Chapter 3

Locally Compact Groupoids

For groupoids , the best analog of fact is to be found in representation of B(G) as a Banach space of completely bounded maps from a C*- algebra associated with G to a C*-algebra associated with the equivalence relation induced by G.

Section (3.1): Complete Positivity

In this section we introduce second and third ways to view elements of $\mathrm{P}(G)$, namely in terms of completely positive mappings. Theorem (3.1.1) is a first step toward getting Banach algebras of completely bounded maps on $M^*(G)$ and on $C^*(G)$. In we obtained $C^*(G)$ by completing $C_c(G)$, and defined ω to be the direct sum of the (cyclic) representations of $C^*(G)$ that arise from normalized positive linear functionals on $C^*(G)$. Let H_ω be the Hilbert space of ω. By a theorem of Renault, that stated,each representation of $C_c(G)$ can be gotten by integrating a unitary representation of G. Thus $\omega|C_c(G)$ is also a direct sum of certain representations π^μ. The process of integration allows us to regard each π^μ, and hence ω, as a representation of either $\mathrm{M}_c(G)$ or $C_c(G)$. We call ω the universal representation of G. We also defined $M^*(G)$ to be the operatornorm closure of $\omega(\mathrm{M}_c(G))$, and notice that $C^*(G)$ is isomorphic to the norm closure of $\omega\left(C_c(G)\right)$. If G is a group, of course $M^*(G) = C^*(G)$, but these two algebras can be different for groupoids.

Theorem (3.1.1)[3]: Let p be a positive definite function on G. Let ω be the universal representation of G, and define $T_p\left(\omega(f)\right) = \omega(pf)$ for $f \in \mathrm{M}_c(G)$. Then T_p extends to a completely positive map of $M^*(G)$ to $M^*(G)$ with completely bounded norm equal to the Q-essential supremum of $\{p(x): x \in X\}$.

Proof : We remind the reader that this Q-essential supremum is the infimum of $\{B: \text{if } \mu \in Q, \text{then } p \leq B\ \mu - \text{a.e.}\}$. Also, in working with ω we will use its construction as adirect sum.

We will need to find a formula for T_p, in order to prove that the map-ping is completely positive. For this we begin with two vectors ξ, η in one summand of H_ω given by an integrated representation π^μ. This means that we begin with a measure $\mu \in Q$ and a Hilbert bundle K over X. The subspace of H_ω in question is $L^2(\mu;\mathrm{K})$, and the restriction of ω to this subspace is the integrated form of a representation, π, of G.

We are using Renault's form here, as described take $\nu = \int \lambda^x \, d\mu(x)$ and $\nu_0 = \Delta_\mu^{-1/2}\nu$. Then for $f \in \mathcal{M}_c(G)$,

$$\left(T_p\omega(f)\xi \middle| \eta\right) = (\omega(pf)\xi|\eta)$$

$$= \int p(\gamma)f(\gamma)\left(\pi(\gamma)\xi(s(\gamma))\middle|\eta(r(\gamma))\right)d\nu_0(\gamma)$$

$$= (\pi^\mu(pf)\,\xi|\eta).$$

By theorem(2.2.7) there are a Hilbert bundle $\mathcal{K}_p$ on X, a (unitary) Borel representation π_p of G on $\mathcal{K}_p$ and a bounded Borel section ξ_p of $\mathcal{K}_p$ such that $p(\gamma) = \left(\pi_p(\gamma)\xi_p \circ s(\gamma)\middle|\xi_p \circ r(\gamma)\right)$ for λ^μ-a.e. $\gamma \in G$. By Theorem (2.2.8), π_p is unique, and the section ξ_p is determined Q-a.e. Thus we can continue the calculation from above as follows:

$$= \int f(\gamma)\left(\pi_p(\gamma)\xi_p \circ s(\gamma)\middle|\xi_p \circ r(\gamma)\right)(\xi \circ s(\gamma)|\eta \circ r(\gamma))\,d\nu_0(\gamma)$$

$$= \int f(\gamma)\left(\left(\xi_p \otimes \pi(\gamma)\right)(\xi_p \otimes \xi)\circ s(\gamma)\middle|(\xi_p \otimes \eta)\circ r(\gamma)\right)d\nu_0(\gamma)$$

$$= \left((\pi_p \otimes \pi)(f)(\xi_p \otimes \xi)\middle|\xi_p \otimes \eta\right)$$

$$= \left((\pi_p - \omega)(f)(\xi_p \otimes \xi)\middle|\xi_p \otimes \eta\right).$$

Here $\xi_p \otimes \xi$ and $\xi_p \otimes \eta$ are in $L^2(\mu;\mathcal{K}_p \otimes \mathcal{K})$. In summary we have

$$\left(T_p\omega(f)\xi\middle|\eta\right) = \left((\pi_p \otimes \omega)(f)V_{p,\mu,\mathcal{K}}\,\xi\middle|V_{p,\mu,\mathcal{K}}\,\eta\right),$$

where $V_{p,\mu,\mathcal{K}} : L^2(\mu;\mathcal{K}) \to L^2(\mu;\mathcal{K}_p \otimes \mathcal{K})$ is defined by $V_{p,\mu,\mathcal{K}}\xi = \xi_p \otimes \xi$. This is a bounded operator because the section ξ_p is bounded and the usual techniques for multiplication operators apply. If we let V_p be the directsum of the operators $V_{p,\mu,\mathcal{K}}$ over all pairs $(\mu,\mathcal{K})$, we have $T_p\omega(f) = V_p^*(\pi_p \otimes \omega)(f)V_p$. A theorem of Stinespring, shows that T_p is completely positive with completely bounded norm equal to $\|V_p\|^2$. But V_p is given by a tensor multiplication which behaves like a scalar multiplication operator, so

$$\|V_p\|^2 = \operatorname{ess\,sup}\left\{\|\xi_p(x)\|^2 : x \in X\right\} = \operatorname{ess\,sup}\{p(x) : x \in X\}.$$

The proof of Theorem (3.1.1) also proves this:

Theorem (3.1.2)[3]: Let p be a positive definite function on G, let $\mu \in \mathcal{Q}$ and let π be a representation of G. Define $T_p'\big(\pi^\mu(f)\big) = \pi^\mu(pf)$ for $f \in \mathsf{M}_c(G)$. Then T_p' extends to a completely positive map of the norm closure of $\pi^\mu(\mathsf{M}_c(G))$ to itself, this being the quotient of the T_p defined in Theorem (3.1.1). The completely bounded norm of T_p as an operator on $\mathsf{cl}(\pi^\mu(\mathsf{M}_c(G)))$ is the μ-essential supremum of $\{p(x): x \in X\}$.

Although the norm on the Fourier-Stieltjes algebra of a groupoid comes from its representation by completely bounded maps rather than as the Banach space dual of the C^*-algebra as it does for groups, the latter fact has a remnant. Here we prove just one lemma regarding that remnant.

Lemma (3.1.3)[3]: Let p be a positive definite function on G, and let μ be a probability measure in $\mathcal{Q}$. Define $\psi_{p,\mu}\big(\omega(f)\big) = \int f(\gamma)p(\gamma)\, d\nu(\gamma)$for $f \in C_c(G)$, where $\nu = \int \lambda^x\, d\mu(x)$. Then $\psi_{p,\mu}$ extends to a positive linear functional on $C^*(G)$ whose norm is at most the $\mathcal{Q}$-essential supremum of p.

Proof: From the definition of π_μ in (See[3]), it follows that the integral in question is equal to $\big(\pi_\mu(f)\xi\big|\xi\big)$, where π is the unitary representation of G derived from p and ξ is the associated of the Hilbert bundle.Thus this linear functional is clearly positive, and its norm is at most $\|\xi\|^2$,the square of the norm of ξ in $\mathsf{H}(\mu)$, but this is at most$\|\xi\|_\infty^2$which is the $\mathcal{Q}$-essential supremum of p.

Next we present a third way to think about $\mathsf{P}(G)$. It depends on using the decomposition described of the Haar system of G over the equivalence relation R associated to G. This decomposition is relative to the mapping $\theta = (r, s)$ of Gonto R. Since G is σ-compact it follows that R is σ-compact in the quotient topology. The decomposition of the Haar system involves two families of measures. First of all there is a measure β_y^yconcentrated on xGy for every pair (x, y)in R, such that each $\beta_y^y$is a Haar measure on yGy and β_y^x is a translate of β_y^y. Then there is a Borel Haar system α for R so that for every $x \in X$ we have

$$\lambda^x = \int \beta_y^z\, d\alpha^x(z, y).$$

There is a Borel homomorphism δ from G to the positive reals such that for every $\mu \in \mathcal{Q}$ the modular homomorphisms Δ_μ for G and $\widetilde{\Delta}_\mu$ for R satisfy$\Delta_\mu = \delta\widetilde{\Delta}_\mu \circ \theta$. For each $x \in$

X let μ^x be the measure on X so that $\alpha^x = \varepsilon^x \times \mu^x$.Then $x \sim y$implies $\mu^x = \mu^y$. Thus $\alpha^{\mu^x} = \mu^x \times \mu^x$, so $\widetilde{\Delta}_{\mu^x} = 1$.

