بسم الله الرحمن الرحيم

Sudan University of Science and Technology

College of Graduate Studies

C^*-Algebras and Topological Dynamical Systems on Nets and for Product Systems over Semigoups

جبريات-C^* والأنظمة الحركية التبولوجية على الشبكات ولأنظمة الضرب فوق شبه الزمر

A thesis Submitted in Partial Fulfillment of the Requirements of the M.Sc. Degree in Mathematics

By

Dalya Ali Mohammed Towfig Khafaga

Supervisor

Prof. Shawgy Hussein Abdalla

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Detication

To my Mother,

Soul of Father,

Husband,

Children,

and sister ...

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*Before of all the praise and thanks be to Allah whom to be scribed all perfection and majesty. The thanks after Allah must be to my supervisor **Prof. Shawgy Hussein Abdalla** who supervised this research and quided me with patience until the result of this research are obtained.*

Abstract

An asymptotic measure expansiveness is introduced and its relationship with dominated splitting is considered. We show recurrence and multiple recurrence results for topological dynamical systems indexed by an arbitrary directed partial semigroup with respect to acoideal basis suitable for this semigroup, but otherwise arbitrary.Extending the work of Cuntz and Vershik, we develop ageneral notion of independence for commuting group endomorphisms. Based on this concept, we initiate the study of irreversible algebraic dynamical systems, which can be thought of as irreversible analogues of the dynamical systems considered by Schmidt. We show a version of uniqueness theorem for Cuntz-Pimsner algebras of discrete product systems over semigroups of Ore type.

الخلاصة

تم إدخال تمدد القياس التقريبي وإعتباره مع الإنشقاق المهيمن. أوضحنا التكرار ونتائج التكرار المضاعف للانظمة الحركية التبولوجية المرقمة بواسطة شبه زمرة جزئية مباشرة إختيارية بالنسبة إلى أساس مثالي مصاحب مناسب لأجل شبه الزمرة هذه ولكنها إختيارية في مكان آخر. مددنا عمل كينتز وفرشيك وطورنا الفكرة العامة غير المستقلة لأجل أندومورفيزمات زمرة التبديل. بناءاً على هذا المفهوم بدأنا دراسة الأنظمة الحركية الجبرية اللا إنعكاسية والتي يمكن أن يفكر فيها كمشابه لا إنعكاسي للأنظمة الحركية المعتبرة بواسطة شميدت. أوضحنا إصدارة لمبرهنة الوحدانية لأجل جبريات كينتز – بيمسنير لأنظمة الضرب المتقطعة فوق شبه زمر نوع أور.

The Contents

Chapter 1

Diffeomorphisms of Asymptotic Measure Expansive

It is proved that if a diffeomorphism admits a co-dimension one dominated splitting then it is asymptotic measure expansive. Also, a diffeomorphism with a homoclinic tangency can be perturbed to a non-asymptotic measure expansive diffeomorphism

Proposition (1.1)[1]:

For a diffeomorphism f having a dominated splitting$E \oplus F$ there exist two continuous maps $\emptyset^s: M \to Emb(D_1^s, M)$ *and* $\emptyset^u : M \to Emb(D_1^u, M)$ *such that for* $W_{loc}^{cs}(x) = \emptyset^s(x)(D_\in^s)$ *and* $W_{loc}^{cs}(x) = \emptyset^u(x)(D_\in^u)$ *we have.*

• $T_x W_{loc}^{cs}(x) = \mathrm{E}(x)$*and* $T_x W_{loc}^{cs}(x) = F(x)$;

• $W_{loc}^{cs}(x) \cap W_{loc}^{cs}(y)$ *is exactly one point denoted by* $\{x, y\}$.

In the case of a partially hyperbolic splitting $E^S \oplus F$, the submanifold W^sintegrating E^sinherits the same behavior. Thus, for any $y, z \in W_{loc}^s(x)$and any positive $n, d(f^n(y), f^n(z)) < \lambda^n$.In fact, backward iterations of the local stable manifolds makes a foliation of the ambient manifold behaving as a contraction on the leaves with respect to the induced Riemannian metric . Any partially hyperbolic diffeomorphism is measure expansive. The idea of proof is quite simple. Take $y \in \Gamma_\delta(x)$, with δ small enough. If $y \notin W_\delta^{cu}(x)$, then, the backward iterations go away from the δ neighborhood. Hence, $y \in W_\delta^{cu}(x)$, and therefore $\Gamma_\delta(x) \subset W_\delta^{cu}(x)$.This implies that $Leb(\Gamma_\delta(x)) = 0$. To prove Theorem (1.3), we need two simple observations. Time first follows from the differentiability of f and the second straightforwardly from the fact that the angles between unit vectors of the two bundles of a dominated splitting are uniformly bounded away from zero. For $y \in \Gamma_\delta(x)$ the projection of y on $W^{cs}_{loc}(x)$ along the cu-leaves is denoted by yE. $yF$is similarly defined. In fact $yE = [x, y]\ and\ yF = [y, x]$.

Lemma (1.2)[1]:

If$\delta$is sufficiently small, then for any $y \in \Gamma_\delta$ (x) **,** there is M such that

$$\frac{d(f^n(x), f^n(yE))}{d(f^n(x), f^n(yf)} \le M\lambda^n$$

Proof: Let us consider a positive number δ such that for any $y, z \in M$with $d(y, z) < \delta$we have $|Df_{E(y)}\| / \|Df_{F(z)}\|_{co} \le \lambda$ **).** In particular, for any two points y, z with $d(f^n(z), f^n(y)) < \delta$, for every $n \in \mathbb{Z}$

$$\|Df^n_{E(y)}\| / \|Df^n_{E(z)}\|_{co} < \lambda^n, \qquad for\ every\ n \in \mathbb{Z} \qquad (1)$$

As$yE = [x.y]$ and $d(f^n(x), f^n(yE)) < \delta, f^n(yE) = [f^n(x), f^n(y)]$ and the same holds for yF. Let γ and η be the curves of minimal length connecting $f^n(x)$ to, $f^n(yE)$ and $f^n(x)$ to, $f^n(yF)$respectively.Choose curves $\gamma_n$and η_n with $f^n(\gamma_n) = \gamma$and $f^n(\eta_n) = \eta$. We should point out that the curves γ_n and η_n are simply the perimages under$f^n$of γ and η respectively. Now, we have

$$\frac{d(f^n(x), f^n(yE))}{d(f^n(x), f^n(yF))} \leq \int \|\gamma^\circ(t)\| dt \,/ \int \|\eta^\circ(t)\| dt$$

$$\leq \int \|Df^n(\gamma_n(t))\gamma_n^\circ(t)\| dt / \int \|Df^n(\eta_n(t))\eta_n^\circ(t)\| dt$$

$$\leq \int \left\|Df^n(\gamma_n(t))_{|_E}\right\| \|\gamma_n^\circ(t)\| dt$$

$$/ \int \left\|Df^n(\eta_n(t))_{|_F}\right\|_{co} \|\eta_n^\circ(t)\| dt \leq \lambda^n \frac{d(x, yE)}{d(x, yF)}$$

last inequality holds by (1.1) and the fact that E is one dimensional.

Theorem (1.3)[1]:

If $f \in Diff^1(M)$ admits a co-dimension one dominated splitting, then f is asymptotic measure expansive.

Since the existence of a dominated splitting is an open property in $Diff^1(M)$, the above result implies that every diffeomorphism admitting a dominated splitting is contained in the C^1 interior of the asymptotic measure expansive diffeomorphisms.

We complete by showing that asymptotic measure expansiveness is far from homoclinic tangency.

Proof: As a result of the above lemma, one can deduce that for$y \in \Gamma_{\delta(x)}$, $d\,(f^n(x), f^n(yE))$tends to 0 exponentially fast. Now, given > 0 , choose η in such a way that the set

$$D_n = \bigcup_{y \in B_{\eta(f^n(x))}} w_{loc}^{cu}(y)$$

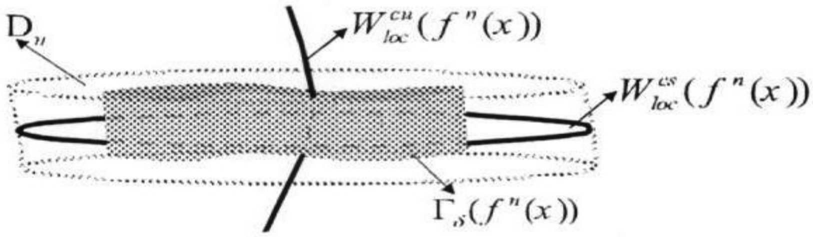


Fig. 1.The local set D_n

has Lebesgue measure less than ϵ. By the above lemma and the continuity of center-unstable manifolds, for large $n, f^n(\Gamma_\delta(x)) \subset D_n$(see Fig.1). In particular,

$$Leb\ (f^n(\Gamma_\delta(x))) \leq\ Leb(D_n)$$

and the proof follows.

Theorem (1.4)[1]:

1f $f \in Diff^1(M)$ has homoclinic tangency, then f can be perturbed in C^1-topology to a diffeomorphism that is not asymptotic measure expansive.

Consequently, time C^1-interior of the asymptotic measure expansive difthomorphisms is formed by diffeomnorphismns far from hornoclinic tangencies.

Proof: It is worth noting that we are working in the C^1-topolor. Bowen, proved that C^2-diffeomorphisms have no horseshoes with positive Lebesgue measure, however, he constructed a C^1-horseshoe with positive Lebesgue measure. In the proof of Theorem **(**1.4**)** we benefit from this construction and the modifications done.

Suppose that f has a periodic point p whose stable and unstable manifolds tangentially intersect each other in a homoclinic point x. The proof will be done straightforwardly using two classic steps borrowed essentially from [1].First, a fiat connection between the stable and unstable manifolds of p at a tangential intersection point is produced;

Lemma (1.5)[1]:

With the assumptions above, there is a diffeomorphisrn$f_1$sufficiently C^1-closed to f such that p is again a periodic point for $f_1$and there is a smooth submanifold C, with $dim(C) = min\{dim(W^s(p)), dim\ (W^u\ (p))\}$ such that $\boldsymbol{x} \in C \subset W^s(p, f_1) \cap W^u(p, f_1)$.

In the second step, using Bowen's construction, a sequence of invariant horseshoes of positive Lebesgue measure are produced. Anon-asymnptotic expansive diffeornorphism is created.

Proposition (1.6)[1] :

There is a C^1*-diffeomorphism* f_2 *arbitrarily near* f_1*(*f_1*as in the above lemma), producing a sequence of horseshoes* H_n *near C with the following properties:*

(i) *for some fired* $k, f_n^k(H_n) = H_n$, *for arty n;*

(ii) *Leb*(H_n) >0;

(iii) $Diarn(H_{n)} \to 0, asn \to \infty$.

Now, *put* $p_n \in Hn, \delta_n = Diarn(H_n)$and $\eta_n = Leb(H_n)$. Since $H_n \subset \Gamma_{\delta_n}(p_n, f_2^k)$one deduces that for any *m*.

$$Leb\ (f_2^{km}(\Gamma_{\delta_n}(p_n))) \geq Leb\ (H_n) > \eta_n.$$

This means that $f_2$is not asymptotic measure expansive.

Chapter 2

Nets and Topological Dynamical Systems

These results are then applied to topological dynamical systems indexed by semigroups possessing digital representation. The theory includes recurrence and multiple recurrence results for topological dynamical systems, indexed by natural numbers, or by finite non-empty subsets of natural numbers.

Section (2.1): Coideal Bases with the (D)-Property:

We introduce the (D)-property of a coideal basis on an infinite directed set $(\Lambda, \prec)$ and we will prove, in Theorem (2.1.7) below, that every net $(x_\lambda)_{\lambda\in\Lambda}$ in a compact metric space has a convergent subnet of the form $(x_\lambda)_{\lambda\in\Lambda}$, where A is an element of an arbitrary coideal basis ß on Λ with the (D)-property. Moreover, we will locate A to be a subset of a given element B of the coideal basis ß. This result will be the starting point in order to prove later recurrence results for topological systems of continuous maps from a compact metric space into itself indexed by an infinite directed set with respect to a coideal basis with the (D)-property.

A coideal on the set of natural numbers appears in [2] and elsewhere. This extended from the set of natural numbers to an arbitrary infinite directed set as follows:

Definition (2.1.1)[2]:

Let Λ be a non-empty infinite set and $\prec$ a relation on Λ satisfying the following conditions:

(i) If $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$, then $\lambda_1 \neq \lambda_2$.

(ii) If $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2$ and $\lambda_2 \prec \lambda_3$, then $\lambda_1 \prec \lambda_3$.

(iii) For every $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \prec \lambda_3$ and $\lambda_2 \prec \lambda_3$

Then $(\Lambda, \prec)$ is a directed set.

Definition (2.1.2)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set, A subset $\mathcal{H}$ of $[\Lambda]^\infty$ is a coideal on $(X, \prec)$ if it satisfies the following three properties:

(i) For every $A \in \mathcal{H}$ and $\lambda_1 \in \Lambda$ there exists $\lambda_2 \in A$ such that $\lambda_1 \prec \lambda_2$.

(ii) If $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.

(iii) If $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{H}$.

Let $(\Lambda, \prec)$ be an infinite directed set and let $A \subseteq \Lambda$ and $\lambda \in A$. Then,

$$A - \lambda = \{ Z \in A : \lambda \prec z \}.$$

Definition (2.1.3)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set. A subset ß of $[\Lambda]^{\infty}$ is a coideal basis on $(\Lambda, \prec)$ if it satisfies the following two properties:

(i) For every $A \in$ ß and $\lambda_1 \in \Lambda$ there exists $\lambda_2 \in \boldsymbol{A}$ such that $\lambda_1 \prec \lambda_2$.

(ii) If $A \cup B \in$ ß, then there exists $C \in$ ß such that either $C \subseteq A$ or $C \subseteq B$.

Obviously, a coideal on $(\Lambda, \prec)$ is a coideal basis on $(\Lambda, \prec)$. The connection between coideals and coideal bases is given in the following proposition.

Proposition (2.1.4)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set. A family $\mathcal{H} \subseteq [\Lambda]^{\infty}$ is a coideal on $(\Lambda, \prec)$ if and only if there exists a coideal basis ß $\subseteq [\Lambda]^{\infty}$ such that

$$\mathcal{H} = \mathcal{L}_{ß} = \{A \subseteq X : \text{ there exists } B \in ß \text{ with } B \subseteq A\}.$$

Definition (2.1.5)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set. A coideal basis ß $\subseteq [\Lambda]^{\infty}$ on $(\Lambda, \prec)$ has the **(P)-property** if for every sequence $(A_n)_{n \in \mathbb{N}}$, with $A_n \in$ ß and $A_1 \supseteq A_2 \supseteq \ldots$, there exists $A \in$ ß such that $A \setminus A_n$ is a finite set for every $n \in \mathbb{N}$.

We will introduce now a weaker property than the (P)-property of a coideal basis on an infinite directed set, which we call (D)-property.

Definition (2.1.6)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set. A coideal basis $ß \subseteq [\Lambda]^{\infty}$ on $(\Lambda, \prec)$ has the **(D)-property** iffor every sequence $(A_n)_{n\in\mathbb{N}}$, with $A_n \in ß$and $A_1 \supseteq A_2 \supseteq \ldots$, there exists $A \in ß$ such that for every $n \in \mathbb{N}$there exists $K_n \in \mathbb{N} \cup \{0\}$ satisfying

$$K_n = max\{ k \in \mathbb{N}: there\ exist\ \lambda_1, \ldots, \lambda_k \in A/A_n\ with\ \lambda_1 \prec \cdots \prec \lambda_k \}.$$

Examples (2.1.7)[2]:

(i) The set $[\mathbb{N}]^{\infty}$ is a coideal on $\mathbb{N}$ with the usual order, according to the pigeon-hole principle, and obviously has the (P)-property and consequently the (D)-property.

(ii) Let $[\mathbb{N}]^{<\infty}_{>0}$ be the set of all the finite non-empty subsets of $\mathbb{N}$. For $F_1, F_2 \in [\mathbb{N}]^{<\infty}_{>0}$ we define $F_1 \prec F_2$if $max\, F_1 < minF_2$. Then $([\mathbb{N}]^{<\infty}_{>0}, \prec)$ is an infinite directed set.

For a sequence $(F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}$ such that $F_n \prec F_{n+1}$ for every $n \in \mathbb{N}$ we set $FU((F_n)_{n\in\mathbb{N}}) = \{U_{i\in\alpha} F_i : \alpha \in [\mathbb{N}]^{<\infty}_{>0}\}$. The family

$$ß = \{ FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}\ with\ F_1 \prec F_2 \prec \cdots \}$$

is a coideal basis on $([\mathbb{N}]^{<\infty}_{>0}, \prec)$, according to the fundamental theorem of Hind- man.

This coideal basis has not the (P)-property, but it has the (D)-property. Indeed, let a sequence $(A_k)_{k\in N}$, with $A_k \in ß$ and $A_1 \supseteq A_2 \supseteq$ .... If $A_k =$

$\mathrm{FU}\left((\mathrm{F}_{\mathrm{n}}^{\mathrm{k}})_{\mathrm{n}\in\mathbb{N}}\right)$, where $(F_n^k)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]_{>_0}^{<\infty}$,with $F_1^k \prec F_2^k \prec \cdots$ for every $\mathrm{k} \in \mathbb{N}$, then we set $A = FU\left((F_k^k)_{n\in N}\right)$.Then $\boldsymbol{A} \in ß$ and for every $\mathrm{k} \in \mathbb{N}$.

$$k-1 = max\{ n \in \mathbb{N}: there\ exist\ \ F_1 \prec \cdots \prec F_n \in A \setminus A_k\ with\ F_1 \prec \cdots \prec F_n\}.$$

(iii) Let $\sum = \{\alpha_1, \alpha_2, \dots\}$ be an infinite countable alphabet and $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. The set of ω-located words over $\sum$ dominated by $\vec{k}$ is

$$L(\sum, \vec{k}) = \left\{w = w_{n_1} \dots w_{n_l}: l \in \mathbb{N}, n_1 < \cdots < n_l \in \mathbb{N}, w_{n_i} \in \left\{\alpha_1, \dots, \alpha_{k_{n_i}}\right\}\ for\ all\ 1 \le i \le l\right\}$$

Let $v \notin \Sigma$ be a variable. The set of variable ω-located wordsover Σ dominated by the sequence $\vec{k}$ is:

$$L(\sum, \vec{k}; v) = \left\{w = w_{n_1} \dots w_{n_l}: l \in \mathbb{N}, n_1 < \cdots < n_l \in \mathbb{N}, w_{n_i} \in \left\{v, \alpha_1, \dots, \alpha_{k_{n_i}}\right\}\ for\ all\ 1 \le i \le l\ and\ there\ exists\ 1 \le i \le l\ \ with\ \ w_{n_i} = v\right\}$$

Let $L(\sum\cup \{v\}, \vec{k}) = L(\sum, \vec{k}) \cup L(\sum, \vec{k}; v)$.

If $w = w_{n_1} \ . \ . \ . \ w_{n_l} \in L(\sum\cup \{v\}, \vec{k})$ then the set $dom\, w = \{ n_1, \dots, n_l\}$ the domain of w For $w, u \in \boldsymbol{L}(\sum\cup \{v\}, \vec{k})$ we define $w \prec u$ if $max\ dom, (w) < min\ dom(u)$. Then $L\left(\sum\cup \{v\}, \vec{k}\right), \prec), (L(\sum, \vec{k}), \prec)$ and $(L(\sum, \vec{k}; v), \prec)$ are infinite directed sets.

For $w = w_{n_1}, \ldots, w_{n_l}, u = u_{m_1}, \ldots, u_{m_l} \in L\left(\sum \cup \{v\}, \vec{k}\right)$ $with\, w \prec u$ we define the concatenating word $w \star u = w_{n_1}, \ldots w_{n_l}, u_{m_1}, \ldots, u_{m_l} \in L\left(\sum \cup \{v\}, \vec{k}\right)$.

For $w = w_{n_1}, \ldots, w_{n_l} \in L\left(\sum \cup \{v\}, \vec{k}\right) L(\sum, \vec{k}; v)$ and $p \in \mathbb{N} \cup \{0\}$ we set $w(0) = w\; and, for\; p \in \mathbb{N}$,

$$w(p) = u_{n_1}, \ldots, u_{n_l} \in L\left(\Sigma, \vec{k}\right)$$

where, for $1 \leq i \leq l, u_{n_i} = w_{n_i} if\; w_{n_i} \in \sum, u_{n_i} = \alpha_p \; if\; w_{n_i} = v$ and $p \leq k_{n_i}$ and finally $u_{n_i} = \alpha_{k_{n_i}} if\; w_{n_i} = v\; and\; p > k_{n_i}$. Let

$$L^{\omega}\left(\Sigma, \vec{k}; v\right) = \left\{(w_n)_{n \in \mathbb{N}} \subseteq L\left(\Sigma, \vec{k}; v\right)\right\} : w_n \prec w_{n+1}\; for\; every \quad n \in \mathbb{N}$$

We will define now the extracted (variable) ω-located words of a sequence $\vec{\omega} = (w_n)_{n \in \mathbb{N}} \in L^{\omega}(\sum, \vec{k}, v)$. An extracted variableω-located word of runhas the form

$$w_{n_1}(p_1) \star \ldots \star w_{n_\lambda}(p_\lambda) \in L\left(\Sigma, \vec{k}; v\right),$$

Where $\lambda \in \mathbb{N}, n_1 < \cdots < n_\lambda \in \mathbb{N}$ and $p_1, \ldots, p_\lambda \in \mathbb{N} \cup \{0\}$ with $0 \leq p_i \leq k_{n_i}$ for every $1 \leq i \leq \lambda$ and $0 \in \{p_1, \ldots, p_\lambda\}$. The set of all the extracted variable ω-located words of $\vec{\omega}$ is denoted by $EV(\vec{\omega})$.

An extractedω-located word of $\vec{\omega}$has the form

$$w_{n_1}(p_1) \star \ldots \star w_{n_\lambda}(p_\lambda) \in L\left(\Sigma, \vec{k}\right),$$

Where $\lambda \in \mathbb{N}, n_1 < \cdots < n_\lambda \in \mathbb{N}$ and $p_1, \ldots, p_\lambda \in \mathbb{N}$with $0 \leq p_i \leq k_{n_i}$ for every $1 \leq i \leq \lambda$ **.** The set of all the extracted ω-located words of $\vec{\omega}$ is denoted by $E(\vec{\omega})$. The families

$$ß = \{E(\vec{\omega}): \vec{\omega} = \{(\boldsymbol{w_n})_{\boldsymbol{n}\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; v)\}\text{and}$$

$$ß_1 = \{EV(\vec{\omega}): \vec{\omega} = \{(\boldsymbol{w_n})_{\boldsymbol{n}\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; v)\}$$

are coideal bases on $(L(\Sigma, \vec{k}), \prec)$ and $(L(\Sigma, \vec{k}; v), \prec)$ respectively, according to a fundamental partition theorem of Carlson proved in [2] for the particular case of a finite alphabet.

These coideal bases have not the (P)-property, but they have the (D)-property. Indeed, let a sequeoce $(A_k)_{n\in\mathbb{N}}$, with $A_k = E(\vec{\omega}_k)$, where $\vec{\omega}_k = (w_n^k)_{\boldsymbol{n}\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; v)$and $A_1 \supseteq A_2 \supseteq \cdots let \vec{\omega} = (w_n^k)_{\mathbf{n}\in\mathbb{N}} \in L^\omega(\sum, \vec{k}; v)$ We set $A = E(\vec{\omega})$. Then $A \in ß$. Moreover, for every $k \in \mathbb{N}$,

$$k - 1 = max\ \{n \in \mathbb{N}: there\ exist\ w_1, \ldots, w_n \in A \setminus \boldsymbol{A_k} with w_1 \prec \cdots \prec w_n\}.$$

Hence, ß has the (D)-property. Analogously, it can be proved that $ß_1$ has the (D)-property.

We give more examples of coideal bases on directed sets with the (D)-property.

After the definition of the coideal bases on directed sets with the (D)property. It is well known that every net $(x_\lambda)_{\lambda\in\Lambda}$in a compact metric space has a convergent subnet. We will prove, in the following theorem, that this suibnet can have the form $(x_\lambda)_{\lambda\in A}$, where A is an element of an arbitrary coideal basis ß on Λ with the (D)-property, and moreover A can be a subset of a given element B of ß.

Let $(\Lambda, \prec)$ be an infinite directed set and $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ be a net in a topological space X. For $x_0 \in X$,we write

$$\lim_{\lambda\in\Lambda} x_\lambda = x_0,$$

if $(x_\lambda)_{\lambda\in\Lambda}$converges to x_0,i.e. if for any neighborhood V ofx_0,there exists$\lambda_0 \equiv \lambda_0(V) \in \Lambda$ such that $x_\lambda \in V$ for every$\lambda \in \Lambda$ with $\lambda_0 \prec \lambda$.

Analogously, we write for an element A of a coideal basis ßon $(\Lambda, \prec)$ and $x_\lambda \in X$

$$\lim_{\lambda\in A} x_\lambda = x_0,$$

If the net $(x_\lambda)_{\lambda\in A}$ converges to x_0, i**.e.** if for any neighborhood Vof x_0, there exists $\lambda_0 \equiv \lambda_0(V) \in A$ such that $x_\lambda \in V$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$.

Theorem (2.1.8)[2]:

Let X, d be a compact metric space, $(\Lambda, \prec)$ an infinite directed set and let $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ be a net in X. For every coideal basis ß on $(\Lambda, \prec)$ with the (D)-property and every $B \in$ ß there exists $A \in$ ß with $A \subseteq B$ such that the subnet $(x_\lambda)_{\lambda\in A}$ of $(x_\lambda)_{\lambda\in\Lambda}$ to converge to some element of X.

Proof: Let ß be a coideal basis on $(\Lambda, \prec)$ with the (D)-property and $B \in$ ß. We set $\hat{B}(x, \epsilon) = \{ y \in X: d(x, y) \leq \epsilon\}$ for every $x \in X$ and $\epsilon > 0$. Since

$(X.d)$ is a compact metric space, we have that $X = \bigcup_{i=1}^{m_1} \hat{B}\left(x_i^1, \frac{1}{2}\right)$for some a$x_1^1, \dots, x_{m_1}^1 \in X$.

Let $A_1 = B$. Since $A_1 = \bigcup_{i=1}^{m_1} C_i$ where $C_i = \left\{\lambda \in A_1 : x_\lambda \in \hat{B}\left(x_i^1, \frac{1}{2}\right)\right\}$, and ß is a coideal basis, there exist $A_2 \in$ ß , $A_2 \subseteq A_1$and $1 \leq i_1 \leq m_1$such that $A_2 \subseteq C_{i_1}$ and consequently $\{x_\lambda : \lambda \in A_2\} \subseteq \hat{B}\left(x_{i_1}^1, \frac{1}{2}\right)$ .We continue analogously. Since $\hat{B}\left(x_{i_1}^1, \frac{1}{2}\right)$ is a compact space, there exist $x_1^2, \dots x_{m_2}^2 \in X$, such that $\hat{B}\left(x_{i_1}^1, \frac{1}{2}\right) \subseteq \bigcup_{i=1}^{m_2} \hat{B}\left(x_{i_2}^2, \frac{1}{4}\right)$ and consequently there exist $A_3 \in$ ß , $A_3 \subseteq A_2$and $1 \leq i_2 \leq m_2$such that $\{x_\lambda : \lambda \in A_3\} \subseteq \hat{B}\left(x_{i_1}^1, \frac{1}{2}\right) \cap \hat{B}\left(x_{i_2}^2, \frac{1}{4}\right)$.

Inductively, we construct a sequence$(A_n)_{n\in \mathbb{N}}$, with $A_n \in$ ß and $A_1 \supseteq A_1 \supseteq \cdots$, and also closed balls $\hat{B}\left(x_{i_j}^n, \frac{1}{2^j}\right)$. for $n \in \mathbb{N}$, such that

$$k_n = max\{k \in \mathbb{N} : there\ exist\ \lambda_1, \dots, \lambda_k \in C \backslash A_n\ with \lambda_1 \prec \cdots \prec \lambda_k\}.$$

This implies that $C \setminus A_1$ does not contain an element of ß. So, Since $C \in$ ß and $C = (C \backslash A_1) \cup (C \cap A_1)$, there exists $A \in$ ß$, A \subseteq C \cap A_1$. Then $A \subseteq C, A \in$ ß $, A \subseteq B = A_1$ and for every $n \in \mathbb{N}, n > 1$ there exists $q_n \in \mathbb{N} \cup \{0\}$ such that

$$q_n = max\{k \in \mathbb{N} : there\ exist\ \lambda_1, \dots, \lambda_k \in A \backslash A_n\ with \lambda_1 \prec \cdots \prec \lambda_k\}.$$

We will prove that $\lim_{\lambda \in A} x_\lambda = x_0$. Indeed, for $\epsilon > 0$ pick $n_0 \in \mathbb{N}$. such that $1 \backslash 2^{n_0} < \epsilon$ Then $d(x_\lambda, x_0) \leq 1 \backslash 2^{n_0} < \epsilon$ for every $\lambda \in A_{n_0+1}$. Let $\lambda_1, \dots, \lambda_{q_{n_0+1}} \in A \backslash A_{q_{n_0+1}} with \lambda_1 \prec \cdots \prec \lambda_{q_{n_0+1}}$Since does not exist $\lambda \in$

$A \setminus A_{q_{n_{0+1}}}$ with $\lambda_{q_{n_{0+1}}} \prec \lambda$, and $A \in ß$, there exists $\lambda_0 \in A \cap A_{n_{0+1}}$ such that $\lambda_{q_{n_{0+1}}} \prec \lambda_0$. Hence, for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ we have that $\lambda \in A_{n_{0+1}}$ and consequently that $d(x_\lambda, x_0) \leq 1 \setminus 2^{n_0} < \epsilon$. This finishes the proof.

The particular case of Theorem (2.1.8) for the directed set $([N]^{<\infty}_{>0}, \prec)$ and the coideal basis ß referred in Example (2.1.7) (ii) was proved by Furstenberg and Weiss .

Also, the particular case of Theorem (2.1.8) for the directed set $\left(L(\sum, \vec{k}; v), \prec\right)$ of ω-located words and the coideal basis ß referred in Example (2.1.7) (iii) was proved by Farmaki and Koutsogiannis.

Section (2.2): Topological Dynamical Systems Indexed by a Directed Partial Semigroup and Applications to Semigroups with Digital Representation:

We will introduce the notion of a directed partial semigroup and consequently the suitable coideal bases on a directed partial semigroup and the topological dynamical systems indexed by a directed partial semigroup. We show recurrence results for topological dynamical systems indexed by a directed partial semigroup with respect to a suitable coideal basis for this semigroup extending the recurrence theorem of Birkhoff, Purstenberg-Weiss.

We start with the following

Definition (2.2.1)[2]:

Let $(\Lambda, \prec)$ be an infinite directed set and let for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$ is defined a unique element $\lambda_1 * \lambda_2 \in \Lambda$. If for every $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2 \prec \lambda_3$ hold $\lambda_1 \prec \lambda_2 * \lambda_3, \lambda_1 * \lambda_2 \prec \lambda_3$ and $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$, than $(\Lambda, \prec, *)$ is called a directed partial semigroup.

We will define the suitable coideal bases on a directed partial semigroup.

Definition (2.2.2)[2]:

Let $(\Lambda, \prec)$ be a directed partial semigroup. A coideal basis ß on $(\Lambda, \prec)$ is suitable for $(\Lambda, \prec, *)$ if every $B \in B$ has the property that $\lambda_1 * \lambda_2 \in B$ for every $\lambda_1, \lambda_2 \in B$ with $\lambda_1 \prec \lambda_2$.

Obviously, if a coideal basis ß is suitable for the directed partial semigroup $(\Lambda, \prec, *)$, then $(B, \prec, *)$ is also a directed partial semigroup for ever $B \in B$.

We will define a topological dynamical system indexed by a directed partial semigroup.

Definition (2.2.3)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigroup. A family $\{T^{\lambda}\}_{\lambda \in \Lambda}$ of continuous functions from a compact metric space X into itself is a Λ-topological dynamical system of X if $T^{\lambda_1} \circ T^{\lambda_2} = T^{\lambda_1 * \lambda_2}$, for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$.

Obviously, if ß is a suitable coideal basis for $(\Lambda, \prec, *)$and $B \in B$, then the family $\{T^{\lambda}\}_{\lambda \in B}$is also a topological dynamical system of X.

Examples (2.2.4)[2]:

Let X be a compact metric space.

(i) Let $T : X \to X$ be a continuous map. Then $\{T^n\}_{n\in\mathbb{N}}$ is a $\mathbb{N}$-topological dynamical system of X.

(ii) According to Example (2.1.7)(2), $([\mathbb{N}]_{>_0}^{<\infty}, \prec)$is an infinite directed set. So, $([\mathbb{N}]_{>_0}^{<\infty}, \prec, U)$ is a directed partial semigroup and the coideal basis ß defined there is a suitable coideal basis for this semigroup. For each $n \in \mathbb{N}$, let $T_n : X \to X$ be a continuous map from a compact metric space X into itself.

For $F = \{n_1 < \cdots < n_k\} \in [\mathbb{N}]_{>_0}^{<\infty}$ $we\ set\ T^F = T_{n_1} \circ \ldots \circ T_{n_k}$.Then $\{\mathrm{T}^{\mathrm{F}}\}_{\mathrm{F}\in[\mathbb{N}]_{>_0}^{<\infty}}$ is a $[\mathbb{N}]_{>_0}^{<\infty}$-topological dynamical system of X. In particular, we can replaceT_n, with T^n for every $n \in \mathbb{N}$, Where $T : X \to X$ is a continuous map.

(iii) Let $\sum = \{\alpha_1, \alpha_1, \ldots\} \subseteq \mathbb{N}$ be an infinite countable alphabet and $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. According to Example (2.1.7)(3), $(\mathrm{L}(\sum, \vec{k}) \prec, \star)$ is a directed partial semigroup and the coideal basis ß defined there is a sinkable coideal basis for this semigroup. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of continuous maps from X into itself and let $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$. For $w = w_{n_1} \ldots w_{n_\lambda} \in L(\sum, \vec{k})$ let

$$T^\omega = T_{n_1}^{l_{n_1} w_{n_1}} \circ \ldots \circ T_{n_\lambda}^{l_{n_\lambda} w_{n_\lambda}}$$

Then $\{T^{\omega}\}_{\omega \in L(\sum, \vec{k})}$ is an $L(\sum, \vec{k})$-topological dynamical system of X.

We define the recurrent points of a topological dynamical system indexed by a directed partial semigroup, with respect to a suitable coideal basis for the semigroup. Consequently, using Theorem (2.1.8), we prove the existence of such points in case the coideal basis has the (D)-property.

Definition (2.2.5)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ-topological dynamical system of a compact metric space (X, d), ß a suitable coideal basis on $(\Lambda, \prec, *)$ and let $B \in$ ß. An element x_0 of X is called B- recurrent if

$$\lim_{\lambda \in A} T^{\lambda}(x_0) = x_0, for\ some\ A \in ß\ with\ A \subseteq B$$

Theorem (2.2.6)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigrup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ-topological dynamical system of a compact metric space (X, d), ß a suitable coideal basis for $(\Lambda, \prec, *)$ with the **(D)-**property and let $B \in$ ß. Then X contains B-recurrent points.

Proof: Let $x \in X$. According to Theorem (2.1.8), there exist $A \in ß, A \subseteq B$ and .$x_0 \in X$ such that

$$\lim_{\lambda \in A} T^{\lambda}(x) = x_0.$$

Let $\varepsilon > 0$. Then, there exists $\lambda_0 \in A$ such that $d(T^{\lambda}(x), x_0) < \varepsilon/2$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ We fix $\lambda_1 \in A$ with $\lambda_0 \prec \lambda_1$. Since T^{λ_1} is a continuous function on X, there exists $\delta > 0$ such that if $y \in X$ with $d(y, x_0) < \delta$ then $\left(T^{\lambda_1}(y), T^{\lambda_1}(x_0)\right) < \varepsilon/2$. Since $\lim_{\lambda \in A} T^{\lambda}(x) = x_0$, there exists $\lambda_2 \in A, \lambda_1 \prec \lambda_2$ such that $d\left(T^{\lambda_2}(x), x_0\right) < \delta.$ Then $d\left(T^{\lambda_1}\left(T^{\lambda_2}(x)\right)(y), T^{\lambda_1}(x_0)\right) < \varepsilon/2$ and consequently $d\left(T^{\lambda_{1*}\lambda_2}(x), T^{\lambda_1}(x_0)\right) < \varepsilon/2$. Since $\lambda_{1*}\lambda_2 \in A$ and $\lambda_0 < \lambda_{1*}\lambda_2$, we have that $d\left(T^{\lambda_{1*}\lambda_2}(x), (x_0)\right) < \varepsilon/2.$ It follows that $d\left(T^{\lambda_{1*}\lambda_2}(x), (x_0)\right) < \varepsilon$. Hence, $\lim_{\lambda \in A} T^{\lambda}(x_0) = x_0.$

The particular case of this theorem, where $\Lambda = \mathbb{N}$, ß $= [\mathbb{N}]^{\infty}$ and the topological dynamical system has the form $\{T^n\}_{n \in \mathbb{N}}$ where T is a continuous function from a compact metric space (X, d) to itself, is Birkhoff's recurrence theorem (Let $T : X \to X$ be a homeomorphism, then there exist a point $x \in X$ such that $\liminf_{n\to\infty} d(T^n x, x) = 0$ [5]).

Corollary (2.2.7)[2]:

(Birkhoffs theorem)[5]: Let X be a compact metric space and $T: X \to X$ a continuous function. There exists $x_0 \in X$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{\lambda \in \mathbb{N}} T^{n_k}(x_0) = x_0$.

We locate recurrent points of a topological dynamical system indexed by a directed partial semigroup, with respect to a suitable coideal basis for this semigroup, in a given subset of the space. Firstly, we will look for almost recurrent points, as their class is wider than the class of recurrent points.

Defection (2.2.8)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ- topological dynamical system of a compact metric space (X, d), ß a suitable coideal basis for $(\Lambda, \prec, *)$ and let $B \in$ ß. An element x_0 of X is called B-almost recurrent if for every $\varepsilon > 0$and $\lambda_0 \in \Lambda$, there exist $\lambda \in B$ and $\lambda_0 \prec \lambda$ such that

$$d\left(T^{\lambda}(x_0), x_0\right) < \varepsilon$$

A closed subset F of X is called B-almost recurrent setif for every $\varepsilon > 0$,$\lambda_0 \in \Lambda$ and $x \in F$ there exist, $y \in F$, $and \lambda \in B \lambda_0 \prec \lambda$ such that

$$d\left(T^{\lambda}(y), x\right) < \varepsilon$$

In the following example we will point out a way to locate almost recurrent subsets of a compact metric space.

Example (2.2.9)[2]:

Let (X, d) be a compact metric space, $(\Lambda, \prec, *)$ a directed partial semigroup, ß a suitable coideal basis on $(\Lambda, \prec)$ with the (D)-property and let B $\in$ ß. Let $F(X)$ he the set of all nonempty closed subsets of X endowed with the Hausdorff metric $\tilde{d}$, where

$$\tilde{d}(K, M) = max\{sup_{x \in K} d(x, M), sup_{x \in M} d(x, K)\}.$$

Then $(F(X), \tilde{d})$ is a compact metric space. Let $\{T^{\lambda}\}_{\lambda \in \Lambda}$ be a Λ-topological dynamical system of (X, d). We define $\tilde{T}^{\lambda} : F(X) \rightarrow F(X)$ with $\tilde{T}^{\lambda}(K) = T^{\lambda}(K)$. Then $\{\tilde{T}^{\lambda}\}_{\lambda \in \Lambda}$ is a Λ-topological dynamical system of $(F(X), \tilde{d})$.

According to Theorem (2.2.6), there exist $A \in ß, A \subseteq B$ and $K \in F(X)$ such that

$$\lim_{\lambda \in A} \tilde{T}^{\lambda}(K) = K.$$

Then K is a B-recurrent element of $F(X)$ and K is a B-almost recurrent subset of *X***.**

We prove now that every almost recurrent subset of *X* contains almost recurrent elements of *X*.

Proposition (2.2.10)[2]:

Let $(\Lambda, \prec, *)$ he a directed partial semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$a Λ-topological dynamical system of a compact metric space $(X.d)$, ß a suitable coideal basis on $(\Lambda, \prec, *)$ and let $B \in ß$. Every B-almost recurrent subset *F* of *X* contains B-almost reccirrent elements of *X*.

Proof: Let *F* be a B-almost recurrent subset of *X*. We fix $\varepsilon > 0$ and $\lambda_0 \in \Lambda$. Inductively, we will construct a sequence $(x_n)_{n\in\mathbb{N}} \subseteq F$a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq B$ with $\lambda_n \prec \lambda_{n+1}$and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < \varepsilon/2$, which satisfy $d\left(T^{\lambda_{n+1}}(x_{n+1}), x_n\right) < \varepsilon_{n+1}$ and $d\left(T^{\lambda_n}(x), x_{n-1}\right) < \varepsilon_n$ whenever $x \in X$ and$d(x, x_n$ for every $n \in \mathbb{N}$.

