

# Chapter 1

## Abel-Tauber Theorems for Fourier Cosine Transforms

*A similar result for fourier cosine series is obtained as corollary .The latter gives an answer to an open problem in Boas book on Fourier series . Application to probability distributions and stationary processes are given .*

We show Abel-Tauber theorems for Fourier cosine series and integrals. For example, we characterize the asymptotic behavior  $f(t) \sim t^{-1}$  as  $t \rightarrow \infty$  in term of Fourier cosine transform of  $f$ , where  $f$  is a locally integrable, eventually non-increasing function on  $[0, \infty)$  such that  $\lim_{t \rightarrow \infty} f(t) = 0$ .

To state our results, we recall and introduce some notation. We denote by  $R_0$  the whole class of slowly varying functions at infinity; that is,  $R_0$  is the class of positive measurable  $l$ , defined on some neighborhood of infinity, satisfying

$$\forall \lambda > 0, \quad \lim_{x \rightarrow \infty} l(\lambda x)/l(x) = 1.$$

For  $l \in R_0$ , the class  $\Pi_l$  is the class of measurable  $g$ , defined on some neighborhood of infinity, satisfying

$$\forall \lambda \geq 1, \quad \lim_{x \rightarrow \infty} \{g(\lambda x) - g(x)\}/l(x) = c \log \lambda$$

for some constant  $c$  called the  $l$ -index of  $g$ . It is useful to name the class of functions of which we define the Fourier cosine transforms. The function  $f: [0, \infty) \rightarrow \mathbb{R}$  belongs to  $D_{loc}^1[0, \infty)$  if it is locally integrable and eventually non-increasing on  $[0, \infty)$ ,  $\lim_{t \rightarrow \infty} f(t) = 0$ . For  $f \in D_{loc}^1[0, \infty)$ , we define the Fourier cosine transform  $F_c$  of  $f$  by

$$F_c(\xi) = \int_0^{\infty-} f(t) \cos t dt \quad (0 < \xi < \infty), \quad (1)$$

where we write  $\int_0^{\infty-}$  to denote an improper integral obtained from  $\int_a^M$  by letting  $M \uparrow \infty$ . Since the improper integral on the right converges uniformly on each  $(\varepsilon, \infty)$  with  $\varepsilon > 0$ ,  $F_c$  is a continuous function on  $(0, \infty)$ .

Here are the main theorems of this [section](#):

**Theorem (1.1)[1]:** Let  $0 < \alpha < 1$ ,  $t \in R_0$  and  $f \in D_{loc}^1[0, \infty)$ . Let  $F_c$  be the Fourier cosine transform of  $f$ . Then the following are equivalent:

$$f(t) \sim t^{-\alpha} l(t) (t \rightarrow \infty), \quad (2)$$

$$F_c(\xi) \sim \xi^{-(1-\alpha)} l(1/\xi) \Gamma(1-\alpha) \sin(\pi\alpha/2) (\xi \rightarrow 0+). \quad (3)$$

We consider Theorem (1.1) (Theorem (1.5) resp.) to be an analogue of Theorem (1.1) for the boundary case  $\alpha = 1$  ( $\alpha = 0$  resp.). From Theorems (1.2) and (1.5), we see that the case  $\alpha = 0, 1$  are critical ones for Fourier cosine transforms, and that they require  $\Pi$ -variation for their characterizations.

For absolutely convergent integral transforms like Laplace transforms, there already exist Abel-Tauber theorems which involve  $\Pi$ -variation. See de Hann, and Bingham and Teugels as well as Bingham et al. Our Theorems (1.2) and (1.5) are different from the previous works in that they involve both  $\Pi$ -variation and improper integrals.

We have the inversion formula for Fourier cosine transforms. but the integral which appears in the formulas is improper. So it is difficult to make direct use of it to prove the Tauberian implications such as (5) $\implies$ (4), and (17) $\implies$ (18). The key to the proofs is to reduce the problem to a completely monotone  $f$ . We use this method in the proofs of Theorem (1.2) and (1.5). We will prove Theorem (1.3) as a corollary of Theorem (1.2).

The plan of this **section** is as follows: we prove Theorems (1.2) and (1.3), we prove Theorem (1.2). we apply Theorem (1.2) and (1.5) to the tail behavior of probability distribution. Finally we apply Theorems (1.2) and (1.5) to stationary processes.

