c(G)$, where $\nu = \int \lambda^x d\mu(x)$ Then $\psi_{p,\mu}$ extends to a positive linear functional on C*(G) whose norm is at most the Q-essential supremum of p.

Proof: From the definition of π_μ, it follows that the integral in question is equal to $(\pi_\mu(f) \xi \mid \xi)$, where π is the unitary representation of G derived from p and ξ is the associated section of the Hilbert bundle.Thus this linear functional is clearly positive, and its norm is at most $\|\xi\|^2$, the square of the norm of ξ in H(μ), but this is at most $\|\xi\|_\infty^2$ which is the Q-essential supremum of p.

Next we present a third way to think about P(G). It depends the decomposition described of the Haar system of G over the equivalence

relation R associated to G. This decomposition is relative to the mapping $\theta = (r, s)$ of G onto R. Since G is σ-compact it follows that R is σ-compact in the quotient topology. The decomposition of the Haar system involves two families of measures. First of all there is a measure β_y^x concentrated on xGy for every pair $(x,$ y) in R, such that each β_y^y is aHaar measure on yGy and β_y^xs a translate of β_y^y. Then there is a Borel Haar system for R so that for every $x \in$ X we have

$$\lambda^x = \int \beta_y^z \, d\alpha^x(z, y) \tag{5}$$

There is a Borel homomorphism δ from G to the positive reals such that for every $\mu \in Q$ the modular homomorphisms Δ_μ for G and $\tilde{\Delta}\mu$ for R satisfy $\Delta_\mu = \delta\, \tilde{\Delta}_\mu \circ \theta$. For each $x \in$ X let μ^x be the measure on X so that $\alpha^x = \varepsilon^x \times \mu^x$.

Then $x \sim$ y implies $\mu^x = \mu^y$. Thus $\alpha^{\mu^x} = \mu^x \times \mu^x$, so $\tilde{\Delta}_{\mu x} = 1$.

Let $\mathbf{M_{oc}}$ (R) be the space of bounded Borel functions on R supported on images under θ of compact subsets of G. Then $\mathbf{M_{oc}}$ (R) is a ∗-algebra under convolution, using the Borel Haar system α .We also extend this algebra to include M(X), as done for $\mathbf{M_c}$(G) and M(X), obtaining $\mathbf{M_{oc}}$ (R, X) in this case.

If μ is a quasi-invariant measure on X, i.e.,$\mu \in Q$, earlier we introduced the notation λ^μ for d$\mu(x)$ and we define α^μ similarly. Now we want to shorten the notation, so we write $v = \lambda^\mu, \tilde{v} = \alpha^\mu, \Delta = \Delta_\mu$, and $\tilde{\Delta} = \tilde{\Delta}_\mu$.

To integrate a unitary representation of G relative to μ to make a*-representation of $\mathbf{M_c}$(G,X), we use the measure $v_0 = \Delta^{-1/2} v$ and to integrate a representation of R we use the measure $\tilde{v}_0 = \tilde{\Delta}^{-1/2} \tilde{v}$ For example, in the first case we have

$$\left(\pi^\mu(f)\xi\middle|\eta = \int f(\gamma)(\pi(\gamma)\,\xi \circ r(\gamma)\middle|\eta \circ s(\gamma))dv_0(\gamma)\right. \tag{6}$$

Whenever $f \in \mathbf{M_c}$ (G) and ξ,η are L^2 sections of the bundle on which π represents G. This is the formulation. From what we have abov it follows

that $v_0 = \int \delta^{-1/2} \beta_y^x d\tilde{v}_0(x,y)$, so there is a convenient relationship between the two measures.

For each unitary representation π of R, and each $\mu \in Q(R)$, we can ask whether the representation π^μ is cyclic, and we can define $\tilde{\omega}$ to be a direct sum formed using for summands one representative from each equivalence class of a cyclic π^μ. Then we can write M*(R) for the norm closure of $M_{oc}(R)$). These $\pi^{\mu\prime}$s extend to $M_{oc}(R, X)$, so $\tilde{\omega}$ does also, and we let M*(R, X) be the norm closure of $\tilde{\omega}$ ($M_{0c}(R, X)$). As stated before, the algebra M*(R, X) is present only for its utility in proving results about G, and the slightly strange definition is just suited to that purpose.

If $p \in P(G)$, we define a pairing of p with an element $f \in M_c(G)$ to give a function on R by

$$\langle f, p \rangle(x, y) = \int fp\delta^{-1/2} \, d\beta_y^x \tag{7}$$

Since p and $\delta^{-1/2}$ are Borel functions and bounded on compact sets we always have $\langle f, p \rangle \in M_{0c}(R)$. We must show that this mapping determined by the equivalence class of p. If $p = p'$ a.e. relative to λ^Q, then for α^μ-almost every pair (x, y) the functions p and p' agree a.e with respect to β_y^x so for every $f \in M_c(G)$ we have $\langle f, p \rangle = \langle f, p' \rangle$ a.e with respect to α^μ. Furthermore, we represent p and p' as matrix entries, and these have restrictions to X that agree a.e. with respect to Q. We will show that the mapping of f to (f, p) gives rise to a completely positive map S_p from M*(G) to M*(R).

There is another property of S_p we use, and its statement requires a little background. That $C_c(G)$ and $M_c(G)$ are bimodules over $C(\overline{X})$, where $h \in C(\overline{X})$ acts via multiplication by $h \circ r$ and $h \circ s$. Recall also that every representation π of the $*$-algebra $C_c(G)$ has an associated representation φ of $C_c(X)$ such that $\pi(hf) = \varphi(h) \pi(f)$ and $\pi(fh) = \pi(f) \varphi(h)$ for all f and h, i.e., so that π is a bimodule map. Hence every representation of $M_c(G)$ also has such an associated representation of $C_c(X)$. We can extend φ to M(X),

getting a representation that preserves monotone limits and hence maintaining the bimodule property.

We notice that M(X) also has natural actions defined the same way on $M_{oc}(R)$ and by pointwise multiplication on each $L^2(\mu;K)$, rendering $\tilde{\omega}$ a bimodule map from $M_{oc}(R)$ to $M^*(R)$. The main properties of S_p are established in the next theorem.

Theorem (2.1.4) [1]: If p $\in$ P(G), there is a completely positive operator $S_p : M^*(G) \to M^*(R)$ that extends the operator defined by $S_p(\omega(f)) = \tilde{\omega}(\langle f, p \rangle))$ for $f \in M_c(G)$. This mapping is an M(X)-bimodule map. If we define $S_p(\omega(g\varepsilon)) = \tilde{\omega}(pg\varepsilon)$ for g $\in$ M(X) and use linearity, we get an extension of the original S_p to a completely positive M(X)-bimodule map of $M^*(G,\overline{X})$ to $M^*(R,\overline{X})$ that takes $\omega(\varepsilon)$ to an element of $\tilde{\omega}(M(X))$. The completely bounded norm of S_p is equal to $\|p\|_\infty$.

Proof:

We need another formula for S_p , first on $M_c(G , \overline{X})$. To find one, we first work with a subrepresentation of $\tilde{\omega}$ acting on a space of the form $L^2(\mu ; K)$

The positive definite function p determines a unitary representation π_p of G on a Hilbert bundle K_p over X, as well as a bounded section ξ_p of K_p for which we have $p(\gamma) = (\pi_p(\gamma) \xi_p \circ s(\gamma) \mid \xi_p \circ r(\gamma))$ for almost all γ relative to λ^ϱ. Then we may replace p by the matrix entry. Indeed, we must make that replacement in order to make sense of the values of p on X. Suppose that ξ and η are in $L^2(\mu, K)$, and compute

$$(S_p(\omega(f))\xi \mid \eta$$

$$= (\tilde{\omega}(\langle f, p \rangle)\xi \mid \eta \tag{8}$$

$$= \int \langle f, p \rangle(x, y)(\tilde{\omega}(x, y)\xi(y) \mid \eta(x) d\tilde{v}_0(x, y)$$

$$= \iint f(\gamma)p(\gamma)\delta(\gamma)^{-1/2} (\tilde{\omega} \circ \theta(\gamma)\xi(y) \mid \eta(x)) d\beta_y^x(\gamma) d\tilde{v}_0(x, y)$$

$$= \int f(\gamma)(\pi_p(\gamma)\xi_p \circ s(\gamma) \mid \xi_p \circ r(\gamma)(\tilde{\omega} \circ \theta(\gamma)\xi \circ s(\gamma) \mid \eta \circ r(\gamma))dv_0(\gamma)$$

$$= \left(\left((\pi_p \otimes \tilde{\omega} \circ \theta)(f)\right)\xi_p \otimes \xi \mid \xi_p \otimes \eta\right).$$

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We also have

$$(\tilde{\omega}(pg\varepsilon)\xi|\eta) = (pg\xi|\eta) \qquad (9)$$

$$= \int \xi_p(x) |\xi_p(x)g(x)(\xi(x)|\eta(x))d\mu(x)$$

$$= ((\pi_p \otimes \tilde{\omega} \circ \theta)(g\varepsilon)\xi_p \otimes \xi | \xi_p \otimes \eta$$

Now define $V_{p,\,\mu,\,K}$: $L^2(\mu;K) \rightarrow L^2(\mu;K_{\mathbf{p}} \otimes K)$ by $V_{p,\mu,K}\xi = \xi_p \otimes \xi$ and let V be the direct sum of all the operators $V_{p,\,\mu,\,K}$. The calculations just done show that for all $f \in M_c(G)$ and $g \in M(X)$ we have

$$S_p\big(\omega(f\lambda + g\varepsilon)\big) = V^*\Big((\pi_p \otimes \tilde{\omega} \circ \theta)(f\lambda + g\varepsilon)\Big)V. \qquad (10)$$

Since $\pi_{\mathbf{b}} \otimes \tilde{\omega} \circ \theta$ is a $*$-representation, Stinespring's Theorem shows that S_p is completely positive. This representation also gives a formula for the extension of $S_{\mathbf{p}}$ to $M^*(G, X)$ and shows that it is an extension by continuity. It is not difficult to show that the norm of S_p is the essential supremum norm of $\xi_{\mathbf{p}}$, and that is the same as $\|p\|_\infty$.

From the definition of V we see that it intertwines the natural of $M(X)$ on $L^2(\mu;K)$ and $L^2(\mu;K_{\mathbf{p}} \otimes K)$. The restrictions of these natural actions to $C_c(X)$ are the representations of $C_c(X)$ associated with the given representations of $C_c(G)$ in the proof of Renault's Theorem. This makes it clear that $S_{\mathbf{p}}$ is also a bimodule map.

Now we want to prepare the way for the proof of the converse of the last theorem. We need less hypothesis than we had conclusion, namely we only need to deal with the transitive quasiinvariant measures on x.

We use the measures μ^x on X such that $\alpha^x = \varepsilon^x \times \mu^x$, as described. For each x we have $\alpha^{\mu^x} = \mu^x \times \mu^x$, which is symmetric, so is trivial. That means that $\Delta_{\mu^x} = \delta$. Since these modular functions are all the same, we will denote them by the single letter.

Let p_x be the representation of I(R, α) gotten by integrating the trivial representation of R on the one-dimensional bundle, relative to the measure x. Since $M_{0c}(R) \subseteq$ I(R, α), the representation p_x can be restricted to $M_{0c}(R)$, and we denote the restriction the same way. Define p_x on M(X) to by the

representation by multiplication on $L^2(\mu^x)$. We combine these two definitions to get a representation p_x of $M_{0c}(R, \bar{X})$ On H_x. Let $\tilde{\omega}_t$ denote the direct sum of all these "transitive" representations p_x, so the representation space of $\tilde{\omega}_t$ is H_X, the direct sum of all the Hilbert spaces H_x. Write $M_t^*(R, X)$ for the norm closure of the image of $M_{0c}(R, \bar{X})$ under .Then $M_t^*(R, X)$ is a quotient of $M^*(R,X)$ as a $C(\bar{X})$ bimodule, as well as a compression of $M^*(R, X)$. We also write $M_t^*(R)$ for the closure of the image of $M_{0c}(R)$.

It is not true that every completely bounded map is a linear combination of completely positive maps, unless the range algebra is injective. The domain and range are closely related and very special. We can circumvent the problems caused by lack of injectivity, but to do so and even to deal with completely positive maps themselves, we need to think of $M_t^*(R, X)$ as acting on a space of Borel sections. We now begin to arrange that.

Observe that the Hilbert spaces H_x are the fibers in a Hilbert bundle over X, i.e., the graph of H, Γ_H, has a natural Borel structure with all the necessary properties. In fact, for each x the space H_x is easily identifiable with $L^2(\alpha^x)$, and we simply transport the usual Borel structure for the latter bundle to H.

If g ϵ M$_{0c}$(R), define a section of Γ_H by letting $\xi_g(x)$ be the class of g $(x, .)$ in $L^2(\mu^x)$. Countably many of these sections can be chosen so that their values at a point x always form a dense set in H_x. Thus we can also choose a countably generated subalgebra of M(X) so that the module of sections over it generated by the countably many ξ_g's determines the Borel structure on Γ_H. Note also that $x \sim y$ implies that $\mu^x = \mu^y$ so $H_x = H_y$.

Theorem (2.1.5) [1]: Let ψ be a completely positive $C(\bar{X})$-bimodule map from C*(G,$\bar{X}$) to M*(R, X), and suppose that $(\psi(\omega(\varepsilon)) \mid H_x \epsilon \tilde{\omega}_t(M(X))$. Then there is a p ϵ p(G) such that $\psi = S_p$, and $\|p\|_\infty \le \|\psi\|_{c,b}$.

Proof:

There is no loss of generality in taking ψ to have completely bounded norm

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at most 1. Next we restrict $\psi \circ \omega$ to $C_c(G, \overline{X})$, getting a completely positive map ψ', of $C_c(G, \overline{X})$ into $M^*(R, X)$. For each $x \in X$, $f \in C_c(G)$, and $g \in C(\overline{X})$ define $\psi'_x(f\lambda + g\varepsilon) = \psi'(f\lambda + g\varepsilon) \mid H_x$. For each x, ψ'_x is a completely positive bimodule map into $L(H_x)$ of bounded norm at most Γ, and $x \sim y$ implies $\psi'_x = \psi'_y$.

The proof consists mainly of accumulating sufficient information about the mappings ψ'_x and objects constructed from them to assemble the desired positive definite function p. Using the Stinespring Theorem for completely positive maps and analyzing the equipment it provides enables us to show that each ψ'_x is of the form S_{px}. Then it is necessary to merge the separate p_x's into one p, using the fact that $x \sim x'$ implies $\psi'_x = \psi'_x$ from which we prove that $p_x = p_{x'}$ a.e. Several more improvements in the behavior of the functions p_x finally allow us to produce a matrix entry that serves as the desired function p.

Step 1. The Borel Behavior of $x \mapsto \psi'_x$,If f, h $\in M_{oc}$ (R) we want to see that
$$x \mapsto p_x(f)(\xi_h(x))$$
is a Borel section of Γ_H. To do this it is sufficient to show that if f, h, k $\in M_{oc}$ (R) then the function $x \mapsto p_x(f)\, \xi_h(x) \mid \xi_K(x))$ is Borel. Such an inner product is given by an integral, according to the definition of p_x, namely
$$\iint f(y,z)h(x.z)\overline{K}(x,y)\,d\mu^x(z)\,d\mu^x(y) \tag{11}$$
This integral defines a Borel function of x since the measure $\mu^x \times \mu^x$ depend on x in a Borel manner. By the definition of $M_t^*(R)$, every p_x is defined on $M_t^*(R)$ and for a $\in M_t^*(R)$ the function $x \mapsto p_x(a)$ is a uniform limit of functions of the form $x \mapsto p_x(f)$ for $f \in M_{0c}(R)$. Hence for a $\in M_t^*(R)$ and h $\in M_{0c}(R)$ the section $x \mapsto p_x(a)(\xi_h(x))$ is Borel.

If we define $\hat{\psi}$ to be the direct sum of all the ψ'_x 's, then $\hat{\psi}$ is also the compression of ψ' to H_x. Thus $\hat{\psi}$ maps $C_c(G, \overline{X})$ into $M_t^*(R, X)$ and $p_x \circ \hat{\psi} = \psi'_x$. From this it follows that if $f \in C_c(G)$ and h $\in M_{0c}(R)$ then the section

$x \mapsto \psi'_{\mathbf{x}}$ (f)($\xi_{\mathbf{h}}$ (x)) of $\Gamma_{\mathbf{H}}$ is Borel. If g $\in$ C($\overline{X}$) there is a function $g_1 \in$ M(X) such that $\hat{\psi}(g\varepsilon) = \tilde{\omega}_{\mathbf{t}}(g_1)$ because $\hat{\psi}(\varepsilon) \in \tilde{\omega}_{\mathbf{t}}$(M(X)) and $\hat{\psi}$ is a C($\overline{X}$)-bimodule map. Hence for a $\in$ C$_{\mathbf{c}}$ (G,$\overline{X}$) and h $\in$ M$_{\mathbf{oc}}$(R) the section $x \mapsto \psi'_{\mathbf{x}}$ (a)($\xi_{\mathbf{h}}(x)$) is Borel .

The fact that $\hat{\psi}$ maps into M_t^*(R, X), and the Borel property derive above are essential for completing the proof.

Step 2. The Stinespring Construction

For each x we represent $\psi'_{\mathbf{x}}$ by Stinespring's Theorem: We get representation π_x of C$_{\mathbf{c}}$(G,$\overline{X}$) on a Hilbert space K_x and an operator V_x from H_x to K_x , such that for a $\in$ C$_{\mathbf{c}}$(G, $\overline{X}$) we have

$$\psi'_{\mathbf{x}}(a) = V_{\mathbf{x}}^* \pi_{\mathbf{x}}(a) V_x \qquad (12)$$

We will use the details of the construction, so we repeat it here. For Stinespring's proof, it suffices to have the domain of the completely positive map to be a*-algebra with identity, so C$_{\mathbf{c}}$(G, $\overline{X}$) can be used. The space K_x is taken to be the Hilbert space constructed from the algebraic tensor product C$_{\mathbf{c}}$(G,$\overline{X}$) $\otimes$ H_x using the semi-inner product whose value on two elementary tensors is given by (a$\otimes\xi$ $|$ b$\otimes$ η) = ($\psi'_{\mathbf{x}}$(b*a) ξ $|$ η). Let q_x be the quotient map from C$_{\mathbf{c}}$(G, $\overline{X}$)$\otimes H_x$ to its quotient modulo vectors of norm 0. The image of q_x is identified with a dense subspace of K_x. (Since C$_{\mathbf{c}}$ (G , $\overline{X}$) and H_x are separable, so is K_x). The representation π_x is determined by having π_x(a)(q(b$\otimes$ ξ)) = q_x(ab$\otimes\xi$) for a , b $\in$ C$_{\mathbf{c}}$(G, $\overline{X}$) and H_x. The operat- or V_x is determined by setting $V_x(\xi) = q_x$(1$\otimes\xi$) for $\xi \in H_x$. Acalculation of inner products shows that $\| V_x \|^2 = \| \psi'_{\mathbf{x}}(1) \|$.

Since ψ_x, π_x, K_x, and V_x are Borel in x and constant on equivalence classes, we get a Hilbert bundle over X that is constant on equivalence classes. The pair (π_x,V_x) is minimal in the sense that π_x(C$_{\mathbf{c}}$(G, $\overline{X}$) is dense inK_x.

Step 3. Getting p_x from the Stinespring Representation. Now we study this structure for a fixed $x \in$ X. By Theorem (1.1.4) we know that π_x can be

obtained by integrating a representation, $\pi'_{\mathbf{x}}$, of G on a bundle K^x relative to a quasiinvariant measure μ_x, i.e., $K_x = L^2(\mu_x ; K^x)$. Let φ_x be the representation of $C(\overline{X})$ on K_x associated with π_x as a nutural representation of $C_c(G, \overline{X})$. In terms of the representation of K_x, φ_x is the representation by multiplication on sections of K^x. We also have $\varphi_x = \pi_x | C(\overline{X})$ where $C(\overline{X})$ is regarded as a subalgebra of $C_c(G, \overline{X})$.

We denote the natural representation of $C(\overline{X})$ on H_x by θ_x; again this is a representation by multiplication.

We need to show that μ_x can be taken to be μ^x. The first step is to show that V_x intertwines θ_x; and φ_x . Take h $\in$ C($\overline{X}$) b $\in$ C$_c$(G, $\overline{X}$) and $\xi, \eta \in H_x$. Then the definition of the inner product and the fact that $\psi'_{\mathbf{x}}$ is $C(\overline{X})$ - bimodule map gives

$$(I \otimes h\xi | b \otimes \eta) = \left(\psi'_x(b^*)h\xi | \eta\right) \tag{13}$$
$$= \left(\psi'_x(b^*h)\xi | \eta\right)$$
$$= (h \otimes \xi | b \otimes \xi).$$

Hence $q_x(\text{h} \otimes \xi) = q_x(1 \otimes \text{h}\xi)$. Using the bimodule property of $\psi'_{\mathbf{x}}$, the definition of $\pi_{\mathbf{x}}$, and the inner product on $K_{\mathbf{x}}$, we compute that

$$(V_x \,(\text{h}\xi) | \, q_x\,(\text{b} \otimes \eta)) = (q_x\,(\text{h} \otimes \xi) | q_x\,(\text{ b} \otimes \eta)).$$
$$= (\pi_x(\text{h}\varepsilon)\, q_x\,(\text{I} \otimes \xi) | q_x\,(\text{ b} \otimes \eta)). \tag{14}$$

Hence, $V_x\,(\text{h}\xi) = \varphi_{\mathbf{x}}(\text{h})\, V_x\,(\xi)$.

From the theory of representations of $C_c(X)$ or of projection valued measures based on X, there is a bounded section of K^x, which we denote by $\zeta_{x'}$ such that for $\xi \in H_x$, the pointwise product $\xi\xi_{\mathbf{x}}$ is a section of K^x representation the element $V_x\,(\xi)$ in K_x. Such a section can be gotten as follows: let g be any strictly positive Borel function on X that represents an element of H_x , let $\zeta^{\mathbf{1}}$ be a section that represents $V_x\,(\text{g})$, and set $\zeta_x = (1/g)\zeta^{\mathbf{1}}$. Then ζ_x need not be a square integrable section, but will be if μ^x is finite so that the function 1 is an element of H_x.

We can write $V_x(\xi) = \xi\xi_x$, using the usual identification of functions with their equivalence classes. Then for $\xi \in H_x$ we have

$$\int |\xi|^2 |\xi_x|^2 \, d\mu_x \leq \int |\xi|^2 \, d\mu^x \qquad (15)$$

Because $\|V_x\| \leq 1$. It follows that μ_x is not singular relative to μ^x, so that μ_x gives positive measure to $[x]$. It also follows that $|\xi_x|$ is zero a.e. off $[x]$, so that $\zeta^1 = g\xi_x$ is in the subspace of $K_x = L^2(\mu_x; K^x)$ consisting of functions that vanish off $[x]$. By the way we integrate representations of G to get representations of $C_c(G)$, we see that this latter subspace is invariant for $C_c(G)$ and hence for $C_c(G, \overline{X})$. From the fact that g is cyclic in H_x, it follows that $g\xi_x = V_x(g)$ is cyclic for $C_c(G, \overline{X})$ in K_x, so the subspace under discussion is in fact all of K_x. That implies that μ_x is in fact equivalent to x, so we may as well take μ_x to be equal to μ^x. That may require multiplying the original ξ_x by some positive function, but now we assume that to have been done. We write v^x for λ^{μ^x}, getting a measure concentrated on $G|[x]$.

In this situation, the inequality (15) implies that $|\xi_x|$ is bounded by 1. We define $p_x(\gamma) =$

$$\left(\pi'_x(\gamma)\zeta_x(s(\gamma)) \big| \zeta_x(r(\gamma))\right) \qquad (16)$$

getting a positive definite function on $G|[x]$. Now the sup-norm of ξ_x is the same as the operator norm of V_x, and that is the same as the square root of the completely bounded norm of ψ'_x, so the sup-norm of p_x is at most the completely bounded norm of ψ'_x.

Step 4. p_x Gives Rise to ψ'_x

We know that $x \sim y$ implies $\psi'_x = \psi'_y$, so $\pi_x = \pi_y$ and $V_x = V_y$. Hence $\pi'_x(\gamma) = \pi'_y(\gamma)$ for v^x-almost every γ, and $\zeta_x(z) = \zeta_y(z)$ for μ^x-almost every z, so that $p_x = p_y$ a.e. relative to v^x, and their restrictions to X agree a.e relative to μ^x.

To see that ψ'_x is the compression of S_{px} to H_x, we begin by setting $v^x = \lambda^{\mu x}$ and $\check{v}^x = \alpha^{\mu x}$, as above, so that $\mathfrak{X} = 1$ and $\Delta = \delta$. Then we calculate for

$f \in C_c(G)$, and $\xi, \eta \in H_x$

$$(\psi'_x(f)\xi|\eta) = (\pi_x(f)V_x\xi\,|\,V_x\eta) \tag{17}$$

$$= \int f(\gamma)(\pi_x\,(\gamma)(\xi\zeta_x))(s(\gamma))\,|\,(\eta\zeta_x)(r(\gamma))\Delta^{-\frac{1}{2}}(\gamma)dv^x(\gamma)$$

$$= \iint f(\gamma)p_x\,(\gamma)\delta^{-\frac{1}{2}}(\gamma)d\beta_z^y(\gamma)\xi(z)\bar{\eta}(y)d\tilde{v}^x(y,z)$$

This shows that $\psi'_x(\,f\,) = S_{px}(\,f\,)\,|\,H_x$. Next we find a formula for $\psi'_x(\varepsilon)$ by computing

$$(\psi'_x(\varepsilon)\xi|\eta) = (\pi_x(\varepsilon)V_x\xi\,|\,V_x\eta) \tag{18}$$

$$= (\xi\zeta_x\,|\,\eta\zeta_x)$$

$$= \int p_x\,(y)\xi(y)\tilde{\eta}(y)d\mu^x(y)$$

from which it follows that $\psi'_x(\varepsilon) = p_x(p_x\,|\,X)$. Since ψ'_x is a $C(\bar{X})$ bimodule map, we see that $\psi'_x(g\varepsilon) = p_x(gp_x)$ for $g \in C(\bar{X})$. This completes the proof that ψ'_x is the compression of S_{px} to H_x. Step 5. Applying Lemma (1.2.1) to the Functions p_x.

Take functions h, k $\in M_{oc}(R)$ from which we make sections ξ_h and ξ_K of H. Let $\xi = \xi_h(x)$ and $\eta = \xi_K(x)$ in the calculations above to see that is $g \in C_c(G)$, then

$$\left(\psi'_x(g)\xi_h(x)\big|\xi_K(x)\right)$$

$$= \int g\,(\gamma)p_x(\gamma)h(x,s(\gamma))\bar{K}(x,r(\gamma))\delta^{-\frac{1}{2}}(\gamma)dv^x(\gamma) \tag{19}$$

If ε is the identity in $C(\bar{X})$ we also get,

$$\left(\psi'_x(\varepsilon)\xi_h(x)\big|\xi_K(x)\right) = \int p_x(\gamma)h(x,y)\bar{K}\,(x,y)d\mu^x(y) \tag{20}$$

Here it is important that the functions of x on the left hand sides of these two formulas are Borel functions.

To apply Lemma (1.2 .1) as it is formulated, we must have a Borel family of finite measures. We begin by considering a compact set K contained in G. The function $y \mapsto \lambda^y(K)$ is bounded on X, and for every $x \in X$ we have $\mu^x(s(xK)) < \infty$ Hence, for $x \in X$ the measure given by the integral

$$\int\limits_{s(x,K)} (\chi_K \lambda^y) d\mu^x(y)$$

is finite.

Notice that a pair $(x, y) \in X \times X$ is in $\theta(K)$ iff $x \in r(Ky)$ iff $y \in s(xK)$. If h is the characteristic function of $\theta(K)$, it follows that $h(x, r(\gamma)) = 1$ iff $r(\gamma) \in s(xK)$, and $h(x, s(\gamma)) = 1$ iff $s(\gamma) \in s(xK)$. Thus the set L, defined to be $\{(x, \gamma) \in X \times G : \gamma \in K$ and $h(x, s(\gamma)) h(x, r(\gamma)) = 1\}$ is a Borel set in $X \times G$, and the same as $\{(x, \gamma) \in X \times G : \gamma \in K, s(\gamma) \in s(xK)$ and $r(\gamma) \in s(xK)\}$. From the preceding paragraph, it follows that every x-section L_x of L has finite measure for v^x.

Choose compact sets $K_1 \subset K_2 \subset \ldots$ whose union is G, and for each n define $h_n = \chi_{0(K_n)}$ and then $L_n = \{(x,\gamma) \in X \times G : \gamma \in K_n, s(\gamma) \in s(x K_n)$ and $r(\gamma) \in s(x K_n)\}$. Define $D_1 = L_1$ and for n-let $D_n = L_n \backslash L_{n-1}$. For each $n \in \mathbb{N}$ and $x \in X$, let $v_x^n = (\chi_{(Dn)x}) v^x$. This gives a Borel family of finite measures on G. Notice that the sets D_n partition $\{(x, \gamma) \in X \times G : \gamma \in G \,|\, [x]\}$.