Indeed, since *F* is B-almost recurrent, for $x_0 \in F$and $\varepsilon_1 = \varepsilon/2$there exist $\lambda_1 \in B$ with $\lambda_0 \prec \lambda_1$ and $x_1 \in F$ such that $d\left(T^{\lambda_1}(x_1), x_0\right) < \varepsilon_1$**.**

Let there exist $x_0, x_1, \dots, x_n \in F \lambda_1, \dots, \lambda_n \in B$with $\lambda_1 \prec \cdots \prec \lambda_n$ and $0 < \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n < \varepsilon/2$ such that $d\left(T^{\lambda_i}(x_i), x_{i-1}\right) < \varepsilon_i$. for every $i = 1, \dots, n$.

Since $T^{\lambda_n}$is a continuous function, there exists $0 < \varepsilon_{n+1} \leq \varepsilon_n$ such that if $x \in X$ and $d(x, x_n)<\varepsilon_{n+1}$, then $d\left(T^{\lambda_n}(x), T^{\lambda_n}(x_n)\right) < \varepsilon_n - d\left(T^{\lambda_n}(x_n), (x_{n-1})\right)$. So, whenever $d(x, x_n)<\varepsilon_{n+1}$ we have that

$$d\left(T^{\lambda_n}(x), (x_{n-1})\right) \leq d\left(T^{\lambda_n}(x), T^{\lambda_n}(x_n)\right) + d\left(T^{\lambda_n}(x), (x_{n-1})\right) < \varepsilon_n.$$

Since F is B-almost recurrent there exists $\boldsymbol{\lambda_{n+1}} \in B$ with $\boldsymbol{\lambda_n} \prec \boldsymbol{\lambda_{n+1}}$ and $\boldsymbol{x_{n+1}} \in F$ such that $d\left(T^{\lambda_{n+1}}(x_{n+1}), x_n\right) < \varepsilon_{n+1}$. This finishes the construction.

We will prove that if $i, j, \in \mathbb{N}$and $i < j$, then

$$d\left(T^{\lambda_i+1*\ldots*\lambda_j}(x_j)x_i\right) < \varepsilon_{i+1}$$

Indeed, since $d\left(T^{\lambda_j}(x_j)x_{j-1}\right) < \varepsilon_j$ we have that $\left(T^{\lambda_{j-1}}\left(T^{\lambda_j}(x_j)\right), x_{j-2}\right) < \varepsilon_{j-1}$, and, since $\lambda_{j-1} \prec \lambda_j$ we have that $d\left(T^{\lambda_j-1*\lambda_j}(x_j), x_{j-2}\right) < \varepsilon_{j-1}$. Repeating the same procedure we obtain that $d\left(T^{\lambda_{i+1}*\ldots*\lambda_j}(x_j), x_i\right) < \varepsilon_{i+1} \leq \varepsilon_1 = \varepsilon/2$

Since $X$is a compact space, there exist $i, j, \in \mathbb{N}$ with $i < j$, such that $d\left(x_i, x_j\right) < \varepsilon/2$.Then,

$$d\left(T^{\lambda_{i+1}*\ldots*\lambda_j}(x_j), x_i\right) \leq d\left(T^{\lambda_{i+1}*\ldots*\lambda_j}(x_j), x_i\right) + d\left(x_i, x_j\right) < \varepsilon$$

For $\boldsymbol{x} = \boldsymbol{x_j}$and $\boldsymbol{\lambda} = \boldsymbol{\lambda_{i+1}} * \ldots * \boldsymbol{\lambda_j} \in B$ we have that $\lambda_0 \prec \lambda$ and $d\left(T^{\lambda}(x), x\right) < \varepsilon$

We define the recurrent subsets of a compact metric space with respect to a topological dynamical system indexed by a directed partial semigroup, in order to locate recurrent elements in them.

Definition (2.2.11)[2]:

Let $(\Lambda, \prec, *)$ he a directed partial semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ-topological dynamical system of a compact metric space (X, d) and ß a suitable cojdeal basis on $(\Lambda, \prec, *)$ .A closed subset F of X is called B-recurrent for $B \in$ ß if for every $\boldsymbol{\varepsilon} > 0$ and $x \in F$, there exist $A \in$ ß with $A \subseteq B$ and $y \in F$ such that

$$\boldsymbol{d}(\boldsymbol{T}^{\boldsymbol{\lambda}}(\boldsymbol{y}), \boldsymbol{x}) < \boldsymbol{\varepsilon}, \qquad for\ every\ \lambda \in A$$

A closed subset F of X is called recurrentifitis B-recurrent for every $B \in$ ß.

Obviously, a B-recurrent subset of X, for $B \in$ ß, is B-almost recurrent and, according to Proposition (2.2.10), it contains B-almost recurrent points. As we will prove in Proposition (2.2.18) below, we can locate B-recurrent points in a homogenous B-recurrent subset of X toward Proposition (2.2.18), we will give the appropriate definitions starting from the definition of a minimal topological dynamical system.

Definition (2.2.12)[2]:

Let X be a compact metric space, $(\Lambda, \prec, *)$ a directed partial semigroup and $\{T^{\lambda}\}_{\lambda \in \Lambda}$a Λ-topological dynamical system of (X, d)**.** This system is minimalif no proper closed subset $Y \subset X is T^{\lambda}$-invariant for every $\lambda \in \Lambda$.

Using Zorn's lemma (Let S be a partially ordered set. If every totally rderedsubset of S has an upper bound, then S contains a maximal element. [6]), can be proved that there exists a closed non-empty subset Y of X such that the system $\{T^{\lambda}\}_{\lambda \in \Lambda}$ restricted to Y to be minimal. There exists the following characterization of minimality in case A is a semigroup.

Proposition (2.2.13)[2]:

Let X be a compact metric space, G a semigroup and let $\{T^{g}\}_{g\in G}$ be a G-topological dynamical system of X. The dynamical system $\{T^{g}\}_{g \in G}$ is minimal if for every open subset U of X, there exist finitely many elements$g_1, g_2, \dots, g_n \in G$ such that

$$\bigcup_{i=1}^{n} (T^{g_i})^{-1} (U) = X.$$

We give a homogenous subset of X with respect to a set $\{T_i\}_{i \in I}$ of transformations acting on X, which introduced by Furstenberg as follows:

Definition (2.2.14)[2]:

Let X be a compact metric space and F a closed subset of X. Then F is called homogeneous with respect to a set of transformations $\{T_i\}_{i \in I}$acting on X if there exists a group of homeomorphisms G of X each of which commutes with each T_i and such that G leaves F invariant and (F, G) is minimal (no proper closed subset of F is invariant under the action of G).

We prove that a homogeneous subset of X is recurrent if satisfies a weaker condition than that in Definition (2.2.11).

Proposition (2.2.15)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial sernigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ-topological dynamical system of a compact metric space (X, d), ß a suitable coideal basis on $(\Lambda, \prec, *)$ and let $B \in$ ß. If a closed subset F of X is homogeneous with respect to the system $\{T^{\lambda}\}_{\lambda \in \Lambda}$ and for every $\boldsymbol{\varepsilon}$>0 there exist .$x, y \in F$ and $A \in$ ß, $A \subseteq B$ such that

$$d\left(T^{\lambda}(y), x\right) < \varepsilon \quad for\ every\ \lambda \in A$$

then, F is B-recurrent.

Proof: Since F is a homogeneous set with respect to the system $\{T^{\lambda}\}_{\lambda \in \Lambda}$ there exists a group G of homeomorphisms each of which commutes with each $T^{\lambda}$and such that G leaves F invariant and (F, G) is minimal.

We claim that for every $\boldsymbol{\varepsilon}$> 0 there exists a finite subset G_0 of G such that, for every $x, y \in F, min_{g \in G_0} d(g(x), y) < \varepsilon/2$.

Indeed, let$\{\boldsymbol{U_i}\}_{\boldsymbol{i=1}}^{\boldsymbol{k}}$ be a finite covering of F by open sets of diameter < $\varepsilon/2$. According to Proposition (2.2.13), we can find a finite set $\{g_1^i, \dots, g_{m_i}^i\}$ for every $1 \leq i \leq k$ such that $U_{j=1}^{m_i}(g_{\mathrm{j}}^i)^{-1}U_i = F$. Let $G_0 = \left\{g_{\mathrm{j}}^i : 1 \leq i \leq k, 1 \leq j \leq k \leq m_i\right\}$. Then for every $x, y \in F$ we have that $y \in U_{i_o}$ forsome$i_o \in \{1, \dots, \kappa\}$ and $x \in (g_{j_o}^{i_o})^{-1}U_{i_o}$ for some $j_o \in \{1, \dots, m_{i_0}\}$. Then $g_{j_o}^{i_o}(x) \in U_{i_o}$ and, since U_{i_o} has diameter < $\varepsilon/2$, we have that $min_{g \in G_0} d(g(x), y) \leq d\left(g_{j_o}^{i_o}(x), y\right) < \varepsilon/2$. This proves the claim.

Let $\boldsymbol{\varepsilon}$>0 and $z \in F$ There exists δ> 0 such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$. then $d(g(x_1), g(x_2)) < \varepsilon/2$ for every $g \in G_0$. According to our hypothesis, there exist $x, y \in F$ and $A \in ß, A \subseteq B$ such that $d(T^{\lambda}(y), x) < \delta$ for ever $\lambda \in A$. Then $d(g(T^{\lambda}(y)), g(x)) < \varepsilon/2$ for every $g \in G_0$.and $\lambda \in A$. Since each $g \in G_0$ commutes with each $T^{\boldsymbol{\lambda}}$ we have that $d(T^{\lambda}(g(y)), g(x)) = d(g(T^{\lambda}(y)), g(x)) < \varepsilon/2$ for every $g \in G_0$ and $\lambda \in A$.

According to our claim, there exists $g \in G_0$ such that $d(g(x), z) < \varepsilon/2$. Then $d(T^{\lambda}(g(y)), z) \leq d(T^{\lambda}(g(y)), g(x)) + d(g(x), z) < \varepsilon$ for every $\lambda \in A$. Hence F is B-recurrent, since $A \in ß, A \subseteq B$ and $g(y) \in F$. Inthesequel of the previous proposition we have the following:

Proposition (2.2.16)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial, semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$a Λ-topological dynamical system of a compact metric space (X, d), ß a suitable coideal basis on $(\Lambda, \prec, *)$ and let $B \in ß$. If a subset F of X is homogeneous with respect to the system, $\{T^{\lambda}\}_{\lambda \in \Lambda}$and B-recurrent, then for every $\boldsymbol{\varepsilon}$>0 there exist an element $x_0 \in F$ and $A \in ß, A \subseteq B$ such that

$$d(T^{\lambda}(x_0)x_0) < \varepsilon, \quad for\, every\, \lambda \in A$$

Proof: Since F is a homogeneous set with respect to the system $\{T^{\lambda}\}_{\lambda \in \Lambda}$, there exists a group G of homeomorphisms each of which commutes with each T^{λ} and such that G leaves F invariant and (F, G) is minimal. Using the homogeneity as in the previous proposition, we have that for every $\boldsymbol{\varepsilon}$> 0 there exists a finite subset G_0 of G such that, for every $x_1, x_2 \in F, min_{g \in G_0} d(g(x_1), x_2) < \varepsilon/2$.

Let $\boldsymbol{\varepsilon} > 0$. There exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$, then $d(g(x_1), g(x_2)) < \varepsilon/2$ for every $g \in G_0$. Let $x \in F$. Since F is B-recurrent, there exist $A \in ßwithA \subseteq B$ and $y \in F$ such that $d(T^\lambda(y), x) < \delta$. for every $\lambda \in A$.

Then, since each $g \in G_0$ commutes with each T^λ, we have that

$$d\left(T^\lambda(g(y)), g(x)\right) = d\big(g(T^\lambda(y)), g(x)\big) < \varepsilon/2$$

for every $g \in G_0$ and $\boldsymbol{\lambda} \in A$.

Let $g \in G_0$ such that $d(g(x), g(y)) < \varepsilon/2$. Then for every $\lambda \in A$ we have

$$d\left(T^\lambda(g(y)), g(y)\right) \leq d\left(T^\lambda(g(y)), g(x)\right) + d(g(x), g(y)) < \varepsilon$$

Set $x_0 = g(y) \in F$.

We will prove, in Proposition (2.2.18) below that, in case the coideal basis ß has the (D)-property and the set $\{T^\lambda : \lambda \in \Lambda\}$ is equicontinuous, the set of all the B-recurrent elements of a homogeneous recurrent subset F of X, for $B \in$ ß, is a dense subset of F.

Definition (2.2.17)[2]:

We say that a set $\{T_i\}_{i \in I}$of continuous functions from a compact metric space (X, d) to itself is equicontinuous**,** iffor every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ with $d(x, y) < \delta$, then $d\big(T^i(x), T^i(y)\big) < \boldsymbol{\varepsilon}$ for every $i \in I$.

Proposition (2.2.18)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^{\lambda}\}_{\lambda \in \Lambda}$ a Λ-topological dynamical system of a compact metric space (X, d) which is equicontniuous**,** ß a suitable coideal basis on $(\Lambda, \prec, *)$ with the **(**D**)**-property and let $B \in$ ß. Then every recurrent homogeneous subset F of X contains B-recurrent points. Moreover the set of all the B-recurrent points of F is a dense subset of F.

Proof: Let V he an open subset of X such that $V \cap F \neq \emptyset$. There exists an open set V' such that $V' \subseteq V, V' \cap F \neq \emptyset$ and $\delta > 0$ such that if $x \in X$ and $d(x, V') < \delta$, then $x \in V$.

Since F is homogeneous with respect to the system $\{T^{\lambda}\}_{\lambda \in \Lambda}$, there exists a group G of homeomorphisms commuting with $\{T^{\lambda}\}_{\lambda \in \Lambda}$ such that G leaves F invariant and (F, G) is minimal. According to Proposition (2.2.13), there exists a finite subset G_0 of G such that $F \subseteq U_{g \in G_0} g^{-1}(V')$.

Let $\boldsymbol{\varepsilon} > 0$ such that if $x_1, x_1 \in X$ with $d(x_1, x_2) < \boldsymbol{\varepsilon}$, then $d(g(x_1), g(x_2) < \delta$ for every $g \in G_0$**.** Since F is a recurrent and homogeneous subset of X, according to Proposition (2.2.16), there exist an element $x_0 \in F$ and $A \in$ ß, $A \subseteq B$ such that

$$d\left(T^{\lambda}(x_0), x_0\right) < \varepsilon, \qquad for\ every \lambda \in A$$

Let $g \in G_0$ such that $g(x_0) \in V'$. Then $d(T^{\lambda}(g(x_0), g(x_0)) < \delta$ for every $\lambda \in A$. Since $g(x_0) \in V'$, we have that $d\left(T^{\lambda}(x_0) \in V\right)$ for every $\lambda \in A$. Hence, for each open set V with $V \cap F \neq \emptyset$ there exist $A \in$ ß, $A \subseteq B$ and $x' = g(x_0) \in V \cap F$ such that $T^{\lambda}(x') \in V$ for every $\lambda \in A$.

Consequently, since $\{T^{\lambda}\}_{\lambda\in\Lambda}$ is equicontinuous, for every open set V with $V \cap F \neq \emptyset$ there exist $A \in ß, A \subseteq B$ and an open set V_1 such that

$$V_1 \cap \boldsymbol{F} \neq \emptyset, \overline{V_1} \subseteq V \text{and } \boldsymbol{T^{\lambda}}(V_1) \subseteq V \quad \text{for every } \lambda \in A$$

Let V_0 be an open subset of X such that $V_0 \cap F \neq \emptyset$. Inductively we construct a sequence $(V_n)_{n\in\mathbb{N}}$ of open sets and also a sequence $(A_n)_{n\in\mathbb{N}} \subseteq$ ß, with $B \supseteq A_1 \supseteq A_2 \supseteq \cdots$ such that for every $n \in \mathbb{N}$

$$\overline{V_n} \subseteq V_{n-1},\ V_n \cap F \neq \emptyset \ and \ T^{\lambda}(V_n) \subseteq V_{n-1} \quad for\ every\ \lambda \in A_n$$

We can also suppose that the diameter of V_n tends to 0. Let $\bigcap_{n\in\mathbb{N}} \overline{V_n} \cap F = \{x_0\}$ Then $x_0 \in V_0$ and we will prove that x_0 is a B-recurrent element of *F*.

Indeed, since the coideal basis ß has the (D)-property, there exists $C \in$ ß, such that for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N} \cup \{0\}$ such that

$$k_n = max\{k \in \mathbb{N}: thereexist \lambda_1, \dots, \lambda_k \in C/A_n \quad with \lambda_1 \prec \cdots \prec \lambda_k\}.$$

This implies that C/A_1 does not contain an element of ß. So, since $C \in$ ß, there exists$A \in ß, A \subseteq C \cap A_1$. Then$A \subseteq C, A \in ß, A \subseteq A_1 \subseteq B$and for every $n \in \mathbb{N}, n > 1$ there exists $q_n \in \mathbb{N} \cup \{0\}$ such that

$$q_n = max\{k \in \mathbb{N}: thereexist \lambda_1, \dots, \lambda_k \in A/A_n \quad with \lambda_1 \prec \cdots \prec \lambda_k\}.$$

We will prove that $\lim_{\lambda\in A} T^{\lambda}(x_0) = x_0$. Let $\boldsymbol{\varepsilon} > 0$. Since the diameter of V_n tends to 0, pick $n_0 \in \mathbb{N}, n_0 > 1$ such that the diameter of V_{n_0} to be less than$\boldsymbol{\varepsilon}$. Let $\lambda_1, \dots, \lambda_{q_{n_0+1}} \in A \backslash A_{n_0+1} with \lambda_1 \prec \cdots \prec \lambda_{q_{n_0+1}}$. Then there exists $\lambda_0 \in A \cap A_{n_0+1}$such that $\lambda_{q_{n_0+1}} \prec \lambda_0$. For every $\lambda \in A$ with $\lambda_0 \prec \lambda$ we have that $\lambda \in A \cap A_{n_0+1}$and consequently that $T^{\lambda}(x_0) \in V_{n_0}$ Since $x_0 \in V_{n_0}$, we have that $d(T^{\boldsymbol{\lambda}}(x_0), x_0) < \boldsymbol{\varepsilon}$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$. Hence, x_0 is a B-recurrent element of F and $x_0 \in V_0$ This finishes the proof.

Finally, using the previous proposition, we will prove, under some additional hypotheses, a multiple recurrence theorem analogous of the starting Theorem (2.2.6).

Theorem (2.2.19)[2]:

Let $(\Lambda, \prec, *)$ be a directed partial semigroup, ß a suitable coideal basis on $(\Lambda, \prec, *)$ with the (D)-property, $m \in \mathbb{N}$, $\{\boldsymbol{T}_1^{\boldsymbol{\lambda}}\}_{\lambda \in \Lambda}$, … , $\{\boldsymbol{T}_m^{\boldsymbol{\lambda}}\}_{\lambda \in \Lambda}$be Λ-topological dynamical systems of a compact metric space (X, d) all contained in o commatative group G of homeomorphisms of X, and let the systems $\{T_i^{\lambda}\}_{\lambda \in \Lambda}, \{(T_i^{\lambda})^{-1}\}_{\lambda \in \Lambda}$be equicontinuous for each$i = 1, \dots, m$. Then, for every $B \in$ ß there exist $A \in$ ß with $A \subseteq B$ and $x_0 \in X$ such that

$$\lim_{\lambda \in A} T_i^{\lambda}(x_0) = x_0 \, for every \, 1 \leq i \leq m \, .$$

Proof: We can assume without loss of generality that $(X. G)$ is minimal, otherwise we can replace X by a G-minimal subset of X. We proceed by induction on m. For $m = 1$ the theorem is valid from Theorem (2.2.6). Assume that the theorem holds for some $m \in \mathbb{N}$. Let $B \in$ ß and $\{T_1^{\lambda}\}_{\lambda \in \Lambda}, \dots, \{T_{m+1}^{\lambda}\}_{\lambda \in \Lambda} be m + 1 \Lambda$-topological dynamical systems satisfying the hypotheses of the theorem. We set

$$S_i^{\lambda} = T_i^{\lambda} \circ \{T_{m+1}^{\lambda}\}^{-1} \quad for \, all \, 1 \leq i \leq m \, .$$

We note that, since G is a commutative group, $S_i^{\lambda_1 * \lambda_2} = S_i^{\lambda_1} \circ S_i^{\lambda_2}$ for every $\lambda_1, \lambda_2 \in B$ with $\boldsymbol{\lambda_1 \prec \lambda_2}$ and $1 \le i \le m$. Hence, $\{S_1^\lambda\}_{\lambda \in \Lambda}, \dots, \{S_m^\lambda\}_{\lambda \in A}$ are Λ-topological dynamical systems of (X, d) satisfying the hypotheses of the theorem. Applying the induction hypothesis, we have the existence of $y_0 \in X$ and $A \in$ ß with $A \subseteq B$ such that:

$$\lim_{\lambda \in A} S_i^\lambda (y_0) = y_0 \text{ for every } 1 \le i \le m .$$

Let $\varepsilon > 0$. For each $i = 1, \dots, m$ there exists $\lambda_i \in A$ such that

$$d\left(T_i^\lambda\left(\left(T_{m+1}^\lambda\right)^{-1}(y_0)\right), y_0\right) = d\left(S_i^\lambda(y_0), y_0\right) < \varepsilon/2 \text{ for every } \lambda \in A \text{ with } \lambda_i \prec \lambda.$$

Let $\lambda_0 \in A$ with $\lambda_1, \dots, \lambda_m \prec \lambda_0$. Then for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ we have that

$$d\left(T_i^\lambda\left(\left(T_{m+1}^\lambda\right)^{-1}(y_0)\right), y_0\right) < \frac{\varepsilon}{2}, \qquad \text{for every } i = 1, \dots, m+1.$$

According to Theorem (2.1.8), there exist $y_1 \in X$ and $A_1 \in$ ß, $A_1 \subseteq A$ such that

$$\lim_{\lambda \in A_1} \left(T_{m+1}^\lambda\right)^{-1}(y_0) = y_1$$

Since $\{T_i^\lambda\}_{\lambda \in \Lambda}$, for $i = 1, \dots, m+1$. are equicontinuous systems, there exists $\delta > 0$ such that if $x, y \in X$ with $d(x, y) < \delta$ then $d\left(T_i^\lambda\left(x\right), T_i^\lambda(y)\right)\big) < \varepsilon/2$, for every $\lambda \in \Lambda$ and $i = 1, \dots, m+1$. Let $\lambda_1 \in A_1$ with $\lambda_0 \prec \lambda_1$ such that $d\left(\left(T_{m+1}^\lambda\right)^{-1}(y_0), y_1\right) < \delta$, for every $\lambda \in$

$A_1$with $\lambda_1 \prec \lambda$. Hence,$d\left(T_i^\lambda\left(\left(T_{m+1}^\lambda\right)^{-1}(y_0)\right), T_i^\lambda(y_1)\right) < \varepsilon/2$, for every $\lambda \in A_1$with $\lambda_1 \prec \lambda$ and every $i = 1, \ldots, m+1$. So, for every $\lambda \in A_1$with$\lambda_1 \prec \lambda$ and every $i = 1, \ldots, m+1$ we have

$$d\left(T_i^\lambda(y_1), y_0\right) \leq d\left(T_i^\lambda(y_1), T_i^\lambda\left(\left(T_{m+1}^\lambda\right)^{-1}(y_0)\right)\right) + d\left(T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0)\right), y_0) < \varepsilon$$

Let $C \in ß, C \subseteq A_1 - \lambda_1 \subseteq B$ Then

$$max\{d\left(T_i^\lambda(y_1), y_0\right): i = 1, \ldots, m+1\} < \varepsilon for every \lambda \in C$$

Let the compact metric space $(X^{m+1}, \tilde{d})$, where

$$\tilde{d}\left((y_1, \ldots, y_{m+1}), (x_1, \ldots, x_{m+1})\right) = ma\,x\{d(y_i, x_{\mathrm{i}}), i = 1, \ldots, m+1\},$$

and the Λ-topological dynamical system $\{\tilde{T}^\lambda\}_{\lambda \in \Lambda}$ of X^{m+1} where $\tilde{T}^\lambda = T_1^\lambda \times \ldots \times T_{m+1}^\lambda$which is equicontinuous. Let $\Delta^{m+1} = \{(x, \ldots, x): x \in X\} \subseteq X^{m+1}$ be the diagonal subset of X^{m+1}. We can assume that G acts onX^{m+1} by replacing each $g \in G$ with $g \times \ldots \times g$. Then the functions $\tilde{T}^\lambda$, for $\lambda \in \Lambda$, commute with the functions of G, G leaves Δ^{m+1}invariant and (Δ^{m+1}, G) is minimal. Hence,Δ^{m+1} is a homogeneous set with respect to $\{\tilde{T}^\lambda\}_{\lambda \in \Lambda}$.

According to Proposition (2.2.18), in order to prove the theorem, it will suffice to prove that Δ^{m+1} is B-recurrent. Hence, according to Proposition (2.2.15), it is enough for a givenε> 0 to find $x, y \in X$ and $C \in ß, C \subseteq B$ such that

$$\left\{\tilde{d}\left(\tilde{T}^{\lambda}\left((y,\dots,y)\right),(x,\dots,x)\right) = max\left\{d\left(T_{\mathrm{i}}^{\lambda}(y),x\right) : i = 1,\dots,m+1\right\}\right.$$
$$< \varepsilon foreevery\lambda \in C$$

.

But we have already proved that, for a given $\varepsilon > 0$ there exist $y_1, y_0 \in X$ and $C \in$ ß, $C \subseteq B$ such that

$$max\left\{d\left(T_{\mathrm{i}}^{\lambda}(y_1),y_0\right) : i = 1,\dots,m+1\right\} < \varepsilon foreevery\lambda \in C.$$

Hence, Δ^{m+1} is a recurrent set. This finishes the proof.

Corollary (2.2.20)[2]:

Let $(\Lambda,\prec,*)$ be a directed partial semigroup, ß a suitable coideal basis on $(\Lambda,\prec,*)$ with the (D)-property, $m \in \mathbb{N}$and $\{T_1^{\lambda}\}_{\lambda\in\Lambda},\dots,\{T_m^{\lambda}\}_{\lambda\in\Lambda}$ be Λ-topological dynamical systems of a, compact metric space (X,d) all contained in a commutative group G of homeomorphisms of X and let $\{\boldsymbol{T_i^{\lambda}}\}_{\lambda\in\Lambda},\{(\boldsymbol{T_i^{\lambda}})^{-1}\}_{\lambda\in\Lambda}$be equicontinuous for each $i = 1,\dots,m$. For every non-empty open subset U of X and $B \in$ ß, there exists $C \in$ ß with $C \subseteq B$ such that

$$\bigcap_{i=1}^{m}(T_i^{\lambda})^{-1}(U) \neq \emptyset \qquad foreevery\lambda \in C$$

Proof: Since G acts minimally on X, according to Proposition (2.2.13), there exists a finite subset G_0 of G such that $X = U_{g\in G_0}\mathrm{g}^{-1}(\mathrm{U})$. According to Theorem (2.2.19), there exist $x_0 \in \mathrm{X}$ and $A \in$ ß with $A \subseteq B$ such that

$$\lim_{\lambda \in A} T_i^{\lambda}(x_0) = x_0 forevery\, 1 \leq i \leq m.$$

Let $g \in G_0$ such that $x_0 \in g^{-1}(U)$. Then, there exists $\lambda_0 \in A$ such that $T_i^{\lambda}(x_0) \in g^{-1}(U)$for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ and for every $1 \leq i \leq m$, Let $C \in ß, C \subseteq A - \lambda_{\mathbf{0}} \subseteq B$. Hence, $g(x_0) \in \bigcap_{i=1}^{m}(T_i^{\lambda})^{-1}(U)$,for every$\lambda \in C$

We will indicate a way to apply the recurrence results for topological dynamical systems or nets proved to systems or nets indexed by sernigroups with digital representation. So, we will define a relation on a semigroup with digital representation in order to make it a directed partial semigroup. In order to define a suitable coideal basis satisfying the (D)-property on a semigroup with digital representation $< D_i >_{i \in I}$we will introduce the $(D_i)_{i\in I}$-located words. Hence, the recurrent results for topological dynamical systems or nets proved can be applied to systems or nets indexed by $(D_i)_{i\in I}$-located words or semigroups with digital representation.

The notion of a semigroup with digital representation introduced by Ferri, Hindman and Strauss as follows:

Definition (2.2.21)[2]:

A semigroup $(X, +)$ has a digital representation$< D_i >_{i \in I}$, where I is a linearly ordered set and D_i is a non-empty finite subset of X for every $i \in I$, if each element of X is uniquely representable as a sum $\sum_{i\in H} x_i$, where H is a finite subset of I, $x_i \in D_i$ D for every $i \in H$ and sums are taken in increasing order of indices. If X has an identity 0_x, then we set $0_x = \sum_{i\in \emptyset} x_i$.

In order to make an infinite semigroup $(X,+)$ with a digital representation $< D_i >_{i \in I}$ a directed partial semigroup, we will endow the set $|\boldsymbol{I}|^{<\infty}$of all the finite subsets of I with an appropriate relation.

Definition (2.2.22)[2]:

Let I be an infinite linearly ordered set. A relation $<_R$ on the set $|I|^{<\infty}$of all the finite subsets of I is called a proper relation on$|I|^{<\infty}$if satisfies:

(i) $\emptyset <_R H, H <_R \emptyset \ for every H \in |I|^{<\infty}_{>0}$.

(ii) If $H_1, H_2 \in |I|^{<\infty}_{>0}$ and $H_1 <_R H_2$, then, for each $i \in H_2$, either $i > max H_1 \, or i < min H_1$.

(iii) $(|I|^{<\infty}, <_R, \cup)$ is a directed partial semigroup.

Let a semigroup $(X, +)$ with a digital representation $< D_i >_{i \in I}$ where I is an infinite linearly ordered set, and let $<_R$ a proper relation on $|I|^{<\infty}$**.**

We define for $\boldsymbol{s_1} = \sum_{\boldsymbol{i \in H_1}} \boldsymbol{x_i}$, $\boldsymbol{s_2} = \sum_{\boldsymbol{i \in H_2}} \boldsymbol{x_i} \in \boldsymbol{X}$ where $H_1, H_2 \in |I|^{<\infty}$,

$$s_1 <_R s_2 \Leftrightarrow H_1 <_R H_2$$

Moreover, for $s_1 = \sum_{i \in H_1} x_i, s_2 = \sum_{i \in H_2} x_i \in X$ with $s_1 <_{\boldsymbol{R}} s_2$we define the concatenation

$$s_1 * s_2 = \sum_{i \in H_1 \cup H_2} x_i .$$

The following proposition holds:

Proposition (2.2.23)[2]:

Let a semigroup $(X, +)$ with a digital representation $< (D_{\boldsymbol{i}}) >_{\boldsymbol{i\in I}}$ where I is an infinite linearly ordered set, and let $<_{\boldsymbol{R}}$ a proper relation on $|I|^{<\infty}$. Then $(X, <_{\boldsymbol{R}}, \star)$ is a directed partial semigroup.

We will give some examples of semigroups with digital representation. We denote by $\mathbb{Z}$ the set of the integer numbers, by $\mathbb{Z}^{-}$ the set of the negative integer numbers and by $\mathbb{Q}$ the set of the rational numbers.

Examples (2.2.24)[2]:

(i) Let $p \in \mathbb{N}, p > 1$. The semigroup $(\mathbb{N}, +)$ has a digital representation $< (D_n) >_{n\in\mathbb{N}}$, where $D_n = \{ip^{n-1}: 1 \le i \le p-1\}$. For $H_1, H_2 \in |\mathbb{N}|^{\le\infty}_{>0}$, we define $H_1 <_R H_2$ if and only if max$H_1 <$ minH_1. Then $(\mathbb{N}, <_{\boldsymbol{R}}, \star)$ is a directed partial semigroup.

(ii) Let a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers. Accordingly, the semigroup $(\mathbb{Z}, +)$ has a digital representation$< (D_n) >_{n\in\mathbb{N}}$where, $D_1 = \{1\}$ and for each $n \in \mathbb{N}, n \ge 2, D_n = \{i(-1)^{n+1}(k_1 + 1) \dots (k_{n-1} + 1) : 1 \le i \le k_n\}$. If $<_R$is the relation on $|\mathbb{N}|^{<\infty}$ defined in the previous example, then $(\mathbb{Z}, <_{\boldsymbol{R}}, \star)$ is a directed partial semigroup.

(iii) More general, if a semigroup $(X, +)$ has a digital representation $< (D_i) >_{i\in I}$, where I is an infinite linearly ordered set and for every $i \in I$ there exists $i \in I$ with $i < j$, then, defining $H_1 <_R H_2$, for $H_1, H_2 \in |I|^{\le\infty}_{>0}$, if and only if max $H_1 <$ min H_2, we can make $(X, <_{\boldsymbol{R}}, \star)$ a directed partial semigroup.

(iv) The semigroup $(\mathbb{Q}, +)$, has a digital representation $< (D_n) >_{n\in\mathbb{Z}}$.

where $D_n = \{i(-1)^n(n+1)!: 1 \leq i \leq n+1\}$ for $n \in \mathbb{N} \cup \{0\}$ and, for $n \in \mathbb{Z}^-, D_n = \left\{i \frac{(-1)}{(-n+1)!}^{-n} : 1 \leq i \leq -n\right\}$. For $H_1, H_2 \in |\mathbb{Z}|_{>0}^{\leq\infty}$, we define $H_1 <_R H_2$ if and only if $H_2 = A_1 \cup A_2$ with $A_1, A_2 \neq \emptyset$ and max $A_1 <$ min H_1, max $H_1 <$ min A_2, Then $(\mathbb{Q}, <_R, \star)$ is a directed partial semigroup.

In order to define a suitable coideal basis satisfying the (D)-property on a semigroup $(X, +)$ with digital representation $< (D_i) >_{i \in I}$ we will introduce the $(D_i)_{i \in I}$-located words.

Definition (2.2.25)[2]:

Let an arbitrary alphabet Σ , an infinite linearly ordered set $(I, <)$and, for each $i \in I$, let $D_i = \{d_{1,i}, \dots, d_{k,i}\}$, be a non-empty finite subset of Σ with cardinality $k_i \in \mathbb{N}$. We define the set of (constant) $(D_i)_{i \in I}$-located words as follows:

$$\mathrm{L}((D_i)_{\mathrm{i \in I}}) = \{\mathrm{w} = w_{i_1,} \dots, w_{i_l}: l \in \mathbb{N}, i_1 < \cdots < i_l \\ \in I \; and \; w_{i_j,} D_{i_j} \forall \, 1 \leq j \leq l\}$$

Let$m \in \mathbb{N}$ and $\vec{v} = (v_1, \dots, v_m), where \; v_1, \dots, v_m \notin \Sigma$ be the variables. We define the set of variable $(D_i)_{i \in I}$-located words as follows:

$$L((D_i)_{i\in I};(v_1,\dots,v_m)$$
$$= \Big\{w = w_{i_1},\dots,w_{i_l}: l \in \mathbb{N}, i_1 < \cdots < i_l \in I\ , w_{i\,j}$$
$$\in D_{i_j} \cup \{v_1,\dots,v_m\} \forall\ 1 \le j \le l\ and\ there\ exist\ \ 1 \le j_1,\dots,j_m$$
$$\le l\ with\ w_{i_{j_1}} = v_{1\,,}\dots,w_{i_{j_m}} = v_m\Big\}.$$

Let $L = L((D_i)_{i\in I}) \cup L((D_i)_{i\in I};\vec{v})$.

For $w = w_{i_1},\dots,w_{i_l} \in L$, the set $dom(w) = \{i_1 < \cdots < i_l\} \subseteq I$ is the domain of w.

We assume that there exists a proper relation $<_R$on the set $|I|^{<\infty}$ of all the finite, subsets of I. Then we define for $w,u \in L$ the relation

$$w <_R u \Leftrightarrow dom(w) <_R dom(u)$$

and also for two words $w = w_{i_1}\dots w_{i_l}, u = u_{t_1}\dots u_{tr} \in L\ with\, w <_R u$ we define the concatenating word

$$w \star u = z_{q_1}\dots z_{q_{r+l}} \in L,$$

Where

$q_1 < \cdots < q_{r+l} = dom(w) \cup dom(u), z_i = w_i if i \in dom(w) and z_i = u_i if i \in dom(u)$.

So, the following proposition holds

Proposition (2.2.26)[2]:

Let an arbitrary alphabet Σ , aninfinite linearly orderedset $(I,<), m \in \mathbb{N}, v_1, \dots, v_m \notin \Sigma$and let D_i be a non-empty finite subset of $\Sigma$for each $i \in I$. If $<_R$ is a proper relation on the set $|I|^{<\infty}$, then $(L((D_i)_{i\in I}; v_1, \dots, v_m), <_R, \star)$ and also $(L((D_i)_{i\in I}), <_R, \star)$ are directed partial semigroups.

Let $w = w_{i_1} \dots w_{i_l} \in L((D_i)_{i\in I}; v_1, \dots, v_m)$, where $D_i = \{d_{1,i}, \dots, d_{k_i,i}\}$, where $k_i \in \mathbb{N}$. For $(p_1, \dots, p_m) \in \mathbb{N}^m \cup \{(0, \dots, 0)\}$ we set$w(0, \dots, 0) = w$ and for $(p_1, \dots, p_m) \in \mathbb{N}^m$

$$w(p_1, \dots, p_m) = u_{i_1} \dots u_{i_l} \in L((D_i)_{i\in I}),$$

where, for $1 \leq j \leq l$,$u_{i_j} = w_{i_j}$, if $w_{i_j} \in D_{i_j}, u_{ij} = d_{p_r,i_j}$ $if w_{i_j} = v_r$, for $1 \leq r \leq m$, and $p_r \leq k_{i_j}$ and finally $u_{i_j} = d_{k_{i_j},i_j}$, if $w_{i_j} = v_r$, for $1 \leq r \leq m$, and $p_r > k_{i_j}$. We set

$$L^\omega((D_i)_{i\in I}; v_1, \dots, v_m) = \{\vec{w} = (w_n)_{n\in\mathbb{N}} : w_n \in L((D_i)_{i\in I}; v_1, \dots, v_m)$$

$$and\ w_n <_R w_{n+1} for every n \in \mathbb{N}\}$$

We fix an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^{\boldsymbol{m}}$ such that $U_{n\in\mathbb{N}} F_n = \mathbb{N}^m$. Let a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega((D_i)_{i\in I}; v_1, \dots, v_m)$.

An extracted variable$(D_i)_{i\in I}$- located word of $\vec{w}$ has the form

$$u = w_{n_1}(\vec{p}_1) \star \dots \star w_{n_\lambda}(\vec{p}_\lambda) \in \mathrm{L}((D_i)_{i\in I}; v_1, \dots, v_m),$$

where$\lambda \in \mathbb{N}, n_1 < \cdots < n_{\boldsymbol{\lambda}} \in \mathbb{N}, \vec{p}_i \in F_{n_i} \cup \{(0, \dots, 0)\}$ for every $1 \leq i \leq \boldsymbol{\lambda}$ and $(0, \dots, 0) \in \{\vec{p}_1, \dots, \vec{p}_{\boldsymbol{\lambda}}\}$. The set of all the extracted variable $(D_i)_{i \in I}$-located words of $\vec{w}$ is denoted by $EV(\vec{w})$.

An extracted $(D_i)_{i \in \boldsymbol{I}}$-located word of $\vec{w}$ has the form

$$z = w_{n_1}(\vec{p}_1) \star \dots \star w_{n_\lambda}(\vec{p}_{\boldsymbol{\lambda}}) \in L((D_i)_{i \in I}),$$

where$\boldsymbol{\lambda} \in \mathbb{N}$, $n_1 < \cdots < n_{\boldsymbol{\lambda}} \in \mathbb{N} and \vec{p}_i \in F_{n_i}$ for every$1 \leq i \leq \boldsymbol{\lambda}$. The set of all the extracted $(D_i)_{i \in \boldsymbol{I}}$-located words of $\vec{w}$ is denoted by $E^m(\vec{w})$.

Theorem (2.2.27)[2]:

Let an arbitrary alphabet $\sum$,an infinite linearly ordered set $(I, <)$**,** a proper relation $<_{\boldsymbol{R}}$on the set $|I|^{<\infty}$, $m \in \mathbb{N}, v_1, \dots, v_m \notin \sum, D_{\boldsymbol{i}}$**,** a non-empty finite subset of $\sum$, for each $i \in I$, and let an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^m$ such that $U_{n \in \mathbb{N}} F_n = \mathbb{N}^{\boldsymbol{m}}$. The families

$$ß_1 = \{E(\vec{w}) : \vec{w} \in L^\omega((D_i)_{i \in I}; v_1, \dots, v_m)\} \text{and}$$

$$ß = \{EV(\vec{w}) : \vec{w} \in L^\omega((D_i)_{i \in I}; v_1, \dots, v_m)\}$$

are suitable coideol bases on $\mathrm{L}((D_i)_{i \in I}; v_1, \dots, v_m), <_R, \star)$ and $(\mathrm{L}((D_i)_{i \in I}), <_R, \star)$ respectively, and satisfy the (D)-property.