**Theorem (1.2)[1]:** Let  $t \in R_0$  and  $f \in D_{loc}^1[0, \infty)$ . Let  $F_c$  be the Fourier cosine transform of  $f$ . Then the following are equivalent:

$$f(t) \sim t^{-1} l(t) (t \rightarrow \infty), \quad (4)$$

$$F_c(1/\cdot) \in \Pi_t \text{ with } l - \text{index } 1. \quad (5)$$

**Proof :** Step 1: Choose  $M > 0$  so large that  $f$  is non-increasing on  $[M, \infty)$ . Set

$$f^M(t) = \begin{cases} f(M) & (0 \leq t < M), \\ f(t) & (M \leq t < \infty). \end{cases}$$

Let  $F_c^M$  be the Fourier cosine transform of  $f^M$ :

$$F_c^M(\xi) = \int_0^{\infty-} f^M(t) \cos t \xi dt \quad (\xi > 0).$$

Then, for any  $\lambda > 1$ ,

$$\begin{aligned}
& |\{F_c^M(1/\lambda x) - F_c^M(1/x)\} - \{F_c(1/\lambda x) - F_c(1/x)\}|/l(x) \\
&= \frac{1}{l(x)} \left| \int_0^M \{f(M) - f(t)\} \{\cos(t/\lambda x) - \cos(t/x)\} dt \right| \\
&\leq \frac{(1 - \lambda^{-1})}{xl(x)} \int_0^M t |f(M) - f(t)| dt \rightarrow 0 (x \rightarrow \infty),
\end{aligned}$$

whence(3) holds if and only if  $F_c^M(1 \cdot) \in \Pi_l$  with  $l$ -index 1. Thus we may assume that  $f$  is finite and non-increasing on  $[0, \infty)$ .

First we prove that the Abelian implication (4) $\implies$ (5),

$$F_c(1/x) - \int_0^M f(t) dt \sim -\gamma l(x) (x \rightarrow \infty),$$

where  $\gamma$  is Euler's constant.  $\int_0^x f(t) dt \in \Pi_l$  in  $x$  with  $l$ -index 1. So for any  $\lambda \geq 1$ ,

$$\begin{aligned}
& \frac{F_c(1/\lambda x) - F_c(1/x)}{l(x)} \\
&= \frac{F_c(1/\lambda x) - \int_0^{\lambda x} f(t) dt}{l(x)} + \frac{\int_0^{\lambda x} f(t) dt - \int_0^x f(t) dt}{\pi l(x)} \\
&\quad - \frac{F_c(1/x) - \int_0^x f(t) dt}{l(x)} \rightarrow \log \lambda (x \rightarrow \infty),
\end{aligned}$$

whence(5).

Step 2: By the second mean-value theorem for integrals,  $\xi F_c(\xi)$  is bounded on  $(0, \infty)$ . (5) implies  $|F_c(1/\cdot)| \in R_0$ , and so  $F_c(\cdot) \in L_{loc}^1[0, \infty)$ . Then we give

$$t^{-1/2} \int_0^t f(u) du = \frac{2}{\pi} \int_0^\infty \frac{F_c(\xi)}{\xi^{1/2}} \cdot \frac{\sin t\xi}{(t\xi)^{1/2}} d\xi (t > 0). \quad (6)$$

Since  $t^{-1/2} \sin t$  is bounded on  $(0, \infty)$  and  $\xi^{-1/2} F_c(\xi)$  is integrable over  $(0, \infty)$ , we have dominated convergence, as  $t \rightarrow \infty$ , in (6), and so

$$\lim_{t \rightarrow \infty} t^{-1/2} \int_0^t f(u) du = 0.$$

Therefore, integrating by parts,

$$\int_1^{\infty} f(t) dt/t = \int_1^{\infty} \left( t^{-1/2} \int_1^t f(u) du \right) t^{-3/2} dt < \infty.$$

Step 3: We define a measure  $\sigma$  on  $(0, \infty)$  by

$$\sigma(d\lambda) = I_{(0,1)}(\lambda) f(1/\lambda) d\lambda/\lambda.$$

Then  $\sigma$  is finite because

$$\sigma(0, \infty) = \int_0^1 f(1/\lambda) d\lambda/\lambda = \int_1^{\infty} f(t) dt/t < \infty.$$

We set

$$g(t) = \int_0^{\infty} e^{-t\lambda} \sigma(d\lambda) \quad (t \geq 0). \quad (7)$$