Let $\mathrm{M}_{Oc}(R)$ be the space of bounded Borel functions on R supported on images under θ of compact subsets of G. Then $\mathrm{M}_{Oc}(R)$ is a $*$-algebra under convolution, using the Borel Haar system .We also extend this algebra to include $\mathrm{M}(X)$, as done for $\mathrm{M}_{c}(G)$ and $\mathrm{M}(X)$, obtaining $\mathrm{M}_{Oc}(R, X)$ in this case.

If μ is a quasi-invariant measure on X, i.e., $\mu \in Q$, earlier we introduced the notation λ^μ for $\int \lambda^x \, d\mu(x)$ and we define α^μ similarly. Now we want to shorten the notation, so we write $\nu = \lambda^\mu$,$\tilde{\nu} = \alpha^\mu$, $\Delta = \Delta_\mu$, and $\widetilde{\Delta} = \widetilde{\Delta}_\mu$.

To integrate a unitary representation of G relative to μ to make a$*$-representation of $\mathrm{M}_{c}(G, X)$, we use the measure $\nu_0 = \Delta^{-1/2}\nu$ and to integrate a representation of R we use the measure $\tilde{\nu}_0 = \widetilde{\Delta}^{-1/2}\tilde{\nu}$. For example, in the first case we have

$$(\pi^\mu(f)\,\xi |\, \eta) = \int f(\gamma)\big(\pi(\gamma)\,\xi\, \circ r(\gamma) | \eta \circ s(\gamma)\big)\, d\nu_0(\gamma)$$

whenever $f \in \mathrm{M}_{c}(G)$ and ξ, η are L^2 of the bundle on which π represents G. This is the formulation of (see[3]). From what we have above,it follows that $\nu_0 = \int \delta^{-1/2} \beta_y^x \, d\tilde{\nu}_0(x, y)$, so there is a convenient relationship between the two measures.

For each unitary representation $\pi$of R, and each $\mu \in Q(R)$, we can ask whether the representation π^μ is cyclic, and we can define $\widetilde{\omega}$ to be a directsum formed using for summands one representative from each equivalence class of a cyclic π^μ. Then we can write $M^*(R)$ for the norm closure of $\widetilde{\omega}(\mathrm{M}_{Oc}(R))$. These π^μ's extend to $\mathrm{M}_{Oc}(R, X)$, so $\widetilde{\omega}$ does also, and we let $M^*(R, X)$ be the norm closure of $\widetilde{\omega}(\mathrm{M}_{Oc}(R, X))$. As stated before, the algebra $M^*(R, X)$ is present only for its utility in proving results about G, and the slightly strange definition is just suited to that purpose.

If $p \in \mathrm{P}(G)$, we define a pairing of p with an element $f \in \mathrm{M}_{c}(G)$ to givea function on R by

$$\langle f, p\rangle(x, y) = \int f p \delta^{-1/2}\, d\beta_y^x.$$

Since p and $\delta^{-1/2}$ are Borel functions and bounded on compact sets, we always have $\langle f, p\rangle \in \mathrm{M}_{Oc}(R)$. We must show that this mapping is determined by the equivalence class of p. If $p = p'$ a.e. relative to λ^{Q}, then for α^{μ}-almost every pair (x, y) the functions p and p' agree a.e. with respect to β_y^x, so for every $f \in \mathsf{M}_c(G)$ we have $\langle f, p\rangle = \langle f, p'\rangle$ a.e. with respect to α^{μ}.Furthermore, we represent p and p' as matrix entries, and these have restrictions to X that agree a.e. with respect to Q. We will show that the mapping of f to $\langle f, p\rangle$ gives rise to a completely positive map S_p from $M^*(G)$to $M^*(R)$.

There is another property of S_p we use, and its statement requires a little background. Recall that $C_c(G)$ and $\mathsf{M}_c(G)$ are bimodules over $C(\overline{X})$, where $h \in C(\overline{X})$ acts via multiplication by $h \circ r$ and $h \circ s$. Recall also that every representation π of the $*$-algebra $C_c(G)$ has an associated representation φ of $C_c(X)$ such that $\pi(hf) = \varphi(h)\,\pi(f)$ and $\pi(fh) = \pi(f)\,\varphi(h)$for all $f$and h, i.e., so that π is a bimodule map. Hence every representation of $\mathsf{M}_c(G)$ also has such an associated representation of $C_c(X)$. We can extend φ to $\mathsf{M}(X)$, getting a representation that preserves monotone limits and hence maintaining the bimodule property.

We notice that $\mathsf{M}(X)$ also has natural actions defined the same way on$\mathsf{M}_{Oc}(R)$ and by pointwise multiplication on each $L^2(\mu; \mathsf{K})$, rendering $\widetilde{\omega}$bimodule map from $\mathsf{M}_{Oc}(R)$to $M^*(R)$. The main properties of S_p are established in the next theorem.

Theorem (3.1.4)[3]: If $p \in \mathsf{P}(G)$, there is a completely positive operator $S_p: M^*(G) \to M^*(R)$ that extends the operator defined by $S_p\big(\omega(f)\big) = \widetilde{\omega}(\langle f, p\rangle)$ for $f \in \mathsf{M}_c(G)$. This mapping is an$\mathsf{M}(X)$-bimodule map. If we define $S_p\big(\omega(\mathrm{g}\varepsilon)\big) = \widetilde{\omega}(p\mathrm{g}\varepsilon)$ for $\mathrm{g} \in \mathsf{M}(X)$ and use linearity, we get an extension of the original S_p to a completely positive $\mathsf{M}(X)$-bimodule map of $M^*\big(G, \overline{X}\big)$ to $M^*\big(R, \overline{X}\big)$ that takes $\omega(\varepsilon)$ to an element of $\widetilde{\omega}(\mathsf{M}(X))$. The completely bounded norm of S_p is equal to $\|p\|_\infty$.

Proof: We need another formula for S_p, first on $\mathsf{M}_c\big(G, \overline{X}\big)$. To find one, we first work with a subrepresentation of $\widetilde{\omega}$ acting on a space of the form $L^2(\mu;\mathsf{K})$.

The positive definite function p determines a unitary representation π_p of G on a Hilbert bundle K_p over X, as well as a bounded section ξ_p of K_p for which we have $p(\gamma) = \big(\pi_p(\gamma)\xi_p \circ s(\gamma) \big| \xi_p \circ r(\gamma)\big)$ for almost all γ relative to λ^{Q}. Then we may replace p by the matrix entry. Indeed, we must make that replacement in order to make sense of the values of p on X. Suppose that ξ and η are in $L^2(\mu, \mathsf{K})$, and compute

$$\left(S_p(\omega(f))\xi|\eta\right) = (\widetilde{\omega}(\langle f,p\rangle)\xi|\eta)$$

$$= \int \langle f,p\rangle(x,y)\left(\widetilde{\omega}(x,y)(y)\xi(x)|\,\eta(x)\right) d\tilde{\nu}_0(x,y)$$

$$= \iint f(\gamma)p(\gamma)\delta(\gamma)^{-1/2}\left(\widetilde{\omega}\circ\theta(\gamma)\xi(y)|\,\eta(x)\right) d\beta_y^x(\gamma)\; d\tilde{\nu}_0(x,y)$$

$$= \int f(\gamma)\left(\pi_p(\gamma)\xi_p\circ s(\gamma)\middle|\xi_p\circ r(\gamma)\right)\left(\widetilde{\omega}\circ\theta(\gamma)\;\xi\circ s(\gamma)|\,\eta\circ r(\gamma)\right) d\nu_0(\gamma)$$

$$= \left(\left((\pi_p\otimes\widetilde{\omega}\circ\theta)(f)\right)\xi_p\otimes\xi\middle|\,\xi_p\otimes\eta\right).$$

We also have

$$(\widetilde{\omega}(p\mathrm{g}\varepsilon)\;\xi|\,\eta) = (p\mathrm{g}\xi|\,\eta)$$

$$= \int \left(\xi_p(x)\middle|\xi_p(x)\right)\mathrm{g}(x)\left(\xi(x)|\,\eta(x)\right) d\mu(x)$$

$$= \left((\pi_p\otimes\widetilde{\omega}\circ\theta)(\mathrm{g}\varepsilon)\xi_p\otimes\xi\middle|\xi_p\otimes\eta\right).$$

Now define $V_{p,\mu,\mathrm{K}} : L^2(\mu;\mathrm{K}) \to L^2(\mu;\mathrm{K}_p\otimes\mathrm{K})$ by $V_{p,\mu,\mathrm{K}}\;\xi = \xi_p\otimes\xi$ and let V be the direct sum of all the operators $V_{p,\mu,\mathrm{K}}$. The calculations just done show that for all $f\in\mathrm{M}_c(G)$ and $\mathrm{g}\in\mathrm{M}(X)$ we have

$$S_p\left(\omega(f\lambda+\mathrm{g}\varepsilon)\right) = V^*\left((\pi_p\otimes\widetilde{\omega}\circ\theta)(f\lambda+\mathrm{g}\varepsilon)\right)V.$$

Since $\pi_p\otimes\widetilde{\omega}\circ\theta$ is a $*$-representation, Stinespring's Theorem shows that S_p is completely positive. This representation also gives a formula for the extension of S_p to $M^*(G,X)$ and shows that it is an extension by continuity. It is not difficult to show that the norm of S_p is the essential supremum norm of ξ_p, and that is the same as $\|p\|_\infty$.