Proof: Let $\vec{w} \in L^\omega((D_i)_{i \in I}; v_1, \dots, v_m)$, and let $EV(\vec{w}) = A_1 \cup A_2$ and $E(\vec{w}) = B_1 \cup B_2$. Firstly, we will define an order on the set $\mathbb{N}^{\boldsymbol{m}}$. For $\vec{p} \in \mathbb{N}^m$we set $i(\vec{p})$to be the least $n \in \mathbb{N}$ such that $(\vec{p}) \in F_n$and then we define $\vec{p}_1 < \vec{p}_2$ for $\vec{p}_1, \vec{p}_2 \in \mathbb{N}^{\boldsymbol{m}}$ if and only if either $i(\vec{p}_1) < i(\vec{p}_2)$ or $i(\vec{p}_1) = i(\vec{p}_2)$ and $\vec{p}_1$ is less than $\vec{p}_2$in the lexicographical ordering.

Let $\mathbb{N}^m = \{\beta_1 <_* \beta_2 <_* \beta_3 <_* \dots\}$. For each $n \in \mathbb{N}$, let $\beta_{k_n} \in \mathbb{N}^m$ be the greatest element of F_n in the lexicographical ordering. Then $\vec{k} = (K_n)_{n \in \mathbb{N}} \subseteq$ $\mathbb{N}$is an increasing sequence. We set $\sum_1 = \{\beta_n : n \in \mathbb{N}\} = \mathbb{N}^m$ and we define the function $h: L(\Sigma_1, \vec{k}) \cup L(\Sigma_1, \vec{k}; v) \rightarrow E(\vec{w}) \cup EV(\vec{w}) with$

$$h\ (t_{n_1} \dots t_{n_1}) = w_{n_1}(p_1^1, \dots, p_m^1) \star \dots \star\ w_{n_\lambda}(p_1^\lambda, \dots, p_m^\lambda),$$

where, for $1 \leq i \leq \lambda, (p_1^i, \dots, p_m^i) = (0, \dots, 0)$ if $t_{n_i} = v$ and $(p_1^i, \dots, p_m^i) = t_{n_i} \in \{\beta_1, \dots, \beta_{k_{n_i}}\} if t_{n_i} \in \Sigma_1$. The function h is onto $E(\vec{w}) \cup$ $EV(\vec{w})$ and moreover $h\left(L(\Sigma_1, \vec{k})\right) = E(\vec{w})$ and $h(L(\Sigma_1, \vec{k}; v)) = EV(\vec{w})$.

According to Carlson's theorem(If $f(z)$ regular and of the form $o(e^{k|z|})$ where $k < \pi$, for $R[z] \geq 0$, and if $f(z) = 0$ for $z = 0,1, \dots$, then $f(z)$ is identically zero)[7], there exist a sequence $\vec{s} = (s_n)_{n \in \mathbb{N}} \in L^\omega(\Sigma_1, \vec{k}; v)$ and $i_0 \in \{1,2\}$, $j_0 \in \{1,2\}$ such that $EV(\vec{s}) \subseteq h^{-1}(A_{i_0})$ and $E(\vec{s}) \subseteq h^{-1}(B_{j_0})$. Set $u_n = h(s_n) \in EV(\vec{w})$ for every $n \in \mathbb{N}$ and $\vec{u} = (u_n)_{n \in \mathbb{N}} \in$ $L^\omega((D_i)_{i\in}; v_1, \dots, v_m)$. Then $EV(\vec{u}) \subseteq h(EV(\vec{s})) \subseteq A_{i_0}$ and $E(\vec{u}) \subseteq$ $h(E(\vec{s})) \subseteq B_{i_0}$. Hence, ß$_1$ and ß are coideal bases on on $(L((D_i)_{i\in I}; v_1, \dots, v_m), <_R, \star)$ and $(L((D_i)_{i\in I}), <_R, \star)$ respectively, and obviously they are suitable. Analogously to Example (2.1.7) (iii), can be proved that ß$_1$ and ß satisfy the (D)-property.

We will go back to semigroups with digital representation. Let a semigroup $(X, +)$ has a digital representation $< D_i >_{i \in I}$and let $<_R$be a proper relation on the set $|I|^{<\infty}$. According to Proposition (2.2.24),$(X, <_R, \star)$ is a directed partial semigroup. Let $g: L((D_i)_{i\in I} \rightarrow X \backslash \{0_x\}$ in case $(X, +)$ has an identity 0_x or $g: L((D_i)_{i\in I} \rightarrow X$ otherwise, with

$$g\left(w_{i_1} \dots w_{i_l}\right) = w_{i_1} + \dots + w_{i_l}.$$

The function g is one-to-one, onto, preserves the order and for $w, u \in L((D_i)_{i\in I})$ with $w <_{\boldsymbol{R}} u$ we have $g(w \star u) = g(w) \star g(u)$. So, using the previous theorem, we can define a suitable coideal basis for $(X, <_R, \star)$ **s**atisfying the (D)-property, via the function g.

Theorem (2.2.28)[2]:

Let a sew semigroup $(X, +)$ with a digital representation $< D_{\boldsymbol{i}} >_{\boldsymbol{i\in I}}$ and let a proper relation $<_{\boldsymbol{R}}$on the set $|I|^{<\infty}$. Fixing an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^{\mathbf{m}}$, for $m \in \mathbb{N}$, such that $U_{n \in \mathbb{N}} F_n = \mathbb{N}^{\boldsymbol{m}}$, the family

$$ß = \{g(E(\vec{w}) : \vec{w} \in (L^{\infty}((D_i)_{i\in I}; v_1, \dots, v_m)\}$$

is a suitable coideal basis on $(X, <_{\boldsymbol{R}}, \star)$ satisfying the (D)-property.

Hence, the recurrent results for topological dynamical systems or nets proved can be applied to systems or nets indexed by $(D_i)_{i\in I}$ -located words or semigroups with digital representation.

Chapter 3

Irreversible Algebraic Dynamical Systems on C*-Algebras

To each irreversible algebraic dynamical system, we associate a universal C*-algebra and show that it is a UCT Kirchberg algebra. We discuss the structure of the core subalgebra, which turns out to be closely related to generalised Bunce-Deddens algebras. We also construct discrete product systems of Hilbertbimodules for irreversible algebraic dynamical systems which allow usto view the associated C*-algebras as Cuntz-Nica-Pimsner algebras. Besides,we show a decomposition theorem for semigroup crossed products of unital C*-algebras by semi direct products of discrete, left can cellativemonoids.

Section (3.1): Irreversible Algebraic Dynamical Systems and Structure of the Associated C*-Algebras:

We familiarize with the primary object of interest called irreversible algebraic dynamical system in itsmostgemieral form**.**

A dynamical system is given by a countably infinite, discrete group G and at most countably many commuting injective, non-surjective group endomorphisms $(\theta_i)_{i\in I}$ of G that are independent in the sense that the intersection of their images is as small as possible. Additionally, we will introduce a minimality condition stating that the intersection of the images of the group endomorphisms from the semigroup generated by $(\theta_i)_{i\in I}$is trivial. In other words, the group endomorphisms $(\theta_i)_{i\in I}$ separate the points in G. At a later stage, this condition is shown to be intimately connected to simplicity of the **C*-**algebra $\mathcal{O}\,[G, P, \theta]$ associated to such a dynamical system.

The following observation is an extension of the concept of independence introduced. We will require neither the group G to be abelian nor the cokernels of the injective group endomorphism's of G to be finite.

Proposition (3.1.1)[3]:

Suppose G is a group. Consider the following statements for two commuting injective group endomorphism's θ_1 and θ_2 of G:

(i) $\theta_1(G)\theta_2(G) = G$.

(ii) The map $\theta_1(G)/(\,\theta_1(G) \cap \theta_2(G)) \to G/\theta_2(\mathrm{G})$ induced by the inclusian $\theta_1(G) \hookrightarrow G$ is a bijection.

(iii) The map $\theta_2(G)/(\,\theta_1(G) \cap \theta_2(G)) \to G/\theta_1(G)$ induced by the inclusiam , $\theta_2(G) \hookrightarrow G$ is a bijection.

(iv) $\theta_1(G) \cap \theta_2(G) = \theta_1\theta_2(G)$.

Then (i), (ii), and (iii) are equivalent and imply (iv). If either of the subgroups $\theta_1(G)$ or $\theta_2(G)$ is of finite index in G, then (i) - (iv) are equivalent.

Proof: Note that we always have $\theta_1(G)\theta_2(G) \subset G$ and $\theta_1(G) \cap \theta_2(G) \supset \theta_1\theta_2(G)$. Moreover, in condition (ii), the inclusion $\theta_1(G) \hookrightarrow G$ induces an injective map $\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \to G/\theta_2(G)$. The corresponding statement holds for (iii).

If (i) holds true, then $G \ni g = \theta_1(g_1)\,\theta_2(g_2)$ for suitable $g_i \in G$. Hence, the left-coset of $\theta_1(g_1)$ maps to the left-coset of g and (ii) follows.

Conversely, suppose (ii) is valid and pick $g \in G$. Then there is $g_1 \in G$ such that $\theta_1(g_1)\,(\theta_1(G) \cap \theta_2(G)) \longmapsto g\theta_2(G)$ via the map from (ii). But since tins map comes from the inclusion $\theta_1(G) \hookrightarrow G$, we have $g\theta_2(G) = \theta_1(g_1)\,\theta_2(G)$. Thus, there is $g_2 \in G$ suchì that $g = \theta_1(g_1)\,\theta_2(g_2)$ showing (i).The equivalence of (i) and (iii) is obtained from the previous argument by swapping θ_1 and θ_2. Given (ii), that is,

$$f_1 : \theta_1(G)/(\theta_1(G) \cap \theta_2(G)) \rightarrow G/\theta_2(G)$$

is a bijection (induced by $\theta_2(G) \hookrightarrow G$), composing ${f_1}^{-1}$with the bijection

$$f_2 : \theta_1(G)/(\,\theta_1\theta_2(G)) \rightarrow G/\theta_2(G)$$

obtained from injectivity of θ_1 yields a bijection

$${f_1}^{-1}f_2 \colon \theta_1(G)/(\theta_1\theta_2(G)) \rightarrow \theta_1(G)/(\,\theta_1(G) \cap \theta_2(G)).$$

Let us assume $\theta_1\theta_2(G) \subsetneqq \theta_1(G) \cap \theta_2(G)$. This means, that there is $g \in \theta_1$(G) such that $g\theta_1\theta_2(G) \neq \theta_1\theta_2(G)$ but $g\theta_1(G) \cap \theta_2(G) = \theta_1(G) \cap \theta_2(G)$. Noting that ${f_1}^{-1}f_2$ maps a left-coset $g'\theta_1\theta_2(G)\, to\, g'\theta_1(G) \cap \theta_2(G)$, this contradicts injectivity of ${f_1}^{-1}f_2$. Hence, we must have $\theta_1(G) \cap \theta_2(G)\; \theta_1\theta_2(G)$. Similarly, (iv) follows from (iii).

Finally, suppose (iv) holds. By injectivity of θ_1, we have

$$\theta_1(G)/(\theta_1(G) \cap \theta_2(G)) = \theta_1(G)/\,\theta_1\theta_2(G) \cong G/\theta_2(G).$$

So if $\{G: \theta_2(G)\}$ is finite, then the injective map from (ii) is necessarily a bijection. If $\{G : \theta_1(G)\}$ is finite, we get (iii) in the same manner.

DefInition (3.1.2)[3]:

Let *G* be a group and θ_1, θ_2 commuting, injective group endomnorphisins of *G*. Then θ_1 and θ_2 are said to be independent, if they satisfy condition (iv) from Proposition (3.1.1) θ_1 and θ_2 are said to be strongly independent, if they satisfy the condition (i) from Proposition (3.1.1).

Note that (strong) independence holds if θ_1 or θ_2 is an autoinorphism.

Lemma (3.1.3)[3]:

Let *G* be a group and suppose $\theta_1, \theta_2, \theta_3$ are commuting, injective group endomorphisms of *G*. θ_1 is (strongly) independent of $\theta_2\theta_2$ if and only if θ_1 is (strongly) independent of both θ_2 and θ_3.

Proof: If θ_1 and $\theta_2\theta_3$ are strongly independent, then

$$\theta_1(G)\,\theta_2(G) \supset \theta_1(G)\,\theta_2(\theta_3(G)) = G$$

shows that θ_1 and θ_2 are strongly independent. As θ_2 and θ_3 commute, θ_1 is also strongly independent of θ_3. Conversely, if θ_1 is strongly independent of both θ_2 and θ_3 then

$$G = \theta_1(G)\theta_2(G) \qquad = \theta_1(G)\theta_2\big(\theta_1(G)\theta_3(G)\big)$$

$$= \theta_1(G\theta_2(G))\,\theta_2(\theta_3(G)) \subset \theta_1(G)\theta_2\theta_3(G)\ ,$$

so θ_1 and $\theta_2\theta_3$ are strongly independent since the reverse inclusion is trivial. If θ_1 and $\theta_2\theta_3$ are independent, then comninutativity of θ_1, θ_2 and θ_3 in comnbination with injectivity of $\theta_3$yield

$$\theta_1(G) \cap \theta_2(G) = \theta_3^{-1}(\theta_1\theta_3(G) \cap \theta_2\theta_3(G) \subset \theta_3^{-1}(\theta_1(G) \cap \theta_2\theta_3(G)$$
$$= \theta_3^{-1}(\theta_1\theta_2\theta_3(G)) = \theta_1\theta_2(G).$$

Since the reverse inclusion is always true, we conclude that θ_1 and θ_2 are independent. Exchanging the role of θ_2 and θ_3 shows independence of θ_1 and θ_3. Finally, if θ_1 is independent of both θ_2 and θ_3, we get

$$\theta_1(G) \cap \theta_2\theta_3(G) = \theta_1(G) \cap \theta_2(G) \cap \theta_2\theta_3(G) = \theta_1\theta_2(G) \cap \theta_2\theta_3(G)$$
$$= \theta_2(\theta_1(G) \cap \theta_3(G)) = \theta_1\theta_2\theta_3(G).$$

by infectivity of θ_2. Thus θ_1 and $\theta_2\theta_3$ are independent.

If $(P, \leq)$is a lattice-ordered monoid with unit 1_P**,** we shall denote the least common multiple and the greatest common divisor of two elements $p, q \in P$ by $p \vee q$ and $p \wedge q$, respectively. p and q are said to be relatively prime (in P**)** if $p \wedge q = 1_p$ or, equivalently, $p \vee q = pq$. Simple examples of such monoids are countably generated free abelian monoids since such monoids are either isomorphic to $\mathbb{N}^k$for some $k \in \mathbb{N}$ or $\bigoplus_{\mathbb{N}} \mathbb{N}$.

Definition (3.1.4)[3]:

An irreversible algebraic dynamical system (G, P, θ) is:

(i) a countably infinite, discrete group G with unit 1_G,

(ii) a countably generated, free abelian monoid P with unit 1_P, and

(iii) a P-action θ on G by injective group endomorphisms for which θ_p and θ_q are independent if and only if p and q are relatively prime.

An irreversible algebraic dynamical system (G, P, θ) is said to be

(i) minimal, if fl $\bigcap_{p \in P} \theta_p$ (G) = $\{1_G\}$,

(ii) commutative, if G is commutative,

(iii) of finite type, if $[G : \theta_p(G)]$ is finite for all $p \in P$ and

(iv) of infinite type, if $[G : \theta_p(G)]$ is infinite for all $p \neq 1_p$.

Examples (3.1.5)[3]:

There are various examples for commutative irreversible algebraic dynamical systems and most of them are of finite type. Let us recall that it suffices to check independence of the endomorphisms on the generators of P according to Lemma (3.1.3).

(i) Choose a family $(p_i)_{i \in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*} = \mathbb{Z} \backslash \{0, \pm 1\}$and let $p = |(p_i)_{i \in I}\rangle$ act on $G = \mathbb{Z}$ by $\theta_{p_i}(g) = p_i g$. Since $\mathbb{Z}$ is an integral domain, each θ_{p_i} is an injective group endomorphisin of G with $[G : \theta_{p_i}(G)] = p_i$. For $i \neq j$, θ_{p_i} and θ_{p_j} are independent if and only if p_i and p_j are relatively prime in $\mathbb{Z}$. Thus, we get a commutative irreversible algebraic dynamical system of finite type if and only if $(p_i)_{i \in I}$ consists of relatively prime integers. Since the number of factors in its prime factorization is finite for every integer, such irreversible algebraic dynamical systems are automatically minimal.

(ii) Let $I \subset \mathbb{N}$, choose relatively prime integers $\{q\} \cup (p_i)_{i \in I} \subset \mathbb{Z}^{\times} \setminus \mathbb{Z}^*$and let $G = \mathbb{Z}[1/g]$. As $\mathbb{Z}[1/g] = \overset{\lim}{\longrightarrow} \mathbb{Z}$ with connecting maps given by multiplication with q, and q is relatively prime to each p_i, the arguments from (i) carry over almost verbatim. Thus we get minimal commutative irreversible algebraic dynamical systems of finite type (G, P, θ).

(iii) Let $\mathbb{K}$ be a countable field and let $G = \mathbb{K}[T]$ denote the polynomial ring in a single variable T over $\mathbb{K}$. Choose non-constant polynomials $p_i \in \mathbb{K}[T], i \in I$. Multiplying by p_i defines an endomorphism $\theta_{p_i}$of G with $[G: \theta_{p_i}(G)] = |\mathbb{K}|^{deg(p_i)}$, where $\deg(p_i)$ denotes the degree of $p_i \in \mathbb{K}[T]$. Thus, if we let $p \coloneqq |(p_i)_{i \in I}\rangle$, then the index of $\theta_p\ (G)$ in G is finite for all $p \in P$ if and only if $\mathbb{K}$ is finite. It is clear that θ_{p_i} and θ_{p_j} are independent if and only if $(p_i) \cap (p_j) = (p_i p_j)$ holds for the principal ideals (whenever $i \neq j$). Since every $g \in \mathbb{K}[T]$ has finite degree, (G, P, θ) is automatically minimal. Thus, provided $(p_i)_{i \in I}$has been chosen accordingly, we obtain a minimal commutative irreversible algebraic dynamical system which is of finite type if and only if $\mathbb{K}$ is finite.

Example (3.1.6)[3]:

For $G = \mathbb{Z}^d$ with $d \geq 1$, the monoid of injective group endomorphisms of G is isomorphic to the monoid of invertible integral matrices $M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$. For each such endomorphismn, the index of its image in G is given by the absolute value of the determinant of the corresponding matrix. In particular, their images always have finite index in G and an endomorphism of G is not surjective precisely if the absolute value of the determinant of the matrix exceeds 1. So let $(T_i)_{i \in I} \subset M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$ be a family of commuting matrices satisfying $|det\, T_i| > 1$ for all $i \in I$ and set $P = |(T_i)_{i \in I}\rangle$as well as $\theta_i(g) = T_i g$. For $i \neq j$, it is easier to check strong independence of θ_i and θ_j instead of independence. Indeed, since we are dealing with a finite type case, the two conditions are equivalent and strong independence takes the form $T_i(\mathbb{Z}^d) + T_j(\mathbb{Z}^d) = \mathbb{Z}^d$, see Proposition (3.1.1). This condition can readily he checked. Moreover, minimality is related to generalised eigenvalues and we note that, in the case where P is singly generated, the generating integer matrix has to be a dilation matrix.

Example (3.1.5)(i) Can be generalised to the case of rings of integers.

Example (3.1.7)[3]:

Let$\mathcal{R}$betheringof integers in a number field and denote by $\mathcal{R}^{\times} = \mathcal{R} \backslash \{0_{\mathcal{R}}\}$ the multiplicative subsemnigroup as well as by $\mathcal{R}^{*} \subset \mathcal{R}^{\times}$the group of units in $\mathcal{R}$ Take $G = \mathcal{R}$ and choose a (countable) family $(P_i)_{i \in I} \subset \mathcal{R}^{\times} \subset \mathcal{R}^{*}$. Ifwe set $p = [(p_i)_{i \in I}\rangle$, then this monoid acts on G by multiplication, i.e. $\theta_p(g) = pg$ for $g \in G, p \in P$. For $i \neq j, \theta_{p_i}$ and θ_{p_j} areindependent if and only if the principal ideals (p_i) and (p_j) in $\mathcal{R}$ have no common prime ideal. If this is the case, (G, P, θ) constitutes a commutative irreversible algebraic dynamical system of finite type. Since the number of factors in the (unique) prime ideal factorization of (g) in $\mathcal{R}$ is finite for every $g \in G$, minimality is once again automatically satisfied.

As a matter of fact, the construction from Example (3.1.7) is applicable to Dedekind domains $\mathcal{R}$. Next, we would like to mention the following example even though, having singly generated P**,** it has nothing to do with independence. The reason is that Joachim Cuntz and Anatoly Vershik, that the **C*** -algebra $\mathcal{O}[G, P, \theta]$ associated to this irreversible algebraic dynamical system is isomorphic to $\mathcal{O}_n$**.**

Ecamp1e (3.1.8)[3]:

For $n \geq 2$, consider the unilateral shift θ_1 acting on $G = \bigoplus_N \mathbb{Z}/n\mathbb{Z}\ by\ (g_0, g_1, \ldots) \mapsto (0, g_0, g_1, \ldots)$. Since θ_1 is an injective group endomnorphismn with $[G: \theta_1(G)] = n$, $(G, P, \theta)$with $p = |\theta_1\rangle$ is a minimal commutative irreversible algebraic dynamical system of finite type.

Example (3.1.9)[3]:

Generalising Example (3.1.8), suppose P is as required in condition (ii) of Definition (3.1.4) and let G_0 he a countable group. Let us assume that G_0 has at least two distinct elements. Then Padmits a shift action θ on $G \coloneqq \bigoplus_P G_0$given by $(\theta_p((g_q)_q \in P))_r = \mathcal{X}_p P(r) g_p{}^{-1} r$ for all $, r \in P$**.** It is apparent that $\theta_p\theta_q = \theta_q\theta_p$ holds for all $p, q \in P$and that θ_p is an injective group endomnorphismn for all $p \in P$**.** The index $\left[G: \theta_p(G)\right]$ is finite for $p \in P \backslash \{1_p\}$ if and only if G_0 is finite and P is singly generated. Indeed, if $p \neq 1_P$, then each element of $\bigoplus_{q \in P \backslash p^P} G_0$ yields a distinct left-coset in $G / \theta_p(G)$**.** Clearly, this group is finite if and only if G_0 is finite and $P$is singly generated. Given relatively prime p and q in $P \backslash \{1_P\}, \theta_p(G)\theta_q(G) \neq G$ since $g_{1_P = }1_{G_0}$for all $(g_r)_{r \in P} \in \theta_p(G)\theta_q(G)$ as $1_p \notin$

$p^P \cup q^P$. Thus, unless P is singly generated, θ does not satisfy the strong independence condition. However, the independence condition is satisfied because $g = (g_r)_{r \in P} \in \theta_p(G) \cap \theta_q(G)$ implies that $g_r \neq 1_{G_0}$ only if $r \in p^P \cap q^P = pq^P$ and thus $g \in \theta_{pq}(G)$.

We have seen in Example (3.1.9) that one cannot expect strong independence for irreversible algebraic dynamical systems of infinite type in general. On the other hand, there are some examples where the subgroups in question have infinite index and the endomorphisms are strongly independent:

Example (3.1.10)[3]:

Given a family $(G^{(i)}, P, \theta^{(i)})_{i \in \mathbb{N}}$ of irreversible algebraic dynamical systems, we can consider $G := \bigoplus_{i \in \mathbb{N}} G^{(i)}$. If Pacts on G component-wise, i.e. $\theta_p(g_i)_{i \in \mathbb{N}} := (\theta_p^{(i)}(g_i))_{i \in \mathbb{N}}$, then (G, F, θ) is an irreversible algebraic dynamical system and $[G: \theta_p(G)]$is infinite unless $p = 1_p$. G is commutative if and only if each $G^{(i)}$is, and $(G, F, \theta)$is minimal if and only if each $(G^{(i)}, P , \theta^{(i)})$ is minimal. If each $(G^{(i)}, P , \theta^{(i)})$ satisfies the strong independence condition, then θ inherits this property as well.

As a final example, we provide more general forms. These examples are neither commutative irreversible algebraic dynamical systems nor of finite type.

Example (3.1.11)[3]:

For $2 \leq n \leq \infty$, let $\mathbb{F}_n$, be the free group in n generators $(a_k)_{1 \leq k \leq n}$. fix $1 \leq d \leq n$ and choose for each $1 \leq i \leq d$ an n-tuple $(m_{i,k})_{1 \leq k \leq n} \subset \mathbb{N}^{\times}$ such that

(i) there exists k such that $\boldsymbol{m_{i,k}} > 1$ for each $1 \leq i \leq d$, and

(ii) $m_{i,k}$and $m_{j,k}$ are relatively prime for all $i \neq j, 1 \leq k \leq n$.

Then $\theta_i(a_k) = a_k^{m_{i,k}}$definesagroupendomorphismof$\mathbb{F}_{\boldsymbol{n}}$, for each $1 \leq i \leq d$. Noting that the length of an element of $\mathbb{F}_{\boldsymbol{n}}$, in terms of the generators $(a_k)_{1 \leq k \leq n}$, and their inverses is non-decreasing under $\theta_{\boldsymbol{i}}$, we deduce that $\theta_{\boldsymbol{i}}$ is injective. It is clear that $\theta_{\boldsymbol{i}}\theta_{\boldsymbol{j}} = \theta_{\boldsymbol{j}}\theta_{\boldsymbol{i}}$holds for all $\boldsymbol{i}$ and $\boldsymbol{j}$. For every $1 \leq i \leq d$, the index $[\mathbb{F}_{\boldsymbol{n}}: \theta_{\boldsymbol{i}}(\mathbb{F}_{\boldsymbol{n}})]$ is infinite. Indeed, take $1 \leq k \leq n$ such that $m_{i,k} > 1$ according to 1) and pick $1 \leq \ell \leq n$ with $\ell \neq k$. Then the family $((a_k a_\ell)^j)_{j \geq 1}$ yields pairwise distinct left-cosets in $\mathbb{F}_n / \theta_i(\mathbb{F}_n)$since reduced words of the form $a_k a_\ell b$ . .. with $b \neq a_\ell^{-1}$are not contained in $\theta_{\boldsymbol{i}}(\mathbb{F}_{\boldsymbol{n}})$. A similar argument shows that $\theta_{\boldsymbol{i}}$and $\theta_{\boldsymbol{j}}$ are not strongly independent for $(i \neq j: by\ 1)$, there are $1 \leq k, \ell \leq n$ such that $m_{i,k} > 1$ and $m_{j,\ell} > 1$. This forces $a_k a_\ell \notin \theta_i(\mathbb{F}_n)\theta_j(\mathbb{F}_n)$. Nonetheless, $\theta_{\boldsymbol{i}}$and $\theta_{\boldsymbol{j}}$ are independent due to 2). Thus, $G = \mathbb{F}_n$ and $P = |(\theta_i)_{1 \leq i \leq d}\rangle$, acting on G in the obvious way constitutes an irreversible algebraic dynamical system which is neither commutative nor of finite type. Minimality of such irreversible algebraic dynamical systems can easily be characterized by:

(iii) *For* each $1 \leq k \leq n$, there exists $1 \leq i \leq d$ satisfying $m_{i,k} > 1$.

In addition to the presented spectrum of examples, we would like to mention that there are also examples of minimal, commutative irreversible algebraic dynamical systems of finite type arising from cellular automata.

We have lemmas which are relevant for the C*-algebraic considerations. The first lemma reflects a crucial feature of the independence assumption

Lemma (3.1.12)[3]:

If (G, P, θ) is an irreversible algebraic dynamical system,

$$g\theta_p(G) \cap h\theta_q(G) = \begin{cases} \{g\theta_p(h')\theta_{p\vee q}(G) & if\ g^{-1}h \in \theta_p(G)\theta_q(G) \\ \emptyset & else \end{cases}$$

holds for all $g, h \in G, p, q \in P$. where h' is uniquely determined by $g\theta_p(h') \in h\theta_q(G)$ up to mnultiplicatiorl from the right by elements from $\theta_{p^{-1}(p\vee q)}(G)$.

Proof: If there exist $g_1, g_2 \in G$ such that $g\theta_p(g_1) = h\theta_q(g_2)$, then $\theta^{-1}h = \theta_p(g_1)\theta_q(g_2^{-1}) \in \theta_p(G)\theta_q(G)$ .follows because *G* is group. Now suppose that $g_3, g_4 \in G$ satisfy $g\theta_p(g_3) = h\theta_q(g_4)$ as well. Since this implies $\theta_p(g_1^{-1}g_3) = \theta_q(g_2^{-1}g_4)$ we deduce $\theta_p(g_1^{-1}g_3) \in \theta_{p\vee q}(G)$.Using injectivity of θ_p this is eqiuvalent to $g_1^{-1}g_3 \in \theta_{p^{-1}(p\vee q)}(G)$. Therefore, $h' = g_1$ is unique lip to right multiplication by elements from $\theta_{p^{-1}(p\vee q)}(G)$.

For the proof of Theorem (3.1.46), we will need the following auxiliary result, which relies on irreversibility of the dynamical system:

Lemma (3.1.13)[3]:

Suppose (G, P, θ) is an irreversible algebraic dynamical system and we have $n \in \mathbb{N}, g_i \in G, p_i \in P\backslash\{1p\}\ for\ 0 \leq i \leq n$.Then. there exist $g \in g_0\theta_{p0}(G), p \in p_0P$ satisfying

$$g\theta_p(G) \subset G\backslash \bigcup_{1\leq i\leq n}\left(\boldsymbol{g_i}\bigcap_{m\in\mathbb{N}}\theta_{p_i^m}(G)\right)$$

Proof: We proceed by induction starting with $n = 1$**.** As $p_1 \neq e$we can find $m \in \mathbb{N}$ such that $p_0 \notin p_1^m p$. Thus we have $p_0 \vee p_1^m \gneqq p_0$. By Lemma (3.1.12),

$$(g_0\theta_{p_0}(G)) \cap (g_1\theta_{p_1^m}(G))$$
$$= \begin{cases} g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0\vee p_1^m}(G) & if\ g_0{}^{-1}g_1 \in \theta p_0(G)\theta_{p_1^m}(G)\}, \\ \emptyset & \text{else}, \end{cases}$$

where$\tilde{g}_1$ is uniquely determined up to $\theta_{p_0^{-1}(p_0\vee p_1^m)}(G)$**.** While $g := g_0$ works in the second case, we need $g \in (g_0\theta_{p_0}(G)\backslash g_0\theta_{p_0}(\tilde{g}_1)\theta_{p_0\vee p_1^m}(G)$ in the first case. Note that such a g exists as $p_0 \vee p_1^m \gneqq p_0$ by the choice of m and we set $p := p_0 \vee p_1^m$.

The induction step from n to $n = 1$ is just a verbatim repetition of tile first step: Assume that the statement holds for fixed n. This means that there exist $h \in g_0\theta_{p_0}(G)$ and $q \in p_0P$ such that

$$h\theta_q(G) \subset G\backslash \bigcup_{1\leq i\leq n}\left(g_i\bigcup_{m\in\mathbb{N}}\theta_{p_i^m}\ (G)\right)$$

As $p_{n+1} \neq e$, we can find $m \in \mathbb{N}$ such that $q \notin p_{n+1}^m P$. In other words, we have $q \vee p_{n+1}^m \gneqq q$. Recall that

$$(h\theta_q(G) \cap (g_{n+1}\theta_{p_{n+1}^m}(G) = \begin{cases} h\theta_q(\tilde{g}_{n+1})\theta_{q\vee p_{n+1}^m}(G) & if h_{g_{n+1}}^{-1} \notin \theta_q(G)\theta_{p_{n+1}^m}(G)\}, \\ \emptyset & \text{else}, \end{cases}$$

where $\tilde{g}_{n+1}$ is uniquely determined up to $\theta_{q^{-1}(q\vee p_{n+1}^m)}(G)$. In the second case, take $g := h$. For the first case, we choose $g \in (h\theta_q(G)) \setminus h\theta_q(\tilde{g}_{n+1})\theta_{q\vee p_{n+1}^m}(G)$. Note that such a g exists as $q \vee p_{n+1}^m \gneqq q$ by the choice of m. Finally, let $p \coloneqq q \vee p_{n+1}^m$. Then, it is clear from the construction that we indeed have

$$g\theta_p(G) \subset G \setminus \bigcup_{1\leq i\leq n+1} \left(g_i \bigcap_{m\in\mathbb{N}} \theta_{p_i^m}(G) \right)$$

We focus to commutative irreversible algebraic dynamical systems (G, P, θ): Injective group endomnorphisins θ_p of a discrete ahelian group G correspond to surjective group endornorphisms $\hat{\theta}_p$ of its Pontryagin dual $\hat{G}$, which is a compact abelian group. Moreover, the cardinality of ker $\hat{\theta}_p$ is equal to the index [G: θ_p (G)]. Via duality, we arrive at a definition of (strong) independence for commuting surjective group endornorphisms $\eta_1$and $\eta_2$of an arbitrary group K.

We then recast the conditions for an irreversible algebraic dynamical system$(G, P, \theta)$with commutative G in terms of its dual model $(\hat{G}, P, \hat{\theta})$. This provides a new perspective on irreversible algebraic dynamical systems: If G

is comnmnutative and (G, P, θ) is of finite type, it can be regarded as an irreversible topological dynamical system. It arises from surjective local homeomorphisms $\hat{\theta}_p$ of the compact Hausdorff space $\hat{G}$.

Recall that a character χ on a locally compact abelian group G is a continuous group homomorphism $\chi: G \to \mathbb{T}$.The set of characters on G forms a locally compact abelian group$\hat{G}$when equipped with the topology of uniform convergence on compact subsets of G. Pontryagin duality states that $\hat{\hat{G}} \cong G$. For this result, we interpret $g \in G$ as a character on $\hat{G}$ via $g(\chi) \coloneqq \chi(g)$. If G is discrete, then $\hat{G}$ is compact and vice versa.

Definition (3.1.14)[3]:

Let G he a locally compact abelian group. For a subset $H \subset G$, the annihilator of H is given by $H^{\perp} := \{ x \in \hat{G} | x|_H = 1 \}$.

Lemma (3.1.15)[3]:

Let G be a locally compact abelian group and $\eta = \mathrm{G} \to G$ a group endomorphism. Then $\hat{\eta}(x)(g) \coloneqq x \circ \eta(g)$ defines a group endomorphisrn $\hat{\eta}: \hat{G} \to \hat{G}$ which is continuous if and only if $\eta$is and we have:

(i) $\hat{\hat{\eta}} = \eta$.

(ii) $\eta(G)^{\perp} = \ker \hat{\eta}$.

(iii) $\hat{\eta}(\hat{G}) \subset \hat{G}$ is dense if and only if η is injective.

(iv) $\widehat{\ker \hat{\eta}} \cong$ coker η if η (G) is closed.

In particular, if G is discrete, then ii) states that $\hat{\eta}: \hat{G} \longrightarrow \hat{G}$is surjective if and only if $\eta = G \to G$ is injective. Moreover**,** $\eta(G)$is always closed. If, in addition, coker η is finite, then $\ker \hat{\eta} \cong \widehat{\ker \hat{\eta}} \cong \text{coker}\, \eta$ follows from iv).

Lemma (3.1.16)[3]:

If G is a locally compact abelian group and $H_1, H_2 \subset G$ are subgroups, then:

(i) $(H_1 \cdot H_2)^{\perp} = H_1^{\perp} \cap H_2^{\perp}$.

(ii) $(H_1 \cap H_2^{\perp})^{\perp} = H_1^{\perp} \cdot H_2^{\perp}$ holds if H_1 and H_2 are closed.

Proposition (3.1.17)[3]:

Let G be a discrete abelian group and θ_1, θ_2 he commuting, injective endomorphisms of G. Then the following statements hold:

(i) θ_1 and $\theta_2$are strongly independent if and only if ker $\hat{\theta}_1$and ker $\hat{\theta}_2$intersect trivially.

(ii) θ_1 and θ_2 are independent if and only if ker $\hat{\theta}_1$ker $\hat{\theta}_2$= ker $\widehat{\theta_1 \theta_2}$.

Proof. For strong independence, we compute

$$(\theta_1(G)\theta_2(G))^{\perp} \overset{(3.1.16)\ (i)}{=} \theta_1(G)^{\perp} \cap \theta_2(G)^{\perp} \overset{(3.1.15)\ (ii)}{=} \ker\hat{\theta}_1 \cap \ker\hat{\theta}_2$$

Therefore, $\theta_1(G)\theta_2(G) = G$ is equivalent to $\ker\hat{\theta}_1 \cap \ker\hat{\theta}_2 = \{1_{\hat{G}}\}$. Similarly, we get

$$(\theta_1(G) \cap \theta_2(G))^{\perp} \overset{(3.1.16)(ii)}{=} \theta_1(G)^{\perp} \cdot \theta_2(G)^{\perp} \overset{(3.1.15)\ (ii)}{=} \ker\hat{\theta}_1 \cdot \ker\hat{\theta}_2$$

On the other hand, Lemma (3.1.15)(ii) gives $ker\widehat{\theta_1\theta_2} = \theta_1\theta_2(G)^{\perp}$.

This motivates the following:

Definition (3.1.18)[3]:

Two commuting, surjective group endomorphisms $\eta_1$and $\eta_2$of a group K are said to be strongly independent, if ker η_1 and ker $\eta_2$intersect trivially. $\eta_1$and $\eta_2$are called independent, if $ker\ \eta_1 \cdot ker\ \eta_2 = ker\ \eta_1\eta_2$.

It is clear that we have an equivalence between the statements:

(i) η_1 and $\eta_2$are strongly independent.

(ii) $\eta_1$is an injective group endornorphism of ker η_2.

(iii) $\eta_2$is an injective group endornorphismn of ker η_1.

If both ker η_1 and ker η_1 are finite, then strong independence and independerice coincide. Therefore, this definition is consistent with, where the case of endomorphisms (of a compact abelian group K) with finite kernels is treated. Note that there is no conflict with (strong) independence for injective group endomorphisms, see Definition (3.1.2), as all these conditions are trivially satisfied by group automorphemes.

With the observations from Lemma (3.1.15) and Lemma (3.1.16) at hands, we can now translate the setup from Definition (3.1.4) for commutative irreversible algebraic dynamical systems:

Proposition (3.1.19)[3]:

For a discrete abeliani group G, a triple (G, P, θ) is a commutative : irreversible algebraic dynamical system if and only if :

(i) $\hat{G}$ is a compact abelian group,

(ii) P is a countably generated, free, abelian monoid (with unit 1_P),and

(iii) $\hat{\theta}$ is an action of P on $\hat{G}$ by surjective group endomorphisms with the property that $\hat{\theta}_p$ and $\hat{\theta}_q$ are independent if and only if p and q are relatively prime in P.

(G, P, θ) is minimal if and only if $\bigcup_{p\in P} \ker\hat{\theta}_p \subset \hat{G}$ is dense. It is of finite (infinite) type if and only if $\ker\hat{\theta}_p$, is (infinite) finite for all $p \in P, p \neq 1_P$.

Proof: Conditions (i) and (ii) of this characterization follow directly from Lemma (3.1.15). Moreover, for any $p \in P$, the equation $\left(\ker\hat{\theta}_p\right)^{\perp} = im\theta_p$yields an isomorphism between coker θ_p and the Pontryagin dual of $\ker\hat{\theta}_p$. Combining Lemma (3.1.15) (iii) and Proposition (3.1.17) yields (iii). Note that we have $\theta_q(\mathrm{G}) \subset \theta_p(\mathrm{G})$ and, correspondingly, $\ker\hat{\theta}_p \subset \ker\hat{\theta}_q$ whenever $q \in pP$. Since P is directed, Lemma (3.1.16)(i) and Lemma (3.1.15)(ii) yield the equivalence between minimality of (G, P, θ) and $\bigcup_{p\in P} \ker\hat{\theta}_p$, being dense in $\hat{G}$. For the last claim, we recall that a locally compact abelian group is finite if and only if its dual group is finite. Thus $\ker\hat{\theta}_p$ is finite if and only if $co\ker\, \theta_p$ is finite.

We will now revisit some of the examples from this section to present their dual models.

Examples (3.1.20)[3]:

The following list corresponds to the one in Example (3.1.5).

(i) For $G = \mathbb{Z}$, a family of relatively prime numbers $(p_i)_{i \in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*}$ generates a monoid $P = |(p_i)_{i \in I}\rangle \subset \mathbb{Z}^{\times}$ which acts by $\theta_{p_i}(g) = p_i g$. In tins case, $\hat{G} = \mathbb{T}$ and $\hat{\theta}_p(t) = t^p$ forall $t \in \mathbb{T}$ and $p \in P$.

(ii) For $I \subset \mathbb{N}$, $0 \in I$, $let\ q$, $(p_i)_{i \in I} \subset \mathbb{Z}^{\times} \backslash \mathbb{Z}^{*}$, be relatively prime numbers and set $P = |(p_i)_{i \in I}\rangle$ as well as $G = Z[1/q] = \lim\limits_{\rightarrow} \mathbb{Z}$ with connecting maps given by multiplication with q. Then this constitutes a minimal commutative irreversible algebraic dynamical system of finite type, see Example (3.1.5)(ii). Then $\hat{G}$ is the solenoid $\mathbb{Z}_q = \lim\limits_{\leftarrow} \mathbb{Z} / q^k \mathbb{Z}$, on which $\hat{\theta}_p$ is given by multiplication with p.