Then  $g$  is finite and non-increasing on  $[0, \infty)$ ,  $\lim_{t \rightarrow \infty} g(t) = 0$ . Since  $\log\{xf(x)\}$  is slowly increasing, by Karamata's theorem, (1.1) is equivalent to

$$g(t) \sim t^{-1} l(t) \quad (t \rightarrow \infty). \quad (8)$$

Let  $G_c$  be the Fourier cosine transform of  $g$ :

$$G_c(\xi) = \int_0^{\infty-} g(t) \cos t\xi d\xi \quad (\xi > 0).$$

Then for  $\xi > 0$ ,

$$G_c(\xi) = \int_0^{\infty} \frac{\lambda}{\lambda^2 + \xi^2} \sigma(d\lambda) = \int_0^{\infty} \frac{f(t)}{1 + t^2 \xi^2} dt - \int_0^1 \frac{f(t)}{1 + t^2 \xi^2} dt.$$

Since

$$\frac{1}{1 + t^2 \xi^2} = \int_0^{\infty} \xi^{-1} e^{-u/\xi} \cos ut du \quad (t > 0, \xi > 0), \quad (9)$$

$$\int_0^{\infty} \frac{f(t)}{1 + t^2 \xi^2} dt = \int_0^{\infty} \xi^{-1} e^{-u/\xi} F_c(u) du \quad (\xi > 0).$$

Hence, for any  $\lambda > 1$  and  $x > 0$ ,

$$\begin{aligned}
& \frac{G_c(1/\lambda x) - G_c(1/x)}{l(x)} \\
&= \int_0^\infty \frac{F_c(1/\lambda x) - F_c(u/x)}{l(x)} e^{-u} du \\
&- \frac{(1 - \lambda^{-2})}{x^2 l(x)} \int_0^1 \frac{t^2 f(t)}{\{1 + (t/x)^2\}\{1 + (t/\lambda x)^2\}} dt.
\end{aligned} \tag{10}$$

The second term on the right clearly tends to zero as  $x \rightarrow \infty$ . Now since  $F_c(\xi) \rightarrow 0$  as  $\xi \rightarrow \infty$ ,  $F_c(1/\cdot)$  can be extended to a continuous function on  $[0, \infty)$ . Therefore we have dominated convergence, as  $x \rightarrow \infty$ , in the first term on the right, so it converges to  $\log \lambda$ . Thus (5) implies

$$G_c(1/\cdot) \in \Pi_t \text{ with } l - \text{index } 1. \tag{11}$$

Therefore, in order to prove (5)  $\Rightarrow$  (8).

Step 4:

$$g(t) \frac{2}{\pi} \int_0^{\infty-} G_c(\xi) \cos t\xi d\xi \quad (t > 0).$$

Since  $\xi G_c(\xi)$  is bounded on  $(0, \infty)$ ,

$$\lim_{\xi \rightarrow \infty} G_c(\xi) \sin t\xi = 0,$$

while, by (11)

$$\lim_{\xi \rightarrow \infty} G_c(\xi) \sin t\xi = 0.$$

So, integrating by parts,

$$tg(t) = \int_{0+}^{\infty-} k(u)L(tu) du \quad (t > 0), \tag{12}$$

where  $\int_{0+}^a \lim_{\varepsilon \downarrow 0} \int_\varepsilon^a$ , and

$$k(t) = \frac{2}{\pi} \cdot \frac{\sin(1/t)}{t} \quad (t > 0),$$

$$L(t) = t\{G_c(1/t)\}' \quad (t > 0).$$

From

$$G_c(1/t) = \int_0^\infty \frac{\lambda}{(\lambda^2 + t^{-2})} \sigma(d\lambda) \quad (t > 0),$$

we get

$$\{G_c(1/t)\}' = \frac{2}{t^3} \int_0^\infty \frac{\lambda}{(\lambda^2 + t^{-2})^2} \sigma(d\lambda) \quad (t > 0).$$

So  $\log\{G_c(1/t)\}'$  is slowly decreasing, (11) implies  $L(t) \sim l(t)$  as  $t \rightarrow \infty$ ,

Now we have

$$L(t) = 2t^2 \int_0^\infty \frac{\lambda}{(1 + t^2\lambda^2)^2} \sigma(d\lambda) \quad (t > 0).$$