From the definition of V we see that it intertwines the natural actions of $\mathrm{M}(X)$ on $L^2(\mu;\mathrm{K})$ and $L^2(\mu;\mathrm{K}_p\otimes\mathrm{K})$. The restrictions of these natural actions to $C_c(X)$ are the representations of $C_c(X)$ associated with the give nrepresentations of $C_c(G)$ in the proof of Renault's Theorem. This makes it clear that S_p is also a bimodule map.

Now we want to prepare the way for the proof of the converse of the last theorem. We need less hypothesis than we had conclusion, namely we only need to deal with the transitive quasiinvariant measures on X.

We use the measures μ^x on X such that $\alpha^x = \varepsilon^x \times \mu^x$, as described. For each x we have $\alpha^{\mu^x} = \mu^x \times \mu^x$, which is symmetric, so $\widetilde{\Delta}_{\mu^x}$ is trivial. That means that $\Delta_{\mu^x} = \delta$. Since these modular functions are all the same, we will denote them by the single letter Δ.

Let ρ_x be the representation of $I(R,\alpha)$ gotten by integrating the trivial representation of R on the one-dimensional bundle, relative to the measure μ^x. Since $\mathsf{M}_{Oc}(R) \subseteq I(R,\alpha)$, the representation ρ_x can be restricted to $\mathsf{M}_{Oc}(R)$, and we denote the restriction the same way. Define ρ_x on $\mathsf{M}(X)$ to be the representation by multiplication on $L^2(\mu^x)$. We combine these two definitions to get a representation ρ_x of $\mathsf{M}_{Oc}(R,\overline{X})$ on H_x. Let $\widetilde{\omega}_t$ denote the direct sum of all these "transitive" representations ρ_x, so the representation space of $\widetilde{\omega}_t$ is H_X, the direct sum of all the Hilbert spaces H_x. Write $M_t^*(R,\overline{X})$ for the norm closure of the image of $\mathsf{M}_{Oc}(R,\overline{X})$ under $\widetilde{\omega}_t$. Then $M_t^*(R,X)$ is a quotient of $M^*(R,X)$ as a $C(\overline{X})$-bimodule, as well as a compression of $M^*(R,X)$. We also write $M_t^*(R)$ for the closure of the image of $\mathsf{M}_{Oc}(R)$.

It is not true that every completely bounded map is a linear combination of completely positive maps, unless the range algebra is injective. In our setting, the domain and range are closely related and very special. We can circumvent the problems caused by lack of injectivity, but to do so and even to deal with completely positive maps themselves, we need to think of $M_t^*(R,X)$ as acting on a space of Borel sections. We now begin to arrange that.

Observe that the Hilbert spaces H_x are the fibers in a Hilbert bundle over X, i.e., the graph of H, $\Gamma\mathsf{H}$, has a natural Borel structure with all the necessary properties. In fact, for each x the space H_x is easily identifiable with $L^2(\alpha^x)$, and we simply transport the usual Borel structure for the latter bundle to H.

If $g \in \mathsf{M}_{Oc}(R)$, define a section of ΓH by letting $\xi_g(x)$ be the class of $g(x,\cdot)$ in $L^2(\mu^x)$. Countably many of these sections can be chosen so that their values at a point x always form a dense set in H_x. Thus we can also choose a countably generated subalgebra of $\mathsf{M}(X)$ so that the module of sections over it generated by the countably many ξ_g's determines the Borel structure on $\Gamma\mathsf{H}$. Note also that $x \sim y$ implies that $\mu^x = \mu^y$ so $\mathsf{H}_x = \mathsf{H}_y$.

Theorem (3.1.5)[3]: Let ψ be a completely positive $C(\overline{X})$-bimodule map from$C^*(G,\overline{X})$ to $M^*(R,X)$, and suppose that $\psi(\omega(\varepsilon))|\mathrm{H}_X \in \widetilde{\omega}_t(\mathrm{M}(X))$. Then there is a $p \in \mathrm{P}(G)$ such that $\psi = S_p$, and $\|p\|_\infty \leq \|\psi\|_{c.b}$.

Proof: There is no loss of generality in taking ψ to have completely bounded norm at most 1. Next we restrict $\psi \circ \omega$to $C_c(G,\overline{X})$, getting a completely positive map, ψ' of $C_c(G,\overline{X})$ into $M^*(R,X)$. For each $x \in X, f \in C_c(G)$, and $\mathrm{g} \in C(\overline{X})$, define $\psi'_x(f\lambda + \mathrm{g}\varepsilon) = \psi'(f\lambda + \mathrm{g}\varepsilon)|\mathrm{H}_x$. For each x,ψ'_x is a completely positive bimodule map into $\mathrm{L}(\mathrm{H}_x)$ of completely bounded norm at most 1, and $x \sim y$ implies $\psi'_x = \psi'_y$.

The proof consists mainly of accumulating sufficient information about the mappings ψ'_x and objects constructed from them to assemble the desired positive definite function p. Using the Stinespring Theorem for completely positive maps and analyzing the equipment it provides enables us to show that each ψ'_x is of the form S_{p_x}. Then it is necessary to merge the separate p_x 's into one p, using the fact that $x \sim x'$ implies $\psi'_x = \psi'_{x'}$ from which we prove that $p_x = p_{x'}$ a.e. Several more improvements in the behavior of the functions p_x finally allow us to produce a matrix entry that serves as the desired function p. We hope that naming the major steps in the proof will help the organization of the proof.

Step (1): TheBorel Behavior of $x \longmapsto \psi_x^x$

If f, $h \in \mathrm{M}_{Oc}(R)$ we want to see that

$$x \longmapsto \rho_x(f)(\xi_h(x))$$

is a Borel section of $\Gamma\mathrm{H}$. To do this it is sufficient to show that if $f,h,k \in \mathrm{M}_{Oc}(R)$ then the function $x \longmapsto (\rho_x(f)\xi_h(x)|\xi_k(x))$ is Borel. Such an inner product is given by an integral, according to the definition of ρ_x,namely

$$\iint f(y,z)\, h(x,z)\bar{k}(x,y)\, d\mu^x(y).$$

This integral defines a Borel function of x since the measures $\mu^x \times \mu^x$depend on x in a Borel manner. By the definition of $M_t^*(R)$, every ρ_x is defined on $M_t^*(R)$ and for $a \in M_t^*(R)$ the function $x \longmapsto \rho_x(a)$ is a uniform limit of functions of the form $x \longmapsto \rho_x(f)$ for $f \in \mathrm{M}_{Oc}(R)$. Hence for $a \in M_t^*(R)$ and $h \in \mathrm{M}_{Oc}(R)$ the section$x \longmapsto \rho_x(a)(\xi_h(x))$ is Borel.

If we define $\hat{\psi}$ to be the direct sum of all the ψ_x''s, then $\hat{\psi}$ is also the compression of ψ' to H_X. Thus $\hat{\psi}$ maps $C_c(G,\overline{X})$ into $M_t^*(R,X)$ and $\rho_x \circ \hat{\psi} = \psi_x'$. From this it follows that if $f \in C_c(G)$ and $h \in \mathsf{M}_{Oc}(R)$ then the section $x \longmapsto \psi_x'(f)(\xi_h(x))$ of Γ_{H} is Borel. If $\mathsf{g} \in C(\overline{X})$ there is a function $\mathsf{g}_1 \in \mathsf{M}(X)$ such that $\hat{\psi}(\mathsf{g}\varepsilon) = \widetilde{\omega}_t(\mathsf{g}_1)$ because $\hat{\psi}(\varepsilon) \in \widetilde{\omega}_t(\mathsf{M}(X))$ and $\hat{\psi}$ is a $C(\overline{X})$-bimodule map. Hence for $a \in C_c(G,\overline{X})$ and $h \in \mathsf{M}_{Oc}(R)$ the section $\mathsf{x} \longmapsto \psi_x'(a)(\xi_h(x))$ is Borel.

The fact that $\hat{\psi}$ maps into $M_t^*(R,X)$, and the Borel property derived above are essential for completing the proof.

Step (2): The Stinespring Construction.

For each x we represent ψ_x' by Stinespring's Theorem: We get a representation π_x of $C_c(G,\overline{X})$ on a Hilbert space K_x and an operator V_x from H_x to K_x, such that for $a \in C_c(G,\overline{X})$ we have

$$\psi_x'(a) = V_x^* \pi_x(a) V_x.$$

We will use the details of the construction, so we repeat it here. For Stinespring's proof, it suffices to have the domain of the completely positive map to be a $*$-algebra with identity, so $C_c(G,\overline{X})$ can be used. The space K_x is taken to be the Hilbert space constructed from the algebraic tensor product $C_c(G,\overline{X}) \otimes \mathsf{H}_x$ using the semi-inner product whose value on two elementary tensors is given by $(a \otimes \xi \,|\, b \otimes \eta) = (\psi_x'(b^* a)\xi | \eta)$. Let q_x be the quotient map from $C_c(G,\overline{X}) \otimes \mathsf{H}_x$ to its quotient modulo vectors of norm 0. The image of q_x is identified with a dense subspace of K_x. (Since $C_c(G,\overline{X})$ and H_x are separable, so is K_x). The representation π_x is determined by having $\pi_p(a)(q(b \otimes \xi)) = q_x(ab \otimes \xi)$ for $a, b \in C_c(G,\overline{X})$ and $\xi \in \mathsf{H}_x$. The operator V_x is determined by setting $V_x(\xi) = q_x(1 \otimes \xi)$ for $\xi \in \mathsf{H}_x$. A calculation of inner products shows that $\|V_x\|^2 = \|\psi_x'(1)\|$.

