Chapter 2

Fourier-Stieltjes Algebras

We give a first step towards extending the theory of Fourier- Stieltjes Algebras from groups to groupids . If G is a locally compact (second countable) groupid. We show that the linear span of the Borel positive definite function on G ,is a Banach algebra when represented as an algebra of completely bounded maps on a C*-algebra associated with G. This necessarily involves identifying equivalent elements. An example shows that the linear span of the continuous positive definite need not be complete .For groups , B(G) is isometric to the Banach space dual of C*(G).

Section (2.1): Back Ground on Groupoids

We mention here that some basic definitions can befound and that we assume locally compact spaces are second countable. More background on groupoids is available . The necessary background on Fourier_Stieltjes algebras can be obtained(see [2]).

In 1963, Mackey , introduced the notion of virtual group as a too land context for several kinds of problems in analysis and geometry. Virtual groups are (equivalence classes of) groupoids having suitable measure theoretic structure and the property of ergodicity. Ergodicity makes a groupoid more group-like, but many results on groupoids do not require ergodicity. Among the structures which fit naturally into the study of groupoids are groups, group actions, equivalence relations (including foliations), ordinary spaces, and examples made from these by restricting to apart of the underlying space.

The original motivation for studying groupoids was provided by Mackey's theory of unitary representations of group extensions. The idea has been applied to that subject: for example see [2]. In his section, Mackey also showed the relevance of the idea for ergodic group actions in general, and a number of applications have been made there, for example see [2].

Most uses of groupoids have been in the study of operator algebras, another approach to understanding and exploiting symmetry. Hahn proved the existence of Haar measures for measured groupoids, whether ergodic or not, and used this to make convolution algebras and study von Neumann algebras associated with measured groupoids .

Feldman and Moore made a thorough analysis of ergodic equivalence relations that have countable equivalence classes, showing that the von Neumann algebras attached to them areexactly the factors that have Cartan subalgebras .Connes intro-duced a variation on the approach of Mackey in particular by working without a chosen invariant measure class. This approach has some advantages for applications to foliations and to C^*-algebras Renault studied C^*-algebras generated by convolution algebras on locally compact groupoids endowed with Haar systems, not using invariant measure classes. It was shown that measured groupoids may be assumed to have locally compact topologies, with no loss in generality. Thus the study of operator algebras associated with groupoid symmetry can always be confined to locally compact groupoids, whether one is interested in C^*-algebras or von Neumann algebras.

Basically one can say that locally compact groupoids occur in situations where there is symmetry that is made evident by the presence of an equivalence relation. Many of these are associated either with group actions or foliations. It can be surprising how group-like both group actions and foliations can be.

In 1964 Eymard , introduced Fourier and Fourier-Stieltjes algebras for non-commutative locally compact groups. Roughly, the Fourier-Stieltjes algebra of a locally compact group, G, denoted $B(G)$, is the unitary representation theory of G equipped with some additional algebraic and geometric structure. More precisely, $B(G)$ is the set of finite linear combinations of continuous positive definite functions on G equipped with a norm, which makes $B(G)$ a commutative Banach algebra (using pointwise addition and multiplication). The elements of $B(G)$ are exactly the matrix entries of unitary representations of G. A primary source of intuition is the fact that when G is abelian, $B(G)$ is the isometric, inverse Fourier-Stieltjes transform of $M^1(\hat{G})$, the convolution, Banach algebra of finite, regular Borel measures on $\hat{G}$, the dual group (of characters) of G. Thus $B(G)$, as a Banach algebra, “is”$M^1(\hat{G})$. The fact that $B(G)$ exists (as a commutative Banach algebra) when G is not abelian leads one to hope that a useful duality theory exists for non-abelian groups which is in spirit similar to the application rich Pontriagin-Van Kampen duality for abelian locally compact groups. That such a duality theory exists has been established byWalter by proving that

(a) $B(G)$ is a complete invariant of G, i.e., $B(G_1)$ and $B(G_2)$are isometrically isomorphic as Banach algebras, if and only if G_1 and G_2 are topologically isomorphic as locally compact groups, and

(b) There is an explicit process for recovering G given its "dualobject", $B(G)$. Exactly how useful this theory will be remains to be seen since all but a few of the hoped for important applications await rigorous proof.

For various reasons it turns out that it may be more fruitful to look at $B(G)$ from a broader perspective than that afforded by the category of locally compact groups. Namely, it is seen that there is a natural duality theory for a "large" collection of Banach algebras that extends in a precise way the Pontriagin duality for abelian groups as well as the above-mentioned duality for non-abelian groups. The theory of C^*-algebras plays a large role both technically and intuitively in this duality theory.

In an effort to understand this new duality theory better, as well as to generate meaningful applications and examples of a concrete nature, we have answered affirmatively the question: Does a locally compact groupoid G have a Fourier-Stieltjes algebra? For groupoids, there is more than one candidate for the Fourier-Stieltjes algebra, and the details are more technical than for groups, but there is an affirmative answer.

The existence of a Fourier-Stieltjes algebra augurs well for future applications. In particular, one example suggests an interesting possibility:The algebra of continuous functions on X vanishing at infinity, $C_0(X)$, is the Fourier algebra of a locally compact space X. This opens up an entire "dual" approach to the currently exploding subject of non-commutative geometry, which at the moment is regarded more or less exclusively interms of the associated C^*-algebras (not the Fourier-Stieltjes algebras).

As for groups, the Fourier-Stieltjes algebra of a groupoid is the linear span of the positive definite functions and the algebra structure is given by pointwise operations. To provide the Banach space structure, we use C^*-algebras attached to G, but we use them in a different way from Eymard, and also use C^*-algebras associated with the equivalence relation that G induces on X.

To describe the various algebras, let us begin with the space $M_c(G)$of compactly supported bounded Borel functions on G, and its subspace$C_c(G)$. Both are algebras under convolution, which is defined by using the Haar system, and have involutions. If R is the equivalence relation on X induced by G, defining $\theta(\gamma) = (r(\gamma), s(\gamma))$ gives a continuous homomorphism of G onto R using the relative product topology on R. The quotient topology on R has some advantages: for example, if θ is one-to-one then θ is a homeomorphism. ($G$is said to be principal.) Under the quotient topology R is σ-compact and we can provide it with a Borel measurable Haar system, which allows us to make a convolution $*$-algebraof the space $\mathsf{M}_{0c}(R)$ of bounded Borel functions on R that are supported by the image of some compact set in G. we show how to make an algebra on G that contains a copy of the space $\mathsf{M}(X)$ of bounded Borel functions on X as well as $\mathsf{M}_{c}(G)$, and this algebra is denoted by $\mathsf{M}_{c}(G,X)$.The analog for R is denoted by $\mathsf{M}_{0c}(R,X)$. Let X- denote the one-point compactification of X. Then $C(\bar{X}) \subseteq \mathsf{M}(X)$, so $\mathsf{M}_{c}(G),X$ contains both $C_c(G)$and $C(\bar{X})$. The span of these two subalgebras is denoted$C_c(G,\bar{X})$.

If ω is the universal representation of G, then ω carries each convolution algebra on G to an algebra of operators and thereby provides the convolution algebra with a norm. The closures of the algebras of operators or the completions under the norms are useful in various ways, so we have notation for them: $C^*(G)$ is the completion of $C_c(G), C^*(G,\bar{X})$ is the completion of $C_c(G,\bar{X}), M^*(G)$ is the completion of $\mathsf{M}_{c}(G)$, and $M^*(G,X)$ is the completion of $\mathsf{M}_{c}(G,X)$. Likewise for R we get $M^*(R)$ and $M^*(R,X)$from$\mathsf{M}_{0c}(R)$ and $\mathsf{M}_{0c}(R,X)$. The algebra $\mathsf{B}(G)$ is isomorphic to a Banach algebra of completely bounded operators on $M^*(G)$, but the functions also correspond to completely bounded bimodule mappings from $C^*(G,\bar{X})$ to$M^*(R,X)$ as bimodules over $C(G,\bar{X})$.

Assuming that background, we define abounded Borel function p on a locally compact groupoid G with Haar system λ to be positive definite if

$$\iint f(\gamma_1)\bar{f}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \geq 0$$

for every $f \in C_c(G)$. The set of these is denoted $\mathsf{P}\ (G)$ and by definition the set $\mathsf{B}(G)$ is the linear span of $\mathsf{P}(G)$. In both sets two elements that agree except on a negligible set need to be identified, though we find it convenient to indulge in the usual carelessness about maintaining the distinction. The primary result is

(I) $\mathsf{B}(G)$ is a Banach algebra.

Results needed to prove this are:

(II) Each $p \in \mathsf{P}(G)$ can be represented in terms of a unitary representation of G and a cyclic “vector” for the representation.

(III) Multiplication by a $b \in \mathrm{B}(G)$ defines a completely bounded operator on $M^*(G)$ whose norm is at least the supremum norm of b.

(IV) The set of operators arising from elements of $\mathrm{P}(G)$ is closed in the space of completely bounded operators on $M^*(G)$.

In fact, $\mathrm{B}(G)$ is a Banach algebra of completely bounded operators on $M^*(G)$, and the elements of $\mathrm{P}(G)$ occur as completely positive operators.In order to prove the completeness of $\mathrm{B}(G)$, we introduce an auxiliary groupoid. Let T_2 denote the transitive equivalence relation on the two point set $\{1, 2\}$, so that functions on T_2 are 2×2 matrices. Thus functionson $G \times T_2$ can be regarded as 2×2 matrices of functions on G. Then each $b \in \mathrm{B}(G)$ appears as a corner entry of a positive definite function on $G \times T_2$whose completely bounded norm is the same as that of b. Furthermore,such a corner entry is always in $\mathrm{B}(G)$. Combining these facts with the com-pleteness of $\mathrm{P}(G \times T_2)$ is what allows us to finish the proof of completeness of $\mathrm{B}(G)$. This groupoid is used in a similar way .

We devoted to background material on three topics: locally compact groupoids, convolution algebras attached to them, and representations of groupoids and the algebras. We showsome measure theoretic technicalities about Haar systems and choosing of Borel functions in prescribed equivalence classes modulo null sets, and the fundamental results about positive definite functions. We give the definition of "positive definite function" and establish the connection between such functions and cyclic unitary representations of G.we show that multiplication by a positive definite function is a completely positive operator on $M^*(G)$, we also show the proof that a positive definite function gives rise to a completely positive operator from $C^*(G, X)$ to $M^*(R, X)$.All of these operators are bimodule maps over $C(\bar{X})$, the algebra of continuous functions on the one-point compactification of the space of units of G. We show results about completely bounded bimodule maps.we are able to complete the proof that the linear combinations of positive definite functions constitute a Banach algebra.

The purpose of this section is to give a source of some essential information about analysis on groupoids needed .

Much of our motivation comes from the fact that group actions give rise to groupoids, and that case was important in the development of the subject. However, we want to present a definition that has a different motivation, hoping to make the idea easier to grasp. Effros suggested this approach.

Start with two sets, X and G, and suppose that X is the set of vertices and G the set of edges of a directed graph. If the structure we are about to describe is present, we say that G is a groupoid on X. Suppose that we havea mapping taking values in G and defined on the set of pairs of edges for which the first edge starts from the vertex where the second edge terminates. (For a groupoid of mappings, we want the operation to be composition and we want the right hand factor to be applied first.) We want the operation to be associative and to have units and inverses.