(iii) For a finite field $\mathbb{K}$, let $p_i \in \mathbb{K}[T], i \in I$ (for an index set I) be polynomials in $G = \mathbb{K}[T]$ with the property that $(p_i) \cap (p_j) = (p_i p_j)$ holds for all $i \neq j$. Then the action $\theta\ of\ P \coloneqq |(p_i)_{i \in I}\rangle$given by multiplication with the polynomial itself yields a commnutative irreversible algebraic dynamical system of finite type, see Example (3.1.5) (iii). Then $\hat{G}$ is the ring of formal power series $\mathbb{K}[[T]]$ over $\mathbb{K}$, and $\hat{\theta}_p$ is given by multiplication with $p\ in\ \mathbb{K}[[T]]$.

Example (3.1.21)[3]:

Recall that, in Example (3.1.6), we considered $G = \mathbb{Z}^d$ for some $d \geq I$, a family of pairwise commuting matrices $(T_i)_{\boldsymbol{i \in I}} \subset M_d(\mathbb{Z}) \cap Gl_d(\mathbb{Q})$ satisfying det$|det\ T_i| > 1\ for\ all\ i \in I$and set $P = |(T_i)_{i \in I}\rangle$ with $\theta_{T_i}(g) =$

$T_i g$ In this case, we have $\hat{G} = \mathbb{T}^d$ and the endomorphism $\hat{\theta}_p$ is given by the matrix corresponding to θ_p interpreted as an endomnorphismn of $\mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{T}^d$.

Example (3.1.22)[3]:

The dual model for the unilateral shift on $G = \bigoplus_N \mathbb{Z}/n\mathbb{Z}\ for\ n \geq 2$ from Example (3.1.8) is given by the shift $(x_k)_{k\in\mathbb{N}} \longmapsto (x_{k+1})_{k\in\mathbb{N}}\ on\ \hat{G} = (\mathbb{Z}/n\mathbb{Z})^N$. The discussion for Example (3.1.9) with the restriction that G_0 be ahelian is analogous, where $\mathbb{N}$ is replaced by P and $\mathbb{Z}/n\mathbb{Z}$ by G_0.

Example (3.1.23)[3]:

In the situation of Example (3.1.10), where we will now require that$(G_n, P, \theta^{(i)})_{i\in\mathbb{N}}$ be a family of commutative irreversible algebraic dynamical systems, $G = \bigoplus_{i\in\mathbb{N}} G_i$turns into $\hat{G} = \prod_{i\in\mathbb{N}} \hat{\boldsymbol{G}}_{\boldsymbol{i}}$. For each $p \in P$**,** the group endornorphism $\hat{\theta}_p$ is given by applying $\theta_p^{(i)}$to the *i*-th component of $\hat{G}$. Ker $\hat{\theta}_p$ is infinite for all $p \in P\backslash\{1_P\}$. If each $\theta^{(\boldsymbol{i})}$ satisfies the strong independence condition from Definition (3.1.2), $\hat{\theta}$satisfies the strong independence condition from Definition (3.1.18) due to Proposition (3.1.17).

We associate a universal C*-algehra $\mathcal{O}[G, P, \theta]$to every irreversible algebraic dynamical system (G, P, θ)**.**The general approach is inspired by the methods for the case of a single group endomorphism with finite cokernel of a discrete ahelian group. Note however, that we are going to use a different spanning family than the one used.

We will examine structural properties of $\mathcal{O}[G,P,\theta]$ as well as of two nested subalgebras: the core $\mathcal{F}$ and the diagonal $\mathcal{D}$. In Lemma (3.1.31), a description of the spectrum G_θ of the diagonal $\mathcal{D}$ is provided, which allows us to regard G_θ as a completion of G with respect to θ in the case where (G,P,θ) is minimal.

Based on the description of G_θ, the action $\hat{\tau}$ of G on $G_\theta$coming from $\tau_g(e_{h,p}) = e_{gh,p}$ is shown to he always minimal. Moreover, we prove that topological freeness of $\hat{\tau}$ corresponds to minirnality of (G,P,θ), see Proposition (3.1.34). As an immediate consequence we deduce that $\mathcal{D} \times_\tau G$ is simple if and only if (G,P,θ) is minimal and $\hat{\tau}$ is amenable, see Corollary (3.1.35). This crossed product is actually isomorphic to $\mathcal{F}$, see Corollary (3.1.39).

The strategy of proof differs because we start by establishing an isomorphism between $\mathcal{O}[G,P,\theta]$and $\mathcal{D} \times (G \times_\theta P)$, by Theorem (3.2.15), we deduce that $\mathcal{O}[G,P,\theta]$ is isomorphic to the semigroup crossed product $\mathcal{F} \times P$. So we get

$$\mathcal{O}[G,P,\theta] \cong \mathcal{D} \times (G \times_\theta P) \cong \mathcal{F} \times P$$

One advantage of this strategy is that we are able to establish these isomorphisms in greater generality, i.e. without minimality of $(G,P,\theta)$and amenability of $\hat{\tau}$ which would give simplicity of both $\mathcal{F}$ and $\mathcal{O}[G,P,\theta]$.

Similarly, we conclude that, whenever (G,P,θ) is minimal and the C-action $\hat{\tau}$ on G_θ is amenable, the C*-algebra $\mathcal{O}[G,P,\theta]$is a unital UCT Kirchherg algebra, see Theorem (3.1.46) and Corollary (3.1.48). Thus

$\mathcal{O}[G, P, \theta]$is classified by its K-theory in this case due to the important classification results of Christopher Phillips and Eberhard Kirschberg.

(G, P, θ) will represent an irreversible algebraic dynamical system unless specified otherwise. $Let\ (\xi_g)_{g\in G}$ denote the canonical orthonormal basis of $\ell^2(\mathrm{G})$. For $g \in G$ and $p \in P$, define operators $U_g$and S_p on $\ell^2(\mathrm{G})$by $U_g(\xi_{g'}) := \xi_{gg'}\ and\ S_p(\xi_{g'}) \coloneqq \xi_{\theta_p(g')}$for $g' \in G$. Then $(U_g)_{g\in G}$ is a unitary representation of the group G and $S_p^*(\xi_{g'}) = \chi_{\theta_p(G)}(g')\xi_{\theta_p^{-1}}(g')$ for $g' \in G$, so $(S_p)_{p\in P}$ is a representation of the semigroup $P$by isometries. Furthermore, these operators satisfy

$$(CNP\ 1) \qquad S_pU_g(\xi_{g'}) = \xi_{\theta_p(gg')} = U_{\theta_p(g)}S_p(\xi_{g'}),$$

and

$$(CNP\ 3) = \sum_{[g]\in G/\theta_p(G)} E_{g,p}(\xi_{g'}) = \xi_{g'} \quad if\ [G:\theta_p(G)] < \infty,$$

where $E_{g,p} = U_gS_pS_p^*U_g^*$. In fact, (CNP 3) holds also in the case of an infinite index $[G:\theta_p(G)]$, as $\left(\sum_{[g]\in F} E_{g,p}\right)_{F\subset G/\theta_p(G)}$ converges to the identity on $\ell^2(\mathrm{G})$ as $F \nearrow G/\theta_p(G)$ with respect to the strong operator topology. But this convergence does not hold in norm because each $E_{g,p}$ is a non-zero projection. In motivation to construct a universal C*-algebra based on this model, it is therefore reasonable to restrict this relation to the case where $[G:\theta_p(G)]$ is finite.

As the numbering indicates, we are interested in an additional relation (CNP 2) which will increase the accessibility of the universal model: If *G* was trivial, this would simply be the condition that S_p and $S_q$doubly commute for all relatively prime *p* and *q* in *P*, i.e. $S_p^* S_q = S_q S_p^*$. This condition has been employed successfully for quasi-lattice ordered groups, for more information. But as *G* is an infinite group, this will not be sufficient.

We want to ensure that, within the universal model to be built, an expression corresponding to $S_p^* U_g S_p$ belongs to $C^*(G)$**.** This property has been used extensively of semigroup crossed products involving transfer operators.

We aim for a better understanding of the structure of the commutative subalgebra $C^*(\{ E_{g,p} | g \in G, p \in P\})$ inside $\mathcal{L}(\ell^2(G))$. In a much more general framework, this has been considered by X in Li**,** and resulted in a new definition of semigroup C*-algebras for discrete left calculative semigroups with identity. One particular strength is the close connection between amenability of semigroups and nuclearity of their C*-algebras.

All of these three instances suggest that a closer examination of the terms $S_p^* U_g S_q$ is in order. For $g = \theta_p(g_1)\theta_q(g_2)$with $g_1, g_2 \in G$ we get $S_p^* U_g S_q = U_{g_1} S_{(p\wedge q)^{-1}q} S_{(p\wedge q)^{-1}p}^* U_{g_2}$.Onthe other hand, $g \notin \theta_p(G)\theta_q(G)$ is equivalent to $g\theta_q(G) \cap \theta_p(G) = \phi$, which forces $S_p^* U_g S_q = 0$. Thus we get

$$(CNP\ 2)\ S_p^* U_g S_q = \begin{cases} U_{g_1} S_{(p\wedge q)^{-1}q} S_{(p\wedge q)^{-1}p}^* U_{g_2} & if\ g = \theta_p(g_1)\theta_q(g_2) \\ 0 & \text{else} \end{cases}$$

for all $g \in G, p, q \in P$. These observations motivate the following definition.

Definition (3.1.24)[3]:

$\mathcal{O}[G, P, \theta]$ is the universal C*-algebra generated by a unitary representation $(u_g)_{g\in G}$ of the group G and a representation $(s_p)_{p\in P}$of the semigroup P by isometries subject to the relations:

$(CNP\ 1) s_p u_g = u_{\theta_p(g)} s_p$

$$(CNP\ 2) \qquad s_p^* u_g s_q = \begin{cases} u_{g_1} s_{(p\wedge q)^{-1}q} s^*_{(p\wedge q)^{-1}p} u_{g_2} & if\ g = \theta_p(g_1)\theta_q(g_2), \\ 0, & \text{else} \end{cases}$$

$$(CNP\ 3) \qquad 1 = \sum_{[g]\in G/\theta_p(G)} e_{g,p} \qquad if\ [G: \theta_p(G)] < \infty,$$

Where $e_{g,p} = u_g s_p s_p^* u_g^*$.

We have the following immediate consequence.

Proposition (3.1.25)[3]:

$\mathcal{O}[G, P, \theta]$ has acanonical non-trivial representation on $\ell^2(G)$ given by $u_g \mapsto U_g, s_p \mapsto S_p$. In particular, $\mathcal{O}[G, P, \theta]$ is non -zero.

Lemma (3.1.26)[3]:

The linear span of $(u_g s_p s_q^* u_h)_{g,h\in G,p,q\in P}$ is dense in $\mathcal{O}[G, P, \theta]$.

Lemma (3.1.27)[3]:

The projections$(e_{\boldsymbol{g},p})_{g\in G,p\in P}$ commute. More precisely, for $g,h\in G$, and $p,q\in P$ we have

$$e_{g,p}e_{h,q}=\begin{cases}e_{g}\theta_p(h'),p\vee q & if\ \ g^{-1}h\in\theta_p(G)\theta_q(G),\\ 0 & \text{else},\end{cases}$$

where $h'\in G$ is determined uniquely up to multiplication from the right by elements of $\theta_{p^{-1}(p\vee q)}(\mathrm{G})$ by the condition that $g\theta_p(h')\in h\theta_q(\mathrm{G})$.

Proof: For$g,h\in G$and$p,q\in P$, the product $e_{\boldsymbol{g},p}e_{\boldsymbol{h},q}$ is non-zero only if $g^{-1}h\in\theta_p(G)\theta_q(G)$ by (CNP 2). Solet us assume that $g^{-1}h\in\theta_p(G)\theta_q(G)$ holds. Then there are $g',h'\in G$ such that $g^{-1}h=\theta_p(h')\theta_q(g')$. As G is a group, this is equivalent to$h\theta_q(g')^{-1}=g\theta_p(h')$. Thus weget

$$e_{g,p}e_{h,q}=u_{g\theta_p}(h')s_p s_{(p\vee q)^{-1}q}s^*{}_{(p\vee q)^{-1}p}s_q^*u^*_{h\theta q(g')^{-1}}=e_{g\theta_p(h'),p\vee q}$$

Clearly, this also proves that the two projections commute. The uniquemless assertion follows from (CNP 2).

Definition (3.1.28)[3]:

The C*-subalgehra $\mathcal{D}$ of $\mathcal{O}[G,P,\theta]$ generated by the commuting projections $(e_{\boldsymbol{g},p})_{g\in G,p\in P}$ is called the diagonal. In addition, let $\mathcal{D}_p\coloneqq C^*\big(\{e_{g,q}\big|[g]\in G/\theta_p(G)p\in qP\}\big)$ denote the C*-subalgebra of $\mathcal{D}$ corresponding to $p\in P$

We have the following.

Lemma (3.1.29)[3]:

For all $p, q \in P, p \in qP$ implies $\mathcal{D}_q \subset \mathcal{D}_p$. $\mathcal{D}$ is the closure of $\bigcup_{p\in P} \mathcal{D}_{p.}$ if $[G : \theta_{p.}(G)]$ is finite, then

$$\mathcal{D}_p = span\{e_{\boldsymbol{g},p} \big| [g] \in G/\theta_p(G)\} \cong \mathbb{C}^{[G:\theta_{p.}(G)]}$$

Let us make the following non-trivial observation:

Lemma (3.1.30)[3]:

Suppose $g \in G, p \in P$ and a finite subset $F$of $G\times P$ are chosen in such a way that $e_{\boldsymbol{g},p} \prod_{(h,q)\in F)}(1 - e_{\boldsymbol{h},q})$ is non-zero. Then there exist g' $\in G$and $p' \in P$ satisfying $e_{\boldsymbol{g}',p'} \leq e_{\boldsymbol{g},p} \prod_{(h,q)\in F)}(1 - e_{\boldsymbol{h},q})$

Proof: If F is empty, then $\prod_{(h,q)\in F)}(1 - e_{\boldsymbol{h},q})$ = 1 by convention, so there is nothing to show. Now let F be non-empty. For $(h, q) \in F)$ let us decompose q uniquely as $q = q^{(fin)} q^{(inf)}$where $[G : \theta_{q^{(fin)}}(G)]$ is finite and we require that, for each $r \in P$ with $q \in rP$, finiteness of $[G : \theta_r(G)]$ implies $q^{(fin)} \in rP$. In other words, $[G : \theta_r(G)]$ is infinite for every $r \neq 1_p$with $q^{(inf)} \in rP$ Using **(**CNP 3**)** for $q^{(fin)}$ and Lemma (3.1.27), we compute

$$1 - e_{\boldsymbol{h},q} = \left(1 - e_{\boldsymbol{h},\boldsymbol{q}^{(fin)}} e_{\boldsymbol{h},\boldsymbol{q}^{(inf)}}\right) \sum_{[k]\in G/\boldsymbol{\theta}_{\boldsymbol{q}^{(fin)}}(\boldsymbol{G})} e_{\boldsymbol{k},\boldsymbol{p}^{(fin)}}$$

$$= e_{\boldsymbol{h},\boldsymbol{q}^{(fin)}} \left(1 - e_{\boldsymbol{h},\boldsymbol{q}^{(inf)}}\right) + \sum_{\substack{[k]\in G/\boldsymbol{\theta}_{\boldsymbol{q}^{(fin)}}(\boldsymbol{G}) \\ [k]\neq[h]}} e_{\boldsymbol{k},\boldsymbol{q}^{(fin)}}$$

Therefore, we can rewrite tine initial product as

$$e_{\boldsymbol{g},q} \prod_{(h,q)\in F} (1 - e_{\boldsymbol{h_i q_i}}) = \sum_{(\tilde{g},\tilde{p})\in\tilde{F}} \ell_{\tilde{g},\tilde{p}} \prod_{(h,q)\in F_{(\tilde{g},\tilde{p})}} (1 - e_{\boldsymbol{h},q}),$$

where

(i) $\tilde{F}$ is a finite subset of $G \times P$,

(ii) $e_{\tilde{g},\tilde{p}} \leq \ell_{g,p}$ for all $(\tilde{g},\tilde{p}) \in \tilde{F}$, ,

(iii) the projections $(e_{\tilde{g},\tilde{p}})_{(\tilde{g},\tilde{p})\in\tilde{F}}$ are mutually orthogonal,

(iv) for each $(\tilde{g},\tilde{p}) \in \tilde{F}$, $\mathsf{F}_{(\tilde{g},\tilde{p})}$ is a finite subset of $G \times P$, and

(v) each $(h,q) \in \mathsf{F}_{(\tilde{g},\tilde{p})}$ satisfies $q = q^{(\text{inf})}$ *and* $\tilde{p} \notin qP$.

Since the product$e_{\boldsymbol{g},p} \prod_{(h,q)\in F}(1 - e_{\boldsymbol{h_i},\boldsymbol{q_i}})$on the left hand side is non-zero, there is$(g_o, p_o) \in \tilde{F}$ suchthat$e_{g_o,p_o} \prod_{(h,q)\in F_{(g_o,p_o)}}(1 - e_{\boldsymbol{h},q})$is non-zero. Without loss of generality, we may assume that $e_{g_o,p_o} e_{h,q}$is non-zero for all $(h,q) \in F_{(g_o,p_o)}$. Consider $F_P = \{p_o \vee q | (h,q) \in F_{(g_o,p_o)}$ for some$h \in G\}$. Pick $p_1 \in F_P$which is minimal in the sense that for any other $r \in F_P, p_1 \in rP$ implies $r = p_1$ Let (h_1,q_1), … ,$(h_n,q_n) \in F_{(g_o,p_o)}$ denotethe elements satisfying $p_0 \vee q_i = p_1$. According to Lemma (3.1.27), we have

$e_{g_o,p_o} e_{\boldsymbol{h_i},\boldsymbol{q_i}} = e_{g_o \theta p_o (g_i'),p_1}$for a suitable $g_i' \in G(for\ i = 1,..,n)$.

Note that $p_0^{-1}p_1 \neq 1_p$ and $q_1 = q_1^{(\text{inf})} \in p_0^{-1}p_1P$, so $[G : \theta_{p_0^{-1}p_1}(G)]$ is infinite. Hence there exists $g_1 \in g_0\theta_{p_0}$ with

$$e_{g_1,p_1} \leq e_{g_o,p_o} and\ e_{g_1,p_1} e_{\boldsymbol{h_i},\boldsymbol{q_i}} = 0 \quad for\ i = 1, \dots, n$$

Setting

$$\mathsf{F}_{(g_1,p_1)} \coloneqq \{(h,q) \in F_{(g_o,p_o)} | e_{h,q} e_{g_1,p_1} \neq 0\} \subsetneq F_{(g_o,p_o)},$$

we observe that

$$e_{g_1,p_1} := \prod_{(h,q)\in F_{(g_1,p_1)}} (1 - e_{\boldsymbol{h},q}) \neq 0$$

follows from the initial statement for $(g_o, p_o)$and F_{g_o,p_o} since we have chosen p_1 in a minimal way. Indeed, if the product was trivial, then there would be $(h,q) \in F_{(g_1,p_1)}$with $e_{h,q} \geq e_{g_1,p_1}$. By Lemma (3.1.27), this would force $p_1 \in qP$ and therefore $p_1 \in (p_1 \vee q)P \subset (p_0 \vee q)P$, which cannot be true since p_1 was chosen in a minimal way.

Thus, we can iterate the process used to obtain (g_1, p_1) and $F_{(g_1,p_1)}$ for $(g_0, p_0)$and $F_{(g_0,p_0)}$. After finitely many steps, we arrive at an element $(g_n, p_n) = (g', p')$with the property that $e_{g\prime,p\prime} \leq e_{(g_0,p_0)}$ is orthogonal to $e_{h,q}$ for all $(h,q) \in F_{(g_0,p_0)}$. This establishes the claimn.

The possibility of passing to smaller subprojections that avoid finitely many defect projections provided through Lemma (3.1.30) will be crucial for the proof of pure infiniteness and simplicity of $\mathcal{O}[G,P,\theta]$, see Theorem (3.1.46) and in particular Lemma (3.1.45). A first application of this observation lies in the determination of the spectrum of $\mathcal{D}$.

Lemma (3.1.31)[3]:

The spectrum of $\mathcal{D}$, denoted by G_θ, is a totally disconnected, compact Hausdorff space. A basis for the topology on G_θ is given by the cylinder' sets

$$Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)} = \{\chi \in G_\theta | \chi(e_{g,p}) = 1, \chi(e_{h_i,q_i}) = 0 \; for \; all \; i\},$$

Where $n \in \mathbb{N}$, $g, h_1, \ldots, h_n \in G$ and $p, q_1, \ldots, q_n \in P$. Moreover,

$$\iota_{(g)} \in Z_{(g',p),(h_1,q_1),\ldots,(h_n,q_n)} \Leftrightarrow g \in g'\theta_p(G) \; and \; g \notin h_i\theta_{q_i}(G) for \; all \; i$$

defines a map $\iota : G \to G_\theta$ with dense image. ι is injective if and only if (G, P, θ) is minimal.

Proof: G_θ is a totally disconnected, compact Hausdorif space since $\mathcal{D}$ is a unital C*-algebra generated by commuting projections. The statement concerning the basis for the topology on G_θ follows from Lemma (3.1.29). To see that ι has dense image, let $\chi \in G_\theta$. As the cylinder sets form a basis for the topology of G_θ, every open neighbourhood of χ containsacylinder set $Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$ With $\chi \in Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$. This means that $e_{g,p}\prod_{i=1}^{n}(1 - e_{\boldsymbol{h_i},\boldsymbol{q_i}})$ is non-zero. Hence we can apply Lemma (3.1.30) to obtain $(g', p') \in G \times P$ satisfying $e_{g',p'} \leq e_{g,p}\prod_{i=1}^{n}(1 - e_{\boldsymbol{h_i},\boldsymbol{q_i}})$. In other words, $\iota(g') \in Z_{(g,p),(h_1,q_1),\ldots,(h_n,q_n)}$ so $\iota(G)$ is a dense subset of G_θ. Now given $g, h \in G$, we observe that $\iota(g) = \iota(h)$ is equivalent to $g^{-1} h \bigcap_{p\in P} \theta_p(G)$ because the cylinder sets form a basis of the topology on the Hausdorif space G_θ. Therefore ι is injective precisely if (G, P, θ) is minimal.

Definition (3.1.32)[3]:

Let X be a topological space and G a group. A G-action on X is said to be topologically free, if the set $X^g = \{x \in X | \, g.x = x\}$ has empty interior for $g \in G \setminus \{1_G\}$.

Definition (3.1.33)[3]:

Let X be a topological space and G a group. A G-action on X is said to be minimal, if the orbit $\mathcal{O}(x) = \{g.x \mid g \in G\}$ is dense in X for every $x \in X$.

Equivalently, an action is minimal if the only invariant open (closed) subsets of X are ϕ and X.

Proposition (3.1.34)[3]:

If (G, P, θ) is an irreversible algebraic dynamical system, then the action G-action $\hat{\tau}$on $G_\theta$isminimal. Itistopologicallyfree if and only if (G, P, θ) is minimal.

Proof: On $\iota(G)$, which is dense in G_θ by Lemma (3.1.31), $\hat{\tau}$is simply given by translation from the left. Hence $\hat{\tau}$ is minimal. For the second part, we note that $\tau_{\boldsymbol{g}} = id_{\mathcal{D}}$ holds for every $g \in \bigcap_{p \in P} \theta_p(G)$. Thus, if (G, P, θ) is not minimal, there is $g \neq 1_G$ such that $G_\theta^g = G_\theta$, so $\hat{\tau}$ is not topologically free. If (G, P, θ) is minimal, then $\hat{\tau}$ acts freely on $\iota(G)$ because ι is injective and G is left-calculative. Since $\iota(G)$ is dense in G_θ, we conclude that $\hat{\tau}$ is topologically free.

Corollary (3.1.35)[3]:

The crossed product $D \times_\tau G$ is simple if and only if (G, P, θ) is minimal and $\hat{\tau}$ is amenable.

Definition (3.1.36)[3]:

The core$\mathcal{F}$ is the C*-subalgebra of $\mathcal{O}[G, P, \theta]$ generated by $\mathcal{D}$ and $(u_g)_{g\in G}$.

Lemma (3.1.37)[3]:

The linear span of $(u_g s_p s_q^* u_h^*)_{g,h\in G,p\in P}$ is dense in $\mathcal{F}$.

Proposition (3.1.38)[3]:

Let $(\boldsymbol{v}_{(\boldsymbol{g},\boldsymbol{p})})_{(\boldsymbol{g},\boldsymbol{p})\in \boldsymbol{G}\rtimes_{\boldsymbol{\theta}}\boldsymbol{P}}$denote the family of isornetries in $\mathcal{D} \rtimes (G \rtimes_\theta P)$ implementing the action of the semnigroup $G \rtimes_\theta P$ on $\mathcal{D}$ given by $(g, p)._{e_{h,q}} = e_{g\theta_p(h),pq}, \quad that\ is\ , v_{(g,p)} e_{h,q} v_{(g,p)}^* = e_{g\theta_p(h),pq},$. Then the map

$$\mathcal{O}\ [G, P, \theta] \xrightarrow{\varphi} \mathcal{D} \rtimes (G \rtimes_\theta P) u_g s_p \quad \mapsto \quad v_{(g,p)}$$

is an isomorphism.

Proof: Recall from Definition (3.1.24) that $\mathcal{O}[G, P, \theta]$ is the universal C*-alqehra generated by a unitary representation $(u_g)_{g\in G}$of the group G and a semi- group of isometries $(s_p)_{p\in P}$ subject to the relations (CNP1)-(CNP3). Hence, in order to show that φ defines a surjective *-homomorphism, it suffices to show that for every $g \in G$, the isometry $v_{(g,1_P)}$is a unitary, and that the families $(\boldsymbol{v}_{(g,\boldsymbol{1_P})})_{g\in G}, (\boldsymbol{v}_{(\boldsymbol{1_G},p)})_{p\in P}$ satisfy (CNP 1)-(CNP 3):

$$v_{(g,1_P)}v_{(g^{-1},1_P)} = v_{(g,1_P)(g^{-1},1_P)} = v_{(1_G,1_P)} = 1$$

$$(CNP\ 1)v_{(1_G,p)}v_{(g,1_P)} = v_{(1_G,p)(g,1_P)} = v_{(\theta_p(g),p)} = v_{(\theta_p(g),1_P)}v_{(1_G,p)}$$

$$(CNP\ 2)v^*_{(1_G,p)}v_{(g,1_P)}v_{(1_G,q)} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)v_{(g_1,(p\wedge\wedge q)^{-1}q)}v^*_{(g_2^{-1},(p\wedge\wedge q)^{-1}p)}$$

$$where\ g = \theta_p(g_1)\theta_q(g_2)$$

$$\Leftrightarrow \quad v_{(1_G,p)}v^*_{(1_G,p)}v_{(g,q)}v^*_{(g,q)} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)v_{(\theta_p,(g_1),p\vee q)}v^*_{(g\theta_q(g_2^{-1}),p\wedge q)}$$

$$\Leftrightarrow \quad e_{1_G,p}e_{g,q} \overset{!}{=} \chi_{\theta_p(G)\theta_q(G)}(g)e_{(g\theta_q(g_2^{-1}),p\vee q)}$$

as $g = \theta_p(g_1)\theta_q(g_1)$ gives $\theta_p(g_1) = g\theta_q({g_2}^{-1})$ This last equation holds by Lemma (3.1.27), so (CNP 2) is satisfied as well. (CNP 3) is a relation that is encoded inside $\mathcal{D}$, so it is satisfied as the range projection of the isometry $v_{(g,p)}$ coincides with $\boldsymbol{e_{g,p}}$. Injectivity of φ follows from the fact that the isometrics $u_g s_p$satisfy the covariance relation for the action of $G_{\rtimes_0}P$ on $\mathcal{D}$ since $u_g s_p e_{h,q}(u_g s_p)^* = e_{g\theta_p(h),pq} = (g,p).e_{h,q}$. Indeed, in this case there is a surjective *-homomorphism from $\mathcal{D} \rtimes (G \rtimes_\theta P)$ to $\mathcal{O}\,[G,P,\theta]$sending $\boldsymbol{v}_{(g,\boldsymbol{p})}$to $u_g s_p$ and the two *- homomorphism are mutually inverse, so φ is an isornorphisrn.

This description of $\mathcal{O}[G,P,\theta]$ allows us to deduce several relevaimt properties of $\mathcal{O}[G,P,\theta]$and its core subalgebra $\mathcal{F}$.

Corollary (3.1.39)[3]:

The isomorphism φ from Proposition (3.1.38) restricts to an isomorphism between $\mathcal{F}$ and $\mathcal{D} \rtimes G$. In particular, we have a canonical isomorphism $\mathcal{O}\,[G, P, \theta] \cong \mathcal{F} \rtimes P$.

Proposition (3.1.40)[3]:

If the G-action $\hat{\tau}$ on G_θ is amenable, then both $\mathcal{F}$ and $\mathcal{O}[G, P, \theta]$ are nuclear and satisfy the universal coefficient theorem ***(UCT***,) (Let *B* be a right R-module. Take a projective resolution of *B*

$$\dots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_n} P_0 \xrightarrow{\epsilon} B \longrightarrow 0 \qquad (1)$$

Then tensor with *D* to obtain

$$\dots \longrightarrow P_n \otimes D \xrightarrow{d_n \otimes 1} P_{n-1} \otimes D \longrightarrow \cdots \xrightarrow{d_1 \otimes 1} P_0 \otimes D \xrightarrow{\epsilon \otimes 1} B \otimes D \longrightarrow 0 \quad (2)$$

Since

$$\mathrm{im}(d_{n+1}\otimes 1) < ker(d_n \otimes 1), (d_n\otimes 1)\circ(d_{n+1}\otimes 1) = (d_n \otimes 1)(im(d_{n+1}\otimes 1)) = 0)[8].$$

Proof : As $\mathcal{F} \cong \mathcal{D} \rtimes_\tau G$ by Corollary (3.1.39) and $\hat{\tau}$ is amenable, $\mathcal{F}$ is nuclear by results of Claire Aniatharaman-Delaroche. Similarly, amenability of $\hat{\tau}$ passes to the corresponding transformation groupoid $\mathcal{G}$. Thus, we can rely on results of Jean-Louis Tu, to deduce that $\mathcal{F} \cong \mathcal{D} \rtimes_\tau G \cong C^*(\mathcal{G})$ satisfies the *UCT*. The class of separable nuclear C*-algebras that satisfy the *UCT* is closed under crossed products by $\mathbb{N}$ and inductive limits. Recall that either $P \cong \mathbb{N}^k$ for some $k \in \mathbb{N}$ or $P \cong \bigoplus_{\mathbb{N}} \mathbb{N}$ according to condition (ii) of Definition (3.1.4). Hence the claims concerning $\mathcal{O}[G, P, \theta]$ follow from $\mathcal{O}[G, P, \theta] \cong \mathcal{F} \rtimes P$, see Corollary (3.1.39).

Corollary (3.1.41)[3]:

The map $E_2\left(u_g s_p s_q^* u_h^*\right) := \delta_{gh} e_{g,p}$, defines a conditional expectation $E_2 : \mathcal{F} \to \mathcal{D}$ which is faithful if and only if $\hat{\tau}$ is amenable.

Proof: Due to Corollary (3.1.39), $\mathcal{F}$ is canonically isomorphic to $\mathcal{D} \rtimes_\tau G$. Since G is discrete, the reduced crossed product $\mathcal{D} \rtimes_{\tau,r} G$ has a faithful conditional expectation given by evaluation at 1_G The map E_2 is nothing but the composition of

$$\mathcal{F} \cong \mathcal{D} \rtimes_\tau G \twoheadrightarrow \mathcal{D} \rtimes_{\tau,r} G \xrightarrow{ev_{1_G}} \mathcal{D}$$

The canonical surjection $\mathcal{D} \rtimes_\tau G \twoheadrightarrow \mathcal{D} \rtimes_{\tau,r} G$ is an isomorphism if and only if $\hat{\tau}$ is amenable.

Corollary (3.1.42)[3]:

The map $E\left(u_g s_p s_q^* u_h^*\right) := \delta_{pq} \delta_{gh} e_{g,p}$ defines a conditional expectation $E : \mathcal{O}[G, P, \theta] \to \mathcal{D}$ which is faithful if and only if $\hat{\tau}$ is amenable.

Proof: Clearly, $E = E_2 \circ E_1$, , so the result follows from Corollary (3.1.41) .

Note that if G happens to be amenable, the faithful conditional expectation$E$ can he obtained directly by showing that the left Ore semigroup $G \rtimes_\theta P$ has an amenable enveloping group. Before we can turn to simplicity of $\mathcal{O}[G, P, \theta]$, we need the following general observations:

Definition (3.1.43)[3]:

Given a family of commuting projections $(E_i)_{i\in I}$ in a unital C*-algebra B and finite subsets $A \subset F$ of I, let

$$Q^E_{F,A} : \prod_{i\in A} E_i \prod_{i\in F\setminus A} (1 - E_i).$$

Products indexed by $\emptyset$ are treated as *1* by convention.

Lemma (3.1.44)[3]:

Suppose $(E_i)_{i\in I}$ is a family of commuting projections in a unital C*-algebra 1 $B, A \subset F$ are finite subsets of I. Then each $Q^E_{F,A}$is a projection, $\sum_{A\in F} Q^E_{F,A}$=1 and, for all **,** $\boldsymbol{\lambda_i} \in \mathbb{C}, i \in F$, we have

$$\sum_{i\in F} \lambda_i E_i = \sum_{A\subset F}\left(\sum_{i\in A} \lambda_i\right) Q^E_{F,A} \text{ and } \left\|\sum_{i\in F} \lambda_i E_i\right\| = \max_{\substack{A\subset F\\ Q^E_{F,A}\neq 0}} \left|\sum_{i\in A} \lambda_i\right|$$

Proof: Since the projections E_i commute, $Q^E_{F,A}$is a projection. The second assertion is obtained via $1 = \prod_{i\in F}(E_i + 1 - E_i) = \sum_{A\subset F} Q^E_{F,A}$. The two equations from the claim follow immediately from this.

Lemma (3.1.45)[3]:

For $d = \sum_{i=1}^{n} \lambda_i\, e_{gi,pi} \in \mathcal{D}_+$with $\lambda_i \in \mathbb{C}$ and $(g_i, p_i) \in G \times P$, there exist $(g, p) \in G \times P$ satisfying $de_{g,p} = \|\mathrm{d}\|_{e_{g,p.}}$

Proof: d is contained in $C^*(\{Q^e_{F,A} | A \subset F = \{(g_i, p_i) | 1 \le i \le n\}\})$, which is commutative by Lemma (3.1.27). Then Lemma (3.1.44) says that there exists $A \subset F$ such that $Q^e_{F,A}$is non-zero and $dQ^e_{F,A} = \|d\|_{Q^e_{F,A}}$. In particular, $\prod_{(g,p)\in A} e_{g,p}$ is non-zero, so Lemma (3.1.27) implies that there exist $g_A \in G$ and $p_A \in P$ such that $\prod_{(g,p)\in A} e_{g,p} = e_{g_A,p_A}$. Thus, we can apply Lemma (3.1.30) to $e_{g_A,p_A} \prod_{(h,q)\in F\setminus A} (1 - e_{h,q}) = Q^e_{F,A} \neq 0$ and the proof is complete.

Note that the hard part of the proof for Lemma (3.1.45)is hidden in Lemma (3.1.30).

Theorem (3.1.46)[3]:

If (G, F, θ) is minimal and the action $\hat{\tau}$ is amenable, then $\mathcal{O}[G, P, \theta]$ is purely infinite and simple.

Proof: The linear span of $(u_g s_p s_q^* u_h^*)_{g,h\in G,p,q\in P}$is dense in $\mathcal{O}[G, P, \theta]$ according to Lemma (3.1.26). Every element z from this linear span is of the form

$$z = \sum_{i=1}^{m1} c_i e_{g_i,p_i} + \sum_{i=m_1+1}^{m2} c_i u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^* + \sum_{i=m_2+1}^{m3} c_i u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^*$$

where $c_i \in \mathbb{C}$,

(i) $g_i \neq h_i for\ m_1 + 1 \le i \le m_2$,and

(ii)$p_i \neq q_i for\ m_2 + 1 \le i \le m_3$.

By Corollary (3.1.42), we have $E(z) = \sum_{i=1}^{m_1} c_i e_{g_i,p_i} \in \mathcal{D}$. If we assume z to be non-zero and positive, which we will do from now on, then $E(z) > 0\, as\, E$ is a faithful conditional expectation. Applying Lemma (3.1.45) to $E(z)$ yields $(g, p) \in G \times P$ such that

(iii) $E(z)e_{g,p} = \|E(z)\|e_{g,p}$

In order to prove simplicity and pure infiniteness of $\mathcal{O}[G, P, \theta]$, it suffices to establish the following claim: There exist $(\tilde{g}, \tilde{p})G \times P$ satisfying

(i) $e_{\tilde{g},\tilde{p}} \le e_{g,p}$

(ii) $e_{\tilde{g},\tilde{p}} u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^* e_{\tilde{g},\tilde{p}} = 0\, for\, m_1 + 1 \le i \le m_2$ and

(iii) $e_{\tilde{g},\tilde{p}} u_{g_i} s_{p_i} s_{q_i}^* u_{h_i}^* e_{\tilde{g},\tilde{p}} = 0\, for\, m_2 + 1 \le i \le m_3$.

Indeed, if this can be done, then we get

$$e_{\tilde{g},\tilde{p}} z_{\tilde{g},\tilde{p}} \overset{\text{(ii),(iii)}}{=} e_{\tilde{g},\tilde{p}} E(z) e_{\tilde{g},\tilde{p}} \overset{\text{(iii),(i)}}{=} \|E(z)\| e_{\tilde{g},\tilde{p}}$$

Now for $x \in \mathcal{O}[G, P, \theta]$ positive and non-zero, let $\boldsymbol{\varepsilon} > 0$ and choose a positive, non-zero element z, which is a finite linear combination of elements $u_{g'} S_{p'} S_{q'}^* u_{h'}^*$to approximate x up to ε. Then $\|E(z)\|$ is a non-zero positive element of $\mathcal{D}$. Thus, choosing $e_{\tilde{g},\tilde{p}}$as above, we see that $e_{\tilde{g},\tilde{p}} z_{\tilde{g},\tilde{p}} = \|E(z)\| e_{\tilde{g},\tilde{p}}$is invertible in $e_{\tilde{g},\tilde{p}}\, \mathcal{O}[G, P, \theta] e_{\tilde{g},\tilde{p}}$. If $\|x - z\|$ is sufficiently small, this implies that $e_{\tilde{g},\tilde{p}}\, x e_{\tilde{g},\tilde{p}}$ is positive and invertible in $e_{\tilde{g},\tilde{p}}\, \mathcal{O}[G, P, \theta] e_{\tilde{g},\tilde{p}}$ as well because $\|E(z)\| \overset{\varepsilon \to 0}{\longrightarrow} \|E(x)\| > 0$. Hence, if we denote its inverse by y, then

$$\left(y^{\frac{1}{2}}\, u_{\tilde{g}} s_{\tilde{p}}\right)^{*} e_{\tilde{g},\tilde{p}}\, x\, e_{\tilde{g},\tilde{p}} \left(y^{\frac{1}{2}}\, u_{\tilde{g}} s_{\tilde{p}}\right) = 1.$$

We claim that there is a pair e $(\tilde{g},\tilde{p}) \in G \times P$satisfying (i)-(iii). Let $(g',p') \in g\theta_p(G) \times pP$ and $m_1 + 1 \leq i \leq m_2$. Noting that $u_{g_i} s_{p_i} s_{p_i}^{*} u_{h_i}^{*} = u_{g_i h_i{}^{-1}} e_{h_i,p_i}$Lemma (3.1.27) implies

$$e_{g'p'} u_{g_i h_i{}^{-1}} e_{h_i,p_i} e_{g'p'} = e_{g'p'} u_{g_i h_i{}^{-1}} e_{h_i,p_i}$$

$$= \chi_{\theta_{p'}(G)}((g')^{-1} g_i h_i{}^{-1} g')\, u_{g_i h_i{}^{-1}} e_{g',p'} e_{h_i,p_i}$$

According to (i), we have $(g')^{-1} g_i h_i{}^{-1} g' \neq 1_G$. Thus, minimality of (G,P,θ)provides $p_i' \in pP$ with the property that $(g')^{-1} g_i h_i{}^{-1} g' \notin \theta_{p_i'}(G)$. So if we take $p^{(b)} \coloneqq \bigvee_{i=m_1+1}^{m_2} p_i'$, then (i) and (ii) of the claim hold for all $(g',p') \in g\theta_p(G) \times p^{(b)}P$. Let us assume that $p' \geq p^{(b)} \vee \bigvee_{i=m_2+1}^{m_2} p_i \vee q_iandg' \in g\theta_{p'}(G)$. Then condition (iii) holds for (g',p') if and only if

$$0 = s_{p'}^{*} u_{(g')^{-1} g_i} s_{p_i} s_{q_i}^{*} u_{h_i{}^{-1} g'} s_{p'}$$
$$= \chi_{\theta_{p_i}(G)}((g')^{-1} g_i)\, \chi_{\theta_{q_i}(G)}(h_i{}^{-1} g')\, s_{p_i^{-1} p'}^{*}\, u_{\theta_{p_i}^{-1}((g')^{-1} g_i)\theta_{q_i}^{-1}(h_i{}^{-1} g')} s_{q_i^{-1} p'}$$

is valid for all $m_2 + 1 \leq i \leq m_3$. This is precisely the case if at least one of the conditions

(i) $(g')^{-1} g_i \in \theta_{p_i}(G)$,

(ii) $(g')^{-1} h_i \in \theta_{q_i}(G)$, or

(iii) $\theta_{p_i}^{-1}((g')^{-1} g_i)\theta_{q_i}^{-1}(h_i{}^{-1} g') \in \theta_{(p_i \vee q_i)^{-1} p'}(G)$

fails for each i. Suppose, we have an index i for which the first two conditions are satisfied. Using injectivity of $\theta_{p_i \vee q_i}$the third condition is equivalent to $\theta_{r_q}((g')^{-1}g_i)\theta_{r_p}(h_i{}^{-1}g') \in \theta_{p'}(G)$ where $r_p \coloneqq (p_i \wedge q_i)^{-1}p_i$ and $r_q \coloneqq (p_i \wedge q_i)^{-1}q_i$. Condition b) implies $r_p \wedge r_q = 1_P \neq r_p r_q$. Moreover, we have

$$\theta_{r_q}((g')^{-1}g_i)\theta_{r_p}(h_i{}^{-1}g') = 1_G \Leftrightarrow \theta_{r_q}(g')\theta_{r_p}(g')^{-1} = \theta_{r_q}(g_i)\theta_{r_p}(h_i{}^{-1}).$$

Let us examine the range of the map $f_i: G \longrightarrow G$ that is defined by $g \mapsto \theta_{r_q}(g)\theta_{r_p}(g)^{-1}$. Note that f_i need not be a group homomorphism unless G is abelian, in which case the following part can be shortened. If $k_1, k_2 \in G$ have the same image under f_i, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$. By $(C1)$ from Definition (3.1.4), this gives $k_2^{-1}k_1 \in \theta_{r_p}(G) \cap \theta_{r_q}(G) = \theta_{r_p r_q}(G)$ But if $k_2^{-1}k_1 = \theta_{r_p r_q}(k_3)$ holds for some $k_3 \in G$, then $\theta_{r_p}(k_2^{-1}k_1) = \theta_{r_q}(k_2^{-1}k_1)$ implies that $\theta_{r_p}(k_3) = \theta_{r_q}(k_3)$ holds as well because P is commutative and $\theta_{q_{i,1}q_{i,2}}$ is injective. By induction, we get $k_2^{-1}k_1 \in \bigcap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$.