Since ψ_x, π_x, K_x and V_x are Borel in x and constant on equivalence classes, we get a Hilbert bundle over X that is constant on equivalence classes. The pair (π_x, V_x) is minimal in the sense that $\pi_x\left(C_c(G,\overline{X})\right) V_x(\mathsf{H}_x)$ is dense in K_x.

Step (3): Getting p_x from the Stinespring Representation.

Now we study this structure for a fixed $x \in X$. By Theorem (2.1.4) we know that π_x can be obtained by integrating a representation, π'_x, of G on a bundle K^x relative to a quasiinvariant measure μ_x, i.e., $\mathsf{K}_x = L^2(\mu_x; \mathsf{K}^x)$. Let φ_x be the representation of $C(\overline{X})$ on K_x associated with π_x as a representation of $C_c(G, \overline{X})$. In terms of the representation of K_x, φ_x is the natural representation by multiplication of K^x. We also have $\varphi_x = \pi_x | C(\overline{C})$, where $C(\overline{X})$ is regarded as a subalgebra of $C_c(G, \overline{X})$.We denote the natural representation of $C(\overline{X})$ on H_x by θ_x; again this is a representation by multiplication.

We need to show that μ_x can be taken to be μ^x. The first step is to show that V_x intertwines $\theta_x$and φ_x. Take $h \in C(\overline{X})$, $b \in C_c(G, \overline{X})$, and $\xi, \eta \in \mathsf{H}_x$ .Then the definition of the inner product and the fact that ψ'_x is a$C(\overline{X})$-bimodule map gives

$$
\begin{aligned}
(1 \otimes h\xi | \, b \otimes \eta) &= (\psi'_x(c^*)h\xi | \, \eta) \\
&= (\psi'_x(b^*h)\xi | \, \eta) \\
&= (h \otimes \xi | \, b \otimes \eta).
\end{aligned}
$$

Hence $q_x(h \otimes \xi) = q_x(1 \otimes h\xi)$. Using the bimodule property of ψ'_x, the definition of π_x, and the inner product on K_x, we compute that

$$
\begin{aligned}
\big(V_x(h\xi) | q_x(b \otimes \eta)\big) &= \big(q_x(h \otimes \xi) | q_x(b \otimes \eta)\big) \\
&= \big(\pi_x(h\varepsilon) q_x(I \otimes \xi) | q_x(b \otimes \eta)\big).
\end{aligned}
$$

Hence, $V_x(h\xi) = \varphi_x(h) V_x(\xi)$.

From the theory of representations of $C_c(X)$ or of projection valued measures based on X, there is a bounded section of K^x, which we denote by ζ_x, such that for $\xi \in \mathsf{H}_x$, the pointwise product $\xi\zeta_x$ is a section of K^x representing the element $V_x(\xi)$ in K_x. Such a section can be gotten as follows: let g be any strictly positive Borel function on X that represents an element of H_x, let ζ^1 be a section that represents $V_x(\mathsf{g})$, and set$\zeta_x = (1/\mathsf{g})\zeta^1$. Then ζ_x need not be a square integrable section, but will be if μ^x is finite so that the function 1 is an element of H_x.

We can write $V_x(\xi) = \xi\zeta_x$, using the usual identification of functions with their equivalence classes. Then for $\xi \in \mathsf{H}_x$ we have

$$\int |\xi|^2 |\zeta_x|^2 \, d\mu_x \leq \int |\xi|^2 \, d\mu^x, \qquad (5)$$

because $\|V_x\| \leq 1$. It follows that μ_x is not singular relative to μ^x, so that μ_x gives positive measure to $[x]$. It also follows that $|\zeta_x|$ is zero a.e. off$[x]$, so that $\zeta^1 = \mathrm{g}\zeta_x$ is in the subspace of $\mathsf{K}_x = L^2(\mu_x;\mathsf{K}^x)$ consisting of functions that vanish off $[x]$. By the way we integrate representations of G to get representations of $C_c(G)$, we see that this latter subspace is invariant for $C_c(G)$ and hence for $C_c(G,\overline{X})$. From the fact that g is cyclic in H_x, it follows that $\mathrm{g}\zeta_x = V_x(\mathrm{g})$ is cyclic for $C_c(G,\overline{X})$ inK_x, so the subspace under discussion is in fact all of K_x. That implies that μ_x is in fact equivalent to μ^x, so we may as well take μ_x to be equal to μ^x. That may require multiplying the original ζ_x by some positive function, but now we assume that to have been done. We write $\nu^x$for λ^{μ^x}, getting a measure concentrated on$G|[x]$.

In this situation, the inequality (5) implies that $|\zeta_x|$ is bounded by 1. We define

$$p_x(\gamma) = \left(\pi'_x(\gamma)\zeta_x s(\gamma) | \zeta_x(r(\gamma))\right)$$

getting a positive definite function on $G|[x]$. Now the sup-norm of ζ_x is the same as the operator norm of V_x, and that is the same as the square root of the completely bounded norm of ψ'_x, so the sup-norm of p_x is atmost the completely bounded norm of ψ'_x.

Step (4):p_x Gives Rise to ψ'_x.

We know that $x \sim y$ implies $\psi'_x = \psi'_y$, so $\pi_x = \pi_y$ and $V_x = V_y$. Hence $\pi'_x(\gamma) = \pi'_y(\gamma)$ for ν^x-almost every γ, and $\zeta_x(z) = \zeta_x(z)$ for μ^x-almost every z,so that $p_x = p_y$ a.e. relative to ν^x, and their restrictions to X agree a.e.relative to μ^x.

To see that ψ'_x is the compression of S_{p_x} to H_x, we begin by setting$\nu^x = \lambda^{\mu^x}$and $\tilde{\nu}^x = \alpha^{\mu^x}$, as above, so that $\tilde{\Delta} = 1$ and $\Delta = \delta$. Then we calculate for $f \in C_c(G)$, and $\xi, \eta \in \mathsf{H}_x$:

$$(\psi'_x(x)\xi \mid \eta) = (\pi_x(f)V_x\xi \mid V_x\eta)$$

$$= \int f(\gamma)\left(\pi_x(\gamma)(\xi\zeta_x)(s(\gamma)) \middle| (\eta\zeta_x)(r(\gamma))\right) \Delta^{-1/2}(\gamma)\, d\nu^x(\gamma)$$

$$= \iint f(\gamma) p_x(\gamma) \delta^{-1/2}(\gamma)\, d\beta_z^y(\gamma)\, \xi(z)\bar{\eta}(y) d\tilde{\nu}^x(y,z).$$

This shows that $\psi'_x(f) = S_{p_x}(f)\big|\mathrm{H}_x$. Next we find a formula for $\psi'_x(\varepsilon)$ by computing

$$(\psi'_x(\varepsilon)\xi \mid \eta) = (\pi_x(\varepsilon)V_x\xi \mid V_x\eta)$$
$$= (\xi\zeta_x \mid \eta\zeta_x)$$
$$= \int p_x(y)\, \xi(y)\bar{\eta}(y)\, d\mu^x(y)$$

from which it follows that $\psi'_x(\varepsilon) = p_x(p_x \mid X)$. Since ψ'_x is a $C(\overline{X})$-bimodule map, we see that $\psi'_x(\mathrm{g}\varepsilon) = p_x(\mathrm{g}p_x)$for $\mathrm{g} \in C(\overline{X})$. This completes the proof that $\psi'_x$is the compression of S_{p_x} to H_x.

Step (5): Applying Lemma (2.2.1) to the Functions p_x.

Take functions $h, k \in \mathrm{M}_{Oc}(R)$ from which we make sections ξ_h and ξ_k of$\mathrm{H}$. Let $\xi = \xi_h(x)$ and $\eta = \xi_k(x)$ in the calculations above to see that if $\mathrm{g} \in C_c(G)$, then

$$\left(\psi'_x(\mathrm{g})\xi_h(x)\right) = \int \mathrm{g}(\gamma)p_x(\gamma)h\big(x, s(\gamma)\big)\bar{k}\big(x, r(\delta)\big)\delta^{-1/2}(\gamma)\ d\nu^x(\gamma).$$

If ε is the identity in $C(\overline{X})$, we also get

$$\left(\psi'_x(\varepsilon)\xi_h(x)|\xi_k(x)\right) = \int p_x(\gamma)\ h(x, y)\bar{k}(x, y)\ \ d\mu^x(\ y).$$

Here it is important that the functions of x on the left hand sides of these two formulas are Borel functions.

To apply Lemma (2.2.1) as it is formulated, we must have a Borel family of finite measures. We begin by considering a compact set K contained in G.The function $y \longmapsto \lambda^y(K)$ is bounded on X, and for every $x \in X$ we have $\mu^x\big(s(xK)\big) < \infty$. Hence, for $x \in X$ the measure given by the integral

$$\int_{s(x,K)} (\chi_K\lambda^y)\, d\mu^x(\ y)$$

is finite.

