Sudan University of Science & Technology

College of graduate studies

A project Submitted in fulfillment for the degree of

M.Sc in Mathematics

Meteorological Modeling and Some applications

نمذجة الأرصاد الجوي وبعض التطبيقات

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Dedication

To the spirit of my father,

To my Mother,

Brothers,

Sisters,

and to who will be my wife in the future

Acknawledgemants

First,Iwould like to thanks Supervised *Dr :Emad Eldeen Abdallh Abderahim* who did not spare us for his information at all and thanks go to all my teachers,Iam deeply indebted to all staff of typists those help me to finish this study.

Abstract

In this research we discuss the thermodynamics of the Dry atmosphere, and the numerical weather prediction issued from the American National Meteorological Center . We also discuss the difference between the shallow water equation s and Semi- Gestrophic equations . We discuss the balanced flow, trajectorics and stream lines .

We used the Kine matic method and Adiabatic method to inferring the vertical motion field . Also we discuss the relationship between the circulation and vorticity by using primary measures of rotation in a fluid with some applications .

الخلاصة

فى هذا البحث ناقشنا الديناميكا الحرارية فى الجو الجاف والتنبؤ العددى للطقس التى نشرت من المركز الوطنى للأرصاد الجوي الأمريكي . وايضا ناقشنا الفرق بين معادلات المياه السطحية ومعادلات شبه الجستروفية كما تناولنا التدفق المتوازن ومسارات خطوط العاصفة .

وكذلك استخدمنا الطريقة الحركية واسلوب ثابت الحرارة لأستنتاج مجال الحركة العمودية كما ناقشنا العلاقة بين الدورانية والدوامية باستخدام المقاييس الاولية للدورانية مع بعض التطبيقات .

The Contents

INTRODICTION

In this research we are dealing with the Governing Equations of the Dynamic Meteorology , and it is organized as follows :-

Firstly we being with a brief introduction to the nature of the forces that influence atmospheric Motions and nature of these apparent forces ,we Also discuss the structure of the static At mosphere and introduce some fundamental physical principles:

- Conservation of Mass for afliud(cotinuty equation),we develops that using two alternative Methods, the first Method is based on an Eulerian control volume , where as the second is based on a Lagrangian control volume.
- Conservation of Energy as applied to moving fluid element.

Also we study the thermodynamics of the Dry atmosphere.

In chapter 2 we study the numerical weather predicton issued from the national Meteorological center, also we indiacated to the numerical weather prediction principally with prediction of large cyclonic storms and anticyclones, Also we study the application of the fundamental lows of physics to the systems and the phanomena hosted by the subtle thin layer of gases and vapors surrounding the planct earth ,we also considered as an application of the fundamental lows because it is assumed that the set of physical concepts so for defined by scince are sufficient,and necessary , to describe all systems belonging to the atmosphere of our planct , also we illustrate simpler Models of weather such as the shallow weater equations and semi-gestrophic equations , and then we study the difference between them.

In chapter 3 we illustrate the basic equations(Momentum , Continuity , Energ) in isobaric coordinates , also we discuss the balanced flow, trajectories and streamlines , and we discuss the thermal wind . Also we used two Methods for inferring the vertical Motion field:

- Kine matic Method , based on the equation of continuity.
- Adiabatic Method , based on the thermodynamic equation.

Also we study three instructive applications of the approach,ranging from are-derivation of the well-known semi-geostraphic theory , via one of the recently derived multi-scale Models for the trapics,to numerical Methods that are well-balan.

Finally we defined the primary measures of rotation in afluid (circulation , vorticity) ,and also discuss the relationship between them . Also we study the vorticity in natural coordinates, potential vorticity equations and scale analysis of the vorticity equation , also we study the vorticity in barotropic fluids,Also we discuss issues Modeling Mathematical for some basic geophysical fluid dynamics(G F D) Models(primitive equations(PEs) of the atmosphere , Ocean , Coupled atmosphere – Ocean).

Chapter(1)
Fundamental Forces and Basic Conservation Laws

Section(1.1):The Fundamental Forces and Static Atmosphere

In this section we deal with the fundamental force and static Atmosphere , and we start with fundamental forces :

(i) Pressure Gradient Force :

We consider an infinitesimal volume element of air, $\delta V = \delta x \delta y \delta z$, centered at the point x_0, y_0, z_0 , as illustrated in Fig.(1.1) Due to random molecular motions, momentum is continually imparted to the walls of the volume element by the surrounding air. This momentum transfer per unit time per unit area is just the pressure exerted on the walls of the volume element by the surrounding air. If the pressure at the center of the volume element is designated by p_0, then the pressure on the wall labeled A in figure(1.1) can be expressed in a Taylor series expansion as

$$p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \text{higer order terms}$$

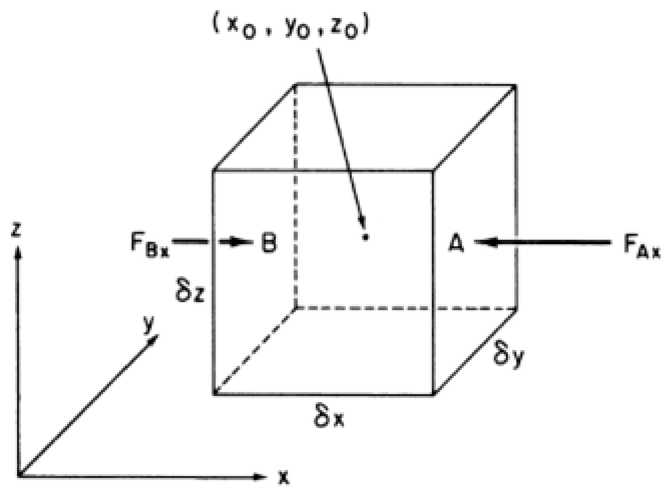


Fig.(1.1) : the x component of the pressure gradient force acting on aflid element

N eglecting the higher order terms in this expansion, the pressure force acting on the volume element at wall A is

$$F_{Ax} = -\left(p_0 + \frac{\partial p \delta x}{2\partial x}\right)\delta y \delta z$$

where $\delta y\ \delta z$ is the area of wall A. Similarly, the pressure force acting on the volume element at wall B is just

$$F_{Bx} = +\left(p_0 - \frac{\partial p \delta x}{2\partial x}\right)\partial y \partial z$$

Therefore, the net x component of this force acting on the volume is

$$F_x = F_{Ax} + F_{Bx} = -\frac{\partial p}{\partial x}\ \delta x\ \delta y\ \delta z$$

Because the net force is proportional to the derivative of pressure in the direction of the force, it is referred to as the pressure gradient force. The mass m of the differential volume element is simply the density ρ times the volume: $m = \rho\ \delta x\ \delta y\ \delta z$ Thus, the x component of the pressure gradient force per unit mass is

$$\frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly, it can easily be shown that the y and z components of the pressure gradient force per unit mass are

$$\frac{F_y}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{and} \quad \frac{F_z}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

so that the total pressure gradient force per unit mass is

$$\frac{\mathbf{F}}{m} = -\frac{1}{\rho}\nabla p \tag{1.1}$$

It is important to note that this force is proportional to the gradient of the pressure field, not to the pressure itself.

(ii) Gravitational Force:

Newton's law of universal gravitation states that any two elements of mass in the universe attract each other with a force proportional to their masses and inversely proportional to the square of the distance separating them. Thus, if two mass elements M and m are separated by a distance $r \equiv |\mathbf{r}|$ (with the vector **r** directed toward m as shown in Fig.(1.2), then the force exerted by mass M on mass m dueto gravitation is

$$\mathrm{F_g} = -\frac{GMm}{r^2}\left(\frac{\mathbf{r}}{r}\right) \quad (1.2)$$

Where G is a universal constant called the gravitational constant.
The law of gravitation as expressed in Equation (1.2) actually applies only to hypothetical "point" masses since for objects of finite extent r will vary from one part of the object to another. However, for finite bodies, Equation (1.2) may still be applied if $|\mathbf{r}|$ is interpreted as the distance between the centers of mass of the bodies. Thus, if the earth is designated as mass M and m is a mass element of the atmosphere, then the force per unit mass exerted on the atmosphere by the gravitational attraction of the earth is

$$\frac{\mathrm{F_g}}{m} \equiv \mathrm{g}^* = -\frac{GMm}{r^2}\left(\frac{\mathbf{r}}{r}\right) \quad (1.3)$$

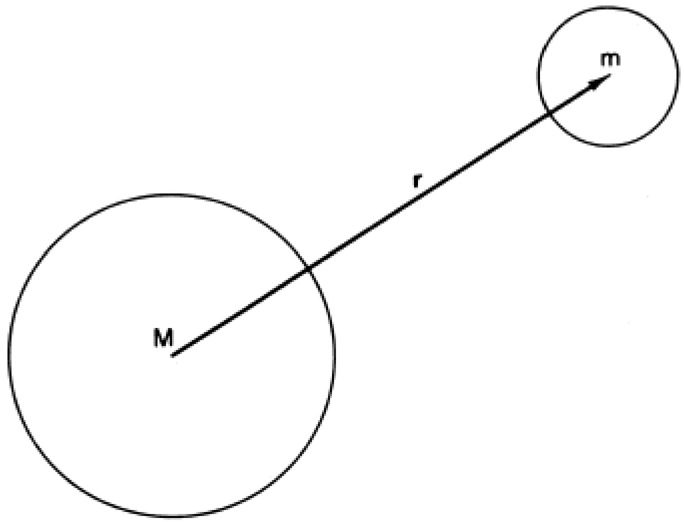


Fig.(1.2) : two spherical masses whose centers are separated by adistance r

In dynamic meteorology it is customary to use the height above mean sea level as a vertical coordinate. If the mean radius of the earth is designated by a and the distance above mean sea level is designated by z, then neglecting the small departure of the shape of the earth from sphericity, $r = a + z$. Therefore, Equation (1.3) can be rewritten as

$$\mathbf{g}^* = \frac{\mathbf{g_0^*}}{(1 + z/a)^2} \tag{1.4}$$

where $g_0^* = -(GM/a^2)(\mathbf{r}/r)$ is the gravitational force at mean sea level. For meteorological applications , $z \ll a$ so that with negligible error we can let $\mathbf{g}^* = \mathbf{g_0^*}$ and simply treat the gravitational force as a constant.

(iii) Viscous Force :

Any real fluid is subject to internal friction (viscosity), which causes it to resist the tendency to flow. Although a complete discussion of the resulting viscous force would be rather complicated, the basic physical concept can be illustrated by a simple experiment. A layer of incompressible fluid is confined between two horizontal plates separated by a distance l as shown in Fig.(1.3). The lower plate is fixed and the upper plate is placed into motion in the x direction at a speed u_0. Viscosity forces the fluid particles in the layer in contact with the plate to move at the velocity of the plate. Thus, at $z = l$ the fluid moves at speed $u(l) = u_0$, and at

$z = 0$ the fluid is motionless. The force tangential to the upper plate required to keep it in uniform motion turns out to be proportional to the area of the plate, the velocity, and the inverse of the distance separating the plates. Thus, we may write $F = \mu A u_0 / l$ where μ is a constant of proportionality, the dynamic viscosity coefficient.

This force must just equal the force exerted by the upper plate on the fluid immediately below it. For a state of uniform motion, every horizontal layer of fluid of depth ∂z must exert the same force F on the fluid below. This may be

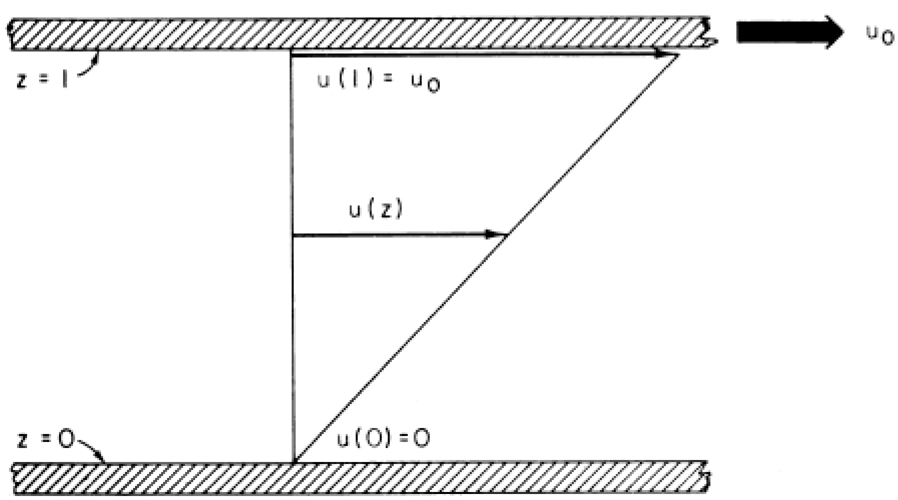


Fig.(1.3) : one-dimensional steady-state viscous shear flow

expressed in the form $F = \mu A \delta u / \delta z$ where $\delta u = u_0 / l$ is the velocity shear across the layer δz. The viscous force per unit area, or shearing stress, can then be defined as

$$\tau_{zx} = \lim_{\delta z \to 0} \mu \frac{\delta u}{\delta z} = \mu \frac{\partial u}{\partial z}$$

where subscripts indicate that τ_{zx} is the component of the shearing stress in the x direction due to vertical shear of the x velocity component.

From the molecular viewpoint, this shearing stress results from a net downward transport of momentum by the random motion of the molecules. Because the mean x momentum increases with height, molecules passing downward through ahorizontal plane at any instant carry more momentum than those passing upward through the plane. Thus, there is a net downward transport of x momentum. This downward momentum transport per unit time per unit area is simply the shearing stress.

In a similar fashion, random molecular motions will transport heat down amean temperature gradient and trace constituents down mean mixing ratio gradients. In these cases the transport is referred to as molecular diffusion. Molecular diffusion always acts to reduce irregularities in the field being diffused.

In the simple two-dimensional steady-state motion example given above there is no net viscous force acting on the elements of fluid, as the shearing stress acting across the top boundary of each fluid element is just equal and opposite to that acting across the lower boundary. For the more general case of nonsteady two-dimensional shear flow in an incompressible fluid, we may calculate the viscous force by again considering a differential volume element of fluid centered at(x, y, z) with sides $\delta x\ \delta y\ \delta z$ as shown in Fig.(1.4). If the shearing stress in the $x$direction acting through the center of the element is designated τ_{zx}, then the stress acting across the upper boundary on the fluid below may be written approximately as

$$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\partial z}{2}$$

while the stress acting across the lower boundary on the fluid *above* is

$$-\left[\tau_{zx} - \frac{\partial \tau_{zx}}{\partial x}\frac{\partial z}{2}\right]$$

(This is just equal and opposite to the stress acting across the lower boundary on the fluid below.) The net viscous force on the volume element acting in the x direction is then given by the sum of the stresses acting across the upper boundary on the fluid below and across the lower boundary on the fluid above:

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x \delta y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial x}\frac{\delta z}{2}\right)\delta x \delta y$$

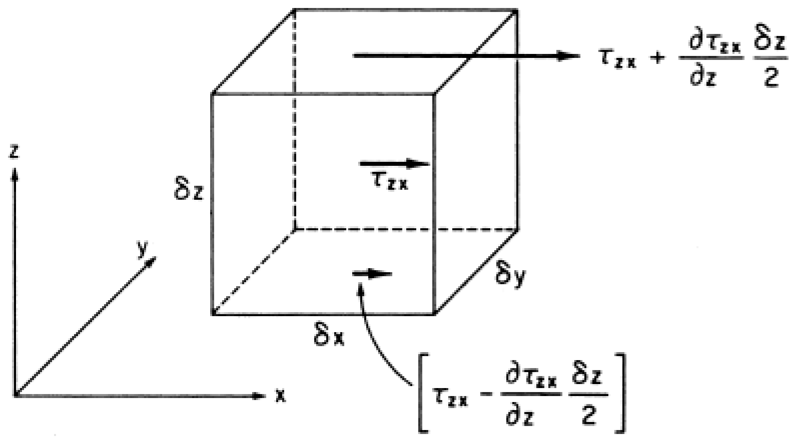


Fig.(1.4) : the x component of the vertical shearing stress on a fluid element .

Dividing this expression by the mass $\rho\delta\, x\delta y\delta z$, we find that the viscous force per unit mass due to vertical shear of the component of motion in the x direction is

$$\frac{1}{\rho}\frac{\partial \tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right)$$

For constant μ, the right-hand side just given above may besimplified to $v\partial^2 u/\partial z^2$, where $v = \frac{\mu}{\rho}$ is the kinematic viscosity coefficient. For standard atmosphere conditions at sea level, $v = 1.46 \times 10^{-5}\mathrm{m^2 s^{-1}}$. Derivations analogous to that shown in Fig.(1.4) can be carried out for viscous stresses acting in other directions. The resulting frictional force components per unit mass in the three Cartesian coordinate directions are

$$F_{rx} = v\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$F_{ry} = v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \qquad (1.5)$$

$$F_{rz} = v\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

For the atmosphere below 100 km, v is so small that molecular viscosity is negligible except in a thin layer within a few centimeters of the earth's surface where the vertical shear is very large. Away from this surface molecular boundary layer, momentum is transferred primarily by turbulent eddy motions.

Now we study the non inertial reference frames and "Apparent " forces. In formulating the laws of atmospheric dynamics it is natural to use a geocentric reference frame, that is, a frame of reference at rest with respect to the rotating earth. Newton's first law of motion states that a mass in uniform motion relative to a coordinate system fixed in space will remain in uniform motion in the absence of any forces. Such motion is referred to as inertial motion; and the fixed reference frame is an inertial, or absolute, frame of reference. It is clear, however, that an object at rest or in uniform motion with respect to the rotating earth is not at rest or in uniform motion relative to a coordinate system fixed in space. Therefore, motion that appears to be inertial motion to an observer in a geocentric reference frame is really accelerated motion. Hence, a geocentric reference frame is a noninertial reference frame. Newton's laws of motion can only be applied in such a frame if the acceleration of the coordinates is taken into account. The most satisfactory way of including the effects of coordinate acceleration is to introduce "apparent" forces in the statement of Newton's second law. These apparent forces are the inertial reaction terms that arise because of the coordinate acceleration. For a coordinate system in uniform rotation, two such apparent forces are required: the centrifugal force and the Coriolis force. In the following we present centripetal acceleration and centrifugal force.

A ball of mass m is attached to a string and whirled through a circle of radius r at a constant angular velocity ω. From the point of view of an observer in inertial space the speed of the ball is constant, but its direction of travel is continuously changing so that its velocity is not constant. To compute the acceleration we consider the change in velocity $\delta \mathbf{V}$ that occurs for a time increment δt during which the ball rotates through an angle $\delta\theta$ as shown in Fig.(1.5). Because $\delta\theta$ is also the angle between the vectors $\mathbf{V}$ and $\mathbf{V} + \delta\mathbf{V}$, the magnitude of $\delta\mathbf{V}$ is just $|\delta\mathbf{V}| = |\mathbf{V}|\ \delta\theta$. If we divide by δt and note that in the limit $\delta t \to 0,\ \delta\mathbf{V}$ is directed toward the axis of rotation, we obtain

$$\frac{D\mathbf{V}}{Dt} = |\mathbf{V}|\frac{D\theta}{Dt}\left(\frac{-\mathbf{r}}{r}\right)$$

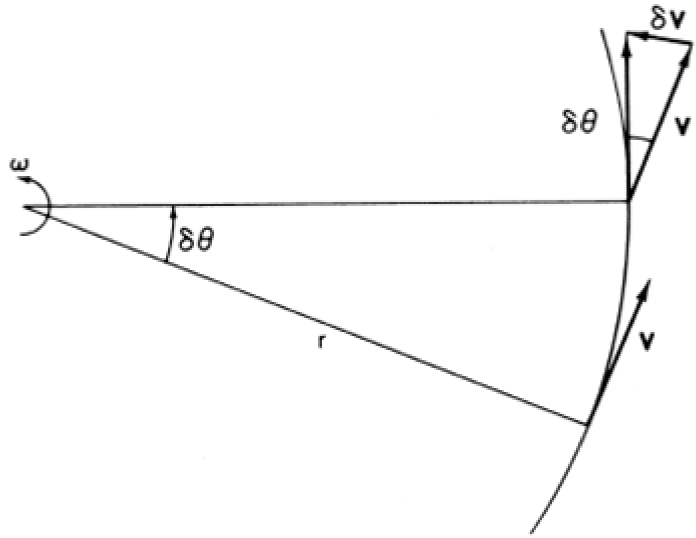


Fig.(1.5) : centripetal acceleration is given by the rate of change of the direction
Of the velocity vector, which is directed toward the axis of rotation,
As illustrated here by δV.

However, $|\mathbf{V}| = \omega r$ and $D\theta / Dt = \omega$, so that

$$\frac{D\mathbf{V}}{Dt} = -\omega^2 r \qquad (1.6)$$

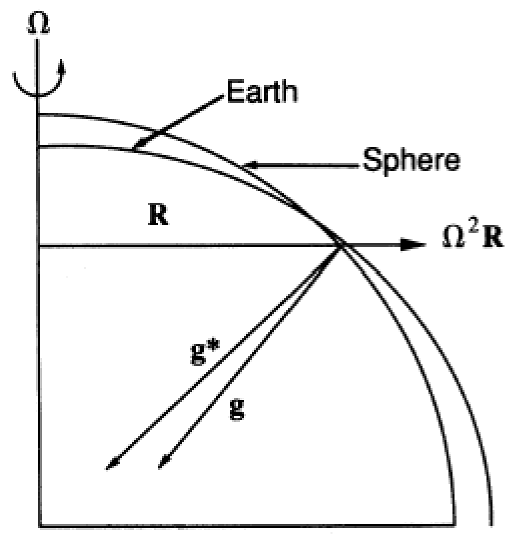
Therefore, viewed from fixed coordinates the motion is one of uniform acceleration directed toward the axis of rotation and equal to the square of the angular velocity times the distance from the axis of rotation. This acceleration is called centripetal acceleration. It is caused by the force of the string pulling the ball.

