Chapter 4

Abel-TauberTheorems for Fourier-Stieltjes Coefficients

The result in the cosine case can be applied to stationary time series with long – time memory. The analogues for Fourier – Stieltjes transforms are also given.

Section (4.1): Proof of Theorems

We are concerned with relations between the asymptotics of a function and its Fourier-Stieltjes coefficients, and the results are Abel-Tauber Theorems of this type. The results in which we pass from the Fourier-Stieltjes coefficients to the original function are Abelian, while the results in the converse direction are Tauberian.The class $BV[0,\pi]$ is that of all right-continuous $f:[0,\pi]\to \boldsymbol{R}$ that have bounded variation on $[0,\pi]$. For $F\in BV[0,\pi]$ we define its Fourier-Stieltjes cosine coefficients (FS cosine coefficients).

$$a_n := \frac{2}{\pi}\int_{[0,\pi]} \cos n\theta \, dF(\theta)(n = 1,2,\dots), \ := \frac{F(\pi)}{\pi}(n = 0), \qquad (1)$$

where $dF\{0\} = F(0)$.

We write R_0 for the class of slowly varying functions at infinity: the class of positive, measurable l, defined on some neighbourhood $[X,\infty)$ of infinity, such that

$$\forall\lambda > 0, \quad \lim_{x\to\infty} l(\lambda x)/l(x) = 1.$$

For $l\in R_0$ the class Π_l is the class of measurable g, defined on some neighbourhood $[X,\infty)$ of infinity, satisfying

$$\forall\lambda > 0, \quad \lim_{x\to\infty} \{\mathrm{g}(\lambda x) - \mathrm{g}(x)\}/l(x) \to c\log\lambda$$

with $c\in\boldsymbol{R}$ called the l-index of g .

A real sequence (c_n) is called slowly decreasing if

$$\lim_{\lambda\downarrow 1}\liminf_{n\to\infty}\inf_{n\le m\le\lambda n}(c_m - c_n) \ge 0 \quad (\text{hences} = 0),$$

slowly increasing if $(-c_n)$ is slowly decreasing. A real sequence (a_n) is said to satisfy the Tauberian condition (T) if

$(a_n)$is eventually positive, and $(\log a_n)$ is either slowly

decreasing or slowly increasing.

For example, (a_n) satisfies (T)if$a_n = n^\rho c_n$, where $\rho \in \boldsymbol{R}$ and (c_n) is eventually positive and monotone.

Definition (4.1.1)[4]: For $l \in R_0$ and $c \in \boldsymbol{R}$, (a_n) is in Π_l with l-index c if for any $\lambda > 0$.

$$\left(a_{[\lambda n]} - a_n\right)/l(n) \to c \log \lambda \; (n \to \infty).$$

Lemma (4.1.2)[4]: Let $l \in R_0$ and $c \in \boldsymbol{R}$. If (a_n) is in Π_l with l-index c, then $(a_{n-1} - a_n)/l(n) \to 0$ as $n \to \infty$.

Proof: Choose an irrational number $\lambda > 1$, say, $\lambda = \sqrt{2}$. Then $[[\lambda n]/\lambda] = n - 1$ for $n = 1,2, \ldots$. Since $l([\lambda n])/l(n) \to 1$ as $n \to \infty$ by the uniform convergence Theorem

$$\frac{a_{n-1} - a_n}{l(n)} = \frac{\left(a_{[[\lambda n]/\lambda]} - a_{[\lambda n]}\right)}{l([\lambda n])} \cdot \frac{l([\lambda n])}{l(n)} + \frac{a_{[\lambda n]} - a_n}{l(n)}$$

$$\to c \log(1/\lambda) + c \log \lambda = 0 \quad (n \to \infty),$$

whence the Lemma.

Theorem (4.1.3)[4]: Let $l \in R_0$ and $c \in \boldsymbol{R}$. Then (a_n) is in Π_l with l-index c if and only if the function $f(x) := a_{[x]}$ is in π_l with l-index c.

Proof: Suppose(a_n) is in Π_l with l-index c. For $\lambda > 0$, we write

$$\frac{f(\lambda x) - f(x)}{l(x)} = \frac{a_{[\lambda n]} - a_{[\lambda[n]]}}{l(n)} + \frac{a_{[\lambda[n]]} - a_{[x]}}{l(x)}.$$

Since $l([x])/l(x) \to 1$as $x \to \infty$, the second term on the right tends to $c \log \lambda$ as $x \to \infty$. Now $0 \leq [\lambda x] - \left[\lambda[x]\right] < \lambda + 1$, so repeated application of Lemma (4.1.2) gives $\left(a_{[\lambda x]} - a_{[\lambda[x]]}\right)/l(x) \to 0$ as $x \to \infty$, hence f is in $\Pi_l$with l-index c. The converse is trivial.

Theorem (4.1.4)[4]: Let $l \in R_0$ and $c \in \boldsymbol{R}$. We write $s_n := \sum_{k=0}^{n} a_k$for $n = 0,1,2, \ldots$. Then

$$a_n \sim c n^{-\alpha} l(n) (n \to \infty) \tag{2}$$

implies

$$(s_n) \in \Pi_l \text{with } l - \text{index } c. \tag{3}$$

Conversely, (3) implies (2) if (a_n) satisfies (T).

We omit the proof, since it is almost the same as that of the function case.