Notice that a pair $(x, y) \in X \times X$ is in $\theta(K)$ iff $x \in r(Ky)$iff $y \in s(xK)$. If h is the characteristic function of $\theta(K)$, it follows that $h\big(x, r(\gamma)\big) = 1$ iff $r(\gamma) \in s(xK)$, and

$h(x, s(\gamma)) = 1$ iff $s(\gamma) \in s(xK)$. Thus the set L, defined to be $\{(x, \gamma) \in X \times G: \gamma \in K$ and $h(x, s(\gamma))h(x, r(\gamma)) = 1\}$ is a Borel set in $X \times G$, and the same as $\{(x, \gamma) \in X \times G: \gamma \in K, s(\gamma) \in s(xK)$ and $r(\gamma) \in s(xK)\}$. From the preceding paragraph, it follows that every x-section L_x of L has finite measure for ν^x.

Choose compact sets $K_1 \subset K_2 \subset \cdots$ whose union is G, and for each n define $h_n = \chi_{O(K_n)}$ and then $L_n = \{(x, \gamma) \in X \times G: \gamma \in K_n, s(\gamma) \in s(xK_n)$ and $r(\gamma) \in s(xK_n)\}$. Define $D_1 = L_1$ and for $n \geq 2$, let $D_n = L_n \backslash L_{n-1}$. For each $n \in \mathbb{N}$ and $x \in X$, let $\nu_n^x = (\chi_{(D_n)_x})\nu^x$. This gives a Borel family of finite measures on G. Notice that the sets Dn partition $\{(x, \gamma) \in X \times G: \gamma \in G \,|[x]\}$.

Now define f_x on G for $x \in X$ by $f_x(\gamma) = p_x(\gamma)\delta^{-1/2}(\gamma)$ for $\gamma \in G \,|[x]$ and 0 for other γ's. If $g \in C_c(G)$ and $x \in X$, then

$$\int g(\gamma) f_x(\gamma) \; d\nu_n^x(\gamma) = \left(\psi_x'(g)\xi_{h_n}(x) \middle| \xi_{h_n}(x)\right),$$

which is a Borel function of x. Hence there is a Borel function F_n on $X \times G$ such that for each x, $F_n(x,\cdot) = f_x$ a.e. relative to ν_n^x. Set

$$F =: \sum_{n \geq 1} \chi_{D_n} F_n.$$

Then F is Borel and for each $x \in X$, $F(x,\cdot) = f_x$ a.e. relative to ν^x.

A similar analysis using μ^x shows that we can also choose F so that $F(x, y) = f_x(y)$ for μ^x-almost every y.

Hence there is a Borel function P on $X \times G$ such that for every x we have $P(x,\cdot) = p_x$ a.e. Also, $x \sim y$ implies that $P(x,\cdot) = P(y,\cdot)$ a.e. relative to either ν^x or ν^y (these are the same measure) and also relative to either μ^x or μ^y when restricted to X. Furthermore, $|P(x,\cdot)|$ is bounded by the completely bounded norm of ψ_x', so $|P|$ is bounded by 1.

Step (6): Improving the Behavior of P

Recall the probability measures $\mu_1^x = s(\lambda_1^x)$ on X obtained from the Borel family of normalized Haar measures on G. We have $\mu_1^x \sim \mu_1^y$. Define a new function P_1 on $X \times G$ by

$$P_1(x,\gamma) = \int P(y,\gamma)\ d\mu_1^x(y).$$

Make a function of three variables from P and use the Borel character of P and the measures $\mu_1^x$to show that P_1 is also Borel. We need to know that P_1 also essentially replicates every function p_x, and is even more invariant than P under changing x to an equivalent point of X.

To begin with we limit ourselves to one orbit, and denote it by S. We write μ^S for a choice of one of the measures $\mu_1^{x_0}$for $x_0 \in S$. We know that for x and y in S the functions $P(x,\cdot)$ and $P(y,\cdot)$ agree a.e. relative to λ^{μ^S},so they agree a.e. relative to λ^z for μ^S-almost every z. Since λ^z and λ_1^z have the same null sets, $P(x,\cdot)$ and $P(y,\cdot)$ agree a.e. relative to λ^z iff the complex measures $P(x,\cdot)\lambda_1^z$ and $P(y,\cdot)\lambda_1^z$ are the same. We have two Borel mappings from S^3 to the standard Borel space of complex Borel measures on $\overline{X}$, so the set E_S on which they agree is Borel, allowing us to use Fubini arguments.

Hence, for every $x \in S$, the set $\{(y,z) \in S^2 : P(y,\cdot) = P(x,\cdot)\text{a.e.}\,d\lambda^z\}$ is a Borel set whose complement has measure 0 for $\mu^S \times \mu^S$. Therefore, there is a conull Borel set Z_x of points z in S such that for μ^S-almost every y we have $P(y,\cdot) = P(x,\cdot)$ a.e. relative to λ^z. Thus, for $z \in Z_x$ it is true that for λ^z-almost every γ we have $P(y,\gamma) = P(x,\gamma)$ for λ^z-almost every y. It followsthat if $z \in Z$, then $P_1(x,\gamma) = P(x,\gamma)$ for λ^z-almost every γ. Hence, for every$x \in S$ we have $P(x,\cdot) = P(x,\cdot)$ a.e. In particular, P_1 also replicates everyp_x, since S is a general orbit.

In the last paragraph, we enountered points $\gamma \in G$ for which $P(y,\gamma)$ is essentially constant in y because it is almost always equal to a particular $P(x,\gamma)$. We need to know more about the set $H = \{\gamma \in G : y \mapsto P(y,\gamma)\text{is essentially constant}\}$. If A is a countable algebra that generates the Borel sets in C, it is not difficult to show that

$$H = \bigcap_{A\in\mathcal{A}} \left\{\gamma \in G : \mu_1^{r(\gamma)} \times \varepsilon^\gamma\left(P^{-1}(A)\right) \in \{0,1\}\right\}.$$

Thus H is a Borel set. Hence the set $C = \{x \in X : \lambda_1^x(H) = 1\}$ is also a Borelset. From the preceding paragraph, it follows that C is conull in every orbit. For $z \in C$, the function $P_1(\cdot,\gamma)$ is constant for λ^z-almost every $\gamma \in zG$.In particular, for $z \in C$ it is true that $x, y \in [z]$ implies that $P_1(x,\cdot)\lambda_1^z = P(y,\cdot)\lambda_1^z$.

The last conclusion is the additional invariance needed, and now we change notation and simply write $P$for P_1, since it does everything we need.

Step (7): Making a Borel Family of Representations from P.

Again, take a particular orbit, S, in X. For every pair $(x,y) \in S^2$, we have $P(x,\cdot) = P(y,\cdot)$ a.e. relative to λ^z for μ^S-almost every z. Take an arbitrary $z \in S$. Then for λ^z-almost every γ_2 it is true that $P(x,\cdot) = P(y,\cdot)$ a.e. relative to $\gamma_2^{-1} \cdot \lambda^z = \lambda^{s(\gamma_2)}$. Hence $P(x, \gamma_2^{-1}\gamma_1) = P(y, \gamma_2^{-1}\gamma_1)$ for $\lambda^z \times \lambda^z$-almost every pair (γ_1, γ_2). (The mapping taking the pair to $\gamma_2^{-1}\gamma_1$ carries $\lambda_1^z \times \lambda_1^z$ to a measure equivalent to λ^{μ^z}).

Now return to studying general points of X. For $f, g \in C_c(G)$ and $(x,y) \in R$, define

$$(f|g)_{(x,y)} = \iint f(\gamma_1)\bar{g}(\gamma_2)\, P(x, \gamma_2^{-1}\gamma_1)\, d\lambda^y(\gamma_1) d\lambda^y(\gamma_2).$$

The formula defines an inner product on $C_c(G)$, and we write $\mathsf{K}(x,y)$ for the resulting Hilbert space. For each $f, g \in C_c(G)$ the function $(x,y) \longmapsto (f\,|g)_{(x,y)}$ is a Borel function on R that is constant on sets of the form $[y] \times \{y\}$, so K defines a Hilbert bundle on R that is constant on the same sets. For $f \in C_c(G)$, let $\sigma(f)$ denote the section of K (or Γ_{K}) that it determines.

For each x, the bundle $\mathsf{K}\,(x,\cdot)$ supports a unitary representation: here we denote it by π_x rather than $\pi_{P(x,\cdot)}$. We know that $x \times x'$ implies that $\pi_x = \pi_{x'}$, which means that for $\gamma \in G\,|[x]$ we have $\pi_x(\gamma) = \pi_{x'}(\gamma)$ (they are on the same space). We want to show that $(x,\gamma) \longmapsto \pi_x(\gamma)$ is Borel on $X \times' G = \{(x,\gamma) : \gamma \in G\,|[x]\}$. It will help to look at $R \times' G = \{(x,y,\gamma)\ \gamma \in G\,|[x]\}$. The function

$$(x,y,\gamma) \longmapsto \iint f(\gamma^{-1}\gamma_1)\bar{g}(\gamma_2)\, P(x, \gamma_2^{-1}\gamma_1)\, d\lambda^y(\gamma_1)\, d\lambda^y(\gamma_2)$$

Is Borel on $R \times' G$, so $(x,\gamma) \longmapsto \Big(\pi_x(\gamma)\sigma(f)\big(x, s(\gamma)\big)\Big|\, \sigma(g)\big(x, r(\gamma)\big)\Big)$ is Borel on $X \times' G$.