Now suppose that we observe the motion in a coordinate system rotating with the ball. In this rotating system the ball is stationary, but there is still a force acting on the ball, namely the pull of the string. Therefore, in order to apply Newton's second law to describe the motion relative to this rotating coordinate system, we must include an additional apparent force, the centrifugal force, which just balances the force of the string on the ball. Thus, the centrifugal force is equivalent to the inertial reaction of the ball on the string and just equal and opposite to the centripetal acceleration.

To summarize, observed from a fixed system the rotating ball undergoes a uniform centripetal acceleration in response to the force exerted by the string. Observed from a system rotating along with it, the ball is stationary and the force exerted by the string is balanced by a centrifugal force.

Now we discuss the Gravity force. Anobject at rest on the surface of the earth is not at rest or in uniform motion relative to an inertial reference frame except at the poles. Rather, an object of unit mass at rest on the surface of the earth is subject to a centripetal acceleration directed toward the axis of rotation of the earth given by $-\Omega\mathbf{R}$, where **R** is the position vector from the axis of rotation to the object and $\Omega = 7.292 \times 10^{-5}$ rad s^{-1} is the angular speed of rotation of the earth. Since except at the equator and poles the centripetal acceleration has a component directed poleward along the horizontal surface of the earth (i.e., along a surface of constant geopotential), there must be a net horizontal force directed poleward along the horizontal to sustain the horizontal component of the centripetal acceleration. This force arises because the rotating earth is not a sphere, but has assumed the shape of an oblate spheroid in which thereis a poleward component of gravitation along a constant geopotential surface just sufficient to account for the poleward component of the centripetal acceleration at each latitude for an object at rest on the surface of the earth. In other words, from the point of view of an observer in an inertial reference frame, geopotential

Fig. (1.6) : Relationship between the true gravitation vector **g*** and gravity **g**. For an idealized homogeneous spherical earth, **g*** would be directed toward the center of the earth. In reality, **g*** does not point exactly to the center except at the equator and the poles. Gravity, **g**, is the vector sum of **g*** and the centrifugal force and is perpendicular to the level surface of the earth, which approximates an oblate spheroid.



surfaces slope upward toward the equator (see Fig.(1.6)). As a consequence, the equatorial radius of the earth is about 21 km larger than the polar radius.

Viewed from a frame of reference rotating with the earth, however,a geopotential surface is everywhere normal to the sum of the true force of gravity, $\mathbf{g}^*$, and the centrifugal force $\Omega^2\mathbf{R}$ (which is just the reaction force of the centripetal acceleration). A geopotential surface is thus experienced as a level surface by an object at rest on the rotating earth. Except at the poles, the weight of an object of mass *m* at rest on such a surface, which is just the reaction force of the earth on the

object, will be slightly less than the gravitational force $m\mathbf{g}^*$ because, as illustrated in Fig. (1.6), the centrifugal force partly balances the gravitational force.
It is, therefore, convenient to combine the effects of the gravitational force and centrifugal force by defining *gravity* **g** such that

$$\mathbf{g} \equiv -g\mathbf{k} \equiv \mathbf{g}^* + \Omega^2\mathbf{R} \tag{1.7}$$

where **k** designates a unit vector parallel to the local vertical. Gravity, g, sometimes referred to as "apparent gravity," will here be taken as a constant ($g = 9.81ms$). Except at the poles and the equator, **g** is not directed toward the center of the earth,but is perpendicular to a geopotential surface as indicated by Fig. (1.6). True gravity $\mathbf{g}^*$, however, is not perpendicular to a geopotential surface, but has a horizontal component just large enough to balance the horizontal component of $\Omega^2 R$.

Gravity can be represented in terms of the gradient of a potential function ∅, which is just the geopotential referred to above:

$$\nabla\Phi = -\mathbf{g}$$

However, because $\mathbf{g} = -g\mathbf{k}$ where $g \equiv |g|$, it is clear that $\Phi=\Phi(z)$ and $d\Phi/dz = g$. Thus horizontal surfaces on the earth are surfaces of constant geopotential. If the value of geopotential is set to zero at mean sea level, the geopotential $\Phi(z)$ at height z is just the work required to raise a unit mass to height z from mean sea level:

$$\Phi = \int_0^z g\, dz \tag{1.8}$$

Despite the fact that the surface of the earth bulges toward the equator, an object at rest on the surface of the rotating earth does not slide "downhill" toward the pole because, as indicated above, the poleward component of gravitation is balanced by the equatorward component of the centrifugal force. However, if the object is put into motion relative to the earth, this balance will be disrupted. Consider a frictionless object located initially at the North pole. Such an object has zero angular momentum about the axis of the earth. If it is displaced away from the pole in the absence of a zonal torque, it will not acquire rotation and hence will feel a restoring force due to the horizontal component of true gravity, which, as indicated above is equal and opposite to the horizontal component of the centrifugal force for an object at rest on the surface of the earth. Letting R be the distance from the pole, the horizontal restoring force for a small displacement is thus $-\Omega^2 R$, and the object's acceleration viewed in the inertial coordinate system satisfies the equation for a simple harmonic oscillator:

$$\frac{d^2R}{dt^2} + \Omega^2 R = 0 \tag{1.9}$$

The object will undergo an oscillation of period $\frac{2\pi}{\Omega}$ along a path that will appear as a straight line passing through the pole to an observer in a fixed coordinate system, but will appear as a closed circle traversed in $1/2$ day to an observer rotating with the earth (Fig.(1.7)). From the point of view of an earthbound observer, there is an apparent deflection force that causes the object to deviate to the right of its direction of motion at a fixed rate.

In the following we illusterate the coriolis force and thecurvature effect. Newton's second law of motion expressed in coordinates rotating with the earthcan be used to describe the force balance for an object at rest on the surface of the earth, provided that an apparent force, the centrifugal force, is included among the forces acting on the object. If, however, the object is in motion along the surface of the earth, additional apparent forces are required in the statement of Newton's second law.

Suppose that an object of unit mass, initially at latitude $\emptyset$ moving zonally at speed u, relative to the surface of the earth, is displaced in latitude or in altitude by an impulsive force. As the object is displaced it will conserve its angular momentum in the absence of a torque in the east–west direction. Because the distance R to the axis of rotation changes for a displacement in latitude or altitude, the absolute angular velocity,$\Omega + u/R$, must change if the object is to conserve its absolut angular momentum.

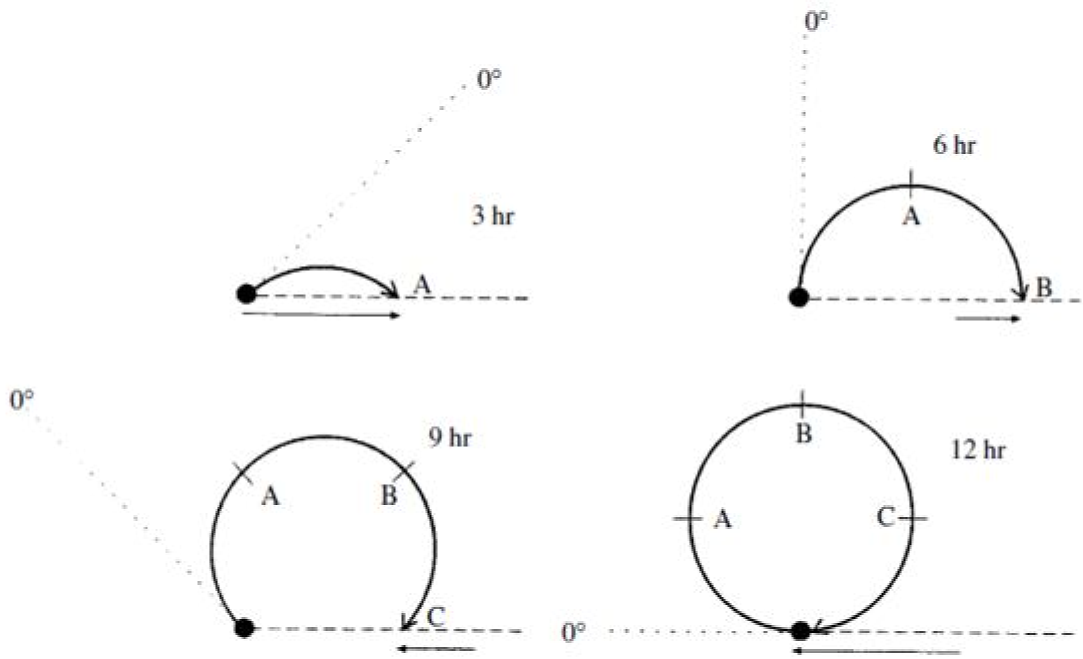


Fig. (1.7) : Motion of a frictionless object launched from the north pole along the 0° longitude meridian at $t = 0$, as viewed in fixed and rotating reference frames at 3, 6, 9, and 12 h after launch. The horizontal dashed line marks the position that the 0° longitude meridian had at $t = 0$, and short dashed lines show its position in the fixed reference frame at subsequent 3 h intervals.Horizontal arrows show 3 h displacement vectors as seen by an observer in the fixed reference frame. Heavy curved arrows show the trajectory of the object as viewed by an observer in the rotating system. Labels A, B and C show the position of the object relative to the rotating coordinates at 3 h intervals. In the fixed coordinate frame the object oscillates back and forth along a straight line under the influence of the restoring force provided by the horizontal component of gravitation. The period for a complete oscillation is 24 h (only 1/2 period is shown) . To an observer in rotating coordinates, however, the motion appears to be at constant speed and describes a complete circle in a clockwise direction in 12 h.

Because Ω is constnt, the relative zonal velocity must change.
Thus, the object behaves as though a zonally directed deflection force were acting on it.

The form of the zonal deflection force can be obtained by equating the total angular momentum at the initial distance R to the total angular momentum at the displaced distance $R + \delta R$:

$$\left(\Omega + \frac{u}{R}\right) R^2 = \left(\Omega + \frac{u + \delta u}{R + \delta R}\right) (R + \delta R)^2$$

where δu is the change in eastward relative velocity after displacement. Expanding the right-hand side, neglecting second-order differentials, and solving for δu gives

$$\delta u = -2\Omega\delta R - \frac{u}{R}\delta R$$

Noting that $R = a\, cos\, \emptyset$, where a is the radius of the earth and i∅s latitude, dividing through by the time increment δt and taking the limit as $\delta t \rightarrow 0$, gives in the case of a meridional displacement in which $\delta R = -\sin\emptyset\delta y$ (see Fig. 1.8):

$$\frac{Du}{Dt} = \left(2\Omega\, sin\emptyset + \frac{u}{a} tan\emptyset\right)\frac{Dy}{Dt} = 2\Omega sin\emptyset + \frac{uv}{a} tan\emptyset \qquad (1.10a)$$

and for a vertical displacement in which $\delta R = +cos\,\emptyset\delta z$:

$$\frac{Du}{Dt} = -\left(2\Omega\, cos\emptyset + \frac{u}{a}\right)\frac{Dz}{Dt} = 2\Omega\omega cos\emptyset - \frac{uw}{a} \qquad (1.10b)$$

where $v = Dy/Ddt$ and $w = Dz/Dt$ are the northward and upward velocity components, respectively. The first terms on the right in (1.10a) and (1.10b) are the zonal components of the Coriolis force for meridional and vertical motions, respectively. The second terms on the right are referred to as metric terms or curvature effects. These arise from the curvature of the earth’s surface.

A similar argument can be used to obtain the meridional component of the Coriolis force. Suppose nowthat the object is set in motion in the eastward direction by an impulsive force. Because the object is now rotating faster than the earth, the centrifugal force on the object will be increased. Letting **R** be the position vector

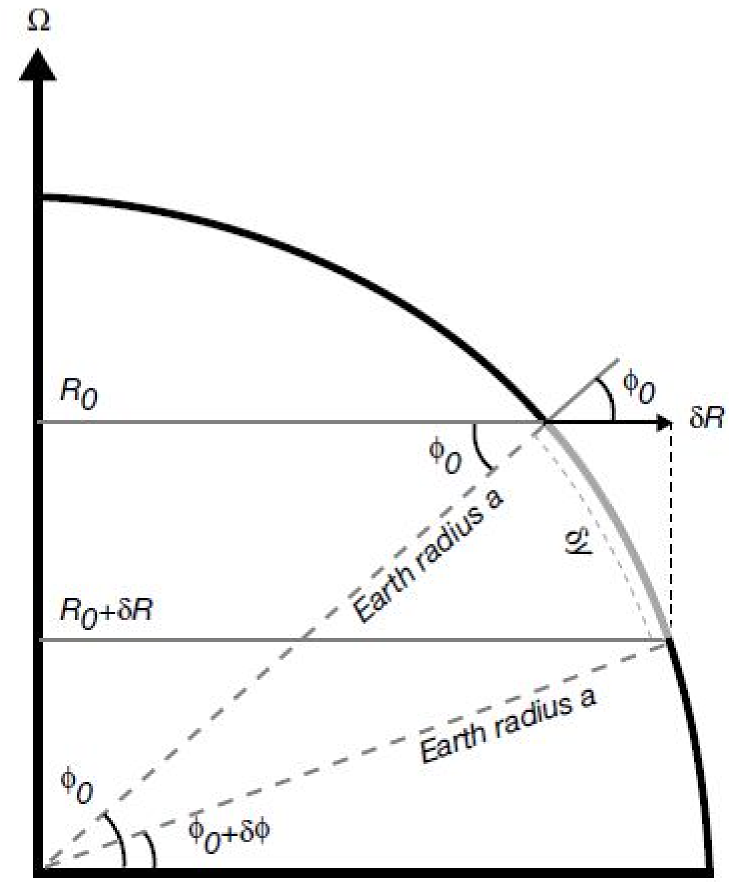


Fig. (1.8) : Relationship of δR and $\delta y = a\delta\phi$ for an equatorward displacement

from the axis of rotation to the object, the excess of the centrifugal force over that for an object at rest is

$$\left(\Omega + \frac{u}{R}\right)^2 \mathbf{R} - \Omega^2\mathbf{R} = \frac{2\Omega u\mathbf{R}}{R} + \frac{u^2\mathbf{R}}{R^2}$$

The terms on the right represent deflecting forces, which act outward along the vector R (i.e., perpendicular to the axis of rotation). The meridional and vertical components of these forces are obtained by taking meridional and vertical components of R as shown in Fig.(1.9) to yield

$$\frac{Dv}{Dt} = -2\Omega u \sin\emptyset - \frac{u^2}{a}\tan\emptyset \qquad (1.11a)$$

$$\frac{Dw}{Dt} = 2\Omega u \cos\emptyset + \frac{u^2}{a} \qquad (1.11b)$$

The first terms on the right are the meridional and vertical components, respectively,of the Coriolis forces for zonal motion; the second terms on the right are again the curvature effects.
For synoptic scale motions $|u| \ll \Omega R$, the last terms in (1.10a) and (1.11a) can be neglected in a first approximation. Therefore, relative horizontal motion produces a horizontal acceleration perpendicular to the direction of motion given by

$$\left(\frac{Du}{Dt}\right)_{c_0} = 2\Omega v sin\emptyset = fv \qquad (1.12a)$$

$$\left(\frac{Dv}{Dt}\right)_{c^0} = -2\Omega sin\emptyset = -fu \qquad (1.12b)$$

where $f \equiv 2\Omega \; sin \; \emptyset$ is the Coriolis parameter.

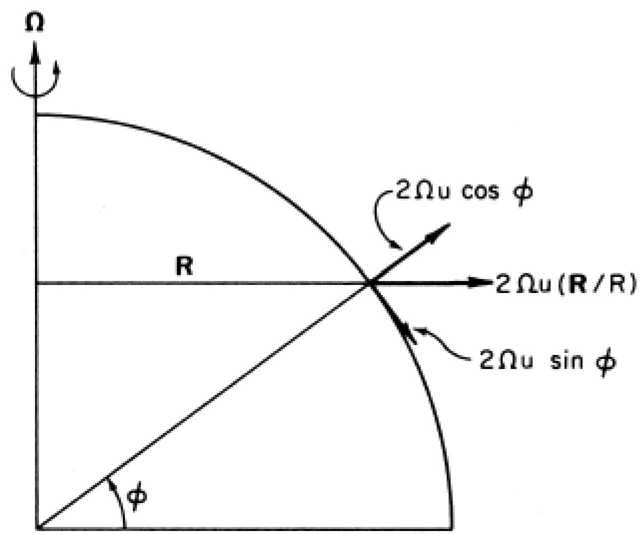


Fig.(1.9) : Components of the Coriolis force due to relative motion along a latitude circle.

The subscript Coindicates that the acceleration is the part of the total acceleration due only to the Coriolis force. Thus, for example, an object moving eastward in the horizontal is deflected equatorward by the Coriolis force, whereas a westward moving object is deflected poleward. In either case the deflection is to the right of the direction of motion in the Northern Hemisphere and to the left in the Southern Hemisphere. The vertical component of the Coriolis force in (1.11b) is ordinarily much smaller than the gravitational force so that its only effect is to cause a very minor change in the apparent weight of an object depending on whether the object is moving eastward or westward.

The Coriolis force is negligible for motions with time scales that are very short compared to the period of the earth's rotation (a point that is illustrated by several problems at the end of the chapter). Thus, the Coriolis force is not important for the dynamics of individual cumulus clouds, but is essential to the understanding of longer time scale phenomena such as synoptic scale systems. The Coriolis force must also be taken into account when computing long-range missile or artillery trajectories.

As an example, suppose that a ballistic missile is fired due eastward at 43^oN latitude $(f = 10^{-4} s^{-1}\ at\ 43^o N)$. If the missile travels 1000 km at a horizontal speed $u_0 = 100 ms^{-1}$, by how much is the missile deflected from its eastward path by the Coriolis force? Integrating (1.12b) with respect to time we find that

$$v = -f u_0 t \qquad (1.13)$$

where it is assumed that the deflection is sufficiently small so that we may let *f* and u_0 be constants. To find the total displacement we must integrate Equation(1.13) with respect to time:

$$\int_0^t v\, dt = \int_{y_0}^{y_0+\delta y} dy = -f u_0 \int_0^t t\, dt$$

Thus, the total displacement is

$$\delta y = -\frac{f u_0 t^2}{2} = -50 \text{ km}$$

Therefore, the missile is deflected southward by 50 km due to the Coriolis effect.

The *x* and *y* components given in Equation (1.12a) and Equation (1.12b) can be combined in vector form as

$$\left(\frac{D\mathbf{V}}{Dt}\right)_{c^0} = -f\mathbf{k} \times \mathbf{V} \qquad (1.14)$$

where $\mathbf{V} \equiv (u, v)$ is the horizontal velocity, $\mathbf{k}$ is a vertical unit vector, and the subscript *C*o indicates that the acceleration is due solely to the Coriolis force. Since $-\mathbf{k} \times \mathbf{V}$ is a vector rotated 90° to the right of $\mathbf{V}$,Equation (1.14) clearly shows the deflection character of the Coriolis force. The Coriolis force can only change the direction of motion, not the speed of motion.

Now we study the Constant Angular Momentum Oscillations .Suppose an object initially at rest on the earth at the point (x_0, y_o) is impulsively propelled along the *x* axis with a speed *V* at time $t = 0$. Then from Equations(1.12a) and (1.12b), the time evolution of the velocity is given by $u = V\cos(f\,t)$ and $v = -V\sin(f\,t)$. However, because $u = Dx/Dt$ and $v = Dy/Dt$, integration with respect to time gives the position of the object at time *t* as

$$x - x_0 = \frac{V}{f}\sin(ft)\ , y - y_0 = \frac{V}{f}(cos\,ft - 1) \qquad (1.15a, b)$$

where the variation of *f* with latitude is here neglected. Equations (1.15a) and (1.15b) show that in the Northern Hemisphere, where *f* is positive, the object orbits clockwise (anticyclonically) in a circle of radius $R = V/f$ about the point $\left(x_0, y_0 - \frac{V}{f}\right)$ with a period given by

$$\tau = \frac{2\pi R}{V} = \frac{2\pi}{f} = \frac{\pi}{\Omega\ sin\ \emptyset} \qquad (1.16)$$

Thus, an object displaced horizontally from its equilibrium position on the surface of the earth under the influence of the force of gravity will oscillate about its equilibrium position with a period that depends on latitude and is equal to one sidereal day at 30° latitude and 1/2 sidereal day at the pole. Constant angular momentum oscillations (often referred to misleadingly as “inertial oscillations”) are commonly observed in the oceans, but are apparently not of importance in the atmosphere.

In the following we present the structure of static atmosphere Thethermodynamic state of the atmosphere at any point is determined by the values of pressure, temperature, and density (or specific volume) at that point. These field variables are related to each other by the equation of state for an ideal gas. Letting p, T, ρ, and $\alpha(\equiv \rho^{-1})$ denote pressure, temperature, density, and specific volume, respectively, we can express the equation of state for dry air as

$$p\alpha = RT \text{ or } p = \rho RT \qquad (1.17)$$

where R is the gas constant for dry air ($R = 287\, J\, \mathrm{kg}^{-1}\mathrm{K}^{-1}$).