To describe this in more detail, we use two functions r and s from G onto X, such that each $\gamma \in G$ is an edge from $s(\gamma)$to $r(\gamma)$. Then for γ and γ'in$G$, the element $\gamma\gamma'$ of G is defined iff $s(\gamma) = r(\gamma')$. We write $G^{(2)} = \{(\gamma, \gamma') \in G \times G : s(\gamma) = r(\gamma')\}$. We also assume there is given a mapping $x \longmapsto i_x$ of X into G and an involution $\gamma \longmapsto \gamma^{-1}$on G. Then we require the following properties:

(a) (associativity) If $s(\gamma_1) = r(\gamma_2)$, then $s(\gamma_1\gamma_2) = s(\gamma_2)$, and $r(\gamma_1\gamma_2) = r(\gamma_1)$. If, also, $s(\gamma_2) = r(\gamma_3)$, then $(\gamma_1\gamma_2)\gamma_3 = \gamma_1(\gamma_2\gamma_3)$.
(b) (units) If $x \in X$, then $r(i_x) = s(i_x) = x$. If $\gamma \in G$, then $\gamma i_{s(\gamma)} = i_{r(\gamma)}\gamma = \gamma$.
(c) (inverses) $r(\gamma^{-1}) = s(\gamma), s(\gamma^{-1}) = r(\gamma), \gamma\gamma^{-1} = i_{r(\gamma)}$, and $\gamma^{-1}\gamma = i_{s(\gamma)}$.

Examples (2.1.1)[2]:

(a) Suppose a group H acts on a set X (on the left). Set $\mathrm{g} = H \times X$, identify $X$with $\{e\} \times X$, and define $r(h, x) = hx, s(h, x) = x$. Then we can define $(h_1, x_1)(h_2, x_2) = (h_1h_2, x_2)$ if $x_1 = h_2x_2, i_x = (e, x)and(h, x)^{-1} = (h^{-1}, hx)$, to make a groupoid. (Right actions work better for left Haar measures as we see below, and then we have $s(x, h) = xh, r(x, h) = x$.)
(b) To make a groupoid from an equivalence relation R on a set X,identify X with the diagonal in $X \times X$, define $r(x, y) = x, s(x, y) = y, (x, y)(y, z) = (x, z)$ and $(x, y)^{-1} = (y, x)$.
(c) Let X be the set of open sets in $\mathbb{R}^n$, and let G be the set of diffeomorphisms between elements of X. For $\gamma \in G$, let $s(\gamma)$ be the domain of the mapping and let $r(\gamma)$ be its range. Let the product be function composition and let the inverse be the inverse of functions.

Every groupoid determines a natural equivalence relation on its set of units, namely $x \sim y$ iff there is a $\gamma : x \to y$. The equivalence class of x is denoted $[x]$ and is called its orbit. As a subset of $X \times X$, this equivalence relation is $R = \{(r(\gamma), s(\gamma)) : \gamma \in G\}$. The function $\theta = (r, s)$ mapping G to R is a groupoid homomorphism and G is called principal iff θ is one-one, i.e.,G is isomorphic to an equivalence relation. If G arises from a group action , G is principal iff the action is free (the only element of the group that has any fixed points is the identity).

If G is a groupoid on X, and $Y \subseteq X$ is non-empty, we call $r^{-1}(Y) \cap s^{-1}(Y)$ the restriction of G to Y, and write $G|Y$ for it. In terms of graphs,$G|Y$ is the set of all edges in G that connect points of Y. $G|Y$is a subgroupoid of G, and a groupoid on Y. For each $x \in X$, $G|\{x\}$ is a group called the stabilizer of x or the isotropy of x.

If A and B are subsets of a groupoid G, we define the product AB of the two sets to be $\{\gamma\gamma' : \gamma \in A, \boldsymbol{\gamma'} \in \boldsymbol{B}, r(\gamma') = s(\gamma)\}$. If A has a single element γ_0,we write $\gamma_0\boldsymbol{B}$ for AB. Thus $YGY = G|Y$ and $xGx = G|\{x\}$ if $Y \subseteq X$ and $x \in X$. We also use the sets $r^{-1}(x) = xG$ and $s^{-1}(x) = Gx$ when $x \in X$.

A groupoid G is a Borel groupoid if G has a Borel structure, X is a Boredset when regarded as a subset of G, and $r, s, (\quad)^{-1}$ and multiplication are Borel functions. We will consider only Borel groupoids which are at least analytic, and then $X = \{\gamma : r(\gamma) = \gamma\}$ is Borel if r is Borel. A groupoid G istopological if it has a topology such that X is closed, and $r, s, (\quad)^{-1}$ and multiplications are continuous, while r and s are open. Again these properties are not independent. It is necessary for r to be open in order to prove that AB is open whenever A and B are open .

We write $\mathrm{M}(G)$ for the space of bounded Borel measurable functions on G, whenever G is a Borel groupoid. If G has a topology in which it is σ-compact (a countable union of compact sets), we write $\mathrm{M}_c(G)$ for the subspace of $\mathrm{M}(G)$ of functions having compact support.

If G is an analytic Borel groupoid, we say a measure μ on G is quasisymmetric if it has the same null sets as its image μ^{-1}under $(\quad)^{-1}$. Thus $\mu$and $(\mu)^{-1}$ are in the same measure class, and the measure class $[\mu]$ (set of measures with the same null sets as μ) is invariant under $(\quad)^{-1}$. For measures on G, this global symmetry is just the same as if G were a group.

In the material following this paragraph, we give the definitions for groupoids that extend the notions of invariance and quasiinvariance of measures under translation on a group or under other actions of the group. Because translation on the left by a groupoid element γ makes sense only on $s(\gamma)G$, and similarly for right translation, the notions of invariance and quasiinvariance are more complicated for groupoids than for groups.

Following Connes we say that a kernel is a function ν assigning a σ-finite (positive) measure ν^x on G to each $x \in X$, so that these two statements are true:

(a) $\nu^x(G \backslash xG)$is always 0. One may say that ν^x is concentrated onxG.
(b) If $f \in \mathrm{M}(G)$, and $f \geq 0$, the function $\nu(f): X \to [x, \infty]$ defined by$\nu(f)(x) = \nu^x(f) = \int f d\nu^x$ is Borel.

Given an element $\gamma \in G$, the mapping $\gamma' \longmapsto \gamma\gamma'$ is a Borel isomorphism of$s(\gamma)G$ onto $r(\gamma)G$ and thus maps $\nu^{s(\gamma)}$ to a measure $\gamma\nu^{s(\gamma)}$on $r(\gamma)G$, for everykernel ν. A kernel ν is called left invariant provided $\nu^{r(\gamma)} = \gamma\nu^{s(\gamma)}$ for all $\gamma \in G$.It is called (left) quasiinvariant if $\nu^{r(\gamma)}$ and $\gamma\nu^{s(\gamma)}$ are equivalent for all $\gamma \in G$.

A left invariant kernel, λ, on a Borel groupoid G is called a Borel Haar system. Then defining λ_x to be the image of λ^x under inversion produces a right Borel Haar system. A Borel Haar system λ on a locally compact groupoid is called a Haar system if supp(λ^*) is always xG and $\lambda(f) \in C_c(X)$ for each $f \in C_c(G)$. In particular, each λ^* is a Radon measure.

When λ is a Haar system, it can be convenient to have a left quasi-invariant kernel λ_1 consisting of probability measures equivalent to the measures λ^*. It is not difficult to show that there is a continuous, strictly positive, function f on G such that for every $x \in X, \int f d\lambda^x = 1$. We choose one such f and write λ_1^x for the measure $f\lambda^*$. We also write μ_1^* for the probability measure $s(\lambda_1^x)$on X; these measures also depend on x in a Borel way.

If λ is a Borel Haar system on a Borel groupoid G and μ is a probability measure on X, we can form a measure $\nu = \int \lambda^x d\mu(x) : \int f d\nu = \iint f(\gamma) f \lambda^x(\gamma) d\mu(x)$. We often write λ^μ for this measure ν. Suppose that $G = X \times H$, where X is a right H-space, and give G the groupoid structure that comes from the group action. Let λ be a left Haar measure on H. For each $x \in X$, let ε^x be the point mass at x, and define $\lambda^x = \varepsilon^x \times \lambda$, to get aBorel left Haar system. If μ is a σ-finite measure on X for this groupoid,then$\nu = \lambda^\mu = \mu \times \lambda$ and the class $[\nu]$ is symmetric iff μ is quasiinvariant under the group action, i.e., for every Borel set $E \subseteq X$ and every group element $h, \mu(E) = 0$ iff $\mu(Eh) = 0$. This follows from the fact that if μ is quasiinvariant under almost all elements of the group,then it is quasiinvariant. Hence, on a general Borel groupoid with Borel Haar systemλ, a σ-finite measure μ on X is called quasiinvariant iff $\lambda^\mu$is quasisymmetric. In that case, a result of Peter Hahn , shows that there is a Borel homomorphism, Δ_μ, of G to the multiplicative positive real numbers such that

$$\Delta_\mu = \frac{d\lambda^\mu}{d(\lambda^\mu)^{-1}}.$$

This homomorphism is called the modular function by analogy with locally compact groups. If μ is quasiinvariant, and Y is a μ-conull Borel set in X,the restriction $G|Y$ is called inessential.

We often refer to the set of all quasiinvariant σ-finite measures on X, and will denote that set by Q. We say a Borel set $N \subseteq X$ is Q-null provided$\mu(N) = 0$ for every $\mu \in Q$. It follows from the existence, and uniqueness upto equivalence, of a quasiinvariant σ-finite measure on each orbit that N is Q-null iff$\lambda^x(GN)$is always 0. The measures μ_1^x introduced above are in this class, and any measure in Q equivalent to such a measure is called transitive because it is concentrated on a single orbit. For a Borel set$N \subseteq G$, we say N is λ^Q-null iff $\lambda^\mu(N) = 0$ whenever $\mu \in Q$. A function f on X is Q-essentially bounded iff the restriction of f to the complement of some Q-null set is bounded, and then $\|f\|_\infty$- is defined to be the smallest element of $\{B: |f| \leq B\mu$-almost everywhere for every $\mu \in Q\}$. The space of Q-essentially bounded functions on X will be denoted by $L^\infty(Q)$. A similar definition is used for the space $L^\infty(\lambda^Q)$ of λ^Q-essentially bounded functions on G, except that the measures λ^μ are used.

Examples (2.1.2)[2]:

(a) If $G = X \times H$, where X and H are locally compact and H is a group, let ε^x denote the unit point mass at x for $x \in X$ and let λ bea left Haar measure on H. Then $\lambda^x = \varepsilon^x \times \lambda$ defines a Haar system for G.

(b) If E is an analytic equivalence relation on X and each equivalence class is countable, we can let λ^x be counting measure on $\{x\} \times [x]$to get a left invariant system of measures.

(c) Here is an example of a locally compact groupoid that has a Borel Haar system but no Haar system. Let $G = [0, 1/2] \times [0] \cup [1/2, 1] \times \boldsymbol{Z}/2$. This is a field of groups. To get a Borel Haar system, we can make each λ^x a multiple of the Haar measure on $\{0\}$or $\mathbb{Z}/2$. Then $\lambda^{1/2}(\{1/2, 0\}) = \lambda^{1/2}([1/2, 1]) > 0$, and if we let f be the characteristic function of $[1/2, 1] \times \{1\}$, then the function $\lambda(f)$ has a jump at $1/2$. We could easily change to another locally compact topology on this G and get a Haar system. In general, it may be necessary to change the topology on G and pass to an inessential restriction in order to get a Haar system.