Theorem (4.1.5)[4]: Let $l \in R_0 0$, $\rho > -1$, and $c \in \boldsymbol{R}$, and let (s_n) be as above. Assume the series $B(x) := \sum_{n=0}^{\infty} a_n e^{-n/x}$ absolutely converges for $x > 0$. Then (16) implies

$$B \in \Pi_l \text{with } l - \text{index } c. \qquad (4)$$

Conversely, (17) implies (16) if $a_n \geq 0$ for all sufficiently large n.

Proof: We write $V(x) = s_{\{x\}}$for $x \geq 0$. By Theorem (4.1.2), (4) holds if and only if V is in Π_l with l-index c. Let $\hat{V}$ be the Laplace-Stieltjes transform of V:

$$\hat{V}(x) := \int_{[0,\infty)} e^{-tx}\, dV(t) = x \int_0^{\infty} e^{-xt} V(t)\, dt \qquad (x > 0).$$

Then $B(x) = \hat{V}(1/x)$for $x > 0$, and so the implication (3) ⇒ (4)follows from the argument Conversely, since $e^{-n/(\lambda x)} - e^{-n/x} = o\big(l(x)\big)$as $x \to \infty$ for any $\lambda > 0$, we may assume $a_n \geq 0$ for all n.Therefore, (4) gives (3) by de Haan’s Theorem.

Next we consider stability of Π-variation under change of variables.

Lemma (4.1.6)[4]: Let $l \in R_0$ and $c \in \boldsymbol{R}$. Assume $\phi\colon (X,\infty) \to (Y,\infty)$ is measurable and satisfies $\phi(x) \sim \alpha x, \alpha > 0$, as $x \to \infty$ If the measurable function $f : (Y,\infty) \to \boldsymbol{R}$ is in Π_l with l-index c, then $f \circ \phi$ is also in Π_l with l-index c.

Proof: Since$\phi(\lambda x)/\phi(x) \to \lambda$as $x \to \infty$, the uniform convergence theorem for Π_l due to Balkema. Gives

$$\frac{f\big(\phi(\lambda x)\big) - f\big(\phi(x)\big)}{l\big(\phi(x)\big)} \to c \log \lambda \;(x \to \infty),$$

while by the uniform convergence theorem for R_0, $l\big(\phi(x)\big)/l(x)$ tends to 1 as $x \to \infty$. Combining,

$$\frac{f\big(\phi(\lambda x)\big) - f\big(\phi(x)\big)}{l(x)} \to c \log \lambda \;(x \to \infty),$$

hence the Lemma.

Proposition (4.1.7)[4]: Let $l \in R_0$, $c \in \boldsymbol{R}$, and $A\colon (0,1) \to \boldsymbol{R}$ be measurable.Write $B(x) := A\big(e^{-1/x}\big)$for $x > 0$ and $C(x) := A((x-1)/(x+1))$for $x > 1$. Then B is in Π_l with l-index c if C is in Π_l with l-index c.

Proof: For

$$\phi_1(x) := \frac{1 + e^{-1/x}}{1 - e^{-1/x}} (x > 0).$$

we have $B \circ \phi_1$. Since $\phi_1(x) \sim 2x$ as $x \to \infty$, we obtain the assertion by Lemma (4.1.7).

Proposition (4.1.8)[4]: Let $l \in R_0$, $c \in \boldsymbol{R}$, and $\phi(x) := 1/\{2 \arctan(1/x)\}$ for $x > 0$. If the function $f: (1/\pi, \infty) \to \boldsymbol{R}$ is in Π_l with l-index c, then $f \circ \phi$ is also in Π_l with l-index c.

Proof: Since $\phi(x) \sim x/2$ as $x \to \infty$, the assertion follows from Lemma (4.1.7). **Theorem (4.1.9)[4]:** Let $l \in R_0$ and $0 < \alpha < 1$. Let $F \in BV[0, \pi]$ with FS cosine coefficients (a_n). Then

$$a_n \sim n^{-\alpha} l(n) (n \to \infty) \tag{5}$$

implies

$$F(\theta) \sim \theta^{\alpha} l(1/\theta) \cdot \frac{\pi}{2\Gamma(\alpha + 1) \cos(\pi\alpha/2)} (\theta \to 0+). \tag{6}$$

Conversely, (3) implies (2) if (a_n) satisfies (T).

Proof :

Since

$$1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} (|r| < 1), \tag{7}$$

Fubini's Theorem yields

$$\sum_{n=0}^{\infty} a_n r^n = \frac{1}{\pi} \int_{[0,\pi]} \frac{1 - r^2}{1 - 2r\cos\theta + r^2} dF(\theta) (|r| < 1) \tag{8}$$

First we prove (2) implies (3). Since (2) implies $a_n \to 0$ as $n \to \infty$,

$$(1 - r) \sum_{n=0}^{\infty} a_n r^n \to 0 \quad (r \uparrow 1).$$

But

$$\int_{[0,\pi]} \frac{1 - r^2}{1 - 2r\cos\theta + r^2} dF(\theta)$$

$$= F(0) + \int_{[0,\pi]} \frac{1}{1 + \{2r(1 - \cos\theta)/(1-r)^2\}} dF(\theta) \to F(0)(r \uparrow 1),$$

Whence (8) gives $F(0) = 0$.