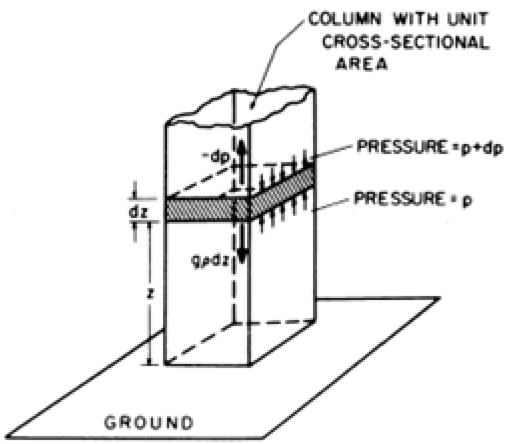


Fig.(1.10): Balance of forces for hydrostatic equilibrium.Small arrows show the upward and downward forces exerted by air pressure on the air mass in the shaded block. The downward force exerted by gravity on the air in the block is given by $\rho g dz$, whereas the net pressure force given by the difference between the upward force across the lower surface and the downward force across the upper surface is $-dp$. Note that dp is negative, as pressure decreases with height. (After Wallace and Hobbs, 1977.)

In The folloing we discauss the Hydrostatic Equation,In the absence of atmospheric motions the gravity force must be exactly balanced by the vertical component of the pressure gradient force. Thus, as illustrated in Fig. (1.10),

$$dp/dz = -\rho g \qquad (1.18)$$

This condition of hydrostatic balance provides an excellent approximation for the vertical dependence of the pressure field in the real atmosphere.
Only for intense small-scale systems such as squall lines and tornadoes is it necessary to consider departures from hydrostatic balance. Integrating Equation (1.18) from a height z to the top of the atmosphere we find that

$$p(z) = \int_{z}^{\infty} \rho \mathrm{g}\, dz \qquad (1.19)$$

so that the pressure at any point is simply equal to the weight of the unit cross section column of air overlying the point. Thus, mean sea level pressure $p(0) = 1013.25\ hPa$ is simply the average weight per square meter of the total atmospheric column. It is often useful to express the hydrostatic equation in terms of the geopotential rather than the geometric height. Noting from Equation (1.8) that $d\Phi = \mathrm{g}\, dz$ and from Equation (1.17) that $\alpha = RT/p$, we can express the hydrostatic equation in the form

$$\mathrm{g}dz = d\,\Phi = -(RT\,/p)dp = -RT\,d\,\ln p \qquad (1.20)$$

Thus, the variation of geopotential with respect to pressure depends only on temperature. Integration of Equation (1.20) in the vertical yields a form of the hypsometric equation:

$$\Phi(z_2) - \Phi(z_1) = \mathrm{g}_0(z_2 - z_1) = R\int_{p_2}^{p_1} Td\,\ln p \qquad (1.21)$$

Here $Z \equiv \Phi(\mathrm{z})/\mathrm{g}_0$, is the *geopotential height*, where $\mathrm{g}_0 \equiv 9.80665 ms^{-2}$ isthe global average of gravity at mean sea level. Thus in the troposphere and lower stratosphere, Z is numerically almost identical to the geometric height z. In terms of Z the hypsometric equation becomes

$$Z_T \equiv z_2 - z_1 = \frac{R}{g^0}\int_{p_2}^{p_1} Td\,\ln p \qquad (1.22)$$

where Z_T is the thickness of the atmospheric layer between the pressure surfaces p_2 *and* p_1 . Defining a layer mean temperature

$$\langle T\rangle = \int_{P_2}^{P_1} Td\ln p\left[\int_{p_2}^{p_1} d\ln p\right]^{-1}$$

and a layer mean scale height $H \equiv R\langle T\rangle / g_0$we have from Equation (1.22)

$$Z_T = H\,\ln\frac{p_1}{p_2} \qquad (1.23)$$

Thus the thickness of a layer bounded by isobaric surfaces is proportional to the mean temperature of the layer. Pressure decreases more rapidly with height in a cold layer than in a warm layer. It also follows immediately from Equation (1.23) that in an isothermal atmosphere of temperature *T*, the geopotential height is proportional to the natural logarithm of pressure normalized by the surface pressure,

$$Z = -H \ln \frac{p}{p_0} \tag{1.24}$$

wherep_0 is the pressure at $Z = 0$. Thus, in an isothermal atmosphere the pressure decreases exponentially with geopotential height by a factor of e^{-1} per scale height,

$$p(Z) = p(0)e^{-z/H}$$

Now we study the Pressure as a Vertical Coordinate, From the hydrostatic Equation (1.18), it is clear that a single valued monotonic relationship exists between pressure and height in each vertical column of the atmosphere. Thus we may use pressure as the independent vertical coordinate and height (or geopotential) as a dependent variable. The thermodynamic state of the atmosphere is then specified by the fields of $\theta(x, y, p, t)$ and $T\ (x, y, p, t)$.

Now the horizontal components of the pressure gradient force given by Equation(1.1) are evaluated by partial differentiation holding z constant. However, when pressure is used as the vertical coordinate, horizontal partial derivatives must be

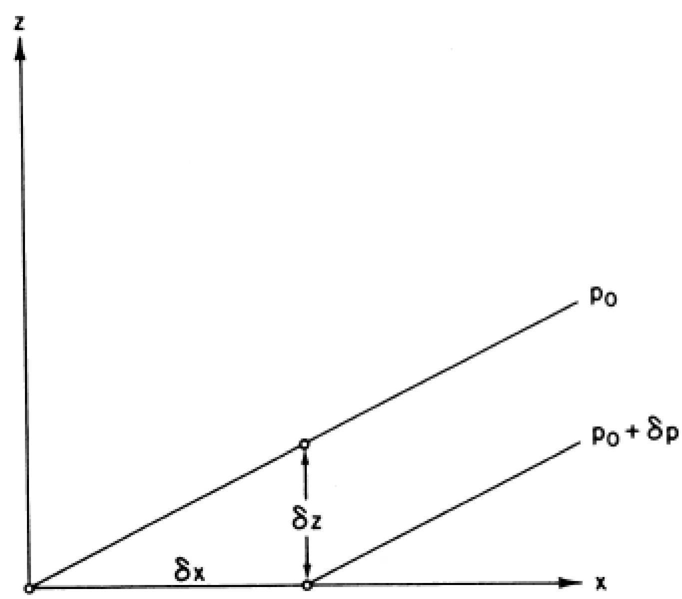


Fig. (1.11): Slope of pressure surfaces in the x, z plane.

evaluated holding p constant. Transformation of the horizontal pressure gradient force from height to pressure coordinates may be carried out with the aid of Fig. (1.11). Considering only the x, z plane, we see from Fig.(1.11) that

$$\left[\frac{(p_0 + \delta p) - p_0}{\delta x}\right]_z = \left[\frac{(p_0 + \delta p) - p_0}{\delta z}\right]_x \left(\frac{\delta z}{\delta x}\right)_p$$

where subscripts indicate variables that remain constant in evaluating the differentials.Thus, for example, in the limit $\delta z \rightarrow 0$

$$\left[\frac{(p_0 + \delta p) - p_0}{\delta z}\right]_x \rightarrow \left(-\frac{\partial p}{\partial z}\right)_x$$

where the minus sign is included because $\delta z < 0$ for $\delta p > 0$.

Taking the limits $\delta x, \delta z \rightarrow 0$ we obtain:

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

which after substitution from the hydrostatic Equation (1.18) yields

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -g\left(\frac{\partial z}{\partial x}\right)_p = -\left(\frac{\partial \emptyset}{\partial x}\right)_p \qquad (1.25)$$

Similarly, it is easy to show that

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \emptyset}{\partial y}\right)_p \qquad (1.26)$$

Thus in the isobaric coordinate system the horizontal pressure gradient force is measured by the gradient of geopotential at constant pressure. Density no longer appears explicitly in the pressure gradient force; this is a distinct advantage of the isobaric system.In the last in this section we illusterate the Generalized Vertical Coordinate, Any single-valued monotonic function of pressure or height may be used as the independent vertical coordinate. For example, in many numerical weather prediction models, pressure normalized by the pressure at the ground $[\sigma \equiv p(x, y, z, t)/p_s(x, y, t)]$ is used as a vertical coordinate. This choice guarantees that the ground is a coordinate surface $(\sigma \equiv 1)$ even in the presence of spatial and temporal surface pressure variations. Thus, this so-called σ coordinate system is particularly useful in regions of strong topographic variations.

We now obtain a general expression for the horizontal pressure gradient, which is applicable to any vertical coordinate $s = s(x, y, z, t)$ that is a single-valued monotonic function of height. Referring to Fig. (1.12) we see that for a horizontal distance δx the pressure difference evaluated along a surface of constant s is related to that evaluated at constant z by the relationship

$$\frac{p_C - p_A}{\delta x} = \frac{p_C - p_B}{\delta z} \cdot \frac{\delta z}{\delta x} + \frac{p_B - p_A}{\delta x}$$

Taking the limits as $\delta x, \delta z \rightarrow 0$ we obtain

$$\left(\frac{\partial p}{\partial x}\right)_s = \frac{\partial p}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s + \left(\frac{\partial p}{\partial x}\right)_z \qquad (1.27)$$

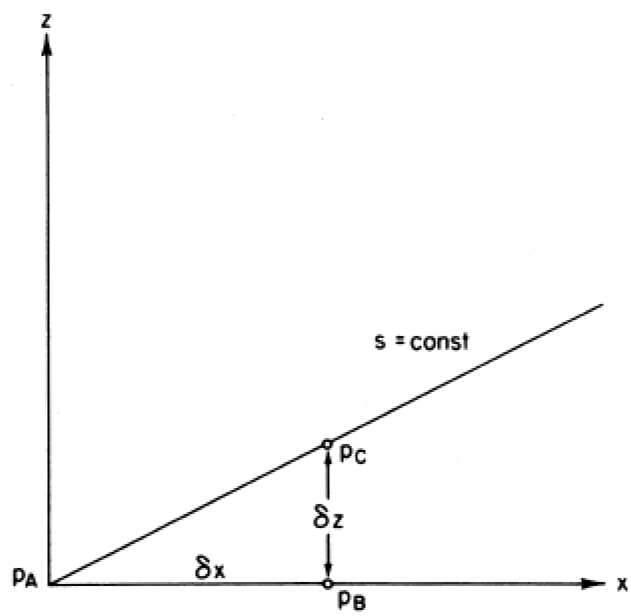


Fig. (1.12): Transformation of the pressure gradient force to s coordinates.

Using the identity $\partial p / \partial z = \left(\frac{\partial s}{\partial z}\right)\left(\frac{\partial p}{\partial s}\right)$, we can express Equation (1.27) in the alternate form

$$\left(\frac{\partial p}{\partial x}\right)_s = \left(\frac{\partial p}{\partial x}\right)_z + \left(\frac{\partial s}{\partial z}\left(\frac{\partial z}{\partial x}\right)_s\right)\left(\frac{\partial p}{\partial s}\right) \qquad (1.28)$$

Section(1.2):The continuity Equation, Energy Equation and Dry Atmosphere

In this section we introduce the some of fundamental conservation principles, conservation of mass, conservation of Energy and thermodynamics of the Dry Atmosphere,and we start with develops the conservation of mass for afluid(the continuity Equation)using two alternative Methods.The first method is based on an Eulerian control volume, whereas the second is based on a Lagrangian control volume:

(i) An Eulerian Derivation

We consider a volume element δx δy δz that is fixed in a Cartesian coordinate frame as shown in Fig.(1.13). For such a fixed control volume the net rate of mass inflow through the sides must equal the rate of accumulation of mass within the volume. The rate of inflow of mass through the left-hand face per unit area is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

whereas the rate of outflow per unit area through the right-hand face is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

Because the area of each of these faces is $\delta y\delta z$, the net rate of flow into the volume due to the x velocity component is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]\delta y\delta z = -\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z$$

Similar expressions obviously hold for the y and z directions. Thus, the net rate of mass inflow is

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$$

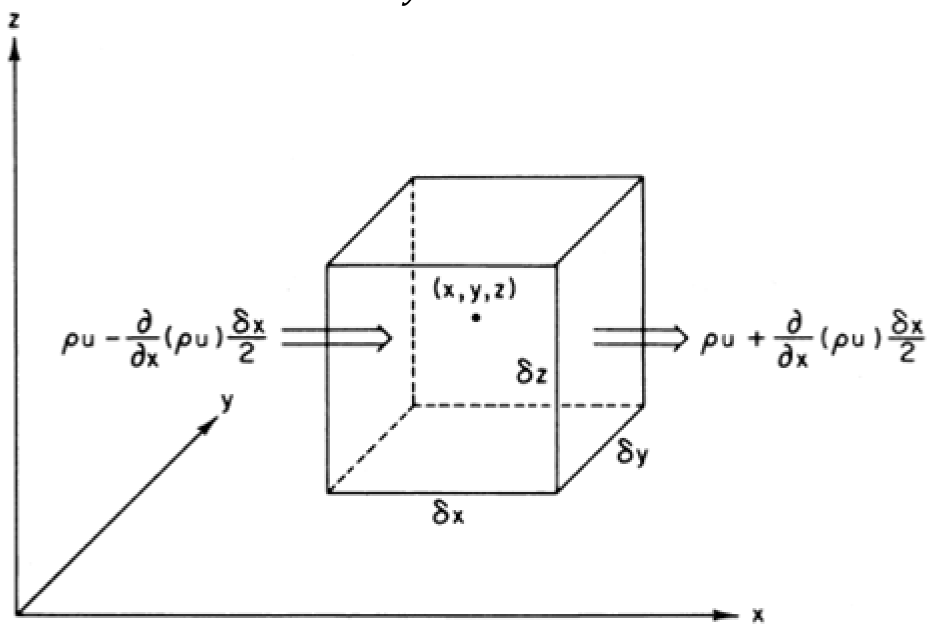


Fig. (1.13): Mass inflow into a fixed (Eulerian) control volume due to motion parallel to the x axis.

and the mass inflow per unit volume is just $-\nabla \cdot (\rho U)$, which must equal the rate of mass increase per unit volume. Now the increase of mass per unit volume is just the local density change $\partial\rho/\partial t$.there for

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{U}) = 0 \qquad (1.29)$$

Equation (1.29) is the mass divergence form of the continuity equation.

An alternative form of the continuity equation is obtained by applying the vector Identity

$$\nabla \cdot (\rho\mathbf{U}) \equiv \rho\nabla \cdot \mathbf{U} \cdot \nabla\rho$$

and the relationship

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

to get:

$$\frac{1}{\rho}\frac{D}{Dt} + \nabla \cdot \mathbf{U} = 0 \qquad (1.30)$$

Equation (1.30) is the velocity divergence form of the continuity equation. It states that the fractional rate of increase of the density following the motion of an air parcel is equal to minus the velocity divergence.This should be clearl distinguished from Equation(1.29), which states that the *local* rate of change of density is equal to minus the mass divergence.

(ii) A Lagrangian Derivation

The physical meaning of divergence can be illustrated by the following alternative derivation of Equation (1.30). Consider a control volume of fixed mass δM that moves with the fluid. Letting $\delta V = \delta x\, \delta y \delta z$ be the volume, we find that because $\delta M = \rho \delta V = \rho \delta x \delta y \delta z$ is conserved following the motion, we can write

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{1}{\rho \delta V}\frac{D}{Dt}(\rho \delta V) = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = 0 \qquad (1.31)$$

But

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x}\frac{D}{Dt}(\delta x) + \frac{1}{\delta y}\frac{D\rho}{Dt}(\delta y) + \frac{1}{\delta z}\frac{D}{Dt}(\delta z)$$

Referring to Fig. (1.14), we see that the faces of the control volume in the y, z plane (designated A and B) are advected with the flow in the x direction at speeds $u_A = Dx/Dt$ and $u_B = D(x + \delta x)/Dt$, respectively. Thus, the difference in speeds of the two faces is $\delta u = uB - uA = D(x + \delta x)/Dt - Dx/Dt$ or $\delta u = D(\delta x)\ /Dt$. Similarly, $\delta v = D(\delta y)/Dt$ and $= D(\delta z)/Dt$. Therefore,

$$\lim_{\delta x, \delta y, \delta z \to 0}\left[\frac{1}{\delta V}\frac{D}{Dt}(\delta V)\right] = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{U}$$

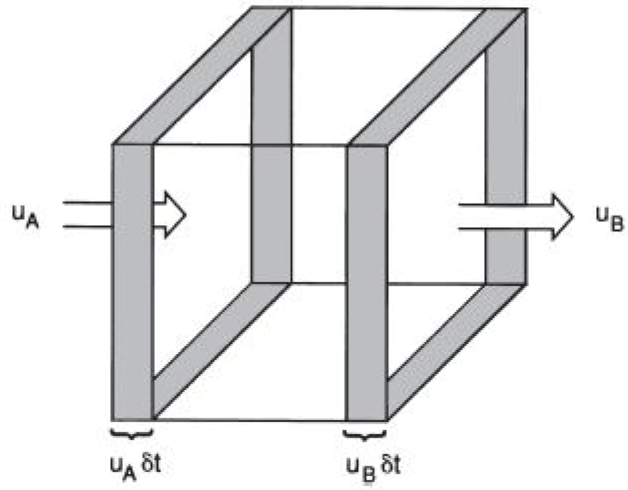


Fig. (1.14): Change in Lagrangian control volume (shown by shading) due to fluid motion parallel to the x axis.

so that in the limit $\delta V \rightarrow 0$,Equation (1.31) reduces to the continuity Equation (1.30); the divergence of the three-dimensional velocity field is equal to the fractional rate of change of volume of a fluid parcel in the limit $\delta V \rightarrow 0$. It is left as a problem for the student to show that the divergence of the horizontal velocity field is equal to the fractional rate of change of the horizontal area δA of a fluid parcel in the limit $\delta A \rightarrow 0$.

Now we study Scale Analysis of the Continuity Equation ,Following the technique developed above,and again assuming that$|\rho'/\rho| \ll 1$, we can approximate the continuity Equation (1.30) as:

$$\underbrace{\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U}\cdot\nabla\rho'\right)}_{A} + \underbrace{\frac{w}{\rho'}\frac{d\rho_0}{dz}}_{B} + \underbrace{\nabla\cdot\mathbf{U}}_{C} \approx 0 \qquad (1.32)$$

where ρ'designates the local deviation of density from its horizontally averaged value, $\rho_0\,(z)$. For synoptic scale motions $\rho'/\rho_0 \sim 10^{-2}$ so that using the characteristic scales given above we find that term A has magnitude

$$\frac{1}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{U}\cdot\nabla\rho'\right) \sim \frac{\rho'}{\rho_0} \approx 10^{-7} S^{-1}$$

For motions in which the depth scale H is comparable to the density scale height,$d \ln \rho_0 / dz \sim H^{-1}$, so that term B scales as:

$$\frac{w}{\rho_0}\frac{d\rho_0}{dz} \sim \frac{w}{H} \approx 10^{-6}S^{-1}$$

Expanding term C in Cartesian coordinates, we have

$$\nabla \cdot \mathbf{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

For synoptic scale motions the terms $\partial u / \partial x$ and $\partial v / \partial y$ tend to be of equal magnitude but opposite sign. Thus, they tend to balance so that

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \sim 10^{-1}\frac{U}{L} \approx 10^{-6}s^{-1}$$

and in addition

$$\frac{\partial w}{\partial z} \sim \frac{W}{H} \approx 10^{-6}s^{-1}$$

Thus, terms B and C are each an order of magnitude greater than term A, and to a first approximation, terms B and C balance in the continuity equation. To a good approximation then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + w\, d \ln \rho_0 = 0$$

or, alternatively, in vector form

$$\nabla \cdot (\rho_0 \mathbf{U}) = 0 \qquad (1.33)$$

Thus for synoptic scale motions the mass flux computed using the basic state density ρ_0 is nondivergent. This approximation is similar to the idealization of incompressibility, which is often used in fluid mechanics. However, an incompressible fluid has density constant following the motion:

$$\frac{D\rho}{Dt} = 0$$

Thus by Equation (1.30) the velocity divergence vanishes $(\nabla \cdot \mathbf{U} = 0)$ in an incompressible fluid, which is not the same as Equation (1.33). Our approximation Equation(1.33) shows that for purely horizontal flowthe atmosphere behaves as though it were an incompressible fluid. However, when there is vertical motion the compressibility associated with the height dependence of ρ_0 must be taken into account.

Now we stud the thermodynamic energy equationWe now turn to the fundamental conservation principle, the conservation of energy as applied to a moving fluid element. The first law of thermodynamics is usually derived by considering a system in thermodynamic equilibrium, that is,a system that is initially at rest and after exchanging heat with its surroundings and doing work on the surroundings is again at rest. For such a system the first law states that the change in internal energy of the system is equal to the difference between the heat added to the system and the work done by the system.

A Lagrangian control volume consisting of a specified mass of fluid may be regarded as a thermodynamic system. However, unless the fluid is at rest, it will not be in thermodynamic equilibrium. Nevertheless, the first law of thermodynamics still applies. To show that this is the case, we note that the total thermodynamic energy of the control volume is considered to consist of the sum of the internal energy (due to the kinetic energy of the individual molecules) and the kinetic energy due to the macroscopic motion of the fluid. The rate of change of this total thermodynamic energy is equal to the rate of diabatic heating plus the rate at which work is done on the fluid parcel by external forces.