We use several convolution algebras , and will introduce them here. There are two basic convolutions, a convolution of functions that can be defined in the presence of a Borel Haar system, and a convolution of kernels that does not depend on any such system. If the groupoid is locally compact and the Haar system is continuous, then $C_c(G)$ is analgebra under the convolution of functions. We will see that convolution of functions can be subsumed under convolution of kernels by replacing each function by the kernel obtained by multiplying the Haar system by the function.

First, let G be a Borel groupoid with a Borel Haar system λ. If f, g are non-negative Borel functions on G, then $\int f(\gamma_1)\mathrm{g}(\gamma_2)d\lambda^{r(\gamma_1)}(\gamma_2)$ is a Borel function of γ_1 , so by taking linear combinations and monotone limits we see that whenever F is a non-negative Borel function on $G \times G$ the integral$\int F(\gamma_1, \gamma_2)d\lambda^{r(\gamma_1)}(\gamma_2)$ depends on γ_1 in a Borel manner. Then for non-negative $f, \mathrm{g} \in \mathrm{M}(G)$, we can let $F(\gamma_1, \gamma_2) = f(\gamma_1)\mathrm{g}(\gamma_1^{-1}\gamma_2)$ when $r(\gamma_2) = r(\gamma_1)$ and $F(\gamma_1, \gamma_2) = 0$ otherwise, and see that $\int f(\gamma_1)\mathrm{g}(\gamma_1^{-1}\gamma_2)d\lambda^{r(\gamma_1)}(\gamma_1)$ is a Borel function of γ_2. Denote this function by $f * \mathrm{g}$, provided that it is always finite valued. Then $f * \mathrm{g} \in \mathrm{M}(G)$. The function $f * \mathrm{g}$ is called the convolution of $f$and g. Convolution can be extended to more general functions using linearity.

Define the space $I_r(G, \lambda)$ to be $\{f \in \mathrm{M}(G): \lambda(|f|)$ is bounded$\}$, and giveit a norm by letting $\|f\|_{I,r}$ be the sup norm of the Borel function $\lambda(|f|)$.As proved before, $I_r(G, \lambda)$ is closed under convolution and the norm $\|\cdot\|_{I,r}$ is an algebra norm. We can define an involution on $\mathrm{M}(G)$as by letting $f^b(\gamma) = \bar{f}(\gamma^{-1})$ for $f \in \mathrm{M}(G), \gamma \in G$. If we set $I(G, \gamma) = I_r(G, \gamma) \cap \left(I_r(G, \gamma)\right)^b$, then we can define $\|f\|_I$ to be the maximum of $\|f\|_{I,r}$ and $\|f^r\|_{I,r}$ for $f \in I(G, \lambda)$, obtaining a normed algebra on which the involution is an isometry.

If G is locally compact and λ is a Haar system, then $C_c(G)$ is a $*$-sub-algebra of $I(G, \lambda)$. In the inductive limit topology, $C_c(G)$ is a topological algebra .

The second kind of convolution can be introduced after the objects are defined: A complex kernel is a function ν assigning a complex measure ν^xon G so that

(a) $\nu^x$is always concentrated on xG.
(b) if$f \in \mathrm{M}(G)$, the function $\nu(f)$ taking $x \in X$ to $\nu^x(f)$ is Borel.

We define $K(G)$ to be the space of bounded complex kernels on G, i.e.,those for which the total variation of ν^x is a bounded function of x.

If $\gamma \in G$ and $\nu \in K(G)$ we can map $\nu^{s(\gamma)}$ to a measure on $r(\gamma)G$, via left translation by γ, as we do in defining Haar systems. Denote this measure by $\gamma\nu^{s(\gamma)}$. If $\nu_1, \nu_2 \in K(G)$ we can define the convolution $\nu = \nu_1 * \nu_2$ by $\nu^x = \int \gamma\nu_2^{s(\gamma)} d\nu_1^x(\gamma)$. We can also define an action of $K(G)$ on $I_r(G,\lambda)$ as follows. If $\nu \in K(G), f \in I_r(G,\lambda)$ and $\gamma' \in G$ set

$$L(\nu)f(\gamma') = \int f(\gamma^{-1}\gamma')d\nu^{r(\gamma')}(\gamma).$$

It is not difficult to verify that $L(\nu)$ is a bounded operator whose norm is at most the essential supremum of the total variation norms of the signed measures ν^x. If ν_1 and ν_2 are in $K(G)$and $f \in I_r(G,\lambda)$, then we can calculate

$$\begin{aligned}\big(L(\nu_1)(L(\nu_2)f)\big)(\gamma) &= \int (L(\nu_2)f)(\gamma_1^{-1}\gamma)d\nu_1^{r(\gamma)}(\gamma_1)\\ &= \iint f(\gamma_2^{-1}\gamma_1^{-1})d\nu_2^{s(\gamma_1)}(\gamma_2)d\nu_1^{r(\gamma)}(\gamma_1)\\ &= \iint f(\gamma_2^{-1}\gamma)d\nu_2^{s(\gamma_1)}d\nu_1^{r(\gamma)}(\gamma_1)\\ &= \int f(\gamma_2^{-1}\gamma)d(\nu_1 * \nu_2)^{r(\gamma)}(\gamma_2),\end{aligned}$$

showing that L takes convolution to composition of operators. Since L is faithful, $K(G)$ is an algebra under convolution. If $f, \mathrm{g} \in I_r(G,\lambda)$ it is not difficult to verify that $f\lambda \in K(G)$and $L(f\lambda)\mathrm{g} = f * \mathrm{g}$:

$$L(f\lambda)\mathrm{g}(\gamma) = \int \mathrm{g}(\gamma_1^{-1}\gamma)f(\gamma_1)d\lambda^{r(\gamma)}(\gamma_1).$$

Since L is faithful and convolution is associative, it follows that $f\lambda * \mathrm{g}\lambda = (f * \mathrm{g})\lambda$. Thus $I_r(G,\lambda)\lambda = \{f\lambda : f \in I_r(G,\lambda)\}$ is a subalgebra of $K(G)$ isomorphic to $I_r(G,\lambda)$. If G is locally compact and has a Haar system λ, the calculations just made also show that $C_c(G)\lambda$ is a subalgebra of $K(G\ )$isomorphic to $C_c(G)$.

Next we want to enlarge $C_c(G)\lambda$ to a subalgebra of $K(G)$ that contains a copy of $C_c(X)$. We denote the one-point compactification of $X$by $\bar{X}$. The mapping $f \to f|X$ takes $C(\bar{X})$ one-one onto the algebra of continuous functions on X that have a limit at infinity. We identify $C(\bar{X})$ with that sub-algebra of $C(X)$ but continue to write $C(\bar{X})$. Notice that there is also a sub-algebra of $K(G)$ isomorphic to $C(\bar{X})$, obtained as follows. First define ε to be the kernel that assigns the point mass at x to each $x \in X$, which we denote by ε^x as above. Next notice that $K(G)$ is closed under multiplication by any bounded Borel function on G, so if $h \in \mathsf{M}(X)$ and $\nu \in K(G)$, we can define $h\nu$ to be $(h \circ r)\nu$, and $\nu h = (h \circ s)\nu$. (These agree with the naturally defined left and right multiplication of $\mathsf{M}(X)$ on $I_r(G, \lambda)$ when the latter is regarded as a space of kernels.) Then $\mathsf{M}(X)\varepsilon$ is a subalgebra of $K(G)$isomorphic to $\mathsf{M}(X)$, and that algebra includes $C(\bar{X})\varepsilon$, which is isomorphic to $C(\bar{X})$.

If we write $C_c(G, \bar{X})$ for the sum of $C(\bar{X})\varepsilon$ and $C_c(G)\lambda$ as subspaces of $K(G)$, it can be seen that $C_c(G, \bar{X})$ is a subalgebra. Also the involution on$C_c(G)$ extends in a natural way to $C_c(G, \bar{X})$. We need the algebra $C_c(G, \bar{X})$because it generates a C^*-algebra that contains $C(\bar{X})$ as a subalgebra ,enabling us to apply a result on completely bounded bimodule mappings.

On the other hand, the algebra $C_c(G)$ has an approximate unit . In order to state the existence theorem, we need to introduce some of their terminology. They call a set L in Gr-relatively compact if KL is relatively compact for every compact set $K \subseteq X$. There exists a decreasing sequence $U_1, U_2, \ldots$ of open r-relatively compact sets whose intersection is X. There also exists an increasing sequence of compact sets in $X, K_1, K_2, \ldots$ whose interiors exhaust X. These come from the second countability of G,and they allow us to make a sequence that is an approximate unit (instead of a more general net). We call a function f in $C_c(G)$symmetric if $f^b = f$.

Theorem (2.1.3)[2]: There is a sequence $e_1, e_2, \ldots$ of symmetric functionsin $C_c^+(G)$ such that for each n we have

(i) $\operatorname{supp}(e_n) \subseteq U_n$, and
(ii) $\int e(\gamma) d\lambda^x(\gamma) \geq 1 - n^{-1}forx \in K_n$ and ≤ 1 for all $x \in X$.

Such a sequence is a two-sided approximate unit for $C_c(G)$ in its inductive limit topology, i.e., for uniform convergence on compact sets.

A (unitary) representation of a locally compact groupoid G is given by a Hilbert G-bundle K over X, the unit space of G; this means we have two functions that have some properties:

(a) a Hilbert space $\mathsf{K}(x)$ for each x. We form $\Gamma_{\mathsf{K}} = \{(x, v): x \in X, v \in \mathsf{K}(x)\}$, called the graph of K, and require that Γ_{K} have a standard Borel structure such that the projection onto X is Borel and there is a countable set of Borel sections of Γ_{K} such that for each x the set of their values at x is dense in $\mathsf{K}(x)$.

(b) a Borel homomorphism π of G into the unitary groupoid of the bundle K, i.e., for each $\gamma, \pi(\gamma): \mathsf{K}\left(s(\gamma)\right) \rightarrow \mathsf{K}\left(r(\gamma)\right)$ is unitary, and π is a Borel function .

This can also be said as follows: (K, π) is a Borel functor on G taking values in the category of Hilbert spaces.

Given a representation $\pi$of G, and a measure $\mu \in Q$, we can obtain from them a $*$-representation of $\mathsf{M}_c(G)$. Before describing the representation,we need another item of notation. We will write $\nu = \lambda^{\mu} = \int \lambda^x d\mu(x)$. Then we take $\Delta = \Delta_{\mu}$ as above and define $\nu_0 = \Delta^{-1/2}\nu$, obtaining a symmetric measure. Next we make a Hilbert space, $L^2(\mu;\mathsf{K})$, of square integrable sections of K. For $f \in \mathsf{M}_c(G)$ we define $\pi^{\mu}(f)$ on $L^2(\mu;\mathsf{K})$ by setting

$$(\pi^{\mu}(f)\xi|\eta) = \int f(\gamma)\big(\pi(\gamma)\xi \circ s(\gamma)|\eta \circ r(\gamma)\big)d\nu_0(\gamma)$$

for$\xi, \eta \in L^2(\mu;\mathsf{K})$. Then π^{μ} is a $*$-representation of$\mathsf{M}_c(G)$ with $\|\pi^{\mu}(f)\| \leq \|f\|_{I,\mu}$, so its restriction to $C_c(G)$ has the same property.We denote the restriction by the same symbol, depending on context to distinguish the two. Later we will also use another method of integratinga unitary representation of G, one that is due to Hahn and does not use the symmetrized measure.