Since $\sum_{n=1}^{\infty} |a/n| < \infty$, we may write

$$C(\theta) := a_0 + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin n\theta \; (\theta \in [0,\pi]).$$

Clearly $F(\pi) = C(\pi)$. By the inversion formula, if x and y are continuity points of F such that $0 < x < y < \pi$, then $F(y) - F(x) = C(y) - C(x)$. Take two sequences of continuity points $(x_n), (y_n)$ such that $x_n \downarrow 0, y_n \downarrow \theta \in [0,\pi]$as $n \to \infty$. Letting

$$F(\theta) = a_0\theta + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin n\theta \; (0 \leq \theta \leq \pi), \qquad (9)$$

where we used $F(0) = 0$. Therefore, by an Abelian result due to Vuilleumier and others,(3) follows.

Next we prove (3) with (T) implies (5). By (6), we have $dF\{0\} = 0$.We write

$$R(x) := \frac{x-1}{x+1} (x > 1).$$

$$\Theta(\xi) := 2 \arctan \xi \; (0 \leq \xi < \infty).$$

$$\mu(d\theta) := I_{(0,\pi/2]}(\theta) dF(\theta),$$

$$F_1(\xi) := \int_{(0,\xi]} (t^2 + 1)\mu \circ \Theta(dt) \; (0 < \xi < \infty),$$

$$\overline{F}_1(x) := xF_1(1/x)(0 < x < \infty),$$

$$k_1(x) := \frac{2}{\pi} \cdot \frac{x^2}{(1 + x^2)^2} (0 < x < \infty).$$

Since $F_1(\xi) = F_1(1)$ for all $\xi > 1$, $\overline{F}_1$ is bounded on each interval $(0, a]$.

For $x > 1$and $\theta \in (\pi/2, \pi]$,

$$1 - 2R(x)\cos\theta + R(x)^2 \geq 1 + R(x)^2,$$

hence

$$\left|\int_{(\pi/2,\pi]} \frac{1 - R(x)^2}{2R(x)\cos\theta + R(x)^2} dF(\theta)\right| \leq \frac{1 - R(x)^2}{1 + R(x)^2} |dF|\big((\pi/2, \pi]\big)$$
$$= O(x^{-1})(x \to \infty). \qquad (10)$$

where $|dF|$ is the total variation measure of F.

Since $\cos\Theta(\xi) = (1 - \xi^2)/(1 + \xi^2)$,

$$\frac{1 - R(x)^2}{1 - 2R(x)\cos\Theta(\xi) + R(x)^2} = \frac{x(\xi^2 + 1)}{\xi^2 x^2 + 1} (x > 1, \xi > 0),$$

and so, for $x > 1$,

$$\frac{1}{\pi}\int_{(0,\pi/2]} \frac{1 - R(x)^2}{1 - 2R(x)\cos\theta + R(x)^2} dF(\theta)$$

$$= \frac{1}{\pi}\int_{(0,\infty)} \frac{x(\xi^2 + 1)}{\xi^2 x^2 + 1} \mu \circ \Theta(d\xi)$$

$$= \frac{1}{\pi}\int_{(0,\infty)} \frac{x}{x^2\xi^2 + 1} dF_1(\xi).$$

By integration by part, the right-hand side is

$$\frac{1}{\pi}\int_0^\infty \frac{2\xi x^2}{(\xi^2 x^2 + 1)^2} F_1(\xi)\, d\xi = k_1 * \overline{F}_1(x)(0 < x < \infty),$$

where $k_1 * \overline{F}_1$ denotes the Mellin convolution of k_1 and $\overline{F}_1$:

$$k_1 * \overline{F}_1(x) := \int_0^\infty k_1(x/t)\overline{F}_1(t)\, dt/t\ (0 < x < \infty).$$

This with (8) and (10) gives

$$\sum_{n=0}^\infty a_n R(x)^n = k_1 * \overline{F}_1(x) + O(x^{-1})(x \to \infty). \qquad (11)$$

The Mellin transform

$$\breve{k}_1(z) := \int_0^\infty t^{-z} k_1(t)\, dt/t = \frac{2}{\pi}\int_0^\infty \frac{t^{2-z}}{(1 + t^2)^2} dt$$

converges absolutely for $-1 < \Re_z < 3$, and is equal to

$$\frac{1}{\pi}\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{1+z}{2}\right).$$

Now

$$F_1(\xi) = F\big(\Theta(\xi)\big) + \int_{(0,\Theta(\xi)]} \tan^2(\theta/2)\, dF(\theta)(0 < \xi \le 1), \qquad (12)$$

and the integral on the right is

$$O\left(\xi^2 \int_{(0,\Theta(\xi)]} |dF(\theta)|\right) = o(\xi^2)(\xi \to 0+). \qquad (13)$$

Hence (3) gives

$$F_1(\xi) \sim F\big(\Theta(\xi)\big) \sim \xi^\alpha l(1/\xi)\frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\cos(\pi\alpha/2)}(\xi \to 0+)$$

or

$$\overline{F}_1(x) \sim x^{1-\alpha} l(x)\frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\cos(\pi\alpha/2)}(x \to \infty).$$

So by Arandelovic′'s Theorem, we obtain

$$k_1 * \overline{F}_1(x) \sim \breve{k}_1(1-\alpha)\overline{F}_1(x) \sim x^{1-\alpha} l(x) 2^{\alpha-1}\Gamma(1-\alpha)(x \to \infty).$$