If we let e designate the internal energy per unit mass, then the total thermodynamic energy contained in a Lagrangian fluid element of density ρ and volume δV is $[e + (1/2)U \cdot U]\,\delta V$. The external forces that act on a fluid element may be divided into surface forces, such as pressure and viscosity, and body forces, such as gravity or the Coriolis force. The rate at which work is done on the fluid element by the x component of the pressure force is illustrated in Fig. (1.15). Recalling that pressure is a force per unit area and that the rate at which a force does work is given by the dot product of the force and velocity vectors, we see that the rate at which the surrounding fluid does work on the element due to the pressure force onthe two boundary surfaces in the y, z plane is given by

$$(pu)_A \delta y \delta z - (pu)_B \delta y \delta z$$

(The negative sign is needed before the second term because the work done on the fluid element is positive if u is negative across face B.) Now by expanding in a Taylor series we can write

$$(Pu)_B = (Pu)_A + \left[\frac{\partial}{\partial x}(Pu)\right]_A \delta x + \cdots$$

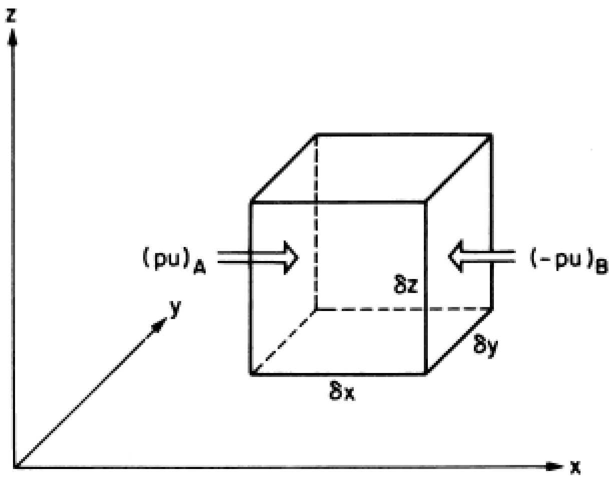


Fig.(1.15): Rate of working on a fluid element due to the x component of the pressure force.

Thus the net rate at which the pressure force does work due to the x component of motion is

$$[(Pu)_A - (Pu)_B]\delta y \delta z = -\left[\frac{\partial}{\partial x}(Pu)\right]_A \delta V$$

where $\delta V = \delta x \delta y \delta z$.
Similarly, we can show that the net rates at which the pressure force does work due to the y and z components of motion are

$$-\left[\frac{\partial}{\partial y}(Pv)\right]\delta V \ and -\left[\frac{\partial}{\partial z}(Pw)\right]\delta V$$

respectively. Hence, the total rate at which work is done by the pressure force is simply

$$-\mathbf{\nabla} \cdot (p\mathbf{U})\delta V$$

The only body forces of meteorological significance that act on an element of mass in the atmosphere are the Coriolis force and gravity. However, because the Coriolis force, $-2\Omega \times \mathbf{U}$, is perpendicular to the velocity vector, it can do no work. Thus the rate at which body forces do work on the mass element is just $\rho\, \mathbf{g} \cdot \mathbf{U}\, \delta V$
Applying the principle of energy conservation to our Lagrangian control volume

(neglecting effects of molecular viscosity), we thus obtain

$$\frac{D}{Dt}\left[\rho\left(e+\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\delta V\right] = -\nabla\cdot(p\mathbf{U})\delta V+\rho\mathbf{g}\cdot\mathbf{U}\,\delta V+\rho J\,\delta V \quad (1.34)$$

Here J is the rate of heating per unit mass due to radiation, conduction, and latent heat release.With the aid of the chain rule of differentiation we can rewrite Equation (1.34) as:

$$\rho\delta V\frac{D}{Dt}\left(e+\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)+\left(e+\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right)\frac{D}{Dt}(\rho\delta V)$$

$$= -\mathbf{U}\cdot\nabla p\delta V - p\nabla\cdot\mathbf{U}\,\delta V - \rho g w\delta V + \rho J\delta V \quad (1.35)$$

where we have used $\mathbf{g} = -g\mathbf{k}$. Now from Equation (1.31) the second term on the left in Equation(1.35) vanishes so that

$$\rho\frac{De}{Dt}+\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - p\nabla\cdot\mathbf{U} - \rho g w + \rho J \quad (1.36)$$

This Equation can be simplified by noting that if we take the dot product of $\mathbf{U}$ with the momentum Equation ($\frac{D\mathbf{U}}{Dt} = -2\Omega\times\mathbf{U} - \frac{1}{\rho}\nabla p + \mathbf{g} + \mathbf{F}_r$) we obtain (neglecting friction)

$$\rho\frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U}\right) = -\mathbf{U}\cdot\nabla p - \rho g w \quad (1.37)$$

Subtracting Equation (1.37) from Equation (1.36), we obtain

$$\rho\frac{De}{Dt} = p\nabla\cdot\mathbf{U} + \rho J \quad (1.38)$$

The terms in Equation(1.36) that were eliminated by subtracting Equation (1.37) represent the balance of mechanical energy due to the motion of the fluid element; the remaining terms represent the thermal energy balance.
Using the definition of geopotential Equation (1.15), we have

$$gw = g\frac{Dz}{Dt} = \frac{D\Phi}{Dt}$$

so that Equation (1.37) can be rewritten as

$$\rho \frac{D}{Dt}\left(\frac{1}{2}\mathbf{U}\cdot\mathbf{U} + \Phi\right) = -\mathbf{U}\cdot\nabla p \qquad (1.39)$$

which is referred to as the mechanical energy equation. The sum of the kinetic energy plus the gravitational potential energy is called the mechanical energy. Thus Equation(1.39) states that following the motion, the rate of change of mechanical energy per unit volume equals the rate at which work is done by the pressure gradient force.

The thermal energy Equation (1.38) can be written in more familiar form by noting from Equation (1.30) that

$$\frac{1}{\rho}\nabla\cdot\mathbf{U} = -\frac{1}{\rho^2}\frac{D\rho}{Dt} = \frac{D\alpha}{Dt}$$

and that for dry air the internal energy per unit mass is given by $e = c_v T$, where $c_v(= 717\text{Jkg}^{-1}K^{-1})$ is the specific heat at constant volume.We then obtain

$$c_v\frac{DT}{Dt} + p\frac{D\alpha}{Dt} = J \qquad (1.40)$$

which is the usual form of the thermodynamic energy equation. Thus the first law of thermodynamics indeed is applicable to a fluid in motion. The second term on the left, representing the rate of working by the fluid system (per unit mass), represents a conversion between thermal and mechanical energy. This conversion process enables the solar heat energy to drive the motions of the atmosphere.

Now we discuss thermodynamics of the dry atmosphere ,Taking the total derivative of the Equation of state (1.14), we obtain

$$p\frac{D\alpha}{Dt} + \alpha\frac{D}{Dt}p = R\frac{DT}{Dt}$$

Substituting for $pD\alpha/Dt$ in Equation (1.40) and using $c_p = c_v + R$, where c_p (= 1004 $\text{Jkg}^{-1}\ \text{K}^{-1}$) is the specific heat at constant pressure, we can rewrite the first law of thermodynamics as

$$c_p\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} = J \qquad (1.41)$$

Dividing through by $T$and again using the equation of state, we obtain the entropy form of the first law of thermodynamics:

$$c_p \frac{D \ln T}{Dt} - R \frac{D \ln p}{Dt} = \frac{J}{T} \equiv \frac{Ds}{Dt} \tag{1.42}$$

Equation (1.42) gives the rate of change of entropy per unit mass following the motion for a thermodynamically reversible process.A reversible process is one in which a system changes its thermodynamic state and then returns to the original state without changing its surroundings. For such a process the entropy s defined by Equation (1.42) is a field variable that depends only on the state of the fluid. Thus Ds is a perfect differential, and Ds/Dt is to be regarded as a total derivative. However, "heat" is not a field variable, so that the heating rate J is not a total derivative.

In the following we present, Potential Temperature For an ideal gas undergoing an adiabatic process (i.e., a reversible process in which no heat is exchanged with the surroundings), the first law of thermodynamics can be written in differential form as

$$c_p D \ln T - RD \ln p = D\left(c_p \ln T - R \ln p\right) = 0$$

Integrating this expression from a state at pressure p and temperature T to a state in which the pressure is p_s and the temperature is θ, we obtain after taking the antilogarithm

$$\theta = T(p_s/p)^{R/c_p} \tag{1.43}$$

This relationship is referred to as Poisson's equation, and the temperature θ defined by Equation(1.43) is called the potential temperature. θ is simply the temperature that a parcel of dry air at pressure p and temperature T would have if it were expanded or compressed adiabatically to a standard pressure p_s (usually taken to be 1000 hPa). Thus, every air parcel has a unique value of potential temperature, and this value is conserved for dry adiabatic motion. Because synoptic scale motions are approximately adiabatic outside regions of active precipitation, θ is a quasi-conserved quantity for such motions. Taking the logarithm of Equation (1.43) and differentiating, we find that

$$c_p \frac{D \ln \theta}{Dt} = c_p \frac{D \ln T}{Dt} - R \frac{D \ln T}{Dt} \tag{1.44}$$

Comparing Equation(1.42) and Equation (1.44), we obtain

$$c_p \frac{D \ln \theta}{Dt} = \frac{J}{T} = \frac{Ds}{Dt} \tag{1.45}$$

Thus, for reversible processes, fractional potential temperature changes are indeed proportional to entropy changes. A parcel that conserves entropy following the motion must move along an isentropic (constant θ) surface.

Now we studyThe Adiabatic Lapse Rate,A relationship between the lapse rate of temperature (i.e., the rate of decrease of temperature with respect to height) and the rate of change of potential temperature with respect to height can be obtained by taking the logarithm of Equation (1.43) and differentiating with respect to height. Using the hydrostatic equation and the ideal gas law to simplify the result gives

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \frac{\partial T}{\partial z} + \frac{g}{c_p} \tag{1.46}$$

For an atmosphere in which the potential temperature is constant with respect to height, the lapse rate is thus

$$-\frac{dT}{dz} = \frac{g}{c_p} \equiv \Gamma d \tag{1.47}$$

Hence, the dry adiabatic lapse rate is approximately constant throughout the lower atmosphere.

Now we illusterate,The Static Stability If potential temperature is a function of height, the atmospheric lapse rate, $\Gamma \equiv -\frac{\partial T}{\partial z}$, will differ from the adiabatic lapse rate and

$$\frac{T}{\theta} \frac{\partial \theta}{\partial z} = \Gamma d - \Gamma \tag{1.48}$$

If $\Gamma < \Gamma d$ so that θ increases with height, an air parcel that undergoes an adiabatic displacement from its equilibrium level will be positively buoyant when displaced downward and negatively buoyant when displaced upward so that it will tend to return to its equilibrium level and the atmosphere is said to be statically stable or stably stratified. Adiabatic oscillations of a fluid parcel about its equilibrium level in a stably stratified atmosphere are referred to as *buoyancy* oscillations. The characteristic frequency of such oscillations can be derived by considering a parcel that is displaced vertically a small distance δz without disturbing its environment. If the environment is in hydrostatic balance, $\rho_0 g = -dp_0/dz$, where p_0 and ρ_0 are the pressure and density of the environment. The vertical acceleration of the parcel is

$$\frac{Dw}{Dt} = \frac{D^2}{Dt^2}(\delta z) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{1.49}$$

where p and ρ are the pressure and density of the parcel. In the parcel method it is assumed that the pressure of the parcel adjusts instantaneously to the environmental pressure during the displacement: $p = p_0$. This condition must be true if the parcel is to leave the environment undisturbed. Thus with the aid of the hydrostatic relationship, pressure can be eliminated in Equation (1.49) to give

$$\frac{D^2}{Dt^2}(\delta z) = g\left(\frac{\rho_0 - \rho}{\rho}\right) = g\frac{\theta}{\theta_0} \tag{1.50}$$

Where Equation (1.43) and the ideal gas law have been used to express the buoyancy force in terms of potential temperature. Here θ designates the deviation of the potential temperature of the parcel from its basic state (environmental) value $\theta_0(z)$. If the parcel is initially at level $z = 0$ where the potential temperature is $\theta_0(0)$, then for a small displacement δz we can represent the environmental potential temperature as

$$\theta_0(\delta z) \approx \theta_0(0) + (d\theta_0/dz)\delta z$$

If the parcel displacement is adiabatic, the potential temperature of the parcel is conserved. Thus, $\theta(\delta z) = \theta_0(0) - \theta_0(\delta z) = -(d\theta_0/dz)\delta z$, and Equation (1.50) becomes

$$\frac{D^2}{Dt^2}(\delta z) = -N^2 \delta z \tag{1.52}$$

Where

$$N^2 = g \frac{d \ln \theta_0}{dz}$$

is a measure of the static stability of the environment. Equation (1.51) has a general solution of the form $\delta z = Aexp(iNt)$. Therefore, if $N^2 > 0$, the parcel will oscillate about its initial level with a period $\tau = 2\pi / N$. The corresponding frequency N is the *buoyancy frequency*.3 For average tropospheric conditions,$N \approx 1.2 \times 10^{-1} s^{-1}$ so that the period of a buoyancy oscillation is about 8 min. In the case of $N = 0$, examination of Equation (1.51) indicates that no accelerating force will exist and the parcel will be in neutral equilibrium at its new level. However, if $N^2 < 0$ (potential temperature decreasing with height) the displacement will increase exponentially in time.We thus arrive at the familiar gravitational or static stability criteria for dry air:

$d\theta 0 / dz > 0$ statically stable,

$d\theta 0 / dz = 0$ statically neutral,

$d\theta 0 / dz < 0$ statically unstable.

On the synoptic scale the atmosphere is always stably stratified because any unstable regions that develop are stabilized quickly by convective overturning.

In the following we study,Scale Analysis of the Thermodynamic Energy Equation If potential temperature is divided into a basic state $\theta_0(z)$ and a deviation $\theta(x, y, z, t\ )$so that the total potential temperature at any point is given by to $\theta_{tot} = \theta_0(z) + \theta(x, y, z, t\ )$, the first law of thermodynamics Equation (1.4 5) can be written approximately for synoptic scaling as

$$\frac{1}{\theta_0}\left(\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) + w\frac{d \ln \theta_0}{dz} = \frac{J}{c_p T} \tag{1.52}$$

where we have used the facts that for $|\theta / \theta_0| \ll 1$,$|d\theta / dz| \ll d\theta_0 / dz$, and

$$\ln \theta_{tot} = \ln[\theta_0(1 + \theta / \theta_0)] \approx \ln \theta_0 + \theta / \theta_0$$

Outside regions of active precipitation, diabatic heating is due primarily to net radiative heating. In the troposphere, radiative heating is quite weak so that typically $J/c_p \leq 1$ °Cd^{-1} (except near cloud tops, where substantially larger cooling can occur due to thermal emission by the cloud particles). The typical amplitudeof horizontal potential temperature fluctuations in a midlatitude synoptic system (above the boundary layer) is $\theta \sim 4$°C. Thus,

$$\frac{T}{\theta_0}\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) \sim \frac{\theta U}{L} \sim 4°\mathrm{Cd}^{-1}$$

The cooling due to vertical advection of the basic state potential temperature (usually called the adiabatic cooling) has a typical magnitude of

$$w\left(\frac{T}{\theta_0}\frac{d\theta_0}{dz}\right) = w(\Gamma d - \Gamma) \sim 4°\mathrm{C}d^{-1}$$

where $w \sim 1$ cm s^{-1} and $\Gamma d - \Gamma$,the difference between dry adiabatic and actual lapse rates, is ~ 4°C km^{-1}. Thus, in the absence of strong diabatic heating, the rate of change of the perturbation potential temperature is equal to the adiabatic heating or cooling due to vertical motion in the statically stable basic state, and Equation(1.52) can be approximated as

$$\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y}\right) + w\frac{d\theta_0}{dz} \approx 0 \tag{1.53}$$

Alternatively, if the temperature field is divided into a basic state $T_0(z)$ and a deviation $T\ (x, y, z, t\ )$, then since $\theta/\theta_0 \approx T/T_0$, Equation (1.53) can be expressed to the same order of approximation in terms of temperature as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) + w(\Gamma d - \Gamma) \approx 0 \tag{1.54}$$

Chapter(2)

Meteorology and Weather Prediction

and Atmosphere

Section (2.1): The Scientific Problem of Weather Prediction

the atmosphere is an approximately spherical thin gaseous shell about the earth, some 13, 000 km in diameter, with 50% of its mass below $5\frac{1}{2}$ - km above the surface, and 90% of its mass below 16 km. In the mean it is rotating with the earth, although in both Northern and Southern Hemispheres in middle latitudes so-called jet streams at about 10 km completely circle the poles as ever-present vast vortices. The jet streams are westerly and slowly undulate north and south in up to 5 or so very long waves. These waves of planetary scale strongly influence the development and motion of systems of smaller scale, and vice versa.

Numerical weather prediction at present deals principally with the prediction Of large cyclonic storms and anticyclones, which are several hundred to several thousand kilometers across, somewhat smaller in scale than the planetary waves. Because of interactions among scales both this storm scale and the planetary Scale must be explicitly predicted. Likewise the effects of even smaller scales must in some way be accounted for to gain maximum achievable forecast skill. This is done by including in the equations turbulence, diffusion, and friction Terms and other supplementary numerical processes. In writing the equations below, we do not show these terms and processes explicitly, but indicate them with appropriate symbols.

Newton's second law of motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = F_x \qquad (2.1a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = F_y \qquad (2.1b)$$

$$g + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \qquad (2.1c)$$

First law of thermodynamics

$$C_v\left[\frac{\partial T}{\partial t}+\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right]$$

$$+p\left[\frac{\partial \rho^{-1}}{\partial t}+u\frac{\partial \rho^{-1}}{\partial x}+v\frac{\partial \rho^{-1}}{\partial y}+w\frac{\partial \rho^{-1}}{\partial z}\right]=H \qquad (2.1d)$$

Conservation of mass

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0 \qquad (2.1e)$$

Conservation of water vapor

$$\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x}+v\frac{\partial q}{\partial y}+w\frac{\partial q}{\partial z}=E \qquad (2.1f)$$

Equation of state for perfect gases

$$\mathrm{p}\rho^{-1}=\mathrm{RT} \qquad (2.1g)$$

where t is time and the known (time independent) variables are

x, y	horizontal Cartesian coordinates
z	height above mean sea level
f	Coriolis parameter
g	acceleration of gravity
c_v	specific heat of air at constant volume
R	gas constant for air

and the unknown (time-dependent) variables are

u, v, w	x, y, z-components of velocity vector
ρ	density
P	pressure

T	temperature
q	specific humidity

F_x and F_y are accelerations due to friction, and drag of topography. H is heat added to the system by radiation, transfer from earth and eoceans, and condensation and evaporation. E is evaporation (positive) into the atmosphere and condensation (negative) of water vapor. The Coriolis forces, fv and $-fu$ are apparent forces due to earth's rotation. Equation (2.1c) states the hydrostatic condition, i. e., that pressure at agiven point equals the weight of air above the point. On the storm scale and larger, the ratio of gravitational acceleration g to vertical acceleraterms that are neglected is about 107, so it is a very accurate approximation;
Note that these equations are in cartesian coordinates, and thus do not account for the spherical shape of the earth and atmosphere. In practice additional, so-called metrical, terms must be included, but they are not necessary for my purpose here, which is merely to illustrate the nature of the computations that need be done.

Now, these equations are for a continuous medium, but our computers are digital. They therefore can be integrated only by numerical estimate. Similarly, the derivatives in space cannot be evaluated except by estimations.
In effect, then, we do not integrate these partial differential equations, but rather solve a set of albegraic so-called partial difference equations whose solution we think approximates the integrals.

The most common way to approach the problem today is to approximate the derivatives in space and time with finite-difference ratios.we should mention, however, that there is a trend toward use of orthogonal functions, such as spherical harmonics, for estimating the partial derivatives in space.

In order to be concrete in illustrating the computations, and still keep the equations and discussion within reasonable bounds, we will here consider a simplified set. The equations (2.1) are thus reduced to the shallow water equations. As written below, the terms containing the effects of very small scale, as well as heat and evaporation, are omitted

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0 \qquad (2.2a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0 \qquad (2.2b)$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (2.2c)$$

where u and v are the vertically mass-averaged horizontal velocity components, and h is a scalar whose gradient times g is the vertically mass-averaged pressure force . In practice, h is reduced by a factor of about 4, to bring the behavior of planetary waves into correspondence with those of the atmosphere.The wind components are interpreted as u and v on the 50 kPa isobaric surface,and h is interpreted as the height of that surface. The 50 kPa surface is roughly at the half-mass level; about half the mass of the atmosphere is above it, and about half below it the set (2) of Equations is referred to by meteorologists as the barotropic model. Considering their simplicity, theygive a remarkably good prediction of the wind field at about 5 1/2 km, from which valuable inferences about surface weather can be made by experienced meteorologists. The first skillful numerical weather predictions were made during the 1950' s with such a model.