It can be convenient to choose μ to be finite, say a probability measure,so we need to know that $\mu' \sim \mu$ implies $\pi^{\mu'}$ is unitarily equivalent to π^{μ}. To prove this implication, take ρ to be a positive Borel function whose squareis the Radon-Nikodym derivative of μ' with respect to μ. Then

$$\rho^2 \circ r = \frac{d\lambda^{\mu'}}{d\lambda^{\mu}}$$

and

$$\rho^2 \circ s = \frac{d\left(\lambda^{\mu'}\right)^{-1}}{d(\lambda^{\mu})^{-1}}$$

so

$$(\rho^2 \circ r)\Delta_{\mu} = (\rho^2 \circ s)\Delta_{\mu'}.$$

Hence we can define $V: L^2(\mu', \mathsf{K}) \rightarrow L^2(\mu, \mathsf{K})$ by $V\xi = \rho\xi$ to get the necessary unitary equivalence. To see that it is indeed an intert wining operator, compute to see that the inner products are equal: $(\pi^{\mu}(f)V\xi|V\eta) = \left(\pi^{\mu'}(f)\xi|\eta\right)$.

It is natural to ask whether every continuous representation of $C_c(G)$can be obtained by integrating a unitary representation of G, as is true for groups. An affirmative answer to this question was provided by an ingenious argument due to Renault , and it follows that every representation of $\mathsf{M}_c(G)$ bounded by $\|\ \ \|_I$ can be obtained by integrating a unitary representation of G.

Theorem (2.1.4)[2]: Let G be a locally compact groupoid that has a Haar system, and let H_0 be a dense subspace of a (separable) Hilbert space H .Suppose that L is a representation of $C_c(G)$ by operators on H_0 such that

(a) L is non-degenerate;
(b) L is continuous in the sense that for every pair of vectors $\xi, \eta \in \mathsf{H}_0$,the linear functional $L_{\xi,\eta}$ defined by $L_{\xi,\eta}(f) = (L(f)\xi|\eta)$ is continuous relative to the inductive limit topology on $C_c(G)$;
(c) L preserves the involution, i.e., $(\xi|L(f^b)\eta) = (L(f)\xi|\eta)$ for $\xi, \eta \in \mathsf{H}_0$ and $f \in C_c(G)$.

Then the operators $L(f)$ are bounded. The representation of $C_c(G)$ on H obtained from L is equivalent to one obtained by integrating a unitary representation of G using a probability measure $\mu \in Q$. In particular, L is continuous relative to $\|\ \ \|_I$.

Renault defined a norm on $C_c(G)$ by $\|f\| = \sup\{\|L(f)\|: L$ is a bounded representation of $C_c(G)\}$. Theorem (2.1.5) shows that we could get the same norm by using the representations π^{μ} in place of the L's. The completion of $C_c(G)$ with respect to the norm just defined is a C^*-algebra denoted $C^*(G)$. Every positive linear functional of norm one on a C^*-algebra gives rise to a representation of the algebra and a cyclic vectorin the Hilbert space of the representation. The direct sum of all these cyclic representations is called the universal representation of the C^*-algebra. We will denote this representation by ω. For $C^*(G)$, we know that every one of the cyclic representations is of the form π^{μ}, so ω can also be regarded as a representation of $\mathsf{M}_c(G)$. We will write $M^*(G)$ for the operator norm closure of $\omega\mathsf{M}_c(G)$. Since ω is an isomorphism on $C^*(G)$, we can regard $C^*(G)$ as a subalgebra of $M^*(G)$. We will also refer to ω as the universal representation of G itself.

In proving that L can be obtained by integration, Renault shows that there is a representation of $C_c(X)$, say ,, associated with L such that for $f \in C_c(G)$ and $h \in C_c(X)$ we have

$$L\big((h \circ r)f\big) = \phi(h)L(f)$$

and

$$L\big(f(h \circ s)\big) = L(f)\phi(h).$$

Then, extends in the obvious way to a unital representation of $C(\bar{X})$ and can be used to extend L to a representation of $C_c(G, \bar{X})$:

$$L(f\lambda + \mathrm{g}\varepsilon) = L(f) + \phi(\mathrm{g}).$$

The reader can verify, easily, that this defines a unital representation of $C_c(G, \bar{X})$. We extend $\omega$to $C(G, \bar{X})$ in this way, and also to $\mathrm{M}_c(G, \bar{X})$. Then we define $C^*(G, \bar{X})$ to be the operator norm closure of $\omega\big(C_c(G, \bar{X})\big)$ and $M^*(G, X)$ to be the closure of $\omega(\mathrm{M}_c(G, X))$.

For some computations we need another norm .Let $\mu \in \mathcal{Q}$, let $f \in \mathrm{M}_c(G)$ and define

$$\|f\|_{II,\mu} = \sup\left\{\int |f(\gamma)\mathrm{g} \circ r(\gamma)h \circ s(\gamma)|\Delta_\mu(\gamma)^{-1/2} d\lambda^\mu(\gamma)\right\},$$

The supremum being taken over unit vectors $\mathrm{g}, h \in L^2(\mu)$. Then define $\|f\|_{II}$to be $\sup\{\|f\|_{II,\mu}: \mu \in \mathcal{Q}\}$. Three facts about this norm should be mentioned. The first is that if π is a unitary representation of G, then $\|\pi^\mu(f)\| \le \|f\|_{II,\mu}$. Thus $\|\omega(f)\| \le \|f\|_{II}$, because $\|\omega(f)\| = \sup\{\pi^\mu(f): \pi$ is a unitary representation and $\mu \in \mathcal{Q}\}$. Next, if π is the one dimensional trivial representation and $f \ge 0$then $\|\pi^\mu(f)\| = \|f\|_{II,\mu}$. It follows that if $0 \le f \in \mathrm{M}_c(G)$then

$$\|\omega(f)\| \le \|f\|_{II}.$$

A third fact is this: if $b \in L^\infty(\lambda^{\mathcal{Q}})$ and $f \in \mathrm{M}_c(G)$, then for any $\mu \in \mathcal{Q}$ we have

$$\|bf\|_{II,\mu} \le \|b\|_\infty \|f\|_{II,\mu},$$

so

$$\|bf\|_{II} \le \|b\|_\infty \|f\|_{II}.$$

Lemma (2.1.5)[2]: If $0 \le f \in M_c(G)$ and $b \in L^\infty(\lambda^{\mathcal{Q}})$, then $\|\omega(bf)\| \le \|b\|_\infty \|\omega(f)\|$.

Proof: Using the three properties of $\| \quad \|_{II,\mu}$ mentioned just above, we have

$$\begin{aligned} \|\omega(bf)\| &\le \sup\{\|bf\|_{II,\mu} : \mu \in \mathcal{Q}\} \\ &\le \sup\{\|b\|_\infty \|f\|_{II,\mu} : \mu \in \mathcal{Q}\} \\ &= \|b\|_\infty \|\omega(f)\|. \end{aligned}$$

Section (2.2): Measure Theoretic Preparations

We show A basic lemma that is needed for the construction of positive definite functions from completely positive maps. After proving that lemma, we also need to prepare some detailed information about Haar systems on locally compact groupoids and how they relate to Borel Haar systems on the associated equivalence relations.

for the proof of the lemma , we recall a basic fact about measures and function spaces. Suppose that (X, B) is a set with a σ-algebra and that A is a subalgebra of B that generates B as a σ-algebra. Let μ be any finite measure defined on B. The measure of the symmetric difference between two sets is the same as the distance between their characteristic functions in $L^1(\mu)$, and hence provides a (pseudo)metric on B. The closure of A in B is a σ-algebra that contains A and hence is B. For us, it is important that the fact of density is independent of μ. This implies similar properties for the set $S(\mathsf{A})$, our notation for the set of linear combinations of characteristic functions of sets in A using coefficients from $\mathbb{Q}[i]$, which is $\mathbb{Q}$ with $\sqrt{-1}$ adjoined. By looking first at simple functions, it is easy to show that $S(\mathsf{A})$ is always dense in $L^1(\mu)$. In the same way, we see that for any $f \in L^1(\mu)$,

$$\|f\|_1 = \sup\left\{\left|\int f\varphi d\mu\right| : \varphi \in S(\mathcal{A}) \text{and} |\varphi| \leq 1\right\},$$

which is a supremum indexed by a family independent of μ. When A can be taken to be countable, as is the case when X is a standard Borel space, these facts are particularly useful.

A similar situation arises if X is locally compact. In that case, there is acountable dense subset $\mathsf{S}(X)$ of $C_c(X)$ that is an algebra over $\mathbb{Q}[i]$, and any such $\mathsf{S}(X)$ is dense in $L^1(\mu)$ for every finite measure μ on X.

The next lemma is a generalization of the fact that for two measure spaces, functions on the product and functions from one measure space to the functions on the other can be identified. The measure on the image space must be allowed to vary.

Lemma (2.2.1)[2]: Let X and Y be standard Borel spaces and let $x \mapsto \nu^x$ be a Borel function from X to finite Borel measures on Y. Suppose that f is a function on X selecting an element $f(x)$ of $L^1(\nu^x)$ for each $x \in X$ so that the function $x \mapsto f(x)\nu^x$ is Borel, taking values in the space of complex valued Borel measures. Then there is a Borel function F on $X \times Y$ such that for every $x \in X$the function $F(x,\cdot)$ is integrable relative to ν^x and in the class $f(x)$. The function F can be chosen so that if $f(x) \in L^\infty(\nu^x)$ then $F(x,\cdot)$ is bounded by$\|f(x)\|_\infty$. It is possible to choose F meeting those conditions and so that if $\nu^x = \nu^{x'}$ and $f(x) = f(x')$ then $F(x,\cdot) = F(x',\cdot)$ (everywhere on Y).

Proof: For the proof we must have a way, that does not depend on x directly, to choose representatives of classes approximating $f(x)$. For this we choose first a countable algebra, A, of Borel sets in Y that generates the σ-algebra of Borel sets, so we can use the facts mentioned before the statement of the lemma. List $S(\mathsf{A})$ as a sequence, $s_1, s_2, \ldots$. For convenience, let us write $x \sim x'$ to mean that $\nu^x = \nu^{x'}$and $f(x) = f(x')$, and say that such points are equivalent.

Now we are ready to describe the basic step which will be used repeatedly in the proof. If $\varepsilon > 0$ and $x \in X$ define $j(x,\varepsilon)$ to be the least element of $\{i{:}\, \|f(x) - s_i\|_{L^1(\nu^x)} < \varepsilon\}$. It is clear that $j(\cdot,\varepsilon)$ takes the same value at equivalent points of X, and we will show that $j(\cdot,\varepsilon)$ is a Borel function.This will follow if we can show that for each bounded Borel function h on Y, $\{x{:}\, \|f(x) - h\|_{L^1(\nu^x)} < \varepsilon\}$is a Borel set. We can get that from the fact that norms can be computed as suprema, because for each $\varphi \in S(\mathsf{A})$, $\int (f(x) - h)\varphi d\nu^x$, is a Borel function of x and hence so is its absolute value.

If we define $\mathsf{g}(x) = s_{j(x,\varepsilon)}$ (as an element of $L^1(\nu^x)$) and $G(x,y) = s_{j(x,\varepsilon)}(y)$, then $\mathsf{g}(x) = \mathsf{g}(x')$ and $G(x,\cdot) = G(x',\cdot)$ (everywhere on Y) whenever $x \sim x'$. Also, both these functions are Borel.