Referring back to (18), this gives

$$\sum_{n=0}^{\infty} a_n R(x)^n \sim x^{1-\alpha} l(x) 2^{\alpha-1}\Gamma(1-\alpha)(x \to \infty)$$

or

$$\sum_{n=0}^{\infty} a_n r^n \sim \left(\frac{1+r}{1-r}\right)^{1-\alpha} l\left(\frac{1+r}{1-r}\right) 2^{\alpha-1}\Gamma(1-\alpha)$$

$$\sim (1-r)^{\alpha-1} l\left(\frac{1}{1-r}\right)\Gamma(1-\alpha)(r \uparrow 1).$$

Since individual terms $a_n r^n$are $o((1-r)^{\alpha-1})$, we may assume $a_n > 0$ for all n, which gives by Karamata's Tauberian Theorem for power series ,

$$\sum_{k=0}^{n} a_k \sim \frac{n^{1-\alpha} l(n)}{1-\alpha}(n \to \infty). \qquad (14)$$

Finally, (T) corresponds to (1.7.10")(see[4]), whence it gives (2). **Theorem (4.1.10)[4]:**Let $l \in R_0$ and $F \in BV[0,\pi]$ with FS cosine coefficients(a_n). We write $\bar{F}(x) := xF(1/x)$for $x \geq 1/\pi$. Then

$$a_n \sim n^{-1} l(n)(n \to \infty) \tag{15}$$

implies

$$\bar{F} \in \Pi_l \text{with } l - \text{index } 1. \tag{16}$$

Conversely, (5) implies (4) if (a_n) satisfies (T).

The Theorems above can be applied to stationary time series. Let $X = (X(n): n \in \boldsymbol{Z})$ be a real, weakly stationary time series with expectation zero, and let R be its correlation function: $R(n) = E[X(n)X(0)]$for$n \in \boldsymbol{Z}$. By the spectral representation Theorem for correlation functions,

$$R(n) = \int_{[0,\pi]} \cos n\theta \, dF(\theta)(n \in \boldsymbol{Z})$$

with non-decreasing $F \in BV[0,\pi]$ called the spectral distribution functionof X. Now X is called long-time memory or long-range dependent if it exhibits the property

$\sum_{n=-\infty}^{\infty} |R(n)| = \infty$ The prototype of such correlation functions is R with

$$R(n) \sim n^{-\alpha} l(n)(n \to \infty),$$

where$0 < \alpha < 1$ and $l \in R_0$. The boundary case a $\alpha = 1$ is delicate; the value of $\sum_{n=-\infty}^{\infty} |R(n)|$ is infinite if and only if $\int^{\infty} l(t)\, dt/l = \infty$. The Theorems above characterize such R in terms of F rather than the spectral density of X, which does not always exist, under the weak condition (T).

To consider the analogues of the Theorems above for sine coefficients, it will be convenient to restrict the class of functions. The class $NBV[0,\pi]$ is the subclass of $BV[0,\pi]$ consisting of all G that are normalized by $G(0) = 0$. For $G \in NBV[0,\pi]$ we define its Fourier-Stieltjes sine coefficients (FS sine coefficients)

$$b_n = \frac{2}{\pi} \int_{[0,\pi]} \sin n\theta \, dG(\theta)(n = 1,2, \dots). \tag{17}$$

Proof: First we prove (17) implies (18). In the same way as above, (17) gives (9).

Write $A(x) := \sum_{j=0}^{[x]} a_j$for $x > 0$. Then for $x > 0$

$$\overline{F}(x) - A(x) = \int_0^\infty f_1(x,t) l_1(t)\, dt + \int_0^\infty f_2(x,t) l_1(t)\, dt,$$

where for $x > 0$ and $t > 0$,

$$f_1(x,t) := \frac{1}{[t]}\left(\frac{\sin([t]/x)}{([t]/x)} - 1\right)(1 \le t < [x]+1),$$

$$:= 0(\text{otherwise}),$$

$$f_2(x,t) := \frac{1}{[t]} \cdot \frac{\sin([t]/x)}{([t]/x)}([x]+1 \le t < \infty),$$

$$:= 0(0 < t < [x]+1),$$

$$l_1(t) := a_{[t]} \cdot [t](1 \le t < \infty), \quad := 1(0 < t < 1).$$

If $0 < \delta < 2$, then as $x \to \infty$,

$$\int_0^x t^{-\delta} |f_1(x,t)|\, dt \le \sum_{j \le x} \frac{1}{j^{1+\delta}}\left(1 - \frac{\sin(j/x)}{(j/x)}\right)$$

$$= O\left(\sum_{j \le x} \frac{(j/x)^2}{j^{\delta+1}}\right) = O(x^{-\delta}).$$

Also

$$\int_0^\infty f_1(x,t)\, dt = \frac{1}{x} \sum_{0 < j/x \le 1} \frac{1}{(j/x)}\left(\frac{\sin(j/x)}{(j/x)} - 1\right)$$

$$\to c_1 := \int_0^1 \frac{1}{u}\left(\frac{\sin u}{u} - 1\right) du \qquad (x \to \infty).$$

So by Vuilleumier'sTheorem, (17) gives

$$\int_0^\infty f_1(x,t) l_1(t)\, dt \sim c_1 l_1(x) \sim c_1 l(x) (x \to \infty).$$