In practice, u, v, and h are given initially on a regular square grid of points in x and y, and their subsequent values are predicted on the same array. Let Δ be the spacing between grid points, the same in each dimension, x and y. Let Δt be the time step used to approximate the partial derivatives in time. Let i, j, k be the serial numbers of points in the x, y, t-dimensions, respectively.That is, $x = i\Delta$, $y = j\Delta$, $t = k\Delta t$. For convenience and economy in writing we introduce a symbolism for finite-difference ratios:

$$u_{2\mathrm{t}} = \frac{u_{\mathrm{i,j,k+1}} - u_{\mathrm{i,j,k-1}}}{2\Delta \mathrm{t}} \cong \frac{\partial u}{\partial \mathrm{t}} \qquad (2.3a)$$

$$u_{2x} = \frac{u_{\mathrm{i+1,j,k}} - u_{\mathrm{i-1,j,k}}}{2\Delta x} \cong \frac{\partial u}{\partial x} \qquad (2.3b)$$

$$u_{2y} = \frac{u_{\mathrm{i,j+1,k}} - u_{\mathrm{i,j-1,k}}}{2\Delta y} \cong \frac{\partial u}{\partial y} \qquad (2.3c)$$

and similarly for the dependent variables v and h. Replacing the derivatives in equations (2.2) with these approximations, we get

$$u_{2\mathrm{t}} + uu_{2x} + vu_{2x} - vf + gh_{2x} = 0 \qquad (2.4a)$$

$$v_{2t} + uv_{2x} + vv_{2y} + fu + gh_{2y} = 0 \quad (2.4b)$$

$$h_{2t} + uh_{2x} + vh_{2y} + h(u_{2x} + u_{2y}) = 0 \quad (2.4c)$$

We note that in the equations (2.4) with the symbols defined as in equations (2.3) all of the terms except the first can be evaluated with values of u ,v , h at the current time step, k. We can therefore easily find u_{2t}, v_{2t}, h_{2t} and knowing the valuesattime step $k - i$, we can find u, v, h at time step $k + l$. We then march forward in time, repeating the process until the desired forecast period is reached. The approximations (2.3) are called centered differences, not involving values at i, j, k Because of the centered differences in time, at the start of the prediction, fields of the dependent variables are required at two time steps ($k = 0$ and $k = l$). Common practice is to input values for time step zero and generate the values for time step one by replacing u_{2t} with

$$\frac{u_{i,j,1} - u_{i,j,0}}{\Delta t}$$

and similarly for v and h. The steps in time cannot be repeated using such approximations, however, because the resulting computational system would be unstable, with exponential growth of energy. This can easily be shown by linear analysis.

Linear analysis also shows that Δt cannot be chosen arbitrarily, but must be related to Δ by

$$\Delta t < \Delta / c$$

otherwise exponential growth rates would be encountered in the solution. The quantity c is the magnitude of largest signal velocity, taken to be the sum of the wind speed and the speed of propagation of gravity waves,

$$c = \sqrt{u^2 + v^2} + \sqrt{gh}$$

For our application $\sqrt{gh} \cong 140\text{ms}^{-1}$ and $\sqrt{u^2 + v^2} < 100\text{ms}^{-1}$, so that

$$c < 240\ \text{ms}^{-1}$$

The leading terms in the equations of motion (2.la) and (2.lb) tend actually to be smaller than the coriolis and pressure force.Thus,

$$v \cong \frac{1}{f\rho}\frac{\partial p}{\partial x}$$

$$u \cong \frac{1}{f\rho}\frac{\partial p}{\partial y}$$

The vector thus defined is called the geostrophic wind.

For initial conditions, independent analyses of wind and pressure fields not only must be quasi geotropically related, but also should exhibit a rather delicate, even subtle, balance with each other.Otherwise, large spurious oscillations of a gravitational nature would occur in the solution, and even more serious errors might result. The subtle balances involved can only be expressed in terms of solutions of differential equations.The balances are sufficiently subtle that errors in the observations mask them.This whole problem is called the initialization problem and is not fully solved yet.

Particularly in the tropical regions and in summer in the temperate zones, small scale vertical overturning results in mixing of momentum, heat, and moisture, which modifies the vertical structure of the atmosphere over large areas. The individual cells are far too small to be carried explicitly in the net of grid points, and their mechanics are at least as complicated as the large-scale systems. To account for their effect on the large-scale systems, they are parameterized.That is to say, attempts are made to relate them to the larger scale fields of variables that are explicitly carried. Radiation effects are also important, both the short-wave radiation received from the sun, and the long-wave radiation emitted by earth and atmosphere. the calculations are far too many for an operational model, which must meet rather strict deadlines for completion. Simpler versions of the radiation equation must suffice for operational models.

Radiation effects should account for the high albedo of clouds and snow and ice cover as well as the more constant variations of albedo over continents and oceans.It is a tricky business to design an abbreviated version of the radiation calculations in order to get as much benefit as possible with as little calculation as possible.

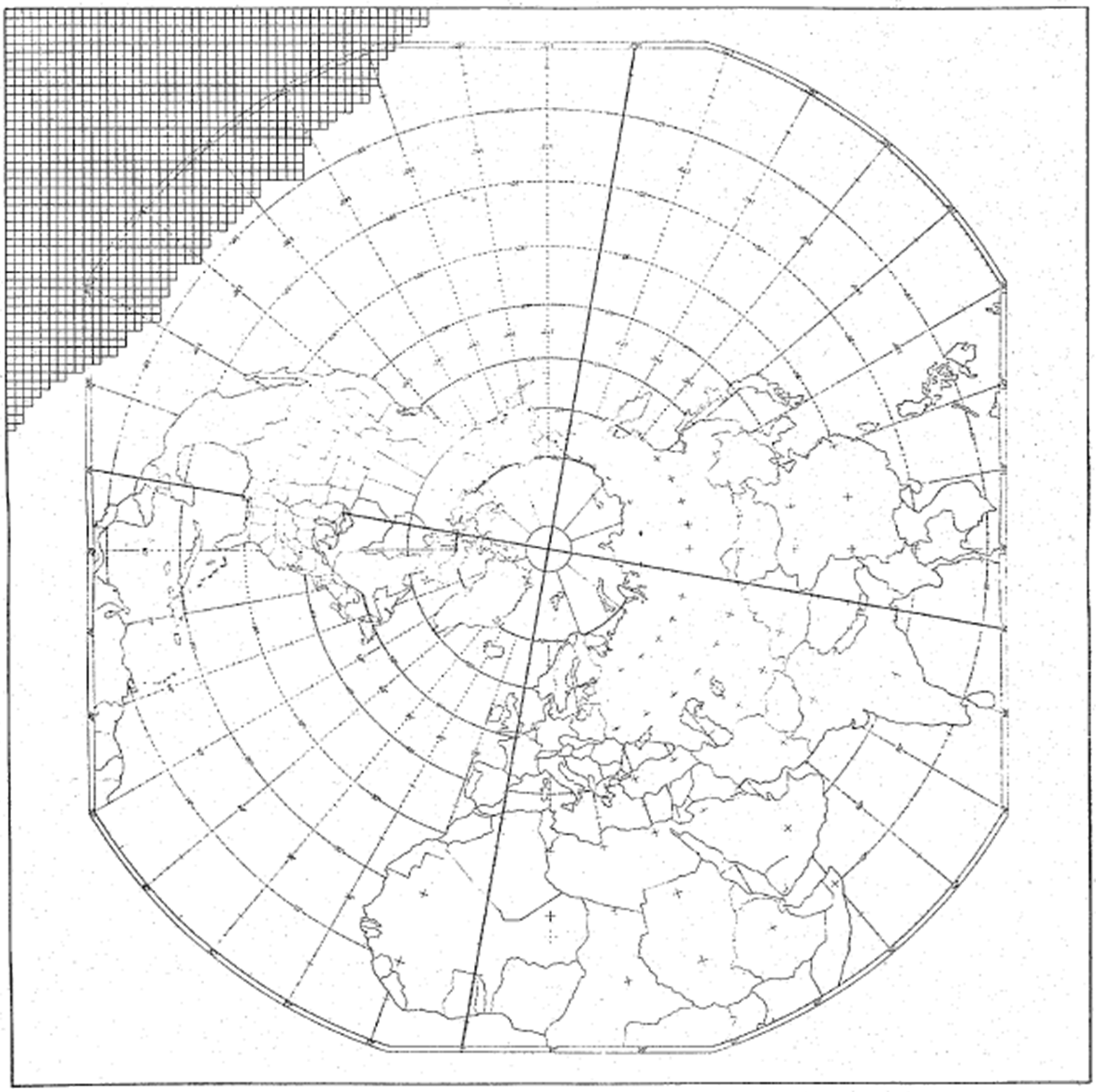


Figure (2.1)

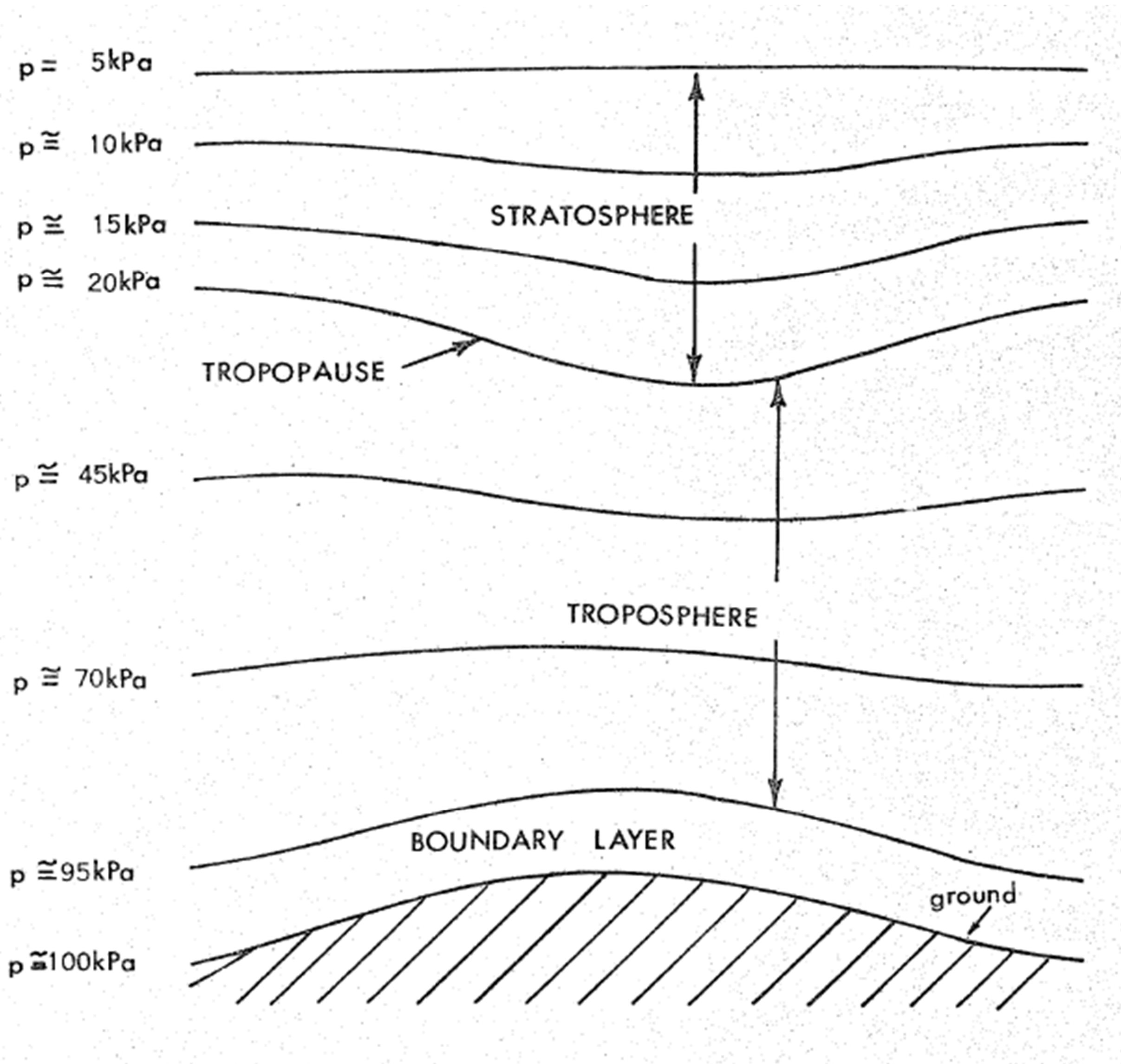


Figure (2.2)

Section (2.2): Meteorology and Weather Prediction

In this section we deal with the impact of mathematics on meteorology and weather prediction, and we start with some simple model of weather. The shallow water equations on a domain of $\mathbb{R}^2$, with local Cartesian coordinates (x, y), and rotating with constant angular frequency $f/2$, are

$$\frac{Du}{Dt} - fv + g\frac{\partial h}{\partial x} = 0, \qquad \frac{Dv}{Dt} + fu + g\frac{\partial h}{\partial y} = 0, \tag{2.5}$$

$$\frac{Dh}{Dt} + h\nabla \cdot u = 0, \tag{2.6}$$

Where$t$denotes time, g is the acceleration due to gravity (a constant) and h(x, y, t) is the depth of the fluid.The horizontal velocity has two components u$(x, y, t) = (u, v)$, and the Lagrangian, or material, derivative is

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla\,. \tag{2.7}$$

The Lagrangian derivative represents the rate of change of a dependent variable as we "follow the flow"; i.e. it is a directional derivative along the trajectory of fluid particle.The derivative u$\cdot\nabla$ is often referred to as the advection, or transport, term. If any variable $A(X, t)$satisfies $DA\,/\,Dt = 0$, then we say that $A(X, t)$ is conserved in the Lagrangian sense. Such conservation laws are of fundamental importance in meteorology and oceanography. The so-called semi-geostrophic equations amount to following approximation to the shallow water equations

$$\frac{Du_g}{Dt} - fv + g\frac{\partial h}{\partial x} = 0, \qquad \frac{Dv_g}{Dt} + fu + g\frac{\partial h}{\partial y} = 0, \tag{2.8}$$

$$\frac{Dh}{Dt} + h\nabla \cdot u = 0, \tag{2.9}$$

Whereu_g and v_g are the two components of the geostrophic wind

$$g\nabla h = (fv_g\,, -fu_g)$$

The geostrophic flow is parallel to contours of constant h (in the context of the Navier-Stokes equations on a rotating domain, the geotropic flow is parallel to the isobars of constant pressure). The difference between the shallow water equations and the semi-geotropic equations is the replacement of the fluid velocity (u, v) with the geostrophic wind (u_g, v_g) in the Lagrangianderivatives of u and v, while leaving the derivative operator (2.7) and the continuity equation (2.9) unchanged. This is known as the geotropic momentum approximation, and it applies to flows in which the rate of change of momentum is much smaller than the Coriolis force. The adverting velocity (that is, the velocity appearing in the Lagrangian Derivative $u \cdot \nabla$) is not approximated in semi-geostrophic theory.

The incorporation of two velocity fields reects the fact that many atmospheric flows, such as jet streams and fronts, have two distinct length scales (for example, a weather front is a relatively sharp discontinuity between air masses, but fronts

extend to distances of the order of 1000km along the interface between air masses). In essence, the geostrophic momentum approximation tells us that it isimportant to represent the adverting velocity accurately, while the quantity being advected (e.g.the geostrophic wind) can be approximated.
The geostrophic momentum approximation is now well understood from the point of view of Hamiltonian mechanics: the canonical momentum is approximated, while the velocity $\dot{q}$ is unchanged.
Equations (2.8) and (2.9) conserve energy and the following form of potential vortices

$$q = \frac{1}{h}\left(f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{1}{f}\frac{\partial(u_g, v_g)}{\partial(x, y)}\right) \tag{2.10}$$

In atmospheric dynamics (with thermodynamics included), potential vortices is proportional to the vector dot product of vortices and stratification that, following the flow, can only be changed by diabatic or frictional processes.
Potential vorticity is a fundamental conceptfor understanding the generation of vorticity in cyclogenesis (the birth and development ofa cyclone), especially along the polar front, and in analyzing flow in the ocean. The use of potential vortices played a key role as a diagnostic in understanding the evolution of Hurricane Sandy.

The semi-geostrophic equations have played a major role in understanding the formation Of fronts (frontogenesis), and the properties of the equations that facilitate such studies arerevealed through a transformation of coordinates. Defining new coordinates (sometimes calledgeostrophic momentum coordinates)

$$X \equiv (X, Y) \equiv \left(x + \frac{v_g}{f}, y - \frac{u_g}{f}\right), \tag{2.11}$$

We find that (2.8) may be replaced by

$$\frac{DX}{Dt} = u_g \equiv (u_g, v_g) \tag{2.12}$$

Hence the motion in these transformed coordinates is exactly geostrophic. The Jacobian isproportional to the potential vorticity (cf. (2.10))

$$\frac{\partial(X, Y)}{\partial(x, y)} = \frac{hg}{f} \tag{2 .13}$$

The vector X may be expressed as the gradient of a scalar function P(x),

$$X = \left(\frac{\partial P}{\partial x}, \frac{\partial p}{\partial y}\right), \quad (2.14)$$

which, to within an arbitrary additive constant, is uniquely defined by

$$P(X,t) = \frac{1}{2}(x^2 + y^2) + \frac{gh(X,t)}{f^2} \quad (2.15)$$

We note, using $g\nabla h = \left(fv_g, -fu_g\right)$ together with (2.11) and (2.15), that (2.13) has the form of aMonge-Ampèere equation for P, given $q(x,y,t)$ and suitable boundary conditions

$$q = \frac{f}{h}\left(P_{xx}P_{yy} - P_{xy}^2\right) \quad (2.16)$$

When the Jacobian (2.13) is non-singular, we express x(X) by introducing a scalar function R(X):

$$X = \left(\frac{\partial R}{\partial X}, \frac{\partial R}{\partial Y}\right), \quad (2.17)$$

where R is given to within an additive constant by

$$R(X) = X \cdot X - P(X) \quad (2.18)$$

(time is a parameter in this transformation). Equation (2.14) is the expression for the Legendre transformation between R(X) and P(x).The semi-geostrophic equations can be integrated in time using the conservation of potential vorticity, expressed in geostrophic momentum coordinates. We write the reciprocal of thePotentialvorticity, q^{-1}, as

$$q(X,t)^{-1} \equiv \rho(X,t) = \frac{h(X,t)}{f}(R_{XX}R_{YY} - R_{XY}^2). \quad (2.19)$$

This may be expressed solely in terms of the geostrophic momentum coordinates by defining $\emptyset(X,t) = \frac{gh(X,t)}{f^2}$ and $\emptyset(X,t) = \frac{1}{2}(X^2 + Y^2) - R(X,t)$, then we note

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \emptyset}{\partial x} = X - x, \frac{\partial \Phi}{\partial Y} = \frac{\partial \emptyset}{\partial y} = Y - y \cdot$$

Hence(3.19) may be written

$$\rho(X,t) = \frac{f}{g}\left(\Phi - \frac{1}{2}(\Phi_X^2 + \Phi_Y^2)\right)\left|Hess\left(\frac{1}{2}(X^2 + Y^2) - \Phi\right)\right|,$$

where Hess(·) is the hessian matrix of the second derivatives of ·with respect to X,Y .We can show that ρ satisfies

$$\frac{\partial \rho}{\partial t} + \dot{X}\frac{\partial \rho}{\partial X} + \dot{Y}\frac{\partial \rho}{\partial Y} = 0 \tag{2.20}$$

Where

$$\dot{X} = f\left(\frac{\partial R}{\partial Y} - Y\right), Y = -f\left(\frac{\partial R}{\partial X} - X\right) \tag{2.21}$$

The integration begins by solving the dual Monge-Amp_ere equation (2.19) for R, and thenusing R in (2.21) to determine the advecting velocities, which are in turn used to update ρvia (2.20). Equations (2.21) can be expressed in Hamiltonian form:

$$\dot{X} = -f\frac{\partial \Phi}{\partial Y}, \qquad \dot{Y} = f\frac{\partial \Phi}{\partial X}.$$

The semi-geostrophic equations involve two important types of geometry: symplecticgeometry associated with the Hamiltonian structure, and contact geometry associated with theLegendre transformation. These two geometries also play a fundamental role in the theory of the Monge-Ampère equation and, in turn, relate to complex geometries, such as Käahler geometry. The transformation properties of the semi-geostrophic equations can be understood within the context of hyper-Käahler geometry, and this geometry has been exploited in the study of a much broader class of models of cyclones .The emergence of a complex structure, which at first sight is somewhat surprising givenThe classical nature of the fluid mechanics, occurs when the Monge-Ampère equation (2.16) is elliptic. This equation is elliptic when q > 0. The ellipticity is related to a convexity condition On the energy and to the stability of the flow . The total energy, which is conserved by the shallow water equations (2.5), (2.6), is

$$E = \int \left(\frac{1}{2}(u^2 + v^2) + \frac{1}{2}gh\right)hdxdy\,. \tag{2.22}$$

This is a functional of u; v and h, and the conditions for E to be minimized corresponds to the stability of a geostrophic flow, viewed as a solution of the unapproximated equations, to perturbations of the form

$$\delta u = f\delta y\,, \qquad \delta v = -f\delta x\,, \qquad \nabla\cdot(\delta x,\delta y) = 0. \qquad (2.23)$$

The second variation of E is greater than zero when the Hessian matrix of the Legendre function P is positive. This corresponds to the ellipticity of theMonge-Ampère equation (2.16).Introducing momentum coordinates, (X',Y'),via $(u,v) \equiv f(y-Y',X'-x)$, wecan show that the perturbations (2.23) imply $\delta\sigma = 0$, wher

$$\sigma = h\frac{\partial(x,y)}{\partial(X',Y')}\,. \qquad (2.24)$$

If we define a distance $d(X,X')$ between x and X' such that

$$d^2 = f^2((X'-x)^2 + (Y'-y)^2),$$

then the energy functional can be rewritten

$$E = \int (\frac{1}{2}d^2(X,X') + \frac{1}{2}gh)hdxdy\,. \qquad (2.25)$$

The proof that E can be uniquely minimized is established by showing that, given σ as a non-negative function of the momentum coordinates, there is a unique mapping from (X',Y') to (x,y) that minimizes E and satisfies (2.24).This is the starting point for expressing the energy minimization as a Monge optimal mass transport problem. Utilizing theorems on the regularity of solutions of the Monge-Ampère equation, together with optimal mass transport theory, it has been shown that the semi-geostrophic equations can be integrated for large times from suitable initial data.