Apply this process first to f with $\varepsilon = 2^{-1}$ to obtain $G_1$and g_1. Then apply it to $f - \mathsf{g}_1$ with $\varepsilon = 2^{-2}$ to obtain $G_2$and g_2, etc. For each n the value of the function $f - (\mathsf{g}_1 + \cdots + \mathsf{g}_n)$ at a point x is an element of $L^1(\nu^x)$ having norm$< 2^{-n}$. Thus for $n \geq 2$, $\|\mathsf{g}_n(x)\|_1 < 3(2^{-n})$. It follows that for each x the sum$\sum_{n\geq 1}|G_n(x,y)|$ is finite for almost all y. Inductively, we see that $G_n(x,\cdot) = G_n(x',\cdot)$if $x \sim x'$. The set $N = \{(x,y) \in X \times Y{:} \sum_{n\geq 1}|G_n(x,y)| = \infty\}$ is a Borel set in $X \times Y$ and the slices of N over x and x' are the same set if $x \sim x'$.Now change each G_n to be 0 on N. Then the sum is always finite and we still have $G_n(x,\cdot) = G_n(x',\cdot)$ if $x \sim x'$.

Define $F(x,y) = \sum_{n\geq 1} G_n(x,y)$. Then F is Borel and satisfies the first and last conclusions of the theorem. Thus the slice of the Borel set$\{(x,y){:}\, |F(x,y)| > \|f(x)\|_\infty\}$over every point of X is of measure 0 and the slices of this set are the same over equivalent points of X. Change F to be 0 on that set, and all the desired conditions are satisfied.

Now we are going to present some results on the fine structure of the Haar system, as developed by Renault. Renault decomposes the Haar system λ over a Borel Haar system α on R, by studying the action of G on a special group bundle, and we summarize the results here. Recall that the isotropy group bundle of G, denoted by G', is defined to be $\{\gamma \in G: r(\gamma) = s(\gamma) = \bigcup xGx: x \in X\}$. This closed in G and hence locally compact, so the space of closed subsets of G' is a compact space in the Fell topology. Let $\Sigma^{(0)}$ be the space of closed subgroups of the fibers in G', which is a closed subset of the space of closed subsets.Then the set $\Sigma = \{(H,\gamma) \in \Sigma^{(0)} \times G' | \gamma \in H\}$ is called the canonical groupbundle of $\Sigma^{(0)}$. G acts on Σ and on $\Sigma^{(0)}$ by conjugation: if $(H_1,\gamma_1) \in \Sigma, \gamma \in G$,and$s(\gamma_1) = r(\gamma)$, then

$$(H_1,\gamma_1)\gamma = (\gamma^{-1}H_1\gamma, \gamma^{-1}\gamma_1\gamma),$$

while if $H \in \Sigma^{(0)}$, say $H \subseteq xGx$, and $r(\gamma) = x$, then $H \cdot \gamma = \gamma^{-1}H\gamma$. We want to make a Borel choice of Haar measures on the groups xGx. One way to do this is to choose a continuous function F_0 on G that is non-negative, 1 at each $x \in X$ and has compact support on each xG. Then for each $x \in X$ choose a left Haar measure β^x on xGx so the integral of F_0 with respect to β^x is 1. Likewise, choose a function F on Σ that is non-negative, 1 at each point (H,e), and has support that intersects every $\{H\} \times H$ in a compactset, and make a similar choice of Haar system on Σ, β^H.

Form the groupoid $\Sigma^{(0)} * G = \{(H,\gamma): s(H) = r(\gamma)\}$arising from the action of G on $\Sigma^{(0)}$. Then the essential uniqueness of Haar measures guarantees the existence of a 1-cocycle, δ, on $\Sigma^{(0)} * G$ so that for every$(H,\gamma) \in \Sigma^{(0)} * G$ we have

$$\gamma^{-1}\beta^H\gamma = \delta(H,\gamma)^{-1}\beta^{\gamma^{-1}H\gamma}.$$

Renault proves that δ is continuous. The cohomology class of δ is determined by G, and Renault calls it the isotropy modulus function of G.

To shorten some formulas in this context, we write $G(x)$for xGx.Renault defines $\delta(\gamma) = \delta\big(G\big(r(\gamma)\big),\gamma\big)$ to get a 1-cocycle, also called δ, on G such that for every $x \in X, \delta|xGx$ is the modular function for β^x. The pre-image in $\Sigma^{(0)} * G$ of a compact set in G is compact, so δ and $\delta^{-1} = 1/\delta$ are bounded on compact sets in G. Renault defines $\beta_y^x = \gamma\beta^\gamma$if $\gamma \in xGy$. If γ' is another element of xGy, then $\gamma^{-1}\gamma' \in yGy$, and since β^y is a left Haar measure on yGy, it follows that β_y^x is independent of the choice of γ. With this apparatus in place, it is possible to describe a decomposition of the Haar system λ for G over the equivalence relation $R = \{\big(r(\gamma), s(\gamma)\big): \gamma \in G\}$.This R is the image of G

under the homomorphism $\theta(= (r,s))$, so it is a σ-compact groupoid. Furthermore, there is a unique Borel Haar system $\alpha$for R with the property that forevery $x \in X$ we have

$$\lambda^x = \int \beta_y^z d\alpha^x(z,y).$$

Now suppose that $\mu \in Q$ so that we can form $\alpha^\mu$and λ^μ, getting quasisymmetric measures. If $\underline{\Delta}= d\alpha^\mu / d(\alpha^\mu)^{-1}$then $\delta\underline{\Delta} \circ \theta$will serve as $d\lambda^\mu / d(\lambda^\mu)^{-1}$,i.e., we can always take $\Delta_\mu = \delta\underline{\Delta}_\mu \circ \theta$. We shall see that sometimes $\underline{\Delta}_\mu = 1so\Delta_\mu = \delta$.

For each x, the measure α^x is concentrated on $\{x\} \times \{x\}$ so there is a measure μ^x on $[x]$ such that $\alpha^\mu = \varepsilon^x \times \mu^x$, where ε^x is the unit point mass at $x \in X \subseteq G$. Since α is a Haar system, we have $\mu^x = \mu^y$if $x \sim y$, and the function $x \mapsto \mu^x$ is Borel. If we take μ' to be the measure μ^z for some $z \in X$,thenμ' is quasiinvarian. We give a different proof.First of all,

$$\alpha^{\mu'} = \int \alpha^x d\mu'(x) = \mu^z \times \mu^z,$$

so $\alpha^{\mu'}$ is symmetric. Hence$\underline{\Delta}_{\mu'} = 1$. Next we consider $\lambda^{\mu'} = \iint \beta_y^x d\mu^z(x) d\mu^z(y)$.Since Gz is locally compact, there is a Borel function $c: [z] \to Gz$ such that for every $x \in [z]$ we have $c(x) \in xGz$. The value of $c(z)$ can be taken to be z. We can use c to define a Borel isomorphism $\psi: G|[z] \to [z] \times G(z) \times [z]$ by

$$\psi(\gamma) = \left(r(\gamma), c(\gamma)\right)^{-1} \gamma c\left(s(\gamma), s(\gamma)\right).$$

By the uniqueness of Haar measure, as above, we see that ψ always carries $\beta_y^x$to a positive multiple of $\varepsilon^x \times \beta^z \times \varepsilon^y$, and that multiple is a Borel function of the pair (x,y). Hence ψ carries $\lambda^{\mu'}$ to a measure equivalent to $\mu^z \times \beta^z \times \mu^y$. It follows that $\lambda^{\mu'}$ is quasisymmetric, as needed.

Since λ is a Haar system, we know that if K is a compact set in G then the function $x \longmapsto \lambda^x(K)$ is bounded. We will use the formula for λ^x in terms of α^x to prove that $x \longmapsto \alpha^x\left(\theta(K)\right)$ is also bounded, and also that μ^x is finite on compact sets for the quotient topology on $[x]$. Let F be the function used above to make a choice of Haar measures β^y. If S is the support of F, then $\beta^y(S) \geq 1$ for every $y \in X$. To prove the boundedness statement above, let K be a compact subset of G and set $K_1 = K(s(K)S)$. Because both factors are compact, so is K_1, so $x \mapsto \lambda^x(K_1)$ is bounded. For $(x,y) \in \theta(K)$, choose $\gamma \in K$ such that $\theta(\gamma) = (x,y)$. Then $\gamma S \subseteq K_1$, so $\beta_y^x(K_1) \geq 1$.Hence

$$\lambda^x(K_1) = \int \beta_y^x(K_1)d\alpha^x(x,y)$$

$$\geq \int_{s(xK)} \beta_y^x(K_1)d\alpha^x(x,y)$$

$$\geq \alpha^x\big(\theta(K)\big).$$

For the second assertion, suppose that C is a compact set in $[x]$ for the quotient topology. Since xG is locally compact and s is continuous and open from xG to $[x]$, there is a compact set K contained in xG whose image contains C. Then $\theta(K) \subseteq xR$, so the boundedness result just proves that $\mu^x\big(s(K)\big) = \alpha^x\big(\theta(K)\big)$ is finite. Hence μ^x is σ-finite.

Define $\mathsf{M}_{0c}(R)$ to be the space of bounded Borel functions on R that vanish off sets of the form $\theta(K)$, where K is a compact subset of G. Nowwe know that $\mathsf{M}_{0c}(R) \subseteq I(R,\lambda)$, and it is not difficult to show that $\mathsf{M}_{0c}(R)$is a $*$-subalgebra of $I(G,\lambda)$. The definition of this algebra is admittedly somewhat unusual, but the algebra will serve a useful purpose in provingthe main step along one way to prove the completeness of the Fourier-Stieltjes algebra of G. The point is that R is a kind of shadow of G, and we need a convolution algebra on it that is a shadow of the same kind.

In this section, we will characterize the functions on a locally compact groupoid that are diagonal matrix entries of unitary representations as the functions that are what we call positive definite. For this to be meaningful,we need a good definition of "positive definite." This is more complicated than for locally compact groups because unitary representations of locally compact groupoids can be Borel functions without being continuous. Thus we make our definition using integrals, and must even identify two functions that agree $\lambda^{\mathcal{Q}}$-almost everywhere, as defined,we will need to construct a positive definite function from a parametrized family of functions, each of which is positive definite on a transitive sub-roupoid. Thus we prove the representation theorem. For a locally compact groupoid that has a Haar system, the notion of positive definite function can be defined in the least restrictive way as follows:

Definition (2.2.2)[2]: Let G be a locally compact groupoid and let λ be a left Haar system on G. Then a bounded Borel function p on G is called positive definite iff for each $x \in X$ and each f in $C_c(G)$ we have

$$\iint f(\gamma_1)\bar{f}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \geq 0. \qquad (1)$$

The set of all such p's will be denoted by $\mathsf{P}(G)$. We say that two elementsof $\mathsf{P}(G)$ are equivalent iff they agree $\lambda^{\mathcal{Q}}$-almost everywhere.

Since we intend to show that positive definite functions are essentially the same as diagonal matrix entries of unitary representations, we begin by showing that such matrix entries are in $\mathsf{P}\,(G)$.

Lemma (2.2.3)[2]: Let π be a unitary representation of G on a Hilbert bundle$\mathsf{H}$,and let ξ be a bounded Borel section of H. Define $p(\gamma) = \big(\pi(\gamma)\xi \circ s(\gamma)|\xi \circ r(\gamma)\big)$for $\gamma \in G$. Then $p \in\mathsf{P}(G)$.