Similarly, if $0 < \delta < 1$, then there exists $C > 0$ such that for $x \ge 1$ and$M \ge 1$,

$$\int_{Mx}^\infty t^\delta |f_2(x,t)|\, dt \le 2^\delta x \sum_{j \ge Mx} \frac{1}{j^{2-\delta}} = C M^{\delta-1} x^\delta. \tag{18}$$

Choose $\epsilon > 0$ small enough; then for large enough M and all $x \geq 1$, the right-hand side with $\delta = 0$ is less than ϵ, while

$$\int_0^{[Mx]+1} f_2(x,t)\,dt = \frac{1}{\pi} \sum_{0<j/x\leq M} \frac{\sin(j/x)}{(j/x)^2}$$

$$\to \int_1^M \frac{\sin u}{u^2} du \quad (x \to \infty),$$

hence

$$\int_0^\infty f_2(x,t)\,dt \to c_2 := \int_1^\infty \frac{\sin u}{u} du \quad (x \to \infty).$$

This and (18) with $M = 1$ imply that the conditions of Vuilleumier's Theorem are satisfied, hence

$$\int_0^\infty f_2(x,t) l_1(t)\,dt \sim c_2 l_1(x) \sim c_2 l(x) (x \to \infty).$$

Combining,

$$\{\overline{F}(x) - A(x)\}/l(x) \to c_1 + c_2 (x \to \infty).$$

Since $l \in R_0$, this gives for any $\lambda > 0$,

$$\{\overline{F}(\lambda x) - A(\lambda x)\}/l(x) \to c_1 + c_2 (x \to \infty).$$

Subtract and use Theorems (4.1.5) and (4.1.6):

$$\{\overline{F}(\lambda x) - \overline{F}(x)\}/l(x) \to \log \lambda \,(x \to \infty),$$

which gives (5).

Next we prove (5) with (T) implies (17). we find$|\overline{F}| \in R_0$, and so $dF\{0\} = F(0) = 0$. Hence as above, we obtain (11). Write

$$D(x) := \frac{F\big(\Theta(1/x)\big)}{\Theta(1/x)} (x > 0),$$

$$a(x) := x\Theta(1/x)(x > 0).$$

Then by (12) and (13),

$$\overline{F}_1(x) = a(x)D(x) + o(x^{-1})(x \to \infty). \quad (19)$$

Proposition (4.1.9) shows that $D \in \Pi_l$ with l-index 1, so that in particular$|D| \in R_0$. Hence, since $a(x) \to 2$ as $x \to \infty$ and there exists $C > 0$ such that for all $\lambda > 1$ and $x \geq 2$

$$|a(\lambda x) - a(x)| \leq C\frac{(1-\lambda^{-1})}{x},$$

we have

$$\{a(\lambda x)D(\lambda x) - a(x)D(x)\}/l(x) = a(\lambda x)\frac{D(\lambda x) - D(x)}{l(x)} + \{a(\lambda x) - a(x)\}\frac{D(x)}{l(x)}$$
$$\to 2\log\lambda\,(x \to \infty).$$

So, by (19), $\overline{F}_1$ is in Π_l with l-index 2.

Since $\breve{k}_1(0) = 1/2, k_1 * \overline{F}_1$is in Π_l with l-index .This and (11) imply that $\sum_{n=0}^{\infty} a_n R(\cdot)^n$ is in Π_l with l-index 1, hence by Proposition (4.1.8) the function $\sum_{n=0}^{\infty} a_n e^{-n/x}$ in x is also in Π_l with l-index1. Applying Theorem (4.1.6) to this, $a_n > 0$ for all sufficiently large n then shows

$$\left(\sum_{k=0}^{n} a_k\right) \in \Pi_l \text{with } l - \text{index } 1. \tag{20}$$

Finally, under (T), Theorem (4.1.5) gives (17).

Theorem (4.1.11)[4]: Let $l \in R_0$ and $0 < \alpha < 2$. Let $G \in NBV[0,\pi]$ with FS sine coefficients (b_n). Then

$$b_n \sim n^{-\alpha} l(n)(n \to \infty) \tag{21}$$

implies

$$C(\theta) \sim \theta^{\alpha} l(1/\theta) \cdot \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)}(\theta \to 0+). \tag{22}$$

Conversely, (22) implies (21) if (b_n) satisfies (T).

Proof: First we prove (7) implies (8). As above, the inversion formula gives

$$G(\theta) = \sum_{n=1}^{\infty} b_n \cdot \frac{1-\cos n\theta}{n}(0 \leq \theta < \pi), \tag{23}$$

hence for $x > 0$,

$$x^{\alpha} G(1/x) = \int_0^{\infty} g_0(x,t) l_2(t)\, dt,$$

where for $x > 0$ and $t > 0$,

$$g_0(x,t) := \frac{1}{[t]} \cdot \frac{1 - \cos([t]/x)}{([t]/x)^{\alpha}} (1 \leq t < [x] + 1), \; := (\text{otherwise}),$$

$$l_2(t) := [t]^{\alpha} \cdot b_{[t]} (1 \leq t < \infty), \quad := 1 (0 < t < 1).$$

By an argument similar to that, Vuilleumier's Theorem gives

$$\int_0^{\infty} g_0(x,t) l_2(t)\, dt \sim c_3 l_2(x) \sim c_3 l(x) (x \to \infty)$$

with

$$c_3 := \int_0^{\infty} \frac{1 - \cos u}{u^{\alpha+1}} du.$$

Since

$$c_3 = \frac{1}{\alpha} \int_0^{\infty -} \frac{\sin u}{u^{\alpha}} du = \frac{\pi}{2\Gamma(\alpha + 1) \sin(\pi\alpha/2)},$$

We obtain (29).