Section (2.3): The Physics of the Atmosphere

Let's consider the momentum equation for a unit volume of fluid whose density is ρ and moving with speed v on a slowly and uniformly rotating planet.

$$\frac{dv}{dt} = -2\Omega \times v - \frac{1}{\rho} + g + v\Delta v \qquad (2.26)$$

Here $\frac{d}{dt}$ denotes the total derivative that is also named material derivative, or substantialderivative, or Lagrangian derivative. Some author refers to itas convective derivative; anyway, since the adjective convective means a typical processfor the energy transfer, that is common in the lower troposphere, it is suggested to avoidthis last name.Ω is the planet angular velocity vector, p is the pressure field and g isthe gravity acceleration vector. Viscosity is expressed in the last term of the (2.26). Forthe standard atmosphere conditions at the sea level v is about $1.510^{-5} m^2 s^{-1}$and for the atmosphere from the bottom up to 100 km it is so small that it can be considerednegligible in comparison with the other terms of the equation (2.26), a part from a very thinlayer of about few centimeters close to the Earth surface, where thevertical wind shear isvery large. In that layer the momentum is transferred mainly by turbulent motions and,within it, important processes are responsible for the exchange of properties between theatmosphere and the ground, soil or water. Centrifugal acceleration $(\Omega \times \Omega \times r)$due to planet rotation has been included in the Gravity term and it identifies the vertical coordinate of the reference frame where themomentum equation is studied. The vector is assumed to be constant and r is thedistance vector of the air parcel with respect to the Earth center. It is worth to note that the equation (2.26) is a partial deferentialequation that in caseof incompressible homogeneous frictionless fluid and without Earth rotation becomes aPoisson equation for pressure field. This means that the pressure is a nonlocal functionof the flow configuration . This is the basic results of the Eulerequations. The Euler equations are of the hyperbolic form .If the friction term is added then the equations become of the parabolic form The addition of the Coriolis component complicates further the equations that make the solution more difficult to achieve.

An important concept adopted to describe the atmospheric systems is the mass con-servation which is expressed by the continuity equation.

$$\frac{d\rho}{dt} + \rho \nabla v = 0 \tag{2.27}$$

In this form the continuity equation states that the density of a moving parcel of fluidchanges if its volume varies and there are no sources or sinks of mass. Furthermore the equation of state of the air parcel links the thermodynamic variables ofthe elementary air parcel. Composing the atmosphere,at all the temperatures and pressure conditions which it experiences on our planet is verywell approximated by the ideal-gas model.

$$p = \rho R T \tag{2.28}$$

R is the gas constant and T is the gas absolute temperature.
Energy conservation is a principle that gives an additional constrain to the atmosphericsystems state and evolution. Combining the first thermodynamic principle and the equation of state (2.28), to express the work done by the gas as a

function of pressure anddensity, for a unit mass of fluid the energy conservation becomes:

$$\frac{d\eta}{dt} = c_p \frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} \tag{2.29}$$

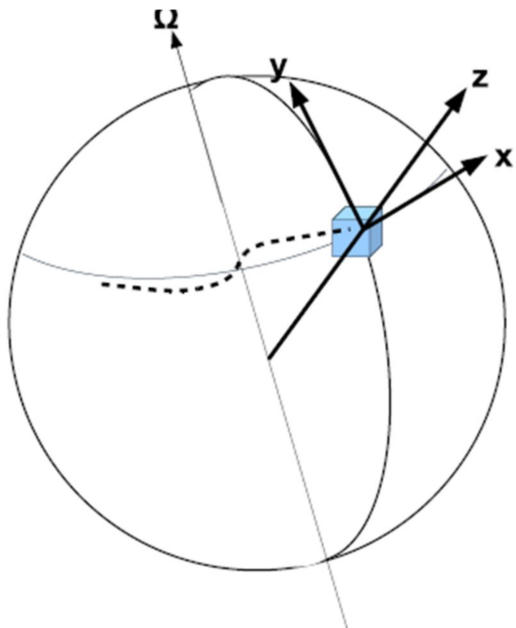


Figure (2.3). The reference frame moving with the air parcel. X and y axes are always placed onaplane parallel to the plane tangent to the Earth surface in the point where the vertical axis Z intersects the Earth surface. The y axis is always pointing towards the North Pole and the y axis is normal to the x one and pointing towards East.

Where$\frac{d\eta}{dt}$ represents the diabetic contribution and c_p is the specific heat at constantPressure, both per unit mass.The vectorial form of the momentum equation (2.26) links the causes of the motionto the acceleration and it is not the result of the application of anyspecific coordinate system.When a coordinate system is selected for the space, the equation (2.26) is equivalent to a set of three scalar equations, one for each of the vector components. To applythe second principle of dynamics, the Newton law, it is suitable to use a reference frame moving with the air parcel, see figure (2.3) , where the x and the y axes are always placed ona plane that is parallel to the plane tangent to the Earth surface in the point where therow containing the vertical coordinate axis z intersects the Earth surface. Furthermorethe y axis is always pointing towards the geographic North Pole and the x axis is normalto the y one and it is pointing towards East. As a consequence the velocity vector v has the scalar components (u, v, w) respectively belonging to the (x, y, z) coordinates.

In scalar form, in the reference frame described in figure (3.3) reporting the latitude of the parcel ∅ and its distance R from the origin of the coordinate system, the momentumequations become:

$$\frac{du}{dt} - \frac{uv\,tan(\emptyset)}{R} + \frac{uw}{R} = 2\Omega v\,sin(\emptyset)$$

$$-2\Omega w\,cos(\emptyset) - \frac{1}{\rho}\frac{\partial p}{\partial x} + (v\Delta v)_x \qquad (2.30)$$

$$\frac{dv}{dt} + \frac{u^2 \tan(\emptyset)}{R} + \frac{vw}{R} = -2\Omega u \sin(\emptyset) - \frac{1}{\rho}\frac{\partial p}{\partial y} + (v\Delta v)_y \qquad (2.31)$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{R} = 2\Omega u \cos(\emptyset) - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + (v\Delta v)_z \qquad (2.32)$$

The Tate equation:

$$p = \rho RT \qquad (2.33)$$

The continuity equation:

$$\frac{d\rho}{dt} + \rho\nabla\cdot v = 0 \qquad (2.34)$$

The energy conservation:

$$\frac{\partial\eta}{\partial t} = C_p\frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} \qquad (2.35)$$

Completes the set of equations which becomes closed and in principle can be solved forthe six unknown's functions u, v, w, p, ρ and T. Note that the scalar equations (2.30), (2.31), (2.32) present the curvature terms in their left side. These terms arise from the acceleration produced by the rotation of the reference frame which has been adopted forthe description of the system. Usually these terms are some order of magnitude less thanthe other contributions involved in the equations, so in simple analytical models they areconsidered negligible.Anyway for reliable numerical model of the atmosphere they areincluded in the solution schemes. It is also worth to note that the Coriolis acceleration acts along the vertical axis too,Look at the term $2\Omega u \cos(\emptyset)$ in equation (2.32). Its contribution makes a bodies movingtowards east lighter and those moving toward west heavier than the bodies at rest, see figure (2.4).

When constituents of the air parcel undergoes a variation of their concentration, for example the water vapor that escapes from the parcel volume because of the condensationin droplets which precipitate, or inversely there is a vapor increase due to the evaporationof liquid water, then sinks and sources of the constituents have to be considered in equation (2.34).

To deal with this process, a new variable is introduced , the specifichumidity q for example, and a new equation is added to the fundamental set to complete the system. The equation (2.34) still maintains its validity for the conserved constituents, while for the q the equation becomes:

$$\frac{d(q\rho)}{dt} + (q\rho)\nabla \cdot v = s \tag{2.36}$$

Where s is the source or the sink on non conserved constituents.

There are some other basic equations that are useful in the physics of the atmosphere.In particular the radiative transfer equation:

$$\frac{dI_v}{d\tau_v} = -I_v + J_v \tag{2.37}$$

where I_v is the specific radiation intensity, at frequency v,J_v is the source function and$\tau_v$is the optical depth. In case in the elementary fluid parcel there are charged particles and the fluid movesin an area of the space affected by electromagnetic fields, then the Lorentz force has tobe added to equation (2.26).

$$f = e\left(E + \frac{v}{c} \times B\right) \tag{2.38}$$

Where e is the charge of the particle moving with speed $v$in the electric field E and ina magneticfield B; c is the speed of light.

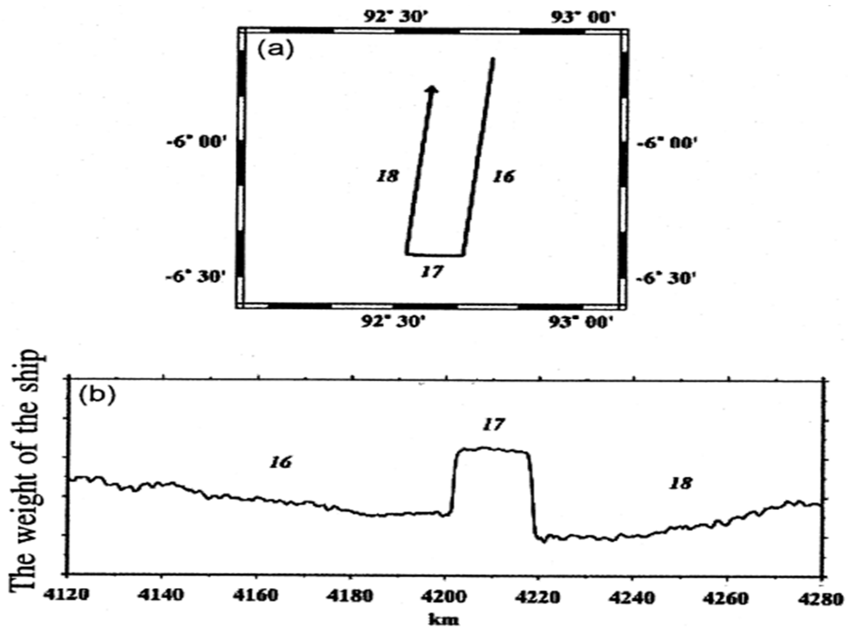


Figure (2.4). Measurements of gravity on a ship sailing in the Pacificocean. The route is reported (a) On a longitude-latitude box, while the gravity data are plotted as a function of the distance Covered by the ship (b) in arbitrary units. The parts of the route parallel to the meridians, 16 and 18 in (a) show a light gravity with respect to the measurements made during the westward Part of the route, 17, due to the Coriolis contribution to the acceleration. Figure taken from Persson(2001).

Chapter (3)

An Application of Basic Equations and Meteorological Modeling

Section(3.1):ElementaryApplication Equations

In this section we deal with the basic equations in isobaric coordinates, and we start with the horizontal momentum equation .

The approximate horizontal momentum Equations:

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} = f(v - v_g)$$

and

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} = -f(u - u_g)$$

may be written in vectorial form as

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times \mathbf{V} = -\frac{1}{\rho}\boldsymbol{\nabla}_{\boldsymbol{p}} \tag{3.1}$$

where $\mathbf{V} = iu + jv$ is the horizontal velocity vector. In order to express Equation (3.1) in isobaric coordinate form, we transform the pressure gradient force using Equations(1.20) and (1.21) to obtain

$$\frac{D\mathbf{V}}{Dt} + f\mathbf{k} \times V = -\nabla_p \Phi \tag{3.2}$$

where $\boldsymbol{\nabla}_p$ is the horizontal gradient operator applied with pressure held constant. Because p is the independent vertical coordinate, we must expand the total derivative as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p}$$

$$= \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{3.3}$$

Here $\omega \equiv Dp/Dt$ (usually called the "omega" vertical motion) is the pressure change following the motion, which plays the same role in the isobaric coordinate system that $w \equiv Dz/Dt$ plays in height coordinates.
From Equation (3.2) we see that the isobaric coordinate form of the geostrophic relationship is

$$f\,\mathbf{V}_g = \mathbf{k} \times \nabla_p \Phi \tag{3.4}$$

One advantage of isobaric coordinates is easily seen by comparing Equation

$$\mathbf{V}_g \equiv \mathbf{k} \times \frac{1}{\rho f} \nabla p$$

and Equation (3.4). In the latter equation, density does not appear. Thus, a given geopotential gradient implies the same geostrophic wind at any height, whereas a given horizontal pressure gradient implies different values of the geostrophic wind depending on the density. Furthermore, if f is regarded as a constant, the horizontal divergence of the geostrophic wind at constant pressure is zero:

$$\nabla_p \cdot \mathbf{V}_g = 0$$

Now we discussThe Continuity Equation, It is possible to transform the continuity Equation (2.31) from height coordinates to pressure coordinates. However, it is simpler to directly derive the isobaric form by considering a Lagrangian control volume $\delta V = \delta x\,\delta y\,\delta z$ and applying the hydrostatic equation $\delta p = -\rho g \delta z$ (note that $\delta p < 0$) to express the volume element as $\delta V = -\delta x \delta y \delta p/(\rho g)$. The mass of this fluid element, which is conserved following the motion, is then $\delta M = \rho \delta V = -\delta x \delta y \delta p/g$. Thus,

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0$$

After differentiating, using the chain rule, and changing the order of the differential operators we obtain

$$\frac{1}{\delta x}\delta\left(\frac{Dx}{Dt}\right) + \frac{1}{\delta y}\delta\left(\frac{Dy}{Dt}\right) + \frac{1}{\delta p}\delta\left(\frac{Dp}{Dt}\right) = 0$$

Or

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

Taking the $limit\ \delta x, \delta y, \delta p \rightarrow 0$ and observing that δx and δy are evaluated at constant pressure, we obtain the continuity Equation in the isobaric system:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0 \qquad (3.5)$$

This form of the continuity equation contains no reference to the density field and does not involve time derivatives. The simplicity of (3.5) is one of the chief advantages of the isobaric coordinate system.

In the following we present The Thermodynamic Energy Equation .The first law of thermodynamics Equation(1.41) can be expressed in the isobaric system by letting $Dp/Dt = \omega$ and expanding DT/Dt by using Equation (3.3):

$$C_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \alpha\omega = J$$

This may be rewritten as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_P\omega = \frac{J}{C_P} \qquad (3.6)$$

where, with the aid of the equation of state and Poisson's Equation (2.43), we have

$$S_P \equiv \frac{RT}{C_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p} \qquad (3.7)$$

which is the static stability parameter for the isobaric system. Using Equation (1.48) and the hydrostatic Equation (3.7) may be rewritten as

$$S_p = (\Gamma d - \Gamma)/\rho g$$

Thus, S_p is positive provided that the lapse rate is less than dry adiabatic. However, because density decreases approximately exponentially with height, S_p

increases rapidly with height. This strong height dependence of the stability measure S_p is a minor disadvantage of isobaric coordinates.

Now we study balanced flow Despite the apparent complexity of atmospheric motion systems as depicted on synoptic weather charts, the pressure (or geopotential height) and velocity distributions in meteorological disturbances are actually related by rather simple approximate force balances. In order to gain a qualitative understanding of the horizontal balance of forces in atmospheric motions, we idealize by considering flows that are steady state (i.e., time independent) and have no vertical component of velocity. Furthermore, it is useful to describe the flow field by expanding the isobaric form of the horizontal momentum Equation (3.2) into its components in a so-called natural coordinate system.

Now we discuss the Natural Coordinates The natural coordinate system is defined by the orthogonal set of unit vectors $\mathbf{t}$, **n**, and k. Unit vector t is oriented parallel to the horizontal velocity at each point; unit vector n is normal to the horizontal velocity and is oriented so that it is positive to the left of the flow direction; and unit vector **k** is directed vertically upward. In this system the horizontal velocity may be written $\mathbf{V} = V\,\mathbf{t}$ where , the horizontal speed, is a nonnegative scalar defined by $V \equiv Ds/Dt$, where $s(x, y, t)$ is the distance along the curve followed by a parcel moving in the horizontal plane. The acceleration following the motion is thus

$$\frac{D\mathbf{V}}{Dt} = \frac{D(V\mathbf{t})}{Dt} = \mathbf{t}\frac{DV}{Dt} + V\frac{D\mathbf{t}}{Dt}$$

The rate of change of t following the motion may be derived from geometrical considerations with the aid of Fig. (3.1):

$$\delta\psi = \frac{\delta S}{|R|} = \frac{|\delta\mathbf{t}|}{|\mathbf{t}|} = |\delta\mathbf{t}|$$

Here R is the *radius of curvature* following the parcel motion, and we have used the fact that $|\boldsymbol{t}| = 1$. By convention, R is taken to be positive when the center of curvature is in the positive **n** direction. Thus, for$R > 0$, the air parcels turn toward the left following the motion, and for $R < 0$ the air parcels turn toward the right following the motion.

Noting that in the limit $\delta s \to 0$, δt is directed parallel to **n**, the above relationship
yields $D\mathbf{t}/Ds = \mathbf{n}/R$. Thus,

$$\frac{D\boldsymbol{t}}{Dt} = \frac{D\boldsymbol{t}}{Ds}\frac{Ds}{Dt} = \frac{\mathbf{n}}{R}V$$

and

$$\frac{D\mathbf{V}}{Dt} = \boldsymbol{t}\frac{DV}{Dt} + \mathbf{n}\frac{V^2}{R} \tag{3.8}$$

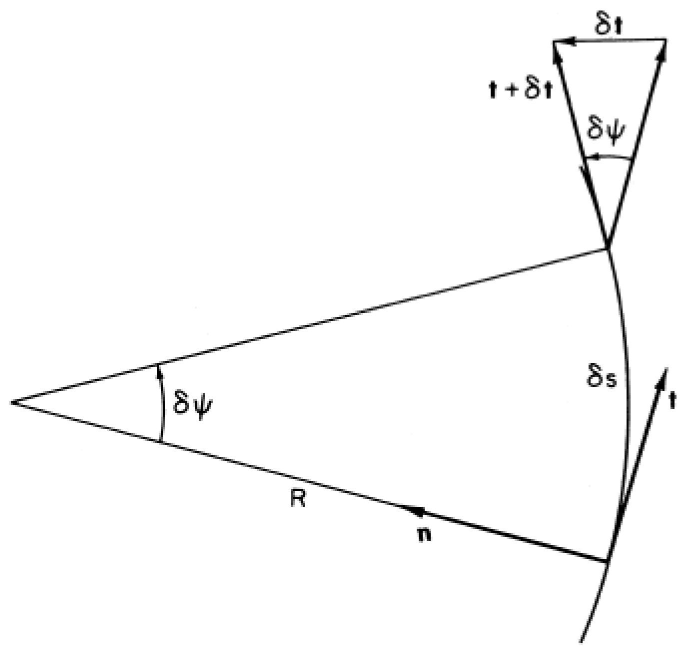


Fig. (3.1): Rate of change of the unit tangent vector **t** following the motion

Therefore, the acceleration following the motion is the sum of the rate of change of speed of the air parcel and its centripetal acceleration due to the curvature of the trajectory. Because the Coriolis force always acts normal to the direction of motion, its natural coordinate form is simply

$$-f\mathbf{k} \times \mathbf{V} = -fV\mathbf{n}$$

whereas the pressure gradient force can be expressed as

$$-\boldsymbol{\nabla}_p\Phi = \mathbf{t}\frac{\partial\Phi}{\partial s} + \mathbf{n}\frac{\partial\Phi}{\partial n}$$

The horizontal momentum equation may thus be expanded into the following component equations in the natural coordinate system:

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s} \tag{3.9}$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \tag{3.10}$$

Equations (3.9) and (3.10) express the force balances parallel to and normal to the direction of flow, respectively. For motion parallel to the geopotential height contours, $\partial\Phi/\partial s = 0$ and the speed is constant following the motion. If, in addition, the geopotential gradient normal to the direction of motion is constant alonga trajectory, (3.10) implies that the radius of curvature of the trajectory is also constant. In that case the flow can be classified into several simple categories depending on the relative contributions of the three terms in (3.10) to the net force balance.

Now we study Geostrophic Flow, Flow in a straight line ($R \to \pm\infty$) parallel to height contours is referred to as geostrophic motion. In geostrophic motion the horizontal components of the Coriolis force and pressure gradient force are in exact balance so that $V=V_g$ where the geostrophic wind V_g is defined by

$$fV_g = -\partial\Phi/\partial n \tag{3.11}$$

This balance is indicated schematically in Fig. 3.2. The actual wind can be in exact geostrophic motion only if the height contours are parallel to latitude circles. the geostrophic wind is generally a good approximation to the actual wind in extratropical synoptic-scale disturbances. However,in some of the special cases treated later this is not true.