Step (8): Finding a Borel That Represents P.

Let D be the set of pairs $(x,y) \in R$ for which the linear functional $f \longmapsto \lambda^y\big(fP(x,\cdot)\big)$ is bounded relative to the seminorm $\|\sigma(f)(x,y)\|$ on $C_c(G)$. The boundedness can be tested using a countable dense subset of $C_c(G)$, so D is Borel, and hence so is the set DC. For each $x \in X$, we have $xD = \{x\} \times D_x$ so that xD is conull with respect to α^x. Notice that $w \sim x$ implies that $C \cap D_x = C \cap D_w$, and this set is conull in the orbit. Hence xDC and wDC have the same conull image in $[x]$ under s.

Now, for $(x, y) \in D$ define $\zeta(x, y)$ to be the vector in $\mathrm{K}(x, y)$ such that $\left(\sigma(f)(x, y) \mid \zeta(x, y)\right) = \lambda^y\left(fP(x,\cdot)\right)$ for every $f \in C_c(G)$, and for $(x, y) \notin D$,let $\zeta(x, y) = 0$. The formula makes it clear that ζ is Borel.

If $y \in C$and $w \sim x \sim y$, then $(w, y) \in D$ iff $(x, y) \in D$, so $y \in C$ implies that $Dy = [y] - \{y\}$. Also, $w, x \in [y]$ implies that $P(w,\cdot)$ and $P(x,\cdot)$ agree a.e.with respect to λ^y and that $\mathrm{K}(w, y) = \mathrm{K}(x, y)$. Together, these imply that $\zeta(w, y) = \zeta(x, y)$. Then for every $\gamma \in G|[y]$,

$$(\pi_x(\gamma)\, \zeta(x, s(\gamma)) \mid \zeta(x, r(\gamma))) = (\pi_w(\gamma)\, \zeta(w, s(\gamma)) \mid \zeta(w, r(\gamma))).$$

Thus both of these functions agree a.e. on $G|[x]$ with $P(x,\cdot)$. Thus we can define

$$p(\gamma) = \left(\pi_{s(\gamma)}(\gamma)\zeta(s(\gamma), s(\gamma)) \mid \zeta(s(\gamma), r(\gamma))\right)$$

for $\gamma \in \theta^{-1}(DC)$ and 1 for other γ's to get a Borel function on G that agree sa.e. with $P(x,\cdot)$ on $G|[x]$.

From Step (4) it follows that $\hat{\psi}$ and the compression of S_p to H_X are the same.

Step (9): The Compression Map from $\mathrm{L}(\mathrm{H}_\omega)$ to $\mathrm{L}(\mathrm{H}_X)$.

To complete the proof, need to show that the compression map C from $\mathrm{L}(\mathrm{H}_\omega)$ to $\mathrm{L}(\mathrm{H}_X)$ is one-one when restricted to $\widetilde{\omega}(\mathrm{M}_{Oc}(R, \overline{X}))$. Then it will follow that ψ and S_p agree on $C_c(G, \overline{X})$, forcing them to be the same.

Suppose that $f\alpha + \mathrm{g}\varepsilon \in \mathrm{M}_{Oc}(R, X)$and $\widetilde{\omega}(f\alpha + \mathrm{g}\varepsilon) \neq 0$. Then there is a representation π of R and a probability measure $\mu \in Q$ such that $\pi^\mu(f\alpha + \mathrm{g}\varepsilon) \neq 0$.We need to use this to find a $z \in X$ such that $\rho_z(f\alpha + \mathrm{g}\varepsilon) \neq 0$, which will imply $\widetilde{\omega}_t(f\alpha + \mathrm{g}\varepsilon) \neq 0$. There is no loss of generality in assuming that there is a probability measure μ' on X such that $\mu = \int \mu_1^x \, d\mu'(x)$. Set$A = \{(x, y) \in R : x \neq y, \text{and } f(x, y) \neq 0\}$, and consider two cases: $\alpha^\mu(A) = 0$ and $\alpha^\mu(A) \neq 0$. In the first case, $\pi^\mu(f) = 0$ unless $\alpha^\mu(X) > 0$, in which case we have $f\alpha = f\varepsilon$ relative to α^μ. Thus there is an $h \in \mathrm{M}(X)$ such that $0 \neq \pi^\mu(f\alpha + \mathrm{g}\varepsilon) = \pi^\mu(h\varepsilon)$. Then $\mu(\{h \neq 0\}) > 0$ so there is a $z \in X$ such that $\mu^z(\{h \neq 0\}) > 0$, and it is easy to show that $\rho_z(h\varepsilon)$, i.e. $\rho_z(f\alpha + \mathrm{g}\varepsilon) \neq 0$.In the second case, there is a $z \in X$ such that $\alpha^{\mu^z}(A) > 0$, and we will show that $\rho_z(f\alpha + \mathrm{g}\varepsilon) \neq 0$. Recall that $\alpha^{\mu^z} = \mu^z \times \mu^z$. Set $R_0 = R \backslash \{(x, x) : x \in X\}$.Then sets of the form

$(E \times F) \cap R_0$, where E and F are disjoint Borel sets in X, generate the Borel sets in R_0, so there must be such a pair for which

$$0 < \int_{E \times F} f d(\mu^z \times \mu^z) < \infty.$$

If we set $h_1 = \chi_F$ and $h_2 = \chi_E$ we get elements of $\mathsf{M}(X)$ which we think of as elements of H_z, and then the displayed integral is $(\rho_z(f)h_1|h_2)$. On the otherhand, $(\rho_z(\mathsf{g}\varepsilon)h_1|h_2) = 0$ because $\mathsf{g}h_1\bar{h}_2 = 0$. Thus $\rho_z(f\alpha + \mathsf{g}\varepsilon) \neq 0$, as needed.

Section (3.2):Completely Bounded Bimodule Maps

Recall that B(G) is defined to be the linear span of $\mathrm{P}\,(G)$. Because we know that $\mathrm{P}\,(G)$ consists of diagonal matrix entries of unitary representations we can form direct sums of representations to show that elements of $\mathrm{B}\,(G)$ are also matrix entries that need not be diagonal. We will provide $\mathrm{B}\,(G)$ a normed algebra structure. One way to compute the norm of an element b of $\mathrm{B}\,(G)$ is in terms of the positive definite functions on a larger groupoid for which b can appear as an "off diagonal part." This is the groupoid version of the well known 2×2 matrix method, and has been exploited by Renault for the same purpose. This permits usingthe completeness of $\mathrm{P}\,(G)$ for a general locally compact groupoid to provethe completeness of $\mathrm{B}\,(G)$.

We can also formulate $\mathrm{B}\,(G)$ as an algebra of completely bounded $C(\overline{X})$-bimodule maps on M*(G), and as a space of completely bounded $C(\overline{X})$-bimodule maps from $C^*(G,\overline{X})$ to $M^*(R,X)$. Since the completely positive elements in the latter set are all given by positive definite functions,and the completely positive bimodule maps form a complete set, we get one way to show that $\mathrm{B}\,(G)$ is complete.

Recall that ω is the direct sum of all cyclic representations of $C^*(G)$. We can construct each cyclic representation as an integrated representation of G, and, as such, it can be taken as a representation of $\mathrm{M}_c(G)$, and we use the same notation. For each $a \in C^*(G)$,$\|\omega(a)\| = \|a\|$ is the same

as $\sup\{\|\pi(a)\|: \pi \text{ is a cyclic representation of } C^*(G)\}$. Also recall, that the norms $\| \quad \|_{II,\mu}$ and $\| \quad \|_{II}$ and their properties.

Theorem (3.2.1)[3]: If $b \in \mathrm{B}\,(G)$, the operator T_b, taking $\omega(f)$ to $\omega(bf)$ for $f \in \mathrm{M}_c(G)$, extends to a completely bounded map of $M^*(G)$ to itself and$\|T_b\|_{cb} \geq \|b\|_Q$.

Proof: ByTheorem (3.1.1), if $p \in \mathrm{P}\,(G)$ then T_b is completely positive, so for $b \in \mathrm{B}\,(G)$ the operator T_b is completely bounded. Set $M = \|b\|_Q$and suppose $0 < \alpha < 1$. Since : is arbitrary, the proof will be complete if we find an $f \in \mathrm{M}_c(G)$ such that $\omega(f) \neq 0$ and $\|T_b\omega(f)\| \geq M\alpha^2\|\omega(f)\|$. To find such an f first notice that there is a $\mu \in Q$ such that the $L^\infty(\lambda^\mu)$-norm of $b$is greater than $M\alpha$, so there exist a $b_0 \in \mathbb{C}$ and $\eta > 0$ such that the set$A = \{\gamma: |b(\gamma) - b_0| < \eta\}$ has positive measure for λ^μ and $|b_0| - \eta > M\alpha$.Then there is a compact set $C \subseteq A$ such that $\lambda^\mu(C) > 0$. We take $f = \chi_C$.

By the definition of $\| \quad \|_{II}$, there is a $\mu' \in \mathcal{Q}$ such that $\|f\|_{II,\mu'} > \alpha \|f\|_{II}$. By the properties of $\| \quad \|_{II}$, if π is the one-dimensional trivial representation of G, we have $\left\|\pi^{\mu'}(f)\right\| > \alpha \|\omega(f)\|$. Now let $\sigma = \pi^{\mu} \otimes \pi^{\mu'}$. We have $\|\sigma(f)\| \geq \left\|\pi^{\mu'}(f)\right\| > \alpha \|\omega(f)\|$.