Next we prove (29) with (T) implies (28). Differentiating both sides of (28) in θ,

$$\sum_{n=1}^{\infty} r^n n \sin n\theta = \frac{r(1 - r^2) \sin \theta}{(1 - 2r \cos n\theta + r^2)^2} (|r| < 1),$$

hence by Fubini's Theorem,

$$\sum_{n=1}^{\infty} n b_n r^n = \frac{2}{\pi} \int_{(0,\pi]} \frac{r(1 - r^2) \sin \theta}{(1 - 2r \cos n\theta + r^2)^2} dG(\theta) (|r| < 1). \qquad (24)$$

Let $R(x)$and (ξ) . Then

$$\left| \int_{(\pi/2,\pi]} \frac{R(x)\{1 - R(x)^2\} \sin \theta}{\{1 - 2R(x) \cos n\theta + R(x)^2\}^2} dG(\theta) \right| \leq \frac{R(x)\{1 - R(x)^2\}}{\{1 + R(x)^2\}^2} |dG|\big((\pi/2, \pi]\big)$$
$$= O(x^{-1}) (x \to \infty). \qquad (25)$$

We write

$$\nu(d\theta) := I_{(0,\pi/2]}(\theta) dG(\theta),$$

$$G_1(\xi) := \int_{(0,\xi]} (t^2+1)\nu\circ\Theta\,(dt)(0<\xi<\infty),$$

$$\tilde{G}_1(x) := x^2 G_1(1/x)(0<\xi<\infty),$$

$$k_1(x) := \frac{1}{\pi}\cdot\frac{3x^5-x^3}{(1+x^2)^3}(0<\xi<\infty).$$

Since $G_1(\xi) = G_1(1)$ for all $\xi > 1$, $\tilde{G}_1$is bounded on each interval $(0,a]$.

Since $\cos\Theta(\xi) = (1-\xi^2)/(1+\xi^2)$, $\sin\Theta(\xi) = 2\xi/(1+\xi^2)$, we find, for $x > 1$,

$$\frac{R(x)\{1-R(x)^2\}\sin\Theta(\xi)}{\{1-2R(x)\cos\Theta(\xi)+R(x)^2\}^2} = \frac{(x^3-x)}{2}\cdot\frac{\xi(\xi^2+1)}{(\xi^2x^2+1)^2}(x>1,\xi>0),$$

so that

$$\frac{2}{\pi}\int_{(0,\pi/2]}\frac{R(x)\{1-R(x)^2\}\sin\theta}{\{1-2R(x)\cos\theta+R(x)^2\}^2}\,dG(\theta)$$

$$= \frac{(x^3-x)}{\pi}\int_{(0,\infty)}\frac{\xi(\xi^2+1)}{(\xi^2x^2+1)^2}\nu\circ\Theta(d\xi)$$

$$= \frac{(x^3-x)}{\pi}\int_{(0,\infty)}\frac{\xi}{(\xi^2x^2+1)^2}\,dG_1(\xi).$$

By integration by parts, the right-hand side is

$$\frac{(x^3-x)}{\pi}\int_0^\infty\frac{(3x^2\xi^2-1)}{(\xi^2x^2+1)^3}G_1(\xi)\,d\xi = (1-x^{-2})k_2*\tilde{G}_1(x)(0<x<\infty),$$

where $k_2*\tilde{G}_1$ is the Mellin convolution of k_2 and $\tilde{G}_1$. Hence

$$\sum_{n=1}^{\infty} nb_nR(x)^n = (1-x^{-2})k_2*\tilde{G}_1(x)+O(x^{-1})(x\to\infty). \tag{26}$$

The Mellin transform $\breve{k}_2(z)$ converges absolutely for $-1<\Re_z<3$, and is equal to

$$\frac{1}{\pi}\int_0^\infty t^{2-z}\frac{3t^2-1}{(t^2+1)^3}dt = \frac{1}{\pi}\int_0^\infty t^{2-z}\frac{d}{dt}\left\{-\frac{t}{(t^2+1)^2}\right\}dt = \frac{(2-z)}{\pi}\int_0^\infty\frac{t^{2-z}}{(t^2+1)^2}dt$$

$$= \frac{(2-z)}{2\pi}\Gamma\left(\frac{3-z}{2}\right)\Gamma\left(\frac{1+z}{2}\right).$$

Now as $\xi \to 0+$.

$$G_1(\xi) = G(\Theta(\xi)) + \int_{(0,\Theta(\xi)]} \tan^2(\theta/2)\, dG(\theta) = G(\Theta(\xi)) + o(\xi^2), \qquad (27)$$

hence by (29),

$$\tilde{G}_1(x) \sim x^2 G(\Theta(1/x)) \sim x^{2-\alpha} l(x) \frac{\pi 2^{\alpha-1}}{\Gamma(\alpha+1)\sin(\pi/2)} (x \to \infty).$$