Now define f_x on G for $x \in X$ by $f_x(\gamma) = p_x(\gamma) \delta^{-1/2}(\gamma)$ for $\gamma \in G|[x]$ and 0 for other γ's. If $g \in C_c(G)$ and $x \in X$, then

$$\int g(\gamma) f_x(\gamma) dv_n^x(\gamma)$$

$$= \left(\psi'_x(g) \xi_{hn}(x) \big| \xi_{hn}(x) \right) \tag{21}$$

which is a Borel function of x. Hence there is a Borel function F_n on $X \times G$ such that for each x, $F_n(x, .) = f_x$ a.e. relative to v_x^n Set

$$F = \sum_{n \geq 1} \chi_{Dn} F_n. \tag{22}$$

Then F is Borel and for each $x \in X$, $F(x, .) = f_x$ a.e. relative to v^x.

A similar analysis using μ^x shows that we can also choose F so that $F(x, y) = f_x(y)$ for μ^x-almost every y.

Hence there is a Borel function P on $X \times G$ such that for every x we have $P(x,.) = p_x$ a.e. Also, $x \sim y$ implies that $P(x,.) = P(y, .)$ a.e. relative to either

ν^x or ν^y (these are the same measure) and also relative to either μ^x or μ^y when restricted to X. Furthermore, $\lvert P(x,.) \rvert$ is bounded by the completely bounded norm of ψ'_x, so $\lvert P \rvert$ is bounded by 1.

Step 6. Improving the Behavior of P

Recall the probability measures $\mu_1^x = s(\lambda_1^x)$ on X obtained from the Borel family of normalized Haar measures on G. We have $\mu_1^x \sim \mu_1^y$ if $x \sim y$. Define a new function P_1 on $X \times G$ by

$$p_1(x,y) = \int p(y,\gamma)d\mu_1^x(y). \tag{23}$$

Make a function of three variables from P and use the Borel character of P and the measures μ_1^x to show that P_1 is also Borel. We need to know that P_1 also essentially replicates every function p_x, and is even more invariant than P under changing x to an equivalent point of X.

To begin with we limit ourselves to one orbit, and denote it by S. We write μ^S for a choice of one of the measures $\mu_1^{x_0}$ for $x_0 \in S$. We know that for x and y in S the functions $P(x,.)$ and $P(y,.)$ agree a.e. relative to λ^{μ^S} so they agree a.e. relative to λ^z for μ^S-almost every z. Since λ^z and λ_1^z have the same null sets, $P(x,.)$ and $P(y,.)$ agree a.e. relative to λ^z iff the complex measures $P(x,.)\lambda^z$ and $P(y,.)\lambda^z$ are the same. We have two Borel mappings from S^3 to the standard Borel space of complex Borel measures on $\overline{X}$, so the set E_s on which they agree is Borel, allowing us to use Fubini arguments.

Hence, for every $x \in S$, th arguments e set $\{(y, z) \in S^2 : P(y,.) = P(x,.)$ a.e. $d\lambda^z\}$ is a Borel set whose complement has measure 0 for $\mu^S \times \mu^S$. Therefore, there is a conull Borel set Z_x of points z in S such that for μ^S-almost every y we have $P(y,.) = P(x,.)$ a.e. relative to λ^z. Thus, for $z \in Z_x$ it is true that for almost every γ we have $P(y,\gamma) = P(x,\gamma)$ for μ^S-almost every y. It follows that if $z \in Z$, then $P_1(x,\gamma) = P(x,\gamma)$ for λ^z-almost every γ. Hence, for every $x \in S$ we have $P_1(x,.) = P(x,.)$ a.e. In particular, P_1 also replicates every p_x, since S is a general orbit.

In the last paragraph, we enountered points $\gamma \in G$ for which P(y, γ) is essentially constant in y because it is almost always equal to a particular P(x, γ). We need to know more about the set H ={$\gamma \in G$: y$\mapsto$P(y, γ) is essentially constant}. If A is a countable algebra that generates the Borel sets in C, it is not difficult to show that

$$H = \bigcap_{A \in \mathcal{A}} \{\gamma \in G : \mu_1^{r(\gamma)} \times \varepsilon^\gamma (p^{-1}(A)) \in \{0,1\}\} \tag{24}$$

Thus H is a Borel set. Hence the set C = {$x \in X : \lambda_1^x$ (H) =1} is also a Borel set . From the preceding paragraph, it follows that C is conull in every orbit. For $z \in$ C, the function P$_1$(., γ) is constant for λ^z-almost every $\gamma \in zG$. In particular for $z \in$ C it is true that x, y $\in$ [z] implies that P$_1$(x,.) λ_1^z= P(y,.) λ_1^z.

The last conclusion is the additional invariance needed, and now we change notation and simply write P for P$_1$, since it does everything we need.

Step 7. Making a Borel Family of Representations from P

Again, take a particular orbit, S, in X. For every pair $(x, y) \in S^2$, we have P(x , .) = P(y , .) a.e. relative to $\lambda^z$for μ^S-almost every z. Take an arbitrary $z \in$ S. Then for λ^z-almost every γ_2 it is true that P(x , .) = P(y , .) a.e. relative to γ_2^{-1}. $\lambda^z = \lambda^{s(\gamma_2)}$. Hense p($x$, $\gamma_2^{-1}\gamma_1$) = p(y, $\gamma_2^{-1}\gamma_1$) for $\lambda^z \times \lambda^z$- almost every pair ($\gamma_1$, γ_2). (The mapping taking the pair to $\gamma_2^{-1}\gamma_1$ carries $\lambda_1^z \times \lambda_1^z$ to a measure equivalent to $\lambda^{\mu z}$).

Now return to studying general points of X. For f, g $\in C_c$(G) and $(x,y) \in$ R defined

$$(f \mid g)_{(x,y)} = \iint f(\gamma_1)\bar{g}(\gamma_2)p(x,\gamma_2^{-1}\gamma_1)d\lambda^y(\gamma_1)d\lambda^y(\gamma_2) \tag{25}$$

The formula defines an inner product on C$_c$(G), and we write K(x, y) for the resulting Hilbert space. For each f, g $\in C_c$(G) the function $(x, y)\mapsto(f \mid g)_{(x, y)}$ is a Borel function on R that is constant on sets of the form [y] $\times$ {y}, so K defines a Hilbert bundle on R that is constant on the same sets.

For $f \in C_c$(G), let σ (f) denote the section of K (or Γ_K) that it determines.

For each x, the bundle $K(x, .)$ supports a unitary representation: here we denote it by π_x rather than $\pi_{P(\mathbf{x},.)}$. We know that $x \sim x'$ implies that $\pi_x = \pi_{x'}$, which means that for $\gamma \in G \mid [x]$ we have $\pi_x(\gamma) = \pi_{\mathbf{x}'}(\gamma)$ (they are on the same space). We want to show that $(x, \gamma) \mapsto \pi_x(\gamma)$ is Borel on $X \times' G = \{(x, \gamma) : \gamma \in G \mid [x]\}$. It will help to look at $R \times' G = \{(x, y, \gamma) : \gamma \in G \mid [x]\}$. The function

$$(x, y, \gamma) \mapsto \iint f(\gamma^{-1} \gamma_1 \bar{g}(\gamma_2)) p(x, \gamma_2^{-1} \gamma_1) d\lambda^y(\gamma_1) d\lambda^y(\gamma_2) \qquad (26)$$

is Borel on $R \times' G$, so $(x,\gamma) \mapsto (\pi_x(\gamma) \sigma(f)(x,s(\gamma)) \mid \sigma(g)(x, r(\gamma)))$ is Borel on $X \times' G$.

Step 8. Finding a Borel Section That Represents P

Let D be the set of pairs $(x, y) \in R$ for which the linear functional $f \mapsto \lambda^y(fP(x,.))$ is bounded relative to the seminorm $\|\sigma(f)(x,y)\|$ on $C_c(G)$. The boundedness can be tested using a countable dense subset of $C_c(G)$, so D is Borel, and hence so is the set DC. For each $x \in X$, we have $xD = \{x\} \times D_x$ so that xD is conull with respect to α^x. Notice that $w \sim x$ implies that $C \cap D_x = C \cap D_w$, and this set is conull in the orbit. Hence xDC and wDC have the same conull image in $[x]$ under s.

Now, for $(x, y) \in D$ define $\xi (x, y)$ to be the vector in $K(x, y)$ such that $(\sigma(f)(x, y) \mid \xi(x, y)) = \lambda^y(fP(x, .))$ for every $f \in C_c(G)$, and for $(x, y) \notin D$. let $\zeta(x, y) = 0$. The formula makes it clear that ξ is Borel.

If $y \in C$ and $w \sim x \sim y$, then $(w, y) \in D$ iff $(x, y) \in D$, so $y \in C$ implies that $Dy = [y] \times \{y\}$. Also, $w, x \in [y]$ implies that $P(w, .)$ and $P(x, .)$ agree a.e. with respect to λ^y and that $K(w, y) = K(x, y)$. To gether, these imply that $\zeta(w, y) = \zeta(x, y)$. Then for every $\gamma \in G \mid [y]$,

$$(\pi_x(\gamma)\zeta(x, s(\gamma)) \mid \zeta(x, r(\gamma)))$$
$$= (\pi_w(\gamma)\zeta(w, s(\gamma)) \mid \zeta(w, r(\gamma))). \qquad (27)$$

Thus both of these functions agree a.e. on $G\,|\,[x]$ with $P(x,.)$. Thus we can define
$$p(\gamma) =$$
$$\left(\pi_{s(\gamma)}(\gamma)\xi\big(s(\gamma),s(\gamma)\big)\big|\varsigma\big(s(\gamma),r(\gamma)\big)\right) \qquad (28)$$

for $\gamma \in \theta^{-1}(DC)$ and 1 for other γ's to get a Borel function on G that agrees a.e. with $P(x, .)$ on $G\,|\,[x]$.

From Step 4 it follows that $\hat{\psi}$ and the compression of S_p to H_x are the same.

Step 9. The Compression Map from $L(H_\omega)$ to $L(H_x)$

To complete the proof, need to show that the compression map C from $L(H_\omega)$ to $L(H_x)$ is one-one when restricted to $\tilde{\omega}(M_{0c}(R,\bar{X}))$. Then it will follow that and S_p agree on $C_c(G, \bar{X})$ forcing them to be the same.

Suppose that $f\alpha+g\varepsilon \in M_{0c}(R,X)$ and $\tilde{\omega}(f\alpha+g\varepsilon) \neq 0$. Then there is a representation π of R and a probability measure $\mu \in \mathcal{Q}$ such that $\pi^\mu(f\alpha+g\varepsilon) \neq 0$.

We need to use this to find a $z \in X$ such that $p_z(f\alpha+g\varepsilon) \neq 0$, which will imply $\tilde{\omega}_t(f\alpha+g\varepsilon) \neq 0$. There is no loss of generality in assuming that there is a probability measure μ' on X such that

$\mu = \int \mu_1^x \, d\mu'(x)$. Set $A=\{(x, y) \in R : x \neq y,$ and $f(x, y) \neq 0\}$, and consider two cases: $\alpha^\mu(A) = 0$ and $\alpha^\mu(A)\neq 0$. In the first case, $\pi^\mu(f) = 0$ unless $\alpha^\mu(X) > 0$, in which case we have $f\alpha = f\varepsilon$ relative to α^μ. Thus there is an $h \in M(X)$ such that $0 \neq \pi^\mu(f\alpha+g\varepsilon) = \pi^\mu(h\varepsilon)$. Then $\mu(\{h \neq 0\}) > 0$ so there is a $z \in X$ such that $\mu^z(\{h \neq 0\}) > 0$, and it is easy to show that $p_z(h\varepsilon) \neq 0$, i.e., $p_z(f\alpha+g\varepsilon) \neq 0$. In the second case, there is a $z \in X$ such that $\alpha^{\mu z}(A) > 0$, and we will show that $p_z(f\alpha+g\varepsilon) \neq 0$. Recall that $\alpha^{\mu z} = \mu^z \times \mu^z$.

Set $R_0 = R\backslash\{(x, x) : x \in X\}$. Then sets of the form $(E \times F) \cap R_0$, where E and F are disjoint Borel sets in X, generate the Borel sets in R_0, so there must be such a pair for which

$$0 < \int\limits_{E\times F} f\,d(\mu^z \times \mu^z) < \infty \qquad (29)$$

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If we set $h_1 = \chi_F$ and $h_2 = \chi_E$ we get elements of $M(X)$ which we think of as elements of H_z, and then the displayed integral is $(p_z)(f)\, h_1\,|\,h_2)$. On the other hand, $(p_z(g\varepsilon)\, h_1\,|\,h_2) = 0$ because $gh_1\,\bar{h}_2 = 0$. Thus $p_z(f\alpha + g\varepsilon) \neq 0$, as needed.

Section (2.2): Completely Bounded Bimodule Maps and Banach Algebra

Recall that B(G) is defined to be the linear span of P(G). Because we know that P(G) consists of diagonal matrix entries of unitary representations we can form direct sums of representations to show that elements of B(G) are also matrix entries that need not be diagonal. We will provide B(G) a normed algebra structure. One way to compute the norm of an element b of B(G) is in terms of the positive definite functions on a larger groupoid for which b can appear as an "off diagonal part." This is the groupoid version of the well known 2×2 matrix method, and has been exploited by Renault for the same purpose. This permits using the completeness of P(G) for a general locally compact groupoid to prove the completeness of B(G).

We can also formulate B(G) as an algebra of completely bounded $C(\overline{X})$-bimodule maps on M*(G), and as a space of completely bounded $C(\overline{X})$-bimodule maps from $C*(G, \overline{X})$ to M*(R, X). Since the Completely positive elements in the latter set are all given by positive definite functions, and the completely positive bimodule maps form a complete set, we get one way to prove that B(G) is complete.

Recall that ω is the direct sum of all cyclic representations of C*(G). We can construct each cyclic representation as an integrated representation of G, and, as such, it can be taken as a representation of Mc(G), and we use the same notation. For each a $\in$ C*(G), $\|\omega(a)\| = \|a\|$ is the same as $\sup\{\|\pi(a)\| : \pi$ is a cyclic representation of C*(G)$\}$. Also recall, the norms $\| \ \|_{II,\mu}$ and $\| \ \|_{II}$ and their properties.

Theorem (2.2.1) [1]: If b $\in$ B(G), the operator T_b , taking $\omega(f)$ to $\omega(bf)$ for $f \in M_c(G)$, extends to a completely bounded map of M*(G) to itself and $\|T_b\|_{cb} \geq \|b\|_\varrho$.

Proof:

By Theorem (2.1.1), if p $\in$ P(G) then T_b is completely positive, so for b $\in$ B(G) the operator T_b is completely bounded. Set M $= \|b\|_Q$ and suppose $0 < \alpha < 1$. Since α is arbitrary, the proof will be complete if we find an $f \in M_c(G)$ such that $\omega(f) \neq 0$ and $\|T_b\,\omega(f)\| \geq M\alpha^2\|\omega(f)\|$. To find such an f first notice that there is a $\mu \in Q$ such that the $L^\infty(\lambda^\mu)$- norm of b is greater than Mα, so there exist a $b_0 \in \mathbb{C}$ and a $\eta > 0$ such that the set A$=\{\gamma: |b(\gamma)-b_0| < \eta\}$ has positive measure for λ^μ and $|b_0| - \eta > M\alpha$ Then there is a compact set C $\subseteq$ Ad such that $\lambda^\mu(C) > 0$. We take $f = \chi_C$.

By the definition of $\|\ \|_{\mathrm{II}}$, there is a $\mu' \in Q$ such that $\|f\|_{\mathrm{II},\mu'} > \alpha\|f\|_{\mathrm{II}}$. By the properties of $\|\ \|_{\mathrm{II}}$, if π is the one-dimensional trivial representation of G, we have $\|\pi^{\mu'}(f)\| > \alpha\|\omega(f)\|$. Now let $\sigma = \pi^\mu \oplus \pi^{\mu'}$. We have $\|\sigma(f)\| \geq \|\pi^{\mu'}(f)\| > \alpha\|\omega(f)\|$.

We can find g_1 and g_2 in $C_c(X) \geq 0$, and > 0 on r(C) $\cup$ s(C). These can be regarded as sections of the bundle for π, and it is clear that $(\pi^\mu(f)\ g_1 | g_2) > 0$ from the integral formula for the inner product. Thus $\pi^\mu(f) \neq 0$, so $\sigma(f) \neq 0$ and $\omega(f) \neq 0$.

Since $\sigma(b_0 f) = b_0\sigma(f)$, it will suffice to show that $\|\sigma((b-b_0)f\| \leq \eta\|\sigma(f)\|$, because then we get $(|b_0| - \eta)\|\sigma(f)\| \leq \|\sigma b(f)\|$, so $(|b_0| - \eta)\alpha\|\omega(f)\| \leq \|\sigma b(f)\| \leq \|\omega(bf)\| = \|T_b\omega(f)\|$, giving the desired inequality. Now f is a characteristic function , So (b$-b_0$) $f = ((b-b_0)\ f)f$. Also, $\|(b-b_0)\ f\|_\infty \leq \eta$, so the inequality we wanted on σ can be obtained by applying the second inequality before Lemma (1.1.5) to both μ and μ'. Thus the proof is complete.

Again we use the algebra C(G , $\overline{X}$) to study B(G), and need the one-one correspondence between its representations and those of $C_c(G)$ and hence those of G. We still use ω for the direct sum of all cyclic representations of C(G, $\overline{X}$) each of them given as an integrated representation of G. We use for the direct sum of all the cyclic representations of $M_c(R, X)$ that can be obtained by integrating a representation of R. Recall that C*(G, $\overline{X}$) is the operator-norm closure of ω(C(G, $\overline{X}$)) and M*(R, $\overline{X}$) is the operator norm

closure of $\tilde{\omega}$ ($M_c(R, X)$). If $x \in X$, use H_x for $L^2(\mu^x)$ as before, and H_x for the direct sum of all the H_x's. Let $\tilde{\omega}_t$ be the subrepresentation of obtained by restricting to H_x.

Theorem (2.2.2) [1]: Let $b \in B(G)$. There is a completely bounded $C(\bar{X})$ bimodule map S_b: $C^*(G, \bar{X}) \to M^*(R, X)$ such that $S_b(\omega(f)) = \tilde{\omega}(\langle f, b \rangle)$ for $f \in C_c(G)$ and $S_b(\omega(g\varepsilon)) = \tilde{\omega}$ ($bg\varepsilon$) for $g \in C(\bar{X})$. For this operator we have

$$\| S_b \|_{cb} \geq \| b \|_\varrho,$$

and

$$S_b(\omega(\varepsilon)) \,\big|\, H_x \in \tilde{\omega}_t(M(X)).$$

Proof:

The operator S_b is a linear combination of four operators S_b for $p \in P(G)$, and these are completely positive bimodule maps by Theorem (2.1.4) .

For the norm inequality, we proceed as in the proof of Theorem (2.2.1) . Let $M = \| b \|_\varrho$ and $0 < \alpha < 1$. It will suffice to find $f \in M_c(G)$ such that $\omega(f) \neq 0$ and $\| S_b \omega(f) \| \geq M\alpha^2 \| \omega(f) \|$. Choose μ, b_0, η, A and C as in Theorem (2.2.1), and take $f = \chi_C$.

We take π to be the trivial one-dimensional representation, and choose μ' and σ as before. The proof that $\omega(f) \neq 0$ used before works here also.

Let $\tilde{\pi}$ denote the one-dimensional trivial representation of R, and form its integral with respect to $\mu, \tilde{\pi}^\mu$. Likewise form $\tilde{\pi}^{\mu'}$, and let $\tilde{\sigma} = \tilde{\pi}^\mu \oplus \tilde{\pi}^{\mu'}$. It will suffice to prove that $\| \tilde{\sigma} (\langle f, 1 \rangle) \| > \alpha \| \omega(f) \|$.

For this purpose, we need to see that $\| \langle f, 1 \rangle \|_{II,\mu} = \| (f) \|_{II,\mu}$. This follows from the fact that $f \geq 0$ together with the relationship between v_0 and $\tilde{v}_0$.

Then we see that

$$\| \langle f, b - b_0 \rangle \|_{II,\mu} \leq \| (b - b_0) f \|_\varrho \| f \|_{II,\mu} < \eta \| f \|_{II,\mu} \tag{30}$$

using the fact that f is a characteristic function.

Both the equality and the inequalities also hold for μ', and since π and are

the one-dimensional trivial representations, they transfer to the corresponding equality and inequalities for σ and $\tilde{\sigma}$.

Hence

$$\begin{aligned}
\left\| \tilde{\omega}(\langle f, b \rangle) \right\| &\geq \left\| \tilde{\sigma}(\langle f, b \rangle) \right\| \\
&\geq \left\| \tilde{\sigma}(\langle f, b_0 \rangle) \right\| - \left\| \tilde{\sigma}(\langle f, b-b_0 \rangle) \right\| \\
&\geq \left| b_0 \right| \left\| \tilde{\sigma}(\langle f, 1 \rangle) \right\| - \eta \left\| \tilde{\sigma}(\langle f, 1 \rangle) \right\| \\
&\geq M\alpha \left\| \tilde{\sigma}(\langle f, 1 \rangle) \right\| \\
&\geq M\alpha^2 \left\| \omega(f) \right\|
\end{aligned} \tag{31}$$

In order to provide the norm on B(G) in a way that will be convenient for proving completeness, we introduce a way to enlarge the groupoid G as it was done. Write T_2 for the transitive equivalence relation on the two element set $\{1, 2\}$, so that T_2 has four elements. It will be convenient to have a shorter notation for matrix coefficients: If π is a unitary representation of G and ξ and η are bounded Borel sections of the bundle H on which π acts, we can write $[\pi, \xi, \eta]$ for the matrix coefficient, namely

$$[\pi, \xi, \eta](\gamma) = \left(\pi(\gamma)\xi \circ s(\gamma) \mid \eta \circ r(\gamma) \right). \tag{32}$$

Theorem (2.2.3) [1]: A bounded Borel function b on G is in B(G) if and only if there is a function p' ϵ P(G × T_2) such that for $\gamma \epsilon$ G we have b(γ) = b'(γ, 1,2)). The function b can be expressed as a matrix coefficient using sections of sup- norm at most 1 if and only if there is an associated p' that can be expressed as a diagonal matrix coefficient using a section of sup norm at most 1.

Proof:

The proof of the first assertion will be given in terms of matrix coefficients and will include the proofs of the facts about sup norms. Let

$$X' = X \times \{1, 2\} \text{ be the unit space of } G' = G \times T_2.$$

Suppose that π is a unitary representation of G on a bundle H and that and η are Borel sections of H of supnorm at most 1 such that b=[π, $\xi.\eta$]. Define a Hilbert bundle H' over X' by setting H'(x,i) = H(x) for i=1, 2. For

$\gamma' = (\gamma, (i, j))$ in G' notice that $s(\gamma') = (s(\gamma), j)$ and $r(\gamma') = (r(\gamma), i)$. That means that we can define a representation π' of G' on H$ by $\pi'(\gamma') = \pi(\gamma)$. Define a section `$ of H by setting $\xi'(x, i) = \eta(x)$ when i =1 and $\xi'(x, i) = \xi(x)$ when I = 2. Then the sup norm of $\xi'$is at most 1.and for every $\gamma \in$ G we have $b(\gamma) = [\pi', \xi', \xi'] (\gamma, (1, 2))$ as required.

For the converse, suppose we begin with H', π', and ξ'. Then for $x \in$ X define $H(x) = H'(x, 1) \oplus H'(x, 2)$ and set $\eta(x) = (\xi'(x, 1), 0)$ and $\xi(x) = (0, \xi'(x, 2))$. For $\gamma \in$ G define $\pi(\gamma)$ to take (ξ_1, ξ_2) to $(\pi'(\gamma,(1,1))\xi_1+\pi'(\gamma,(1,2))\xi_2, \pi'(\gamma,(2,1))\xi_1+\pi'(\gamma,(2,2))\xi_2)$, thus acting as a matrix by left multiplication on column vectors. The sections ξ and η have sup norm at most 1, and we have $b = [\pi, \xi.\eta]$.

Because of the results, we can now complete the task we set ourselves at the beginning of the, as indicated by the section heading. Recall that for $b \in$ B(G), T_b is the operator on M*(G) determined by multiplication by b on M_c(G), and that we sometimes work with B(G) as an algebra of functions, even though the elements are actually equivalence classes.

Theorem (2.2.4)[1]: B(G) is a Banach algebra with pointwise operations for the algebraic structure and with the norm defined by $\|b\| = \|T_b\|_{cb}$ for b $\in$ B(G).

Proof:

Theorem(1.2.9) shows that B(G) is an algebra under pointwise operations, and equals $P(G) - P(G) + iP(G) - iP(G)$. Any function that is 0 for λ^Q-almost every point of G represents the 0 element of M*(G), so for b $\in$ B(G) the operator $T_b$depends only on the equivalence class of b.Thus $b \mapsto T_b$ is well defined from the space of equivalence classes of functions in B(G) to the space of completely bounded operators on M*(G). Since $\|T_b\|_{cb} \geq \|b\|_\infty$, we see that $b \mapsto T_b$ is also one-one.Thus the norm makes B(G) a commutative normed algebra.

To prove that B(G) is complete, let b_1, b_2, . . . be a sequence in B(G)

such that the norms $\|T_{b^n}\|_{cb}$ are summable. Then Theorem (2.2.3) says that we can construct positive definite functions p'_1 , p'_2 , . . . on the groupoid G'=G × T_2 such that for every γ ∈ G and every n we have $b_n(\gamma) = p'_n(\gamma,(1, 2))$, and for every n we have $\|p'_n\|_\infty = \|p_n\|_\infty$. Two forms of the completeness of P(G') can be used to complete the proof. We let $c_n = b_1 + \ldots + b_n$.

In the first proof, we notice that the sequence $S_{p'_1}$, $S_{p'_2}$,... of completely positive C($\bar{x}$')- bimodule maps from C*(G', $\bar{X}$') to M*(R'X') is summable. The sum is also a completely positive C($\bar{X}$')- bimodule map, so by Theorem (2.1.5) it is of the form $S_{p'}$ for a p' ∈ P(G'). Then the function b defined on G by b = p'(. , (1, 2)) is in B(G) by Theorem (2.2.3). We also get $\|S_{p'_n - p'}\|_{cb}$ $\geq \|S_{c_n - b}\|_{cb} \geq \|c_n - b\|_\infty$ by Theorem (2.2.3) and Theorem (2.2.2), so $\|c_n - b\|_\infty \to 0$. We need to prove that $\|c_n - b\| \to 0$ as n→∞.

To do this begin with $f \geq 0$ in Mc(G). Then Lemma (1.1.5) say that

$$\|\omega((c_n - b)f)\| \leq \|c_n - b\|_\infty \|\omega(f)\| \qquad (33)$$

Hence $T_{c_n}(\omega(f)) \to T_b(\omega(f))$ in M*(G). The f's span a dense set in M*(G), and the T_{c_n}'s are uniformly bounded, so it follows that $T_{c_n} \to T_b$ pointwise on M*(G). Now the fact that the completely bounded operators on M*(G) are complete implies that the sequence T_{c_n} has a limit, T' in the completely bounded sense, which is automatically also a pointwise limit on M*(G).

Hence T'=T_b, so that $\|T_{c_n - b}\|_{cb} \to 0$ and by Theorem (2.2.1) that is equivalent to saying $\|c_n - b\| \to 0$ as $n \to \infty$.

For the other proof of completeness, we notice that p'$_1$, p'$_2$,. . . is summable in the Q-essential supremum norm as functions on G'. Hence there is a Borel function p' that is the sum in that norm. By the Dominated Convergence Theorem, p' ∈ P(G'). Again we take b =p'(. , (1, 2)). Theorems (2.2.3) and (2.2.2) once again show that $\|c_n - b\|_\infty \to 0$ and we complete the proof as before.

Since B(G) is a Banach algebra, any closed subalgebra of it is a Banach

algebra. Convergence in the completely bounded norm implies convergence in $L^\infty(\lambda)^\varrho$, so certain subalgebras are easily seen to be closed. Among these are B(G), defined to be {b $\in$ B(G) : b is continuous}, and B(G, X), defined to be the set of elements b $\in$ B(G) such that b$\mid$X is continuous and vanishes at ∞. The subalgebra B(G, X) is defined to be B(G) $\cap$ B(G,X).

Theorem (2.2.5) [1]: B(G), $\mathcal{B}$(G,X), and B(G,X) are closed subalgebras of $\mathcal{B}$(G) and hence Banach algebras.

The first example is a groupoid on which the linear span of the continuous positive definite functions is not complete and there exist continuous elements of $\mathcal{B}$(G) that cannot be expressed as a difference of continuous positive definite functions.