Proof: Fix $x \in X$and $f \in C_c(G)$. Then for $\eta \in\mathsf{H}(x)$,

$$\left|\int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma)|\eta)d\lambda^x(\gamma)\right| \leq \|f\|_1\|\xi\|_\infty\|\eta\|,$$

so there is an element $\xi(x) \in\mathsf{H}(x)$ such that for all $\eta \in\mathsf{H}(x)$ we have

$$\int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma)|\eta)d\lambda^x(\gamma) = (\xi(x)|\eta).$$

Indeed, this defines a Borel section, ξ, of H. The Borel character of ξfollows from the fact that $\big(\pi(\gamma)\xi_1 \circ s(\gamma)|\eta_1 \circ r(\gamma)\big)$ is a Borel function of γ whenever ξ_1 and η_1 are Borel sections of H. For this section ` the integral involved in the condition (1) is equal to $\big(\xi(x)|\xi(x)\big)$, which is certainly non-negative.

Lemma (2.2.4)[2]: If $p \in\mathsf{P}(G)$, the formula

$$(f|\mathrm{g})_x = \iint f(\gamma_1)\bar{\mathrm{g}}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^x(\gamma_1)d\lambda^x(\gamma_2) \qquad (2)$$

defines a semi-inner product on $L^1(\lambda^x)$. Let $\mathsf{H}(x)$ denote the Hilbert space completion of the resulting inner product space. Then H is a Hilbert bundle over X. For $\gamma_1 \in G$, define $\pi(\gamma_1)$ from $L^1\big(\lambda^{s(\gamma_1)}\big)$ to $L^1\big(\lambda^{r(\gamma_1)}\big)$ by $(\pi(\gamma_1)f)(\gamma) = f(\gamma_1^{-1}\gamma)$. Then π determines a unitary representation, also denoted by π, on the bundle H.

Proof: The form $(f|\mathrm{g})_x$ is clearly linear in f and conjugate linear in g.Since the vector space is complex the Hermitian symmetry follows from positive definiteness.

Let $\mathcal{N}(x) = \{f \in L^1(\lambda^x): (f|f)_x = 0\}$ and set $\mathsf{F}(x) = L^1(\lambda^x)/\mathcal{N}(x)$, the corresponding inner-product space. Write $\mathsf{H}(x)$ for the completion of $\mathsf{F}(x)$. Let $|\ \ |_x$ be the norm (or semi-norm) arising from $(\ |\ )_x$. For $f, g \in L^1(\lambda^x)$, $|(f|\mathrm{g})_x| \le \|p\|_\infty \|f\|_1 \|\mathrm{g}\|_1$, so $|f|_x \le \|p\|_\infty^{1/2} \|f\|_1$. It follows that the image of $C_c(xG)$, which is the image of $C_c(G)$, is dense in $\mathsf{H}(x)$.

Now we want to make a Borel structure on the graph of H, denoted by $\Gamma = \Gamma_{\mathsf{H}} = \{(x, \xi): x \in X, \xi \in \mathsf{H}(x)\}$. The process used is fairly standard. First, if $f \in C_c(G)$ and $x \in X$, define $\sigma(f)(x)$ to be the element of $\mathsf{H}(x)$ represented by $f|xG$. This defines a section $\sigma(f)$ of the graph of H. We want all $\sigma(f)$'s to be Borel sections, and that tells us how to define the Borel structure. For $f \in C_c(G)$ define ψ_f on Γ by $\psi_f(x, \xi) = (\sigma(f)(x)|\xi)_x$.

Then give Γ the smallest Borel structure relative to which the projection to X is Borel along with all the functions $\psi_f\big(f \in C_c(G)\big)$. It follows from the fact that p is Borel and bounded that each section $\sigma(g)$ for $\mathrm{g} \in C_c(G)$ is indeed a Borel section. Since G is second countable, there is a countable set dense in $C_c(G)$. For any countable dense set of f's, the ψ_f's would determine the same Borel structure as $\{\psi_f: f \in C_c(G)\}$, so the latter is standard: Apply the Gram-Schmidt process in a pointwise manner to a dense sequence of sections of the form $\sigma(f)$ to get a sequence $\mathrm{g}_1, \mathrm{g}_2, \ldots$ of Borel functions such that

(a) $\mathrm{g}_n|xG$ is always in $L^1(\lambda^x)$.
(b) if $f \in C_c(G)$ and $n \ge 1$, then $x \to \big(\sigma(f)|\sigma(\mathrm{g}_n)\big)_x$ is a Borel function.
(c) for each x the non-zero elements of $\{\sigma(\mathrm{g}_n)(x): n \ge 1\}$ form an orthonormal basis of $\mathsf{H}(x)$.

Then it is easy to show that Γ is isomorphic to the disjoint union of a sequence of product bundles $X_n \times \mathsf{K}_n$, where $\{X_\infty, X_1, X_2, \ldots\}$ is a Borel partition of X and each K_n is a Hilbert space of dimension n. Thus Γ is standard because each $X_n \times \mathsf{K}_n$ is standard.

If $f \in L^1(\lambda^x)$ and $\gamma_1: x \to y$ is in G, define $\pi(\gamma_1)f$ by $\big(\pi(\gamma_1)(f)(\gamma)\big) = f(\gamma_1^{-1}\gamma)$ for $\gamma \in yG$. Since λ is left invariant, $\pi(\gamma_1)f \in L^1(\lambda^y)$. Notice that $\pi(\gamma_1^{-1})$ is the inverse of $\pi(\gamma_1)$. If g is another element of $L^1(\lambda^x)$, then

$$(\pi(\gamma_1)f|\pi(\gamma_1)\mathrm{g})_y = \iint f(\gamma_1^{-1}\gamma_2)\bar{\mathrm{g}}(\gamma_1^{-1}\gamma_3)p(\gamma_3^{-1}\gamma_2)\,d\lambda^y(\gamma_2)\,d\lambda^y(\gamma_3)$$

$$= \iint f(\gamma_2)\bar{g}(\gamma_3)p(\gamma_3^{-1}\gamma_2)d\lambda^x(\gamma_2)d\lambda^x(\gamma_3)$$

$$= (f|g)_x.$$

Hence $\pi(\gamma_1)$ extends to a unitary operator from $\mathrm{H}(x)$ to $\mathrm{H}(y)$, for which we use the same notation.

To work with the bundle and with the representation we need to restrict to subsets of product spaces where the various operations are defined. There are two fibered products, $\Gamma \times' \Gamma = \{(x,\xi,x,\xi'): x \in X, \xi,\xi' \in \mathrm{H}(x)\}$, a subset of $\Gamma \times \Gamma$, and $G \times' \Gamma \subseteq G \times \Gamma$, defined to be $\Gamma \times' \Gamma = \{(\gamma,x,\xi): s(\gamma) = x, \xi \in \mathrm{H}(x)\}$. Let us show that $(\gamma,x,\xi) \longmapsto (r(\gamma),\pi(\gamma)\xi)$ is Borel from $G \times' \Gamma$ to Γ. The composition of this map with the projection to X is clearly Borel. Let $f \in C_c(G)$ and compose the map with ψ_f. The value of the composition at (γ,x,ξ) is $\psi_f(r(\gamma),\pi(\gamma)\xi) = \left(\pi(\gamma)^{-1}\left(\sigma(f)(r(\gamma))\right)|\xi\right)_x$. This is the value of another composition,

$$G \times' \Gamma \to \Gamma \times' \Gamma \to \mathbb{C},$$

where the first function takes (γ,x,ξ) to $\left(s(\gamma),\pi(\gamma^{-1})\left(\sigma(f)(r(\gamma))\right);x,\xi\right)$ and the second is the inner product function. The first function is Borel if each component is, so let us see that the first component is a Borel function of γ. Composition of it with projection is s and hence Borel. If $g \in C_c(G)$,

$$\psi_g\left(s(\gamma),\pi(\gamma^{-1})\sigma(f)(r(\gamma))\right) = \iint \bar{f}(\gamma\gamma_2)g(\gamma_1)p(\gamma_2^{-1}\gamma_1)d\lambda^{s(\gamma)}(\gamma_1)d\lambda^{s(\gamma)}(\gamma_2)$$

$$= \int_{G\times G\times G} F\, d\left(\varepsilon^\gamma \times \lambda^{s(\gamma)} \times \lambda^{s(\gamma)}\right),$$

where F is the function that is 0 at $(\gamma_0,\gamma_1,\gamma_2)$ unless $s(\gamma_0) = r(\gamma_1) = r(\gamma_2)$ and then its value is $\bar{f}(\gamma_0\gamma_1)g(\gamma_2)p(\gamma_2^{-1}\gamma_1)$ (F is Borel). A fairly standard argument then shows that $\gamma \mapsto \psi_g(s(\gamma)),\pi(\gamma^{-1})\left(\sigma(f)(r(\gamma))\right)$ is Borel, as desired.

To show that the inner product is Borel on $\Gamma \times' \Gamma$, we use the functions g_n used to show that the bundle is standard. Indeed,

$$((x,\xi)|(x,\eta))_x = \sum_{n\geq 1} \psi_{g_n}(x,\xi)\bar{\psi}_{g_n}(x,\eta)$$

which is a Borel function. It follows that $(\gamma, x, \xi) \mapsto \psi_f(r(\gamma), \pi(\gamma)\xi)$ is Borel, as needed.

This completes the construction of a unitary representation from a positive definite function. From now on, subscripts will be used on inner products and norms associated with such bundles only when necessary tomake clear which space is involved. Our next task is to find a (cyclic) section ξ_p such that $p(\gamma) = \left(\pi(\gamma)\xi_p\big(s(\gamma)\big)|\xi_p\big(r(\gamma)\big)\right)$ for $\lambda^{\mathcal{Q}}$-almost every $\gamma \in G$.

The argument can be outlined as follows. Given $\mu \in \mathcal{Q}$ we let $\mathsf{H}\ (\mu)$denote the Hilbert space of square integrable sections of H, which is sometimes written $L^2(\mu,\mathsf{H})$. There is no loss of generality in assuming that μ is a probability measure, since changing to an equivalent measure produces an equivalent representation. The representation π of G can be integrated to give a representation of $C_c(G)$ on $\mathsf{H}\ (\mu)$, denoted by π_μ , using the formulation of Hahn, rather than that of Renault. The definition is given below. If $u_1, u_2, \ldots$ is a symmetric approximate unit for $C_c(G)$, the sequence of sections $\sigma(u_1), \sigma(u_2), \ldots$ has a subsequence that converges weakly to a section ξ_μ such that for $f \in C_c(G)$ we have $\pi_\mu(f)\xi_\mu = \sigma(f)$, and the matrix entry made from π and ξ_μ agrees with p a.e. relative to λ^μ. To get a sectionξ_p not depending on μ, we observe that if we had such a ξ_p , then for$f \in C_c(G)$ we would get $\int fp\, d\lambda^x = \big(\sigma(f)|\xi_p\big)_x$. Thus we consider the set $D(p)$ of those $x \in X$ for which $\int fp\, d\lambda^x$, as a linear function of f in $C_c(G)$, "extends" to a bounded linear functional on $\mathsf{H}\ (x)$. We need to know that $D(p)$ is conull for every $\mu \in \mathcal{Q}$, and this follows from the existence of ξ_μ. We let $\xi_p(x)$ be the vector representing that linear functional, and verify that ξ_p is the section we wanted.