By Arandelovic′'s theorem,

$$k_2 * \tilde{G}_1(x) \sim \check{k}_2(2-\alpha)\tilde{G}_1(x) \sim x^{2-\alpha} l(x) 2^{\alpha-2}\Gamma(2-\alpha)(x \to \infty).$$

Referring back to (26), this gives

$$\sum_{n=1}^{\infty} n b_n R(x)^n \sim x^{2-\alpha} l(x) 2^{\alpha-2}\Gamma(2-\alpha)(x \to \infty)$$

or

$$\sum_{n=1}^{\infty} n b_n r^n \sim (1-r)^{\alpha-2} l\left(\frac{1}{1-r}\right)\Gamma(2-\alpha)(r \uparrow 1).$$

Therefore by Karamata's Tauberian Theorem for power series,

$$\sum_{k=1}^{n} k b_k \sim \frac{n^{2-\alpha} l(n)}{2-\alpha} (x \to \infty).$$

Since the series $(n b_n)$ also satisfies gives (26).

Theorem (4.1.12)[4]: Let $l \in R_0$, and $G \in NBV[0,\pi]$ with FS sine coefficients (b_n). We write $\tilde{G}(x) := (1/x)$for $x \geq 1/\pi$. Then

$$b_n \sim n^{-2} l(n)(n \to \infty) \qquad (28)$$

implies

$$\bar{G} \in \Pi_l \text{with } l - \text{index}\, 1/2. \qquad (29)$$

Conversely, (29) implies (28) if (b_n) satisfies (T).

If c_n decreases to zero as $n \to \infty$, then the Fourier cosine series $f(\theta) := \sum_{n=}^{\infty} c_n \cos n\theta$ converges for any $\theta \in (0,2\pi)$. For this, we have Abel-Tauber Theorems which link the asymptotics of (c_n) and $f(1/\cdot)$, and similarly for Fourier sine series; see Aljančic′ et aland Yong.Here monotonicity of (c_n) fills the two roles of a sufficient condition for convergence and a Tauberian condition. However, though monotonicity is simple, it is far from best-possible in each of these conditions. In contrast,[T] is a consequence of each of the final assertions, hence it does not restrict the class of FS coefficients that the Theorems cover; We refer to Bingham for Tauberian Theorems for Fourier and Jacobi series with such weak Tauberian conditions.

First we consider Π-variation for sequences. For $x \in \boldsymbol{R}$, we write $[x]$ for its integer part. In what follows, $(a_n)_{n=0}^{\infty}$ is a real sequence.

Proof: First we prove (9) implies (10). In the same way as above, (9) gives (23). Write $B(x) := \frac{1}{2}\sum_{j=1}^{[x]} jb_j$for $x > 0$. Then for $x > 0$,

$$\tilde{G}(x) - B(x) = \int_0^{\infty} g_1(x,t) l_2(t)\, dt + \int_0^{\infty} g_2(x,t) l_2(t)\, dt,$$

where for $x > 0$ and $t > 0$,

$$g_1(x,t) := I_{[1,[x]+1)}(t) \cdot \frac{1}{[t]} \cdot \frac{1 - (1/2)([t]/x)^2 - \cos([t]/x)}{([t]/x)^2},$$

$$g_2(x,t) := I_{[[x]+1,\infty)}(t) \cdot \frac{1}{[t]} \cdot \frac{1 - \cos([t]/x)}{([t]/x)^2},$$

$$l_2(t) := [t]^2 \cdot b_{[t]}(1 \le t < \infty), \quad := (0 < t < 1).$$

By Vuilleumier’s Theorem,

$$\int_0^{\infty} g_1(x,t) l_2(t)\, dt \sim c_3 l_2(x) \sim c_2 l(x) (x \to \infty),$$

where

$$c_3 := \int_0^1 \frac{1 - (1/2)u^2 - \cos u}{u^3}\, du \quad (x \to \infty).$$

Similarly,

$$\int_0^{\infty} g_2(x,t) l_2(t)\, dt \sim c_4 l_2(x) \sim c_4 l(x) (x \to \infty),$$

where

$$c_4 := \int_0^\infty \frac{1 - \cos u}{u^2}\, du.$$

Combining,

$$\{\tilde{G}(x) - B(x)\}/l(x) \to c_3 + c_4 (x \to \infty),$$

which implies (3).

Next we prove (3) with (T) implies (2). We set $\tilde{l}(x) := |\tilde{G}(x)|$ for $x > 1/\pi$. Then(3)shows $\tilde{l} \in R_0$. By integration by parts, for some $C > 0$ and all $\xi \in (0,1)$,

$$\left|\int_{(0,\Theta(\xi)]} \tan^2(\theta/2)\, dG(\theta)\right| = \left|\xi^2 G(\Theta(\xi)) - \int_{(0,\Theta(\xi)]} \frac{\sin(\theta/2)}{\cos^3(\theta/2)} G(\theta)\, d\theta\right|$$

$$\leq \xi^2 \Theta(\xi)^2 \tilde{l}(1/\Theta(\xi)) + C \int_{(0,\Theta(\xi)]} \theta^3 \tilde{l}(1/\theta)\, d\theta,$$

which is $O(\xi^3)$ as $\xi \to 0$. Write

$$E(x) := \frac{G(\Theta(1/x))}{\Theta(1/x)^2} (x > 0),$$

$$b(x) := x^2 \Theta(1/x)^2 (x > 0).$$

Then by the estimate above

$$\tilde{G}_1(x) = b(x)E(x) + O(x^{-1})(x \to \infty).$$

By Proposition (4.1.12), E is in Π_l with l-index $1/2$, hence, arguing , $\tilde{G}_1$ is in Π_l with l-index 2. Since $\breve{k}_2(0) = 1/2$ shows that $k_2 * \tilde{G}_1$ is in Π_l with l-index 1. By (26), this implies that $\sum_{n=1}^{\infty} n b_n R(\cdot)^n$ is in Π_l with l-index 1. So under (T), Proposition (4.1.11) and Theorems (4.1.9) and (4.1.8) give(2).