We set  $L(0) = 0$ . Then we easily see that  $L \in C^1[0, \infty)$ ; in particular,  $L$  is locally of bounded variation on  $[0, \infty)$ . We define two positive, non-decreasing functions  $\phi_1, \phi_2$  by

$$\begin{aligned} \phi_1(t) &= 2t^2 \quad (t > 0), \\ \phi_2(t) &= 1 / \left\{ \int_0^\infty \frac{\lambda}{(1 + t^2\lambda^2)^2} \sigma(d\lambda) \right\} \quad (t > 0). \end{aligned}$$

Then we have

$$\begin{aligned} \phi_i(2t) &= O(\phi_i(t)) \quad (i = 1, 2) \quad (t \rightarrow \infty), \\ L(t) &= \phi_1(t)\phi_2(t) \quad (t > 0), \end{aligned}$$

Therefore, by the Quasi-Monotonicity Theorem (see [1]),  $L$  is quasi-monotone; that is, for some  $\delta > 0$ ,

$$\int_0^t u^\delta |dL(u)| = O(t^\delta L(t)) \quad (t \rightarrow \infty),$$

where  $|dL(u)|$  is the variation measure of  $L$ . Since

$$\begin{aligned} \int_t^\infty k(u) du &= O(1/t) \quad (t \rightarrow \infty), \\ \int_{0+}^t k(u) du &= O(t) \quad (t \rightarrow 0+), \end{aligned}$$

by applying the theorem of Bojanic and Karamata (see [1]) to (12) we obtain

$$tg(t) \sim l(t) (t \rightarrow \infty),$$

whence(8). This completes the proof.

**Theorem (1.3)[1]:** Let  $t \in R_0$ . Suppose that the real sequence  $\{a_n\}$  is eventually non-increasing, and tends to 0 as  $n \rightarrow \infty$ . We set

$$\hat{a}_c(\xi) = \sum_{n=0}^{\infty} a_n \cos n\xi \quad (0 < \xi < 2\pi). \quad (13)$$

Then the following are equivalent:

$$a_n \sim n^{-1}l(n) (n \rightarrow \infty), \quad (14)$$

$$\hat{a}_c(1/\cdot) \in \Pi_l \text{ with } l - \text{index } 1. \quad (15)$$

Let  $K$  be a positive constant. If we see  $l(t) \equiv K$  in Theorem (1.3), then we obtain an answer to an open problem in Boas .

Now we recall the Abel-Tauber theorem of Pitman , and Soni and Soni , which is closely related to Theorems (1.1.1) and (1.1.2).

**Proof :** We set

$$f(t) = \begin{cases} 2a_0 & (0 \leq t \leq \frac{1}{2}), \\ a_n & (n - \frac{1}{2} < t \leq n + \frac{1}{2}, n = 1, 2, \dots). \end{cases}$$

Let  $F_c$  be the Fourier cosine transform of  $f$ . Then

$$F_c(\xi) = \frac{\sin(\xi/2)}{(\xi/2)} \hat{a}_c(0 < \xi < 2\pi). \quad (16)$$

By (16) and [1], we easily see that (15) holds if and only if  $F_c(1/\cdot) \in \Pi_l$  with  $l$ -index 1. Therefore, by Theorem (1.1), we obtain the theorem.

To prove Theorem (1.2), we need the following version of the Abel-Tauber theorem of de Haan .

**Theorem (1.4)[1]: (A Version of de Haan's Abel Tauber Theorem)** Let  $l \in R_0$  and  $c \geq 0$ . Let  $U$  be a non-decreasing right-continuous function on  $[0, \infty)$ . Assume its

Laplace-Stieltjes transform  $\widehat{U}(s) := \int_{[0, \infty)} e^{-s\lambda} dU(\lambda)$  is finite for any  $s > 0$ . Then the following are equivalent:

(i)  $U(1/\cdot) \in \Pi_l$  with  $l$ -index  $-c$ ,

(ii)  $\widehat{U} \in \Pi_l$  with  $l$ -index  $-c$ .

The proof of Theorem 3.1 is almost the same as that of (see [1]).

**Theorem (1.5)[1]:** Let  $t \in R_0$  and  $f \in D_{loc}^1[0, \infty)$ . Let  $F_c$  be the Fourier cosine transform of  $f$ . Then

$$F_c(\xi) \sim \xi^{-1} l(1/\xi) \frac{\pi}{2} (\xi \rightarrow 0+) \quad (17)$$

implies

$$f \in \Pi_l \text{ with } l\text{-index } -1. \quad (18)$$

Conversely, if  $F_c$  is non-negative and non-increasing in a neighborhood of zero, then (18) implies (17).