Hence $f_i^{-1}(\theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1}))$ is either empty, in which case there is nothing to do, or it is of tile form $\tilde{g}_i \bigcap_{n\in\mathbb{N}} \theta_{(r_p r_q)^n}(G)$ for a suitable $\tilde{g}_i \in G$. But for the collection of those i for which the preimage in question is non-empty, we can apply Lemma (3.1.13) to obtain $\tilde{g} \in g\theta_{p'}(G)$ such that $f_i(\tilde{g}) \neq \theta_{r_p}(h_i)\theta_{r_q}(g_i^{-1})$ for all relevant i.

By condition (C2) from Definition (3.1.4), we can choose $\tilde{p} \geq p'$ large enough so that these elements are still different modulo $\theta_{(p_i \vee q_i)^{-1}\tilde{p}}(G)$ for all i. In this case, we get

$$\theta_{p_i}^{-1}(\tilde{g}^{-1} g_i)\theta_{q_i}\left(h_i{}^{-1}\tilde{g}\right) \notin \theta_{(p_i \vee q_i)^{-1}\tilde{p}}(G) \text{ for all } m_2 + 1 \leq i \leq m_3,$$

so $(\tilde{g}, \tilde{p})$satisfies (iii). In other words, we have proven that the pair $(\tilde{g}, \tilde{p})$ satisfies (i)-(iii). Thus, $\mathcal{O}[G, P, \theta]$ is purely infinite and simple.

From this result, we easily get tile following corollaries:

Corollary (3.1.47)[3]:

If (G, P, θ) is minimal and $\hat{\tau}$is amenable, then the representation $\lambda : \mathcal{O}[G, P, \theta] \rightarrow \mathcal{L}(\ell^2(G))$from Proposition (3.1.25) is faithful.

Proof. This follows readily from Proposition (3.1.25) and simplicity of $\mathcal{O}[G, P, \theta]$**.**

Combining Lemma (3.1.26), Theorem (3.1.46) and Proposition (3.1.40),we get:

Corollary (3.1.48)[3]:

If (G, P, θ) is minimal and $\widehat{\mathrm{T}}$is amenable, then $\mathcal{O}[G, P, \theta]$ is a unital *UCT* Kirchherg algebra.

Thus, minimal irreversible algebraic dynamical systems $(G, P, \theta)$for which the action $\hat{\tau}$ is amenable yield C*-algebraic $\mathcal{O}[G, P, \theta]$that are classified by their K-theory. Let us come back to some of the examples from this section and briefly describe the structure of the **C*-**algebraic obtained in the various cases:

Examples (3.1.49)[3]:

(i) Let $G = \mathbb{Z}, (p_i)_{i\in I} \subset \mathbb{Z}\backslash\{0, \pm 1\}$ be a family of relatively prime integers, and set $P = |(p_i)_{i\in I}\rangle \subset \mathbb{Z}^{\times}$, which acts on G by $\theta_i(g) = p_i g$. We know from the considerations in Example (3.1.5) (i) that (G, P, θ) is minimal, so $\mathcal{O}[G, P, \theta]$ is a unital *UCT* Kirchberg algebra. If we denote $p := \prod_{i\in I}|p_i| \in \mathbb{N} \cup \{\infty\}$, then G_θ can be identified with the p-adic completion $\mathbb{Z}_p = \lim\limits_{\leftarrow}(\mathbb{Z}/q\mathbb{Z}, \theta_q)_{q\in P}\ of\ \mathbb{Z}$. Moreover, $\mathcal{F}$ is the Bunce-Deddens algebra of type p^{∞} for the classification of Bunce-Deddens algebras by supernatural numbers.

(ii) Let $I \subset \mathbb{N}$, choose $\{q\} \cup (p_i)_{i\in I} \subset \mathbb{Z}\backslash\mathbb{Z}^{*}$relatively prime, $P = |(p_i)_{i\in I}\rangle$, set $G = \mathbb{Z}[1/q]$, and let $\theta_p(g) = pg$ for $g \in G, p \in P$. As in (i), $\mathcal{O}[G, P, \theta]$ is a *UCT* Kirchberg algebra by the considerations in Example (3.1.5) (ii) and Corollary (3.1.48). If $p := \prod_{i\in I}|p_i| \in \mathbb{N} \cup \{\infty\}$, then G_θ can be thought of as a p-adic completion of $\mathbb{Z}[1/q]$ and we obtain $\mathcal{F} \cong G(G_\theta) \rtimes_\tau \mathbb{Z}[1/q]$.

Eamp1e (3.1.50)[3]:

We have seen in Example (3.1.8) that for $n \geq 2$, the dynamnical system given by the unilateral shift on $G = \bigoplus_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$ is a minimal commutative irreversible algebraic dynamical system of finite type. It has been observed that $\mathcal{O}[G, P, \theta]$is isomorphic to$\mathcal{O}_n$ in a canonical way: If $e_1 = (1, 0, 0, \dots,) \in G, s \in \mathcal{O}[G, P, \theta]$ denotes the generating isometry for P

and $s_1, \dots, s_n$ are the generating isometries of $\mathcal{O}_n$then this isomorphism is given by $u_{ke_1}s \mapsto s_k \quad for\ k = 1, \dots, n$. In particular, $\mathcal{F}$ is the *UHF* algebra of type n^{∞} and G_θ is homeomorphic to the space of infinite words using the alphabet $\{1, \dots, n\}$.

Ecamp1e (3.1.51)[3]:

Given a family $(G^{(i)}, p, \theta^{(i)})_{i\in\mathbb{N}}$, where each $(G^{(i)}, p, \theta^{(i)})$ an irreversible algebraic dynamical system, we can consider $G = \bigoplus_{i\in\mathbb{N}} G^{(i)}$, on which Pacts component-wise. Assume that each $(G^{(i)}, p, \theta^{(i)})$ and hence (G, P, θ) is minimal, compare Example (3.1.10). We have $G_\theta \cong \prod_{i\in I} G^{(i)}_{\theta(i)}$. Thus the G-action $\hat{\tau}$ on G_θ is amenable if and only if each G_i-action $\hat{\tau}_i$ on $G^{(i)}_{\theta(i)}$ is amenable. As *G* is commutative (amenable) if and only if eac-h $G^{(i)}$ is, there are various cases where amenability of $\hat{\tau}$is for granted. In such situations, $\mathcal{O}[G, P, \theta]$ is a unital *UCT* Kirchberg algebra.

Example (3.1.52)[3]:

For the examples arising from free group $\mathbb{F}_n\ with\ 2 \leq n \leq \infty$, see Example (3.1.11), we are able to provide criteria (i)-(iii) to ensure that we obtain minimal irreversible algebraic dynamical systems. Hence G_θ can be interpret as a certain completion of $\mathbb{F}_n$ with respect to θ. Now $\mathbb{F}_n$, is far from being amenable, but the action $\hat{\tau}$could still be amenable: The free groups are known to be exact. By a famous result of Narutaka Ozawa, exactness of a discrete group is equivalent to amenability of the left translation action on its Stone-Cech compactification . Recently, Mehrdad Kalantar and Matthew Kennedy have shown that exactness of a discrete group is also determined completely by amenability of the natural action on its Furstenberg boundary. The latter space is usually substantially smaller than the Stone-Cech compactification and their methods may give some insights into the question of amenability of the examples presented here. -

Section (3.2): A product Systems Perspective and Cross Prouducts by Semidirect Products:

We provide a more detailed presentation of the case where $(G,P,\theta)$is of finite type. We exhibit additional structural properties of the spectrum G_θ of the diagonal $\mathcal{D}$ in $\mathcal{O}[G,P,\theta]$. The assumption that $\theta_p(G) \subset G$ is normal for every $p \in P$ causes G_θ to inherit the group structure from *G*. This turns G_θ into a profinite group. If, in addition, (G,P,θ) is minimal and G is amenable, then $\mathcal{F}$ falls into the class of generalised Bunce-Deddens algebras they belong to a large class of C*-algebras that can be classified by K-theory.

We are particularly interested in the case where *G* is abelian. For such dynamical systems, the situation is significantly easier as $\theta_p(\mathrm{G}) \subset \mathrm{G}$ is normal for all $p \in P$ and the action $\hat{\tau}$ is always amenable. In fact, the structure of $\mathcal{D}$ and $\mathcal{F}$ is quite similar to the one discovered in the singly generated case: G_θ is a compact abelian group and we have a chain of isomorphisms $\mathcal{F} \cong C(G_\theta) \rtimes_\tau G \cong C(\hat{G}) \rtimes_\tau \hat{G}_\theta$. We will assume that (G,P,θ) is an irreversible algebraic dynamical system of finite type.

Proposition (3.2.1)[3]:

Suppose (G,P,θ) is minimal and G is amenable. Then $\mathcal{F}$ is a generalised Bunce-Deddens algebra.

Proof: This follows directly from the construction of the generalised Bunce Deddens algebras presented: Choose an arbitrary, increasing, cofinal sequence $(p_n)_{n\in\mathbb{N}} \subset P$, where cofinal means that, for every $q \in P$ there exists an $n \in \mathbb{N}$ such that $p_n \in qP$. Then $(\theta_{p_n}(G))_{n\in\mathbb{N}}$ is a family of nested, normal subgroups of finite index in G. This family is separating for G by minimality of (G,P,θ).

These assumptions force $\mathcal{F}$ to be unital, nuclear, separable, simple, quasidiagonal, and to have real rank zero, stable rank one, strict comparison for projections as well as a unique tracical state. As the combination of real rank zero and strict comparison for projections yields strict comparison, so $\mathcal{F}$ also has finite decomposition rank. This establishes the remaining step to achieve classification of the core $\mathcal{F}$ by means of its Elliott invariant $(K_0(\mathcal{F}), K_0(\mathcal{F})_+, [1_{\mathcal{F}}], K_1(\mathcal{F}))$.

Corollary (3.2.2)[3]:

Let (G_i, P_i, θ_i) he minimal and G_i be amenable for i = 1,2. If $\mathcal{F}_1$ and $\mathcal{F}_2$ denote the respective cores, then $\mathcal{F}_1 \cong \mathcal{F}_2$ holds if and only if

$$(K_0(\mathcal{F}_1), K_0(\mathcal{F}_1)_+, [1_{\mathcal{F}_1}], K_1(\mathcal{F}_1)) \cong (K_0(\mathcal{F}_2), K_0(\mathcal{F}_2)_+, [1_{\mathcal{F}_2}], K_1(\mathcal{F}_2)).$$

We presenting an intriguing isomorphism of group crossed products on the level of $\mathcal{F}$.

Corollary (3.2.3)[3]:

Let (G, P, θ) be commutative and minimal. Then there is a $\hat{G}_\theta$-action $\hat{\tau}$ on C($\hat{G}$) for which $\mathcal{F} \cong C(G_\theta)\rtimes_\tau G \cong C(\hat{G})_{\rtimes_{\hat{\tau}}}\hat{G}_\theta$.

Proof: The first isomorphism has been achieved in Corollary 3.19. For the second part, let $\hat{\tau}_{\chi_\theta}(\chi)(g) \coloneqq \chi_\theta(\iota(g))\chi(g)\ for\ \chi_\theta \in \hat{G}_\theta, \chi \in \hat{G}$and $g \in G$. Since $\iota : G \to G_\theta$ is a group homomorphism, $\hat{\tau}_{\chi_\theta}(\chi)$ defines a character of G. Clearly $\hat{\tau}$ is compatible with the group structure on $\hat{G}_\theta$. The group homomorphism ι identifies G with a dense subgroup of G_θ. In this case the characters on G_θ are in one-to-one correspondence with the characters on G. Note that this correspondence is precisely given by regarding characters on G_θ as characters on G using ι. Therefore, $\hat{\tau}$defines an action of $\hat{G}_\theta$ by homneornorphisms of the compact space $\hat{G}$. Once we know that $\hat{\tau}$defines an action, we readily see that there is a canonical surjective *-hornomnorphism $C(G_\theta) \rtimes_{\hat{\tau}} G \twoheadrightarrow C(\hat{G}) \rtimes_{\hat{\tau}} \hat{G}_\theta$. As $C(G_0) \rtimes_{\hat{\tau}} G$ is simple, this map is an isomorphism.

We provide a product system of Hilbert bimodules for each irreversible algebraic dynamical system (G, P, θ). The features of (G, P, θ) result in a particularly well-behaved product system χ. Therefore, it is possible to obtain a concrete presentation of $\mathcal{O}_X$**.** from the data of the dynamical system. In the case of irreversible algebraic dynamical systems of finite type, this algebra is shown to be isomorphic to $\mathcal{O}[G, P, \theta]$.

The corresponding result in the general case, that is, allowing for the presence of group endomorphisms $\theta_p$of G with infinite index, requires a more involved argument. The reason is that the prerequisites are not met, so one has to deal with Nica covariarice of representations. Since this is more closely related to the Nica-Toeplitz algebra $\mathcal{NT}_{x}$ we will only treat the finite type case for the strategy in the general case. More precisely, it shows that, for $x$associated to (G, P, θ), Nica covariance boils down to its original form. A representation φ of the product system χ is Nica covariant if and only if $\varphi_p(1_{C^*(G)})$ and $\varphi_q(1_{C^*(G)})$ are doubly commuting isometrics whenever p and q are relatively prime in P.

We start with a brief recapitulation of the necessary definitions for product systems and Cuntz-Nica-Pimnsner covariance.

Definition (3.2.4)[3]:

A product system of Hilbert bimodules over a monoid P with coefficients in a C*-algebra A is a monoid x together with a monoidal homomorphism $\rho: x \to P$ such that:

(i) $x_p := \rho^{-1}(p)$ is a Hilbert bimodule over A for each $p \in P$,

(ii) $x_{1_p} \cong {}_{id}A_{id}$as Hilbert himnodules and

(iii) for all $p, q \in P$ we have $x_p \otimes_A x_q \cong x_{pq}\ if\ p \neq 1_P$, and $x_{1_P} \otimes_A x_q \cong \overline{\phi_q(A) x_q}$.

Definition (3.2.5)[3]:

Let $\mathcal{H}$ be a Hilbert bimodule over a C*- algebra A and. $(\xi_i)_{i\in I} \subset \mathcal{H}$ Consider the following properties:

(i) $\langle \xi_i, \xi_j \rangle = \delta_{ij} 1_A$for all $i, j \in I$.

(ii) $\eta = \sum_{i \in I} \xi_i \langle \xi_i, \eta \rangle \ for \ all \ \eta \in \mathcal{H}$.

If $(\xi_i)_{i \in I}$satisfies (i) and (ii), it is called an orthonormal basis for $\mathcal{H}$.

Lemma (3.2.6)[3]:

Let $\mathcal{H}$ be a Hilbert bimodule. If $(\xi_i)_{i \in I} \subset \mathcal{H}$ is an orthonormal basis, then $(\Theta_{\xi_i, \xi_j})_{i,j \in I}$ is a system of matrix units and $\sum_{i \in I} \Theta_{\xi_i, \xi_i} = 1_{\mathcal{L}(\mathcal{H})}$ if $\mathcal{H}$ admits a finite orthonormal basis, then $K(\mathcal{H}) = \mathcal{L}(\mathcal{H})$.

This lemma is a reformulation implies that product systems whose fibres have finite orthonormal bases are compactly aligned. An explicit proof of this fact is presented .

Definition (3.2.7)[3]:

Let χ be a product system over P and suppose B is a C*-algebra. A map $\varphi : \chi \to B$, whose fibre maps $\chi_p \to B$ are denoted by φ_p, is called a Toeplitz representation of χ, if :

(i) φ_{1_P} is a *-homnomnorphisrn.

(ii) φ_p is linear for all $p \in P$.

(iii) $\varphi_p(\xi)^* \varphi_p(\eta) = \varphi_{1_P}(\langle \xi, \eta \rangle)$ for all $p \in P$ and $\xi, \eta \in \chi_p$.

(iv)$\varphi_p(\xi) \varphi_q(\eta) = \varphi_{pq}(\xi\eta)$ for all $p, q \in P$ and $\xi \in \chi_p$,$\eta \in \chi_q$.

A Toeplitz representation will be called a representation whenever there is no ambiguity. Given a representation φ of χ in B, it induces

*-homomorphisms $\psi_{\varphi,p}$: $\mathrm{K}(\chi_p) \to B$ for $p \in P$ characterised by $\Theta_{\xi,\eta} \to \varphi_p(\xi)\varphi_q(\eta)^*$. If χ is compactly aligned, the representation φ is said to be Nica covariant, if $\psi_{\varphi,p}(k_p)\psi_{\varphi,q}(k_q) = \psi_{\varphi,p\vee q}\left(\iota_p^{p\vee q}(k_p)\iota_q^{p\vee q}(k_q)\right)$holds for all $p, q \in P$ and $k_p \in K(\chi_p), k_q \in K(\chi_q)$. Concerning the choice of an appropriate notion of Cuntz-Pimnsner covariance for product systems, there have been multiple attempts:

Definition (3.2.8)[3]:

Let B be a C*-algebra and suppose χ is a compactly aligned product system of Hubert bimodules over P with coefficients in A.

(CP_F) A representation $\varphi: \chi \to B$ is called Cuntz-Pimsner covariant in the sense of [3], if it satisfies

$$\psi_{\varphi,p}(\emptyset_p(a)) = \varphi_{1_P}(a) \text{ for all } p \in P \text{ and } a \in \emptyset_p^{-1}(K(\chi_p)) \subset A.$$

(CP) A representation $\varphi: \chi \to B$ is called Cuntz-Pimsner covariant, if the following holds:

Suppose $F \subset P$ is finite and we fix $k_p \in K(\chi_p)$ for each $p \in F$. If, for every $r \in P$, there is $s \geq r$ such that

$$\sum_{p \in F} \iota_p^t(k_p) = 0 \text{ holds for all } t \geq s,$$

then $\sum_{p \in F} \psi_{\varphi,p}(k_p) = 0$ holds true.

(CNP) A representation $\varphi : \chi \to B$ is said to be Cuntz-Nica-Pimsner' covariant, if it is Nica covariant and (CP)-covariant.

Fortunately, it was observed that the different notions are closely related that (CP_F) implies Nica covariance in the cases of interest to us.

Proposition (3.2.9)[3]:

Suppose (G, P, θ) is an irreversible algebraic dynamical system. Let $(u_g)_{g\in G}$ denote the standard uniaries generating $C^*(G)$ and α be the action of P on C*(G) induced by $\theta, i.e. \alpha_p(u_g) = u_{\theta_p(g)}$ for $p \in P$ and $g \in G$. Then $\chi_p := C^*(G)_{\alpha_p}$, with left action ϕ_p given by multiplication in $C^*(G)$ and inner product $\langle u_g u_h \rangle_p = \chi_{\theta_p(G)}(g^{-1}h) u_{\theta_p^{-1}(g^{-1}h)}$ is an essential Hilbert bimodule. The union of all χ_p forms a product system χ over P with coefficients in $C^*(G)$. χ is a product system with orthonormal bases. it is of finite type if (G, P, θ) is of finite type.

Proof: It is straightforward to show that χ defines a product system of essential Hilbert bimodules and we omit the details. For $p \in P$, we claim that every complete set of representatives $(g_i)_{i\in I}$ for $G/\theta_p(G)$ gives rise to an orthonormal basis of χ_p. Indeed, if we fix such a transversal $(g_i)_{i\in I}$ and pick $g \in G$, then $\langle u_{g_i}, u_g \rangle_p = \chi_{\theta_p(G)}(g_i^{-1} g) u_{\theta_p^{-1}(g_i^{-1} g)}$ equals 0 for all but one $j \in I$, namely the one representing the left-coset [g] in $G/\theta_p(G)$. Thus, the family $(u_{g_i})_{i\in I} \subset \chi_p$ consists of orthonormnal elemnents with respect to $\langle .,. \rangle_p$, and $u_{g_i} \alpha_p(\langle u_{g_i}, u_g \rangle) = \delta_{ij} u_g$ so $(u_{g_i})_{i\in I}$ satisfies (3.2.5) (2).

Lemma (3.2.10)[3]:

Suppose (G, P, θ) is an irreversible algebraic dynamical system and χ denotes the associated product system from Proposition (3.2.9). Then the rank-one projection $\Theta_{u_g,u_g} \in K(\chi_p)$depends only on the equivalence class of g in $G/\theta_p(G)$. Moreover, if φ is a Nica covariant ant representation of χ, then

$$\psi_{\varphi,p}\left(\Theta_{u_{g_1},u_{g_1}}\right)\psi_{\varphi,q}\left(\Theta_{u_{g_2},u_{g_2}}\right)$$
$$= \begin{cases} \psi_{\varphi,p\vee q}\left(\Theta_{u_{g_3},u_{g_3}}\right) if\ g_1^{-1}g_2 = \theta_p(g_3)\theta_q(g_4) \text{for some } g_3, g_4 \in G, \\ 0 \qquad \text{else.} \end{cases}$$

holds for all $g_1, g_2 \in G$ and $p, q \in P$.

Proof: If $g_1 = g0_p(g_2)$for some $g_2 \in G$, then

$$\Theta_{u_{g_1},u_{g_1}}(u_h) = \chi_{\theta_{p(G)}}(\theta_p(g_2^{-1})g^{-1}h)u_h = \chi_{\theta_{p(G)}}(g^{-1}h)u_h = \Theta_{u_g,u_g}(u_h)$$

for all $h \in G$ and hence $\Theta_{u_{g_1},u_{g_1}} = \Theta_{u_g,u_g}$. For the second claim, Nica covariance of $\iota_{\mathcal{O}_\chi}$implies

$$\psi_{\varphi,p}\left(\Theta_{u_{g_1},u_{g_1}}\right)\psi_{\varphi,q}\left(\Theta_{u_{g_2},u_{g_2}}\right) = \psi_{\varphi,p}\left(\iota_p^{p\vee q}\left(\Theta_{u_{g_1},u_{g_1}}\right)\iota_q^{p\vee q}\left(\Theta_{u_{g_2},u_{g_2}}\right)\right).$$

If we denote $p' := (p \wedge q)^{-1}p$ and $q' := (p \wedge q)^{-1}q$, then

$$\iota_p^{p\vee q}\left(\Theta_{u_{g_1},u_{g_1}}\right) = \sum_{[g_3]\in G/\theta_{q'}(G)} \Theta_{u_{g_1}\theta_p(g_3),u_{g_1}\theta_p(g_3)} \in \mathcal{L}\left(\chi_{p\vee q}\right)$$

and

$$\iota_p^{p\vee q}\left(\Theta_{u_{g_2},u_{g_2}}\right) = \sum_{[g_4]\in G/\theta_{p'}(G)} \Theta_{u_{g_2}\theta_q(g_4),u_{g_2}\theta_q(g_4)} \in \mathcal{L}\left(\chi_{p\vee q}\right)$$

hold. We observe that

$$\Theta_{u_{g_1}\theta_p(g_3),u_{g_1}\theta_p(g_3)}\Theta_{u_{g_2}\theta_q(g_4),u_{g_2}\theta_q(g_4)}$$

is non-zero if and only if $[g_1\theta_p(g_3)] = [g_2\theta_q(g_4)] \in G/\theta_{p\vee q}(G)$. In particular, this is always zero if $g_1^{-1}g_2 \notin \theta_p(G)\theta_q(G)$. Let us assume that there are $g_3, \dots, g_8 \in G$ such that

$$\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) = \theta_{p\vee q}(g_7)$$

and

$$\theta_p(g_5^{-1})g_1^{-1}g_2\theta_q(g_6) = \theta_{p\vee q}(g_8)$$

Rearranging the first equation to insert it into the second, we get

$$\theta_p(g_5^{-1}g_3)\theta_{p\vee q}(g_7)\,\theta_q(g_4^{-1}g_6) = \theta_{p\vee q}(g_8)$$

By injectivity of $\theta_{(p\wedge q)}$ this is equivalent to

$$\theta_{p'}(g_5^{-1}g_3)\theta_{(p\wedge q)^{-1}(p\vee q)}(g_7)\,\theta_{q'}(g_4^{-1}g_6) = \theta_{(p\wedge q)^{-1}(p\vee q)}(g_8)$$

From this equation we can easily deduce $g_5^{-1}g_3 \in \theta_{q'}(G)$ and $g_4^{-1}g_6 \in \theta_{p'}(G)$ from independence of$\theta_{p'}$and $\theta_{q'}$, see Definition (3.1.4)(iii). Thus, if there are $g_3, g_4 \in G$ such that $\theta_p(g_3^{-1})g_1^{-1}g_2\theta_q(g_4) \in \theta_{p\vee q}(G)$, then they are unique up to $\theta_{q'}(G)$ and $\theta_{p'}(G)$ respectively. This completes the proof.

Theorem (3.2.11)[3]:

Let (G, P, θ) be an irreversible algebraic dynamical system of finite type and χ the product system from Proposition (3.2.9). Then $u_g s_p \mapsto \iota_{\mathcal{O}_{\chi},p}(u_g)$ defines an isomnorphisms $\varphi: \mathcal{O}[G, P, \theta] \to \mathcal{O}_{\chi}$.

Proof: The idea is to exploit the respective universal property on both sides. We begin by showing that $(\iota_{\mathcal{O}_{\chi},1_P}(u_g))_{g\in G}$is a unitary representation of *G* and $(\iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)}))_{p\in P}$ is a representation of the inonoid *P* by isomnetries satisfying (CNP1)-(CNP3), compare Definition (3.1.24). $\iota_{\mathcal{O}_{\chi},1_P}$ is a ***-**homomorphism, so we get a unitary representation of *G*. In addition,

$$\iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)})^* \iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)}) = \iota_{\mathcal{O}_{\chi},1_P}(\langle 1_{C^*(G)}, 1_{C^*(G)}\rangle p) = \iota_{\mathcal{O}_{\chi},1_P}(1_{C^*(G)})$$
$$= 1_{\mathcal{O}_{\chi}}$$

and

$$\iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)}) \iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{q}}(1_{C^*(G)}) = \iota_{\mathcal{O}_{\chi,Pq}}(1_{C^*(G)}) \alpha_p(1_{C^*(G)}) = \iota_{\mathcal{O}_{\chi,Pq}}(1_{C^*(G)})$$

show that we have a representation of P by isometries. (CNP 1) follows from

$$\iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)}) \iota_{\mathcal{O}_{\chi},1_P}(u_g) = \iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}\left(u_{\theta_{p(g)}}\right) = \iota_{\mathcal{O}_{\chi},1_P}\left(u_{\theta_{p(g)}}\right) \iota_{\boldsymbol{\mathcal{O}}_{\chi},\boldsymbol{p}}(1_{C^*(G)})$$

Let $p, q \in P$ and $g \in G$. Then (CNP 2) follows easily from applying Lemma (3.2.10) to

$$\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}(1_{C^*(G)})^* \iota_{\mathcal{O}_{\chi},1_P}(u_g)\iota_{\mathcal{O}_{\chi},\boldsymbol{q}}(1_{C^*(G)})$$
$$= \iota_{\mathcal{O}_{\chi},\boldsymbol{p}}(1_{C^*(G)})^* \psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{q}}}(\Theta_{1.1})\psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{q}}}\left(\Theta_{u_g,u_g}\right)\iota_{\mathcal{O}_{\chi},\boldsymbol{q}}(u_g).$$

Finally, we observe that,

$$\iota_{\mathcal{O}_{\chi},1_P}(u_g)\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}(1_{C^*(G)})\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}(1_{C^*(G)})^* \iota_{\mathcal{O}_{\chi},1_P}(u_g)^* = \psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}}\left(\Theta_{u_g,u_g}\right)$$

and tile computation

$$\sum_{[g_4]\in G/\theta_{p'}(G)} \psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}}\left(\ominus_{u_g,u_g}\right) = \psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}}(1_{\mathcal{L}(x)}) = \psi_{\iota_{\mathcal{O}_{\chi},\boldsymbol{p}}}\left(\emptyset_p(1_{C^*(G)})\right)$$
$$= \iota_{\mathcal{O}_{\chi},1_P}(1_{C^*(G)}) = 1_{\mathcal{O}_x}$$

yield (CNP 3). Thus we conclude that $\varphi: \mathcal{O}[G,P,\theta] \to \mathcal{O}_\chi$ defines a surjective *-hornomnorphismn. For the reverse direction, we show that

$$\varphi_{CNP} : \chi \to \mathcal{O}[G,P,\theta]$$

$$\xi_{p,g} \mapsto u_g s_p$$

defines a (CNP)-covariarit representation of χ, where $\xi_{p,g}$denotes the represeritative for u_g in $\chi_{\boldsymbol{p}}$. To do so, we have to verify (i)-(iv) from Definition (3.2.7) and the (CNP)-covariance condition. (i) and (ii) are obvious. Using (CNP 2) to compute

$$\varphi_{CNP,p}(\xi_{p,g_1})^* \varphi_{CNP,p}(\xi_{p,g_2}) = s_p^* u_{g_1^{-1}g_2} s_p$$

$$= \chi_{\theta_p(G)}(g_1^{-1}g_2)u_{\theta_p^{-1}(g_1^{-1}g_2)}$$

$$= \varphi_{CNP,1_P}(\xi_{p,g_1}, \xi_{p,g_2})$$

we get (iii). (iv) follows from (CNP 1) as

$$\varphi_{CNP,p}(\xi_{p,g_1})\varphi_{CNP,q}(\xi_{q,g_2}) = u_{g_1}s_p u_{g_2}s_q$$

$$= u_{g_1}\theta_p(g_2)s_{pq}$$

$$= \varphi_{CNP,pq}(\xi_{p,g_1}\alpha_p(\xi_{q,g_2})).$$

Thus, we are left with the (CNP)-covariance condition. But since χ is a product system of finite type, see Proposition (3.2.9), we only have to show that φ_{CNP} is (CP_F)-covariant due to [3]. Noting that

$\varphi_p^{-1}\left(k(\chi_p)\right) = C^*(G)$for all $p \in P$, we obtain

$$\psi_{\varphi_{CNP,p}}\left(\phi_p(u_g)\right) = \psi_{\varphi_{CNP,p}}\left(\sum_{[h]\in G/\theta_p(G)} \Theta_{u_{gh},u_h}\right)$$

$$= u_g \sum_{[h]\in G/\theta_p(G)} e_{h,p}$$

$$= u_g = \varphi_{CNP,1_P}(\xi_{1_P,g}).$$

Thusφ_{CNP} is a (CNP)-covariant representation of χ. By the universal property of $\mathcal{O}_\chi$ there exists a *-homomorphism $\bar{\varphi}_{CNP}: \mathcal{O}_\chi \to \mathcal{O}[G, P, \theta]$ such that $\bar{\varphi}_{CNP} \circ o_{\iota_{\mathcal{O}_\chi}} = \varphi_{CNP}$. It is apparent that $\bar{\varphi}_{CNP}$ and φ are inverse to each other, so φ is an isomorphism.

We establish a result about viewing a crossed product of a C*-algebra by a semidirect product of discrete, left cancellative monoids as an iterated crossed product, see Theorem (3.2.15). This extends the well-known result for semidirect products of locally compact groups in the discrete case and is essential for the proof of Corollary (3.1.39).

We restrict our attention to the case of unital coefficient algebras and include the basic definitions for semigroup crossed products based on covariant pairs of representations.

All semigroups will be left cancellative and discrete. In the following, let Isom (B) denote the semigroup of isometries in a unital C*-algebra B.

Definition (3.2.12)[3]:

Let S be a semigroup and A a unital C*-algebra with an S-action α by endomorphisms. A covariant pair (π_A, π_S) for (A, S, α) is given by a unital C*-algebra B together with a unital C*-homomorphism $\pi_A: A \to B$ and a sernigroup homomorphism $\pi_S: S \to$ Isom (B) subject to the covariance condition:

$$\pi_S(s)\pi_A(a)\,\pi_s(s)^* = \pi_A(\alpha_s(a))\; for\ all\ a\ \in A, s \in S$$

Definition (3.2.13)[3]:

Let S be a semnigroup and A a unital C*-algebra with an S-action α by endomorphisms. The crossed product for (A, S, α), denoted by $A \rtimes_\alpha S$, is the C*-algebra generated by a covariant pair (ι_A, ι_S) which is universal in the sense that whenever (π_A, π_S) is a covariant pair for (A, S, α), it factors through (ι_A, ι_S). That is to say, there is a surjective C*-homomorphism $\bar{\pi}: A \rtimes_\alpha S \to C^*(\pi_A(A), \pi_S(S))$ satisfying $\pi_A = \bar{\pi} \circ \iota_A$ and $\pi_s = \bar{\pi} \circ \iota_S$. $A \rtimes_\alpha S$ is uniquely determined up to canonical isomorphism by this universal property.

This crossed product may be 0. But it is known that the coefficient algebra A embeds into $A \rtimes_\alpha S$ provided that S acts by injective endoimorphisms and is right-reversible, i.e. $Ss \cap St \neq \emptyset$ for all $, t \in S$.

Suppose that T is a semigroup which acts on another semnigroup S by semigroup homomorphisms θ_t. Then we can form the semidirect product $S \rtimes_\theta T$, which is the semigroup given by $S \times T$ with $ax +$ b-composition rule:

$$(s,t)(s',t') = (s\theta_t(s'), tt')$$

Now suppose further that S and T are monoids and that α is an action of $S \rtimes_\theta T$ on a unital C*-algebra A. Then the semigroup crossed product $A \rtimes_\alpha (S \rtimes_\theta T)$ is given by a unital *-homomorphism

$$\iota_A, S \rtimes_\theta T: A \to A \rtimes_\alpha (S \rtimes_\theta T)$$

and a semigroup homomorphism

$$\iota_{S \rtimes_\theta T}: S \rtimes_\theta T \to Isom(A \rtimes_\alpha (S \rtimes_\theta T))$$

Of course, we can also consider $A_{\times_{\alpha|S}} S$ given by a unital *-homomorphism $\iota_{A,S}: A \to A_{\rtimes_{\alpha|S}} S$ and a homomorphism ts $\iota_S: S \to$ Isom($A \rtimes_{\alpha|S} S$). A natural question in this situation is whether α and θ give rise to a T-action $\tilde{\alpha}$ on $A \rtimes_{\alpha|S} S$. The next lemma provides a positive answer for the case where α satisfies $\{1_A - \alpha_{(s,1_T)}(1_A) | s \in S\} \subset \bigcap_{t\in T} \ker \alpha_{(1_S,t)}$. For the sake of readability, let $p_{(s,t)} := \iota_{A,S}(\alpha_{(s,t)}(1_A))$ for $s \in S, t \in T$ and we will simply write p_t for $p_{(1s,t)}$. We observe that the aforementioned condition is equivalent to $p_{(\theta_t(s),t)} = p_t$ for all $s \in S, t \in T$.

Lemma (3.2.14)[3]:

Suppose that S and T are monoids with α T-action θ on S by semigroup homomorphisms. Let α be an action of $S \rtimes_\theta T$ on a unital C*-algebra A by endomorphisms. For $t \in T$, let

$$\tilde{\alpha}_t(\iota_{A,S}(a)\iota_S(s)) := \iota_{A,S}(\alpha_{(1_{s,t})}(a))\iota_S(\theta_t(s)) \; for \; a \in A, s \in S$$

$\tilde{\alpha}_t$is an endomorphism from $A \rtimes_{\alpha|S} S \to p_t(A \rtimes_{\alpha|S} S)p_t$ provided that

$$1_A - \alpha_{(s,1_T)}(1_A) \in \ker \alpha_{(1_S,t)} \; for \; all \; s \in S$$

In particular, if this holds for all $t \in T$, i.e.

$$1_A - \alpha_{(s,1_T)}(1_A) \in \bigcap_{t\in T} \ker \alpha_{(1_S,t)} \; for \; all \; s \in S \; .$$

therm $\tilde{\alpha}$ defines an action of T on $A \rtimes_{\alpha|S} S$.

Proof: Note that $\tilde{\alpha}_t(\iota_S(s)) = \tilde{\alpha}_t(\iota_{A,S}(1_A)\iota_S(s) = p_t\iota_S(\theta_t(s))$ is valid for all $s \in S, t \in T$ since $\iota_{A,S}$ is unital. Suppose $t \in T$ satisfies

$$1_A - \alpha_{(s,1_T)}(1_A) \in \ker \alpha_{(1_S,t)} \; for \; all \; s \in S$$

This is equivalent to $p_{(\theta_t(s),t)} = p_t$. Hence, $p_t$commutes with $\iota_S(\theta_t(s))$ since

$$\iota_S(\theta_t(s))p_t = \iota_S(\theta_t(s))p_t\iota_S(\theta_t(s))^*\iota_S(\theta_t(s)) = p(\theta_t(s),t)\iota_S(\theta_t(s)) = p_t\iota_S(\theta_t(s)).$$

To prove that $\tilde{\alpha}_t$ is an endomorphism of $A \rtimes_{\alpha|S} S$ we show that

$$(\iota_{A,S} \circ \alpha_{(1_S,t)}, p_t(\iota_S \circ \theta_t(.)))$$

is a covariant pair for $(A, S, \alpha|S\ )$.Itis then easy to see that the induced map coming from the universal property of the crossed product is precisely $\tilde{\alpha}_t$ and maps $A \rtimes_{\alpha|S} S$ onto the corner $p_t(A \rtimes_{\alpha|S} S)p_t$.

$\iota_{A,S} \circ \alpha_{(1_S,t)}$ is a unital *-homomorphism from A to $p_t(A \rtimes_{\alpha|S} S)p_t$ In addition, $p_t(\iota_S \circ \theta_t(.))$ maps S to the isomnetries in $p_t(A \rtimes_{\alpha|S} S)p_t$ because

$$p_t\iota_S(\theta_t(s))^* p_t\iota_S(\theta_t(s)) = \iota_S(\theta_t(s))^* p_t\iota_S(\theta_t(s)) = \iota_S(\theta_t(s))^*\iota_S(\theta_t(s))p_t = p_t.$$

This map turns out to be a semigroup homomorphism as

$$p_t\iota_S(\theta_t(s_1))p_t\iota_S(\theta_t(s_2)) = p_t^2\iota_S(\theta_t(s_1))\iota_S(\theta_t(s_2)) = p_t\iota_S(\theta_t(s_1s_2)).$$

Finally, for $a\ \in A\ \ and\ s \in S$, we compute

$$p_t\iota_S(\theta_t(s))\iota_{A,S}(\alpha_{(1_s,t)}(a))(p_t\iota_S(\theta_t(s)))^* = p_t\iota_{A,S}\left(\alpha_{(\theta_t(s),t)}(a)\right)p_t$$

$$= \iota_{A,S}(\alpha_{(1_s,t)}(\alpha_{(s,1_T)}(a)).$$

Thus, $(\iota_{A,S} \circ \alpha_{(1_S,t),}p_t(\iota_S \circ \theta_t(.)))$ forms a covariant pair for $(A, S, \alpha|S\ )$. In particular, the induced map $\tilde{\alpha}_t$ is an endomnorphism of $A \rtimes_{\alpha|S} S$.