Section (4.2):Fourier-Stieltjes Transforms

In this section, we show the analogues of Theorems (4.1.1)-(4.1.4) for Fourier-Stieltjes transforms. The classes $BV\ [0,\infty)$ and $NBV\ [0,\infty)$ are defined similarly. In particular, each function in $BV\ [0,\infty)$ is bounded on$[0,\infty)$.For $F\in BV\ [0,\infty)$, we define its Fourier-Stieltjes cosine transform (FS cosine transform)

$$f(t) := \frac{2}{\pi}\int_{[0,\infty)} \cos t\xi\, dF(\xi)(0\le t<\infty).$$

where as above $dF\{0\} = F(0)$. Similarly, for $G\in NBV[0,\infty)$, we define its Fourier-Stieltjes sine transform (FS sine transform)

$$\mathrm{g}(t) := \frac{2}{\pi}\int_{[0,\infty)} \sin t\xi\, dG(\xi)(0\le t<\infty).$$

The function $h\colon [0,\infty)\to \boldsymbol{R}$ is called slowly decreasing if

$$\lim_{\lambda\downarrow 1}\liminf_{x\to\infty}\inf_{t\in[1,\lambda]}\big(h(tx)-h(x)\big)\ge 0\quad (\text{hences} = 0),$$

slowly increasing if $-h$ is slowly decreasing. The function $f\colon [0,\infty)\to \boldsymbol{R}$issaid to satisfy the Tauberian condition (T) if f is eventually positive, and $\log f$ is either slowly decreasing or slowly increasing . First we consider the cosine case.

Theorem (4.2.1)[4]: Let $l\in R_0$and $0<\alpha<1$. Let $F\in BV[0,\infty)$ with FS cosine transform f. Then

$$f(t)\sim t^{-\alpha}l(t)(t\to\infty) \tag{30}$$

implies

$$F(\xi)\sim\xi^{\alpha}l(1/\xi)\cdot\frac{\pi}{2\Gamma(\alpha+1)\cos(\pi\alpha/2)}(\xi\to 0+). \tag{31}$$

Conversely, (31) implies (30) if f satisfies (T).

Theorem (4.2.2)[4]: Let $l\in R_0$. Let $F\in BV[0,\infty)$ with FS cosine transform f.

We write $\overline{F}(x) := xF(1/x)$ for $x>0$. Then

$$f(t)\sim t^{-1}l(n)(t\to\infty) \tag{32}$$

implies

$$\bar{F}\in\Pi_l \text{with } l-\text{index } 1. \tag{33}$$

Conversely, (33) implies (32) if f satisfies (T).

The Theorems above can be applied to stationary processes. Let $X = (X(t)\colon t\in\boldsymbol{R})$be a real, centered, weakly stationary process with correlation function $R(t) := E[X(t)X(0)]$ and spectral distribution function F:

$$R(t) = \int_{[0,\infty)} \cos t\xi\, dF(\xi)(t\in\boldsymbol{R}).$$

Then the Theorems above link the asymptotics of R and $F(1/\cdot)$.

Next we consider the sine case.

Theorem (4.2.3)[4]: Let $l \in R_0$and $0 < \alpha < 2$. Let $G \in NBV[0,\infty)$ with FS sine transform g. Then

$$\mathrm{g}(t) \sim t^{-\alpha} l(t) (t \to \infty) \tag{34}$$

implies

$$C(\xi) \sim \theta^{\alpha} l(1/\xi) \cdot \frac{\pi}{2\Gamma(\alpha+1)\sin(\pi\alpha/2)} (\xi \to 0+). \tag{35}$$

Conversely, (35) implies (34) if g satisfies (T).

Theorem (4.2.4)[4]: Let $l \in R_0$ and $G \in NBV[0,\infty)$ with FS sine transform g. We write $\tilde{G}(x) := x^2 G(1/x)$ for $x > 0$. Then

$$\mathrm{g}(t) \sim t^{-2} l(t) (t \to \infty) \tag{36}$$

implies

$$\tilde{G} \in \Pi_l \text{with } l - \text{index } 1/2. \tag{37}$$

Conversely, (37)implies (36) if g satisfies (T).

The proofs of the Theorems above are similar to and even easier than those of Theorems (4.1.1)-(4.1.4), hence we omit the details. We only note that the following equalities are keys to the proofs:

$$\int_0^{\infty} e^{-xt} f(t)\, dt = \frac{2}{\pi} \int_{[0,\infty)} \frac{x}{x^2+\xi^2} dF(\xi),$$

$$\int_0^{\infty} e^{-xt} t\mathrm{g}(t)\, dt = \frac{4}{\pi} \int_{(0,\infty)} \frac{x\xi}{(x^2+\xi^2)^2} dG(\xi).$$

List of symbol

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