Let X= {$(x, y) : (x, y)$ has polar coordinates (r, θ) with $0 \le r \le 1$, $\theta \in$ {0,1,1/2,1/3,…}} and set G = X $\times$ $\mathbb{Z}$. This is a bundle of groups, and $(x, n)+(x', n')$ is defined iff $x = x'$, and then it equals $(x, n+n')$. Write P(G) for the set of Borel positive definite functions on G and P(G) for the set of continuous elements of P(G). Let B(G) be the linear span of P(G), let B_1(G) be the linear span of P(G) and let B(G) be the set of continuous elements of $\mathcal{B}$(G). A bounded function p is in P(G) iff it is a Borel function and p(r, θ, .) is positive definite on $\mathbb{Z}$ for each point of X. Since positive definite functions on $\mathbb{Z}$ are in one-one correspondence with positive measures on $\mathbb{T}$ via the Fourier transform, we can also think of P(G) as consisting of Borel functions from X to the positive measures on $\mathbb{T}$. Define

$$p(r,\theta,n) = \begin{cases} e^{i\theta(1+r)n} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

and

$$q(r,\theta,n) = \begin{cases} e^{i\theta(1-r)} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

We can also think of these as taking values that are point masses at $e^{i\theta(1+r)}$ and $e^{i\theta(1-r)}$, or the 0 measure at the origin. We have p$-$q $\in$ B(G).

Suppose that u $\in$ P(G) and $-u \le p - q \le u$ where the inequalities indicate the pointwise order in the space of measure-valued functions. This is the same as the natural order in B(G) in which elements of P(G) are positive. Since p(r, θ), .) is the point mass at $e^{i0(1+r)}$, u(r, θ, .) dominates the point mass at that point. By continuity, u(0, 0, .) dominates the point mass at $e^{i\theta}$. This means that $u(0, 0, .)$ has infinite norm, so there is no such u. Thus we have a continuous element of $\mathcal{B}$(G) that is not a difference of continuous positive definite functions.

With more effort, a worse example can be made. Choose n angles, and begin with p and q restricted to the radii with those angles. The limit at the origin of both of them exists, the limits are the same, and it is a sum of n point masses. To make elements of P(G) we take that value at the origin and at all other points of X. Let b be the difference of these elements of P(G). Any element of P(G) that dominates b must have a value at the origin that dominates that sum of n point masses. Observe that b is 0 except on the original chosen radii, and that the total variation norm of each value of b is at most 2.

Now partition the angles in X into sets with 2^K elements, for k=1, 2,…, and use the construction just described to make elements b_K in B_1(G). Then let b $= \sum_{K=1}^{\infty} 2^{-K} b_K$. This converges in the completely bounded norm sinceeach b_K has completely bounded norm 2. Hence it also converges inuniform norm, so that b $\in$ B(G). Also b is in the closure of B_1(G). However, the domination arguments used above show that b is not in B_1(G).

The next example shows that locally compact groupoids can have unitary representations that are Borel but not continuous.

Consider an action of the integers on the circle by an irrational rotation and form the transformation group groupoid, G $= \mathbb{T} \times \mathbb{Z}$. If u is a unitary valued Borel function on $\mathbb{T}$, there is a unitary representation U such that for all $\tau \in \mathbb{T}$, u(τ) = U(τ, 1). If u is not continuous, neither is U.

Chapter 3

Measured Groupoid and Multipliers of Fourier Algebra

Dualities are established between B(G) and A(G) and the convolution algebras $C_\mu^*(G)$ and VN(G) in the framework of operator modules. They are used to generalize results of Varopoulos and Pisier about Littelwood functions and completely bounded multipliers.

Section (3.1): Fourier Algebras of Measured Groupoid and Duality

G. Pisier has recently given a new proof of a theorem of N.Varopoulos about Schur multipliers and Littlewood tensors. In fact, he has a more general result which he specializes both to the group case and to the case studied by Varopoulos.

We shall adapt Pisier's proof to the case of an arbitrary measured r-discrete groupoid. We introduce the Fourier− Stieltjes algebra B(G) and the Fourier algebra A(G) of a measured groupoid, i.e. a locally compact groupoid G equipped with a continuous Haar system λ and a quasi-invariant measure μ. The elements of B(G) are defined as bounded coefficients of unitary representations, or more precisely, functions of the form γ ↦ (ξ ∘ r(γ), L(γ)η ∘ s(γ)), where ξ , η are essentially bounded measurable sections of a measurable G-Hilbert bundle H. The elements of A(G) are the bounded coefficients of the regular representation H = L^2(G, λ), with a possible multiplicity. In the case of the trivial groupoid G = X × X, the algebra B(G) already appears in Krein and in Grothendieck, A. Ramsay and M. Walter have defined in the Fourier−Stieltjes of a topological groupoid rather than of a measured groupoid; while some parts of the theory overlap, the measure theoretical setting adopted here is simpler and better suited to our goal; to explain the difference, one can say that they study in the case of a space X the bounded Borel functions on X while we study the essentially bounded measurable functions on X.

Just as for groups, the Fourier−Stieltjes and the Fourier algebras play acrucial role in the duality theory of the convolution algebras $C_\mu^*(G)$ and VN(G). To express this duality, one has to take into account the presence of the unit space $X = G^{(0)}$. The main result identifies B(G) as the dual of $L^2(X)^* \otimes_{hX} C_\mu^*(G) \otimes_{hX} L^2(X)$ and VN(G) as the dual of $L^2(X)^* \otimes_{hx} A(G) \otimes_{hx} L^2(X)$. Here, we use the framework of operator spaces and $\otimes_{hX}$ means the Haagerup tensor product over $L^\infty(X)$ .The crux of the proof is anow standard application of the Hahn Banach theorem. This result also provides an interpretation of B(G) as a space of completely bounded linear maps. For example, in the case G = X × X, B(G) can be viewed as the space of Schur multipliers. In the general case, the elements of B(G) are exactly the functions which define by pointwise multiplication a bounded or, equivalently, a completely bounded linear map of $C_\mu^*(G)$ into it self.

The studies the multiplier algebras MA(G) and M_0 A(G). Just as in the case of groups, the elements of MA(G) (resp. M_0A(G)) are the functions which define by pointwise multiplication abounded (resp. completely bounded) linear map of VN(G) into itself. These multiplier algebras of VN(G), contrarily to those of C*(G), may differ. The algebras B(G), MA(G) and M_0A(G) all coincide when G is amenable and it is likely that B(G) and M_0A(G) coincide only in that case (this has been proved by M. Bozejko in the case of a discrete group). An important observation, due to Bozejko and Fendler in the group case, is that the canonical map from M_0 A(G) into B(G $*$ G) is an isometry. When they are viewed as the G-invariant elements of B(G $*$ G),the elements of M_0 A(G) are the Herz-Schur multipliers. The main results concern an arbitrary r-discrete measured groupoid and are on one hand, which characterizes the elements φ of $L^\infty(G)$ with the property εφ ∈ B(G) for every ε ∈ $L^\infty(G)$, which characterizes the elements φ of $L^\infty(G)$ with the property εφ ∈ M_0 A(G) for

 every ε ∈ $L^\infty(G)$. The first ones are the Littlewood functions, i.e. the functions which admit a decomposition φ = φ_r + φ_s such that

$$\sup_{r(\gamma)=x} \sum |\varphi_r(\gamma)|^2 < \infty$$

and   (1)

$$\sup_{s(\gamma)=x} \sum |\varphi_s(\gamma)|^2 < \infty$$

while the second ones are the Varopoulos functions, i.e. the measurable functions for which there exists a finite M such that

$$\sum_{(\gamma,\gamma')\in A^x \times B^x} |\varphi(\gamma^{-1}\gamma')|^2 \leq M \, max(|A^x| \times |B^x|) \qquad (2)$$

for all measurable subsets A, B of G and almost every x. The maining redient of the proof is the version of the non-commutative Grothendieckin equality presented which we apply to the von Neumann algebra VN(G) and translate into a statement about the Fourier algebra A(G) via duality theory.

The data will consist here of a second countable locally compact groupoid G, endowed with a faithful transverse measure Λ. We assume that G possesses a Haar system λ. The disintegration of Λ with respect to λ provides a quasiinvariant measure μ with modular function δ (by definition, the measures $\mu \circ \lambda$ and $\mu \circ \lambda^{-1}$ are equivalent and δ is the RadonNikodym derivative of $\mu \circ \lambda$ with respect to ($\mu \circ \lambda^{-1}$). We shall also use the symmetric measure $\nu = \delta^{-1/2}(\mu \circ \lambda)$ on G. The triple (G, λ, μ) will be called a measured groupoid. For $1 \leq p \leq \infty$ we shall write $L^p(G)$ and $L^p(G^{(0)})$ instead of $L^p(G, \nu)$ and $L^p(G^{(0)},\mu)$.

The extension of the classical theories of positive type functions on groups and of positive type kernels to more general groupoids is straightforward.

Proposition (3.1.1) [2]: Let (G, λ, μ) be a measured groupoid and let φ be an element of $L^\infty(G)$. Then the following conditions are equivalent:

(i) For every positive integer n and every ζ_1 , ..., $\zeta_n \in C$, the inequality

$$\sum_{i,j} \varphi(\gamma_i^{-1}\gamma_j)\bar{\zeta}_i\zeta_j \geq 0$$

holds for μ-almost every $x \in G^{(0)}$ and λ^x-almost every $\gamma_1 , ..., \gamma_n \in G^x$.

(ii) For every $\xi \in C_c(G)$, the inequality

$$\iint \varphi(\gamma_1^{-1}\gamma_2)\overline{\zeta(\gamma_1)}\zeta(\gamma_2)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \geq 0$$

holds for almost every $x \in G^{(0)}$.

(iii) For every $f \in C_c(G)$, we have the inequality

$$\int \varphi(\gamma)(f^* * f)(\gamma)dv(\gamma) \geq 0$$

(iv) There exists a measurable G-Hilbert bundle H and ξ in $L^\infty(G^{(0)},H)$ such that

$$\varphi(\gamma) = (\xi,\xi)(\gamma) \overset{\text{def}}{=} (\xi \circ r(\gamma), L(\gamma)\xi \circ s(\gamma)$$

Definition (3.1.2) [2]: An element φ of $L^\infty(G)$ satisfying the above conditions will be said to be of positive type. The set of (classes of essentially bounded) functions of positive type on G will be denoted by P(G).

Proposition (3.1.3) [2]: (i) Let G and H be measured groupoids. If φ is in P(G) and ψ is in P(H), then $\varphi \otimes \psi$ is in P(G $\times$ H).

(ii) Let G and H be measured groupoids and π:G$\rightarrow$H be a measurable homomorphism. If φ is in P(H) then $\varphi \circ \pi$ is in P(G).

(iii) The sum of two functions of positive type on G is a function of positive type.

(iv) The (pointwise) product of two functions of positive type on G is a function of positive type.

In the next proposition and in the sequel, I_2 denotes the trivial groupoid on the set $\{1, 2\}$.

Proposition (3.1.4) [2]: Let φ be an element of $L^\infty(G)$. Then the following conditions are equivalen:

(i) φ is a linear combination of elements of P(G) .

(ii)There exists a measurable G-Hilbert bundle H and sections $\xi, \eta \in$ $(G^{(0)},H)$ such that $\varphi = (\xi, \eta)$, where$(\xi, \eta)(\gamma) \overset{\text{def}}{=} (\xi \circ r(\gamma), L(\gamma)\eta \circ s(\gamma))$

(iii) There exists ρ and τ in P(G) such that $\begin{pmatrix} p & \varphi \\ \varphi* & \tau \end{pmatrix}$ is an element of P(G $\times$ I_2).

Proof:

(i)$\Longrightarrow$(ii). One can write $\lambda(\xi, \xi) + \mu(\eta, \eta)$ as $(\lambda\xi\oplus\mu\eta, \xi\oplus\eta)$ by using the direct sum of the G-Hilbert bundles.

(ii)$\Longrightarrow$(i). The polarization identity expresses (ξ, η) as a linear combination of four functions of positive type.

(ii)$\Longrightarrow$(iii).Suppose $\varphi = (\xi, \eta)$. Define $\rho = (\xi, \xi)$, $\tau = (\eta, \eta)$ and $F=\begin{pmatrix} p & \varphi \\ \varphi* & \tau \end{pmatrix}$,Then F= (ζ, ζ) where $\zeta = \xi\otimes e_1+\eta\otimes e_2$ is a section of the G $\times$ I_2-Hilbert bundle $H\otimes C^2$, where $(H\otimes C^2)_{(x, i)} = H_x\otimes C^2$ and $L(\gamma, (i, j)) = L(\gamma)\otimes e_{i,j}$. More explicitly, if we write $\xi_1 = \xi$, $\xi_2 = \eta$ and $(x, i) = \xi_i (x)\otimes e_i$, then we have $(\zeta, \zeta)(\gamma, (i, j)) = (\xi_i, \xi_j)(\gamma)$.

(iii)$\Longrightarrow$(ii). We may write $F=\begin{pmatrix} p & \varphi \\ \varphi* & \tau \end{pmatrix}$ as (ζ, ζ) where ζ is a section the G $\times$ I_2-Hilbert bundle H. We define the following G-Hilbert module H' as follows: $H'_x= H_{(x,1)}\oplus H_{(x,2)}$ and

$$L'(\gamma)=(1/2)\begin{pmatrix} L(\gamma, (1,1)) & L(\gamma, 1,2)) \\ L(\gamma, (2,1)) & L(\gamma, (2,2)) \end{pmatrix} \tag{3}$$

We define the following sections of H': $\xi(x) = (\zeta(x, 1), 0)$ and $\eta(x) = (0, \zeta(x, 2)$. Then $\varphi(\gamma) = (\zeta, \zeta)(\gamma, (1, 2)) = (1/2)(\xi,\eta)(\gamma)$.

Definition (3.1.5) [2]: An element φ of L^∞(G) satisfying the above conditions will be called a (unitary and essentially bounded) coefficient. The set of (unitary and essentially bounded) coefficients on G will be denoted by B(G).

The set B(G) of coefficients of L^∞(G) is clearly a linear subspace Moreover, it is closed under pointwise multiplication; this reflects the operation of tensoring representations: $(\xi_1, \eta_1)(\xi_2,\eta_2) =(\xi_1\otimes\xi_2,\eta_1\otimes\eta_2)$. It is also closed under the involution $\varphi*(\gamma) = \bar\varphi(\gamma^{-1})$; this reflects the unitarity of representations: if $\varphi = (\xi, \eta)$, then $\varphi* = (\eta, \xi)$. This makes B(G) into an

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involutive commutative algebra. It remains to define a norm to turn it into an involutive Banach algebra. The following lemma completes the proposition.

Lemma (3.1.6) [2]: Let φ belong to B(G). Then the following conditions are equivalent:

(i) There exists a measurable G-Hilbert bundle H and measurable sections ξ,η essentially bounded by 1 in norm such that $\varphi = (\xi,\eta)$.

(ii) There exists elements of P(G) ρ and τ essentially bounded by1 such that

$$\begin{pmatrix} \rho & \varphi \\ \varphi* & \tau \end{pmatrix}$$ belongs to P(G × I₂).

Corollary (3.1.7) [2]: For $\varphi \in$ B(G), the following numbers are equal:

(i) $\|\varphi\|_{B(G)} = \inf \|\xi\|_\infty \|\eta\|_\infty$ where the infimum is taken over all the representations $\varphi = (\xi,\eta)$.

(ii) $\inf \|F\|_{L^\infty(G)}$ where the infimum is taken over all the functions F in P(G × I₂) such that $\varphi(\gamma) = F(\gamma, (1, 2))$.

Proposition (3.1.8) [2]: The function $\| \|_{B(G)}$ as defined above is a norm on B(G) which makes it into an involutive Banach algebra.

Proof:

If $\varphi = (\xi,\eta)$, then $| \varphi(\gamma) | \leq \|\xi \circ r(\gamma)\| \|\eta \circ s(\gamma)\|$, hence $\|\varphi\|_{L^\infty(G)} \leq \|\varphi\|$. In particular $\|\varphi\| = 0 \Longleftrightarrow \varphi = 0$.

If $\|\varphi\| < \alpha$, there exists F $\in$ P(G × I₂) such that F(1, 2) = φ and $\|F\|_{L^\infty(G)} < \alpha$. Similarly, if $\|\varphi'\| < \alpha'$, there exists F'$\in$ P(G × I₂) such that F(1,2) = φ' and $\|F'\|_{L^\infty(G)} < \alpha'$. Then F+F' $\in$ P(G × I₂), $\|F + F'\|_{L^\infty(G)} < \alpha+\alpha'$ and F(1,2) + F'(1,2) = $\varphi + \varphi'$, hence $\|\varphi + \varphi'\| < \alpha+\alpha'$.

The representation $(\xi \otimes \xi', \eta \otimes \eta')$ of $\varphi \varphi'$, where $\varphi = (\xi,\eta)$ and $\varphi'=(\xi',\eta')$ gives $\|\varphi \varphi'\| \leq \|\varphi\| \|\varphi'\|$.

The representation $\varphi*=(\eta, \xi)$ of $\varphi = (\xi, \eta)$ gives $\|\varphi*\|=\|\varphi\|$. Suppose that the sequence (φ_n) in B(G) satisfies $\|\varphi_n\| < \alpha_n$ with $\sum \alpha_n < \infty$.

Then there exists a sequence (F_n) in P(G × I₂) such that $\varphi_n = F_n(1,2)$ and $\| F_n \|_{L^\infty(G)} < \alpha_n$. Then $F = \sum F_n$ exist in L^∞(G × I₂) is of positive type by, for example, characterization (iii) of Proposition (3.1.1). We know that $\varphi = \sum \varphi_n$ exists in L^∞(G) because $\| \varphi_n \|_{L^\infty(G)} \leq \| \varphi_n \| < \alpha_n$.

Since φ =F(1, 2), φ belongs to B(G). Moreover the series $\sum \varphi_n$ converges to φ in B(G) because $\| \varphi - \sum_1^n \varphi_i \| \leq \| F - \sum_1^n F_i \|_{L^\infty(G)}$ tends to zero.

Definition (3.1.9) [2]: The involutive Banach algebra B(G) is called the Fourier−Stieltjes algebra of the measured groupoid G.

Example (3.1.10) [2]: When G = X is a groupoid reduced to its unit space, then B(X) = L^∞(X).

Example (3.1.11) [2]: When G is a locally compact group, B(G) is the usual Fourier−Stieltjes algebra of the group, as defined by P.

Example (3.1.12) [2]: When G = X × X is the trivial groupoid with unit space X , where (X, μ) is a measure space, the elements of B(G) are the Hilbertian functions, as defined by A. Indeed φ ∈ L^∞(X × X, μ × μ) belongs to B(G) if and only if there exists a Hilbert space H in fact , L^2(X, μ) will do and bounded measurable functions ξ,η: X→H such that the equality $\varphi(x, y) = (\xi(x), \eta(y))$ holds for almost every (x, y) ∈ X × X. The algebra B(X × X), with a different but equivalent norm.

The regular representation Reg of the measured groupoid (G, λ, μ) is given by the regular G-Hilbert bundle L^2(G, λ), its fiber at x is $L^2(G,\lambda^x)$; the action of G is given by left translation: $(\gamma\xi)(\gamma') = \xi(\gamma^{-1}\gamma')$ and C_c(G) provides a fundamental family of sections. We shall also consider theregular representation with multiplicity L^2(G, λ, H), where H is a Hilbert space.

Definition (3.1.13) [2]: The Fourier algebra A(G) of the measured groupoid(G, λ, μ) is defined as

the closed linear span in B(G) of the coefficients of the regular

representation.

Lemma (3.1.14) [2]: Every element of A(G) can be written as a coefficient of the regular representation with infinite multiplicity $L^2(G, \lambda, I^2)$.

Proposition (3.1.15) [2]: The Fourier algebra A(G) is a norm-closed involutive ideal of B(G).

Proof:

Let (χ, η) be a coefficient of $L^2(G, \lambda)$ and (χ', η') a coefficient of an arbitrary G-Hilbert bundle H. Then $(\chi, \eta)(\chi', \eta')$ is a coefficient of the G-Hilbert bundle $L^2(G, \lambda) \otimes H = L^2(G, \lambda, H)$ which is isomorphic to a subbundle of $L^2(G, \lambda, I^2)$, the regular G-Hilbert bundle whose fiber at x is $L^2(G, \lambda^x, I^2)$. Indeed, let s*H be the induced Hilbert bundle over G via the source map s. Then the fundamental isomorphism U from $L^2(G, \lambda, H)$ onto $L^2(G, \lambda, s*H)$ defined by $\tilde{\xi}(\gamma) = \gamma^{-1}\xi(\gamma)$ trivializes the action of G on H: the action on $L^2(G, \lambda, H)$ is given by $L(\gamma) \xi(\gamma') = \gamma\xi(\gamma^{-1}\gamma')$ and the action on $L^2(G, \lambda, s*H)$ is just the regular action $\tilde{L}(\gamma) \tilde{\xi}(\gamma') = \gamma \tilde{\xi}(\gamma^{-1}\gamma')$.

Finally, s*H can be embedded into the trivial Hilbert bundle $G \times I^2$. This shows that $(\chi, \eta)(\chi', \eta')$ is in A(G). Hence A(G) is an ideal of B(G).

Lemma (3.1.16) [2]: Given any compact set K in G, one can find $f, g \in C_c(G)$ such that $f^* * g(\gamma) = 1$ for $\gamma \in K$.

Proof:

We first pick $g \in C_c(G)$ such that $\int g(\gamma^{-1}) d\lambda^x(\gamma) = 1$ for $x \in s(K)$. Then we pick $f \in C_c(G)$ such that $f(\gamma) = 1$ for $\gamma \in K$ supp g^{-1}.

Proposition (3.1.17) [2]: The elements φ of B(G) which have an r-compact sport (that is, for every compact subset K of $G^{(0)}$, supp(φ) has a compact intersection with $r^{-1}(K)$) form a dense involutive subalgebra of the Fourier algebra A(G).

Proof: Let φ be an element of B(G) with r-compact support . Since A(G) is a left $L^\infty(G^{(0)})$-module, we can use a partition of the unity in in $L^\infty(G^{(0)})$

to reduce the problem to the case when φ has compact support.The lemma gives f,g ∈ C$_c$(G) such that the coefficient (f, g) of the regular representation satisfies (f, g)(γ) =1 for γ in the support of φ. Hence φ = φ(f, g) is in A(G). On the other hand, ξ,η ∈ G$^{(0)}$, L^2(G, λ)) can be approximated in L^∞(G$^{(0)}$, L^2(G, λ)) by elements ξ$_n$, η$_n$ with r-compact support. Then the coefficient (ξ$_n$, η$_n$) approximate (ξ,η) and has an r-compact support.

In the case of a locally compact group G, it is well known that the Banach space B(G) is the dual of the full C*-algebra C*(G) and that the von Neumann algebra VN(G) has A(G) as its predual. To obtain a duality between Fourier algebras and convolution algebras of more general groupoids, one has to take into account the unit space.we consider a measured groupoid (G, λ ,μ).

Let us recall that every (continuous and involutive) representation of the convolution algebra C$_c$(G, λ) in a separable Hilbert space is obtained by integrating a representation of G: given a quasi-invariant measure μ and a measurable G-Hilbert bundle H, the integrated representation L is given by the Hilbert space L^2(G$^{(0)}$,H) and the coefficients

$$(\xi, L(f)\eta) = \int \big(\xi \circ r(\gamma), L(\gamma)\eta \circ s(\gamma)\big) f(\gamma) dv(\gamma) \qquad (4)$$

where ξ,η are in L^2(G$^{(0)}$,H) and f is in C$_c$(G). The quasi-invariant measure μ is fixed throughout this work and we shall only consider those re presentations which are absolutely continuous with respect to μ . Any such representation L extends to the involutive Banach algebra L^1((G) $\overset{\text{def}}{=}$ L^∞(G$^{(0)}$, L^1(G, λ)) ∩ L^∞(G$^{(0)}$, L^1(G,λ^{-1})) defined as the space of functions { f: G→**C**which are measurable and such that the maps $x \mapsto \int |f|\, d\lambda^x$ are essentially bounded. It is normed by

$$\|f\|_I = max\left\{\sup \int |f|\, d\lambda^x, \sup \int |f|\, d\lambda_x\right\} \qquad (5)$$

We shall denote by $C_\mu^*(G)$ the completion of $L^1(G)$ for the C*-norm $\|f\|=\sup\|L(f)\|$ where L ranges over all representations which are absolutely continuous with respect to μ.

The regular representation Reg is given by the regular G- $L^2(G, \lambda)$. The reduced C*-algebra $C_{red}^*(G)$ is obtained by completing $C_c(G)$ for the norm $\|f\|_{\text{red}} = \|\text{Reg}(f)\|$, where Reg is the regular representation and the von Neumann algebra VN(G) is the bicommutant Reg($C_c(G)$)" of the regular representation on the Hilbert space $L^2(G^{(0)}, L^2(G, \lambda)) = L^2(G, \mu \circ \lambda)$.

It is convenient to use the framework of operator spaces. We recall that a Hilbert space H has an operator space struture given by H ⊂ B(C, H) and that its dual H* has the operator space structure given by H* ⊂ B(H, C). The scalar product provides a conjugate linear isometry $\xi \mapsto \xi^*$ from H onto H*. Given two Hilbert spaces H and K, an operator T ∈ B(H, K) and a vector η ∈ K, we shall use the notation $\eta^*T \overset{\text{def}}{=} (T^*\eta)^*$ (t = T(η*)) which defines the transpose of T. Given an operator space E and x_1, ..., x_n ∈ E, we shall denote by $[x_{i1}]$ ∈ M_{n1}(E) (resp. $[x_{1i}]$ ∈ M_{1n}(E)) the column (resp. row) vector it defines.

As before, we are given a measured groupoid (G, λ, μ). Besides their operator space structures, the spaces $C_\mu^*(G)$, $L^2(G^{(0)})$ and $L^2(G^{(0)})$* are modules over the algebra A = $L^\infty(G^{(0)})$. For the sake of readibility, we shall write X = $G^{(0)}$. The actions are defined by

$hfk(\gamma) = h(r(\gamma)) f(\gamma) k(s(\gamma))$ for h, k ∈ $L^\infty(X)$ and f ∈ $L^1(G)$

$ha(x) = h(x)a(x)$ for h ∈ $L^\infty(X)$ and a ∈ $L^2(X)$

$a^*h = (\overline{h}a)^*$ for h ∈ $L^\infty(X)$ and a ∈ $L^2(X)$

These operations make $C_\mu^*(G)$, $L^2(X)$ and $L^2(X)$* into completely contractive operator $L^\infty(X)$-modules in the sense of [2].

Given an operator algebra A, a right A-operator module E and a left A-operator module F, one can define the module Haagerup tensor product of E and F over A, denoted by E $\otimes_{hA}$ F. It is the quotient of the Haagerup

tensor product E$\otimes_h$ F by the closed subspace spanned by the tensors ea$\otimes f-e\otimes$af. It is also the completion of the algebraic tensor product E $\otimes_A$ F with respect to the semi-norm $\|u\|$ = inf $\|e\| \|f\|$ where e ranges over M$_{1l}$ (E), f ranges over M$_{l1}$(F) and u = e $\odot_A$ f = $\sum$e$_i$ $\otimes_A$ f_i .In our case, the algebra A is L^∞(X) and we write $\otimes_{hX}$ instead of $\otimes_{hL^\infty(X)}$.

Definition (3.1.18) [2]: Given a measured groupoid (G, λ, μ), we define the space X(G) =L^2(X)*$\otimes_{hX}$ C_μ^*(G)$\otimes_{hX}L^2$(X). Its positive part X(G)$^+$ is defined as the image of the closed convex cone generated by the elements $a^*\otimes$T$\otimes$ a with a ϵ L^2(X) and T is a positive element of C_μ^*(G). It has aconjugate linear involution defined by $(a^*\otimes f\otimes$b)* = b*$\otimes f$ *$\otimes a$ and self-adjoint real linear subspace defined by X(G)$_{s.a.}$={u ϵ X(G): u = u*}.
The image of $a^*\otimes$T$\otimes$b in X will be written a^*Tb.

Proposition (3.1.19) [2]: Let u be an arbitrary element of X(G).

(i) It admits a representation u = a^*Tb with a^* ϵ L^2(X)*, b ϵ L^2(X) and T ϵ C_μ^*(G).

(ii) Its norm is given by $\|u\|$ = inf $\| a^*\|_2 \|$T$\| \|$b$\|_2$ where the infimum is taken over all the possible representations u = a^*Tb

(iii) It belongs to X(G)$_{s.a.}$ if and only if it admits a representation u = a^*Ta where a is in L^2(X) and T is a self-adjoint element of C_μ^*(G).

Proof:

(i) and (ii). The element u admits the representation U = $\sum_{ij}$ a_i*T$_{ij}$b$_j$ with [a_{i1}], [b$_{j1}$] ϵ L^2(X)$\otimes$ l^2 and [T$_{ij}$] ϵ B(H$\otimes l^2$), where C_μ^*(G) has been realized as a subalgebra of B(H). One sets a = $(\sum_i| a_i|^2)^{1/2}$, b = $(\sum_j| bj|^2)^{1/2}$.Then one can write a_i = h$_i$a, b$_j$ = k$_j$b with T=$\sum_{i,j}$$\bar{h}_iT_{ij}K_j$. Moreover,one has $\| a^*\|_2 = \|[a^*_{1i}]\| , \|b\|_2 = \|$[b$_{j1}$]$\|$ and
$\|$T$\|$ = $\|[\bar{h}_{1i}][T_{ij}][K_{j1}]\| \leq \| [h_{1i}][T_{ij}][K_{j1}]\| \leq \| [T_{ij}]\|$.　　　　　(6)

(iii) Suppose that u = a^*Sb is self-adjoint. We define c = $(| a|^2+| b|^2)^{1/2}$. Then we have a = hc, b = kc with h, k ϵ L^∞(X) we may write u = c*$\bar{h}$SKc. Then T= $(\bar{h}$SK+ $\bar{K}$S*h)/2 is selfadjoint and u=c*Tc.