Conversely, assume that $\bar{\alpha}_t$ defines an endomorphism of $A \rtimes_{\alpha|S} S$. Then $(\tilde{\alpha}_t \circ \iota_{A,S}, \tilde{\alpha}_{t}, \circ \iota_S)$ forms a covariant pair for $(A, S, \alpha|S)$ mapping A and S tothe C*-algebra $B := \tilde{\alpha}_t\left(A \rtimes_{\alpha|S} S\right)$. Note that the unit inside this C*-algebra is p_t. In particular, we have a semi group homomorphism $\tilde{\alpha}_t \circ \iota_s: S \to Isom(B)$. This forces

$$p_t = \tilde{\alpha}_t(\iota_S(s))^* \tilde{\alpha}_t(\iota_S(s)) = \iota_S\left(\theta_t(s)\right)^* p_t \iota_S\left(\theta_t(s)\right) = p_{(\theta_t(s),t)}$$

for all $s \in S$, which is equivalent to

$$\left\{1_A - \alpha_{(s,1_T)}(1_A) \middle| \, s \in S \right\} \subset \ker \alpha_{(1_S,t)}$$

Since $\alpha|T$ and $\theta$are semigroup homomorphisms , $\tilde{\alpha}$ defines an action of T on $A \rtimes_{\alpha|S} S$ provided that the imposed condition holds for every $t \in T$.

Theorem (3.2.15)[3]:

Suppose S and T are monoids together with a T-action θ on S by semigroup homomorphisms, and an action α of $S \rtimes_\theta T$ on a urtital C*-aigebra A by endomorphisms. If

$$\left\{1_A - \alpha_{(s,1_T)}(1_A) \middle| \, s \in S \right\} \subset \bigcap_{t \in T} \ker \alpha_{(1_S,t)}$$

holds true, then there is a canonical isomorphism

$$A \rtimes_\alpha (S \rtimes_\theta T) \xrightarrow{\pi} \left(A \rtimes_{\alpha|S} S\right) \rtimes_{\tilde{\alpha}} T \text{ , } \iota_{A,S\rtimes_\theta T}(a) \longmapsto \iota_{A\rtimes S} \circ \iota_{A,s}(a)$$
$$\iota_{S\rtimes_\theta T}(s,t) \longmapsto (\iota_{A\rtimes S} \circ \iota_s)(s)\iota_T(t)$$

where $\tilde{\alpha}$ is given by $\tilde{\alpha}_t(\iota_{A,S}(a)\iota_S(s)) = \iota_A(\alpha_{(1_s,t)}(a))\iota_S(\theta_t(s))$

Proof : Recall that $(\iota_A, S \rtimes_\theta T,\ \iota_{S\rtimes_\theta}T)$, $(\iota_{A,}S\ ,\iota_{S,})$ and $(\iota_{A\rtimes S},\ \iota_T)$ denote the uiversal covariant pairs for $(A, S \rtimes_\theta T, \alpha)$, $(A, S, \alpha|S)$ and $\left(A \rtimes_{\alpha|S} S, T, \tilde{\alpha}\right)$, respectively. The strategy is governed by the following claims:

(i) $(\ \iota_{A\rtimes S}\ \circ\ \iota_{A,S}, (\ \iota_{A\rtimes S}\circ\iota_S) \times \iota_T)$ forms acovariant pair for $(A, S \rtimes_\theta T, \alpha)$.

(ii) $(\iota_{A,S_{\rtimes_\theta}T}\ \times \iota_{S_{\rtimes_\theta}T}|S, \iota_{S_{\rtimes_\theta}T}|_T)$ forms acovariant pair for $(A \rtimes_{\alpha|S} S, T, \tilde{\alpha})$.

If we assume (i) and (ii), then (i) and the universal property of $A \rtimes_\alpha (S \rtimes_\theta T)$ give a *-homomorphism

$$A \rtimes_\alpha (S \rtimes_\theta T) \overset{\pi}{\twoheadrightarrow} \left(A \rtimes_{\alpha|S} S\right) \times_{\tilde{\alpha}} T \iota_{A,S\rtimes_\theta T}(a) \qquad \mapsto\ \iota_{A\rtimes S} \circ \iota_{A,s}(a)$$

$$\iota_{S\rtimes_\theta T}(s,t) \qquad \mapsto (\iota_{A\rtimes S} \circ \iota_s)(s)\iota_T(t)$$

Since S and T both have an identity, the induced map equals π . Note that the pair from (ii) is the natural candidate to provide an inverse for π. Indeed, if (ii) is valid, then the two induced *-homomorphisms are mutually inverse on the standard generators of the C*-algebras on both sides. Thus it remains to establish (i) and (ii).

For step (i), note that $\iota_{A\rtimes S}\ \circ\ \iota_{A,s}$is a unital *-homomorphism and $\iota_{A\rtimes s} \circ \iota_s$ defines a semigroup homomorphism from S to the isometries in $\left(A \rtimes_{\alpha|S} S\right) \rtimes_{\tilde{\alpha}} T$. The covariance condition for $(\mathsf{T}, \tilde{\alpha})$ yields

$$\iota_T(t)\iota_{A\rtimes S} \circ \iota_s(s) =\ \tilde{\alpha}(\iota_{A\rtimes S}\ \circ \iota_s(s)\iota_T(t) = \iota_{A\rtimes S}\ \circ \iota_s(\theta_t(s))\iota_T(t).$$

Therefore, $(\iota_{A\rtimes S}\circ\iota_s)\times\iota_T$ is well-behaved with respect to the semnidirect product structure on S × T coming from θ, so we get a semigroup homomorphism $(\iota_{A\rtimes S}\circ\iota_s)\times\iota_T: S_{\rtimes_\theta}T\to Isom\,((A\rtimes_{\alpha|S}S)\rtimes_{\tilde{\alpha}}T)$. Now let $a\in A\,,s\in S\ and\ t\in T$. Then we compute

$$((\iota_{A\rtimes s}\circ\iota_s)\times\iota_T)(s,t)\iota_{A\rtimes S}\circ\iota_{A,S}(a)((\iota_{A\rtimes S}\circ\iota_{A,S})\times\iota_T)(s,t)^*$$

$$=\iota_{A\rtimes S}\circ\iota_s(s)\iota_T(t)\iota_{A\rtimes S}\circ\iota_{A,S}(a)\iota_T(t)^*\iota_{A\rtimes S}\circ\iota_S(s)^*$$

$$=\iota_{A\rtimes S}\circ\iota_s(s)\iota_{A\rtimes S}\circ\iota_{A,S}\left(\alpha_{(1_{s,t})}(a)\right)\iota_{A\rtimes S}\circ\iota_S(s)^*$$

$$=\iota_{A\rtimes S}\circ\iota_{A,S}\left(\alpha_{(s,1_T)(1_{s,t})}(a)\right)$$

$$=\iota_{A\rtimes S}\circ\iota_{A,S}\left(\alpha_{(s,t)}(a)\right),$$

which completes (i). For part (ii), we remark that $(\iota_{A,S\rtimes_\theta T},\iota_{S\rtimes_\theta T}|S)$ is a covariant pair for $(A,S,\alpha|S)$. Since $\iota_{A,S\rtimes_\theta T}$ and $\iota_{A,S}$ are unital, the induced map is unital as well. Moreover, $\iota_{S\rtimes_\theta T}|T$ is a semigroup homomorphism mapping T to the isometries in$A\rtimes_\alpha(S\rtimes_\theta T)$). Thus, we are left with the covariance condition. Note that it suffices to check the covariance condition on the standard generators of $A\rtimes_{\alpha|S}S$. For $a\in A\,,s\in S\ and\ t\in T$, we get

$$\iota_{S\rtimes_\theta T}(1_{S,}t)\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(s,1_T)\iota_{S\rtimes_\theta T}(1_{S,}t)^*$$
$$=\iota_{S\rtimes_\theta T}(1_{S,}t)\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(1_{S,}t)^*\iota_{S\rtimes_\theta T}(1_{S,}t)\iota_{S\rtimes_\theta T}(s,1_T)\iota_{S\rtimes_\theta T}(1_{S,}t)^*$$
$$=\iota_{A,S\rtimes_\theta T}\left(\alpha_{(1_{s,t})}(a)\right)\iota_{S\rtimes_\theta T}(\theta_t(s),1_T)p_t$$
$$=\iota_{A,S\rtimes_\theta T}\left(\alpha_{(1_{s,t})}(a)\right)\iota_{S\rtimes_\theta T}(\theta_t(s),1_T)$$
$$=\tilde{\alpha}_t\left(\iota_{A,S\rtimes_\theta T}(a)\iota_{S\rtimes_\theta T}(s,1_T)\right)$$

Hence (i) and (ii) are both valid, so the proof is complete.

Chapter 4

Product Systems over Semigroups of Ore Type

We introduce Doplicher-Robertspicture of Cuntz-Pimsner algebras, and the semigroup dual to a product system of ’regular’ C*-correspondences. Under a certain aperiodicity condition on the latter, we obtain the uniquenesstheorem and a simplicity criterion for the algebras. These results generalize thecorresponding ones for crossed products by discrete groups, we give interesting conditionsfor topological higher rank graphs and P-graphs, and apply to the new Cuntz C*-algebra $\mathcal{Q}_{\mathbb{N}}$arising from the "$ax + b$"-semigroup over $\mathbb{N}$.

Section (4.1): Regular Product Systems of C*-Correspondences and their C*-Algebras with Dual Objects

We first introduce and discuss certain product systems of C*-correspondences satisfying additional regularity conditions, and then construct their associated Cimtz-Pimnsner algebras and their reduced versions in the spirit of the Doplicher-Roberts algebras. The construction involves an object that may be viewed as a right tensor C*-precategory over P**.** Regular product systems introduced and their C*-algebras will play a central role.

Regular product systems and their right tensor C*-precategories.

Definition (4.1.1)[4]:

Let X be a C*-correspondence with coefficients in A. We say X is regular if its left action is injective and via compact operators, that is

$$ker\,\emptyset = \{0\} \quad and \quad \emptyset(A) \subseteq k(X). \qquad (1)$$

We say that a product system $X := \coprod_{p\in P} X_p$ over a semigroup P is regular if each fiber X_p , $p \in P$, is a regular C*-correspondence.

The notions of regularity and tensor product are compatible in the sense that the tensor product of two regular C*-correspondences is automatically regular below. We will need the following .

Lemma (4.1.2)[4]:

Let Y he a regular C*-corresponderice with coefficients in A and let X, Z be right Hilbert A-modules.

(i) For each $x \in X$, the mapping

$$Y \ni y \xrightarrow{T_x} x \otimes y \in X \otimes Y$$

is compact, that is $T_x \in k(y, X\otimes Y)$. Furthermore, we have $\|T_x\| = \|x\|$.

(ii) For each $S \in k(X, Z)$ we have $S \otimes 1_Y \in k(X\otimes Y, Z\otimes Y)$ and the mapping

$$k(X, Z) \ni S \longmapsto S\otimes 1_Y \in k(X\otimes Y, Z\otimes Y) \qquad (2)$$

is isometric. It is surjective whenever $\emptyset_Y\colon A \to k(Y)$ is.

Proposition (4.1.3)[4]:

Tensor product of regular C*-correspondences is a regular C*-correspondence.

Proof: If X and Y are C*-correspondences over A then the left action of A on $X \otimes Y$ is $\emptyset_{X\otimes Y} = \emptyset_X \otimes 1_Y$. Hence if X and Y are regular, then $\emptyset_{X\otimes Y}$ is injective and acts by compacts, by Lemma (4.1.2) part (ii).

Now, let X be a regular product system over P. The family

$$k_X \coloneqq \{K(X_q, X_p)\}_{p,q\in P}$$

forms in a natural manner a C*-precategory. We will describe a right tensoring structure on k_X by introducing a family of mappings $\iota_{p,q}^{pr,qr}\colon k(X_q, X_p) \to k(X_{qr}, X_{pr}), p, q, r \in P$, which extends the standard family of diagonal homomorphisms ι_q^{qp} defined . If $q \neq e$ we put

$$\iota_{p,q}^{pr,qr}(T)(xy) \coloneqq (Tx)y, \qquad where\ x \in X_q, y \in X_r \ \ and \ \ T \in k(X_q, X_p)$$

Note that under the canonical isomorphisrn $X_{pq} \cong X_p \otimes_A X_q$ operator $\iota_{p,q}^{pr,qr}(T)$ corresponds to $T\otimes 1_{X_r}$. Hence by part (ii) of Lemma (4.1.2),$\iota_{p,q}^{pr,qr}(T) \in k(X_{qr}, X_{pr})$ and $\iota_{p,q}^{pr,qr}$ is isometric. Similarly, in the case q = e, the formula

$$\iota_{p,e}^{pr,r}(t_x)(y) \coloneqq xy\ , \qquad where\ y \in X_r\ \ and\ \ t_x \in k(X_e, X_p), x \in X_p$$

yields a well defined map. By Lemma (4.1.2)part(i), this is an isometry from $k(X_e, X_p)$ into $k(X_r, X_{pr})$. Note that $\iota_{p,p}^{pr,pr} = \iota_p^{pr}$.

Definition (4.1.4)[4]:

The C*-precategory $k_x \coloneqq \{K(X_q, X_p)\}_{p,q\in P}$ equipped with the family of maps $\{\iota_{p,q}^{pr,qr}\}_{p,q,r\in P}$ defined above is called a right tensor C*-precategory associated to the regular product system *X*.

Lemma (4.1.5)[4]:

Let ψ be a representation of a regular product system X over a semigroup *P* in a C*-algebra *B*. For each $p, q \in P$we have a contractive linear map $\psi_{p,q}: K(X_q, X_p) \to B$ determined by the formula

$$\psi_{p,q}(\Theta_{x,y}) = \psi_p(x)\psi_q(\boldsymbol{y})^* for x \in X_p \;, y \in X_q \qquad (3)$$

Mappings $\{\psi_{p,q}\}_{p,q\in P}$ satisfy

$$\psi_{p,q}(S)\psi_{q,r}(T) = \psi_{p,r}(ST) \text{for } S \in K(X_q, X_p), T \in K(X_r, X_q), p, q. r \in P \qquad (4)$$

and are all isometric if ψ is injective. If ψ is Cuntz-Pimsner covariant, then

$$\psi_{p,q}(S) = \psi_{pr,qr}(\iota_{p,q}^{pr,qr}(S)) \quad for\ all\ p, q. r \in P\ and\ S \in K(X_q, X_p) \qquad (5)$$

Proof: It is not completely trivial but quite well known that (3) defines a linear contraction which is isometric if ψ_e is injective. One readily sees that (4) holds for ‘rank one operators $S = \Theta_{x,y}\ T = \Theta_{u,w}$and thus it holds ingeneral. Suppose that ψ is Cuntz-Pimsner covariant representation on

Hilbert space H and let $p, q, r \in P$. To see (5), it suffices to consider the case when $S = \Theta_{x,y}$with $x \in X_p$, and $y \in X_q$. We may writing $x = x'a$ where $x' \in X_p$ and $a \in A$. We get

$$\psi_{p,q}(S) = \psi_p(x')\pi(a)\psi_q(y)^* = \psi_p(x')\psi^{(r)}(\emptyset_r(a))\psi_q(y)^*$$
$$\in \psi_{pr,qr}(K(X_{qr}, X_{pr}))$$

Hence both $\psi_{p,q}(S)$ and $\psi_{pr,qr}(\iota_{p,q}^{pr,qr}(S))$ act as zero on the orthogonal complement of the space $\psi_{pr}(X_{qr})H = \psi^{(qr)}(K(X_{qr}))H$. Since the linear span of elements of the form $\psi_{qr}(x_0 y_0)h, x_0 \in X_q, y_0 \in X_r, h \in H$, is dense in $\psi_{pr}(X_{qr})H$, (5) follows from the following computation:

$$\psi_{pr,qr}\left(\iota_{p,q}^{pr,qr}(\Theta_{x,y})\right)\psi_{qr}(x_0 y_0) = \psi_{pr}(\iota_{p,q}^{pr,qr}(\Theta_{x,y})x_0 y_0)$$
$$= \psi_{pr}((\Theta_{x,y}x_0)y_0) = \psi_{pr}(x\langle y, x_0\rangle y_0)$$
$$= \psi_p(x)\psi_q(y)^*\psi_q(x_0)\psi_r(y_0) = \psi_{p,q}(\Theta_{x,y})\psi_{qr}(x_0 y_0).$$

Doplicher-Roherts picture of a Cuntz-Pimsner algebra and its reduced version .

We assume that *X* is a regular product system over a semigroup of Ore type. We need the following lemma.

Lemma (4.1.6)[4]:

Suppose ψ is a Cuntz-Pimsner covariant representation of a regular product system *X* over a semigroup *P* of Ore type.

(i) For all $x \in X_p, y \in X_q$ and $s \geq p, q$ we have

$$\psi_p(x)^*\psi_q(y) \in \overline{span}\{\psi(f)\psi(h)^*: f \in X_{p^{-1}s}, f \in X_{q^{-1}s}\}$$

(ii) We have the equality

$$\overline{span}\{\psi(x)\psi(y)^*: x, y \in X, [d(x), d(y)] = [p, q]\} =$$
$$\overline{span}\{\psi(x)\psi(y)^*: x \in X_{pr}, y \in X_{qr}, r \in P\}.$$

(iii) $C^*(\psi(X)) = \overline{span}\{\psi(x)\psi(y)^*: x, y \in X\}$.

Furthermore, there is a dense subspace of $C^*(\psi(X))$ consisting of elements of the form

$$\psi^{(q)}(S_q) + \sum_{p\in F} \psi_{p,q}(S_{p,q}) \qquad (6)$$

where $q \in P$ and $F \subseteq P$ is a finite set such that $q \nsim_R p$ for all $p \in F$.

Proof: Ad(i). Write $x = Sx'$ with $S \in K(X_p)$ and $x' \in X$, and similarly $y \in Ty'$ with $T \in K(X_q), y' \in X_q$. Then we get

$$\psi_p(\boldsymbol{x})^*\psi_q(y) = \psi_p(x')^*\psi^{(p)}(S^*)\psi^{(q)}(T)\psi_q(y')$$
$$= \psi_p(x')^*\psi^{(s)}(\iota_p^s(S^*)\iota_p^s(T))\psi_q(y')$$

Since $\iota_p^s(S^*)\iota_p^s(T) \in K(X_s)$ we may approximate $\psi^{(s)}(\iota_p^s(S^*)\iota_p^s(T))$ with finite sums of operators of the form $\psi_s(f'f)\psi_s(h'h)^*$ where $f' \in X_P, f \in X_{P^{-1}s}$ and $h' \in X_q, h \in X_{q^{-1}s}$. Hence $\psi_p(x)^*\psi_q(y)$ can be approximated by finite sums of elements of the form

$$\psi_p(x')^*\psi_s(f'f)\psi_s(h'h)^*\psi_q(y') = \psi_{p^{-1}s}(\langle x', f'\rangle_p f)\psi_{q^{-1}s}(\langle y', h'\rangle h)^*$$

This proves claim (i).

Ad(ii). Clearly, $\overline{span}\{\psi(x)\psi^*(y): x, y \in X, [d(x), d(y)] = [p, q]\}$ contains $\overline{span}\{\psi(x)\psi(y)^*: x \in X_{pr}, y \in X_{qr}, r \in P\}$. To see the converse inclusion, we use the mappings introduced in Lemma (4.1.5) and assume that $[p', q'] = [p, q]$ that is $p'r' = pr$ and $q'r' = qr$ for some $r, r' \in P$. Then for $T \in K(X_{q'}, X_{p'})$ we have

$$\psi_{p',q'}(T) = \psi_{p'r',q'r'}(\iota_{p',q'}^{p'r',q'r'}(T)) = \psi_{pr,qr}(\iota_{p',q'}^{p'r',q'r'}(T))$$
$$\in \overline{span}\{\psi(x)\psi(y)^*: x \in X_{pr}, y \in X_{qr}\},$$

which proves our claim.

Ad (iii). Part (i) implies that $C^*(\psi(X))$is the closure of elemnents of the form

$$\sum_{i=1}^{n} \psi_{p_i}(x_i)\,\psi_{q_i}(y_i)^*, \tag{7}$$

where $p_i, q_i \in P, x_i \in X_{p_i}, y_i \in X_{q_i}, i = 1, .., n$. Moreover, taking any $q_0 \in P$ that dominates all $q_i, i = 1, \ldots, n$, and writing $y_i = y_i' \ a_i$ with $y_i' \in X_{q_i}, a_i \in A$, we get

$$\psi_{p_i}(x_i)\,\psi_{q_i}(y_i)^* = \psi_{p_i}(x_i)\psi^{(q_i^{-1}q_0)}\left(\emptyset_{q_i^{-1}q_0}(a_i^*)\right)\psi_{q_i}(y_i')^*, \qquad i = 1, \ldots, n$$

Approximating $\psi^{(q_i^{-1}q_0)}\left(\emptyset_{q_i^{-1}q_0}(a_i^*)\right)$ by finite sums of elemnemits of the form $\psi_{q_i^{-1}q_0}(u_i)\psi_{q_i^{-1}q_0}(v_i)^*$ we see that $\psi_{p_i}(x_i)\,\psi_{q_i}(y_i)^*$ can be approximated by finite sums of elements of the form

$$\psi p_i(x_i)\psi_{q_i^{-1}q_0}q(u_i)\psi_{q_i^{-1}q_0}(v_i)^*\psi_{q_i}(y_i')^* = \psi_{p_iq_i^{-1}q_0}(x_iu_i)\psi_{q_0}(y_i'v_i)^*.$$

Thus we see that tile element (7) can be presented in the form

$$\sum_{p\in F'} \psi_{p,q_0}(S_{p,q_0}) \qquad (8)$$

where $F' = \{p_iq_i^{-1}q_0 : i = 1, \dots, n\} \subseteq P$ is a finite set. Let $F_0 = \{p \in F' : q_0 \sim_R p\}$ and for each $p \in F_0$ choose r_p ∈P such that $pr_p = q_0r_p$. Let $r \in P$ be such that $r \geq r_p$ for all $p \in F_0$, and put

$$q := q_0r \quad and \quad F := \{pr : p \in F' \backslash F_0\}.$$

Then $pr = q$ for all $p \in F_0$, and $p \nsim_R q$ for all $p \in F$. By (4) we have $\psi_{p,q_0}(S_{p,q_0}) \in \psi_{pr,q_0r}\left(K\left(X_{q_0r}, X_{pr}\right)\right) = \psi_{pr,q}\left(K\left(X_q, X_{pr}\right)\right)$ and hence the element (8) can be presented inthe form (6).

We are ready to prove the main theorem. It gives a direct constructionof the Cuntz-Pimsner algebra $\mathcal{O}_X$ of a regular product system *X*as the full cross-sectional C*-algebra of a suitable Fell bundle corresponding to the limits of directed systems of the compact operators arising from *X*.

Theorem (4.1.7)[4]:

Let X be a regular product system over a semigroup P of Ore type and let *G(P)* be the enveloping group of *P*. For each $[p,q] \in G(P)$we define

$$B_{[p,q]} := \varinjlim k\,(X_{qr}, X_{pr})$$

to be the Banach space direct limit of the directed system $\left(\left\{k\left(X_{qr}, X_{pr}\right)\right\}_{r\in p}, \{\iota_{pr,qr}^{ps,qs}\}_{\substack{r,s\in P\\ r\leq s}}\right)$. The family $ß = \{B_t\}_{t\in G(P)}$ is in a natural manner equipped with the structure of a Fell bundle over *G(P)* and we have a canonical isomnorphismn

$$\mathcal{O}_x \cong C^*\left(\left\{B_g\right\}_{g\in G(P)}\right)$$

from the Cuntz-Pimsner algebra $\mathcal{O}_X$ onto the full cross-sectional C*-algebra $C^*(\{B_g\}_{g\in G(P)}$. In particular,

(i) the universal representation $j_X : X \to \mathcal{O}_X$ is injective,

(ii) $\mathcal{O}_X$ has a natural grading $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ over *G(P)*, such that

$$(\mathcal{O}_X)_g = \overline{span}\{j_X(x)j_X(y)^* : x, y \in X, [d(x), d(y)] = g\} \qquad (9)$$

(iii) for every injective representation $\psi$of X, the integrated representation $\prod_\psi$ of $\mathcal{O}_X$ is isometric on each Banach space $(\mathcal{O}_X)_g, g \in G(P)$, and thus it restricts to an isomorphism of the core C*-subalgebra of $\mathcal{O}_X$, namely

$$(\mathcal{O}_X)_{\boldsymbol{e}} = \overline{span}\{j_X(x)j_X(y)^*: x, y \in X, d(x) = d(y)\}.$$

Proof: As the direct limit $\underset{\rightarrow}{lim}\, k\,(X_{qr}, X_{pr})$ depends only on 'sufficiently large r', it follows immediately that the limit does not depend on the choice of a representative of $[p, q]$ and thus $B_{[p.q]}$ is well defined. Let $\varphi_{p,q}$: $k(X_q, X_p) \rightarrow B_{[p.q]}$ denote the natural embedding Of $k(X_q, X_p)$ into $B_{[p.q]}$. It is isometric because all the connecting maps $\{\iota_{pr,qr}^{ps,qs}\}$ $r \le s$ are. Using the (inductive) properties of the mappings $\varphi_{p,q}$ and (right tensoring) properties of the mappings $\iota_{p,q}^{pr,qr}$ one sees that the formula

$$\varphi_{p_1,p_2}(S) \circ \varphi_{q_1,q_2}(T) \coloneqq \varphi_{p_1}(p_2^{-1}s), q_2(q_1^{-1}s)(\iota_{p_1,p_2}^{p_1,(p_2^{-1}s),\boldsymbol{s}}(S)\iota_{q_1,q_2}^{s,q_2(q_1^{-1}s)}(T)),$$

where $s \ge p_2, q_1, S \in k\big(X_{p_2}, X_{p_1}\big), T \in k(X_{q_2}, X_{q_1})$, yields well defined bilinear maps

$$\circ: B_{[p_1,p_2]} \times B_{[q_1,q_2]} \rightarrow B_{[p_1,p_2]\circ[q_1,q_2]}$$

These maps establish an associative multiplication $\circ$ on $\{B_t\}_{t\in G(P)}$, satisfying

$$\|a \circ b\| \le \|a\|.\|b\|.$$

Hence $\{B_t\}_{t\in G(P)}$becomes a Banach algebraic bundle. Simnilarly, formula

$$\varphi_{p_1,p_2}(S)^* \coloneqq \varphi_{p_2,p_1}(S^*), \qquad S \in k(X_{p_2}, X_{p_1}),$$

defines a **'*'** operation that satisfies axioms and hence we get a Fell bundle structure on $\{B_g\}_{g\in G(P)}$ (we omit straightforward but tedious verification of the details).

Now, we view $C^*(\{B_g\}_{g\in G(P)})$ as a maximal C*-completion of the direct sum $\bigoplus_{g\in G(P)} B_g$. Using the maps , we define mappings

$$\Psi: X = \coprod_{p\in P} X_p \to C^*(\{B_g\}_{g\in G(P)})$$

by

$$X_p \ni x \to \varphi_{p,e}(t_x), \qquad p \in P \tag{10}$$

is an isomorphism of C^*-correspondences, it follows that Ψ restricted to each summand X_p, is an injective representation of a C^*-correspondence. Moreover, for $x \in X_p, Y \in X_q$ we have $t_{xy} = i_{p,e}^{pq,q}(t_x)t_y$ and thus

$$\Psi(x)\Psi(y) = \varphi_{p,e}(t_x) \circ \varphi_{q,e}(t_y) = \varphi_{pq,e}(i_{p,e}^{pq,q}(t_x)t_y) = \varphi_{pq,e}(t_{xy}) = \Psi(xy).$$

Hence Ψ is a faithful representation of the product system X in $C^*(\{B_t\}_{t\in G(P)})$. We recall that $\iota_{e,e}^{p,p}(t_a) = \iota_e^p(a) = \emptyset_p(a)$ and hence

$$\Psi(a) = \varphi_{e,e}(t_a) = \varphi_{p,p}\left(\iota_{e,e}^{p,p}(t_a)\right) = \varphi_{p,p}\left(\emptyset_p(a)\right) = \Psi\left(\emptyset_p(a)\right), \qquad a \in A, p \in P,$$

that is Ψ is Cuntz-Pimsner covariant. Since Ψ is injective, so is j_X and claim (i) holds. Now, considering the integrated representation $\prod_\Psi: \mathcal{O}_X \to C^*(\{B_g\}_{g\in G(P)})$, for $x \in X_p, y \in X_q$ we have

$$\prod\nolimits_\Psi (j_X(x)j_X(y)^*) = \Psi(x) \circ \Psi(y)^* = \varphi_{p,e}(t_x) \circ \varphi_{e,q}(t_y^*) = \varphi_{p,q}(t_x t_y^*)$$
$$= \varphi_{p,q}(\Theta_{x,y}). \tag{11}$$

It follows that $\prod_\Psi$ maps

$$(\mathcal{O}_x)_{[p,q]} := \overline{span}\{j_X(x)j_X(y)^*: x \in X_{pr}, y \in X_{qr}, r \in P\}$$

onto $B_{[p,q]}$ Putting $g = [p,q]$ and using Lemma (4.1.6) part (iii), we see that $(\mathcal{O}_X)_{\boldsymbol{g}}$ is given by (9). We claim that $\prod_{\Psi}$, is injective on $(\mathcal{O}_X)_{\boldsymbol{g}}$. To see this, let $j_{p,q}$ denote the mappings from Lemma (4.1.5) associated to the universal representation j_X and note that we have

$$j_{ps,qs} \circ \iota_{pr,qr}^{ps,qs} = j_{pr,qr} \qquad for\ r \leq s$$

by (5). By the universal property of inductive limits, there is a mapping

$$B_{[p,q]} \ni \emptyset_{pr,qr}(T) \longmapsto j_{pr,qr}(T) \in (\mathcal{O}_X)_{[p,q]},$$

winch is inverse to $\prod_{\Psi} \ |_{[p,q]}$ Accordingly $\prod_\Psi$ is an epimorphism injective on each $(\mathcal{O}_X)_{\boldsymbol{g}}$. Since the spaces $B_g, g \in G(P)$, are linearly independent, so are $(\mathcal{O}_X)_{\boldsymbol{g}}, g \in G(P)$. Consequently, in view of Lemma (4.1.6) we have

$$\mathcal{O}_X = \overline{\bigoplus_{g \in G(P)} (\mathcal{O}_X)_{\boldsymbol{g}}}$$

and claim (ii) follows. In particular $\prod_\Psi: \bigoplus_{g \in G(P)} (\mathcal{O}_X)_g \to \bigoplus_{g \in G(P)} B_g$ is an isomorphism and as $C^*(\{B_g\}_{g \in G(P)})$ is the closure of $\bigoplus_{g \in G(P)} B_g$ in a maximal C*-norm we see that $\prod_\Psi$ actually yields the desired isomnorphism $\mathcal{O}_X \cong C^*(\{B_g\}_{g \in G(P)})$.

For the proof of part (iii), notice that we have just showed that $(\mathcal{O}_X)_{[p,q]}$is the closure of the increasing union $\cup_{r \in P} j_{pr,qr}\left(k\left(X_{qr}, X_{pr}\right)\right)$, where $j_{pr,qr} : k(X_{qr}, X_{pr}) \to (\mathcal{O}_X)_{[p,q]}$ are isometric maps. Similarly, if ψ is an injective covariant representation of X, then $\prod_{\psi}((\mathcal{O}_X)_{[p,q]})$ is the closure of the increasing union $\bigcup_{r\in P} \psi_{pr,qr}\left(k(X_{qr}, X_{pr})\right)$, and by Lemma (4.1.5) mappings $\psi_{pr,qr}: k(X_q, X_p) \to \prod_{\Psi}((\mathcal{O}_X)_{[p,q]})$ are isometric. Since $\prod_{\psi} \circ j_{pr,qr} = \psi_{pr,qr}, p, q, r \in P$ it follows that surjection $\prod_{\psi}: (\mathcal{O}_X)_{[p,q]} \to \psi((\mathcal{O}_X)_{[p,q]})$ is an isometry, since it is isometric on a dense subset.

Let A be a C*-algebra. We denote by $\simeq$ the unitary equivalence relation between representations of A, and by $[\pi]$ the corresponding equivalence class of $\pi: A \to ß(H)$. Spectrum $\hat{A} = \{[\pi] : \pi \in Irr(A)\}$ consists of the equivalence classes of all irreducible representations of A, equipped with the Jacobson topology. The relation $\leq$ of being a sub representation factors through$\simeq$ to a relation $\preccurlyeq$on $\widehat{\mathrm{A}}$. Namely if $\pi: \mathrm{A} \to ß(\mathrm{H}_\pi)$and $\rho: A \to ß(H_\rho)$ are representations of **A**, then

$$[\pi] \preccurlyeq [\rho] \Leftrightarrow \exists\ isometry\ U: H_\pi \to H_\rho\ s.t\ (\forall a \in A)\pi(a) = U^* \rho(a) U.$$

Let $\alpha: \mathrm{A} \to \mathrm{B}$ be a homomorphism between two C*-algebras. It is useful to think of the dual map we aim to define as a factorization of a multivalued map $\hat{\alpha}_0 : \mathrm{Irr}(\mathrm{B}) \to \mathrm{Irr}(\mathrm{A})$ given by

$$\hat{\alpha}_0(\pi_B) = \{\pi_A \in \mathrm{Irr}(\mathrm{A}): \pi_A \leq \pi_B \circ \alpha\} \qquad (12)$$

The set $[\hat{\alpha}_0(\pi_B)] = \{[\pi_A] \in \widehat{\mathrm{A}}: \pi_A \leq \pi_B \circ \alpha\}$does not depend on the choice of a representative of the class $[\pi_B]$ and thus the following definition make sense.

Definition (4.1.8)[4]:

The dual map to a homomorphism $\alpha: \mathsf{A} \to \mathsf{B}$ is a multivalued map $\hat{\alpha}: \widehat{\mathsf{B}} \to \widehat{\mathsf{A}}$ given by the formula

$$\hat{\alpha}([\pi_B]) \coloneqq \{[\pi_A] \in \widehat{\mathsf{A}} : [\pi_A] \preccurlyeq [\pi_B \circ \alpha]\}$$

$$= \{[\pi_A] \in \widehat{\mathsf{A}} : \pi_A \leq \pi_B \circ \alpha\}$$

Time range of $\hat{\alpha}$ behaves exactly as one would expect. But for non-liminal B the map $\hat{\alpha}$, and inparticular its domain, has to be treated with care. Let us explain it with help of the following proposition and an example.

Proposition (4.1.9)[4]:

For every homomorphism $\alpha: \mathsf{A} \to \mathsf{B}$ between two C*-algebras. its image

$$\hat{\alpha}(\widehat{\mathsf{B}}) \coloneqq \{[\pi_A] \in \widehat{\mathsf{A}}: \ker \pi_A \supseteq \ker \alpha\}$$

is a closed subset of $\widehat{\mathsf{A}}$. Its domain $D(\hat{\alpha})$ is contained in an open subset $\{[\pi_B] \in \widehat{\mathsf{B}}: \ker \pi_B \not\supseteq \mathsf{B}\,\alpha\,(\mathsf{A})\mathsf{B}\}$of $\widehat{\mathsf{B}}$. Moreover, if B is liminal, then

$$D(\hat{\alpha}) \coloneqq \{[\pi_B] \in \hat{B}: ker\, \pi_B \not\supseteq B\, \alpha\, (A)B\}$$

and $\hat{\alpha}: \widehat{\mathsf{B}} \to \widehat{\mathsf{A}}$ is continuous.

Proof: If $[\pi_A] \in \hat{\alpha}(\hat{B})$, then $\pi_A \leq \pi_B \circ \alpha$ for some $\pi_B \in Irr(B)$, and hence $ker\, \pi_A \supseteq ker\, \alpha$. Conversely, if $[\pi_A] \in \hat{A}$ is such that $ker\, \pi_A \supseteq ker\, \alpha$, then $\pi_A$factors through to the irreducible representation of $A/ker\, \alpha \cong \alpha(A)$.Thus the formula $\pi(\alpha(a)) \coloneqq \pi_A(a), a \in A$, yields a well defined element of $Irr(\alpha(A))$. Extending π to any $\pi_B \in Irr(B)$ onehas $\pi_A \leq \pi_B \circ \alpha$.

Now, let J be an ideal of A. Then $\hat{J} = \{[\pi_A] \in \hat{A}: ker\, \pi \not\supseteq J\}$ is open and we have

$$[\pi_B] \in \hat{\alpha}^{-1}(\hat{J}) \Leftrightarrow \exists_{\pi_A \in Irr(A)} \pi_A \leq \pi_B \circ \alpha,$$
$$ker\, \pi_A \not\supseteq J \Longrightarrow ker(\pi_B \circ \alpha) \not\supseteq J$$

$$\Leftrightarrow ker\, \pi_B \not\supseteq \alpha(J)$$

$$\Leftrightarrow ker\, \pi_B \not\supseteq B\alpha(J)B.$$

That is $\hat{\alpha}^{-1}(\hat{J}) \subseteq \{\pi_B \in \hat{B}: ker\, \pi_B \not\supseteq B\alpha(J)B\}$ and in particular $D(\hat{\alpha}) = \hat{\alpha}^{-1}(\hat{A}) \subseteq \{\pi_B \in \hat{B}: ker\, \pi_B \not\supseteq B\alpha\,(A)B\}$.

If we additionally assume that B is liminal, then for $\pi_B \in Irr(B)$ the representation $\pi_B \circ \alpha$ decomposes into a direct sum of irreducibles. Namely, there is a subset K of $\hat{\alpha}_0(\pi_B)$such that $\pi_B \circ \alpha = \bigoplus_{\pi_A \in K} \pi_A \oplus 0$ (where 0 stands for the zero representation and is vacuous if $\pi_B \circ \alpha$is nondegenerate). Hence the implication

$$ker(\pi_B \circ \alpha) \not\supseteq J \Longrightarrow \exists_{\pi_A} \in K \subseteq Irr(A)\ s.t. \pi_A \leq \pi_B \circ \alpha, \qquad ker\, \pi_A \not\supseteq J$$

holds true. This combined with the preceding argument yields $\hat{\alpha}^{-1}(\hat{J}) = \{\pi_B \in \hat{B}: ker\, \pi_B \not\supseteq B\alpha(J)B\}$ and the second part of the assertion follows.

Example (4.1.10)[4]:

Let $H = L^2_\mu[0,1]$ with $\mu$the Lebesgue measure. Put $B := ß(H), A := L^\infty[0,1]$ and let $\alpha: A \to B$ bethemonomorphism sending $\alpha \in A$to the operator of multiplication by a. Then $\pi_B = id$ is irreducible and $\pi_B \circ \alpha$faithful but $\widehat{\alpha}([\pi_B]) = \emptyset$. Accordingly,

$$D(\widehat{\alpha}) \neq \{[\pi_B] \in \widehat{B}: \ker \pi_B \not\supseteq B\alpha(A)B\} = \widehat{B}$$

Let X be a regular C^*-correspondence with coefficients in A. We may treat X as $K(X) - \langle X, X\rangle_A$-imprimitivity bimodule and therefore the induced representation functor$X - \mathrm{Ind}: \mathrm{Irr}(\langle X, X\rangle_A) \to \mathrm{Irr}(K(X))$factors through to the homeomorphism $[X - \mathrm{Ind}]: \widehat{\langle X, X\rangle_A} \to (\widehat{K(X)}$which in turn may be viewed as a multivalued map $[X - \mathrm{Ind}]: \widehat{A} \to (\widehat{K(X)}$ with domain D([X-Ind]) = $\widehat{\langle X, X\rangle}_A$.

Definition (4.1.11)[4]:

Let X be a regular C^*-correspondence over A**.** We define dual map $\widehat{X}: \widehat{A} \to \widehat{A}$ to X as the following composition of multivalued maps

$$\widehat{X} = \widehat{\emptyset} \circ [X - lnd],$$

where $\widehat{\emptyset}: \widehat{K(X)} \to \widehat{A}$is dual to the left action $\emptyset: A \to K(X)$ of Aon X**.** Alternatively, $\widehat{X}$ is a factorization of the map $\widehat{X}_0: \widehat{\emptyset}_0 \circ X - \mathrm{Ind}: \mathrm{Irr}(A) \to \mathrm{Irr}(A)$.

Proposition (4.1.12)[4]:

The multivalued map dual to a regular **C*** -correspondence X is always surjective, that is $\hat{X}(\hat{A}) = \hat{A}$. The domain of $\hat{X}$ satisfies the following inclusion

$$D(\hat{X}) \subseteq (\langle X, \emptyset(A)X\rangle_A)^{\frown} \quad (13)$$

Note here that $\langle X, \emptyset(A)X\rangle_A$ is an ideal in A. If, in addition, A is liminal, then $\hat{X}$ is a continuous rnultivalued map and we have equality in (13); in particular, if X is full and essential, then $\hat{X}: \hat{A} \to \hat{A}$ is a continuous multivaluedsurjection with full domain, $D(\hat{X}) = \hat{A}$.