Before giving details, we introduce the space $L^{1,2}(\lambda, \mu)$, consisting of those Borel functions f for which

$$\|f\|_{1,2}^2 = \int \left(\int |f(\gamma)|\, d\lambda^x(\gamma)\right)^2 d\mu(x) < \infty.$$

Now, begin by taking $\mathsf{H}\ (\mu)$ as defined above, and observe that for $f \in L^{1,2}(\lambda, \mu)$, the section $\sigma(f)$ is in $\mathsf{H}\ (\mu)$, and $\|\sigma(f)\| \leq \|p\|_\infty^{1/2}\|f\|_{1,2}$, so that $f_n \to f$ in $L^1(\lambda, \mu)$ implies $\sigma(f_n) \to \sigma(f)$ in $\mathsf{H}\ (\mu)$. To make the proofwork, we must integrate the representation π to get π_μ having the property that for $f, \mathsf{g} \in C_c(G), \pi_\mu(f)\big(\sigma(\mathsf{g})\big) = \sigma(f * \mathsf{g})$. This can be done if we use the method, which applies to $I(G, \lambda)$, which is a subspace of $L^1(\lambda, \mu)$because μ is a probability measure.

For $f \in I(G,\lambda)$ we define $\pi_\mu(f)$ by saying that for sections ξ,η in $\mathsf{H}(\mu)$ we have

$$(\pi_\mu(f)\xi|\eta) = \int f(\gamma)\left(\pi(\gamma)\xi(s(\gamma))|\eta(r(\gamma))\right)d\lambda^\mu(\gamma).$$

The integral defines a bounded sesquilinear form, so the formula defines a bounded operator $\pi_\mu(f)$. It is proved that π_μ is a bounded representation of $I(G,\lambda)$. If $f \in I(G,\lambda)$ and $\xi \in \mathsf{H}(\mu)$, then $\pi_\mu(f)\xi$ is represented by a section whose value at almost every x is $\int f(\gamma)\pi(\gamma)\xi(s(\gamma))d\lambda^\mu(\gamma)$, where the integral is defined weakly. If $\mathrm{g} \in I(G,\lambda)$, we have

$$\begin{aligned}(\pi_\mu(f)\sigma(\mathrm{g})\eta) &= \int f(\gamma)\left(\pi(\gamma)\left(\sigma(\mathrm{g})(s(\gamma))\right)|\eta(r(\gamma))\right)d\lambda^\mu(\gamma)\\ &= \iint f(\gamma)\left(\pi(\gamma)\left(\sigma(\mathrm{g})(s(\gamma))\right)|\eta(r(\gamma))\right)d\lambda^\mu(\gamma)\,d\mu(x)\\ &= \int \left(\sigma(f*\mathrm{g})\right)(r(\gamma))|\eta d\lambda^\mu(\gamma),\end{aligned}$$

because $\pi(\gamma)\left(\sigma(\mathrm{g})(s(\gamma))\right)$ is represented by a function on $r(\gamma)G$ whose value at a point γ_1 is $g(\gamma^{-1}\gamma_1)$.

Lemma (2.2.5)[2]: Let G be a locally compact groupoid with a Haar system λ.Suppose that $\mu \in Q$ is a probability measure. If p is a positive definite functionon G and (H,π) is constructed from p as in the proof of Lemma (2.2.4), then there is a section $\xi_\mu \in \mathsf{H}(\mu)$ such that

(a) $\left|\xi_\mu(x)\right|_x^2 \le \|p\|_\infty$ for $x \in X$
(b) $\pi_\mu(f)\xi_\mu = \sigma(f)$ for $f \in C_c(G)$
(c) $p(\gamma) = \left(\pi(\gamma)\xi_\mu(s(\gamma))|\xi_\mu(r(\gamma))\right)$ a. e. $d\lambda^\mu(\gamma)$

Proof: Let $u_1,u_2,\ldots$ be a symmetric approximate unit for G. Then $|\sigma(u_i)(x)| \le \|p\|_\infty^{1/2}$ for each x and i, so $\|\sigma(u_i)(x)\| \le \|p\|_\infty^{1/2}$ for each i. Thus $\sigma(u_1),\sigma(u_2),\ldots$ has a subsequence converging weakly to a vector $\xi_\mu \in \mathsf{H}(\mu)$.We may suppose that subsequence is $\sigma(u_1),\sigma(u_2),\ldots$. If, for every Borel set E in X, $P(E)$ is the projection of $\mathsf{H}(\mu)$ onto the subspace determined by sections that vanish off E, then $P(E)\sigma(u_n)$ converges weakly to $P(E)\xi_\mu$,which has norm at most $\left(\|p\|_\infty\mu(E)\right)^{1/2}$. It follows that

$|\xi_\mu(x)|_x^2 \le \|p\|_\infty$for a.e. x, and we can change ξ_μ to make it true for all x. For $f \in C_c(G) f * u_i \to f$ uniformly and all these functions vanish off a fixed compact set.Thus $f * u_i \to f$in $L^{1,2}$, and $\sigma(f * u_i) \to \sigma(f)$ in $\mathsf{H}\ (\mu)$. Hence $\pi_\mu(f)\sigma(u_i)$converges to $\sigma(f)$. We also know that $\pi_\mu(f)$ is a bounded operator, so $\pi_\mu(f)\sigma(u_i)$ converges weakly to $\pi_\mu(f)\xi_\mu$. Hence $\pi_\mu(f)\xi_\mu = \sigma(f)$, as elements of $\mathsf{H}\ (\mu)$.

It follows from this that if $f, \mathrm{g} \in C_c(G)$, then $(\sigma(f)|\sigma(\mathrm{g})) = (\pi_\mu(f)\xi_\mu|\pi_\mu(\mathrm{g})\xi_\mu)$ and this can be written as

$$\iiint \left(\pi(\gamma_1)\xi_\mu(s(\gamma_1))|\pi(\gamma_2)\xi_\mu(s(\gamma_2))\right) f(\gamma_1)\bar{\mathrm{g}}(\gamma_2)\, d\lambda^x(\gamma_1)\, d\lambda^x(\gamma_2)\, d\mu(x)$$

which is equal to

$$\iiint \left(\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1))|\xi_\mu(s(\gamma_2))\right) f(\gamma_1)\bar{\mathrm{g}}(\gamma_2)\, d\lambda^x(\gamma_1)\, d\lambda^x(\gamma_2)\, d\mu(x).$$

If $h \in C_c(G)$ we can replace f in this calculation by $hf = (h \circ r)f$. From this it follows that if $f, \mathrm{g} \in C_c(G)$ then for μ-almost every x we have

$$(\sigma(f)|\sigma(\mathrm{g}))_x = \iint \left(\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1))|\xi_\mu(s(\gamma_2))\right) f(\gamma_1)\bar{\mathrm{g}}(\gamma_2)\, d\lambda^x(\gamma_1)\, d\lambda^x(\gamma_2).$$

By the definition of $(\ |\ )_x$ this shows that for μ-almost every x,

$$p(\gamma_2^{-1}\gamma_1) = \left(\pi(\gamma_2^{-1}\gamma_1)\xi_\mu(s(\gamma_1))|\xi_\mu(s(\gamma_2))\right) \qquad (3)$$

is true for $\lambda^x \times \lambda^x$-almost all pairs (γ_1, γ_2). For each such x, for λ^x-almost every γ_2, the formula (3) is true for λ^x-almost every γ_1, i.e., $p(\gamma) = \left(\pi(\gamma)\xi_\mu(s(\gamma))|\xi_\mu(r(\gamma))\right)$ for $\lambda^{s(\gamma_2)}$-almost all γ. Indeed, the set $\{s(\gamma_2)$ (3)holds for λ^x-almost every $\gamma_1\}$ is conull in $[x]$.

Theorem (2.2.6)[2]: Let G be a locally compact groupoid and let λ be a Haar system on G. If p is a positive definite function on G and (H, π) is the associated unitary G-bundle over X, then there is a bounded section ξ_p of H such that if $\mu \in \mathcal{Q}$, then

(a) $p(\gamma) = \left(\pi(\gamma)\xi_p(s(\gamma))|\xi_p(r(\gamma))\right)$ a.e. $d\lambda^\mu(\gamma)$
(b) if $f \in C_c(G)$, then $\pi_\mu(f)\xi_p = \sigma(f)$ in $\mathsf{H}\ (\mu)$.

If p is continuous, then ξ_p can be chosen to be continuous and $p(\gamma) = \left(\pi(\gamma)\xi_p(s(\gamma))|\xi_p(r(\gamma))\right)$ for all γ.

Proof: Define $D = D(p) = \{x \in X: f \mapsto \int fp\, d\lambda^x = \lambda^x(fp)$ extends from $C_c(G)$ to give a bounded linear functional on H(x) of norm at most $\left\{\|p\|_\infty^{1/2}\right\}$. For each $f \in C_c(G), \lambda(fp)$ and $x \longmapsto (f|f)_x$ are Borel functions, and boundedness can be tested on a countable dense set, so D is a Borelset.For x # D, there is a unique vector $\xi_p(x) \in \mathrm{H}\,(x)$ such that $\left(\sigma(f)(x)|\xi_p(x)\right)_x = \lambda^x(fp)$for $f \in C_c(G)$, and if we let $\xi_p(x) = 0$ for $x \notin D, \xi_p$ is a Borel section ofH, bounded by $\|p\|_\infty^{1/2}$. We need to show that D is Q-conull, i.e., conull for each $\mu \in Q$.

Let $\mu \in Q$. Then there is a $\xi_\mu \in \mathrm{H}\ (\mu)$ satisfying (a), (b), (c) of Lemma (2.2.5)and thus for each $f \in C_c(G)$ we have

$$\left(\sigma(f)(x)|\xi_\mu(x)\right)_x = \left(\pi_\mu(f)\xi_\mu(x)|\xi_\mu(x)\right)_x = \lambda^x(fp)$$

for μ-a.e. x. Since bounded linear functionals are determined by their values on a countable dense set, and since boundedness of a linear functional can be tested on a countable dense set, there is a μ-conull set D_μ such that for$x \in D_\mu$and $f \in C_c(G)$,

$$\left(\sigma(f)|\xi_\mu(x)\right)_x = \lambda^x(fp).$$

Thus $D_\mu \subseteq D$, from which it follows that D is μ-conull and $\xi_p(x) = \xi_\mu(x)\mu$-a.e. This fact and Lemma (2.2.4) combine to establish the truth of statements(a) and (b) in the theorem. By the definitions of $D$and ξ_p , it follows thatξ_p is bounded by $\|p\|_\infty^{1/2}$.