We can find g_1 and $g_2$in $C_c(X)$,≥ 0, and > 0 on $r(C) \cup s(C)$. These can be regarded as sections of the bundle for π, and it is clear that$(\pi^{\mu}(f)g_1|g_2) > 0$ from the integral formula for the inner product. Thus $\pi^{\mu}(f) \neq 0$, so $\sigma(f) \neq 0$and $\omega(f) \neq 0$.

Since $\sigma(b_0 f) = b_0 \sigma(f)$, it will suffice to show that $\|\sigma(b - b_0)f\| \leq \eta \|\sigma(f)\|$,because then we get $(|b_0| - \eta)\|\sigma(f)\| \leq \|\sigma(bf)\|$, so $(|b_0| - \eta)\, \alpha \|\omega(f)\| \leq \|\sigma(bf)\| \leq \|\omega(bf)\| = \left\|T_b\big(\omega(f)\big)\right\|$, giving the desired inequality. Now f is a characteristic function, so $(b - b_0)\, f = \big((b - b_0)\, f\big)\, f$. Also, $\|(b - b_0)\, f\|_{\infty} \leq \eta$,so the inequality we wanted on σ can be obtained by applying the second inequality before Lemma (2.1.6) to both μ and μ'. Thus the proof is complete.

Again we use the algebra $C\big(\overline{X}\big)$ to study $\mathrm{B}\,(G)$, and need the one-one correspondence between its representations and those of $C_c(G)$ and hence those of G. We still use ω for the direct sum of all cyclic representations of $C\big(G, \overline{X}\big)$, each of them given as an integrated representation of G. We use $\widetilde{\omega}$ for the direct sum of all the cyclic representations of $\mathrm{M}_{\,c}(R, X)$ that can be obtained by integrating a representation of R. Recall that $C^*\big(G, \overline{X}\big)$ is the operator norm closure of $\omega\left(C\big(G, \overline{X}\big)\right)$ and $M^*\big(R, \overline{X}\big)$ is the operator norm closure of $\widetilde{\omega}(\mathrm{M}_{\,c}(R, X))$. If $x \in X$, use $\mathrm{H}_{\,x}$ for $L^2(\mu^x)$ as before, and $\mathrm{H}_{\,X}$ for the direct sum of all the $\mathrm{H}_{\,x}$'s. Let $\widetilde{\omega}_t$be the subrepresentation of $\widetilde{\omega}$obtained by restricting to $\mathrm{H}_{\,X}$.

Theorem (3.2.2)[3]: Let $b \in \mathrm{B}\,(G)$. There is a completely bounded $C\big(\overline{X}\big)$-bimodule map $S_b\colon C\big(G, \overline{X}\big) \to M^*(R, X)$ such that $S_b\big(\omega(f)\big) = \widetilde{\omega}(\langle f, b\rangle)$ for $f \in C_c(G)$ and $S_b\big(\omega(g\varepsilon)\big) = \widetilde{\omega}(bg\varepsilon)$for $g \in C\big(\overline{X}\big)$. For this operator we have

$$\|S_b\|_{cb} \geq \|b\|_{\mathcal{Q}},$$

and

$$S_b\big(\omega(\varepsilon)\big)\big|\mathrm{H}_{\,X} \in \widetilde{\omega}_t(\mathrm{M}(X)).$$

Proof: The operator S_b is a linear combination of four operators $S_p$for $p \in \mathrm{P}(G)$, and these are completely positive bimodule maps byTheorem (3.1.4).

For the norm inequality, we proceed as in the proof of Theorem (3.2.1). Let $M = \|b\|_Q$and $0 < \alpha < 1$. It will suffice to find $f \in \mathrm{M}_c(G)$ such that $\omega(f) \neq 0$and $\|S_b(f)\| \geq M\alpha^2\|\omega(f)\|$. Choose μ, b_0, η, A and C as in Theorem (3.2.1),and take $f = \chi_C$.

We take π to be the trivial one-dimensional representation, and choose μ' and σ as before. The proof that $\omega(f) \neq 0$ used before works here also.

Let $\tilde{\pi}$ denote the one-dimensional trivial representation of R, and form its integral with respect to $\mu, \tilde{\pi}^\mu$. Likewise form $\tilde{\pi}^{\mu'}$, and let $\tilde{\sigma} = \tilde{\pi}^\mu \otimes \tilde{\pi}^{\mu'}$. It will suffice to prove that $\|\tilde{\sigma}(\langle f, 1\rangle)\| > \alpha\|\omega(f)\|$.

For this purpose, we need to see that $\|\langle f, 1\rangle\|_{II,\mu} = \|f\|_{II,\mu}$. This follows from the fact that $f \geq 0$ together with the relationship between ν_0 and $\tilde{\nu}_0$.Then we see that

$$\|\langle f, b - b_0\rangle\|_{II,\mu} \leq \|(b - b_0)\, f\|_Q\|f\|_{II,\mu} < \eta\|f\|_{II,\mu}$$

using the fact that f is a characteristic function.

Both the equality and the inequalities also hold for μ', and since π and $\tilde{\pi}$ are the one-dimensional trivial representations, they transfer to the corresponding equality and inequalities for σ and $\tilde{\sigma}$. Hence

$$\begin{aligned}
\|\tilde{\omega}(\langle f, b\rangle)\| &\geq \|\tilde{\sigma}(\langle f, b\rangle)\| \\
&\geq \|\tilde{\sigma}(\langle f, b_0\rangle)\| - \|\tilde{\sigma}(\langle f, b - b_0\rangle)\| \\
&\geq |b_0|\|\tilde{\sigma}(\langle f, 1\rangle)\| - \eta\|\tilde{\sigma}(\langle f, 1\rangle)\| \\
&\geq M\alpha\|\tilde{\sigma}(\langle f, 1\rangle)\| \\
&\geq M\alpha^2\|\omega(f)\|.
\end{aligned}$$

In order to provide the norm on $\mathrm{B}(G)$ in a way that will be convenient for proving completeness, we introduce a way to enlarge the groupoid G asit was done. Write T_2 for the transitive equivalence relation on the two element set $\{1,2\}$, so that T_2 has four elements. It will be convenient to have a shorter notation for matrix coefficients: If π is a unitary representation of G and ξ and η are bounded Borel sections of the bundle H on which π acts, we can write $[\pi, \xi, \eta]$ for the matrix coefficient, namely

$$[\pi, \xi, \eta](\gamma) = \big(\pi(\gamma)\, \xi \circ s(\gamma) \,|\, \eta \circ r(\gamma)\big).$$

Theorem (3.2.3)[3]: A bounded Borel function b on G is in $\mathrm{B}(G)$ if and only if there is a function $p' \in \mathrm{P}(G \times T_2)$ such that for $\gamma \in G$ we have $b(\gamma) = p'\big(\gamma, (1,2)\big)$. The function b can be expressed as a matrix coefficient using sections of sup norm at most 1 if and only if there is an associated p' that can be expressed as a diagonal matrix coefficient using a section of sup norm at most 1.

Proof: The proof of the first assertion will be given in terms of matrix coefficients and will include the proofs of the facts about sup norms. Let $X' = X \times \{1,2\}$ be the unit space of $G' = G \times T_2$.

Suppose that π is a unitary representation of G on a bundle H and that ξ and η are Borel sections of H of sup norm at most 1 such that $b = [\pi, \xi, \eta]$. Define a Hilbert bundle H' over X' by setting $\mathrm{H}'(x, i) = \mathrm{H}(x)$ for $i = 1,2$. For $\gamma' = \big(\gamma, (i, j)\big)$ in G' notice that $s(\gamma') = (s(\gamma), j)$ and $r(\gamma') = (r(\gamma), i)$. That means that we can define a representation π' of G' on H' by $\pi'(\gamma') = \pi(\gamma)$. Define a section ζ' of H' by setting $\zeta'(x, i) = \eta(x)$ when $i = 1$ and $\zeta'(x, i) = \xi(x)$ when $i = 2$. Then the sup norm of ζ' is at most 1 and for every $\gamma \in G$ we have $b(\gamma) = [\pi', \zeta', \zeta']\big(\gamma, (1,2)\big)$ as required.

For the converse, suppose we begin with H' π', and ζ'. Then for $x \in X$ define $\mathrm{H}(x) = \mathrm{H}'(x, 1) \oplus \mathrm{H}'(x, 2)$ and set $\eta(x) = (\zeta'(x, 1), 0)$ and $\xi(x) = \big(0, \zeta'(x, 2)\big)$. For $\gamma \in G$ define $\pi(\gamma)$ to take (ξ_1, ξ_2) to

$$\big(\pi'\big(\gamma, (1,1)\big)\xi_1 + \pi'\big(\gamma, (1,2)\big)\xi_2, \pi'\big(\gamma, (2,1)\big)\xi_1 + \pi'\big(\gamma, (2,2)\big)\xi_2\big),$$

thus acting as a matrix by left multiplication on column vectors. The sections ξ and η have sup norm at most 1, and we have $b = [\pi, \xi, \eta]$.