Proof: As [X-Ind]: $\hat{A} \to \widehat{K(X)}$ is surjective and $\ker \emptyset = \{0\}$ we get $\hat{X}(\hat{A}) = \hat{A}$ by Proposition (4.1.9). Since [X- Ind]:$\widehat{\langle X, X\rangle_A} \to \widehat{K(X)}$is a homeomorphism, it follows from Proposition (4.1.9) that

$$D(\hat{X}) \subseteq [X - \mathrm{Ind}]^{-1}(K(X)\widehat{\emptyset(A)}K(X)) \quad (14)$$

with equality if A is liminal (note that if A is liminal then $K(X)$ is also liminal being Morita—Rieffel equivalent to the liminal C*-algebra $\langle X, X\rangle_A \subseteq A$). Hence it suffices to show that the sets in the right hand sides of (13) and (14) coincide. However, for any representation π of A and any C*-subalgebra $B \subseteq K(X)$ we have

$$B \subseteq ker(X - \mathrm{Ind}(\pi)) \Leftrightarrow \pi(\langle BX, BX\rangle_A) = 0 \Leftrightarrow (X, BX)_A \subseteq ker\pi.$$

thus the assertion follows from the equality

$$\langle X, K(X)\emptyset(A)K(X)X\rangle_A = \langle K(X)X, \emptyset(A)K(X)X\rangle_A = \langle X, \emptyset(A)X\rangle_A.$$

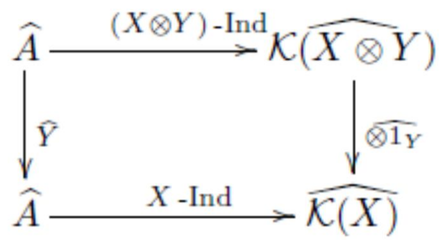
In view of Proposition (4.1.3), if X and Y are regular C*-correspondences with coefficients in A, then the tensoring on the right by the identity 1_Yin Y yields a homomorphism $\otimes 1_y : K(X) \to K(X \otimes Y)$. With help of its dual map we are able to analyze the relationship between the spectra of compact operators on the level of spectrum of A.

Proposition (4.1.13)[4]:

Let X and Y be regular C*-correspondences with coefficients in A. Then W have

$$[X. - \mathrm{Ind}] \circ \hat{Y} = \otimes 1_Y \circ [(X \otimes Y) - \widehat{\mathrm{Ind}]} \tag{15}$$

In other words, the diagram of multivalued maps



is commutative, and in particular

$$D([X - \mathrm{Ind}] \circ \hat{Y}) = D(\widehat{\otimes 1_Y} \circ [(X \otimes Y) - \mathrm{Ind}]) = \hat{Y}^{-1}(\widehat{\langle X, X\rangle_A})$$

Proof: Let $\pi_A : A \to ß(H)$ be an irreducible representation. If $\pi \in \hat{Y}_0(\pi_A)$, then H_π in is a closed suhspace of $Y \otimes_{\pi_A} H$ irreducible under the left multiplication by elements of A, or more precisely, irreducible for $(Y - \mathrm{Ind}(\pi_A))(\emptyset_Y(A))$. Since the tensor product of C*-correspondences is both associative and distributive with respect to direct sums, we may naturally identify $X \otimes_\pi H_\pi$ with a closed subspace of $X \otimes Y \otimes_{\pi_A} H$. Since for $a \in K(X)$ we have

$$\big((X \otimes Y) - \mathrm{Ind}(\pi_A)\big)(a \otimes 1_Y)\big(x \otimes y \otimes_{\pi_A} h\big) = ax \otimes y \otimes_{\pi_A} h,$$

we see that, the action of $\big((X \otimes Y) - \mathrm{Ind}(\pi_A)\big)(a \otimes 1_Y)$ On$X \otimes_\pi H_\pi$coincides with the action of (X-Ind(π))(a). In particular, the subspace $X \otimes_\pi H_\pi$ is either {0}, when it $\pi \notin \widehat{\langle X, X\rangle}_A$ or is irreducible for $\big((X \otimes Y) - \mathrm{Ind}(\pi_A)\big)(K(X) \otimes 1_Y)$. Consequently,

$$(X - \mathrm{Ind}) \circ \hat{Y}_0(\pi_A) \subseteq \widehat{\otimes 1_{Y_0}} \circ (X \otimes Y) - \mathrm{Ind}\ (\pi_A).$$

To show the reverse inclusion, let $\rho \in \big(\widehat{\otimes 1_Y}\big)_0 \circ (X \otimes Y) - \mathrm{Ind}\ (\pi_A)$. Then $\rho$is an irreducible subrepresentation of the representation $\pi_{K(X)}: K(X) \to ß\big(X \otimes Y \otimes_{\pi_A} H\big)$where $\pi_{K(X)}(a) = \big((X \otimes Y) - \mathrm{Ind}(\pi_A)\big)(a \otimes 1_Y)$. We may consider the dual C*-correspondence $\tilde{X}$ (not to he confused with the dual $\hat{X}$ to the C*-correspondence X) as an$\langle X, X\rangle_A - K(X) -$imprimitivity bimnodule. Then using the natural isomnorpinsmn

$$\big(\tilde{X} \otimes_{K(X)} \otimes X\big) \otimes Y \otimes_{\pi_A} H \cong Y \otimes_{\pi_A} H,$$

we see that $\tilde{X}$- Ind($\pi_{K(X)}$) is equivalent to $Y - \mathrm{Ind}(\pi_A) \circ \emptyset_Y : A \to ß\big(Y \otimes_{\pi_A} X\big)$. Since induction respects direct sums, $\tilde{X}$- Ind (ρ) is equivalent to an irreducible subrepresentation π of $Y - \mathrm{Ind}(\pi_A) \circ \emptyset_Y$. Then πbelongs to both $\widehat{\langle X, X\rangle}_A$ and $\hat{Y}_0(\pi_A)$, and we have

$$\rho \cong X - \mathrm{Ind}(\tilde{X} - \mathrm{Ind}(\rho)) \cong X - \mathrm{Ind}(\pi).$$

Consequently, $\widehat{\otimes 1_{Y_0}} \circ (X \otimes Y) - \text{Ind}\ (\pi_A) \subseteq X - \text{Ind}\ \circ \widehat{Y}_0(\pi_A)$.

Corollary (4.1.14)[4]:

The composition of duals to C*-correspondences coincides with the dual of their tensor product:

$$\widehat{X} \circ \widehat{Y} = \widehat{X \otimes Y}.$$

Proof: We showed in the proof of Proposition (4.1.13) that $X - \text{Ind}\ \circ \widehat{Y}_0 = \widehat{\otimes 1_{Y_0}} \circ (X \otimes Y) - \text{Ind}$ and all subspaces of $X \otimes Y \otimes_{\pi_A} H$ irreducible for $\big((X \otimes Y) - \text{Ind}(\pi_A)\big)(K(X) \otimes 1_Y)$are of the form $X \otimes_\pi H_\pi$, where $\pi \in \widehat{Y}_0(\pi_A) \cap \widehat{\langle X, X\rangle}_A$. Since $\emptyset_{X \otimes Y}(A) \subseteq K(X) \otimes 1_Y$, the action of $(X \otimes Y) - \text{Ind}(\pi_A)\left(\emptyset_{X \otimes Y}(a)\right) a \in A$, coincides on $X \otimes_\pi H_\pi$ with X-Ind(π) $(\emptyset_X(a)$. Thus we have

$$\widehat{X}_0 \circ \widehat{Y}_0 = \left(\widehat{\emptyset_{X_0}} \circ X - \text{Ind}\right) \circ \widehat{Y}_0 = \widehat{\emptyset_{X \otimes Y_0}} \circ (X \otimes Y) - \text{Ind} = \widehat{X \otimes Y}_0.$$

Let X be a product system over P. By Corollary (4.1.14), the family $\{\hat{X}_p\}_{p \in P}$ofdual maps to C*-correspondences $X_p, p \in P$, forms a sernigroup of multivalued maps on $\widehat{A}$, that is

$$\hat{X}_e = id, \quad and \quad \hat{X}_p \circ \hat{X}_q = \hat{X}_{pq}, \qquad p, q \in P.$$

If A is liminal then these mnutivalued maps are continuous by Proposition (4.1.12).

Definition (4.1.15)[4]:

We call the semigroup $\hat{X} := \{\hat{X}_p\}_{p \in P}$ dual to the product system X.

We prove certain technical facts concerning the interaction among Cuntz-Pimsner representations, dual maps and the process of induction.

Lemma (4.1.16)[4]:

Let X be a product system over a left calculative semi group P. If $p, q, s \in P$ are such that $s \geq p, q$, then

$$\hat{X}_{q^{-1}s}\hat{X}_{p^{-1}s}^{-1} = [X_q - \mathrm{Ind}]^{-1} \circ \hat{\iota}_q^s \circ \hat{\iota}_p^{s\,-1} \circ [X_p - \mathrm{Ind}].$$

Proof: Applying Proposition (4.1.13) to $Y = X_{p^{-1}s}, X = X_p and\, Y = X_{q^{-1}s}, X = X_q$, respectively, we get

$$\left[X_p - Ind\right]\hat{X}_{p^{-1}s} = \hat{\iota}_p^s[X_s - Ind] \quad and \left[X_q - Ind\right]\hat{X}_{q^{-1}s} = \hat{\iota}_q^s[X_s - Ind].$$

As $\left[X_p - \mathrm{Ind}\right]$ and $\left[X_q - \mathrm{Ind}\right]$arehomeomorphisms, this is equivalent to

$$\hat{X}_{p^{-1}s} = \left[X_p - \mathrm{Ind}\right]^{-1}\hat{\iota}_p^s[X_s - \mathrm{Ind}] \quad and \;\; \hat{X}_{q^{-1}s}$$
$$= \left[X_q - \mathrm{Ind}\right]^{-1}\hat{\iota}_q^s[X_s - \mathrm{Ind}],$$

and the assertion follows.

Lemma (4.1.17)[4]:

Suppose Y is an imprimitivity Hilbert A-B-bimodule and (π_A, π_Y, π_B) is its representation on a Hilbert space H. Thus $\pi_A: A \to ß(H), \pi_B: B \to ß(H)$, are representations and with the map $\pi_Y: Y \to ß(H)$ they satisfy

$$\pi_A(a)\pi_Y(y)\pi_B(b) = \pi_Y(ayb), \qquad \pi_Y(x)\pi_Y(y)^* = \pi_A(A\langle x, y\rangle),$$

$$\pi_Y(x)^*\pi_Y(y) = \pi_B(\langle x, y\rangle B),$$

$a \in A, b \in B, x, y \in Y$. If $\pi$is an irreducible subreprsentation of $\pi_B$then the restriction $\rho(a) := \pi_A(a)|_{\pi_Y(y)H_\pi}$, yields an irreducible sub representation of π_A such that $[\rho] = [Y - \mathrm{Ind}(\pi)]$.

Proof: Let it $\pi \leq \pi_B$ be a representation of B on a Hilbert space $H_\pi \subset H$.The Hilbert space $\pi_Y(Y)H_\pi \subset H$ is invariant for elements of $\pi_A(A)$and therefore $\rho(a) := \pi_A(a)|_{\pi_Y(Y)H_\pi}, a \in A$ defines a representation of A. Since

$$\left\|\sum_{i=1}^{n} \pi_Y(y_i)h_i\right\|^2 = \sum_{i,j=1}^{n} \langle \pi_Y(y_i)h_i, \pi_Y(y_j)h_j\rangle = \sum_{i,j=1}^{n} \langle h_i, \pi_A(\langle y_i, y_j\rangle_A)h_j\rangle$$

$$= \left\|\sum_{i=1}^{n} y_i \otimes_\pi h_i\right\|^2,$$

the mapping $\pi_Y(y)h \mapsto y \otimes_\pi h, y \in Y, h \in H_\pi$, extends by linearity and continuity to a unitary operator $V: \pi_Y(y)H_\pi \to Y \otimes_\pi H_\pi$, which intertwines ρ and Y-Ind(π) because

$$V\rho(a)\, \pi_Y(y)h = V\,\pi_Y(ay)h = (ay \otimes_\pi h) = Y - \mathrm{Ind}(\pi)(a)V\,\pi_Y(y)h.$$

Accordingly, if π it is irreducible then ρ, being unitary equivalent to the irreducible representation Y-Ind(π), is also irreducible.

Lemma (4.1.18)[4]:

Suppose ψ is a Cuntz- Pimsner covariant representation of a regular' product system X Over ρ on a Hilbert space H. Let $p, q \in P$ and let π he an irreducible summnand of ,$\psi^{(q)}$ acting on a subspace K of H. Then the restriction

$$\pi_p(T) := \psi^{(p)}(a)\big|_{\psi_p(X_p)\psi_q(X_q)^* K}, \qquad T \in K(X_p), \tag{16}$$

yields a representation $\pi_p: K(X_p) \to ß(\psi_p(X_p)\psi_q(X_q)^* K)$ which is either zero or irreducible. and such that

$$[\pi_p] = [(X_p - Ind)((X_q - Ind)^{-1}(\pi))].$$

Proof: The dual C*-correspondence $\tilde{X}_q$ to X_q is an imprirnitivity $\langle X_q, X_q\rangle_A - K(X_q)$-bimodule and $(\psi_e, \tilde{\psi}_q, \psi^{(q)})$, where $\tilde{\psi}_q(\mathfrak{b}(x)) = \psi_q(X)^*$, is its representation. Thus, by Lemma (4.1.17), the restriction $\pi_e(a) := \psi_e(a)|_{\psi_q(X_q)^* K}, a \in A$ yields an irreducible subrepresentation $\pi_e: A \to ß(\psi_q(X_q)^* K)$ of ψ_e such that $[\pi_e] = [\tilde{X}_q - \mathrm{Ind}(\pi)] = [X_q - \mathrm{Ind}^{-1}(\pi)]$, if $\pi_e(\langle X_p, X_p\rangle_A) = 0$, then (16) is a zero representation. Otherwise we may apply Lemma (4.1.17) to π_e and the representation $(\psi^{(p)}, \psi_p, \psi_e)$ of the imprimitivity $K(X_q) - \langle X_p, X_p\rangle_A$-bimnodule X_p. Then we see that (16) yields an irreducible representation such that $[\pi_p] = [X_p - \mathrm{Ind}(\pi_e)] = [X_p - \mathrm{Ind}\ (X_q - \mathrm{Ind})^{-1}(\pi))]$.

Section (4.2): A Uniqueness Theorem and Simplicity Criteria for Cuntz-Pimsner Algebras with Applications

We consider a directed, left cancellative semigroup ***P*** and a regular product system *X* over *P* with coefficients in an arbitrary C*-algebra A. We recall from Theorem (4.1.7) that the Cuntz-Pimsner algebra $\mathcal{O}_X$is graded over the enveloping group *G(P)*with fibers

$$(\mathcal{O}_x)_g = \overline{span}\{j_X(x)j_X(y)^* : x, y \in X, [d(x), d(y)] = g\}, \qquad g \in G(P).$$

Moreover, $\mathcal{O}_X$ may be viewed as a full cross-sectional algebra $C^*(\{(\mathcal{O}_X)_g\}_{g\in G(p)})$ of the Fell bundle$\{(\mathcal{O}_X)_g\}_{g\in G(p)}$, and the reduced Cuntz—Pimsner algebra

$$\mathcal{O}_X^r := C_r^*(\{(\mathcal{O}_X)_g\}_{g\in G(p)})$$

is cletined as the reduced cross-sectional algebra of $\{(\mathcal{O}_X)_g\}_{g\in G(p)}$. There exists a canonical epimorphsmn

$$\lambda: \mathcal{O}_X \to \mathcal{O}_X^r. \qquad (17)$$

This epimorphism may not be injective. However, λ is always injective whenever the group *G(P)*is amenable or more generally when the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(p)}$has the approximation property be defined.

By now, several conditions implying amenability of the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(p)}$are known. That is, conditions which guarantee the identity $\mathcal{O}_X = \mathcal{O}_X^r$. These conditions seem to be independent of aperiodicity we want to investigate, and thus we decided not to assume any of them. Accordingly, we seek an intrinsic condition on the product system X (or on the dual sernigroup $\hat{X}$**)** which would guarantee that every Cuntz-Pimsner representation of X injective on the coefficient algebra A generates a C*-algebra lying in between $\mathcal{O}_X$ and $\mathcal{O}_X^r$. Before proceeding further, we summarize a few known facts useful in the aforementioned context.

Proposition (4.2.1)[4]:

Suppose that ψ is an injective Cuntz-Pimsner representation of a regular product system X. Consider the following conditions:

(i) The canonical epimorphism $\prod_{\boldsymbol{\psi}}: \mathcal{O}_X \to C^*(\psi(X))$, where $i_X(x) = \psi(x), x \in X$, is an isomnorphism.

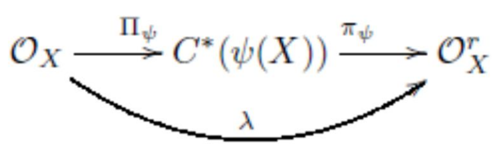
(ii) There is a coaction $\beta\ of\ G = G(P)$on $C^*(\psi(X))$ such that $\beta\big(\psi(x)\big) = \psi(x) \otimes i_G\big(d(x)\big), x \in X$

(iii) There is a conditional expectation $E_{\boldsymbol{\psi}}$ from $C^*(\psi(X))$onto

$$\mathcal{F}_\psi = \overline{span}\{\psi(x)\psi(y)^* : x, y \in X, d(x) \sim d(y)\}$$

vanishing on elements $\psi(x)\psi(y)^*$with $d(x) \nsim d(y)$.

We have the implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$, and (iii,) is equivalent to existence of a unique epimorphism $\pi_{\boldsymbol{\psi}}: C^*(\psi(X)) \to \mathcal{O}_X^r$such that the following diagram



(18)

is commutative. In particular, if the epimorphism λ from (17) is an isornorphisrn, then the conditions (i), (ii) ,(iii) areequivalent**.**

Proof: Implication (i) $\Rightarrow$ (ii) is obvious because we know that $\mathcal{O}_X$ is equipped with the coaction in the prescribed form. Suppose (ii) holds. Using the contractive projections onto the spectral subspaces for the coaction β, and the fact that elements of the form $\psi(x)\psi(y)^*$ span a dense suhspace of $C^*(\psi(E))$, Lemma (4.1.6), we get

$$[C^*(\psi(X))]_g^\beta = \{c \in C^*(\psi(X)) : \beta(c) = c \otimes i_G(g)\} = \overline{span}\{\psi(x)\psi(y)^* : [d(x), d(y)] = g\}.$$

In particular, the projection onto $[C^*(\psi(X))]_e^\beta = \mathcal{F}_\psi$, is the conditional expectation described in (iii). If we assume (iii), then $\{\prod_\psi ((\mathcal{O}_X)_g)\}_{g\in G}$ is a Fell bundle which yields a topological grading of $C^*(\psi(X))$**.** Hence by [4**]** there exists a desired epimorphism $\pi_\psi: C^*(\psi(X)) \to \mathcal{O}_X^r$. Conversely, if such an epimorphism $\pi_\psi: C^*(\psi(X)) \to \mathcal{O}_X^r$ exists, then composing it with the canonical conditional expectation on $\mathcal{O}_X^r$ one gets the conditional expectation described in (ii).

Definition (4.2.2)[4]:

We say that a representation $\psi: X \to B$ of a product system X is topologically graded if it has the property described in part (iii) of Proposition (4.2.1).

Thus, to conclude our discussion, by uniqueness theorem for $\mathcal{O}_X$ we understand a result which guarantees that for every injective Cuntz-Pimsner covariant representation ψ of X there is a map π_{ψ}, making the diagram (18) commutative. By Proposition (4.2.1), this is equivalent to ψ being topologically graded. We now introduce a dynamical condition which entails such a result.

Definition (4.2.3)[4]:

We say that a regular product system X, or the dual semnigroup $\{\hat{X}_p\}_{p\in P}$, is topologically a periodic if for each nonempty open set $U \subseteq \hat{A}$ each finite set $F \subseteq P$ and element $q \in P$ such that $q \nsim_{\boldsymbol{R}} p$ for $p \in F$, there exists a $[\pi] \in U$ such that for some enumeration of elements of $F = \{p_1, \ldots, p_n\}$and some elements $s_1, \ldots, s_n \in P$ with $q \leq s_1 \leq \ldots \leq s_n$ and $p_i \leq s_i$ We have

$$[\pi] \notin \hat{X}_{q^{-1}s_i}\left(\hat{X}^{-1}_{p_i^{-1}s_i}([\pi])\right) \quad for\ all\ i = 1, \ldots, n. \qquad (19)$$

Proposition (4.2.4)[4]:

If condition (19) holds for some sequence $q \leq s_1 \leq \ldots \leq s_n$,then it also holds for any sequence $q \leq s'_1 \leq \ldots \leq s'_n$such that

$$p_i \leq s'_i \leq s_i \quad for\ all\ i = 1, \ldots, n.$$

Moreover , we have the following.

(i) If *(G(P), P)* is a quasi-lattice ordered group then in Definition (4.2.3) one can always take

$$s_1 = p_1 \vee q \qquad and\ s_i = p_i \vee s_{i-1} \qquad for\ all\ i = 2, \dots, n.$$

(ii) Topological aperiodicity of X implies that for any open nonempty set $U \subseteq \hat{A}$ and any finite set $F \subseteq P$ such that $p \nsim_{\boldsymbol{R}} e$ for $p \in F$, there is a $[\pi] \in U$ satisfying

$$[\pi] \notin \widehat{X}_p([\pi]) \qquad for\ all\ p \in F. \tag{20}$$

If $(P, \leq)$ is linearly ordered then the converse implication also holds.

(iii) In the case of a product system $\{X^{\otimes n}\}_{n\in\mathbb{N}}$ arising from a single regular C*-correspondence X, the topological aperiodicity is equivalent to that for each $n > 0$ the set

$$F_n = \{[\pi] \in \hat{A} : \pi \in \hat{X}^n([\pi])\}$$

has empty interior. (In this case we will say that the C*-correspondence X is topologically a periodic.)

Proof: Let us notice that if $q, p_i \leq s_i' \leq s_i$, then using the semnigroup property of $\hat{X}$ (Corollary (4.1.14)), surjectivity of mappings $\hat{X}_p, p \in P$ (Proposition (4.1.12)) we get

$$\hat{X}_{q^{-1}s_i} \circ \hat{X}^{-1}_{p_i^{-1}s_i} = \hat{X}_{q^{-1}s_i'} \circ \hat{X}_{s_i'^{-1}s_i} \circ \left(\hat{X}_{p_i^{-1}s_i'} \circ \hat{X}_{s_i'^{-1}s_i}\right)^{-1}$$
$$= \hat{X}_{q^{-1}s_i'} \circ \hat{X}_{s_i'^{-1}s_i} \circ \hat{X}^{-1}_{s_i'^{-1}s_i} \circ \hat{X}^{-1}_{p_i^{-1}s_i} \supseteq \hat{X}_{q^{-1}s_i'} \circ \hat{X}^{-1}_{p_i^{-1}s_i}$$

Hence $[\pi] \notin \hat{X}_{q^{-1}s_i}(\hat{X}^{-1}_{p_i^{-1}s_i}([\pi]))$ implies $[\pi] \notin \hat{X}_{q^{-1}s_i'}(\hat{X}^{-1}_{p^{-1}s_i}([\pi]))$. This proves the initial part of the assertion.

Ad (i). It follows immediately from what we have just shown.

Ad (ii). If $F = \{p_1, \ldots, p_n\} \subseteq P$and $p \nsim_{\boldsymbol{R}} e$ for all $p \in F$, then putting $q = e$ we see that topological aperiodicity of X implies that for any nonempty open set $U \subseteq \hat{A}$ there are elements $s_1, \ldots, s_n \in P, p_i \leq s_i, i = 1, \ldots, n$ and a point $[\pi] \in U$ such that

$$[\pi] \notin \widehat{X}_{q^{-1}s_i}\left(\hat{X}^{-1}_{p_i^{-1}s_i}([\pi])\right) = \widehat{X}_{s_i}\left(\hat{X}^{-1}_{p_i^{-1}s_i}([\pi])\right) \quad \text{for all } i = 1, \ldots, n.$$

By the inclusion noticed above we have $\widehat{X}_{s_i} \circ \hat{X}^{-1}_{p_i^{-1}s_i} = \widehat{X}_{p_i} \circ \widehat{X}_{q^{-1}s_i} \circ \hat{X}^{-1}_{p_i^{-1}s_i} \supseteq \widehat{X}_{p_i}$ and thus condition (20) follows.

Conversely, suppose $(P, \leq)$ is linearly ordered and the condition described in (ii) is satisfied. Let $U \subseteq \hat{A}$ be open and nonempty, $F \subseteq P$ finite and $q \in P$ such that $q \nsim_{\boldsymbol{R}} p$ for $p \in F$. Enumerating elements of $F = \{p_1, \ldots, p_n\} \subseteq P$ in a lion-increasing order we have

$$p_1 \leq p_2 \leq \cdots \leq p_{k_0} \leq q \leq p_{k_0+1} \leq \cdots \leq p_n$$

for certain $k_0 \in \{0, 1, \ldots, n\}$. Defining

$$s_i := \begin{cases} q, & i \leq k_0 \\ p_i & i \geq k_0 + 1 \end{cases}$$

we see that, $q \leq s_i \leq \ldots \leq s_n$ and

$$\hat{X}_{q^{-1}s_i} \circ \hat{X}^{-1}_{p_i^{-1}s_i} := \begin{cases} \hat{X}^{-1}_{p_i^{-1}q}, & i \leq k_0 \\ \hat{X}_{q^{-1}p_i} & i \geq k_0 + 1 \end{cases}$$

Put $F' : \{p_i^{-1}q : i = 1, .. k_0\} \cup \{q^{-1}p_i : i = k_0 + 1, ... n\}$and note that $p \nsim_{\boldsymbol{R}} e$ for all $p \in F'$. Thus we may apply condition described in (ii) to F' and then we obtain a $[\pi] \in U$ satisfying (19).

Ad (iii). By part (ii) above, topological aperiodicity implies the condition described in (iii). To see the converse, again by part (ii), it suffices to show (20) for a finite set $F \subseteq \mathbb{N} \setminus \{0\}$. The latter follows from condition described in (iii) applied to $n = m!$ where $m = \max\{k : k \in F\}$.

Theorem (4.2.5)[4]:

(Uniqueness theorem).Suppose that a regular product system X is topologically aperiodic. Then every injective Cuntz-Pimsner representation of X is topologically graded. if the canonical epimorphism $\leftthreetimes : \mathcal{O}_X \to \mathcal{O}_X^r$ is injective then there is a natural isomorphisrn

$$\mathcal{O}_X \cong C^*(\psi(X))$$

for every injective Cuntz-Pimsner representation $\psi$of X.

Proof: Suppose that ψ is an injective Cuntz-Pimsner representation of X in a C*-algebra B. Then $\psi^{(p)} : k(X_p) \to B$ is injective for all $p \in P$. Let us consider an element of the form

$$\psi^{(q)}(S_q) + \sum_{p \in F} \psi_{p,q}(S_{p,q})$$

where $q \in P, F \subseteq P$ is a finite set such that $p \nsim_{\boldsymbol{R}} e$ for all $p \in F$, and $S_q \in k(X_q), S_{p,q} \in k(X_q, X_p)$. By Lemma (4.1.6) part (iii), such elements form a dense subspace of $C^*(\psi(X))$. Thus existence of the appropriate conditional expectation will follow from the inequality

$$\|S_q\| = \|\psi^{(q)}(S_q)\| \le \left\|\psi^{(q)}(S_q) + \sum_{p\in F} \psi_{p,q}(S_{p,q})\right\| \qquad (21)$$

To prove this inequality, we fix $\varepsilon > 0$ and recall that for any $a \in A$ the mapping $\hat{A} \ni [\pi] \longmapsto \|\pi(a)\|$ is lower semicontinuous and attains its maximum equal to $\|a\|$, [4]. Thus, since $X_q - \mathrm{Ind}: \hat{A} \to \widehat{k(X)}$is a homeomorphism, we deduce that there is an open nonemnpty set $U \subseteq \hat{A}$ such that

$$\|X_q - \mathrm{Ind}(\pi)(S_q)\| > \|S_q\| - \varepsilon \qquad for\ every\ [\pi] \in U$$

Let $F = \{p_1, \dots, p_n\}$. By topological aperiodicity of X, there are elements $s_1, \dots, s_n \in P$ such that $q \le s_1 \le \dots \le s_n$ and $p_i \le s_i, i = 1, \dots, n,$ and there exists a $(\pi) \in U$satisfying (19). Let us fix these objects.

We recall that if $p < s,$ then $i_p^s\left(k(X_p)\right) \subseteq k(X_s)$ and thus $\psi^{(p)}\left(k(X_p)\right) \subseteq \psi^{(s)}(k(X_s))$, c cf. Lemma (4.1.5). In particular, we have the increasing sequence of algebras

$$\psi^{(q)}\left(k(X_q)\right) \subseteq \psi^{(s_1)}\left(k(X_{s_1})\right) \subseteq \cdots \subseteq \psi^{(s_n)}\left(k(X_{s_n})\right) \subseteq C^*(\psi(X)).$$

We construct a relevant sequence of representations of these algebras as follows. We put

$$v_q\colon \psi^{(q)}\left(k(X_q)\right) \to ß(H_q) \quad \text{defined as} \quad v_q\left(\psi^{(q)}(s)\right) = X_q - \mathrm{Ind}(\pi)(S)$$

Then v_q is an irreducible representation because so is π. We let $v_{s_1}\colon \psi^{(s_1)}\left(k(X_{s_1})\right) \to ß(H_{s_1})$ to be any irreducible extension of v_q, andfor$i = 2,3,\ldots,n$we take $v_{s_i}\colon \psi^{(s_i)}\left(k(X_{s_i})\right) \to ß(H_{s_i})$ to be any irreducible extension of $v_{s_{i-1}}$. Finally, we let $v\colon C^*(\psi(X)) \to ß(H)$ to be any extension of v_{s_n}. In particular, we have

$$H_q \subseteq H_{s_1} \subseteq \cdots \subseteq H_{s_n} \subseteq H.$$

Let $P_q \in ß(H)$ be the projection onto the suhspace H_q. Clearly

$$\left\|P_q v\left(\psi^{(q)}(S_q)\right)P_q\right\| = \left\|v_q\left(\psi^{(q)}(S_q)\right)\right\| = \|X_q - \mathrm{Ind}(\pi)(S_q)\| \geq \|S_q\| - \varepsilon$$

and as $\varepsilon$is arbitrary we can reduce the proof to showing that

$$P_q v\left(\psi_{p,q}(S_{p,q})\right)P_q = 0 \qquad for\ p \in F. \tag{22}$$

To this end, we fix a $p_i \in F$. Let P_{s_i} be the projection onto H_{s_i} and consider the space

$$H_{p_i} \coloneqq v(\psi_{p_i}(X_{p_i})\psi_q(X_q)^*)H_q$$

We claim that $P_{s_i}H_{p_i} = \{0\}$. Since $H_q \subseteq H_{s_i}$, this implies (22) and finishes the proof. Suppose to the contrary that $P_{s_i}H_{p_i} \neq \{0\}$ By Lemma (4.1.18) and the definitions of v and H_{p_i} the mapping

$$k(X_{p_i}) \ni S \to v\left(\psi^{(p_i)}(S)\right)\Big|_{H_{p_i}}$$

is an irreducible representation equivalent to X_{p_i}- Ind(π). In particular, H_{p_i} is irreducible for $v\left(\psi^{(p_i)}\left(k(X_{p_i})\right)\right)$. Since

$$v\left(\psi^{(p_i)}\left(k(X_{p_i})\right)\right) \subseteq v\left(\psi^{(s_i)}\left(k(X_{s_i})\right)\right) \text{and } p_{s_i} \in v\left(\psi^{(s_i)}\left(k(X_{s_i})\right)\right)',$$

we see that $P_{s_i}H_{p_i}$ is an irreducible subspace for $v\left(\psi^{(p_i)}\left(k(X_{p_i})\right)\right)$. Thus, since H_{p_i} and $P_{s_i}H_{p_i}$ are both irreducible subspaces for $v\left(\psi^{(p_i)}\left(k(X_{p_i})\right)\right)$, either $H_{p_i} = P_{s_i}H_{p_i}$ or $H_{p_i} \perp P_{s_i}H_{p_i}$. However, (as $P_{s_i}H_{p_i} \neq \{0\}$) the latter is clearly impossible. Thus $H_{p_i} \subseteq H_{s_i}$ and denoting by $\pi_{\boldsymbol{s_i}}$ the representation

$$k(X_{s_i}) \ni S \overset{\pi_{s_i}}{\to} v\left(\psi^{(s_i)}(S)\right)\Big|_{H_{p_i}}$$

we get $[\pi_{s_i}] \in \widehat{\iota_{p_i}^{s_i}}^{-1}([X_{p_i} - \mathrm{Ind}(\pi)])$. Denoting by π_q the representation

$$k(X_q) \ni S \to v\left(\psi^{(q)}(S)\right)\Big|_{H_q}$$

we have $[\pi_{\boldsymbol{q}}] \in \widehat{\iota_q^{s_i}}([\pi_{\boldsymbol{s_i}}])$ and $\pi_q = X_q - \mathrm{Ind}(\pi)$. Hence we get

$$[\pi] = \left[\left(X_q - \mathrm{Ind}\right)^{-1}(\pi_q)\right] \in \left[X_q - \mathrm{Ind}^{-1}\right](\widehat{\iota_q^{s_i}}([\pi_{s_i}]))$$
$$\subseteq \left[X_q - \mathrm{Ind}^{-1}\right](\widehat{\iota_q^{s_i}}(\widehat{\iota_{p_i}^{s_i}}^{-1}([X_{p_i} - \mathrm{Ind}(\pi)])))$$

Thereby in view of Lemma (4.1.16) we arrive at

$$[\pi] \in \hat{X}_{q^{-1}s_i}(\hat{X}^{-1}_{p_i^{-1}s_i})([\pi])$$

which contradicts the choice of π.

As an application of Theorem (4.2.5), we obtain simplicity criteria for the reduced Cuntz- Pimsner algebra $\mathcal{O}_X^r$ To this end, we first introduce the indispensable terminology.

Definition (4.2.6)[4]:

Let X be a regular product system over a semigroup P with coefficients in a C*-a1gebra A. We say that an ideal J in A is X-invariant if and only if for each $p \in P$ the set

$$X_p^{-1}(J) := \{a \in A : \langle X_p, aX_p\rangle_p \subseteq J\}$$

is equal to J. We say $X$is minimal if there are no nontrivial **X**-invariant ideals in A, that is if for any ideal J in A we have

$$(\forall p \in P) X_p^{-1}(J) = J \Longrightarrow J = \{O\} \quad or \; J = A.$$

Theorem (4.2.7)[4]:

(Simplicity of $\mathcal{O}_X^r$) If a regular product system X is topologically a periodic and minimal, then $\mathcal{O}_X^r$is simple.

Proof: Suppose$I$ is an ideal in $\mathcal{O}_X^r$ and put $J = (j_X^r)^{-1}(I) \cap A = \{a \in A : j_X^r(a) \in I\}$. Then J is an ideal in A**.** We claim that J is X-invariant. Indeed, for $p \in P$ we have

$$j_X^r(\langle X_p, JX_p\rangle_A) = j_X^r(X_p)^* j_X^r(JX_p) = j_X^r(X_p)^* j_X^r(J) j_X^r(X_p) \subseteq I$$

That is $\langle X_p, JX_p\rangle_A \subseteq J$ and hence $J \subseteq X_p^{-1}(J)$. Onthe other hand, if $a \in X_p^{-1}(J)$. then we have

$$\emptyset_p(a) = \sum_i \Theta_{x_{i,}y_i,j_i} \quad where \;\; x_{i,}y_i \in X_p \;\; and \;\; j_i \in J.$$

Since $j_X^r : X \to \mathcal{O}_X^r$ is Cuntz-Pimsner covariant, we get

$$j_X^r(a) = j_X^{r(p)}\left(\emptyset_p(a)\right) = \sum_i j_X^{r(p)}(\Theta_{x_{i,}y_i,j_i}) = \sum_i j_X^r(x_i) j_x^r(y_i, j_i)^*$$
$$= \sum_i j_X^r(x_i) j_x^r(j_i^*) j_X^r(y_i)^* \in I.$$

Thus $X_p^{-1}(J) \subseteq J$ and this proves our claim. In view of minimality of X, either $J = A$or $J = \{0\}$. In the former case, $\mathcal{O}_X^r = C^*(j_X^r(X)) = \boldsymbol{I}$because $j_X^r(X_p)$= $j_X^r(AX_p)$= $j_X^r(A) j_X^r(X_p) \subseteq I$ for each $p \in P$**.** In the latter case, the composition of $j_X^r : X \to \mathcal{O}_X^r$with the quotient map $\theta: \mathcal{O}_X^r \to \mathcal{O}_X^r / I$ yields a Cuntz Pimsner representation$k_x := \theta \circ j_X^r \;\; of \;\; X \; in \; \mathcal{O}_X^r / I$ which is injective on A. Thus by Theorem (4.2.5) we have an epimorphism

$$\pi_{k_X}: \mathcal{O}_X^r / I \to \mathcal{O}_X^r$$

such that $\pi_{k_x}(q(j_X^r(x)) = j_X^r(x), x \in X$.Hence $j_X^r(x) \cap I = \{0\}$and therefore $I = \{0\}$**.**

Schweizer found a necessary and sufficient condition for simplicity of Cuntz-Pimsner algebras associated with single C*-correspondences, improving similar results. Namely, if X is a left essential and full C*-correspondence with coefficients in a unital C*-algebra A, then $\mathcal{O}_X$ is simple if and only if X is minimal and nonperiodic, meaning that $X^{\otimes n} \approx \ {}_A A_A$ implies $n = 0$, where $\approx$ denotes the unitary equivalence of C*-correspondences. This result suggests that topological aperiodicity of a product system X should imply nonperiodicity of X, and this is indeed the case.

Proposition (4.2.8)[4]:

Suppose that X is a topologically a periodic regular product system over a semigroup P of Ore type. Then $X_p \approx X_e$ implies $p \sim_{\boldsymbol{R}} e$, and if in addition $(G(P), P)$ is a quasi-lattice ordered group, then $X_{q^{-1}(pVq)} \approx X_{p^{-1}(pVq)}$ implies $p = q$.

Proof: In view of Proposition (4.2.4) parts (i) and (ii), it suffices to note that $X_p \approx X_q$implies that $[\pi] \in \hat{X}_p(X_q^{-1}(\pi))$ for all $[\pi] \in \hat{A}$. To this end, let $V: X_p \to X_q$bea bimodule unitary implementing the equivalence $X_p \approx X_q$. Let $[\pi] \in \hat{A}$ be arbitrary and take any $[\rho] \in \hat{X}_q^{-1}$ $([\pi])$ (such ρ exists because $\hat{X}_q$is surjective). In other words, $[\pi] \preccurlyeq [X_q - \mathrm{Ind}(\rho) \circ \emptyset_q]$. Then **V** gives rise to a unitary map

$$\tilde{V} : X_p \otimes_\rho H_\rho \to X_q \otimes_\rho H_\rho , \qquad such\ that \quad \tilde{V}(x \otimes h) = (Vx) \otimes h.$$

Indeed, this follows from the following simple computation:

$$\left\|\sum_{i=1}^{n} x_i \otimes h_i\right\|^2 = \sum_{i,j=1}^{n} \langle x_i \otimes_\rho h_i, x_j \otimes_\rho h_j \rangle = \sum_{i,j=1}^{n} \langle h_i, \rho(\langle x_i, x_j \rangle_A) h_j \rangle$$

$$= \sum_{i,j=1}^{n} \langle h_i, \rho\big(\langle V x_i, V x_j \rangle_A\big) h_j \rangle = \sum_{i,j=1}^{n} \langle (V x_i) \otimes_\rho h_i, (V x_j) \otimes_\rho h_j \rangle$$

$$= \left\|\sum_{i=1}^{n} (V x_i) \otimes_\rho h_i\right\|^2$$

where $x_i \in X_p, h_i \in H_\rho, \mathrm{i} = 1, \ldots, \mathrm{n}$. Since V is a left A-module morphism, we see that $\tilde{V}$ establishes a unitary equivalence between $X_p - \mathrm{Ind}(\rho) \circ \emptyset_p$ and $X_q - \mathrm{Ind}(\rho) \circ \emptyset_q$. Hence we have both $[\pi] \preccurlyeq [X_q - \mathrm{Ind}(\rho) \circ \emptyset_q]$ and $[\pi] \preccurlyeq [X_p - \mathrm{Ind}(\rho) \circ \emptyset_p]$.

We give several examples and applications of the theory developed above. We discuss algebras associated with saturated Fell bundles, twisted C *-dynamical systems, product systems of topological graphs and the Cuntz algebra $Q_{\mathbb{N}}$.