We note that in Theorem (1.1) the Abelian implication (4)  $\implies$  (5) is essentially due to Pitman.

The analogue of Theorem (1.1) for Fourier cosine series is

**Proof:** In the same way as the proof of Theorem (1.1), we may assume that  $f$  is finite, positive, and non-increasing on  $[0, \infty)$ . We may also assume that  $f$  is left-continuous. We set  $U(\lambda) = f(1/\lambda)$  for  $\lambda > 0$ , and  $U(0) = 0$ . Then  $U$  is a finite, non-decreasing, and right-continuous function on  $[0, \infty)$ . We define a finite measure  $\sigma$  on  $[0, \infty)$  by

$$\sigma(d\lambda) = dU(\lambda).$$

We define  $g$  by (7). Then, by Theorem 3.1, (18) holds if and only if

$$g \in \Pi_t \text{ with } l\text{-index } -1. \quad (19)$$

Let  $G_c$  be the Fourier cosine transform of  $g$ . Then integration by parts yields, for any  $\xi > 0$ ,

$$G_c(\xi) = \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2} \sigma(d\lambda) = \int_0^\infty \frac{1 - t^2 \xi^2}{(1 + t^2 \xi^2)^2} f(t) dt.$$



By (9),

$$\frac{1 - t^2 \xi^2}{(1 + t^2 \xi^2)^2} = \frac{\partial}{\partial t} \left( \frac{t}{1 + t^2 \xi^2} \right) = \int_0^\infty \xi^{-2} u e^{-u/\xi} \cos ut \, du \quad (t > 0, \xi > 0),$$

whence [see[1]] gives

$$G_c(\xi) = \xi^{-2} \int_0^\infty F_c(u) u e^{-u/\xi} \, du \quad (\xi > 0). \quad (20)$$

Since  $uF_c(u)$  is bounded on  $(0, \infty)$ , (17) implies

$$G_c(\xi) \sim \xi^{-1} l(1/\xi) \frac{\pi}{2} \quad (\xi \rightarrow 0+). \quad (21)$$

Therefore, in order to prove (17)  $\Rightarrow$  (18), it is enough to prove (21)  $\Rightarrow$  (19).

By the representation

$$-g(t) = \int_0^\infty e^{-t\lambda} \lambda \sigma(d\lambda) \quad (t > 0),$$

we see that  $-g$  is locally integrable and non-increasing on  $[0, \infty)$ ,  $\lim_{t \rightarrow \infty} g(t) = 0$ .

Therefore, by de Haan's theorem (see [1]), (5) is equivalent to

$$-g(t) \sim t^{-1} l(t) \quad (t \rightarrow \infty). \quad (22)$$

Now integration by parts yields

$$\xi G_c(\xi) = \int_0^{\infty-} \{-g(t)\} \sin t\xi \, dt \quad (\xi > 0).$$

So by [see[1]], (21) and (22) are equivalent. Thus (17)  $\Rightarrow$  (18).

Finally, if  $F_c$  is non-negative and non-increasing in a neighborhood of 0, then, by Karamata's Tauberian theorem (21) conversely implies (17), whence (18)  $\Rightarrow$  (17). This completes the proof.

Let  $X$  be a real random variable defined on a probability space  $(\Omega, \mathcal{F}, P)$ . We define the tail difference  $D$  of  $X$  by

$$D(x) = P(X > x) - P(X \leq -x) \quad (x \geq 0).$$

Let  $V$  be the imaginary part of the characteristic function of  $X$ :

$$V(\xi) = E[\sin \xi X](\xi \in \mathbb{R}).$$

Then we have

$$\frac{V(\xi)}{\xi} = \int_0^{\infty-} D(x) \cos \xi x dx \quad (\xi > 0).$$

Therefore, by Theorems (1.1.1) and (1.1.2), we obtain the following theorem:

**Theorem (1.6)[1]:** Let  $l \in R_0$ . Assume that  $D$  is eventually non-increasing on  $[0, \infty)$ . Then

- (i)  $D(x) \sim x^{-1}l(x)$  as  $x \rightarrow \infty$  if and only if  $xV(1/x)F_c \in \Pi_l$  in  $x$  with  $l$ -index 1.
- (ii) If

$$V(\xi) \sim l(1/\xi) \frac{\pi}{2} (\xi \rightarrow 0+), \quad (23)$$

then

$$D \in \Pi_t \text{ with } l - \text{index } 1. \quad (24)$$

Conversely, if  $\xi^{-1}V(\xi)$  is non-negative and non-increasing in  $\xi$  in a neighborhood of 0, then (24) implies (23)

In this section, we apply Theorems (1.1) and (1.2) to stationary processes. Let  $X = (X(t): t \in \mathbb{R})$  be a real, weakly stationary process with zero expectation, and let  $R$  be the correlation function of  $X: R(t) = E(X(t)X(0))$  for  $t \in \mathbb{R}$ . Let  $\mu_x$  be the spectral measure of  $X: R(t) = \int_{-\infty}^{\infty} e^{-it\xi} \mu_x(d\xi)$  for  $t \in \mathbb{R}$ . If  $\mu_x$  is absolutely continuous with respect to the Lebesgue measure  $d\xi$ , then we call the density  $\Delta$  the spectral density of  $X: \mu_x(\xi) = \Delta(\xi)d\xi$ . The spectral density  $\Delta$  is non-negative, even, and integrable function on  $\mathbb{R}$ .

**Proposition (1.7)[1]:** Assume that the correlation function  $R$  is eventually monotone on  $[0, \infty)$ ,  $\lim_{t \rightarrow \infty} R(t) = 0$ . Then the spectral measure of  $X$  is absolutely continuous with respect to the Lebesgue measure, and the spectral density  $\Delta$  is given by

$$\Delta(\xi) = \frac{1}{\pi} \int_0^{\infty-} R(t) \cos t\xi dt \quad (t \in \mathbb{R}). \quad (25)$$

**Proof:** By the assumption,  $R$  is even, continuous, and either eventually non-negative and non-increasing, or eventually non-positive and non-decreasing. By Lévy's inversion formula, if  $a$  and  $b$  are continuity points of  $\mu_x$  such that  $0 < a < b$ , then

$$\mu_x(a, b) = \lim_{M \rightarrow \infty} \frac{1}{2\pi} \int_{-M}^M \frac{e^{itb} - e^{ita}}{it} = F(b) - F(a),$$

where

$$F(\xi) := \frac{1}{\pi} \int_0^{\infty-} \frac{R(t)}{t} \sin \xi t \, dt \quad (\xi > 0).$$

Since the improper integral  $\int_0^{\infty-} R(t) \cos \xi t \, dt$  converges uniformly in  $\xi > \varepsilon$  for any  $\varepsilon > 0$ , the function  $F$  is of  $C^1$ -class in  $(0, \infty)$  and satisfies

$$F'(\xi) = \frac{1}{\pi} \int_0^{\infty-} R(t) \cos \xi t \, dt \quad (\xi > 0).$$

Therefore  $\mu_x$  is absolutely continuous in  $(0, \infty)$ , and the density there is equal to the derivative  $F'$ . If we put  $\Delta(\xi) := F'(\xi)$  for  $\xi > 0$ , then we obtain

$$R(t) = \mu_x\{0\} + 2 \int_0^{\infty-} \Delta(\xi) \cos \xi t \, d\xi \quad (t \in \mathbb{R}).$$

By the Riemann-Lebesgue Lemma, the second term on the right converges to zero as  $t \rightarrow \infty$ , so that  $\mu_x\{0\} = 0$ . This completes the proof.

By Proposition (1.7), and Theorems (1.1) and (1.2), we immediately obtain the following theorem:

**Theorem (1.8)[1]:** Let  $l \in R_0$ . Assume that the correlation function  $R$  is eventually non-increasing on  $[0, \infty)$ ,  $\lim_{t \rightarrow \infty} R(t) = 0$ . Then

- (i)  $R(t) \sim t^{-1} l(t)$  as  $t \rightarrow \infty$  if and only if  $\Delta(1/\cdot) \in \Pi_l$  with  $l$ -index  $\pi^{-1}$ .
- (ii) If

$$\Delta(\xi) \sim \xi^{-1} (1/\xi) \frac{1}{2} \quad (\xi \rightarrow 0+), \quad (26)$$

then

$$R \in \Pi_t \text{ with } l\text{-index } 1. \quad (27)$$

Conversely, if  $\Delta$  is non-increasing in a neighborhood of 0, the (27) implies (26).