Now we study the Inertial Flow, If the geopotential field is uniform on an isobaric surface so that the horizontal pressure gradient vanishes, (3.10) reduces to a balance between Coriolis force and centrifugal force:

$$V^2/R + fV = 0 \tag{3.12}$$

Equation (3.12) may be solved for the radius of curvature

$$R = -V/f$$

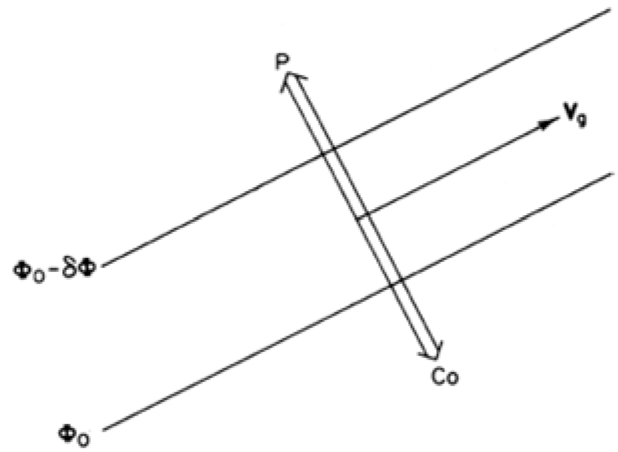


Fig. (3.2): Balance of forces for geostrophic equilibrium. The pressure gradient force is designated by P and the Coriolis force by C_0

Since from(3.9), the speed must be constant in this case, the radius of curvature is also constant (neglecting the latitudinal dependence of f). Thus the air parcels follow circular paths in an anticyclonic sense.3 The period of this oscillation is

$$P = \left|\frac{2\pi}{V}\right| = \frac{2\pi}{|f|} = \frac{\frac{1}{2}\, day}{|\sin \emptyset|} \qquad (3.13)$$

P is equivalent to the time that is required for a Foucault pendulum to turn through an angle of 180◦. Hence, it is often referred to as one-half pendulum *day*. Because both the Coriolis force and the centrifugal force due to the relative motion are caused by inertia of the fluid, this type of motion is traditionally referredto as an inertial oscillation, and the circle of radius $|R|$ is called the inertia circle.It is important to realize that the “inertial flow” governed by Equation (3.12) is not the sameas inertial motion in an absolute reference frame. Inthis flow the force of gravity, acting orthogonal to the plane of motion,keeps theoscillation on a horizontal surface. In true inertial motion, all forces vanish andthe motion maintains a uniform absolute velocity.

In the atmosphere, motions are nearly always generated and maintained by pressure gradient forces; the conditions of uniform pressure required for pure inertial flow rarely exist. In the oceans, however, currents are often generated by transient winds blowing across the surface, rather than by internal pressure gradients. As a result, significant amounts of energy occur in currents that oscillate with near inertial periods. An example recorded by a current meter near the island of Barbados is shown in Fig. 3.3.

In the following we illusterate the Cyclostrophic Flow, If the horizontal scale of a disturbance is small enough, the Coriolis force may be neglected in (3.10) compared to the pressure gradient force and the centrifugal force. The force balance normal to the direction of flow is then

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}$$

If this equation is solved for , we obtain the speed of the cyclostrophic wind

$$V = \left(-R\frac{\partial \Phi}{\partial n}\right)^{1/2} \quad (3.14)$$

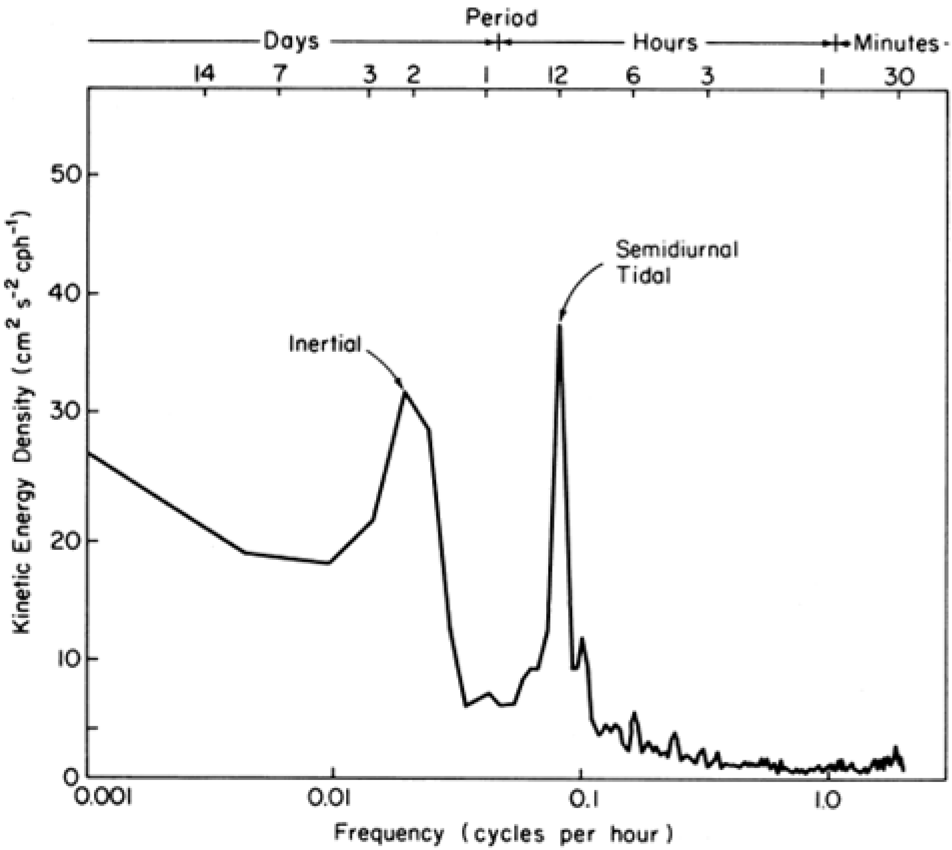
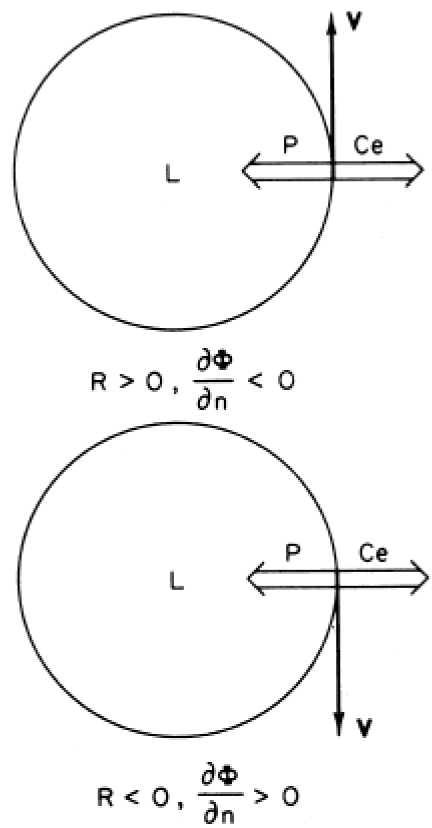


Fig. (3.3): Power spectrum of kinetic energy at 30-m depth in the ocean near Barbados (13°N). Ordinate shows kinetic energy density per unit frequency interval (cph^{-1}designates cycles per hour). This type of plot indicates the manner in which the total kinetic energy is partitioned among oscillations of different periods. Note the strong peak at 53 h, which is the period of an inertial oscillation at 13° latitude. [After Warsh et al., (1971.) Reproduced with permission of the American Meteorological Society.]

As indicated in Fig.(3.4), cyclostrophic flow may be either cyclonicor anticyclonic. In both cases the pressure gradient force is directed toward the center of curvature, and the centrifugal force away from the center of curvature.The cyclostrophic balance approximation is valid provided that the ratio of the centrifugal force to the Coriolis force is large. This ratio $V/(fR)$ is equivalent to the Rossby number discussed in Chapter one. As an example of cyclostrophic scale motion we consider a typical tornado. Suppose that the tangential velocity is 30 m s^{-1} at a distance of 300 m from the center of the vortex. Assuming that $f = 10^{-4}$ s^{-1}, the Rossby number is just Ro = $V/|fR| \approx 10^3$, which implies that the Coriolis force can be neglected in computing the balance of forces for a tornado. However, the majority of tornadoes in the Northern Hemisphere are observed to rotate in a cyclonic (counterclockwise) sense. This is apparently because they are embedded in environments that favor cyclonic rotation .
Smaller scale vortices, however, such as dust devils and water spouts, do not have

Fig. (3.4): Force balance in cyclostrophic flow: P designates the pressure gradient and C_e the centrifugal force.



a preferred direction of rotation. According to data collected by Sinclair (1965), they are observed to be anticyclonic as often as cyclonic.

Now we studyThe GradientWind Approximation,Horizontal frictionless flow that is parallel to the height contours so that the tangential acceleration vanishes $(DV/Dt = 0)$ is called gradient flow. Gradient flow is a three-way balance among the Coriolis force, the centrifugal force, and the horizontal pressure gradient force. Like geostrophic flow, pure gradient flow can exist only under very special circumstances. It is always possible, however, to define a gradient wind, which at any point is just the wind component parallel to the height contours that satisfies Equation (3.10). For this reason,Equation (3.10) is commonly referred to as the gradient wind equation. Because Equation (3.10) takes into account the centrifugal force due to the curvature of parcel trajectories, the gradient wind is often a better approximation to the actual wind than the geostrophic wind. The gradient wind speed is obtained by solving Equation (3.10) for V to yield

$$V = -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}\right)^{1/2}$$

$$= -\frac{fR}{2} \pm \left(\frac{f^2R^2}{4} + fRV_g\right)^{1/2} \quad (3.15)$$

where in the lower expression Equation (3.11) is used to express $\partial\Phi/\partial n$ in terms of the geostrophic wind.Not all the mathematically possible roots of Equation (3.15) correspond to physically possible solutions, as it is required that V be real and nonnegative. In Table (3.1) the various roots of Equation (3.15) are classified according to the signs of R and $\partial\Phi/\partial n$ in order to isolate the physically meaningful solutions The force balances for the four permitted solutions are illustrated in Fig. (3.5). Equation (3.15) shows that in cases of both regular and anomalous highs the pressure gradient is limited by the requirement that the quantity under the radical be nonnegative; that is,

$$|fV_g| = \left|\frac{\partial\Phi}{\partial n}\right| < \frac{|R|f^2}{4} \quad (3.16)$$

Thus, the pressure gradient in a high must approach zero as $|R| \rightarrow 0$. It is for this reason that the pressure field near the center of a high is always flat and the wind gentle compared to the region near the center of a low.The absolute angular momentum about the axis of rotation for the circularly symmetric motions shown in Fig. (3.5) is given by $VR + fR^2/2$. From Equation (3.15) it is verified readily that regular gradient wind balances have positive absolute angular momentum in the Northern Hemisphere, whereas anomalous cases have negative absolute angular momentum. Because the only source of negative absolute angular momentum is the Southern Hemisphere, the anomalous cases are unlikely to occur except perhaps close to the equator. In all cases except the anomalous low (Fig. (3.5c)) the horizontal components of the Coriolis and pressure gradient forces are oppositely directed. Such flow is

Table(3.1): Classification of Roots of the Gradient Wind Equation in the Northern Hemisphere

Sign $\partial\Phi/\partial n$	R > 0	R < 0
Positive ($V_g < 0$)	Positive root:[a] unphysical	Positive root: antibaric flow (anomalous low)
	Negative root: unphysical	Negative root: unphysical
Negative ($V_g > 0$)	Positive root: cyclonic flow (regular low)	Positive root: ($V > -fR/2$): anticyclonic flow (anomalous high)
	Negative root: unphysical	Negative root: ($V < -fR/2$): anticyclonic flow (regular high)

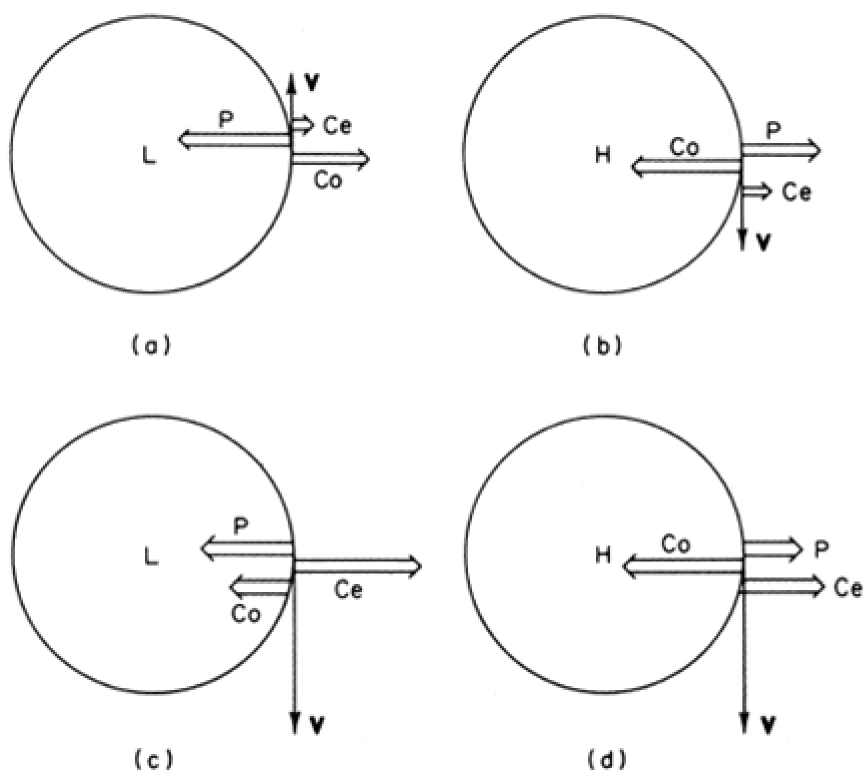


Fig. (3.5): Force balances in the Northern Hemisphere for the four types of gradient flow: (a) regular low (b) regular high (c) anomalous low (d) anomalous high.

called baric. The anomalous low is antibaric; the geostrophic wind V_g defined in Equation (3.11) is negative for an anomalous lowand is clearly not a useful approximation tothe actual speed.4 Furthermore, as shown in Fig. 3.5, gradient flow is cyclonic only when the centrifugal force and the horizontal component of the Coriolis force have the same sense $(Rf > 0)$; it is anticyclonic when these forces have the opposite sense $(Rf < 0)$. Since the direction of anticyclonic and cyclonic flow is reversed in the Southern Hemisphere, the requirement that $Rf > 0$ for cyclonic flow holds regardless of the hemisphere considered.

The definition of the geostrophic wind Equation (3.11) can be used to rewrite the force balance normal to the direction of flow (3.10) in the form

$$\frac{V^2}{R} + fV - fV_g = 0$$

Dividing through by fV shows that the ratio of the geostrophic wind to the gradient wind is

$$\frac{V_g}{V} = 1 + \frac{V}{fR} \tag{3.17}$$

For normal cyclonic flow $(fR > 0)$, V_g is larger than V , whereas for anticyclonic flow (fR < 0), V_g is smaller than V . Therefore, the geostrophic wind is an overestimate of the balanced wind in a region of cyclonic curvature and an underestimate in a region of anticyclonic curvature. For midlatitude synoptic systems, the difference between gradient and geostrophic wind speeds generally does not exceed 10–20%. [Note that the magnitude of $V/(fR)$ is just the Rossby number.] For tropical disturbances, the Rossby number is in the range of 1–10, and the gradient wind formula must be applied rather than the geostrophic wind. Equation (3.17) also shows that the antibaric anomalous low, which has $Vg < 0$, can exist only when $V/(fR) < -1$. Thus, antibaric flow is associated with small-scale intense vortices such as tornadoes.

Now we discuss the trajectories and streamlines ,In the natural coordinate system used in the previous section to discuss balanced flow, $s(x, y, t)$ was defined as the distance along the curve in the horizontal plane traced out by the path of an air parcel. The path followed by a particular air parcel over a finite period of time is called the trajectory of the parcel. Thus, the radius of curvature R of the path s referred to in the gradient wind equation is the radius of curvature for a parcel trajectory. In practice, R is often estimated by using the radius of curvature of a geopotential height contour, as this can be estimated easily from a synoptic chart.

However, the height contours are actually streamlines of the gradient wind (i.e., lines that are everywhere parallel to the instantaneous wind velocity).It is important to distinguish clearly between streamlines, which give a “snapshot” of the velocity field at any instant, and trajectories, which trace the motion of individual fluid parcels over a finite time interval. In Cartesian coordinates, horizontal trajectories are determined by the integration of

$$\frac{D_s}{D_t} = V_{(x,y,t)} \tag{3.18}$$

over a finite time span for each parcel to be followed, whereas streamlines are determined by the integration of

$$\frac{dy}{dx} = \frac{v(x, y, t_0)}{u(x, y, t_0)} \tag{3.19}$$

with respect to x at time t_0. (Note that since a streamline is parallel to the velocity field, its slope in the horizontal plane is just the ratio of the horizontal velocity components.) Only for steady-state motion fields (i.e., fields in which the local rate of change of velocity vanishes) do the streamlines and trajectories coincide. However, synoptic disturbances are not steady-state motions. They generally move at speeds of the same order as the winds that circulate about them. In order to gain an appreciation for the possible errors involved in using the curvature of

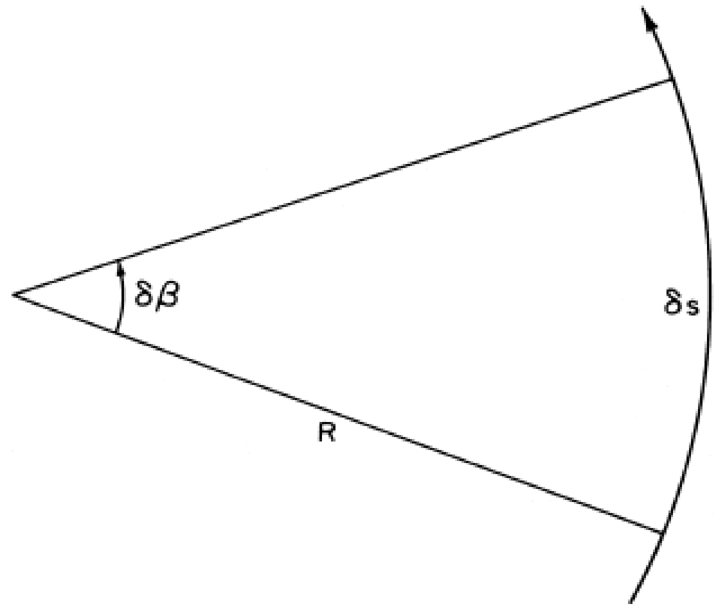


Fig. (3.6): Relationship between the change in angular direction of the wind δβ and the radius of curvature R.

the streamlines instead of the curvature of the trajectories in the gradient wind equation, it is necessary to investigate the relationship between the curvature of the trajectories and the curvature of the streamlines for a moving pressure system.

We let β(x, y, t) designate the angular direction of the wind at each point on an isobaric surface, and R_t and R_s designate the radii of curvature of the trajectories and streamlines, respectively. Then, from Fig. (3.6), $\delta s = R\,\delta\beta$ so that in the limit $\delta s \rightarrow 0$

$$\frac{D\beta}{Ds} = \frac{1}{R_t} \quad and \quad \frac{\partial\beta}{\partial s} = \frac{1}{R_S} \tag{3.20}$$

where $D\beta/Ds$ means the rate of change of wind direction along a trajectory (positivefor counterclockwise turning) and $\partial\beta/\partial s$ is the rate of change of wind directionalong a streamline at any instant. Thus, the rate of change of wind direction following the motion is

$$\frac{D\beta}{Dt} = \frac{d\beta}{Ds}\frac{Ds}{Dt} = \frac{V}{R_t} \tag{3.21}$$

or, after expanding the total derivative,

$$\frac{D\beta}{Dt} = \frac{\partial\beta}{\partial t} + V\frac{\partial\beta}{\partial s} + \frac{V}{R_S} \tag{3.22}$$

Combining Equation (3.21) and Equation (3.22), we obtain a formula for the local turning of the wind:

$$\frac{\partial\beta}{\partial t} = V\left(\frac{1}{R_t} - \frac{1}{R_s}\right) \tag{3.23}$$

Equation (3.23) indicates that the trajectories and streamlines will coincide only when the local rate of change of the wind direction vanishes.

In general, midlatitude synoptic systems move eastward as a result of advection by upper level westerly winds. In such cases there is a local turning of the wind due to the motion of the system even if the shape of the height contour pattern remains constant as the system moves. The relationship between R_t and $R_s$in such a situation can be determined easily for an idealized circular pattern of height contours moving at a constant velocity C. In this case the local turning of the wind is entirely due to the motion of the streamline pattern so that

$$\frac{\partial\beta}{\partial t} = -C\cdot\nabla\beta = -C\frac{\partial\beta}{\partial s}cos\gamma = -\frac{C}{R_S}cos\gamma$$

where γ is the angle between the streamlines (height contours) and the direction of motion of the system. Substituting the above into Equation (3.23) and solving for R_t with the aid of Equation(3.20), we obtain the desired relationship between the curvature of the streamlines and the curvature of the trajectories:

$$R_t = R_s\left(1 - \frac{cos\gamma}{V}\right)^{-1} \tag{3.24}$$

Equation (3.24) can be used to compute the curvature of the trajectory anywhere on a moving pattern of streamlines. In Fig.(3.7) the curvatures of the trajectories for parcels initially located due north, east, south, and west of the center of a cyclonic system are shown both for the case of a wind speed greater than the speed of movement of the height contours and for the case of a wind speed less than the speed ofmovement of the height contours. In these examples the plotted rajectories are based on a geostrophic balance so that the height contours are equivalent to streamlines. It is also assumed for simplicity that the wind speed does not depend on the distance from the center of the system. In the case shown in Fig. (3.7b) there is a region south of the low center where the curvature of the trajectories is opposite that of the streamlines. Because synoptic-scale pressure systems usually move at speeds comparable to the wind speed, the gradient wind speed computed on the basis of the curvature of the height contours is often no better an approximation to the actual wind speed than the geostrophic wind. In fact, the actual gradient wind speed will vary along a height contour with the variation of the trajectory curvature .