Remark (3.1.20) [2]: One defines similarly the space $L^2(X)^* \otimes_{hX} B \otimes_{hX}$ $L^2(X)$, where B is any C*-algebra on which $L^\infty(X)$ acts by multipliers, for example, one can take B $= L^\infty(X)$, or M $C^*_\mu(G)$ (the multiplier algebra of $C^*_\mu(G)$), or the C*-subalgebra $\widetilde{C^*_\mu(G)}$ generated by $L^\infty(X)$ and $C^*_\mu(G)$. The space obtained for this last choice of A is particularly useful and will be denoted by $\widetilde{X(G)}$. Note that it contains the space $L^1(X) = L^2(X)^* \otimes_{hX}$ $L^\infty(X) \otimes_{hx} L^2(X)$. We shall use implicitly the fact that a representation L of $C^*_\mu(G)$ extends uniquely to $\widetilde{C^*_\mu(G)}$ and to M $C^*_\mu(G)$.In particular, a coefficient $\varphi = (\xi, \eta) \in B(G)$ of a representation L defines a linear functional on any of these spaces $L^2(X)^* \otimes_{hx} A \otimes_{hx} L^2(X)$ according to the formula

$$\varphi(a^*Tb) = (a\xi, L(T) b \eta) \tag{7}$$

We show that B(G) is the dual of the space X(G). The duality between the spaces B(G) and X(G) is given by the formula (7).

Lemma (3.1.21) [2]: With above notation

(i) For every $\varphi \in B(G)$, there exists a unique bounded linear functional Φ on X(G) such that

$$\Phi (a^*fb) = \int ((\overline{a \circ r})) \varphi f(b \circ s) \, dv$$

for a, b $\in L^2(X), f \in L^1(G)$.

(ii) The functional Φ is positive if and only if φ is of positive type

(iii) The map $\varphi \mapsto \Phi$ respects the involution: $\Phi(u^*) = \Phi^*(u)$.

Proof:

(i) For any $\varphi \in L^\infty(G)$, the integral on the right handside is well defined and satisfies

$$\left| \int \overline{a \circ r} \, \varphi f(b \circ s) \, dv \right| \leq \|a\|_2 \|\varphi\|_\infty \|f\|_I \|b\|_2. \tag{8}$$

If moreover $\varphi = (\xi, \eta)$, where $\xi, \eta \in L^\infty(X,H)$ belongs to B(G), this integral can be written $(a\xi, L(f) b\eta)$ where L is the integrated representation and $a\xi \in L^2(X,H)$ is defined by $a\xi(x) = a(x) \xi(x)$. Let us first define Φ on the Haagerup tensor product $L^2(X)^* \otimes_h C^*_\mu(G) \otimes_h L^2(X)$. Suppose that the

element $U \in L^2(X)^* \otimes_h C_\mu^*(G) \otimes_h L^2(X)$ is written as $U = a^* \odot f \odot b$, with $a \in L^2(X, C^p)$, $f \in M_{p,q}(C_\mu^*(G))$, $b \in L^2(X, C^q)$ (this means that $U = \sum_{i,j} a_i^* \otimes f_{i,j} \otimes b_j$ then we define

$$\Phi(U) = (a\xi, (L \otimes I)(f) b\eta) \qquad (9)$$

where $a\xi \in L^2(X, H^p)$, $b\xi \in L^2(X, H^q)$ and $(L \otimes I)(f)_{i,j} = L(f_{i,j})$. This yield

$$|\Phi(U)| \leq \|a\|_2 \|\xi\|_\infty \|f\| \|b\|_2 \|\eta\|_\infty \qquad (10)$$

hence $|\Phi(U)| \leq \|\xi\|_\infty \|\eta\|_\infty \|U\|$ and $\|\Phi\| \leq \|\varphi\|_{B(G)}$. Finally, since Φ vanishes on the elements $a^* h \otimes f \otimes kb - a^* \otimes hfk \otimes b$ as above, it factors through the quotient.

(ii) If $\varphi = (\xi, \xi)$ is of positive type, then

$$\Phi(a^* \otimes T \otimes a) = (a\xi, L(T) a\xi) \qquad (11)$$

is positive when $T \in C_\mu^*(G))$ is positive. By continuity, it is positive on $X(G)_+$. Conversely, if Φ is positive, then

$\int \varphi(f^* * f)\, d\nu$ is positive for every $f \in C_c(G)$ and, according to Proposition (3.1.1), φ is of positive type.

 (iii) One checks the equality on $u = a^* fb$ and uses the symmetry of the measure ν.

　　We want to show that the map $\varphi \mapsto I_\varphi = \Phi$ is isometric and onto. The case of a positive linear functional is easily handled.

Lemma (3.1.22) [2]: Let X(G) be as above

　　(i) Every element $u \in X(G)_+$ can be written $u = a^* T a$, where $a \in L^2(X)$ and T is a positive element of $C_\mu^*(G))$.

　　(ii) Let Ψ be a positive linear functional on X(G) of norm not greater than one. Then there is a unique $\psi \in P(G)$ such that $\Psi = l_\Psi$. Moreover, $\|\psi\| \leq 1$.

Proof:

 For (i), we first observe that the set of elements of X(G) of the form $u = a^* T a$, where $a \in L^2(X)$ and T is a positive element of $C_\mu^*(G))$ is closed under positive scalar multiplication and finite sums. Let us show that it is

closed for the norm. Suppose that u is the limit of $u_n = a_n^*T_n a_n$ with T_n positive. We write $u = a^*Ta$ with a cyclic for $L^\infty(X)$. By continuity, we obtain that for every $\varphi = (\xi, \xi) \in P(G)$, $(a\xi, L(T) a\xi) = \Phi(u) \geq 0$. There fore, $L(T) \geq 0$. for every representation L and $T \geq 0$.

For (ii), we choose a function $c \in L^2(X)$ which is continuous and does not vanish. The linear functional Φ_1 on $C^*(G)$ defined by $\Psi_1(T) =$ $\Psi(c^*\otimes T\otimes c)$ is positive and can therefore be written $\Psi_1(T) = (\xi_1, L(T) \xi_1)$ where L is a representation of $C^*(G)$ in a Hilbert space H and $\xi_1 \in H$. According to the result already quoted, there exists a quasi-invariant measure and a measurable G-Hilbert bundle H disintegrating the representation L. Since the representation of $C_0(X)$ provided by Ψ_1 is absolutely continuous with respect to μ, we may use μ as the quasiinvariant measure of this disintegration. Thus we may assume that $H = L^2(X, H)$ and write

$$\Psi_1(f) = \int (\xi_1 \circ r(\gamma), L(\gamma)\xi_1 \circ s(\gamma) f(\gamma) dv(\gamma) \tag{12}$$

For $f \in C_c(G)$. One deduces the equality $\Psi((hc)^*\otimes f \otimes(hc)) = (h\xi_1$, $L(f)h\xi_1)$ for every $h \in C_c(X)$ and every $f \in C_c(G)$. Using an approximate unit in $C^*(G)$, the norm estimate on Ψ gives $\|h\xi_1\|_2 \leq \|hc\|_2$ for every $h \in C_c(G)$. Thus we may write $\xi_1 = c\xi$ where $\xi \in L^\infty(X, H)$ has norm less than one. This gives

$$\Psi((hc)^*\otimes f\otimes(Kc)) = \int (\xi, \xi)\overline{hc} f(Kc) dv \tag{13}$$

for every h, k $\in C_c(X)$. By density, this implies that $\Psi((a^*\otimes T\otimes b) = (a\xi, L(T)b\xi)$ for every a, b $\in L^2(X)$ and T $\in C_\mu^*(G)$. Thus we have written $\Psi = \widetilde{\psi}$ with $\psi = (\xi, \xi) \in p(G)$. The uniqueness is clear : Let $\psi_1, \psi_2 \in B(G)$ be such that $\Psi = \widetilde{\psi}_1 = \widetilde{\psi}_2$ .then $\int \psi_1 f\, dv = \int \psi_2 f\, dv$ for every $f \in C_c(G)$ and $\psi_1 = \psi_2$.

The main result is an application of the Hahn-Banach theorem. The proof given below is modelled after Haagerup.

Theorem (3.1.23) [2]: The map $\varphi \mapsto \Phi$ defined above is an isometry from B(G)onto the dual of

$L^2(X)^* \otimes_{hX} C_\mu^*(G) \otimes_{hL^\infty(X)} L^2(X)$.

Proof:

Let Φ be a bounded linear functional on X(G) of norm not greater than one. We introduce the groupoid $G \times I_2$ and the space $\tilde{X}(G \times I_2)$ corresponding to the algebra $\widetilde{C_\mu^*}(G)(G \times I_2)$ generated by $C_\mu^*(G \times I_2)$ and $L^\infty(X \times \{1, 2\})$ as in Remark (3.1.20). We shall construct a positive linear functional Ψ on $\bar{X}(G \times I_2)$, with Φ as the right upper corner. The elements of $X(G \times I_2)$ will be written as (2×2)-matrices with coefficients in $\bar{X}(G)$, for example $[a^*(i) T(i, j) b(j)]$. We fix vectors $c(1)$ and $c(2)$ in $L^2(X)$ cyclic for $L^\infty(X)$. Let E be the linear subspace of $\widetilde{C_\mu^*}(G \times I_2)$ consisting of the matrices of the form $[T(i, j)]$, i $=1,2$, where $T(1,2)$ and $T(2, 1)$ belong to $C_\mu^*(G)$ and $T(i, i) = T(i)$ belongs to $L^\infty(X)$. We define on E the linear functional Ψ_1 by the formula

$$\Psi_1(T) = \sum_{i=1,2} \int \bar{c}\,(i, x)T(i)(x)c(i, x)d\mu(x)$$

$$+ \Phi\big(c(2)^* T(2,1)c(1)\big) + \Phi^*\big(c(1)^* T(1,2)c(2)\big). \qquad (14)$$

I claim that Ψ_1 positive on positive elements. By continuity, it is enough to check the positivity on the positive matrices $T=[T(i, j)]$ such that $T(1)$ and $T(2)$ are bounded from below by a strictly positive number. Then, the positivity of T is equivalent to the positivity of

$$\begin{pmatrix} 1 & T(1)^{-1l2}T(1,2)T(2)^{-1l2} \\ T(2)^{-1l2}T(1l2)^* T(1)^{-1/2} & 1 \end{pmatrix}$$

But the positivity of this latter matrix is equivalent to the norm condition

$$\big\| T(1)^{-1/2}T(1,2)T(2)^{-1/2} \big\| \le 1 \qquad (15)$$

We write S $=T(1)^{-1/2}T(1,2)T(2)^{-1/2}$.Then

$$\big|\Phi(c(I)^* T(I,2)c(2))\big| = \big|\,\Phi(c(1)^* T(1)^{1/2} ST(2)^{1/2} c(2))\,\big|$$

$$\|T(I)^{1/2}c(I)\|_2 \|T(2)^{1/2}c(2)\|_2 \tag{16}$$

Hence

$$\psi_1(T) \geq \sum_{1,2} \left\|T(i)^{\frac{1}{2}}c(i)\right\|_2^2 - 2\left\|T(1)^{\frac{1}{2}}c(1)\right\|_2 \left\|T(2)^{\frac{1}{2}}c(2)\right\|_2 \geq 0 \tag{17}$$

By Krein theorem (is operation about convex sets in topological vector spaces. A particular case of this theorem, which can be easily visualized, states that given aconvex polygon, one only needs the corners of the polygon to recover the polygone shape. The statement of the theorem is false if the polygone is not convex, as then there can be many ways of drawing apolygonhaving given points as corners.

Formally, let X be alocally convex topological vector space (assumed to be Hausdorff), and let K be acompact convex subset of X. Then, the theorem states K is the closed convex hull of its extreme points) [6]. it extends to a positive linear functional, $\widetilde{\Psi}$ on $\widetilde{C_\mu^*}(G \times I_2)$ of norm $\widetilde{\Psi}(1) = \|C\|_2^2 = \|C(1)\|_2^2 + \|C(2)\|_2^2$. Since c = (c(1), c(2)) is cyclic for $L^\infty(X \times \{1, 2\})$, there exists by extension by continuity a positive linear functional Ψ on $\tilde{X}(G \times I_2)$ of norm not greater than one such that $\Psi(c^*Tc) = \widetilde{\Psi}_1(T)$ for T $\in$ $\widetilde{C_\mu^*}(G \times I_2)$. By construction, for a matrix T which has T(1,2) as only nonzero entry, $\Psi(c^*Tc) = \Phi(c(1)^*T(1,2)c(2))$. Therefore, $\Psi(1, 2) = \Phi$, which is what we wanted.

Now, according to part (ii) of the previous lemma, there exists $\psi \in P(G \times I_2)$ of norm not greater than one such that $\Psi = l_\psi$. If we define ϕ as the right upper corner $\phi = \psi(1,2)$, we have $\Phi = l_\phi$. Moreover, we know that This concludes the proof that $\|\phi\| \leq 1$. This concludes the proof that I: B(G)→X(G)* is isometric and onto.

Remark (3.1.24) [2]: There is a norm decreasing inclusion from $L^1(G)$ into X(G). Indeed, every

$f \in C_c(G)$ admits a representation $f(\gamma) = \overline{a \circ r(\gamma)}g(\gamma)$ b $\circ$ s(γ) with a, b $\in$ $L^2(X)$ and g $\in$ $C_c(G)$. The element a*gb $\in$ X(G) depends on f only

since for every $\varphi \in B(G)$, $\varphi(a*gb) = \int f\varphi \, dv$. Moreover, this show that $\|a*gb\| \leq \|f\|_1$. In the case $G = X \times X$, where X is a measure space, this is the well known inclusion of $L^1(X \times X) = L^1(X) \otimes_\pi L^1(X)$ into $L^1(X) \otimes_{\mathbf{H'}} L^1(X)$ (which is equivalent, according to the Grothendieckin equality , to $L^1(X) \otimes_\varepsilon L^1(X)$). In the case of a locally compact group, this inclusion can be written $f \mapsto \int f(\gamma) \, L(\gamma) \, dv(\gamma)$, where L is the universal representation of G.

We may combine the theorem with the following result:

Proposition (3.1.25) [2]: There is acomplete isometry:

$$(X(G))^* = CB_{X,X}(C_\mu^* (G), B(L^2(X)))$$

where the right-hand side is the operator space of completely bounded linear maps from $C_\mu^*(G)$ into $B(L^2(X))$ which commute with the left and right actions of $L^\infty(X)$.

Proof:

It is known that the standard dual of $H^* \otimes_{\mathbf{h}} E \otimes_{\mathbf{h}} H$ is completely isometric to the operator space $CB(E, B(H))$. The isometry $\Phi \mapsto \phi_1$ is defined by

$$(a. \Phi_1(T)b) = \Phi(a^* \otimes T \otimes b \tag{18}$$

where a, b $\in H = L^2(X)$ and T $\in E = C_\mu^*(G)$. It is clear from this formula that Φ factors thru X(G) if and only if Φ_1 is a bimodule map for the left and right actions of $L^\infty(X)$.

There is a fourth interpretation of the Fourier−Stieltjes algebra B(G) which has been already observed notably, where it is used to define the norm, and in the case of an r-discrete principal groupoid.

The following proposition is essentially Theorem (2.1.1).

Proposition (3.1.26) [2]: Let φ be an element of B(G). Then pointwise multiplication by φ defines a completely bounded linear map from $C_\mu^*(G)$ into itself. Moreover, its completely bounded norm is not greater than $\|\varphi\|$.

Proof:

Let us write $\varphi = (\xi, \eta)$ with $\xi, \eta \in L^\infty G^{(0)}, K)$ and (M, K) a representation of G. Given an integer n, an arbitrary representation (L, H) of $G \times I_n$ and $\alpha, \beta \in L^2(G^{(0)} \times \{1, ..., n\}, H)$, we may write, for $f \in C_c(G \times I_n)$

$$(\alpha, L(\varphi f)\beta) = (\tilde{\alpha}, \tilde{L}(f)\tilde{\beta}) \tag{19}$$

where $\tilde{L}$ is the representation $L \otimes M$ on the bundle $H \otimes K$ (with fiber $H_{x,i} \otimes K_x$ and $\tilde{\alpha}$ (resp. $\tilde{\beta}$) is the square-integrable section $\tilde{\alpha}(x,i) = \alpha(x,i) \otimes \xi(x)$ (resp. $\tilde{\beta}(x, i) = \beta(x, i) \otimes \eta(x)$). Since $\|\tilde{\alpha}\|_2 \leq \|\alpha\|_2 \|\xi\|_\infty$ and $\|\tilde{\beta}\|_2 \leq \|\beta\|_2 \|\eta\|_\infty$, we deduce that

$$|(\alpha, L(\varphi f)\beta| \leq \|\alpha\|_2 \|\varphi\| \|f\|_{c^*_{\mu(G \times In)}} \|\beta\|_2. \tag{20}$$

This shows that $\|\varphi f\|_{c^*_{\mu(G \times In)}} \leq \|\varphi\| \|f\|_{c^*_{\mu(G \times In)}}$ \hfill (21)

Let us summarize the various characterizations of the elements of the Fourier−Stieltjes algebra B(G) that we have encountered.

Theorem (3.1.27) [2]: Let φ be an element of $L^\infty(G)$. Then the following conditions are equivalent:

(i) φ belongs to B(G) and has norm not greater than one.

(ii) For every a, b $\in L^2(X)$ and every $f \in C_c(G)$,

$$\left| \int \tilde{a} \circ r(\gamma) \, \varphi(\gamma) \, f(\gamma) \, b \circ s(\gamma) \, d\nu(\gamma) \right| \leq \|a\|_2 \|f\| \|b\|_2.$$

(iii) φ defines by pointwise multiplication a completely bounded linear map from $C^*_\mu(G)$ into $B(L^2(X))$ of completely bounded norm not greate than one.

(iv) φ defines by pointwise multiplication a bounded linear map from $C^*_\mu(G)$ into $B(L^2(X))$ of norm not greater than one.

(v) φ defines by pointwise multiplication a completely bounded linear map from $C^*_\mu(G)$ into $C^*_\mu(G)$ of completely bounded norm not greater than one.

(vi) φ defines by pointwise multiplication a bounded linear map from $C^*_\mu(G)$ into $C^*_\mu(G)$ of norm not greater than one. Moreover the corresponding notions of positivity all coincide.

Proof:

The equivalence of (i), (ii) and (iii) has been already established. The above proposition shows that (i) implies (v). The implications (v)$\Longrightarrow$(vi), (vi)$\Longrightarrow$(iv) and (iv)$\Longrightarrow$(ii) are all obvious.

The interpretation of B(G) as a space of completely bounded linear operators , namely $CB_{X,X}(C_\mu^*(G), B(L^2(X)))$, provides it with two matrix norm structures:

$$B(G, M_n) \overset{\text{def}}{=} CB_{X,X}(C_\mu^*(G), B(L^2(X)) \otimes M_n) \tag{22}$$

$$B(G, M_n^*) \overset{\text{def}}{=} CB_{X,X}(C_\mu^*(G) \otimes M_n, B(L^2(X))). \tag{23}$$

We shall give an alternate definition of these spaces.

In order to define B(G, M_n), we introduce operator-valued functions. Given an auxiliary measurable G-Hilbert bundle K, an essentially bounded measurable function φ: $\gamma \in G \mapsto \varphi(\gamma) \in B(K_{s(\gamma)}, K_{r(\gamma)})$ will be said of positive type if for every positive integer m, μ- almost every $x \in G^{(0)}$ and λ^x-almost every $\gamma_1, \ldots, \gamma_m \in G^x$ the matrix $[\gamma_i \varphi(\gamma_i^{-1} \gamma_j) \gamma_j^{-1}]$ defines positive operator in $B(K_x^m)$. The condition (iv) of Proposition (3.1.1) says in this context that φ is of positive type if and only if there exists a representation (L, H) of G and an essentially bounded measurable function $\xi : x \in G^{(0)} \mapsto \xi(x) \in$ $B(K_x, H_x)$ such that φ(γ) = (ξ ∘ r(γ))*L(γ)ξ ∘ s(γ). We shall write simply φ = ξ*ξ. The set of these operator-valued positive type functions will be denoted by P(G, B(K)) or by P(G, M_n) in the case of the trivial bundle K= $G^{(0)} \times C^n$. The Banach space B(G, B(K)) is the linear span of P(G, B(K)). More generally, one can define the Banach space B(G, B(H₂ ,H₁)), where H$_i$, I =1,2 are two fixed measurable Hilbert bundles over X. Its elements are the functions of the form φ(γ) $\overset{\text{def}}{=}$ $\xi_1^* \xi_2(\gamma)$ = (ξ₁∘ r(γ))*L(γ) ξ₂ ∘ s(γ) ,where (L,H) is a representation of G and ξ$_i$ belong to $L^\infty(G^{(0)}, B(H_i, H))$. Its norm is given by $\|\varphi\| = \|\xi_1\|_\infty \|\xi_2\|_\infty$ where the infimum is taken over all the representations φ = $\xi_1^* \xi_2$. The duality results which have been established for B(G) carry over to B(G, B(H₂ , H₁)). Indeed this space is the dual of

$$L^2(X, H_1)^* \otimes_{hx} C_\mu^*(G) \otimes_{hx} L^2(X, H_2),$$

where the duality is given, for $\xi_i \in L^\infty(X,B(H_i,H))$, $a_i \in L^2(X, H_i)$, $T \in C_\mu^*(G)$ by $\xi_1^* \xi_2(a_1^* T a_2) = (\xi_1 a_1, L(T) \xi_2 a_2)$, where $\xi_1 a_i \in L^2(X, H)$ is defined by $\xi_i a_i(x) = \xi_i(x) a_i(x)$. In this case, it is usually impossible to write an arbitrary element of the tensor product as a single elementary tensor since the representation of $L^\infty(X)$ is no longer cyclic. On the other hand, the dual of the above tensor product is easily identified as the space $CB_{X,X}(C_\mu^*(G),$ $B(L^2(X, H_2), L^2(X,H_1)))$. When one specializes $H_1=H_2$ to the trivial bundle $X \times C^n$, this says that both definitions of $B(G,M_n)$ agree.

Let us now define $B(G,M_n^*)$ or more generally $B(G, L^1(H_2,H_1))$ where H_1, H_2 are two fixed Hilbert spaces. Given a representation (L, H) of G and $\xi_i \in L^\infty(X, H \otimes_h H_i)$, i=1, 2, one defines $\varphi : \gamma \in G \mapsto (\gamma) \in H_1^* \otimes_h H_2$ by $\varphi(\gamma)$ $\overset{\text{def}}{=} (\xi_1, \xi_2)(\gamma) = (\xi_1 \circ r(\gamma)), (L(\gamma) \otimes 1)\xi_2 \circ s(\gamma))$ where (,) is the canonical map $H \otimes H_1 \times H \otimes H_2 \to H_1^* h \otimes H_2$. We define $B(G, H_1^* \otimes_h H_2)$ as the space of these functions and provide it with the norm $\|\varphi\| = \inf \|\xi_1\|_\infty \|\xi_2\|_\infty$, where the infimum is taken over all the representations $\varphi = (\xi_1, \xi_2)$ In the case $H_1 = H_2 = C^n$, we write $B(G,M_n^*)$. A proof similar to the one given above shows that this space is the dual of

$$L^2(X)^* \otimes_{hx} (C_\mu^*(G) \otimes K(H_2, H_1)) \otimes_{hx} L^2(X).$$

The duality is given, for $\xi_i \in L^\infty(X, H \otimes_h H_i)$, a_1, $a_2 \in L^2(X)$, $T \in C_\mu^*(G) \otimes K(H_2, H_1)$ by

$$(\xi_1, \xi_2)(a_1^* T a_2) = (a_1\xi_1, (L \otimes I)(T) a_2\xi_2) \tag{24}$$

On the other hand, the dual of this tensor product is easily identified as the space $CB_{X,X}(C_\mu^*(G)K(H_2, H_1), B(L^2(X))$. In particular, $B(G,M_n^*)$ can be identified with $CB_{X,X}(C_\mu^*(G) \otimes M_n, B(L^2(X))$.

We equip B(G) and A(G) with the operator space structure $A(G, M_n) \subset B(G, M_n)$.

Proposition (3.1.28) [2]: Let (G, λ, μ) be a measured groupoid with unit space $G^{(0)} = X$. Then

(i) Every element u of $L^2(X)^* \otimes_{hx} B(G) \otimes_{hx} L^2(X)$ can be written u = a*φ b with a, b $\in L^2(X)$ and φ $\in$ B(G).

(ii) For u $\in L^2(X)^* \otimes_{hx} B(G) \otimes_{hx} L^2(X)$, $\|u\| = \inf \|a\|_2 \|\varphi\| \|b\|_2$ over all the possible representations u = a*φb.

(iii) The above statements hold with A(G) in place of B(G)

Proof :

Every element U $\in L^2(X)^* \otimes_h B(G) \otimes_h L^2(X)$ admits a representation U=$\sum_{ij} a_i^* \varphi_{ij} b_j$ with [a_{i1}], [b_{j1}] $\in L^2(X, I^2)$ and [φ_{ij}] $\in$ B(G, B(I^2)). Let us assume that $\|U\| < 1$. Then, we may impose that $\|[a_{i1}]\| = \|[b_{j1}]\| = 1$ and $\|[\varphi_{ij}]\| < 1$. By definition of B(G, B(I^2)), there exist a representation (L, H) of G and ξ,η $\in L^\infty(X)$, B(I^2, H)) such that $\varphi_{ij} = (\xi\varepsilon_i, \eta\varepsilon_j)$, where ($\varepsilon_i$) is the canonical basis of I^2 and we may impose that $\|\xi\|_\infty < 1$ and $\|\xi\|_\infty < 1$. As before, we introduce a = $(\sum |a_i|^2)^{1/2}$, b= $(\sum |b_j|^2)^{1/2}$ and we write $a_i = h_i a$, $b_j = k_j b$. We have a, b $\in L^2(X)$, $\|a\|_2 = \|b\|_2 = 1$, $h_i, K_j \in L^\infty(X)$ and $\sum |h_i|^2 (x) = \sum |K_j|^2 (x) = 1$ for almost every x. We introduce the vectors ξ'=$\sum h_i \varepsilon \xi_i$, $\eta' = $ They satisfy $\|\xi'\|_\infty < 1$ and $\|\eta'\|_\infty < 1$. Moreover, the image u of U in $L^2(X)^* \otimes_{hx} B(G) \otimes_{hx} L^2(X)$ can be written u =$\sum_{ij} (h_i a)^* \varphi_{ij}(k_j b) = $ a*φb, where φ = (ξ', η'). We have realized the representation u = a*φb with $\|a\|_2 \|\varphi\| \|b\|_2 < 1$. The proof for A(G) is identical. In this case, the representation (L, H) is a multiple of the regular representation.

We have established above a duality between X(G) =$L^2(X)^* \otimes_{hx}$ $C_\mu^*(G) \otimes_{hx} L^2(X)$ and B(G) given by φ(a*Tb) = (aξ, L(T) bη) for a, b $\in L^2(X)$, T $\in C_\mu^*(G)$ and φ = (ξ, η) a coefficient of the representation L. If L is a multiple of the regular representation, the above expression is well defined for T $\in$ VN(G) and T↦(aξ, L(T)bη) is a normal linear functional on VN(G). As we shall see, this provides another description of the predual of VN(G).

Lemma (3.1.29) [2]: Let H_i, I =1, 2 and H be measurable Hilbert bundles over X.

(i) Given a $\in B(C^n, L^2(X, H_1))$ and $\xi \in L^\infty(X, B(H_1, H_2))$, we can define $V = \xi a \in B(C^n, L^2(X, H_2))$ by $V\lambda = \xi(a\lambda)$. Moreover $\|V\| \leq \|\xi\|_\infty \|a\|$. Conversely, given $V \in B(C^n, L^2(X, H))$, there exists a $\in B(C^n, L^2(X, C^n))$ and $\xi \in L^\infty(X, B(C^n, H))$ such that $V = \xi a$ and $\|V\| = \|\xi\|_\infty \|a\|$.

Proof:

(i) Since ξ is a diagonal operator and $a\lambda$ belongs to $L^2(X, H_1)$, $\xi(a\lambda)$ belong to $L^2(X, H_2)$ and $\|\xi(a\lambda)\|_2 \leq \|\xi\|_\infty \|a\lambda\|_2 \leq \|\xi\|_\infty \|a\| \|\lambda\|_2$.