Because of the results we can now complete the task. Recall that for $b \in \mathrm{B}(G)$, T_b is the operator on $M^*(G)$ determined by multiplication by b on $\mathrm{M}_c(G)$, and that we sometimes work with $\mathrm{B}(G)$ as an algebra of functions, even though the elements are actually equivalence classes.

Theorem (3.2.4)[3]: $\mathrm{B}(G)$ is a Banach algebra with pointwise operations for the algebraic structure and with the norm defined by

$$\|b\| = \|T_b\|_{cb}$$

for $b \in \mathrm{B}(G)$.

Proof: Theorem(2.2.9) shows that $\mathsf{B}\ (G)$ is an algebra under pointwise operations, and equals $\mathsf{P}\ (G) - \mathsf{P}\ (G) + i\mathsf{P}\ (G) - i\mathsf{P}\ (G)$. Any function that is 0 for λ^{Q}-almost every point of G represents the 0 element of $M^*(G)$, so for $b \in \mathsf{B}\ (G)$ the operator T_b depends only on the equivalence class of b. Thus $b \mapsto T_b$ is well defined from the space of equivalence classes of functions in $\mathsf{B}\ (G)$ to the space of completely bounded operators on $M^*(G)$. Since $\|T_b\|_{cb} \geq \|b\|_\infty$, we see that $b \longmapsto T_b$ is also one-one. Thus the norm makes $\mathsf{B}\ (G)$ a commutative normed algebra.

To prove that $\mathsf{B}\ (G)$ is complete, let $b_1, b_2, \ldots$ be a sequence in $\mathsf{B}\ (G)$such that the norms $\left\|T_{b_n}\right\|_{cb}$ are summable. Then Theorem (3.2.3) says that we can construct positive definite functions $p_1', p_2', \ldots$ on the groupoid $G' = G \times T_2$ such that for every $\gamma \in G$ and every n we have $b_n(\gamma) = p_n'(\gamma, (1,2))$, and for every n we have $\|p_n'\|_\infty = \|b_n\|_\infty$. Two forms of the completeness of $\mathsf{P}\ (G')$ can be used to complete the proof. We let $c_n = b_1 + \cdots + b_n$.

In the first proof, we notice that the sequence $S_{p_1'}, S_{p_2'}, \ldots$ of completely positive $C\left(\overline{X}'\right)$-bimodule maps from $C^*\left(G', \overline{X}'\right)$ to $M^*(R'X')$is summable.The sum is also a completely positive $C\left(\overline{X}'\right)$-bimodule map, so by Theorem (3.1.5) it is of the form $S_{p'}$ for a $p' \in \mathsf{P}(G')$. Then the function b defined on G by $b = p'(\cdot, (1,2))$ is in $\mathsf{B}\ (G)$ by Theorem (3.2.3). We also get$\left\|S_{p_n' - p'}\right\|_{cb} \geq \left\|S_{c_n - b}\right\|_{cb} \geq \|c_n - b\|_\infty$ by Theorem (3.2.3) and Theorem (3.2.2), so$\|c_n - b\|_\infty \to 0$. We need to prove that $\|c_n - b\| \to 0$as $n \to \infty$.

To do this begin with $f \geq 0$ in $\mathsf{M}\ _c(G)$. Then Lemma (2.1.6) says that

$$\left\|\omega\big((c_n - b)f\big)\right\| \leq \|c_n - b\|_\infty \|\omega(f)\|.$$

Hence $T_{c_n}\big(\omega(f)\big) \to T_b\big(\omega(f)\big)$in $M^*(G)$. The f's span a dense set in $M^*(G)$,and the T_{c_n}'s are uniformly bounded, so it follows that $T_{c_n} \to T_b$ pointwise on $M^*(G)$. Now the fact that the completely bounded operators on $M^*(G)$are complete implies that the sequence T_{c_n} has a limit, T' in the completely bounded sense, which is automatically also a pointwise limit on $M^*(G)$.Hence $T' = T_b$, so that $\left\|T_{c_n - b}\right\|_{cb} \to 0$, and by Theorem (3.2.1) that is equivalent to saying $\|c_n - b\| \to 0$ as $n \to \infty$.

For the other proof of completeness, we notice that $p_1', p_2', \ldots$ is summable in the Q-essential supremum norm as functions on G'. Hence there is a Borel function p' that is the sum in that norm. By the Dominated Convergence Theorem, $p' \in \mathrm{P}(G')$. Again we take $b = p'(\cdot,(1,2))$. Theorems (3.2.3) and (3.2.2) once again show that $\|c_n - b\|_\infty \to 0$, and we complete the proof as before.

Since $\mathrm{B}(G)$is a Banach algebra, any closed subalgebra of it is a Banach algebra. Convergence in the completely bounded norm implies convergence in $L^\infty(\lambda^{\mathcal{Q}})$, so certain subalgebras are easily seen to be closed. Among these are $B(G)$, defined to be $\{b \in \mathrm{B}(G) : b \text{ is continuous}\}$, and $\mathrm{B}(G,X)$, defined to be the set of elements $b \in \mathrm{B}(G)$ such that $b\,|\,X$ is continuous and vanishes at ∞. The subalgebra $B(G,X)$ is defined to be $B(G) \cap \mathrm{B}(G,X)$.

Theorem (3.2.5)[3]:$B(G), \mathrm{B}(G,X)$, and $B(G,X)$ are closed subalgebras of $\mathrm{B}(G)$ and hence Banach algebras.

The first example is a groupoid on which the linear span of the continuous positive definite functions is not complete and there exist continuous elements of $\mathrm{B}(G)$ that cannot be expressed as a difference of continuous positive definite functions.

Let $X = \{(x,y) : (x,y) \text{ has polar coordinates} (r,\theta) \text{with } 0 \le r \le 1, \theta \in \{0,1,1/2,1/3,\dots.\}\}$ and set $G = X \times \mathbb{Z}$. This is a bundle of groups, and$(x,n) + (x',n')$ is defined iff $x = x'$, and then it equals $(x,n+n')$. Write$\mathrm{P}(G)$ for the set of Borel positive definite functions on G and $\mathrm{P}(G)$ for the set of continuous elements of $\mathrm{P}(G)$. Let $\mathrm{B}(G)$ be the linear span of $\mathrm{P}(G)$, let $B_1(G)$ be the linear span of $\mathrm{P}(G)$ and let $\mathrm{B}(G)$ be the set of continuous elements of $\mathrm{B}(G)$. A bounded function p is in $\mathrm{P}(G)$ iff it is a Borel function and $p(r,\theta,\cdot)$ is positive definite on $\mathbb{Z}$ for each point of X. Since positive definite functions on $\mathbb{Z}$ are in one-one correspondence with positivemeasures on $\mathbb{T}$ via the Fourier transform, we can also think of $\mathrm{P}(G)$as consisting of Borel functions from X to the positive measures on $\mathbb{T}$.

Define

$$p(r,\theta,n) = \begin{cases} e^{iO(1+r)n} & \text{if } r > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

and

$$q(r,\theta,n) = \begin{cases} e^{iO(1-r)n} & \text{if } r > 0 \\ 0 & \text{if } r = 0. \end{cases}$$

We can also think of these as taking values that are point masses at $e^{iO(1+r)}$and $e^{iO(1-r)}$, or the 0 measure at the origin. We have $p-q \in B(G)$. Suppose that $u \in P(G)$ and $-u \leq p-q \leq u$ where the inequalities indicate the pointwise order in the space of measure-valued functions. This is the same as the natural order in $\mathrm{B}\ (G)$ in which elements of $\mathrm{P}\ (G)$ are positive. Since $p\big((r,\theta)\cdot\big)$ is the point mass at $e^{iO(1+r)}$, $u(r,\theta,\cdot)$ dominates the point mass at that point. By continuity, $u(0,0,\cdot)$ dominates the point mass at e^{iO}. This means that $u(0,0,\cdot)$ has infinite norm, so there is no such u. Thus we have a continuous element of $\mathrm{B}\ (G)$ that is not a difference of continuous positive definite functions.

With more effort, a worse example can be made. Choose n angles, and begin with p and q restricted to the radii with those angles. The limit at the origin of both of them exists, the limits are the same, and it is a sum of n point masses. To make elements of $P(G)$ we take that value at the origin and at all other points of X. Let b be the difference of these elements of $P(G)$. Any element of $P(G)$ that dominates b must have a value at the origin that dominates that sum of n point masses. Observe that b is 0 except on the original chosen radii, and that the total variation norm of each value of b is at most 2.

Now partition the angles in X into sets with 2^k elements, for $k = 1,2,\ldots$,and use the construction just described to make elements b_k in $B_1(G)$. Then let $b = \sum_{k=1}^{\infty} 2^{-k} b_k$. This converges in the completely bounded norm since each b_k has completely bounded norm 2. Hence it also converges in uniform norm, so that $b \in B(G)$. Also b is in the closure of $B_1(G)$. However,the domination arguments used above show that b is not in $B_1(G)$.

The next example shows that locally compact groupoids can have unitary representations that are Borel but not continuous.

Consider an action of the integers on the circle by an irrational rotation,and form the transformation group groupoid, $G = \mathbb{T} \times \mathbb{Z}$. If u is a unitary valued Borel function on $\mathbb{T}$, there is a unitary representation U such that for all $\tau \in \mathbb{T}, u(\tau) = U(\tau,1)$. If u is not continuous, neither is U.