To complete the proof, we show first that if p is continuous, then $D = X$.Again take a $\mu \in Q$ and the section ξ_p. We have $\lambda^x(fp) = \left(\sigma(f)(x)|\xi_\mu(x)\right)_x$ for μ-a.e. x, and for such x's,

$$|\lambda^x(fp)| \le |\sigma(f)(x)|_x \left\|\xi_\mu(x)\right\|$$
$$\le |\sigma(f)(x)|_x \|p\|_\infty^{1/2}.$$

Since p is continuous, both $\lambda(fp)$ and $x \longmapsto \left(\sigma(f), \sigma(f)\right)_x$ are continuous, so this estimate holds on the support of μ. The supports of the μ's in Q fill X,so

$$|\lambda^x(fp)| \leq |\sigma(f)(x)| \|p\|_\infty^{1/2}$$

for all f in $C_c(G)$ and all x. Thus $D = X$. Now $p(\gamma) = \big(\pi(\gamma)\xi_p(s(\gamma))|\xi_p(r(\gamma))\big)$ a.e. $d\lambda^\mu(\gamma)$ for every μ, so it will end the proof if we can show that the second function is continuous. By a partition of unity argument, this will follow if we can show that $\big(\pi(\gamma)\xi(s(\gamma))|\xi(r(\gamma))\big)$ is a continuous functionof γ for every continuous section ξ of compact support. In fact we can reduce to considering $\xi = \sigma(f)$for $f \in C_c(G)$, by using partitions of unity and uniform limits. Then we have

$$\big(\pi(\gamma)\xi(s(\gamma))|\xi(r(\gamma))\big) = \iint f(\gamma^{-1}\gamma_1)\bar{f}(\gamma_2)p(\gamma_2^{-1}\gamma_1)d\lambda^{r(\gamma)}(\gamma_1)\, d\lambda^{r(\gamma)}(\gamma_2).$$

Continuity of this function of γ can be deduced by applying the following easy lemma and a variant of it using the second coordinate projection ,because the integrands can be extended to functions satisfying the hypotheses of the lemma.

Lemma (2.2.7)[2]: Suppose G is a locally compact groupoid with a Haar system λ and let F be a continuous complex valued function on $G \times G$. Let $p_1: G \times G \to G$ be the first coordinate projection. Suppose that for every compact set $C \subseteq G$ the set $p_1^{-1}(C) \cap \text{supp}(F)$ is compact. Then, $\int F(\gamma, \gamma_2)d\lambda^{r(\gamma)}(\gamma_2)$ is a continuous function of γ.

We have an existence theorem, but we should show that the results are essentially the same for any two equivalent elements of $\mathrm{P}\ (G)$.

Theorem (2.2.8)[2]: Suppose that $p, q \in \mathrm{P}\ (G)$ and that $p = q\lambda^Q$-a.e. Then the associated representations $(\mathrm{H}_{\ p}, \pi_p)$ and $(\mathrm{H}_{\ q}, \pi_q)$ are the same, and the sections ξ_p and ξ_q agree Q-a.e.

Proof: Let $z \in X$ and consider the inner products on $L^1(\lambda^z)$ defined using p and q. Denote them by $(\ |\ )_p$ and $(\ |\ )_q$ respectively. To prove they are the same, it will suffice to show that $p(\gamma_2^{-1}\gamma_1) = q(\gamma_2^{-1}\gamma_1)$ for $\lambda^z \times \lambda^z$-almost every pair (γ_1, γ_2), because the inner products are defined by double integrals using these functions and measures.

Let μ_1^z be a quasiinvariant probability measure equivalent to the measureμ^z that was associated with the orbit $[z]$ near the end . Let Ebe the set of $x \in X$ for which $p = q$ a.e. relative to λ^x. Then $\mu^z(E) = 1$, so$\{\gamma: s(\gamma) \in E\} = GE$ is λ^z-conull. If $\{\gamma_1 \in zG: p(\gamma^{-1}\gamma_1) = q(\gamma^{-1}\gamma_1)\}$ is conull relative to λ^z by translation invariance of the Haar system. By theFubini Theorem, we get the desired agreement a.e.

This shows that the Hilbert bundles H_p and H_q are identical, and since the formula for the representation is just left translation in each case, the representations are the same.

To show that the sections ξ_p and ξ_q agree $\mathcal{Q}$-a.e., we resort to the definitions, namely, $\xi_p(x)$ and $\xi_{q(}(x)$ are determined by the fact that for $f \in C_c(G)$

$$\left(\sigma(f)|\xi_p(x)\right) = \lambda^x(fp)$$

and

$$\left(\sigma(f)|\xi_q(x)\right) = \lambda^x(fq).$$

Let F be the set of $x \in X$ for which $\xi_p(x) = \xi_q(x)$. Since the two sections are Borel, F is a Borel set. We need to show that if $\mu \in \mathcal{Q}$, then $\mu(X\backslash F) = 0$. We know that for each $f \in C_c(G)$ the two functions $\lambda(fp)$ and $\lambda(fq)$ agree almost everywhere relative to μ. Let Cbe a countable dense set in $C_c(G)$, and let N be a μ-null set such that $x \notin N$ and $f \in \mathsf{C}$ imply $\lambda^x(fp) = \lambda^x(fq)$.Since p and q are bounded, this equality is preserved under limits in $C_c(G)$, so it holds for $x \notin N$ and all $f \in C_c(G)$. Thus F contains the complement of N, as desired.

Theorem (2.2.9)[2]: Sums and products of positive definite functions are positive definite.

Proof: Now let us consider the enlarging the space from which we construct the fibers of the Hilbert bundle H_p using the positive definite function p. In later sections it will be convenient to replace the algebra $C_c(G)$ by the larger algebra $C_c(G,\bar{X})$, an algebra of kernels introduced , and we will need to know that using the latter in our construction does not change the fibers in that bundle.

Definition (2.2.10)[2]: Let G be a locally compact groupoid and let λ be a left Haar system on G. Then a bounded Borel function p on G is called strictly positive definite if for each $x \in X$ and each ν in $C_c(G,\bar{X})$ we have

$$\iint p(\gamma_2^{-1}\gamma_1)d\nu^x(\gamma_1)\, d\bar{\nu}^x(\gamma_2) \geq 0. \qquad (4)$$

The set of all such p's will be denoted by $\mathrm{P}'(G)$. Two functions $p, q \in \mathrm{P}'(G)$will be called equivalent iff they agree λ^{Q}-almost everywhere on G and their restrictions to X agree Q-almost everywhere.

we have $\mathrm{P}'(G) \subseteq \mathrm{P}(G)$, and would like to know that the sets are equal. Strictly speaking, this is not true because a function p can satisfy condition (1) and be negative everywhere on X unless there is a $\mu \in Q$ such that $\lambda^{\mu}(X) > 0$. Actions by non-discrete groups give rise to groupoids for which$Q$ contains no such μ. However, we have proved that every equivalence class in $\mathrm{P}\ (G)$ contains a diagonal matrix entry. Thus a kind of reverse of the containment would follow from the analog of Lemma (2.2.3),namely below showing that diagonal matrix entries are in$\mathrm{P}'(G)$. This meaning of the reverse containment would be that every class in $\mathrm{P}\ (G)$ contains an element of $\mathrm{P}'(G)$, or that diagonal matrix entries are in $\mathrm{P}'(G)$.

However, there is another natural question that also should be answered.If two diagonal matrix entries are equivalent in $\mathrm{P}(G)$, are they equivalent in $\mathrm{P}'(G)$? The affirmative answer is given in Lemma (2.2.14).

Lemma (2.2.11)[2]: Let π be a unitary representation of Gon the Hilbert bundle K, and let ξ be a bounded Borel section of K. Define $p(\gamma) = \big(\pi(\gamma)\xi \circ s(\gamma) | \xi \circ r(\gamma)\big)$for $\gamma \in G$. Then $p \in \mathrm{P}'(G)$.

Proof: As in the proof of Lemma (2.2.3), for $f \in C_c(G)$, there is a section, ξ, of the bundle such that for each $x \in X$ and every $\eta \in \mathrm{K}\ (x)$ we have

$$\int f(\gamma)(\pi(\gamma)\xi \circ s(\gamma)|\eta)\, d\lambda^{x}(\gamma) = (\xi(x)|\eta).$$

If $\mathrm{g} \in C(\bar{X}), x \in X$, and $\eta \in \mathrm{K}(x)$, then

$$\int \mathrm{g}(\gamma)(\pi(\gamma)\xi \circ s(\gamma)|\eta)\, d\varepsilon^{x}(\gamma) = \mathrm{g}(x)(\xi(x)|\eta).$$

If $\nu = f\lambda + \mathrm{g}\varepsilon$, these show that the integral involved in the condition (4) is equal to $\big(\zeta(x) + \mathrm{g}(x)\zeta(x) | \zeta(x) + \mathrm{g}(x)\xi(x)\big)$, which is certainly non-negative.

Corollary (2.2.12)[2]: Every equivalence class in $\mathrm{P}(G)$ contains an element of $\mathrm{P}'(G)$.

Lemma (2.2.13)[2]: Let (H, π) be a representation of G, let $u_1, u_2, \ldots$ be asymmetric approximate, as described in Theorem (2.1.3), and let ξ be a bounded Borel section of

H. Suppose that $\mu \in \mathcal{Q}$, and let π_μ be the integrated form of π as defined just before the statement of Lemma (2.2.4). Then $\pi_\mu(u_n)\xi \to \xi$ as $n \to \infty$.

Proof: By construction of the functions u_n, $\|u_n\|_I \leq 1$ for each n, so every $\pi_\mu(u_n)$ has norm at most 1. Hence it suffices to find a dense set of vectors satisfying the conclusion. Each vector of the form $\pi_\mu(g)\eta$ satisfies the conclusion, and hence vectors in the linear span of the set of such vectors do also. That linear span is dense.

Lemma (2.2.14)[2]: Suppose that π and π_1 are representations of G on bundles K and K$_1$, and that ξ and ξ_1 are bounded Borel sections of K and K$_1$. If $\big(\pi(\gamma)\xi \circ s(\gamma) | \xi \circ r(\gamma)\big) = \big(\pi_1(\gamma)\xi_1 \circ s(\gamma) | \xi_1 \circ r(\gamma)\big)$ for almost every γ relative to $\lambda^{\mathcal{Q}}$, then $\big(\xi(x)|\xi(x)\big) = \big(\xi_1(x)|\xi_1(x)\big)$ for almost every $x \in X$ relative to $\mathcal{Q}$.

Proof: Since ξ and ξ_1 are Borel sections the set E of $x \in X$ for which $\big(\xi(x)|\xi(x)\big) = \big(\xi_1(x)|\xi_1(x)\big)$ is a Borel set. We need to prove that for $\mu \in \mathcal{Q}$, $\mu(E) = 1$. The hypothesis implies that for $f \in$ M$_c(G)$ we have $\big(\pi_\mu(f)\xi|\xi\big) = \big(\pi_{1,\mu}(f)\xi_1|\xi_1\big)$, these being inner products associated with the integrated forms using Hahn's method (cf. Lemma (2.2.5), and the paragraph before it).Let φ and φ_1 be the representations of $C(\bar{X})$ by multiplication on the sections of K and K$_1$. Then it follows from the discussion following the statement of Theorem (2.1.4) that for $h \in C(\bar{X})$ and $f \in C_c(G)$

$$\big(\varphi(h)\pi_\mu(f)\xi \mid \xi\big) = \big(\varphi_1(h)\pi_{1,\mu}(f)\xi_1 \mid \xi_1\big).$$

Now for the $f \in C_c(G)$ take the terms of a symmetric approximate unit, to see that for all $h \in C(\bar{X})$,

$$(\varphi(h)\, \xi | \, \xi) = (\varphi_1(h)\xi_1 \, |\xi_1).$$

This means that for all $h \in C(\bar{X})$,

$$\int h(x)\big(\xi(x) \mid \xi(x)\big) d\mu(x) = \int h(x)\big(\xi_1(x) \mid \xi_1(x)\big) d\mu(x).$$

Thus E is indeed μ-conull.

After this, we will always take elements of P(G) or P$'(G)$ to be diagonal matrix entries, and understand that they are determined a.e. on X as well as on G.