For (ii), let V be in $B(C^n, L^2(X, H))$. Then we can define for every $x \in X$ a bounded operator $V(x) \in B(C^n, H_x)$ such that the equality $(V\lambda)(x) = V(x)\lambda$ holds for every $\lambda \in C^n$ and every x in the complement of a set of measure 0. Let $V(x) = \xi(x) a(x)$ be the polar decomposition of $V(x)$ with $a(x) = (V^*(x) V(x))^{1/2} \in B(C^n, C^n)$ and $\xi(x) \in B(C^n, H_x)$ a Partial isometry. This defines $\xi \in L^\infty(X, B(C^n, H))$ and a $\in B(C^n, L^2(X, C^n))$ by the formula $a(\lambda)(x) = a(x)\lambda$.

We have $V = \xi a$, $\|\xi\|_\infty = 1$ and $\|a\| = \|V\|$.

Theorem (3.1.30) [2]: The predual $VN(G)_*$ is completely isometric to the module Haagerup tensor product $L^2(X)^* \otimes_{hx} A(G) \otimes_{hx} L^2(X)$.

Proof:

Given integers $p, q, r, s, a \in M_{qp}(L^2(X)) = B(C^p, L^2(X, C^q))$, $b \in M_{rs}(L^2(X))$

$= B(C^s, L^2(X, C^r))$ and $\varphi = \xi^* \eta \in A(G, M_{qr})$, where as before $\xi \in L^\infty(X, B(C^n, H))$, $\eta \in L^\infty(X, B(C^r, H))$ and (L, H) is amultiple of the regular representation of

G, we define a linear map u: $VN(G) \to M_{ps}$ by

$$(\lambda, u(T)\mu = ((\xi a)\lambda, L(T)(\eta b)\mu) \qquad (25)$$

where $\lambda \in C^p$, $\mu \in C^s$ and we have used the notation of the lemma. The elements $(\xi a)\lambda$ and $(\eta b)\mu$ belong to $L^2(X, H)$ and satisfy the norm estimates $\|((\xi a)\lambda\|_2 \leq \|\xi\|_\infty \|a\| \|\lambda\|_2$ and $\|(\eta b)\mu\|_2 \leq \|\eta\|_\infty \|b\| \|\mu\|_2$. one deduces that u is normal and completely bounded with $\|u\|_{cb} \leq \|\xi\| \|a\| \|\eta\| \|b\|$ and therefore that $\|u\|_{cb} \leq \|a^* \odot \varphi \odot b\|$. Moreover, the map

a*⊙φ⊙b↦u is $L^\infty(X)$-linear. We thus obtain a completely contractive linear map from $L^2(X)^* \otimes_{hx} A(G) \otimes_{hx} L^2(X)$ into $VN(G)_*$.

Let us show that it is completely isometric and onto. Let n be an integer and let u be a completely bounded linear map from $VN(G)$ into M_n. It admits the Stinespring's representation $u(T) = V^* L(T) W$, where (L, H) is a representation of $VN(G)$ and V, W are bounded operators from C^n into H. We may also assume that $\|u\|_{cb} = \|V\|\|W\|$. If moreover u is normal, we can assume that (L, H) is the regular representation. We are using here the fact $VN(G)$ is in standard form in $H = L^2(G, \mu \circ \lambda)$.

We write $H = L^2(X, H)$ where $H = L^2(G, \lambda)$ is the regular G-Hilbert bundle and apply the lemma to V and W. Thus we may write $V = \xi a$ and $W = \eta b$ with $a, b \in B(C^n, L^2(X, C^n))$ and $\xi, \eta \in L^\infty(X, B(C^n, H))$ and $\|V\| = \|\xi\|_\infty \|a\|, \|W\| = \|\eta\|_\infty \|b\|$. Therefore u is the image of $a^* \odot \xi^* \eta \odot b \in M_n(L^2(X)^* \otimes_{hx} A(G \otimes_{hx} L^2(X)))$ and this element has norm not greater than $\|u\|_{cb}$.

Proposition (3.1.31) [2]: Every element of $A(G)^+$ is of the form $\varphi = (\xi, \xi)$ with $\xi \in L^\infty(G^{(0)}, L^2(G, \lambda))$

Proof:

We choose a $\in L^2(G^{(0)})$ strictly positive. As $VN(G)$ is in standard form in $H = L^2(G, \mu \circ \lambda)$, there exists $\xi_1 \in L^2(G, \mu \circ \lambda) = L^2(G^{(0)}, L^2(G, \lambda))$ such that $a^* \varphi a = (\xi_1, \xi_1)$. This implies that ξ_1 can be written $\xi_1 = a\xi$ with $\xi \in L^\infty(G^{(0)}, L^2(G, \lambda))$ and that $\varphi = (\xi, \xi)$.

Section (3.2): Fourier Algebras and Their Multipliers

Since A(G) contains a bounded approximate unit for L^∞(G), the multiplier algebra MA(G) of the Fourier algebra A(G) is naturally identified with an involutive subalgebra of L^∞(G). It is endowed with the norm $\|\varphi\|_{MA(G)} = \sup\{\|\varphi\psi\|_{A(G)}, \|\psi\|_{A(G)} \leq 1\}$. Since A(G) is an ideal in B(G) there is a norm decreasing inclusion of B(G) into MA(G).

The multipliers of A(G) have a C*-algebraic interpretation. Let φ be in MA(G). By transposition, pointwise multiplication by φ defines a bounded linear map of VN(G) into itself which has the same norm.

Proposition (3.2.1) [2]: Let (G, λ, μ) be a measured groupoid. For $\varphi \in L^\infty$(G) the following conditions are equivalent:

(i) Pointwise multiplication by φ defines a bounded linear map from A(G) into itself of norm less than one.

(ii) Pointwise multiplication by φ defines a bounded linear map from VN(G) into itself of norm less than one.

Proof:

(i)$\Longrightarrow$(ii). Let M_φ be the multiplier defined by φ. Because of the L^∞(G$^{(0)}$)-linearity of M_φ and of the norm estimate of Proposition (3.1.31), $1 \otimes M_\varphi \otimes 1$ defines a bounded linear map of $L^2(X)^* \otimes_{hx} A(G) \otimes_{hx} L^2(X)$ into itself of norm less than one. Its transpose is a bounded linear map of VN(G) into itself of norm less than one, which is given by pointwise multi plication by φ on C$_c$(G) (viewed as a subalgebra of VN(G)).

(ii)$\Longrightarrow$(i). We will first show that for every $\psi \in$ A(G), $\varphi\psi$ belongs to B(G) According to Theorem (3.1.24), it suffices to check that it defines a bounded linear functional on $L^2(X)^* \otimes_{hx} C_\mu^*(G) \otimes_{hx} L^2(X)$. Choose $f \in$ C$_c$(G) and a, b $\in L^2$(G$^{(0)}$). Then $\langle \varphi\psi, a^*f\, b \rangle = \langle a^*\psi b, \varphi f \rangle$ and

$$|\langle a^*\psi b, \varphi f \rangle| \leq \|a^*\psi b\|_{VN(G)_*} \|\varphi f\|_{VN(G)} \qquad (26)$$

$$\leq \|a\varphi b\|_{VN(G)_*} \|f\|_{VN(G)}$$

$$\leq \|a\|_2 \|\varphi\|_{A(G)} \|b\|_2 \|f\|_{C^*_{\mu(G)}}$$

This shows that φψ belongs to B(G) and has norm less than one. Since φψ has r-compact support if ψ has, multiplication by φ maps A(G) into itself.

Definition (3.2.2) [2]: An element of $L^\infty(G)$ satisfying the above equivalent properties is called a multiplier of the Fourier algebra A(G); its multiplier norm is the norm of the bounded linear map from A(G) (or from VN(G)) into itself it defines. The space of multipliers of A(G) is denoted by MA(G).

Proposition (3.2.3) [2]: Let φ be an element of MA(G). Then the following conditions are equivalent:

(i) φ defines a CB map from A(G) into itself.

(ii) φ defines a CB map from VN(G) into itself.

Moreover the CB-norms coincide.

Proof:

(i)⟹(ii). Let us assume that φ defines by multiplication a CB map from A(G) into itself of CB norm less than one: for every integer p and every $\psi \in M_p(A(G))$, we have $\|\varphi\psi\|_p \leq \|\psi\|_p$. Hence for every p, q every a, b ∈ $M_{pq}(L^2(X))$ and every $\psi \in M_p(A(G))$, we have

$$\|\varphi(a^*\odot\psi\odot b)\|_q = \|a^* \odot \varphi\psi \odot b\|_q$$
$$\leq \|a\|_{pq} \|\varphi\psi\|_p \|b\|_{pq}$$
$$\leq \|a\|_{pq} \|\psi\|_p \|b\|_{pq}. \tag{27}$$

This shows that φ defines a CB map from $L^2(X)^* \otimes_{hx} A(G)_{hx} L^2(X)$ into itself of norm less than one. transpose is a CB map from VN(G) into itself of norm less than one.

(ii)⟹(i). We now assume that for every p, q, every a_1 , b_1 ∈ $M_{pq}(L^2(X))$ and every $\psi \in M_p(A(G))$, we have

$$\|a_1^* \odot\varphi\psi\odot b_1\|_q \leq \|a_1\|_{pq} \|\psi\|_p \|b_1\|_{pq} \tag{28}$$

We fix p and $\psi \in M_p(A(G))$ and we show that $\|\varphi\psi\|_p \leq \|\psi\|_p$. We compute these norms by identifying $M_p(A(G))$ as a subspace of $CB(C^*_\mu(G),$ $B(L^2(X)^p))$. We have to show that for every integer n, every a, b $\in$ $L^2(X)^{pn}$ and every $f \in C_c(G)_n$, we have

$$(a,(\varphi\psi \otimes I_n)(f)b)| \leq \|a\|\|\psi\|_p\|f\|_n\|b\|. \tag{29}$$

We identify $L^2(X)^{pn}$ as $L(X)^p \otimes_h C^n$ and write a = $a_1 \odot a_2$, b = $b_1 \odot b_2$ with a_1 , $b_1 \in M_{1,q}(L^2(X)^p) = M_{p,q}(L^2(X)$ and a_2 , $b_2 \in M_{q,1}(C^n) = M_{n,1}(C^q)$. Then we can write

$$(a,(\varphi\psi \otimes I_n)(f)b) = \langle a_1^* \odot \varphi\psi \odot b_1, a_2^* \odot f \odot b_2 \rangle \tag{30}$$

where $a_1^* \odot \varphi\psi \odot b_1$ is an element of $M_q(L^2(X)^* \otimes_{hx} A(G) \otimes_{hx} L^2(X)) =$ $M_q(VN(G)*)$ and $a_2^* \odot f \odot b_2$ is viewed as an element of $C^q \otimes_h C^q$, which is a space which contains the dual of $M_q(VN_*(G))$ as an isometric subspace. Therefore the absolute value of this quantity is less than $\|a_1^* \odot \varphi\psi \odot b_1\|$ $\|a_2^* \odot f \odot b_2\|$. Because of our hypothesis and of the definition of the Haagerup norm, this is less than $\|a_1\|_{pq}\|\psi\|_p\|b_1\|_{pq}\|a_2\|_{n1}\|f\|_n\|b_2\|_{n1}$. Taking the infimum over the possible representations of a and b, we obtain the required inequality.

Definition (3.2.4) [2]: An element of MA(G) satisfying the above equivalent properties is called a
CB multiplier of the Fourier algebra A(G); its CB multiplier norm is the
norm of the CB linear map from A(G) (or fromVN(G)) into itself it
defines. The space of CB multipliers of A(G) is denoted by $M_0A(G)$.

Proposition (3.2.5) [2]: Let (G, λ, μ) be a measured groupoid. We have the following norm decreasing inclusions:

$$B(G) \subset M_0A(G) \subset MA(G).$$

Proof:

Consider $\varphi \in B(G)$ and $\varphi_n = \varphi \times I_n \in B(G \times I_n)$. The interpretation of φ as an element of $CB(C^*_\mu(G), B(L^2(X)))$ shows that $\|\varphi_n\| = \|\varphi\|$. On the other

hand, since $A(G \times I_n)$ is an ideal in $B(G \times I_n)$, we have that for $\psi \in A(G \times I_n)$, $\varphi_n \psi$ is an element of $A(G \times I_n)$ and $\|\varphi_n \psi\| \leq \|\varphi_n\| \|\psi\|$.

This shows that φ belongs to $M_0 A(G)$ with a norm less than $\|\varphi\|$. The other norm decreasing inclusion is clear.

We have seen that the elements of $L^\infty(G)$ which multiply $C_\mu^*(G)$ into itself are automatically completely bounded (and are precisely the elements of $B(G)$). This is no longer the case for the multipliers of $VN(G)$. We shall give a condition ensuring that every multiplier of $VN(G)$ is completel bounded.

Proposition (3.2.6) [2]: Let (G, λ, μ) be a measured groupoid. Then, the follow ing conditions are equivalent:

(i)The trivial representation is weakly contained in the regular representation.

(ii) The regular representation is faithful on $C_\mu^*(G)$

(iii) A(G) is dense in B(G) in the weak* topology.

(iv) There exists a net (e_i) in $A(G)^+$ which converges to $1 \in B(G)$ in the topology $\sigma(B(G), \widetilde{X(G)})$

(v) There exists a net (ξ_i) in $L^\infty G^{(0)}$, $L^2(G, \lambda)))$ such that

(i) $x \mapsto \|\xi_i(x)\|^2$ tends to 1 in the weak* topology of $L^\infty(G^{(0)})$.

(ii) the coefficients (ξ_i, ξ_i) converge to 1 in the weak* topology of $L^\infty(G)$.

Proof:

It is known that (i) and (ii) are equivalent: if the regular representation is faithful, any state of $C_\mu^*(G)$ is a weak limit of a net consisting of vector states of the regular representation. Conversely, if the trivial representation is weakly contained in the regular representation tensoring by the trivial representation shows that every representation is weakly contained in a multiple of the regular representation. Therefore the regular representation is faithful.

(iii)$\Longrightarrow$(ii). Let T ϵ $C_\mu^*(G)$ be such that Reg(T) = 0. Then for every a, b ϵ $L^2(X)$ and φ ϵ A(G), $\langle$a*Tb, $\varphi\rangle$= (Reg(T), a*φb) = 0, where a*Tb is viewed as an element of B(G)$_*$ and a*φb as an element of VN(G)$_*$. By density of A(G), this implies that a*Tb = 0. One deduces that for every representation L living on μ, L(T) = 0 and therefore T = 0.

(ii)$\Longrightarrow$(iv). Assume that (ii) holds. Then the regular representation is also faithful on $\widetilde{C_\mu^*(G)}$ If l = a*Ta ϵ $\widetilde{X(G)}^+$, with a cyclic and T $\geq$ 0 vanishes on A(G)$^+$, then Reg(T) = 0, hence T= 0 and l = 0. This shows that A(G)$^+$ is dense in B(G)$^+$ in the topology σ(B(G), $\widetilde{X(G)}$). In particular, there is a net (e$_i$) in A(G)$^+$ converging to I in this topology.

(iv)$\Longrightarrow$(iii). Let φ be an arbitrary element of B(G). Then φ e$_i$ belongs to A(G) and weak* converges to φ. Indeed, for every a*Tb ϵ B(G)$_*$, (φei) (a*Tb) = e$_i$ (a*φTb) converges to I(a*φTb) = φ(a*Tb). We have used the fact that B(G) multiplies $C_\mu^*(G)$.

(iv)$\Longrightarrow$(v). We have seen that every element e$_i$ ϵ A(G)$^+$ is of the form e$_i$=(ξ_i , ξ_i) where ξ_i ϵ L^∞(G$^{(0)}$, L^2(G, λ)). Moreover, the dualities which we are using embed L^1(G$^{(0)}$) and L^1(G) into $\widetilde{X(G)}$).

(v)$\Longrightarrow$(i). Let a*1a be a vector state of the trivial representation, with a ϵ L^2(G$^{(0)}$) and $\|a\|_2$=1.Then $a*$e$_i$a =($a\xi_i$, $a\xi_i$) is a positive linear functional associated with the regular representation. Since, by (a), the norms $\|a\xi_i\|_2$ tend to 1, we may assume that they are bounded. Therefore, in order to check the convergence of $a*$e$_i$ a to $a*$1a, it suffices to check the convergence on the elements of the dense subalgebra L^1(G). This convergence results from (b).

Definition (3.2.7) [2]: The measured groupoid (G, λ, μ) is called amenable if it satisfies the above equivalent conditions.

Proposition (3.2.8) [2]: Let (G, λ, μ) be an amenable measured groupoid, then MA(G) = B(G).

Proof:

Let φ be a multiplier of A(G) of norm less than M. Then pointwise multiplication by φ defines a bounded linear map from VN(G) into itself, which has the same norm. Since the norms of VN(G) and of $C_\mu^*(G)$ coincide, it also defines a bounded linear map from $C_\mu^*(G)$ into itself with the same norm. According to Theorem (3.1.27), φ belongs to B(G) and has anorm less than M.

Let G and H be groupoids and π: H→G be a homomorphism. For afunction φ defined on G, the function φ ∘ π will also be denoted by π*φ. We shall study this transposed map π* on various algebras of functions on the groupoid G .

Suppose that G and H are measured groupoids with respective measures (V_G, μ_G) and (V_H, μ_H). If π is measurable and $\pi_* V_H$ is absolutely continuous with respect to V_G, then π* is a norm decreasing homomorphism from $L^\infty(G)$ into $L^\infty(H)$. If we only assume that $\pi_*^{(0)} \mu_H$ is absolutely continuous with respect to μG, then we can still define π* as a norm decreasing homomorphism from B(G) into B(H). Indeed the element φ of B(G) can be written as a coefficient (ξ, η) where ξ and η are essentially bounded sections of a G-Hilbert bundle H. Then ξ ∘ $\pi^{(0)}$ and η ∘ $\pi^{(0)}$ are essentially bounded sections of the induced H-Hilbert bundle π*H and we define π*φ as the coefficient (ξ ∘ $\pi^{(0)}$, η ∘ $\pi^{(0)}$). The element of B(H) so obtained does not depend of the representation of φ as a coefficient and satisfies $\|\pi^*\varphi\|_{B(H)} \leq \|\varphi\|_{B(G)}$. Moreover, if π is r-proper, in the sense that the inverse image of an r-compact subset of G by π is r-compact, then π* maps A(G) into A(H).

In the case π: G × H→G is the first projection, π*φ is denoted by φ × 1. As before, I_n denotes the trivial groupoid on a set with n elements.

Lemma (3.2.9) [2]: Let I be a trivial groupoid and π: G × I→G the first projection, then

π*: B(G)→B(G × I) is a complete isometry.

Proof:

We show that π^* is an isometry and leave the general case to the reader φ

Suppose that φ ∈ B(G) is written as a coefficient (ξ, η) where ξ and η are essentially bounded sections of the G × I-Hilbert bundle H. Since G and G × I are Morita equivalent, H is, up to isomorphism, an induced bundle π^*H', where H' is a G-Hilbert bundle. Thus, for a.e $(\gamma,(i, j) \in G \times I$, $\varphi(\gamma) = (\xi(r(\gamma), i), L(\gamma) \eta(s(\gamma), j)$. By Fubini's theorem, there exists $(i_0, j_0) \in I$ such that the equality above holds for a.e. γ. This shows $\|\varphi\|_{B(G)} \leq \|\xi\|_\infty \|\eta\|_\infty$, hence $\|\varphi\|_{B(G)} \leq \|\varphi \times I\|_{B(G \times I)}$.

Proposition (3.2.10) [2]: Let φ be an element of $L^\infty(G)$. Then the following conditions are equivalent:

(i) φ belongs to $M_0A(G)$ and has norm less than one

(ii) For every measured groupoid H, φ × 1 belongs to MA(G × H) with norm less than one.

Proof:

(i)⟹(ii). Pointwise multiplication m_ϕ by φ defines a σ-weakly continuous **CB** map from VN(G) into itself with **CB**-norm less than one. It extends uniquely to a σ-weakly continuous map $\tilde{m}_\phi$ from VN(G)$\hat{\otimes}$VN(H) (the spatial tensor product) into itself of norm less than one such that

$$\tilde{m}_\phi(a \otimes b) = m_\phi(a) \otimes b \tag{31}$$

for every $a \in$ VN(G) and b ∈ VN(H). We can identify VN(G)$\hat{\otimes}$VN(H) and VN(G × H). Thus (ii) results from Proposition (3.2.1). The reverse implication has been observed (it suffices to consider H $=I_n$ for all n's).

The following result emphasizes the good behaviour of the **CB** multiplier algebra with respect to groupoid homomorphism .

Proposition (3.2.11) [2]: Let G and H be measured groupoids and π: H→G be a measurable homomorphism such that $\pi_*^{(0)} \mu_H \langle \mu_G$. Then, π^*: B(G)→B(H) extends to a completely contractive homomorphism π^*: $M_0A(G) \to M_0A(H)$.

Proof:

Let $\phi \in M_0A(G)$ be given. Propositions (3.2.10) and (3.2.1) tell us that pointwise multiplication by $1 \times \phi$ defines a σ-weakly continuous linear map m from VN(H $\times$ G) into itself. On the other hand, the homomorphism id $\times$ π: H→H $\times$ G induces an injective homomorphism L: VN(H)→VN(H $\times$ G). One checks that m sends the image of VN(H) into itself. Therefore, there exists a unique σ-weakly continuous linear map n of norm less than $\|\phi\|_{M_0A(G)}$ from VN(H) into itself such that $m \circ L = L \circ n$. One checks that the transposed map from A(H) into itself is a multiplier , denoted by $\pi^*\varphi$. Replacing G by G $\times$ I_n shows that $\pi^*\varphi$ is completely bounded with norm less than $\|\phi\|_{M_0A(G)}$. This defines the map $M_0A(G) \to M_0A(H)$. In case $\pi_*\nu_H \langle \nu_G$,π^* is the restriction to $M_0A(G)$ of the well defined homomorphism π^*: $L^\infty(G) \to L^\infty(H)$.

Recall that, given a groupoid G, a (left) G-space X comes equipped witha surjection r = $\pi^{(0)}$: X→$G^{(0)}$ and a multiplication G $*$ X→X, where G $*$ X denotes the set of composable pairs (γ, x) where $s(\gamma) = r(x)$. The associated semi-direct product is the set H = G $*$ X with unit space X, multiplication law $(\gamma', \gamma x) (\gamma,x) = (\gamma'\gamma, x)$ and inverse $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma x)$. We assume that X and G are locally compact and that these operations are continuous. The Haar system of G defines the Haar system of H. We assume that G and H are equipped with quasi-invariant measures μ_G and μ_H such that r$_*\mu_H \langle \mu_G$.

Proposition (3.2.12) [2]: Let X be a right G-space as above, H = G $*$ X the semidirect product

groupoid and π the projection homomorphism of H onto G.

(i) Let ϕ be in $L^\infty(G)$ and $\pi^*\varphi$, be its image in $L^\infty(H)$. Then, ϕ belongs to $M_0A(G)$ if and only if $\pi^*\phi$, belongs to $M_0A(H)$. Moreover they have then the same norm.

(ii) The induced homomorphism π^*:$M_0A(G) \to M_0A(H)$ is an isometry.

Proof:

We assume that $\pi^*\phi$ is in MA(H) and call m the σ-weakly continuous linear map m from VN(H) into itself it defines. The action of G on H by left multipliers induces an injective homomorphism L from VN(G) into VN(H). One checks that m maps the image of VN(G) into itself. Thus there exists a unique σ -weakly continuous linear map n of norm less than $\|\pi^*\phi\|_{MA(H)}$ from VN(H) into itself such that $m \circ L = L \circ n$. One checks that n agrees with pointwise multiplication by ϕ. Therefore π is in MA(G) and $\|\phi\|_{MA(H)} \leq \|\pi^*\phi\|_{MA(H)}$. The reverse assertion has already been shown. The property (ii) results from (i).

The most fundamental G-space is the space G itself, where G acts by left translations. The corresponding semi-direct product is the groupoid $G^{(2)}$. This groupoid is amenable in any possible sense, since it is the principal groupoid associated with the quotient map r: G→G$^{(0)}$. Its C*-algebra C*($G^{(2)}$) = $C_{red}^*(G^{(2)})$ is the continuous trace C*-algebra associated with the continuous field of Hilbert spaces $x \mapsto L^2(G,\lambda^x)$. We choose the realization.

$$G^{(2)} = G * G = \{(\gamma, \gamma') \in G \times G: r(\gamma) = r(\gamma')\} \tag{32}$$

Then, the fundamental homomorphism π: G * G→G is given by $\pi(\gamma,\gamma') = \gamma^{-1}\gamma'$. We assume as usual that G is equipped with a Haar system and a quasi-invariant measure μ .Then G * G is equipped with the Haar system $\lambda^\gamma = \delta_\gamma \times \lambda^{r(\gamma)}$ and the invariant measure μ ∘ λ.

M. Krein has put forward ,in the case when G is a group, the relation between B(G) and B(G × G) and proved that π^* is isometric when G is amenable.. The following result which generalizes, clarifies this relation.

Theorem (3.2.13) [2]: Let (G,λ,μ) be a measured groupoid and let $\pi: G * G \to G$ be the

fundamental homomorphism. Then π^* is an isometry from $M_0A(G)$ onto the G-invariant elements of B($G * G$).

Proof:

We know that $\pi^*: M_0A(G) \to M_0A(G * G)$ is isometric. But we have $M_0A(G * G) = B(G * G)$ because $G * G$ is amenable.

Definition (3.2.14) [2]: The G-invariant elements of $B(G * G)$ are called the HerzSchur multipliers of G.

F. Lust-Picquart and G. Pisier have established the following version of the non-commutative Grothendieck theorem. The following statement is taken.

Theorem (3.2.15) [2]: Let A be a C*-algebra, n an integer, T_1 , ..., $T_n \in A$ and n and $\omega_1, \ldots \omega_n \in A^*$. Then the following inequality holds:

$$\left| \sum_1^n \omega_i (T_i) \right| \leq \int_{T^n} \left\| \sum_1^n z_i \omega_i \right\|_* dz \left[\left\| \sum_i^n T_i T_i^* \right\|^{\frac{1}{2}} + \left\| \sum_i^n T_i^* T_i \right\|^{\frac{1}{2}} \right]$$

We recall that $\left\| \sum_i^n T_i T_i^* \right\|^{1/2}$ is the norm of the row vector $[T_{1n}] = [T_1, \ldots, T_n]$ and $\left\| \sum_i^n T_i^* T_i \right\|^{1/2}$ is the norm of the column vector $[T_{n1}] = {}^t[T_1, \ldots, T_n]$.

We are going to apply this result in the case where $A = C_\mu^*(G)$ is the C*-algebra of the measured groupoid (G, λ, μ) and deduce the following theorem.

Theorem (3.2.16) [2]: Let (G, λ, μ) be a measured groupoid, n an integer f_1 , ..., $f_n \in C_c(G)$ and φ_1 , ..., $\varphi_n \in A(G)$. Then the following inequality holds:

$$\left| \sum_1^n \int \varphi_i (\gamma) f_i(\gamma) dv(\gamma) \right|$$

$$\leq 2 \int_{T^n} \left\| \sum_1^n z_i \varphi_i \right\|_{A(G)} dz \quad [\alpha_r(f) + \alpha_c(f)]$$

where α_r[resp. α_c] denotes the norm of $L^2(X)^* \otimes_{hx} M_{1n}(C_\mu^*(G)) \otimes_{hx} L^2(X)$ [resp. $L^2(X)^* \otimes_{hx} M_{n1}(C_\mu^*(G)) \otimes_{hx} L^2(X)$] and $f = (f_1$, ..., $f_n)$ is viewed as an element of both spaces.

Proof:

We proceed. By homogeneity, we may assume that $\alpha_r(f) + \alpha_c(f) = 1$. Given $\varepsilon > 0$, there exist non-vanishing a, b $\in L^2(G^{(0)})$, and R_1 , ..., $R_n \in C_c(G)$ such that $f_i = aR_ib$ $\|a\|_2 = 1$ $\|b\|_2 = 1$ and $\|[R_1, ..., R_n]\|_{M1nC^*_\mu(G))} \leq (1+\varepsilon)\,\alpha_r(f)$.

Similarly, one can find non-vanishing c, d $\in L^2(G^{(0)})$ and S_1 , ..., $S_n \in C_c(G)$ such that

$f_i = cS_id$ $\|c\|_2 = \|d\|_2 = 1$ and $\|{}^t[S_1, ..., S_n]\|_{Mn1C^*_\mu(G))} \leq (1+\varepsilon)\,\alpha_c(f)$.