We consider a regular product system X over a semigroup $P$of Ore type, with the additional property that each C*-correspondence $X_p, p \in P$**,** is a Hilbert bimodule equipped with left **A**-valued inner product $p\langle \cdot,\cdot \rangle : X_p \times X_p \to A$. We call such an X regular product system of Hilbert bimnodules. With help of for instance, one can show that a regular product system is a product system of Hilbert bimodules if and only if each left action homomorphism $\emptyset_p : A \to K(X_p)$ is surjective. In this case,$\emptyset_p : A \to K(X_p)$ is an isomorphism and

$$p\langle x,y\rangle = \emptyset_p^{-1}(\Theta_{x,y}), \qquad x,y \in X_p$$

The following Proposition (4.2.9) gives another characterization of regular product systems of Hilbert bimodules in terms of the Fell bundle structure in $\mathcal{O}_X$ identified in Theorem (4.1.7) above.

Proposition (4.2.9)[4]:

A regular product system X over a semigroup P of Ore type is a product system of Hilbert bimodules if and only if the algebra of coefficients Aembeds into $\mathcal{O}_X$ as the core subalgebra $(\mathcal{O}_X)_{[e,e]}$, that is

$$j_X(A) = (\mathcal{O}_X)_{[e,e]}$$

In this case, each space X_p**.** embeds into $\mathcal{O}_X$ as the fiber $(\mathcal{O}_X)_{[p,e]}$ In particular, $j_X(X_p) = (\mathcal{O}_X)_{[p,e]}$, for all $p \in P$. and

$$(\mathcal{O}_X)_{[p,q]} = \overline{span}\,\{j_X(x)j_X(y)^*: x \in X_p, y \in X_q\}, \qquad p,q \in P. \qquad (23)$$

Proof: If all the maps $\emptyset_p: A \to K(X_p)$are isomorphisms, it follows from Lemma (4.1.2) part (ii) that all the maps $\iota_{p,q}^{pr,qr}: K(X_q, X_p) \to K(X_{qr}, X_{pr})$ are (Banach space) isomorphisms. Hence

$$\varinjlim K(X_{qr}, X_{pr}) = \varphi_{p,q}\left(K(X_q, X_p)\right)$$

where $\varphi_{p,q}$ denotes the natural embedding of $K(X_q, X_p)$ into the inductive limit $\varinjlim K(X_{qr}, X_{pr})$. As the isomnorphismn from Theorem (4.1.7) sends

$j_X(x)j_X(y)^*$ to $\varphi_{p,q}(\Theta_{x,y}), x \in X_p, y \in X_q,$ we get [4]. In particular, we have$j_X(A) = (\mathcal{O}_X)_{[e,e]}$.

Conversely, if we assume that $\emptyset_p: A \to K(X_p)$is not onto for certain $p \in P$. Then

$$\varphi_{e,e}(K(A)) = \varphi_{p,p}\left(\emptyset_p(A)\right) \subseteq \varphi_{p,p}\left(K(X_p)\right) \varinjlim K(X_r, X_r),$$

and hence $j_X(A) \subseteq (\mathcal{O}_X)_{[e,e]}$.

Definition (4.2.10)[4]:

A partial action of a group G on a topological space Ω consists of a pair $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$, where D_g's are open subets of Ω and $\theta_g: D_{g^{-1}} \to D_g$ are homeomorphisms such that

$(PA1) D_e = \Omega \ and \ \theta_e = id,$
$(PA2) \theta_t(D_{t^{-1}} \cap D_s) = D_t \cap D_{ts},$
$(PA3)\ \theta_s(\theta_t(x)) = \theta_{st}(x), for\ x \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}.$

the partial action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologicaily free if for every open nonempty $U \subseteq \Omega$ and finite $F \subseteq G \setminus \{e\}$ there exists . $x \in U$ such that $x \in D_{t^{-1}}$implies $\theta_t(x) \neq x$ for all $t \in F$.

Proposition (4.2.11)[4]:

Suppose X is a regular product system of Hilbert bimodules and the underlying semigroup $\boldsymbol{P}$ is of Ore type. The formulas

$$D_{[q,p]} := \hat{X}_q\big(\langle \widehat{X_p, X_p}\rangle_p\big),$$

$$\hat{X}_{[p,q]}([\pi]) := \hat{X}_p \hat{X}_q^{-1}([\pi]), \qquad [\pi] \in D_{[q,p]}, p, q \in \mathrm{P},$$

yield a well defined family of open sets $\{D_g\}_{g\in G(P)}$and homeomorphisms $\hat{X}_g: D_g^{-1} \to D_g$ such that $(\{D_g\}_{g\in G(P)}, \{\hat{X}_g\}_{g\in G(P)})$ is a partial action of *G(P)*on $\hat{A}$. Moreover,

(i) $\{X_g\}_{g\in G(P)}$ is a semigroup dual to $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$, where we treat $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$asa pruduct system, and $\hat{X}_g$are viewed as multivalued maps on $\hat{A}$ with $\hat{X}_g(\hat{A}\backslash D_{g^{-1}}) = \{\emptyset\}$

(ii) *We* have the following implication:

$$\left(\{D_g\}_{g\in G(P)}, \{\hat{X}_g\}_{g\in G(P)}\right) is\ topologically\ free$$
$$\Rightarrow\ X is\ topologically\ aperiodic, \qquad (24)$$

and if P is both left and right Ore. (so for instance it is a group or a cancellative abelian semi group) then the above implication is actually an equivalence.

Proof: Tobegin with, let us note that for an ideal IinA and $p \in P$we have

$$\hat{X}_p(\hat{I}) = {}_p\langle \widehat{X_p I, X_p}\rangle, \hat{X}_p^{-1}(\hat{I}) = \big(\langle \widehat{X_p, I X_p}\rangle_p\big) \qquad (25)$$

Now, let $[\pi] \in \hat{A}$ and $r \in P$be arbitrary. Natural representatives of the classes $\hat{X}_p \hat{X}_q^{-1}([\pi])$ and $\hat{X}_{pr} \hat{X}_{qr}^{-1}([\pi])$ act by multiplication from the left on the spaces

$$X_p \otimes \tilde{X}_q \otimes_\pi H_\pi, \qquad X_{pr} \otimes \tilde{X}_{qr} \otimes_\pi H_\pi,$$

respectively. The obvious C*-correspondence isomorphisms

$$X_{pr} \otimes \tilde{X}_{qr} \cong X_p \otimes \left(X_r \otimes \tilde{X}_r\right) \otimes \tilde{X}_q \cong X_p \otimes A \otimes \tilde{X}_q \cong X_p \otimes \tilde{X}_q$$

yield a unitary equivalence between the aforementioned representations. Hence $\hat{X}_p \hat{X}_q^{-1}([\pi]) = \hat{X}_{pr} \hat{X}_{qr}^{-1}([\pi])$ and thus $\hat{X}_p \hat{X}_q^{-1}$ does mint depend on the choice of represen tative of $[p, q]$. It follows from (25) that the natural domain of $\hat{X}_p \hat{X}_q^{-1}$ is $\hat{X}_q\left(\widehat{\langle X_p, X_p \rangle_p}\right)$which coincides with the spectrum of ${}_q\langle X_q \langle X_p, X_p \rangle_p, X_q \rangle$. This shows that the formulas above indeed define homeomorphisms $\hat{X}_g : D_{g^{-1}} \to D_g, g \in G(P)$.

Condition (PAl) is obvious. To show (PA2), let $t = [t_1, t_2], s = [s_1, s_2]$ and $r \geq t_2, s_1$. Putting $q = t_1(t_2^{-1} r), p = s_2(s_1^{-1} r)$, we have $t = [t_1(t_2^{-1} r), t_2(t_2^{-1} r)] = [q, r]$ and $s = [s_1(s_1^{-1} r), s_2(s_1^{-1} r)] = [r, p]$. Hence

$$\hat{X}_t(D_s) = \hat{X}_{[q,r]}\left(D_{[\boldsymbol{r},\boldsymbol{p}]}\right) = \hat{X}_q \hat{X}_r^{-1} \left(\hat{X}_r\left(D_{[\boldsymbol{e},\boldsymbol{p}]}\right)\right) = \hat{X}_q\left(D_{[\boldsymbol{e},\boldsymbol{p}]} \cap D_{[\boldsymbol{e},\boldsymbol{r}]}\right).$$

On the other hand, since $st = [t_1, t_2] \circ [s_1, s_2] = [t_1(t_2^{-1} r), s_2(s_1^{-1} r)] = [q, p]$, we have

$$D_{ts} \cap D_t = D_{[q,p]} \cap D_{[q,r]} = \hat{X}_q\left(D_{[e,p]}\right) \cap \hat{X}_q(D_{[\boldsymbol{e},\boldsymbol{r}]}) = \hat{X}_q\left(D_{[e,p]} \cap D_{[\boldsymbol{e},\boldsymbol{r}]}\right).$$

This proves condition (PA2).

To show (PA3), let $t = [t_1, t_2]$, $s = [s_1, s_2]$, $r \geq t_1, s_2$ and $[\pi] \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}$. Then a natural representative of $\hat{X}_{st}([\pi]) = \hat{X}_{[s_1,s_2]\circ[t_1,t_2]}([\pi]) = \hat{X}_{s_1 s_2^{-1} r} \hat{X}^{-1}_{t_2 t_1^{-1} r}([\pi])$ acts by left multiplication on the space

$$X_{s_1 s_2^{-1} r} \otimes \tilde{X}_{t_2 t_1^{-1} r} \otimes_\pi H_\pi = X_{s_1} \otimes X_{s_2^{-1} r} \otimes \tilde{X}_{t_1^{-1} r} \otimes \tilde{X}_{t_2} \otimes_\pi H_\pi$$

Similarly, a representative of $\hat{X}_s\left(\hat{X}_t([\pi])\right) = \left(\hat{X}_{s_1} \circ \hat{X}_{s_2}^{-1} \circ \hat{X}_{s_2} \circ \hat{X}_{s_2}^{-1}\right)([\pi])$ acts by left multiplication on the space

$$X_{s_1} \otimes \hat{X}_{s_2} \otimes X_{t_1} \otimes \hat{X}_{t_2} \otimes_\pi H_\pi.$$

The latter can be considered an invariant subspace of the former with help of the following natural isomorphisms of C*-correspondences:

$$\begin{aligned} X_{s_1} \otimes \tilde{X}_{s_2} \otimes X_{t_1} \otimes \tilde{X}_{t_2} &\cong X_{s_1} \otimes \tilde{X}_{s_2} \otimes \left(X_r \otimes \tilde{X}_r\right) \otimes X_{t_1} \otimes \tilde{X}_{t_2} \\ &\cong X_{s_1} \langle X_{s_2}, X_{s_2} \rangle_{s_2} \otimes X_{s_2^{-1} r} \otimes \tilde{X}_{t_1^{-1} r} \otimes \langle X_{t_1}, X_{t_1} \rangle_{t_1} \tilde{X}_{t_2}. \end{aligned}$$

By the choice of $[\pi]$ and property (PA2), we see that $\hat{X}_s \hat{X}_t$ *([π])* is nonzero and thus equals $\hat{X}_{st}([\pi])$, as irreducible representations have no non-trivial subrepresentations.

Ad (i). This follows from our description of $\hat{X}_{[p,q]}$and the form of $\mathcal{O}_{[p,q]}$given in (23).

Ad (ii). Implication (24) is straightforward. For the converse, let us additionally assume that *P*is right cancellative and right reversible (then *P*is both left arid right Ore). Take any $g_1, \ldots, g_n \in G(P) \setminus \{[e, e]\}$. Using left reversibility of *P*we may represent these elements in the form $g_1 = [t, r_1], \ldots, g_n = [t, r_n]$, *where* $t, r_1, \ldots, r_n \in P$ *and* $t \neq r_i$ *for* $i = 1, \ldots, n$.

By right reversibility of P, one can inductively find elements $q_1, \ldots, q_n, p'_1, \ldots, p'_n \in P$such that

$$q_1 t = p'_1 r_1.$$
$$q_2 q_1 t = p'_2 p'_1 r_2,$$
$$\ldots q_n \cdots q_2 q_1 t = p'_n \cdots p'_2 p'_1 r_n.$$

Then defining

$$q := q_n \ldots q_1, \qquad s := qt \quad and \quad p_i \coloneqq q_n \ldots q_{i+1} p'_i \cdots p'_1 \quad for\ i = 1, \ldots, n,$$

we get $S = p_i r_i$ and $p_i \neq q$ for $i = 1, \ldots, n$. Hence $q^{-1}s = t$ and $p_i^{-1}s = r_i$ for every $i = 1, \ldots, n$. Thus

$$\hat{X}_{g_i} = \hat{X}_{[t, r_i]} = \hat{X}_t \hat{X}_{r_i}^{-1} = \hat{X}_{q^{-1}s} \hat{X}_{p_i^{-1}s}^{-1}$$

Since $\hat{X}_{q^{-1}s} \hat{X}_{p_i^{-1}s}^{-1} = \hat{X}_{[q^{-1}s, p_i^{-1}s]}$ does not depend on the choice of $s \geq q, p_i$, we see that the aperiodicity condition applied to q and $p_1, \ldots, p_n$yields the topological freeness condition for $q_1, \ldots, q_n$.

We do not know if the converse to implication (24) holds in general. Nevertheless, applying Proposition (4.2.1) and Theorems (4.2.5) and (4.2.7), we obtain the following.

Corollary (4.2.12)[4]:

Suppose $\{B_g\}_{g \in G}$is a saturated Fell bundle. Treating its fibers as imprimitivity Hilbert bimodules over B_e, the dual semigroup $\{\hat{B}_g\}_{g \in G}$ is a group of genime homeomorphisms of $\hat{B}_e$.

(i) The action $\{\hat{B}_g\}_{g\in G}$ is topologically free if and only if the product system $X = \coprod_{g\in G} B_g$ is topologically aperiodic. If this is the case, then every C*-norm on $\oplus \coprod_{g\in G} B_g$ is topologically graded.

(ii) If the action $\{\hat{B}_g\}_{g\in G}$ is topologically free and has no invariant non-trivial open subsets then the reduced cross-sectional C*-algebra $C_r^*(\{B_g\}_{g\in G})$ is simple.

Suppose α is an action of a semigroup P by endomorphisms of A such that each, $\alpha_s, s \in P$**,** extends to a strictly continuous endomorphism $\bar{\alpha}_s$ of the multiplier algebra M(A)**.** Let ω be a circle-valued multiplier on P**.** That is $\omega: P \times P \to \mathbb{T}$ is such that

$$\omega(p,q)\omega(pq,r) = \omega(p,qr)\omega(q,r), \qquad p,q,r \in P.$$

Then (A,α,P,ω) is called a twisted semigroup C*-dynamical system. A twisted crossed product $A \times_{\alpha,\omega} P$**,** is the universal C*-algebra generated by $\{ i_A(a)i_P(s) : a \in A, s \in P\}$,where (i_A, i_p) is a universal covariant representation of (A,P,α,ω). That is, $i_A : A \to A \times_{\alpha,\omega} P$is a homomorphism and $\{i_P(p) : p \in P\}$are isometrics in M($A \times_{\alpha,\omega} P$**)** such that

$$i_P(p)i_P(q) = \omega(p,q)i_P(pq) \quad and \quad i_P(p)i_A(a)i_P(p)^* = i_A\big(\alpha_p(a)\big),$$

for $p,q \in P$ and $a \in A$. A necessary condition for i_A to be injective is that all endomorphisms $\alpha_p, p \in P$, are injective. We apply Theorem (4.1.7) to show that when P is of Oretypethis condition is also sufficient. Additionally, we reveal a natural Fell bundle structure in $A \times_{\alpha,\omega} P$.

We associate to (A, α, P, ω) a product system $X = \coprod_{p\in P^{\circ p}} X_p$over the opposite semigroup $P^{\circ p}$ We equip the linear space $X_p := \mathrm{\alpha_p}(A)A$ with the following C*-correspondence operations

$$a \cdot x = \mathrm{\alpha_p}(a)x, \qquad x \cdot a = xa, \qquad \langle x, y\rangle_p = x^*y,$$

$a \in A, x, y \in X_p$. The multiplication in$X$ is defined by

$$x \cdot y = \overline{\omega(p,q)}\mathrm{\alpha_p}(x)y, for\ x \in X_p = \alpha_p(A)A\ and\ y \in X_q = \mathrm{\alpha_q}(A)A.$$

X is a product system and the left action of A on each of its fibers is by compacts. Accordingly, X is a regular product system if and only if all the endomorphisms $\alpha_p, p \in P$, are injective. Moreover, there is an isomnorpinsm

$$A \rtimes_{\alpha,\omega} P \cong \mathcal{O}_X$$

given by the mapping that sends an element $i_P(p)^* i_A(a) \in A \rtimes_{\alpha,\omega} P$ to the image of the element $a \in X_p = \alpha_p(A)A$ in $\mathcal{O}_X$. Using this isomnorphisrn and Theorem (4.1.7) one immediately gets the following.

Proposition (4.2.13)[4]:

Suppose that (A, α, P, ω) is atwisted semigroup C*-dynamical system, where P is of Ore type and all the endomorphisms $\alpha_p, p \in P$, are injective. Then the following hold.

(i) The algebra A embeds via i_A into the crossed product $A \rtimes_{\alpha,\omega} P$.

(ii) The crossed product $A \rtimes_{\alpha,\omega} P$ is naturally graded over the group of fractions *G(P)* by the subspaces of the form

$$B_g := \overline{span}\{i_P(p)^* i_A(a) i_P(q) : a \in \alpha_p(A)A\alpha_q(A), [p,q] = g\}, \qquad g \in G(P).$$

Morvover $A \rtimes_{\alpha,\omega} P$ can he identified with the cross-sectional C*algebra $C^*\left(\{B_g\}_{g\in G(P)}\right)$.

We keep the assumptions of Proposition (4.2.13). It is natural to define a reduced twisted crossed product $A \times^r_{\alpha,\omega} P$ to be the reduced crosse sectional algebra of the Fell bundle $\{B_g\}_{g\in G(P)}$. Let $\lambda : A \times_{\alpha,\omega} P \to A \times^r_{\alpha,\omega} P$ be the canonical epimorphism, and

$$I_\lambda \coloneqq ker\lambda.$$

We wish to generalize the main results to the case of twisted semnigroup actions. Let *X* be a product system associated to (A, α, P, ω)as above. One can see, cf., for instance, that a fiber $X_p, p \in P$, is a Hilbert bimodule if and onlyif the range of α_p is a hereditary sub algebra of A. If this is the case, then $\alpha_p(A)$ is a corner in A:

$$\alpha_p(A) = \alpha_p(A)A\alpha_p(A) = \bar{\alpha}_p(1)A\bar{\alpha}_p(1),$$

and the left inner product in $X_p$is defined by

$${}_p\langle x, y\rangle = \alpha_p^{-1}(xy^*), \qquad x, y \in X_p = \alpha_p(A)A.$$

The spectrum of $\alpha_p(A)$ can be identified with an open subset of $\hat{A}$. Then the homeomorphism $\hat{\alpha}_p: \widehat{\alpha_p(A)} \to \hat{A}$ dual to the isomorphism $\alpha_p: A \to$

$\alpha_p(A)$can be naturally treated as a partial homeomorphism of $\hat{A}$. The following Lemma (4.2.14) is based dealing with interactions on unital algebras.

Lemma (4.2.14)[4]:

If the monomorphism α_p, has a hereditary range, then the homeomorphisms $\hat{\alpha}_p: \widehat{\alpha_p(A)} \to \hat{A}$ and $\hat{X}_p: \langle \widehat{X_p, X_p} \rangle_p \to \hat{A}$ coincide.

Proof: With our identifications, we have

$$\widehat{\alpha_p(A)} = \left\{[\pi] \in \hat{A}: \pi\left(\alpha_p(A)\right) \neq 0\right\} = \langle \widehat{X_p, X_p} \rangle_p$$

Let $\pi : A \to ß(H)$ be an irreducible representation such that $\pi\left(\alpha_p(A)\right) \neq 0$. Then $\hat{\alpha}_p([\pi])$is the equivalence class of the representation $\pi \circ \alpha_p: A \to ß\left(\pi\left(\alpha_p(A)\right)H\right)$. Since $\pi\left(\alpha_p(A)\right)H = \pi(\alpha_p(A)A)H$ and

$$\left\|\sum_i a_i \otimes_\pi h_i\right\|^2 = \left\|\sum_{i,j} \langle h_i, \pi(a_i^* a_j) h_j \rangle_p\right\| = \left\|\sum_i \pi(a_i) h_j\right\|^2,$$

$a_i \in X_p = \alpha_p(A)A, h_i \in H, i = 1, \dots, n,$ we see that $a \otimes_\pi h \mapsto \pi(a)h$ yields a unitary operator $U : X_p \otimes_\pi H \to \pi\left(\alpha_p(A)\right)H$. Furthermore, for $a \in A, b \in \alpha_p(A)$ *and* $h \in H$ we have

$$\left[X_p - Ind(\pi)(a)U^*\right]\pi(b)h = X_p - Ind(\pi)(a)b \otimes_\pi h = \left(\alpha_p(a)b\right) \otimes_\pi h$$
$$= \left[U^*(\pi \circ \alpha_p)(a)\right]\pi(b)h.$$

Hence U intertwines $X_p - \mathrm{Ind}$ and $\pi \circ \alpha_p$. This proves that $\widehat{X}_p = \hat{\alpha}_p$.

Before stating our criterion of simplicity for semigroup crossed products, we need to define minimality for semigroup actions.

Definition (4.2.15)[4]:

Let α be an action of a semigroup P on a C*-algebra A. We say that α is minimal if for every ideal J in A such that $\alpha_p^{-1}(J) = J$ for all $p \in P$ we have $J = A \; or \; J = \{0\}$.

Let us note that if X is the product system associated to a twisted semigroup C*-dynamical system (A, α, P, ω) then minimality of α in the sense of Definition (4.2.15) is equivalent to minimality of X in the sense of Definition (4.2.6).

Proposition (4.3.16)[4]:

Suppose (A, α, P, ω) is a twisted semi group C*-dynamical system with P of Ore type. We assume that each endomorphism $\alpha_p, p \in P$, is injective and has hereditary range. As above, we regard $\hat{\alpha}_p, p \in P$, as partial homeomorphisms of $\hat{A}$. The formulas

$$D_{[q,p]} := \widehat{\alpha}_q\left(\widehat{\alpha_p(A)}\right), \qquad \widehat{\alpha}_{[p,q]}([\pi]) := \widehat{\alpha}_p\left(\widehat{\alpha}_q^{-1}([\pi])\right), \qquad [\pi] \in D_{[q,p]}, p, q \in P$$

yield a well defined partial action $\left(\{D_g\}_{g\in G(P)}, \{\hat{\alpha}_g\}_{g\in G(P)}\right)$ which coincides with the partial action induced by the Fell bundle $\{B_g\}_{g\in G(P)}$ described in Proposition (4.2.13) part (ii). Moreover,

$$\left(\{D_g\}_{g\in G(P)}, \{\hat{\alpha}_g\}_{g\in G(P)}\right) \textit{ is topologically free}$$
$$\Longrightarrow \{\hat{\alpha}_p\}_{p\in P} \textit{ is topologically aperiodic},$$

and

(i) if the semigroup $\{\hat{\alpha}_p\}_{p\in P}$ is topologically aperiodic, then for any ideal I in $A \times_{\alpha,\omega} P$ such that $I \cap A = \{0\}$ we have $I \subseteq I_\lambda$.

(ii) if the sernigroup $\{\hat{\alpha}_p\}_{p\in P}$ is topologically aperiodic and α is minimal, then the reduced twisted crossed product $A \rtimes^r_{\alpha,\omega} P$ is simple.

Proof: With the identification of $A \rtimes^r_{\alpha,\omega} P$ with $\mathcal{O}_X$, for each $g \in G(P)$ we have the correspondence between $(\mathcal{O}_X)_g$ and B_g. Thus Lemma (4.2.14) and Proposition (4.2.11) imply the initial part of the assertion. The remaining claims (i) and (ii) follow from Lemma (4.2.14) and Theorems (4.2.5) and (4.2.7).

Let $E = (E^0, E^1, s, r)$ be a topological graph as introduced. This means we assume that vertex set E^0 and edge set E^1 are locally compact Hausdoff spaces, source map $s : E^1 \to E^0$ is a local homeomorphism, and range map $r : E^1 \to E_0$ is a continuous map.

A C*-correspondence X_E of the topological graph E is defined in the following manner. The space X_E consists of functions $x \in C_0(E^1)$ for which

$$E^0 \ni v \longmapsto \sum_{\{e\in E^1 : s(e)=v\}} |x(e)|^2$$

belongs to $A := C_0(E^0)$. Then X_E is a C*-correspondence over **A** with the following structure.

$$(x \cdot a)(e) := x(e)a(s(e)) \quad for\ e \in E^1,$$
$$\langle x, y\rangle_A(v) := \sum_{\{e\in E^1 : s(e)=v\}} \overline{x(e)}y(e) \ \ for\ v \ \in \ E^0, and$$
$$(a \cdot x)(e) := \ a(r(e))x(e) \quad for\ e \in E^1.$$

C*-correspondence X_E generates a product system over ℕ. It follows that this product system (or simply, this C*-correspondence X_E).is regular if and only if

$\overline{r(E^1)}$
$= \ E^0\ and\ every\ v\ \ E^0$ has a neighborhood V such that $r^{-1}(V)$is compact. (26)

In particular, (26) holds whenever $r: E^1 \to \ E^0$ is a proper surjection. If both E^0 and E^1 are discrete then E is just a usual directed graph and then (26) says that every vertex in E^0 receives at least one and at most finitely mnany edges (in other words, graph E is row-finite and without sources). Accordingly , the C*-algebra of E is

$$C^*(E) := \ \mathcal{O}_{X_E}.$$

Let $e = \ (e_n, \ldots, e_1), r(e_i) = s(e_{i+1}), i = 1, .., n-1$, beapath in E. Then e is a cycle if $r(e_n) = s(e_1)$, and vertex $s(e_1)$ is called tile base point of e. A cycle e is said to be without entries if $r^{-1}(r(e_k)) = \ e_k\ for\ all\ k \ = \ 1, .., n$. Graph E is topologically free, if base points of all cycles without entries in Ehave empty interiors. It is known, that topological freeness of E is equivalent to the uniqueness property for *C*(E)***.**

In general, topological aperiodicity of X_E is stronger than topological freeness of E. However, when $E = (E^0, E^0, s, id)$ is a graph that comes from a mapping $s : E^0 \to E^0$, these two notions coincide.

Proposition (4.2.17)[4]:

Suppose $X_E$is a C*-correspondence of a topological graph E satisfying (26). The dual C* -correspondence acts on E^0 (identified with the spectrum of $A = C_0(E^0)$) via the formula

$$\hat{X}_E(v) = r\left(s^{-1}(v)\right). \tag{27}$$

In particular,

(i) X_E is topologically aperiodic if and only if the set of base points for periodic paths in E has empty interior;

(ii) If r is injective, topological aperiodicity of X_E is equivalent to topological freeness of E;

(iii) If E is discrete, then X_E is topologically aperiodic if and only if E has no cycles, and this in turn is equivalent to *C*(E)* being an AF-algebra.

Proof: We identify $\hat{A}$ with $E^0$by putting $v(a) := a(v)\ for\ v \in E^0, a \in A = C_0(E^0)$. We fix $v \in E^0$and an orthonormnal basis $\{x_e\}_{e\in s^{-1}(v)}$ in the Hilbert space $\mathbb{C}^{|s^{-1}(v)|}$. Let us consider the representation $\pi_v = A \to$ ß$(\mathbb{C}^{|s^{-1}(v)|})$ given by

$$\pi_v(a) = \sum_{e\in s^{-1}(v)} a\big(r(e)\big)\boldsymbol{x_e}, \qquad a \in A = C_0(E^0).$$

One readily checks that the mapping

$$X_E \otimes_{\boldsymbol{v}} \mathbb{C} \ni x \otimes_{\boldsymbol{v}} \lambda \longmapsto \sum_{e\in s^{-1}(v)} \lambda x(e)\boldsymbol{x_e} \in \mathbb{C}^{|s^{-1}(v)|}$$

gives rise to a unitary which establishes equivalence $X_E - \mathrm{Ind}(v) \cong \pi_v$. Furthermore, we have

$$\{\omega \in E^0 : \omega \leq \pi_v\} = \{\omega \in E^0 : \omega = r(e) for\ some\ e \in s^{-1}(v)\} = r(s^{-1}(v)).$$

This yields (27). Claim (i) follows from (27), part (iii) of Proposition (4.2.4) and the Baire category theorem. Claims (ii) and (iii) are now straightforward.

Corollary (4.2.18)[4]:

Keeping theassumptions of Proposition (4.2.17), let $V \subseteq E^0$ be closed. Then ideal $J = C_0(E^0 \setminus V)$ is X_E-invarant if and only if $\hat{X}_E(V) = V$.

Proof: It is known, that ideal $J = C_0(E^0 \setminus V)$ is X_E-invariant if and only if V satisfies the following two conditions

(i) $(\forall e \in E^1) s(e) \in V \Longrightarrow r(e) \in V, \quad and$

(ii) $v \in V \Longrightarrow (\exists_e \in r^{-1}(v)\, s(e) \in V.$

in view of (27), conditions (i) and (ii) are respectively equivalent to the inclusions $\hat{X}_E(V) \subseteq V\ and\ V \subseteq \hat{X}_E(V)$.

Example (4.2.19)[4]:

(Excel's crossed product for a proper local homeomorphism). Let $A = C_0(M)$ for a locally compact Hausdoff space M and let $\alpha: A \to A$be the operator of composition with a proper surjective local homeomorphism $\sigma : M \to M$. Then α is an extendible monomorphism possessing a natural left inverse transfer operator $L: A \to A$, defined by

$$L(a)(t) = \frac{1}{|\sigma^{-1}(t)|} \sum_{s \in \sigma^{-1}(t)} a(s),$$

Let $X_L$be the C*-correspondence with coefficients in A, constructed as follows. X_L is the completion of A with respect to the norm given by the inner-product below, and with the following structure:

$$x \cdot a = x\alpha(a), \quad \langle x, y \rangle = L(x^* y), \quad a \cdot x = ax,$$

where $a \in A, x, y \in X_L$. Clearly, the left action of A on X_L is injective. One can also show that it is by compacts. Hence X_L is a regular C*-correspondence. It is known that is naturally isomorplic to a C*-correspondence associated to the topological graph $E = (M, M, \sigma, id)$. Thus, by Proposition (4.2.17), the dual C*-correspondence to X_L acts on M, identified with the spectrum of $A = C_0(M)$, via the formula

$$\hat{X}_L(t) = \sigma^{-1}(t). \tag{28}$$

It is observed that

$$C_0(M) \rtimes_{\alpha,L} \mathbb{N} \coloneqq \mathcal{O}_{X_L}$$

is a natural candidate for Exel's crossed product when $A = C_0(M)$ is non-unital. When M is compact, $C(M) \times_{\alpha,L} \mathbb{N}$ coincides with the crossed product introduced and can be effectively described in terms of generators and relations.

Now, combining Proposition (4.2.17), we see that the following conditions are equivalent.

(i) X_L is topologically aperiodic;

(ii) the set of periodic points of σ has empty interior;

(iii) σ is topàlogically free in the sense of Exel and Vershik .

(iv) every non-trivial ideal in $C_0(M) \rtimes_{\alpha,L} \mathbb{N}$ intersects $C_0(M)$ non-trivially.

Consequently, in view of Corollary (4.2.18), the crossed product $C_0(M) \times_{\alpha,L} \mathbb{N}$ is simple if and only if in addition to the above equivalent conditions there is no nontrivial closed subset Y of M such that $\sigma^{-1}(Y) = Y$.

We introduce topological P-graphs which generalize both topological k-graphs and (discrete) P-graphs. Within the framework of a general approach to product systems proposed, the reasoning shows that a topological P-graph defined below is simply a product system over P with values in a groupoid of topological graphs. In the sequel P is a semigroup of Ore type. We treat elements of P as morphisms in a category with single object e.

Definition (4.2.20)[4]:

By a topological P-graph we mean a pair (Λ, d) consisting of:

(i) a small category Λ endowed with a second countable locally compact Hnusdorff topology under which the composition map is continuous andopen, the range map ***r*** is continuous and the source map ***s*** is a local homeomorphism;

(ii) a continuous functor $d: \Lambda \to P$, called degree map, satisfying the factorization property: if $d(\lambda) = pq$ then there exist unique μ, v with $d(\mu) = p, d(v) = q \text{ and } \lambda = \mu v$.

Elements (morphisms) of Λ are called paths. $\Lambda^p := d^{-1}(p)$ stands for the set of paths of degree $p \in P$. Paths of degree e are called vertices.

We associate to a topological P-graph (Λ, d) a product system inthe same manner as it is done for topological k-rank graphs. That is, for each $p \in P$ we let $X_p = X_{E_p}$ be the standard C*-correspondence associated to the topological graph

$$E_p = \left(\Lambda^e, \Lambda^p, s|\Lambda_p, r|_{\Lambda_p}\right)$$

so that $A := C_0(\Lambda^e)$ and X_p, is the completion of the pre-Hilbert A-module $C_e(\Lambda^p)$ with the structure

$$\langle f, g\rangle_p(v) = \sum_{\eta \in \Lambda^p(v)} \overline{f(\eta)}\, g(\eta) \;\; and \;\; (a \cdot f \cdot b)(\lambda) = a(r(\lambda)) f(\lambda) b(s(\lambda)).$$

The proof works in our more general setting and shows that the formula

$$(fg)(\lambda) := f(\lambda(e,p))g(\lambda(p,pq))$$

defines a product $X_p \times X_q \ni (f,g) \to fg \in X_{pq}$ that makes $X = \coprod_{p\in P} X_p$, into a product system. In view of (26), we see that the product system X is regular if and only if for every $p \in P$ we have

$\overline{r(\Lambda^p)} = \Lambda^0$and

every $v \in E^0$ has a neighborhood V such that $r^{-1}(V) \cap \Lambda^p$ is compact in Λ^p.

If the above condition holds, we say that the topological P-graph (Λ, d) is regular. It follows that if (Λ, d) is a regular topological k-rank graph (that is, if $P = \mathbb{N}^k$), then the Cuntz-Krieger algebra of (Λ, d) defined coincides with $\mathcal{O}_X$ Hence it is natural to coin the following definitions.

Definition (4.2.21)[4].

Suppose (Λ, d) is a regular topological P-graph, where P is a semi group of Ore type. We define a C*-algebra $C^*(\Lambda, d)$ and a reduced C*algebra $C_r^*(\Lambda, d)$ of (Λ, d) to be respectively the Cuntz- Pimsner algebra $\mathcal{O}_X$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}_X^r$ where $\boldsymbol{X}$ is the regular product system defined above.

Proposition (4.2.22)[4]:

Suppose (Λ, d) is a regular topological P-graph. The C*-algebras $C^*(\Lambda, d)$ and $C_r^*(\Lambda, d)$ are non-degenerate in the sense that they are generated by the images of injective Guntz-Pimsner representations of $X = \coprod_{p\in P} X_p$,. Moreover,

(i) X is topologically a periodic if and only if for every nonempty open set $U \subseteq \Lambda^e$, each finite set $F \subseteq P$ and an element $q \in P$ with $q \nsim_R p$ for all $p \in F$, there is an enumeration $\{p_1, \ldots, p_n\}$of elements of F and there are elements $s_1, \ldots, s_n \in P$such that $q \leq s_1 \leq \cdots \leq s_n, p_i \leq s_i, for\ i = 1, \ldots, n$, and the union

$$\bigcup_{i=1}^{n} \{v \in \Lambda^e : \mu \in \Lambda^{p_i^{-1}s_i}, v \in \Lambda^{q^{-1}s_i}\ s(\mu) = s(v)\ and\ r(\mu) = r(v) = v\} \quad (29)$$

does not contain ***U***.

(ii) ***X***is minimal if and only if there is no nontrivial closed set $V \subseteq \Lambda^e$ such that

$$r\big(\Lambda^P \cap s^{-1}(V)\big) = V \quad for\ all\ p \in P. \quad (30)$$

In particular, if the equivalent conditions in (i) hold, then any non-zero ideal in $C_r^*(\Lambda, d)$ has non-zero intersection with $C_0(\Lambda^e)$. If the conditions described in (i,) and (ii) hold, then $C_r^*(\Lambda, d)$ is simple.

Proof: The initial claim of tins proposition follows from Theorem (4.1.7) above. To see that the equivalence in part (i) holds, it suffices to apply formula (27) to the C*-correspondences $X_p = X_{E_p}, p \in P$**.** Similarly, using (27) and Corollary (4.2.18), we see that X-invariant ideals in$C_0(\Lambda^e)$ are in one-to-one correspondence with closed sets V satisfying (30). This proves part (ii). The final claim of the proposition now follows from Theorems (4.2.5) and (4.2.7) above.

Cuntz- introduced $\mathcal{Q}_{\mathbb{N}}$, the universal C*-algebra generated by a unitary u and isometrics $s_n, n \in \mathbb{N}^{\times}$, subject to the relations

$(Q1) s_m s_n = s_{mn}$,
$(Q2) s_m u = u^m s_m$, and
$(Q3) \sum_{k=0}^{m-1} u^k s_m s_m^* u^{-k} = 1$,

for all $m, n \in \mathbb{N}^{\times}$ Cuntz proved that $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite. Now we deduce the simplicity of $\mathcal{Q}_{\mathbb{N}}$ from our genergl result - Theorem (4.2.7) above.

It was shown that $\mathcal{Q}_{\mathbb{N}}$ may be viewed as the Cuntz-Pimnsner algebra of a certain product system. We recall an explicit description of that product system given.

The product system X is over the seimgroup $\mathbb{N}^{\times}$and its coefficient algebra is $A = C(S^1)$. We denote by $\boldsymbol{Z}$ the standard unitary generator of A. Each fiber $X_m, m \in \mathbb{N}^{\times}$, is a C*-correspondence over A associated to the classical covering map $S^1 \ni z \to z^m \in S^1$ as constructed in Example (4.2.19). Each X_m as left A-module is free with rank 1, and we denote the basis element by 1_m Hence, each element of X_m may be uniquely written as$\xi 1_m$ with $\xi \in A$. We have

$$(\xi 1_m) \cdot a = \xi \alpha_m(a) 1_m,$$
$$\langle \xi 1_m \eta 1_m \rangle_m = L_m(\xi^* \eta),$$

$$a \cdot \xi 1_m = (a\xi)1_m,$$

For $\xi, a \in A$ then

$$X := \coprod_{m \in \mathbb{N}^\times} X_m$$

becomes a product system with multiplication $X_m \times X_r \to X_{mr}$ given by

$$(\xi 1_m)(\eta 1_r) := (\xi \alpha_m(\eta))1_{mr}$$

for $m, r \in \mathbb{N}^\times$. We have

$$\mathcal{O}_X \cong \mathcal{Q}_{\mathbb{N}}$$

Now, let $E_{i,j} i, j = 0,1, \ldots, m-1$, be a system of matrix units in $M_m(\mathbb{C})$. There is an isomorphism

$$C(S^1) \otimes M_m(\mathbb{C}) \cong K(X_m)$$

such that

$$f \otimes E_{i,j} \leftrightarrow \Theta_{Z^i \alpha_m(f) 1_m, Z^j \ 1_m}.$$

Thus $\widehat{K(X_m)}$ may be identified with the circle S^1. With these identifications, we have

$$\emptyset_m(Z) = Z \otimes E_{0,m-1} + \sum_{j=0}^{m-2} 1 \otimes E_{j+1,j},$$

and hence the multivalued map $\widehat{\emptyset_m}: S^1 \to S^1$ is such that

$$\widehat{\emptyset_m}(\boldsymbol{z}) = \{\omega \in S^1 | \omega^m = z\}.$$

Furthermore, $[X_m - \mathrm{Ind}\,]$ is identified with the identity map on S^1, and consequently the multivalued map $\widehat{X_m} = \widehat{\emptyset_m} \circ [X_m - \mathrm{Ind}]: S^1 \to S^1$is

$$\widehat{X_m}(\boldsymbol{z}) = \{\omega \in S^1 | \omega^m = z\}.$$

For in$m \neq n$the set $\{z \in S^1 |\, z \in \widehat{X_m}(\widehat{X_n}^{-1}(z))\}$ is finite, while every nonempty open subset of S^1 is infinite. It follows that the product system X is topologically aperiodic.

Now, we see that A does not contain any non-trivial invariant ideals. Indeed, suppose J is an X-invariant ideal in A. Then $L_m(J) \subseteq J$ for all $m \in \mathbb{N}^{\times}$. There exists an open subset U of S^1 and a function $f \in J$ such that $f \geq 0$ and $f(t) \neq 0$ for all $t \in U$. If m is sufficiently large then for each $z \in S^1$there is a $w \in U$ such that $w^m = z$. Then $L_m(f)$ is strictly positive onS^1 and hence invertible. Since $L_m(f) \in J$, we conclude that $J = A$.

List of Symbols

Symbol		Page No
$\oplus$	Orthogonal Sum	1
loc	local	1
Leb	lebesgue	1
Dif.	diffeomorphism	3
dim	dimension	5
min	minimum	5
max	maximum	10
dom	domain	12
deg	degree	50
ker	kernel	58
im	imaginary	61
ℓ^2	Hilbert space for the sequences	65
UCT	Universal coefficient theorem	76
$\otimes$	Tensor product	76
$\ominus$	Direct difference	96
isom	isomorphism	98
Irr	irreducible	117
L^2	Hilbert space	120

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