Finally, we define a' and b' by $a'^2 = \alpha_r(f)a^2 + \alpha_c(f)c^2$ and $b'^2 = \alpha_r(f)b^2 + \alpha_c(f)d^2$. Then $\|a'\|_2 = \|b'\|_2 = 1$ and we may write $a = ha'$ and b = kb' with h and k bounded by $1/\sqrt{\alpha_r(f)}$. Writing $R'_i = hR_ik$, we obtain

$$f_i = a'R'_ib' \qquad \|a'\|_2 = 1 \qquad \|b'\|_2 = 1 \text{ and } \|\textstyle\sum R'_i R'^*_i\|^{1/2} \leq 1 + \varepsilon$$

and similarly

$$f_i = a'S'_ib' \qquad \|a'\|_2 = 1 \qquad \|b'\|_2 = 1 \text{ and } \|\textstyle\sum S'^*_i S'_i\|^{1/2} \leq 1 + \varepsilon$$

Then, we may assume that $R'_i = S'_i$ The desired inequality results from Theorem (3.2.15) with $T_i = R_i = S_i$ and $\omega_i = a'\varphi_ib'$. Indeed,

$$\left| \sum_1^n \int \varphi_i(\gamma) f_i(\gamma) dv(\gamma) \right| = \left| \sum_1^n a'\varphi_ib'(T)_i \right|$$

$$\leq 2(1+\varepsilon) \int \left\| \sum_1^n z_i a'\varphi_ib' \right\|_* dz$$

and

$$\left\| a'\left(\sum_1^n z_i\varphi_i\right)b' \right\|_* \leq \|a'\|_2 \left\| \sum_1^n z_i\varphi_i \right\|_{A(G)} \|b'\|_2 \left\| \sum_1^n z_i\varphi_i \right\|_{A(G)}$$

We assume from now on that G is r-discrete. The following lemma gives an estimate of the norms $\alpha_r(f)$ and $\alpha_c(f)$ when the functions f_1 , ..., f_n are supported on bisections, that is, sets on which the restrictions of the maps r and s are one-to-one.

Lemma (3.2.17) [2]: Assume that G is r-discrete and that μ has module δ. Let f be in $C_c(G)$ and let S_1 , ..., S_n be a cover of supp f by open disjoint

bisections. Let f_i be the restriction of f to S_i. Then, for every $\varphi_1 , ..., \varphi_n \in$ B(G), we have:

$$| \sum_1^n \int \varphi_i f_i dv | \leq \|\varphi\|_{B(G,M_{1n}^*)} \int \left(\int |f|^2 \delta d\lambda_x \right) d\mu(x)$$

and

$$| \sum_1^n \int \varphi_i f_i dv | \leq \|\varphi\|_{B(G,M_{n1}^*)} \int \left(\int |f|^2 \delta^{-1} d\lambda^x \right) d\mu(x)$$

Proof:

We write φ_i as a coefficient: $\varphi_i = (\xi, \eta_i)$ with $\xi, \eta_i \in L^\infty(G^{(0)},H)$ We then have:

$$\sum_1^n \int \varphi_i f_i dv = \sum_1^n \int (\xi \circ r(\gamma), L(\gamma)\eta_i (x) f_i(\gamma)\delta^{1/2}(\gamma) d\lambda_x(\gamma) d\mu(x)$$

$$= \sum_1^n \int (\xi \circ r(S_i x), L(S_i x)\eta_i (x)) f_i(S_i x)\delta^{1/2}(S_i x) d\mu(x)$$

We have taken into account the fact that for a given i and a given x, there is at most one element $\gamma = S_i x$ in S_i with source x. Therefore,

$$| \sum_1^n \int \varphi_i f_i dv | \leq \sum_1^n \int \|\xi \circ r(S_i x)\| \| \eta_i (x) | f_i(S_i x) | \delta^{1/2}(S_i x) d\mu(x)$$

$$\leq sup_y \|\xi(y)\| \int \sum_1^n \|\eta_i(x)\| f_i(S_i x)\delta^{1/2}(S_i x) d\mu(x)$$

$$\leq sup_y \|\xi(y)\| \int \left(\sum_1^n \|\eta_i(x)\|^2 \right)^{1/2}$$

$$\times \left(\sum_1^n |f_i(S_i x)|^2 \delta(S_i x) \right)^{1l2} d\mu(x)$$

$$\leq sup_y \|\xi(y)\| sup_y \int \left(\sum_1^n \|\eta_i(y)\|^2 \right)^{1l2}$$

$$\times \int \left(\sum_1^n |f_i(S_i x)|^2 \, \delta(S_i x) \right)^{1/2} d\mu(x)$$

$$\leq \sup_y \|\xi(y)\| \sup_y \int \left(\sum_1^n \|\eta_i(y)\|^2 \right)^{1/2}$$

$$\times \int \left(\sum_{s(\gamma)=x} |f_i(\gamma)|^2 \, \delta(\gamma) \right)^{1/2} d\mu(x)$$

This gives the first inequality. The other inequality is proved in the same function.

Combining this lemma with the previous theorem, one obtains the following result:

Proposition (3.2.18) [2]: Assume that G is r-discrete and that μ has module δ. Let f be in $C_c(G)$

with compact support K, and let $S_1, \ldots, S_n$ be a cover of K by open disjoint bisections. Then, for every bounded measurable function φ on G, we have:

$$\left| \int \varphi f \, dv \right| \leq 2 \int_{T^n} \|z_s \varphi\|_{A(G)} dz \left[\int \left(\int |f|^2 \, \delta \, d\lambda_x \right)^{1/2} d\mu(x) \right.$$

$$\left. + \int \left(\int |f|^2 \, \delta^{-1} d\lambda_x \right)^{1/2} d\mu(x) \right]$$

where $z_s(\gamma) = z_i$ if $\gamma \in S_i$ and $z_s(\gamma) = 0$ otherwise.

Proof:

We apply the theorem to $f_i = f_{|S_i}$ and $\varphi_i = \varphi_{|S_i}$ and use the estimate of $\alpha_r(f)$ and $\alpha_c(f)$ provided by the lemma.

Corollary (3.2.19) [2]: Let φ be a bounded measurable function on G. Then the following conditions are equivalent:

(i) There exists a partition of G by a countable family of open bisections S_i such that $\varepsilon\phi$ belongs to B(G) for every bounded function ε constant on each S_i.

(ii) φ admits a decomposition $\varphi = \varphi^{(1)} + \varphi^{(2)}$ where $\varphi^{(1)}$ and $\varphi^{(2)}$

satisfy, respectively,

$$sup_x \sum_{s(\gamma)=x} |\varphi^{(1)}(\gamma)|^2 < \infty \qquad and \qquad sup_x \sum_{r(\gamma)=x} |\varphi^{(2)}(\gamma)|^2 < \infty$$

(iii) For every bounded measurable function ε, εφ belongs to B(G).

Proof:

(i)$\Longrightarrow$(iii). By the closed graph theorem, there exists a constant M such that for every $\varepsilon \in I^\infty(\mathbf{N})$, we have $\left\| \Sigma \varepsilon_i \; \varphi_{|Si} \right\|_{B(G)} \leq M \|\varepsilon\|_\infty$. Then for every finite subfamily $S_\mathbf{F} = (S_\mathbf{i})_{i \in F}$ and every $z \in \mathbf{T^F}$, we have $\left\| z_{S_F}\varphi \right\|_{A(G)} \leq M$. Hence for every $f \in C_c(G)$, it holds that

$$\left| \int \varphi f dv \right| \leq 2M \times \left[\int \left(\int |f|^2 \delta d\lambda_x \right)^{1/2} d\mu(x) \right. \qquad (33)$$

$$\left. + \int \left(\int |f|^2 \delta^{-1} d\lambda_x \right)^{1/2} d\mu(x) \right]$$

This says that φ defines a bounded linear functional of norm $\leq 2M$ on the space $L^1(G^{(0)}, L^2(G,\delta\lambda^{-1})) \cap L^1(G^{(0)}, L^2(G, \delta^{-1}\lambda))$. Notice that in the duality $\langle\varphi,f\rangle = \int \varphi f dv$,the dual space of $L^1(G^{(0)}, L^2(G, \delta\lambda^{-1}))$ is $L^\infty(G^{(0)}, L^2(G, \lambda^{-1}))$ while the dual space of $L^1(G^{(0)}, L^2(G, \delta^{-1}\lambda))$ is $L^\infty(G^{(0)}, L^2(G, \lambda))$. Therefore φ belongs to the dual space $L^\infty(G^{(0)}, L^2(G, \lambda^{-1})) + L^\infty(G^{(0)}, L^2(G, \lambda))$.

(ii)$\Longrightarrow$(iii). It suffices to consider the cases $\varphi = \varphi^{(i)}$, $i =1, 2$. Suppose for example that φ belongs to $L^\infty(G^{(0)}, L^2(G, \lambda))$. Then so does $\varepsilon\varphi$ For every bounded measurable function ε. To conclude, observe that every element $\varphi \in L^\infty(G^{(0)}, L^2(G, \lambda))$ can be written as the coefficient $\varphi = (\tilde{\varphi},\eta)$. where η is the characterist function of $G^{(0)}$ (and it has a norm not greater than $\|\varphi\|_\infty$). The other case is similar.

(iii)$\Longrightarrow$(i) is trivial.

Theorem (3.2.20) [2]: The space of multipliers $M(L^\infty(G), B(G))$ and the space of Little wood functions $LT(G) = L^\infty(G^{(0)}, L^2(G,\lambda^{-1})) + L^\infty(G^{(0)}, L^2(G, \lambda))$ Coincide and their norms are equivalent

$$(1/2)\|\varphi\|_{LT} \leq \|\varphi\|_{M(L^{\infty}(G),B(G))} \leq \|\varphi\|_{LT}$$

We study next the functions on G which multiply $L^{\infty}(G)$ into the space of completely bounded multipliers $M_0A(G)$ (we call them the absolute Fourier multipliers). Let π: G $*$ G→G be the fundamental homomorphism $\pi(\gamma,\gamma') = \gamma^{-1}\gamma'$. A function ψ on G $*$ G is of the form $\pi^*\varphi = \varphi \circ \pi$ iff it is invariant under the diagonal action of G onto G $*$ G, i.e. iff it satisfies $\psi(\gamma\gamma_1,\gamma\gamma_2) = \psi(\gamma_1,\gamma_2)$. Recall that π^* is an isometry from $M_0A(G)$ onto the subspace $B(G * G)^G$ of $B(G * G)$ consisting of the invariant elements.

Proposition (3.2.21) [2]: The following space coincide and have equivalent norms:

(i)The space of multipliers M $(L^{\infty}(G), M_0A(G))$

(ii) The space LT $(G * G)^G$ of Littlewood functions φ on $(G * G)$ invariant under G.

Proof:

Let φ be a bounded measurable function on G which multiplies $L^{\infty}(G)$ into $M_0A(G)$. Choose a countable partition (S_i) of G by open bisections.

Then $(\pi^{-1}(S_i))$ is a countable partition of G $*$ G by open bisections and Corollary (3.2.19) applies to the function $\pi^*\varphi$. Therefore $\pi^*\varphi$ is aLittlewood function on G $*$ G. Conversely, if ψ is a Littlewood function on G $*$ G invariant under G, it is of the form $\pi^*\varphi$ and it multiplies $L^{\infty}(G*G)$ into $B(G * G)$. In particular, it multiplies $\pi^*L^{\infty}(G)$ nito $B(G * G)^G$ .This says that φ multiplies $L^{\infty}(G)$ into $M_0A(G)$.

There is a characterization, introduced by N. Varopoulos, of the absolute Fourier multipliers which does not use the groupoid G $*$ G.

Definition (3.2.22) [2]: A measurable function φ: G→C will be called aVaropoulos function if there exists M > 0 such that for every measurable subsets E, F $\subset$ G and for μ-almost every x,

$$\int\limits_{E\times F} |\varphi(\gamma^{-1}\gamma')|^2 d\lambda^x(\gamma)d\lambda^x(\gamma') \le M^2 max(\lambda^x(E),(\lambda^x(F)).$$

The space of Varopoulos functions is denoted by Va(G) and the norm $\|\varphi\|_{Va}$ is the least M satisfying above condition.

Proposition (3.2.23) [2]: Let (G, λ, μ) be a measured groupoid. Then every Littlewood function is a Varopoulos function. Moreover we have $\|\varphi\|_{Va} \le \sqrt{2}\,\|\varphi\|_{LT}$.

Proof:

Suppose that $\varphi^{(1)}$ satisfies $\sup_x \int |\phi^{(1)}(\gamma)|^2\,d\lambda^x(\gamma) \le M_1^2$. Let E be a measurable subset of G. Then,

$$\int\limits_{E\times E} |\varphi^{(1)}(\gamma^{-1}\gamma')|^2 d\lambda^x(\gamma)d\lambda^x(\gamma') \le M_1^2\lambda^x(E) \tag{34}$$

One obtains similarly that if $\varphi^{(2)}$ satisfies $\sup_x \int |\varphi^{(2)}(\gamma)|^2 d\lambda_x(\gamma) \le M_2^2$, then

$$\int\limits_{E\times E} |\varphi^{(2)}(\gamma^{-1}\gamma')|^2 d\lambda^x(\gamma)d\lambda^x(\gamma') \le M_2^2\lambda^x(E) \tag{35}$$

Hence, if ϕ admits the Littlewood decomposition $\varphi = \varphi^{(1)} + \varphi^{(2)}$, then

$$\int\limits_{E\times E} |\varphi(\gamma^{-1}\gamma')|^2 d\lambda^x(\gamma)d\lambda^x(\gamma') \le 2(M_1^2 + M_2^2\lambda^x(E) \tag{36}$$

In the case of the groupoid G ∗ G, Varopoulos and Littlewood functions coincide.

Proposition (3.2.24) [2]: Let X,Y,Ω be locally compact spaces, π_x[resp. π_Y]be a local homeomorphism from X [resp.Y] onto Ω and Λ be a measure on Ω. Suppose that φ: X $\times_\Omega$ Y→C is a measurable function satisfying for almost every ω

$$\sum\limits_{A\omega\times B\omega} |\varphi(x,y)|^2 \le (|A_\omega|, |B_\omega|)$$

For every measurable set $A \subset X$, $B \subset Y$ and where A_ω, B_ω are the ω-sections for every measurable set Then, one can write $\varphi = \varphi^{(1)} + \varphi^{(2)}$ where $\varphi^{(1)}$, $\varphi^{(2)}$: $X \times {}_\Omega Y \to C$ are measurable functions satisfying for almost ωevery

$$\sum_{\pi Y(y) = \omega} |\varphi^{(1)}(x, y)|^2 \le 1 \qquad and \qquad \sum_{\pi X(x) = \omega} |\varphi^{(2)}(x, y)|^2 \le 1$$

Proof:

This is a version of Varopoulos' result with parameter. Let us fix ω and denote by φ_ω the restriction of φ to $X_\omega \times Y_\omega$. one can write $\varphi_\omega = \varphi^{(1)} + \varphi_\omega^{(2)}$,where

$\varphi^{(1)}$, $\varphi^{(2)}$: $X_\omega \times Y_\omega \to C$ satisfy

$$\sum_{\pi Y(y) = \omega} |\varphi_\omega^{(1)}(x, y)|^2 \le 1 \qquad and \qquad \sum_{\pi X(x) = \omega} |\varphi_\omega^{(2)}(x, y)|^2 \le 1$$

One deduces that for every $f \in C_c(X \times {}_\Omega y)$,

$$\left| \sum_{X_\omega \times Y_\omega} \varphi(x, y) f(x, y) \right|$$

$$\le \sum_{x \in X_\omega} \left[\sum_{y \in Y_\omega} |f(x, y)|^2 \right]^{1/2}$$

$$+ \sum_{y \in Y_\omega} \left[\sum_{x \in X_\omega} |f(x, y)|^2 \right]^{1/2} \tag{37}$$

We integrate over Ω to obtain

$$\left| \int \varphi(x, y) f(x, y) dv(x. y) \right| \le \int \left[\sum_y |f(x, y)|^2 \right]^{1/2} d\mu_X(x)$$

$$+ \int \left[\sum_y |f(x, y)|^2 \right]^{1/2} d\mu_Y(y) \tag{38}$$

where v, μ_X, μ_Y are the lifts of Λ to $X \times {}_\Omega Y$, X, Y. Therefore, φ defines abounded linear functional on the space $L^1(X, L^2(\pi_x^* Y)) \cap L^1(Y, L^2(\pi_y^* X)$ and admits the announced representation.

To conclude the discussion, we make the following observation.

Proposition (3.2.25) [2]: Let φ be a bounded measurable function on G. Then the following assertions are equivalent:

(i) φ is a Varopoulos function on G with V aropoulos norm less than M.

(ii) π*φ is a Varopoulos function on G ∗ G with Varopouls norm less than M.

Proof:

Assume (i) By definition, for every measurable subsets E, F of G and for almost every x, we have

$$\int_{E\times F} |\varphi(\gamma^{-1}\gamma')|^2 d\lambda^x(\gamma) d\lambda^x(\gamma') \leq M^2 max(\lambda^x(E), (\lambda^x(F)). \qquad (39)$$

But this is equivalent to

$$\int_{G*E\times G*F} |\pi^*\varphi(\gamma_1,\gamma_2)|^2 d\lambda^\gamma(\gamma,\gamma_1) d\lambda^\gamma(\gamma,\gamma_2) \leq M^2(\lambda^\gamma(G*E) \qquad (40)$$

for almost every γ. Hence the required property is satisfied for the subsets of G ∗ G of the form G ∗ E, where E is a measurable subset of G. One deduces that it is satisfied for every measurable subset E' of G ∗ G. The above also shows that the converse is true.

Chapter 4

L^p-Fourier For Locally Compact Groups

We investigate how these spaces reflect properties of the underlying group and study the structural properties of these algebras. As an application of this theory, we characterize the Fourier-Stieltjes ideals of SL(2,$\mathbb{R}$).

Section (4.1): L^p-representations of C*-Algebra with L^p-Fourier Algebra

The theory of Banach algebras is motivated by examples, and many of the important examples in the field of Banach algebras arise from locally compact groups. The most classicly studied Banach algebra associated to a locally compact group G is the group algebra L^1(G) with multiplication given by convolution. showed that L^1(G) complete invariant for locally compact groups G in the sense that $L^1(G_1)$ is isometrically isomorphic to $L^1(G_2)$ as Banach algebras if and only if G_1 is homeomorphically isomorphic to G_2. Hence, we can expect that many properties of the group may be reflected in the group algebra. For example, it is easily checked that G is abelian if and only if L^1(G) is commutative and G is a discrete group if and only if L^1(G) is unital. A much less obvious property shown by Barry Johnson is that G is amenable if and only if L^1(G) is amenable as a Banach algebra.

Since the group algebra L^1(G) of a locally compact group G is an involutive Banach algebra, it is natural to consider operator algebras containing a copy of L^1(G) as a dense subspace. The most heavily studied of these are the full and reduced group C*-algebras C*(G) and C_r^*(G), and the group von Neumann algebra VN(G) which contain norm and weak*-dense copies of L^1(G), respectively. Unlike the group algebra L^1(G), these operator algebras fail to completely determine the group G but are still able to encode many useful properties of the underlying group G .

Related to the group von Neumann algebra and the full group C*-algebra, we have the Fourier algebra A(G) and the Fourier-Stieltjes algebra B(G) which naturally identify with the predual of VN(G) and the dual of C*(G), respectively. The Fourier and Fourier-Stieltjes algebras are can be viewed as subalgebras of C_0(G) and C_b(G), respectively, endowed with a norm dominating the uniform norm. Despite always being commutative Banach algebras even when G is nonabelian, Martin Walter demonstrated that these Banach algebras A(G) and B(G) are complete invariants . In many ways A(G) is analagous to the group algebra L^1(G), however it is not the case that A(G) is amenable if and only if G is amenable. In fact Brian Forrest and Volker Runde showed that A(G) is amenable if and only G is almost abelian, i.e., if and only if G contains an open abelian subgroup of finite index Recall that A(G) is the predual of VN(G) and, hence, has a canonical operator space structure. By taking this observation into account, Zhong-Jin Ruan demonstrated that G is amenable if and only if A(G) is operator amenable.The amenability of G has also been characterized by the existance of a bounded approximate identity in A(G) by Leptinand in terms of the multipliers of A(G) by Losert.

Nate Brown and Erik Guentner defined the concept of L^p-representations and their associated C*-algebras. Let G be a locally compact group and $1 \leq$ p $\leq \infty$. A (continuous unitary) representation
π : G $\rightarrow$ B($\mathcal{H}$) is said to be an L^p-representation roughly speaking, the matrix coefficient functions s$\mapsto\langle\pi(s)x,x\rangle$ are in L^p(G) for sufficiently many $x \in \mathcal{H}$. As examples, the left regular representation π is an L^p-representation of G for each $1 \leq$ p $< \infty$ and the trivial representation is an L^p-representation if and only if G is compact. When G is the group SL(2,$\mathbb{R}$), it is an immediate consequence of the work of Ray Kunze and Elias Stein that each nontrivial irreducible represenation of G is an L^p-represenation of G for some p $\in$ [2,∞). The C*-algebra $C^*_{L^p}$(G) is defined to be the completion of L^1(G) with respect to a C*-norm arising from L^p-represenations of the

group G. When p ∈ [1, 2], $C_{L^p}^*$(G) is simply the reduced group C*-algebra C_r^*(G), but this need not be the case for p > 2. Indeed, Rui Okayasu showed that the C*-algebras $C_{\ell^p}^*$($\mathbb{F}_d$) are distinct for every $2 \leq p < \infty$ where $\mathbb{F}_d$ denotes the free group on $2 \leq d < \infty$ generators. We observe that the analogous result holds for SL (2, $\mathbb{R}$).

In Michael Brannan and Zhong-Jin Ruan defined and developed some basic theory of L^p-Fourier and Fourier-Stieltjes algebras, denoted A_{L^p}(G) and B_{L^p}(G). Evidencing their usefulness, the L^p-Fourier−Stieltjes algebras were used to find many intermediate C*-norms on tensor products of group C*-algebras. The L^p-Fourier and Fourier-Stieltjes algebras are ideals of the Fourier−Stieltjes algebra corresponding to coefficient functions of L^p-representations. Similar to the case of the C*-algebras, the L^p-Fourier algebra coincides with the Fourier algebra A(G) and the L^p-Fourier-Stieltjes algebra with the reduced Fourier−Stieltjes algebra when p ∈ [1, 2]. This is not the case necessarily for p > 2. In fact we demonstrate rich classes of groups G so that A_{L^p}(G) is distinct for every p ∈ [2, ∞) and B_{L^p}(G) is distinct for each p ∈ [2,∞). As an application of the theory developed, we characterize the Fourier-Stieltjes ideals of SL(2, $\mathbb{R}$) in terms of L^p-Fourier-Stieltjes algebras .

Similar to the Fourier algebra, we show the L^p-Fourier algebra is a complete invariant for locally compact groups. Unlike the Fourier algebra, the L^p-Fourier algebra can lack many nice properties even when G is a very nice group. For example, when G is a noncompact abelian group, then A_{L^p}(G) is not even square dense for each p ∈ (2,∞) and, hence, lacks any reasonable notion of amenability. So the analogues of Ruan's and Leptin's characterizations of amenability fail for the L^p-Fourier algebras. Though the analogues of these characterizations of amenability fail for the L^p-Fourier algebra,we show that the analogue of Losert's characterization of amenability in terms of multipliers holds for A_{L^p}(G) and RundeSpronk's characterization of amenability in terms of operator Connes amenability

holds for $B_{L^p}(G)$.

We provide an overview of the theory of Fourier and Fourier-Stieltjes spaces as developed by Pierre Eymard and Gilles Arsac.

Let G be a locally compact group. The Fourier-Stieltjes algebra is defined to be the set of coefficient functions $s \mapsto \pi_{x,y} := \langle \pi(s)x, y \rangle$ as $\pi : G \to B(\mathcal{H}_\pi)$ ranges over the (continuous unitary) representations of G and x, y over $\mathcal{H}_\pi$. Then B(G) identifies naturally with the dual of the full group C*-algebra C*(G) via the identification $\langle u, f \rangle \int_G u(s) f(s) ds$ for $f \in L^1(G)$ and u $\in$ B(G). When endowed with the norm attained from this identification with C*(G)*, the Fourier-Stieltjes algebra becomes a Banach algebra under pointwise operations. When G is abelian, B(G) is isometrically isomorphic as a Banach algebra to the measure algebra M($\widehat{G}$) and the isomorphism is given by the Fourier-Stieltjes transform.

Let S be a collection of representations of G. The Fourier space A$_S$ is defined to be the closed linear span of coefficient functions $\pi_{x,y}$ in B(G) as π ranges over representations in S and x, y over $\mathcal{H}_\pi$. If S consists of a single representations π, then A_S is denoted by A_π. These Fourier spaces A_S are translation invariant (under both left and right translation) subspaces of B(G) and, conversely, every closed translation invariant subspace of B(G) is realizable as a Fourier space A_π for some representation π of G. A fortiori, for every collection S of representations of G, A_S is equal to A_π for some representation π of G .

As a distinguished Fourier space, the Fourier algebra A (G) is defined to be A_λ where λ denotes the left regular representation of G. Although not obvious, it is a consequence of Fell's absorption principle that A(G) is a subalgebra (and in fact an ideal) of B (G). When G is abelian A(G) is isometrically isomorphic to the group algebra $L^1(\widehat{G})$ via the Fourier transform.

Fix some representation π of G. The Fourier space A_π is exactly the set of infinite sums $\sum_{n=1}^\infty \pi_{x_n,y_n}$ with $\{x_n\}, \{y_n\} \subset \mathcal{H}_\pi$ satisfying the condition that $\sum_{n=1}^\infty \|x_n\| \|y_n\| < \infty$. Moreover the norm of an element $u \in A_\pi$ is given by

$$\|u\| = \inf\left\{\sum_{n=1}^\infty \|x_n\| \|y_n\| : u = \sum_{n=1}^\infty \pi_{x_n,y_n}\right\} \tag{1}$$

and this infimum is attained.

For each represnation π of G, define VN_π to be the von Neuman algebra $\pi(L^1(G))'' = \pi(G)'' \subset B(\mathcal{H}_\pi)$. The Fourier space A_π naturally identifies with the predual of VN_π via the pairing $\langle u, T \rangle = \sum_{n=1}^\infty \langle Tx_n, y_n \rangle$, for u $\in$ B(G) and T$\in$ VN_π, where $u = \sum_{n=1}^\infty \pi_{xn,yn}$ and $\sum_{n=1}^\infty \|x_n\| \|y_n\| < \infty$. In particular, this implies that the Fourier algebra A(G) is the predual of the group von Neumann algebra VN(G) = VN_λ. For representations π and σ of G, the Fourier space A_π is contained in A_σ if and only if π is quasi-contained in σ, i.e., if and only if π is contained in some amplification $\sigma^{\oplus\alpha}$ of σ for some cardinal α, which occurs if and only if the map $\sigma(f) \mapsto \pi(f)$ for $f \in L^1(G)$ extends to a normal *-isomorphism from $VN_\sigma \to VN_\pi$. Hence, there is a one-to-one correspondence between group von Neumann algebras VN_π of G and Fourier spaces A_π.

Let **S** be a collection of representations of G. The Fourier-Stieltjes space $B_{\mathbf{S}}$ is defined to be the closure of $A_{\mathbf{S}}$ in the weak*-topology $\sigma(B(G),C^*(G))$. Since every Fourier space is realizable as a space A. for some representation π of G, every Fourier-Stieltjes space is also realizable as B_σ for some representation σ.

Let π be a representation of G. Then the Fourier-Stieltjes space B_π. can be identified with the C*-algebra C_π^*: $= \overline{\pi(L^1(G))}^{\|\cdot\|} \subset B(\mathcal{H}_\pi)$ via the pairing $< u, \pi(f) > = \int_G u(s)f(s)ds$.

Let S and S' be two collections of representations of G. Then $B_{\mathbf{S}} \subset B_{\mathbf{S'}}$. if and only if $\sup_{\pi\in S}\|\pi(f)\| \leq \sup_{\sigma\in S'}\|\sigma(f)\|$ for every $f \in L^1(G)$, i.e., if and

only if **S** is weakly contained in S'. Hence, there is a one-to-one correspondence between group C*-algebras C_π^* of G and Fourier-Stieltjes spaces B_π.

We study the Fourier and Fourier-Stieltjes spaces associated to the L^p-representations of a locally compact group G.

The theory of L^p-representations and their corresponding C*-algebras for discrete groups was recently developed by Nate Brown and Erik Guentner. Though Brown and Guentner defined L^p-representations of discrete groups, their definitions and basic results generalize immediately of locally compact groups. Rather than making explicit notes of this, we will simply state their results in the context of

locally compact groups.

Let G be a locally compact group and D a linear subspace of $C_b(G)$. A representation $\pi:G \rightarrow B(\mathcal{H}_\pi)$ is said to be a D-representation if there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}_\pi$ so that $\pi_{x,x} \in D$ for every $x \in \mathcal{H}_0$. The following facts are noted, and are easily checked:

(i) The D-representations are closed under arbitrary direct sums.

(ii) If D is a subalgebra of $C_b(G)$, then the tensor product of two D-representations remains a D-representation.

(iii) If D is an ideal of $C_b(G)$, then the tensor product of a D-representation with any representation is a D-representation.

For our purposes, we will be most interested in studying the case when D = $L^p(G) \cap C_b(G)$ for p $\in$ [1,∞). In this case, the left regular representation λ of G is an L^p-representation since taking the dense subspace of $L^2(G)$ to be $C_c(G)$ clearly satisfies the required condition.

To each linear subspace D of $C_b(G)$ define a C*-seminorm $\|\cdot\|_\mathbf{D} : L^1(G)$ $\in$ [0,∞) by

$$\| f \|_\mathbf{D} = sup\{ \| \pi (f) \| : \pi \text{ is a D-representation} \}. \qquad (2)$$

The C*-algebra $C_D^*(G)$ is defined to be the "completion" of $L^1(G)$ with respect to this C*- seminorm. When D = $L^p(G) \cap$ B(G), we write $C_{Lp}^*(G) =$

$C_D^*(G)$ and $\|.\|_{L^p} = \|\cdot\|_D$. This process of building C*-algebras was originally completed in the case when D was an ideal of $\ell^\infty(\Gamma)$ of a discrete group Γ, and was called an ideal completion . We note that in the case when $D = L^p$ for some p $\in$ [1,∞), then $\|.\|_{L^p}$ dominates the reduced C*-norm since λ is an L^p-representation. A fotiori $\|.\|_{L^p}$ is a norm on $L^1(G)$ and the identity map on $L^1(G)$ extends to a quotient map from $C_{Lp}^*(G)$ onto $C_r^*(G)$.

In general, it is desirable that the space D $\subset$ $c_b(G)$ used in this construction is translation invariant (under both left and right translation). Indeed, this guarantees that if u is a positive definite function on G which lies in D, then the GNS representation of u is a D-representation and, hence, u extends to a positive linear functional on $C_D^*(G)$.

The subspaces D of $C_b(G)$ which have been most heavily studied in the context of D-representations are $C_0(G)$ and $L^p(G)$. Brown and Guentner recognized both these cases in their section and developed much of the basic theory for the associated C*-algebras in their original section. In the case when D = L^p, Brown and Guentner demonstrated that $C_{Lp}^*(G) = C_r^*(G)$ for every p $\in$ [1,2] and that if $C_{Lp}^*(G)$ = C*(G) for some p $\in$ [1,∞) then G is amenable.

Brown and Guentner demonstrated that this construction can produce an intermediate C*-algebra between the reduced and full by showing that $C_{Lp}^*(\mathbb{F}_d) \neq C_r^*(\mathbb{F}_d)$ for some p $\in$ (2,∞) where $\mathbb{F}_d$ is the free group on $2 \leq d < \infty$ generators . Subsequently, Okayasu showed that the C*-algebras $C_{Lp}(\mathbb{F}_d)$ are distinct for every p $\in$ [2,∞) (this was also independently shown by both Higson and Ozawa).

Let G be a locally compact group and H an open subgroup of G and suppose that H $\rightarrow$ B($\mathcal{H}$) is an L^p-representation of H. Then ind$_H^G \pi$ is an L^p-representation of G. We proved this for subgroups of discrete groups, but the proof holds for any open subgroup of a locally compact group. Since every L^p-representation of G clearly restricts to an L^p-representation of H,

it follows that $\|.\|_{L^p(G)} |_{L^1(H)} = \|.\|_{L^p(H)}$. Hence, in the case when Γ is a discrete group containing a copy of a non commutative free group, it follows that the C*-algebras $C^*_{Lp}(\Gamma)$ are distinct for every p $\in$ [2,∞).

Recall that the C_0-representations and the L^p-representations are the two most heavily studied types of D-representations. Brannan and Ruan define the D-Fourier algebra $A_D(G)$ and D-Fourier-Stieltjes algebra $B_D(G)$ when D is a subalgebra of $C_b(G)$. When D = $C_0(G)$, the D-Fourier algebra $A_D(G)$ is already well studied and is known as the Rajchman algebra. In contrast, very little has been done in regards to the L^p-Fourier and L^p-Fourier-Stieltjes algebras. We recall the definitions and prove some basic properties of these spaces.

Let D be a linear subspace of $C_b(G)$. The D-Fourier space is defined to be

$$A_D(G) = A_D := \{\pi_{x,y}: \text{a D-representation}, x, y \in \mathcal{H}_\pi\}.$$

Similarly, the D-Fourier-Stieltjes space $B_D(G) = B_D$ is defined to be the closure of A_D with respect to the weak*-topology σ(B(G),C*(G)). When the subspace D of $C_b(G)$ is a subalgebra (resp., ideal) of $C_b(G)$, then Brannan and Ruan noted that $A_D(G)$ and $B_D(G)$ are subalgebras (resp.,ideals) of B(G). In these cases, we may call $A_D(G)$ and $B_D(G)$ the D-Fourier algebra and D-Fourier-Stieltjes algebra, respsectively. Note that since L^p is an ideal in $C_b(G)$, $A_{L^p}(G)$ and $B_{L^p}(G)$ are ideals in B(G).

Let D be a subspace of C_b (G). we defined the Fourier space A$_S$ and the Fourier-Stieltjes space B$_S$ when S is a collection of representations of G. As an immediate consequence of the next proposition, we get that A_D= A$_S$ and B_D = B$_S$ when S is taken to be the collection of D-representations of G.

Proposition (4.1.1) [3]: Let D be a subspace of $C_b(G)$. Then A_D is a closed translation invariant subspace of B(G). Moreover ,

$$\|u\|_{B(G)} = inf\{\|x\| \|y\| : u = \pi_{x,y} \text{ and } \pi \text{ is a D-representation }\} \qquad (3)$$

and this infimum is attained for some D-representation π and x, y $\in \mathcal{H}_\pi$.

Proof:

For every $u \in A_D$, choose a D-representation $\pi_u : G \to B(\mathcal{H}_u)$ so that $u = (\pi_u)_{x,y}$ for some x, $y \in \mathcal{H}_u$. Then $\pi := \oplus_{\pi \in A_D} \pi_u$, being a direct sum of D-representations, is also representation. Then $A_\pi \ \Box \ A_D$ since every element $u \in A_D$ is a coefficient function of π.

Now let $u \in A_\pi$. Then we can find sequences $\{x_n\}$, $\{y_n\}$ in $\mathcal{H}_\pi$ so that $u = \sum_{n=1}^{\infty} \pi_{x_n, y_n}$ and $\|u\| = \sum_{n=1}^{\infty} \|x_n\| \|y_n\|$. Let $\tilde{\pi} : G \to B(\mathcal{H}_\pi^{\oplus \infty})$ be the infinite amplification $\infty . \pi$ Then, since $\tilde{\pi}$ is a D-representation and $u = \tilde{\pi}_{(xn),(yn)}$, we arrive at the desired conclusions.

These observations allow us to identify B_D with the dual of $C_D^*(G)$.

Proposition (4.1.2) [3]: Let D be a subspace of $C_b(G)$. Then B_D is identified with the dual space of $C_D^*(G)$ via the dual pairing $< u, f > = \int u(s) f(s) \, ds$ for $f \in L^1(G)$.

Proof:

Let D be the representation from the proof of the previous proposition and note that we demonstrated that $A_D = A_\pi$. Hence, it suffices to check that $C_D^* = C_\pi^*$.

Since π is a D-representation, $\|\pi(f)\| \leq \|f\|_D$ for every $f \in L^1(G)$. Now let σ be a D-representation of G. Then $A_\sigma \ \Box \ A_D = A_\pi$ implies that $B_\sigma \ \Box \ B_\pi$ and, hence, that $\|\sigma(f)\| \leq \|\pi(f)\|$ for every $f \in L^1(G)$. Thus, $\|f\|_D = \|\pi(f)\|$ for every $f \in L^1(G)$.

Let P(G) denote the set of positive definite functions on G. Then A_D has a very nice description in terms of the linear span of positive definite functions when D is a translation invariant subspace of $C_b(G)$.

Proposition (4.1.3) [3]: Suppose that D is a translation invariant subspace of $C_b(G)$. Then A_D is the closed linear span of P(G) $\cap$ D in B(G).

Proof:

Let u $\in$ P(G) $\cap$ D. Then, since the GNS representation of u is a D-representation, u is clearly in A_D. As A_D is a closed subspace of B(G), we conclude that A_D contains the closed linear span of P(G) $\cap$ D.

Now let $u \in A_D$. Then we can write $u = \pi_{x,y}$ for some D-representation π of G and x, y $\in \mathcal{H}_\pi$. Let $\mathcal{H}_0$ be a dense subspace of $\mathcal{H}_\pi$ so that $\pi_{z,z} \in$ D for every z . H$_0$ and choose sequences { x_n }, {y_n} in $\mathcal{H}_0$ converging in norm to x and y, respectively. Then

$$\pi_{x_n,y_n} = \sum_{K=0}^{3} i^K \pi_{x_n + i^K y_n, x_n + i^K y_n} \tag{4}$$

converges to $u = \pi_{x,y}$ in norm. Hence, A$_D$ is the closed linear span of P(G) $\cap$ D.

For the remainder of this section, we will focus specifically on L^p-Fourier and Fourier-Stieltjes algebras. We begin by identifying cases when these spaces are familiar subspaces of B(G).

Proposition (4.1.4) [3]: Let G be a locally compact group.

(i) A_{L^p} (G) = A(G) for every p $\in$ [1, 2].

(ii) If G is compact, then $A_{L^p}(G)$ = B(G) for every p $\in$ [1,∞)

(iii) If G is amenable, then $B_{L^p}(G)$ = B(G) for every p $\in$ [1,∞).

(iv) If $B_{L^p}(G)$ = B(G) for some p $\in$ [1,∞), then G is amenable.

Proof:

(i) Recall that the Fourier algebra A(G) is both the closed linear span of P(G)$\cap$C$_c$(G) and of P(G)$\cap$ L^2(G). Since P(G)$\cap$C$_c$(G) $\square$ P(G)$\cap$ L^p(G) $\square$ P(G)$\cap$ L^2(G) for every p $\in$ [1, 2], we arrive at the desired conclusion.

(ii) Let π be a representation of G and $x \in \mathcal{H}_\pi$. Then $\pi_{x,x}$ is bounded in uniform norm by $\|x\|^2$. Since $x \in \mathcal{H}_\pi$ was arbitrary, we conclude that every representation π of G is an L^p-representation and, hence, that $A_{L^p}(G)$ = B(G).

(iii) Since G is amenable, $C_r^*(G)$ = C*(G). Hence, the reduced Fourier-Stieltjes algebra B_λ = B(G). Since $A_{L^p}(G)$ $\square$ A(G), we conclude that $B_{L^p}(G)$ must also be all of B(G).

(iv) If $B_{L^p}(G)$ = B(G), then $C_{Lp}^*(G)$ = C*(G) and, hence, G is amenable.

Proposition (4.1.5) [3]: Let G be a locally compact group and p, q, r $\in$ [1,∞) be such that $\frac{1}{P} + \frac{1}{q} = \frac{1}{r}$. Then $uv \in A_{L^r}(G)$ for every $u \in A_{L^p}(G)$ and $v \in A_{L^q}(G)$. Similarly, $uv \in B_{L^r}(G)$ for all $u \in B_{L^p}(G)$ and $v \in B_{L^q}(G)$.

Proof:

Let $u \in A_{L^p}(G)$ and $v \in A_{L^q}(G)$. By Proposition (4.1.3), we can approximate u and v well in norm by linear combinations $a_1 u_1 + \ldots + a_n u_n$ and $b_1 v_1 + \ldots + b_m v_m$ of positive definite elements in $L^p(G)$ and $L^q(G)$, respectively. Then the product

$$\sum_{i,j} a_i b_j u_i v_j$$

is a linear combination of elements in P(G) $\cap L^r(G)$ approximating uv well in norm. Hence, $uv \in A_{L^r}(G)$ by Proposition (4.1.3).

Note that since multiplication in B(G) is separately weak*-weak* continuous, it follows that $uv \in B_{L^r}(G)$ for all $u \in B_{L^p}(G)$ and $v \in B_{L^q}(G)$.

Proposition (4.1.6) [3]: Suppose H is an open subgroup of a locally compact group G and 1≤ p< ∞ Then $A_{L^p}(H) = A_{L^p}(G)|_H$ and $B_{L^p}(H) = B_{L^p}(G)|_H$.

Proof:

The first part of the statement is deduced by similar reasoning as used in the previous section. Indeed, the equality $A_{L^p}(G)|_H = A_{L^p}(H)$ follows from the definition of L^p-Fourier spaces since $\pi|_H$ is an L^p-representation for every L^p-representation π of G, and ind $_H^G \sigma$ is an L^p-representation for every L^p-representation σ of H.

We now proceed to prove the second part of the statement. Notice that since H is an open subgroup of G, $L^1(H)$ embeds naturally into $L^1(G)$.

A similar argument as above shows that this extends to a natural embedding of $C^*_{Lp}(H)$ into $C^*_{Lp}(G)$. Let $u \in C^*_{Lp}(H)^* = B_{L^p}(H)$. Then, by the Hahn-Banach theorem, there is an elemente $\tilde{u} \in C^*_{Lp}(G)^* = B_{L^p}(G)$ extending u as a linear functional. Then for $f \in L^1(H) \square L^1(G)$,

$$\int_H \tilde{u}(s)f(s)ds = \int_G \tilde{u}(s)f(s)ds = <\tilde{u}, f> = <u, f>$$

$$= \int_H u(s)f(s)ds \tag{5}$$

So $u = \tilde{u} \mid H$ almost everywhere. Since u and $\tilde{u}$ are each continuous functions, this impliesthat $u = \tilde{u} \mid H$ and, hence, that $B_{L^p}(H) \square B_{L^p}(G) \mid_H$. A similar but simpler argument shows that $B_{L^p}(H) \square B_{L^p}(G) \mid_H$.

The above proposition can fail, even for A_{L^p}, when H is a non-open closed subgroup of G.

We finish by giving a first class of examples of groups G which show that $A_{L^p}(G)$ and $B_{L^p}(G)$ are interesting subspaces of B(G) for

$2 < p < \infty$.

Proposition (4.1.7) [3]: Let Γ be a discrete group containing a copy of a non commutative free group. Then $B_{\ell^p}(\Gamma)$ is distinct for every p $\in$ [2,∞). Hence, $A_{\ell^p}(\Gamma)$ is also distinct for every p $\in$ [2,∞).

Proof:

The first statement is immediate from previous comments since $C^*_{\ell p}(\Gamma)$ is distinct for each p $\in$ [2,∞). The second statement follows from the first since $B_{\ell^p}(\Gamma)$ is the weak*-closure of $A_{\ell^p}(\Gamma)$.

We show that the algebras $A_{L^p}(G)$ are distinct for every p $\in$ [2,∞) when G is a non compact locally compact abelian group. We will later see that this phenomena does not generalize to the setting of general noncompact locally compact groups which shows that we are required to use tools from commutative harmonic analysis. Before entering into proofs.

Let Γ be a discrete abelian group. A subset Θ of Γ is said to be dissociate if every element $\omega \in \Gamma$ can be written in at most one way as a product

$$\omega = \prod_{j=1}^{n} \theta_j^{\epsilon j}$$

where $\theta_1, \ldots, \theta_n \in \Theta$ are distinct elments, $\epsilon_j = \pm 1$ if $\theta_j \neq 1$, and $\epsilon_j = 1$ if $\theta_j^2 = 1$. As anexample, if $\Gamma = \mathbb{Z}$ then the set $\{3^j : j \geq 1\}$ is dissociate. As in the case of the integers, every infinite discrete abelian group admits an infinite dissociate set.

Let G be a compact abelian group with normalized haar measure and $\Gamma = \hat{G}$ be the dual group of G. If γ is a group element of Γ such that $\gamma^2 \neq 1$ and $a(\gamma)$ is a constant with $|a(\gamma)| \leq 1/2$, then the trigonometric polynomial

$$q_\gamma := 1 + a(\gamma)\,\gamma + \overline{a(\gamma)}\,\bar{\gamma} \tag{6}$$

is a positive function on G with $\|q_\gamma\|_1 = 1$. Similarly, if $\gamma \in \Gamma\backslash\{1\}$ has the property that and $0 \leq a(\gamma) < 1$, then q$\gamma := 1 + a(\gamma)\gamma$ is a positive function which integrates over G to 1. We will consider weak*limits of products of polynomials of this type.

Let $\Theta \ \Box\ \Gamma$ be a dissociate set.To each $\theta \in \Theta$ assign a value a(θ) $\in \mathbb{C}$ with the imposed restrictions from above. For each finite subset $\phi \ \Box\ \Theta$ define $P_\phi = \prod_{\theta \in \Phi} q_\theta$. This being a product of positive functions is a positive function on G with Fourier transform

$$\widehat{P}_\phi(\gamma) = \begin{cases} \prod_{\theta\in\Phi} a(\theta)^{(\in\theta)}, & \gamma = \prod_{\theta\in\Phi} \theta^{\in\Phi} \\ 0, & otherwis \end{cases} \tag{7}$$

where Γ range over $\{-1, 0, 1\}$ and

$$a(\theta)^{(\epsilon_\theta)} = \begin{cases} 1, & \epsilon_\theta = 0 \\ a(\theta), & \epsilon_\theta = 1 \\ \overline{a(\theta)}, & \epsilon_\theta = -1 \end{cases} \tag{8}$$

It follows that as $\Phi \nearrow \Theta, P_\Phi$ converges weak* to a measure μ on G where

$$\hat{\mu}(\gamma) = \begin{cases} \prod_{\theta\in\Theta} a(\theta)^{(\in\theta)}, \\ 0, & otherwise \end{cases} \gamma = \prod_{\theta\in\Theta} \theta^{\in\theta}, \epsilon_\theta = 0 \text{ for all but finitely many } \theta$$

The measure μ is said to be based on Θ and a. This method of constructing measures is called the Riesz Product construction and the set of all such constructions is denoted R(G).

Therefore Ameasure μ ∈ R($\mathbb{T}$) based on Θ and a is an element of $L^1(\mathbb{T})$ if and only if a ∈ $\ell^2(\Theta)$. This result was extended to all compact abelian groups G by Hewitt and Zuckerman. If Γ is the dual of a compact group G and μ ∈ R(G) is based on Θ and a, then $\hat{\mu} \in A(\Gamma) = A\ell^2(\Gamma)$ if and only if $a \in \ell^2(\Theta)$. We will demonstrate that the analogue of this theorem holds when 2 is replaced with p for $2 \leq p < \infty$. Towards this goal, we begin by proving an elementary lemma.

Lemma (4.1.8) [3]: Suppose that $0 < \alpha < 1$ and $\{x_n\}$ is a bounded sequence but $\{x_n\} \notin \ell^p$. Then

there exists a bounded sequence $\{y_n\}$ so that $\{x_n y_n\} \in \ell^p$ but $\{x_n y_n^{\alpha}\} \notin \ell^p$.

Proof:

Clearly it suffices to consider the case when p = 1. We first focus our attention to the case when $\{x_n\} \in c_0$. Then we can choose mutually disjoint subsets $I_1, I_2, \ldots$ of $\mathbb{N}$ so that

$\sum_{n \in I_k} |x_n| = 1$ for each k. Define

$$y_n = \begin{cases} K^{-\frac{1}{\alpha}} & if \, n \in I_K \\ 0 & otherwise \end{cases} \qquad (9)$$

Then

$$\sum_{n \in \mathbb{N}} |x_n y_n| = \sum_K \sum_{n \in I_K} |x_n y_n| = \sum_K \sum_{n \in I_K} |x_n| K^{-1/\alpha} = \sum_K K^{-1/\alpha} < \infty,$$

$$\sum_{n \in \mathbb{N}} |x_n y_n^\alpha| = \sum_K \sum_{n \in I_K} |x_n y_n^\alpha| = \sum_K \sum_{n \in I_K} |x_n| K^{-1} = \sum_K K^{-1} < \infty.$$

Now assume that $\lim \sup |x_n| > 0$. Then we can find δ > 0 and a subsequence $\{x_{n_K}\}$ so that $|x_{n_K}| \geq \delta$ for every k. Defining

$$y_n = \begin{cases} K^{-\frac{1}{\alpha}} & if\, n \in n_K \\ 0 & otherwise \end{cases} \tag{10}$$

gives the desired result.

We are now prepared to show the following.

Theorem (4.1.9) [3]: Let G be a compact abelian group with dual group Γ and $\mu \in R(G)$ be based on Θ and a. Then $\hat{\mu} \in A_{\ell^p}(\Gamma)$ if and only if $a \in \ell^p(\Theta)$.

Proof:

First we suppose that $\sum_{\theta \in \Theta} |a(\theta)|^p < \infty$ and let

$$\Omega(\Theta) = \{\theta_1 \ldots \theta_n \mid \theta_1^{\epsilon_1} \ldots \theta_n^{\epsilon_n} \in \Theta \text{ distinct}, \epsilon_1, \ldots, \epsilon_n = \pm 1, n \geq 0\}.$$

Then

$$\|\tilde{\mu}\|_p^p = \sum_{\omega \in \Omega(\Theta)} |\tilde{\mu}(\omega)|^p \tag{11}$$

$$\leq 1 + \sum_{n=1}^{\infty} \sum_{\Phi \subset \Theta} 2^n \prod_{\theta \in \Phi} |a(\theta)|^p$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{2^n}{n!} \left(\sum_{\theta \in \Theta} |a(\theta)|^p \right)^n$$

$$= exp\left\{ 2 \sum_{\theta \in \Theta} |a(\theta)|^p \right\} < \infty$$

Hence, $\hat{\mu} \in A_{\ell^p}(\Gamma)$.

Now suppose that $\sum_{\theta \in \Theta} |a(\theta)|^p = \infty$ but $\hat{\mu} \in A_{\ell^p}(\Gamma)$. Hewitt and Zuckerman showed this is not possible for p = 2, so we will assume without loss of generality that p > 2. Choose a sequence $\{b(\theta)\} \in \ell^{\infty}(\Theta)$ with $\|b\|_{\infty} \leq 1$ so that $\{a(\theta)b(\theta)\} \in \ell^p(\Theta)$ but $\{a(\theta)b(\theta)^{\alpha}\} \notin \ell^p(\Theta)$ for $\alpha = \frac{p-2}{2p}$. Let $v \in R(G)$ be based on Θ and $c := \left\{ a(\theta)b(\theta)^{=\frac{p-2}{2}} \right\}$. Define $q = \frac{2p}{p-2}$ (this is chosen so that 1/p + 1/q = 1/2). Then

$$\sum_{\theta \in \Theta} |c(\theta)|^q = \sum_{\theta \in \Theta} |a(\theta)b(\theta)|^p < \infty \tag{12}$$

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implies that $\hat{v} \in \ell^q(\Theta)$. So $\hat{\mu} \cdot \hat{v} \in A\ell^2(\Gamma) = A(\Gamma)$ by Proposition (4.1.5). Observe that $\mu * v$ is the element in R(G) generated by Θ and a $\cdot$ c. So a $\cdot$ c $\in \ell^2(\Theta)$. But, by our assumption on b,

$$\sum_{\theta \in \Theta} \left| a(\theta)c(\theta) \right|^2 = \sum_{\theta \in \Theta} \left| a(\theta)b(\theta)^{=\frac{p-2}{2p}} \right|^p = \infty \tag{13}$$

a contradiction. Therefore, $\hat{\mu}$ $A_{\ell^p}(\Gamma)$ iff a $\in \ell^p(\Theta)$.

Corollary (4.1.10) [3]: Let Γ be an infinite discrete Abelian group. The subspaces $A_{\ell^p}(\Gamma)$ of B(Γ)are distinct for every p $\in [2,\infty)$.

Our next step is show that $A_{\ell^p}(G)$ is distinct for each $2 \leq p < \infty$ for another class of locally compact abelian group s $\in$ G.

Suppose Γ is a lattice in a locally compact abelian group G. Further, suppose that $v \in A(G)$ is a normalized positive definite function with the property that supp $v \cap \Gamma = \{e\}$ and (s + supp v) $\cap \Gamma$ is finite for every s $\in$ G. Then the map $J = J_v$ from B(Γ) into B(G) defined by

$$Ju(s) = \sum_{\xi \in \Gamma} u(\xi)v(s - \xi) \tag{14}$$

is a well defined isometry with the following properties:

(i) Ju $\in$ P(G) if and only if $u \in$ P (Γ)

(ii) Ju $\in$ A(G) if and only if $u \in A$ (Γ)

Lemma (4.1.11) [3]: Let G =$\mathbb{R}^n \times$ K for some compact abelian group K and $n \geq 1$. Choose a

normalized $\upsilon \in A(G) \cap P(G)$ so that supp $\upsilon \subset [-1/3, 1/3]^n \times$ K, and suppose $\mu \in R(\mathbb{Z}^n)$ is based on Θ and a. Then $J_v\hat{u} \in A_{\ell^p}(\mathbb{R}^n)$ if and only if $a \in \ell^p(\Theta)$.

Proof:

We leave it as an exercise to the reader to check that if supp $\upsilon \subset [-1/3, 1/3]^n \times$ K, then supp $\upsilon \cap \mathbb{Z}^n = \{e\}$ and (s + supp υ) $\cap \mathbb{Z}^n$ is finite for every s $\in$ G.

Let $u \in P(\Gamma) \cap \ell^p(\Gamma)$. For $(x_1, \ldots, x_n, k) \in \mathbb{R}^n \times K$, write $x_i = m_i + y_i$ for some $m_i \in \mathbb{Z}$

and $y_i \in [-1/2, 1/2]$ $(1 \le i \le n)$. Then

$J_v u(x_1, \ldots, x_n, K)$

$$= \sum_{(\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n} u(\ell_1, \ldots, \ell_n) v(m_1 + y_1 - \ell_1, \ldots, m_n + y_1 - \ell_n, K)$$

$$= u(m_1, \ldots, m_n) v(y_1, \ldots, y_n, K) \tag{15}$$

For each $m_1, \ldots, m_n \in \mathbb{Z}^n$, define $M_{m1,\ldots,mn} = [m_1 - 1/2, m_1 + 1/2] \times \cdots \times [m_n - 1/2, m_n + 1/2] \times K$. Then

$$\int_G |J_v u|^p$$

$$= \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} \int_{M_{m1,\ldots,mn}} |J_v u|^p \tag{16}$$

$$= \sum_{(m_1, \ldots, m_n) \in \mathbb{Z}^n} \int_{[-1/2, 1/2]^n \times K} |u(m_1, \ldots, m_n) v(y_1, \ldots, y_n, K)|^p d(y_1, \ldots, y_n, K)$$

$$= \|u\|_p^p \int_{[-1/2, 1/2]^n \times K} |v(y_1, \ldots, y_n, K)|^p d(y_1, \ldots, y_n, K) < \infty$$

Hence, $J_v u \in L^p(G)$. As J is an isometry mapping $P(\Gamma)$ into $P(G)$ and $A_{\ell p}(\Gamma)$ is the closed linear span of $P(\Gamma) \cap \ell^p(\Gamma)$, it follows that J_v maps $A_{\ell p}(\Gamma)$ into $A_{\ell p}(G)$.

Let $\mu \in R(\mathbb{Z}^n)$ be based on Θ and a, and suppose that $a \notin \ell^p(\Theta)$. Let c be chosen as in the proof of Theorem (4.1.9) and $v \in R(\mathbb{Z}^n)$ be based on Θ and c. Then $v \in A_{\ell q}(\Gamma)$ and, hence $J_v v \in A_{\ell q}(G)$ where q satisfies $1/p + 1/q = 1/2$. For $m_1, \ldots, m_n \in \mathbb{Z}$, $y_1, \ldots, y_n \in [-1/2, 1/2]$ and $k \in K$,

$$J_v \hat{\mu}(m_1 + y_1, \ldots, m_n + y_n, k) J_v \hat{v}(m_1 + y_1, \ldots, m_n + y_n, k)$$

$$= \hat{\mu}(m_1, \ldots, m_n) \hat{v}(m_1, \ldots, m_n) v(y_1, \ldots, y_n, k)^2$$

$$= J_{v^2} \widehat{\mu * v} \tag{17}$$

Since v^2 is a positive definite function with support contained in $[-1/3, 1/3]^n \times K$, $J_v \widehat{\mu * v} \in A(G)$ if and only if $\widehat{\mu * v} \in A(G)$. But $\mu * v$ is the element of R(G) based on Θ and a · c and, as in the proof of Theorem (4.1.9) ,a·c $\notin \ell^2(\Theta)$. So $\widehat{\mu * v} \notin A(\Gamma)$ and, hence, $J_v \hat{\mu}$. $J_v \hat{v}$ is not in A(G). It follows that $J_v \mu \notin A_{\ell^p}(G)$.

Corollary (4.1.12) [3]: $A_{\ell^p}(G)$ is distinct for every p $\in$ [2,∞) when G = $\mathbb{R}^n \times K$ where K is some compact abelian group and $n \geq 1$.

Proof:

It suffices to check that there is a nonzero positive definite function v whose support is contained in $[-1/3, 1/3]^n \times K$. Observe that

$$\omega(x) := \chi_{[-1/6, 1/6]} * \chi_{[-1/6, 1/6]}(x) = \begin{cases} 1 - 3|x| \leq {}^1\!/_3, & |x| \leq {}^1\!/_3 \\ 0, & other\ wise \end{cases} \quad (18)$$

is a positive definite function on $\mathbb{R}$ with support contained in $[-1/3, 1/3]$. Taking $\upsilon = \omega \times \ldots \times \omega \times 1_K$ clearly does the trick.

We now prove one last lemma before we show that $A_{\ell^p}(G)$ is distinct for each p $\in$ [2,∞) when G is any non compact locally compact abelian group.

Lemma (4.1.13) [3]: Suppose K is a compact subgroup of a locally compact group G. Then

$A_{\ell^p}(G: K) := \{u \in A_{\ell^p}(G) : u(sk) = u(s)$ for all s $\in$ G, k $\in$ K$\}$ is isometrically isomorphic to $A_{\ell^p}(G/K)$.

Proof:

Let m_K denote the normalized Haar measure for K and note that m_K is a central idempotent measure. Denote the universal representation of G by ϖ and define $p_K = \varpi(m_K)$.

Observe that if π is a representation of G, then pK^π is constant on cosets of K and, hence defines a representation $\pi_K : G/K \to U(\mathcal{H}_\pi)$ by $\pi_K(sK) = pK^\pi(s)$ for s $\in$ G .

Suppose π is an L^p-representation of G and H_0 is a dense subspace of $\mathcal{H}_\pi$ so that $\pi_{x,x} \in L^p(G)$ for all $x \in \mathcal{H}_0$. Let q : G $\to$ G/K be the canonical quotient map. Then

$$(\pi_K)_{pK^x, pK^x} \circ q = \langle pK^\pi(.)x, x \rangle = m_K * \pi_{x,x} \in L^p(G)$$

for all $x \in \mathcal{H}_0$. Since $m_K * L^p(G) \cong L^p(G/K)$, it follows that π_K is an application of G/K.

Conversely, suppose that $\tilde{\pi}$ is an L^p-representation of G/K. Then Weyl's integral formula implies that $\tilde{\pi} \circ q$ is an L^p-representation of G. Furthermore, $m_K * (\tilde{\pi} \circ q)_{x,y} = (\tilde{\pi} \circ q)_{x,y}$ for all $x,y \in \mathcal{H}_{\tilde{\pi}}$.

Theorem (4.1.14) [3]: Let G be a noncompact locally compact abelian group. Then $A_{\ell^p}(G)$ is distinct for every p $\in$ [2,∞).

Proof:

By the structure theorem for locally compact abelian groups, G has an open subgroup of the form $\mathbb{R}^n \times$ K where $n \geq 0$ and K is compact. If $n > 0$, then the result follows from Lemma (4.1.11). Otherwise, it follows from Lemma (4.1.13) that $A_{L^p}(\mathbb{R}^n \times$ K) is distinct for every p $\in$ [2,∞) and, hence, $A_{L^p}(G)$ is distinct for every p $\in$ [2,∞) by Proposition (4.1.6).

We finish by showing that this same phenomenon which occurs for abelian groups also occurs in almost connected SIN groups.

Theorem (4.1.15) [3]: Let G be a noncompact almost connected SIN group. Then $A_{L^p}(G)$ is
distinct for every p $\in$ [2,∞).

Proof:

By the structure theorem for almost connected SIN groups, G contains an open subgroup of finite index which is of the form $\mathbb{R}^n \times$ K for some $n \geq 0$ and compact group K. Then, since G is noncompact, it is necessarily the case that $n \geq 1$. So it suffices check this for groups G of the form $\mathbb{R}^n \times$ K for some $n \geq 1$. As this follows from Lemma we conclude that $A_{L^p}(G)$ is distinct for all p $\in$ [2,∞).

Section (4.2): The Structure of L^p-Fourier Algebras and Ideals of SL(2,IR)

We investigate the structural properties of the L^p-Fourier and Fourier-Stieltjes algebras with an emphasis on the former. Similar to the Fourier algebra, we find that the L^p-Fourier algebra completely determines the group. However, armed with our knowledge of these spaces in the cases when G is either an abelian locally compact group or a discrete group containing a copy of a noncommutative free group, we observe that many nice properties which hold for Fourier algebras fail for L^p-Fourier algebras. We begin by determining the spectrum of the L^p-Fourier algebras.

Proposition (4.2.1) [3]: Let G be a locally compact group. Then the spectrum of A_{L^p}(G) is G where we identify elements of G with their point evaluations.

Proof:

Clearly we have that G $\subset \sigma(A_{L^p}$(G)), so it suffices to check that σA_{L^p}(G)) $\subset$ G. Let $\chi \in \sigma(A_{L^p}$(G)) and choose an integer n so that p/$n \leq 2$. Then, since u^n is in A(G) for every $u \in A_{L^p}$(G), there exists s $\in$ G so that $< \chi, u^n >= u(s)^n$ for all $u \in A_{L^p}$(G). As $< \chi, u > < \chi, u^n> = < \chi, u^{n+1} >= u(s)^{n+1}$, it follows that χ is evaluation at s. Hence, we conclude that

$\sigma(A_{L^p}$(G)) = G.

Recall that a linear functional D on a Banach algebra $\mathcal{A}$ is said to be a point derivation if there exists some multiplicative linear functional χ on $\mathcal{A}$ so that D(ab) = χ(a)D(b)+D(a)χ(b) for all a, b $\in \mathcal{A}$. The existence of nonzero point derivations is an obstruction to the (operator) weak amenability of $\mathcal{A}$. Since the Fourier algebra is always operator weakly amenable, the Fourier algebra does not admit any nonzero point derivations. As acorollary to the above proposition, we show that the L^p-Fourier-Stieltjes algebras admit no nonzero point derivations either. This corollary was pointed out to us by Nico Spronk.

Corollary (4.2.2) [3]: Let G be a locally compact group and p $\in$ [1,∞). Then $A_{L^p}(G)$ does not admit any nonzero point derivations.

Proof:

Suppose that $A_{L^p}(G)$ admits a nonzero point derivation D and choose a multiplicative linear functional χ on $A_{L^p}(G)$ so that $D(uv) = D(u)\chi(v) + D(v)\chi(u)$ for all $u, v \in A_{L^p}(G)$. By the above proposition, χ is the point evaluation functional at some point s $\in$ G. Choose u $\in A_{L^p}(G)$ and $v \in$ A(G) so that $D(u) \neq 0$ and $v(s) \neq 0$. Then

$$D(uv) = D(u)\chi(v) + \chi(u)D(v) = v(s)D(u) \neq 0 \tag{19}$$

since $v \in$ A(G) implies that $D(v) = 0$. But since A(G) is an ideal in B(G) and A(G) admits no nonzero point derivations, we must have that $D(uv) = 0$. This contradicts the above calculation and, therefore, we conclude that $A_{L^p}(G)$ admits no point derivations.

One of the most coveted properties of the Fourier algebra A(G) is that it completely determines the underlying locally compact group G. We now show that the analogue of this theorem holds for $A_{L^p}(G)$. The proof is similar to that given by Martin Walter.

Theorem (4.2.3) [3]: Let G_1 and G_2 be locally compact groups and suppose $A_{L^p}(G_1)$ is isometrically isomorphic to $A_{L^q}(G_2)$ as Banach algebras for some p, q $\in$ [2,∞). Then G_1 is homeomor phically isomorphic to G_2.

Proof:

Most of this proof is identical to that given by Walter and a careful read of his section reveals that the only detail that is left to be verified is that the identification of G with $\sigma(A_{L^p}(G))$ is a homeomorphic one when $\sigma(A_{L^p}(G))$ is equipped with the weak*-topology.

Let $VN_{L^p}(G)$ be the von Neumann algebra dual to $A_{L^p}(G)$. Then the canonical embedding of G into $VN_{L^p}(G)$ is continuous in the weak*-topologies. Denote this map by ρ. Then, since A(G) is contained in $A_{L^p}(G)$, the map $\rho(s) \mapsto \lambda(s)$ from is continuous in the weak*-topologies from

$VN_{L^p}(G)$ and VN(G), respectively. Finally, Eymard showed that the map $\lambda(s) \mapsto s$ from $\lambda(G)$ to G is continuous. Hence, we conclude the identification of G with $\sigma(A_{L^p}(G))$ is a homemorphic one.

The Fourier algebra admits many beautiful properties and it natural to wonder whether analogues of these continue to hold for the L^p-Fourier algebra. In many cases, such as with Walter's theorem, analogues do exist, but we will now see that this is not always the case.

We have found several classes of noncompact groups G so that $A_{L^p}(G)$ is distinct for every p ϵ [2,∞). The following example shows in a strong way that this need not happen in general.

Example (4.2.4) [3]: Let G be the ax + b group. that the Fourier algebra A(G) coincides with its Rajchman algebra $B_0(G)$: = B(G) ∩ C_0(G). Since elements B(G) are uniformly continuous, if u ϵ B(G) is L^p-integrable then $u \epsilon C_0$(G).

As $A_{L^p}(G)$ is the closed linear span of P(G) ∩ L^p(G) and the norm on B(G) dominates the uniform norm, it follows that $A_{L^p}(G) \subset B_0$(G) .Therefore $A_{L^p}(G)$ = A(G) for every $1 \leq p < \infty$.

As previously mentioned, a locally compact group G is amenabile if and only if A(G) is operator amenable if and only if A(G) admits a bounded identity. These theorems fail attrociously when A(G) is replaced with $A_{L^p}(G)$ for p > 2. Indeed, our next example shows that $A_{L^p}(G)$ need not even be square dense even when G is abelian.

Example (4.2.5) [3]: Let G be a noncompact abelian group and p > 2. Then $uv \epsilon A_{L^{p/2}}(G)$ for all $u,v \epsilon A_{L^p}(G)$ implies that $A_{L^p}(G) \cdot A_{L^p}(G) \subset A_{L^{p/2}}(G)$. By Theorem (4.1.14) we know that $A_{L^{p/2}}(G)$ is strictly contained in $A_{L^p}(G)$. So $A_{L^p}(G)$ is not squaxre dense.

As a consequence of this observation, we find that $A_{L^p}(G)$ is never an amenable Banach algebra when G is noncompact and p > 2.

Proposition (4.2.6) [3]: Let G be a locally compact group and p > 2. If $A_{L^p}(G) \neq A(G)$, then $A_{L^p}(G)$ is not (operator) weakly amenable.

Proof:

Without loss of generality, we may assume that $A_{L^{p/2}}(G) \neq A_{L^p}(G)$. Indeed, if not we define

$$\tilde{p} = \inf \{q \in [2,\infty) : A_{L^q}(G) = A_{L^P}(G)\} \tag{20}$$

and replace p with $\tilde{p} + \epsilon$ for some $0 < \epsilon < \min\{1, p - \tilde{p}\}$. Then the space $A_{L^p}(G)$ has not changed and $A_{L^p}(G) \neq A_{L^{p/2}}(G)$ since $\tilde{p} > 1$ implies $p/2 < (1+\tilde{p})/2 < \tilde{p}$. So indeed we may assume that $A_{L^p}(G) \neq A_{L^{p/2}}(G)$. Then a similar argument as in the previous example shows that $A_{L^p}(G)$ is not square dense and, there for, is not (operater) weakly amenable.

Corollary (4.2.7) [3]: Let G be a noncompact locally compact group and p > 2. Then $A_{L^p}(G)$ is a nonamenable Banach algebra.

Proof:

By the above proposition, we may assume without loss of generality that $A_{L^p}(G) = A(G)$. Then G does not contain an open abelian subgroup of finite index by Proposition(4.1.6) and Theorem (4.1.14) since such a subgroup is necessarily noncompact. In particular, this implies that G is not almost abelian. Hence, $A_{L^p}(G) = A(G)$ is nonamenable.

Let G_1 and G_2 be locally compact groups. The Effros-Ruan tensor product formula implies that $A(G_1)\widehat{\otimes}A(G_2) = A(G_1 \times G_2)$ where $\widehat{\otimes}$ denotes the operator projective tensor product and $u \otimes v \in A(G_1)\widehat{\otimes}A(G_2)$ is identified with $u \times v \in A(G_1 \times G_2)$. The next example shows that the analogue of this formula fails for A_{L^p}. Before this, we observe that the algebraic tensor product $A_{L^p}(G_1) \otimes A_{L^p}(G_2)$ embeds in $A_{L^p}(G_1 \times G_2)$ via the above identification.

Proposition (4.2.8) [3]: Let G_1 and G_2 be locally compact groups and p > 2. Then $u \times v \in A_{L^P}$

$(G_1 \times G_2)$ for all $u \in A_{L^P}(G_1)$ and $v \in A_{L^P}(G_2)$.

Proof:

First suppose that u and v are positive definite functions which are L^p-integrable Then $u \times v$ is a positive definite function on $G_1 \times G_2$ and

$$\int_{G_1 \times G_2} |u \times v|^P = \int_{G_1} \int_{G_2} |u(s)v(t)|^p ds dt = \|u\|_p \|v\|_p < \infty. \quad (21)$$

Similar arguments as used previously in the paper now show that $u \times v \in A_{L^P}(G_1 \times G_2)$ for all $u \in A_{L^P}(G_1)$ and $v \in A_{L^P}(G_2)$.

Example (4.2.9) [3]: Let Γ_1 and Γ_2 be discrete groups containing copies of non abelian free groups and $p > 2$. Then $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$ is not norm dense in $A_{\ell^p}(\Gamma_1 \times \Gamma_2)$. Indeed, identify copies of $\mathbb{F}_2$ in both Γ_1 and Γ_2 and let Δ be the diagonal subgroup of $\mathbb{F}_2 \times \mathbb{F}_2 \subset \Gamma_1 \times \Gamma_2$. Then $u \times v \big|_\Delta \in A_{\ell^{p/2}}(\Delta)$ for all $u \in A_{\ell^p}(\Gamma_1)$ and $v \in A_{\ell^p}(\Gamma_2)$ by Proposition (4.1.5) and Proposition (4.1.6). But $A_{\ell^p}(\Gamma_1 \times \Gamma_2)\big|_\Delta = A_{\ell^p}(\Delta)$. As $A_{\ell^{p/2}}(\Gamma_1 \times \Gamma_2)$ is a proper subspace of $A_{\ell^p}(\Gamma_1 \times \Gamma_2)$, we conclude that $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$ is not norm dense in $A_{\ell^p}(\Gamma_1) \otimes A_{\ell^p}(\Gamma_2)$.

The observations made in this previous example have applications to finding intermediate C*-norms between the spatial and maximal tensor product norms.

Theorem (4.2.10) [3]: Let Γ_1 and Γ_2 be discrete groups containing copies of noncommutative free groups and $p > 2$. Then $C^*_{\ell p}(\Gamma_1 \times \Gamma_2)$ gives rise to a C*-norm on the algebraic tensor product $C^*_{\ell p}(\Gamma_1) \otimes C^*_{\ell p}(\Gamma_2)$ in the natural way. This norm is distinct from the minimal and maximal tensor product norms.

Before proving this theorem, we recall a result which we will make use of. Let G_1 and G_2 be locally compact groups with representations π_1 and π_2. We showed that there is a one-to-one correspondence between C*-norms on the algebraic tensor product $C^*_{\pi 1}(G_1) \otimes C^*_{\pi 2}(G_2)$ and Fourier-Stieltjes spaces B_σ of $G_1 \times G_2$ such that $B_\sigma \big| G_1 = B_{\pi_1}$, $B_\sigma \big| G_2 = B_{\pi_2}$ and $B_\sigma \supset B_{\pi_1 \times \pi_2}$. The C*-norm on $C^*_{\pi 1}(G_1) \otimes C^*_{\pi 2}(G_2)$ corresponding to B_σ is the natural

one, i.e., for $f_1, \ldots, f_n \in L^1(G_1)$ and $g_1, \ldots, g_n \in L^1(G_2)$, the norm of $\sum_{K=1}^n \pi_1(f_K) \otimes \pi_2(\mathrm{gk})$ is given by

$$\left\| \sum_{K=1}^n \pi_1(f_K) \otimes \pi_2(\mathrm{gk}) \right\| = \left\| \sum_{K=1}^n \sigma(f_K \times g_K) \right\|. \tag{22}$$

Proof: Let π_1 and π_2 be faithful ℓ^p-representations for $C_{\ell p}^*(\Gamma_1)$ and $C_{\ell p}^*(\Gamma_2)$, respectively. It follows from Proposition (4.2.8) that $B_{\pi_1 \times \pi_2} \subset B_{\ell p}(\Gamma_1 \times \Gamma_2)$ and, by Proposition $B_{\ell p}(\Gamma_1 \times \Gamma_2)\big|\, \Gamma_1 = B_{\ell p}(\Gamma_1) = B_{\pi_1}$ and $B_{\ell p}(\Gamma_1 \times \Gamma_2)\big|\, \Gamma_2 = B_{\ell p}(\Gamma_2) = B_{\pi_2}$. So $C_{\ell p}^*(\Gamma_1 \times \Gamma_2)$ indeed induces a C*-norm on $C_{\ell p}^*(\Gamma_1) \otimes (\Gamma_2)$ in the natural way. From the observations in the previous example, we have that $B_{\ell p}(\Gamma_1) \neq B_{\pi_1 \times \pi_2}$ and, hence, that the norm coming from $C_{\ell p}^*(\Gamma_1 \times \Gamma_2)$ is not the spatial tensor product norm.

Identify copies of $\mathbb{F}_2$ in $\Gamma_1 \times \Gamma_2$ and let Δ denote the diagonal subgroup of $\mathbb{F}_2 \times \mathbb{F}_2 \subset \Gamma_1 \times \Gamma_2$. In the proof of a Fourier-Stieltjes space B_σ satisfying the above conditions is constructed with the property that $B_\sigma\big|_\Delta$ contains the constant function 1. Then $B_{\ell p}(\Gamma_1 \times \Gamma_2)$ does not contain B_σ since otherwise $B_{\ell p}(\Delta)$ would contain the constant 1 and, hence, $B_{\ell p}(\Delta)$ would be all of $B(\Delta)$. This would be a contradiction since $\Delta \cong \mathbb{F}_2$ is nonamenable.

In a previous example we observed that characterizations of amenability in terms of the Fourier algebra can fail when $A(G)$ is replaced with $A_{L^p}(G)$. We finish this section by identifying some characterizations of amenability which do translate over.

Let $\mathcal{A}$ be a Banach algebra. A linear operator $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier of $\mathcal{A}$ if $T(ab) = aT(b) = T(a)b$ for all $a, b \in \mathcal{A}$. In the context of Fourier algebras $A(G)$, every multiplier is bounded and is realizable as multiplication by some function on G. Viktor Losert characterized the amenability of a locally compact group G in terms of multipliers by showing that G is amenable if and only if $M(A(G))$, the set of multipliers of $A(G)$, is exactly B(G) if and only if the norm on $A(G)$ is equivalent to the

norm it attains as a multiplier on itself. We show that the analogue of this theorem holds for L^p-Fourier algebras.

Theorem (4.2.11) [3]: The following are equivalent for a locally compact group G and $1 \le p < \infty$.

(i) G is amenable.

(ii) $M(A_{\ell^p}(G)) = B(G)$

(iii) $\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A_{L^p(G)})}$ on B(G)

(iv) $\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A_{L^p(G)})}$ on A(G)

(v) $\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A_{L^p(G)})}$ on A(G)

Proof:

(i) $\implies$ (ii): It is an application of the closed graph theorem that every element of $M(A_{L^p}(G))$ is bounded and given by a multiplication operator. Suppose that $v \in C_b$(G) is amultiplier of A_{L^p}(G). Since G is amenable, A(G) admits a bounded pointwise approximate identity $\{ u_\alpha \}$.Then $\{u_\alpha v\}$ is a bounded sequence converging pointwise to v and, hence, $v \in$ B(G) .

(ii) $\implies$ (iii): Standard application of the open mapping theorem

(iii) $\implies$ (iv): Clear

(iv) $\implies$ (v): Suppose that there exists c > 0 so that

$$\sup \{ \|uv\|_{B(G)}: v \in A_{L^P}(G), \|v\|_{B(G)} \le 1\} > c\|u\|_{B(G)} \qquad (23)$$

for every $u \in A$(G) and choose n sufficiently large so that p/n < 2. Fix $u \in A_{L^p}$(G) and choose a unit vector v_1 in A_{L^p}(G) so that $\|uv_1\|_{B(G)} > c\|u\|_{B(G)}$. Next choose $v_2 \in A_{L^p}$(G) so that

$$\|(uv_1)v_2\|_{B(G)} > c \|uv_1\|_{B(G)} > C^2\|u\|_{B(G)}. \qquad (24)$$

Repeat this process until we arrive at n unit vectors $v_1, \ldots, v_n \in A_{L^p}$(G) and define $v = v_1 \cdots v_n$. Then $v \in A$(G) has norm at most 1 and $\|uv\|_{B(G)} > C^n\|u\|_{B(G)}$.Hence, $\|\cdot\|_{B(G)}$ is equivalent to $\|\cdot\|_{M(A(G))}$ on A(G).

(v) $\implies$(i): As mentioned above, this was shown by Losert.

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We now prove a characterization of amenability in terms of the L^p-Fourier-Stieltjes algebra. Recall that

Volker Runde and Nico Spronk introduced the notion of operator Connes amenability and showed that a locally compact group G is amenable if and only if the reduced Fourier-Stieltjes algebra B⋋ is operator Connes amenable. Since $B_{L^p}(G)$ is the dual space of $C^*_{\ell p}(G)$, it also has a natural operator space structure.We finish by showing that this characterization holds for L^p-Fourier-Stieltjes algebras.

Theorem (4.2.12) [3]: Let G be a locally compact group and p ∈ [1,∞). Then G is amenable if and only if $B_{L^p}(G)$ is operator Connes amenable.

Proof:

First suppose that G is amenable. Then $B_{L^p}(G) =$ B⋋(G) = B(G) is operator Connes amenable.

Next suppose that $B_{L^p}(G)$ is operator Connes amenable.Then, $B_{L^p}(G)$ has an identity.So $B_{L^p}(G) =$ B(G) and, hence, G is amenable by Proposition (4.1.4).

We study the L^p-Fourier-Stieltjes algebras for SL(2,R) and characterize the Fourier-Stieltjes ideals of SL(2,ℝ). The representation theory of SL(2,ℝ) is very well understood, and this knowledge is used intimately. The irreducible representations of SL(2,ℝ) fall into the following five categories:

$$
\begin{array}{ll}
\text{Trivial representation} & : \quad \tau \\
\text{Discrete series} & : \quad \{T_n : \text{n} \in \mathbb{z},\ |n| \geq 2\}, \\
\text{Mock discrete series} & : \quad T_{-1}, T_1, \\
\text{Principal series} & : \quad \{\pi_{it,\epsilon} : \text{t} \in \mathbb{R}, \epsilon = \pm 1\}. \\
\text{Complementary series} & : \quad \{\pi_r : -1 < r < 0\}.
\end{array}
$$

Ray Kunze and Elias Stein studied the integrability properties of the coefficients of irreducible representations SL(2,ℝ) and demonstrated the

remarkable fact that for every nontrivial irreducible representation π of SL(2,$\mathbb{R}$), there exists a p $\in$ [2,∞) so that $\pi_{x,x} \in L^p$ for every $x \in \mathcal{H}_\pi$. In fact,

for an irreducible representation π of SL(2,$\mathbb{R}$) they showed:

(i) π is an element of the discrete series if and only if every coefficient function of π is L^2-integrable,

(ii) π is an element of the mock discrete series or the continuous principal series if and only if every coefficient function of π is $L^{2+\epsilon}$-integrable for every $\epsilon > 0$, but not every coefficient function is L^2-integrable,

(iii) π is an element of the complementary series with parameter $r \in (-1, 0)$ if and only if every coefficient function of r is $L^{2/(1+r)+\epsilon}$-integrable for every $\epsilon > 0$, but not every coefficient function is $L^{2/(1+r)}$-integrable.

A fortiori, every nontrivial irreducible representation of SL(2,$\mathbb{R}$) is an L^p-representation for some p $\in$ [2,∞). We use this and a result of Repka to show that the spaces $B_{L^p}(\text{SL}(2,\mathbb{R}))$ are distinct for every p $\in$ [2,∞).

Lemma (4.2.13) [3]: Let G be the group SL(2,$\mathbb{R}$). Then

(i) The discrete series, mock discrete series, and principal series are weakly contained in the L^p-representations for every p $\in$ [2,∞).

(ii) The complementary series representations π_r is weakly contained in the L^p-representations (for p $\in$ [2,∞) if and only if $r \in [2/p - 1, 0)$

Proof:

Let π be a representation of SL(2,$\mathbb{R}$). Then, immediately implies that if π is an L^p-representation for some p > 2, then the direct integral decomposition of π does not include the representations π_r for $-1 < r < 2/p-1$ (apart from on a null set). Hence, π does not weakly contain π_r for any $-1 < r < 2/p-1$ since the set { π_r : $-1 < r < 2/p-1$} is open in the Fell topology.

Note that by the results of Kunze and Stein mentioned above, π_r is an L^p-representation for every $2/p - 1 < r < 0$. Hence, the L^p-representations weakly contain π_r if and only if $2/p - 1 \leq r < 0$.

To complete our proof we must note that the mock discrete series and principal seriesare weakly contained in the left regular representation. But this is given by the Cowling Haagerup-Howe theorem since they are each $L^{2+\epsilon}$- representations for every $\epsilon > 0$.

Corollary (4.2.14) [3]: Let G = SL(2,$\mathbb{R}$). Then the Fourier-Stieltjes spaces $B_{L^p}(G)$ are distinct for every p $\in$ [2,∞). Equivalently, the C*-algebras $C^*_{\ell p}(G)$ are distinct for every p $\in$ [2,∞).

We now proceed to prove the following result acharacterization of the Fourier-Stieltjes ideals of SL(2,$\mathbb{R}$).

Theorem (4.2.15) [3]: Let I be a nontrivial Fourier-Stieltjes ideal of SL(2, $\mathbb{R}$). Then I = B(G) or I= $B_{L^p}(G)$ for some p $\in$ [2,∞).

Proof:

Write I =B_π for some representation π of G. Then, since $\pi \otimes \lambda$ is unitarily equivalent to an amplification of λ by Fell's absorption principle, it is an easy exercise to see that $B_\pi \supset B_\lambda = B_{L^2}(G)$.

Consider the case when the trivial representation τ is weakly contained in π. Then I contains the unit and, hence, is all of B(G).

Next consider the case when π does not contain the complementary representation $\pi_r$for any $r \in (-1, 0)$. Then, by Lemma (4.2.13), B. is a subset of $B_{L^2}(G)$. Since we already know the reverse inclusion, we conclude that $I= B_{L^2}(G)$.

Finally, we consider the case when π weakly contains some element of the complementary series. Let

$$r = \inf \{r' \in (-1, 0) : \pi_{r'} \text{ is weakly contained in } \pi\}.$$

Then $r > -1$ since π_r converges to the trivial representation τ in the Fell topology as $r \to -1$. Also notice that π weakly contains $\pi_r$since $\pi_{r'} \to \pi_r$ in the Fell topology as $r' \to r$. Puk´anszky showed that if r_1, $r_2 \in (-1, 0)$ with $r_1 + r_2 < -1$, then $\pi_{r_1+r_2+1}$is a subrepresentation of $\pi_{r_1} \otimes \pi_{r_2}$ Since $r + r' < -1$ for $-1 < r' < -r -1$ and π weakly contains $\pi_r \otimes \pi_{r'}$. for every

$-1 < r < 0$, it follows that π weakly contains $\pi_{r'}$ for each $r \leq r' < r$. Therefore, by Lemma (4.2.13), we conclude that $I = B_{L^p}(G)$ where $p = 2/(1+r)$.

It is natural to wonder which other groups are the Fourier-Stieltjes ideals characterizable as above. Unfortunately this characterization does not hold for arbitrary locally compact groups G.

Example (4.2.16) [3]: Consider the free group $\mathbb{F}_\infty$ on countably many generators $a_1, a_2, \ldots$ and let $\mathbb{F}_d$ denote the subgroup generated by $a_1, \ldots$, a_d for some $2 \leq d < \infty$. For each $p \in [1,\infty)$, define

$$D_p = \{f \in \ell^\infty(\mathbb{F}_\infty): f\big|_{s\mathbb{F}_d t} \in \ell^p(s\mathbb{F}_d^t) \text{ for all s, t} \in \mathbb{F}_\infty\}. \tag{25}$$

Then D_p is an ideal of $\ell^\infty(\mathbb{F}_\infty)$ which implies that B_{D^p} is an ideal of B(G). Moreover, it was shown that $C^*_{Dp}(\mathbb{F}_\infty) \neq C^*_{\ell q}(\mathbb{F}_\infty)$ for any $1 \leq q < \infty$ and that $C^*_{Dp}(\mathbb{F}_\infty)$ is distinct for each $p \in [2,\infty)$. Hence, $\mathbb{F}_\infty$ has a continuum of Fourier-Stieltjes ideals which are not of the form $B_{L^p}(\mathbb{F}_\infty)$ for some $p \in [2,\infty)$.

List of Symbols

Symbol		Page
L^∞	essential lebesque space	11
Supp	Support	15
L^2	Hilbert Space	16
Sup	Supremum	17
L^2	Lebesque on the red line	20
a.e	olmost everywhere	33
$\otimes$	tensor product	36
ess	essential	41
$\oplus$	orthogonal sum	62
max	maximum	68
L^p	Lebesque space	68
red	redused	75
Reg	regular	75
inf	infimum	76
MA	multiplier Algebra	89
ℓ^∞	essential Hilbert spsce of sequences	136

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