In the following we present the thermal wind The geostrophic wind must have vertical shear in the presence of a horizontal temperature gradient, as can be shown easily from simple physical considerations based on hydrostatic equilibrium. Since the geostrophic wind Equation (3.4) is proportional to the geopotential gradient on an isobaric surface, a geostrophic wind directed along the positive y axis that increases in magnitude with height requires that the

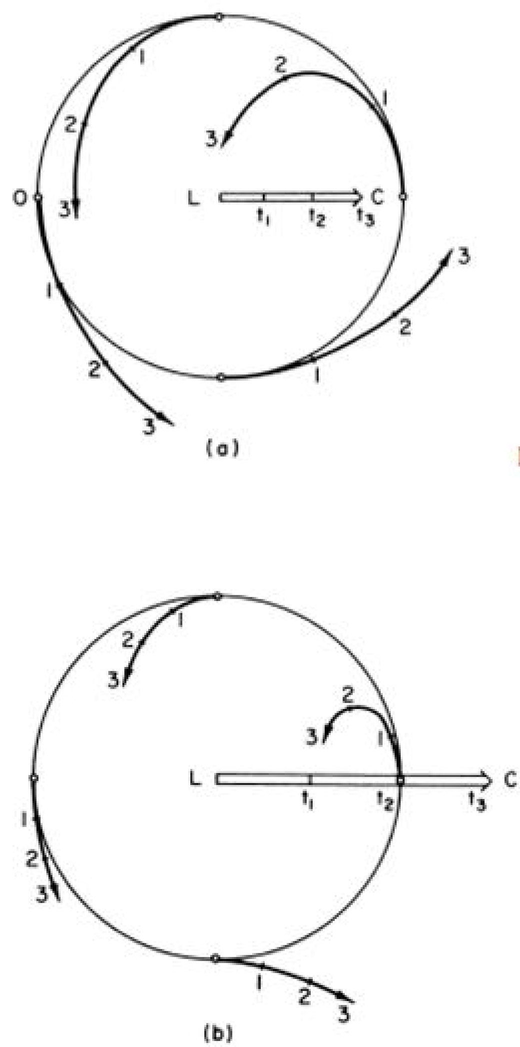


Fig. (3.7): Trajectories for moving circular cyclonic circulation systems in the Northern Hemisphere with (a) $V = 2C$ and (b) $2V = C$. Numbers indicate positions at successive times. The L designates a pressure minimum.

slope of the isobaric surfaces with respect to the x axis also must increase with height as shown in Fig.(3.8). According to the hypsometric Eq. (1.21), the thickness δz corresponding to a pressure interval δp is

$$\delta z \approx -g^{-1} R T \delta \ln p \qquad (3.25)$$

Thus, the thickness of the layer between two isobaric surfaces is proportional to the temperature in the layer. In Fig. (3.8) the mean temperature T_1 of the column denoted by δz_1 must be less than the mean temperature T_2 for the column denoted by δz_2 Hence, an increase with height of a positive x directed pressure gradient must be associated with a positive x directed temperature gradient. The air in a vertical column at x_2, because it is warmer (less dense), must occupy a greater depth for a given pressure drop than the air at x_1

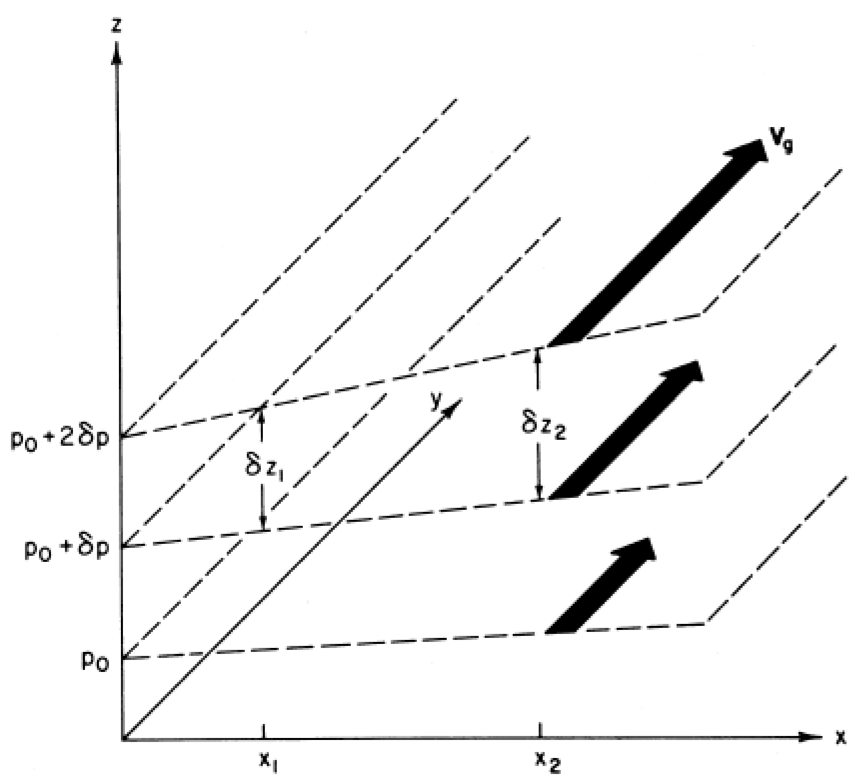


Fig. (3.8): Relationship between vertical shear of the geostrophic wind and horizontal thickness gradients. (Note that $\delta p < 0$.)

Equations for the rate of change with height of the geostrophic wind components are derived most easily using the isobaric coordinate system. In isobaric coordinates the geostrophic wind (3.4) has components given by

$$v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x} \quad \text{and} \quad u_g = \frac{1}{f}\frac{\partial \Phi}{\partial y} \tag{3.26}$$

where the derivatives are evaluated with pressure held constant. Also, with the aid of the ideal gas law we can write the hydrostatic equation as

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p} \tag{3.27}$$

Differentiating Equation (3.26) with respect to pressure and applying Equation (3.27), we obtain

$$p\frac{\partial v_g}{\partial p} \equiv \frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p \tag{3.28}$$

$$p\frac{\partial u_g}{\partial p} \equiv \frac{\partial u_g}{\partial \ln p} = \frac{T}{f}\left(\frac{\partial T}{\partial y}\right)_P \tag{3.29}$$

or in vectorial form

$$\frac{\partial \mathbf{V}_g}{\partial \ln p} = -\frac{R}{f}\mathbf{k} \times \nabla_p T \tag{3.30}$$

Equation (3.30) is often referred to as the thermal wind equation. However,it is actually a relationship for the vertical wind shear (i.e., the rate of change of the geostrophic wind with respect to ln p). Strictly speaking, the term thermal wind refers to the vector difference between geostrophic winds at two levels. Designatingthe thermal wind vector by V_T, we may integrate Equation(3.30) from pressure level P_0 to level $(P_1 < P_2)$ to get

$$V_T \equiv V_g(p_1) - V_g(p_0) = -\frac{R}{f}\int_{p_0}^{p_1}(k \times \nabla_p T)\, d\ln p \tag{3.31}$$

Letting $\langle T \rangle$denote the mean temperature in the layer between pressure $p_0$and p_1, the x and y components of the thermal wind are thus given by

$$u_T = -\frac{R}{f}\left(\frac{\partial\langle T\rangle}{\partial y}\right)_p \ln\left(\frac{p_0}{p_1}\right);\; v_T = \frac{R}{f}\left(\frac{\partial\langle T\rangle}{\partial x}\right)_p \ln\left(\frac{p_0}{p_1}\right) \tag{3.32}$$

Alternatively, we may express the thermal wind for a given layer in terms of the horizontal gradient of the geopotential difference between the top and the bottom of the layer:

$$u_T = -\frac{1}{f}\frac{\partial}{\partial y}(\Phi_1 - \Phi_0)\,; \qquad v_T = \frac{1}{f}\frac{\partial}{\partial x}(\Phi_1 - \Phi_0) \tag{3.33}$$

The equivalence of Equation (3.32) and Equation (3.33) can be verified readily by integrating thehydrostatic Equation (3.27) vertically from $p_0$to p_1 after replacing T by the mean $\langle T\rangle$The result is the hypsometric Equation (1.22):

$$\Phi_1 - \Phi_0 \equiv gZ_T = R\langle T\rangle \ln\frac{p_0}{p_1} \tag{3.34}$$

The quantity Z_T is the thickness of the layer betweenp_0 andp_1 measured in units of geopotential meters. From (3.34) we see that the thickness is proportional to the mean temperature in the layer. Hence, lines of equal Z_T(isolines of thickness) are equivalent to the isotherms of mean temperature in the layer.

The thermal wind equation is an extremely useful diagnostic tool, which is often used to check analyses of the observed wind and temperature fields for consistency. It can also be used to estimate the mean horizontal temperature advection in a layer as shown in Fig.(3.9). It is clear from the vector form of the thermal wind relation:

$$\mathbf{V}_T = \frac{1}{f}\mathbf{k}\times\nabla(\Phi_1 - \Phi_0) = \frac{g}{f}\mathbf{k}\times\nabla Z_T = \frac{R}{f}\mathbf{k}\times\nabla\langle T\rangle \ln\left(\frac{p_0}{p_1}\right) \tag{3.35}$$

that the thermal wind blows parallel to the isotherms (lines of constant thickness) with the warm air to the right facing downstream in the Northern Hemisphere.

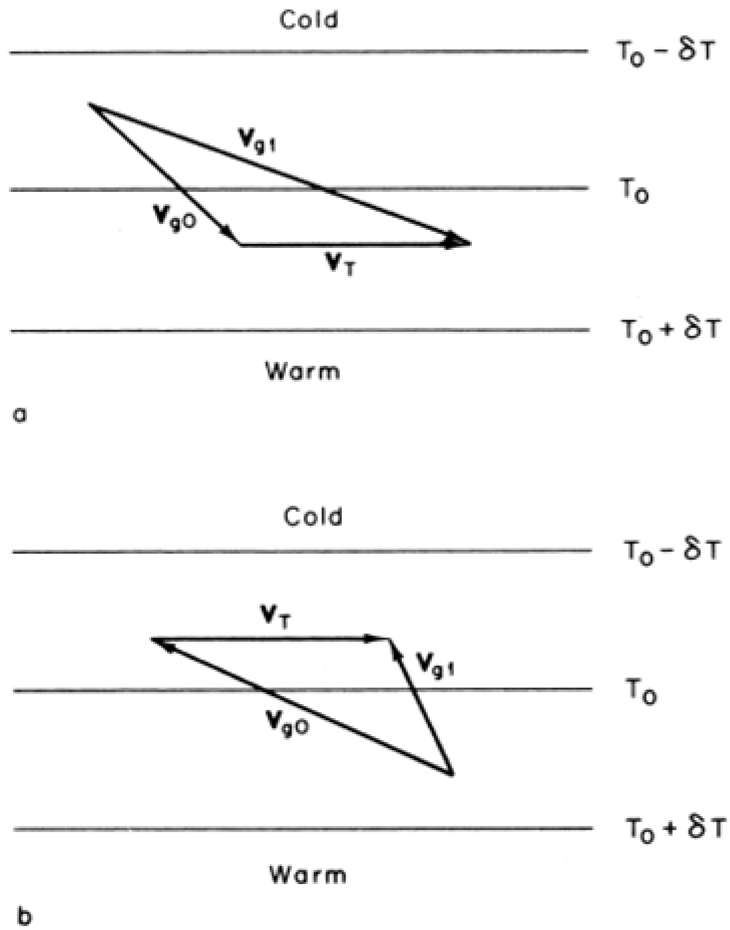


Fig. (3.9): Relationship between turning of geostrophic wind and temperature advection: (a) backing of the wind with height and (b) veering of the wind with height.

Thus, as is illustrated in Fig. (3.9a), a geostrophic wind that turns counterclockwise with height (backs) is associated with cold-air advection. Conversely, as shown in Fig. (3.9b), clockwise turning (veering) of the geostrophic wind with height implies warm advection by the geostrophic wind in the layer. It is therefore possible to obtain a reasonable estimate of the horizontal temperature advection and its vertical dependence at a given location solely from data on the vertical profile of the wind given by a single sounding. Alternatively, the geostrophic wind at any level can be estimated from the mean temperature field, provided that the geostrophic velocity is known at a single level. Thus, for example, if the geostrophic wind at 850 hPa is known and the mean horizontal temperature gradient in the layer 850–500 hPa is also known, the thermal wind equation can be applied to obtain the geostrophic wind at 500 hPa.

Now we study Barotropic and Baroclinic Atmospheres , A barotropic atmosphere is one in which the density depends only on the pressure, $\rho = \rho(p)$, so that isobaric surfaces are also surfaces of constant density. For an ideal gas, the isobaric surfaces will also be isothermal if the atmosphere is barotropic. Thus, $\nabla_p T = 0$ in a barotropic atmosphere, and the thermal wind Equation (3.30) becomes $\partial \mathbf{V}_g / \partial \ln p = 0$, which states that the geostrophic wind is independent of height in a barotropic atmosphere. Thus, barotropy provides a very strong constraint on the motions in a rotating fluid; the large-scale motion can depend only on horizontal position and time, not on height.

An atmosphere in which density depends on both the temperature and the pressure, $\rho = \rho(p, T)$, is referred to as a baroclinic atmosphere. In a baroclinic atmosphere the geostrophic wind generally has vertical shear, and this shear is related to the horizontal temperature gradient by the thermal wind Equation (3.30). Obviously, the baroclinic atmosphere is of primary importance in dynamic meteorology.However, as shown in later chapters, much can be learned by study of the simpler barotropic atmosphere.

Section(3.2): Instructive Applications

In this section we deal with the instructive Applications ,and we start with the Vertical motion .As mentioned previously, for synoptic-scale motions the vertical velocity component is typically of the order of a few centimeters per second. Routine meteorological soundings, however, only give the wind speed to an accuracy of about a meter per second. Thus, in general the vertical velocity is not measured directly but must be inferred from the fields that are measured directly. Two commonly used methods for inferring the vertical motion field are the kinematic method, based on the equation of continuity, and the adiabatic method, based on the thermodynamic energy equation. Both methods are usually applied using the isobaric coordinate system so that $\omega(p)$ is inferred rather than $w(z)$. These two measures of vertical motion can be related to each other with the aid of the hydrostatic approximation.

Expanding Dp/Dt in the *(x, y, z)* coordinate system yields

$$\omega \equiv \frac{D_p}{D_t} = \frac{\partial p}{\partial t} + V \cdot \nabla_p + w\left(\frac{\partial p}{\partial z}\right) \qquad (3.36)$$

Now, for synoptic-scale motions, the horizontal velocity is geostrophic to a first approximation. Therefore, we can write $\mathbf{V} = \mathbf{V}_g + \mathbf{V}_a$, where V_a is the ageostrophic wind and $|\mathbf{V}_a| \ll |\mathbf{V}_g|$|. However, $\mathbf{V}_g = (\rho f)^{-1} k \times \nabla_p$, so that $\mathbf{V}_g \cdot \nabla_p = 0$
Using this result plus the hydrostatic approximation (3.36) may be rewritten as

$$\omega = \frac{\partial p}{\partial t} + \mathbf{V}_a \cdot \boldsymbol{\nabla}_p - g\rho w \quad (3.37)$$

Comparing the magnitudes of the three terms on the right in Equation(3.37), we find that for synoptic-scale motions

$$\partial p/\partial t \sim 10\text{hPa d}^{-1}$$

$$V_g \cdot \nabla_p \sim (1 \text{ m s}^{-1})(1\text{Pa km}^{-1}) \sim 1\text{hPa d}^{-1}$$

$$g\rho^w \sim 100 \text{ hPa d}^{-1}$$

Thus, it is quite a good approximation to let

$$\omega = -\rho g^w \quad (3.38)$$

(i) The Kinematic Method

One method of deducing the vertical velocity is based on integrating the continuity equation in the vertical. Integration of Equation (3.5) with respect to pressure from a reference level $p_s$to any level p yields

$$\omega(p) = \omega(p_s) - \int_{p_s}^{p} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p dp = \omega(p_s) + (p_s - p)\left(\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y}\right)_p \quad (3.39)$$

Here the angle brackets denote a pressure-weighted vertical average

$$\langle\,\rangle \equiv (p - p_s)^{-1} \int_{p_s}^{p} (\,)dp$$

With the aid of Equation (3.38), the averaged form of Equation (3.39) can be rewritten as

$$w(z) = \frac{\rho(p_s)w(z_s)}{\rho(z)} - \frac{p_s - p}{\rho(z)g}\left(\frac{\partial\langle u\rangle}{\partial x} + \frac{\partial\langle v\rangle}{\partial y}\right) \quad (3.40)$$

where z and z_s are the heights of pressure levels p and p_s , respectively.

Application of Equation (3.40) to infer the vertical velocity field requires knowledge of the horizontal divergence. In order to determine the horizontal divergence, the partial derivatives $\partial u/\partial x$ and $\partial v/\partial y$ are generally estimated from the fields of u and v by using finite difference approximations (see Section 13.3.1). For example, to determine the divergence of the horizontal velocity at the point x_0, y_0 in Fig. (3.10) we write

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\approx\frac{u(x_0+d)-u(x_0-d)}{2d}+\frac{v(y_0+d)-v(y_0-d)}{2d} \qquad (3.41)$$

However, for synoptic-scale motions in midlatitudes, the horizontal velocity is nearly in geostrophic equilibrium. Except for the small effect due to the variation of the Coriolis parameter (see Problem 3.19), the geostrophic wind is nondivergent; that is, $\partial u/\partial x$ and $\partial v/\partial y$ are nearly equal in magnitude but opposite in sign. Thus, the horizontal divergence is due primarily to the small departures of the wind from geostrophic balance (i.e., the ageostrophic wind).A 10% error in evaluating one of the wind components in Equation (3.41) can easily cause the estimated divergence to be in error by 100%. For this reason, the continuity equation method is not recommended for estimating the vertical motion field from observed horizontal winds.

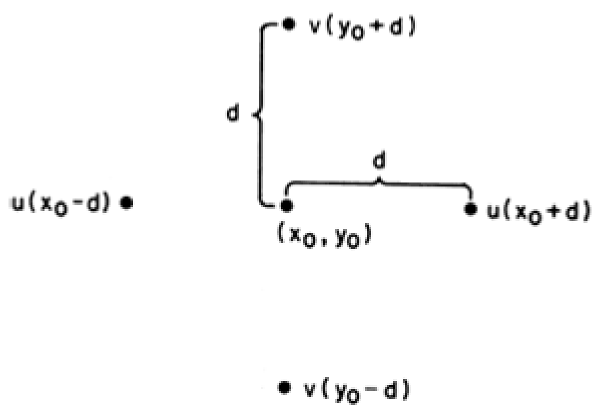


Fig. (3.10): Grid for estimation of the horizontal divergence.

(ii) The Adiabatic Method

A second method for inferring vertical velocities, which is not so sensitive to errors in the measured horizontal velocities, is based on the thermodynamic energy equation. If the diabatic heating J is small compared to the other terms in the heat balance,Equation (3.6) yields

$$\omega = S_p^{-1}\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial t}\right) \tag{3.42}$$

Because temperature advection can usually be estimated quite accurately in midlatitudes by using geostrophic winds, the adiabatic method can be applied when only geopotential and temperature data are available. A disadvantage of the adiabatic method is that the local rate of change of temperature is required. Unless observations are taken at close intervals in time, it may be difficult to accurately estimate $\partial T\ /\partial t$ over a wide area. This method is also rather inaccurate in situations where strong diabatic heating is present, such as storms in which heavy rainfall occurs over a large area.

In the following we study the Surface Pressure Tendency.The development of a negative surface pressure tendency is a classic warning of an approaching cyclonic weather disturbance.A simple expression that relates the surface pressure tendency to the wind field, and hence in theory might be used as the basis for short-range forecasts, can be obtained by taking the limit $p \rightarrow 0$ in Equation(3.39) to get

$$\omega(p_s) = -\int_0^{p_s} (\nabla \cdot V)dp \tag{3.43}$$

followed by substituting from Equation (3.37) to yield

$$\frac{\partial p_s}{\partial t} \approx -\int_0^{p_s} (\nabla \cdot V)dp \tag{3.44}$$

According to Equation (3.44), the surface pressure tendency at a given point is determined by the total convergence of mass into the vertical column of atmosphere above that point. This result is a direct consequence of the hydrostatic assumption, which implies that the pressure at a point is determined solely by the weight of the column of air above that point.Temperature changes in the air column will affect the heights of upper level pressure surfaces, but not the surface pressure.In addition, there is a strong tendency for vertical compensation. Thus, when there is convergence in the lower troposphere there is divergence aloft, and vice versa. The net integrated convergence or divergence is then a small residual in the vertical integral of a poorly determined quantity.

Nevertheless, Equation (3.44) does have qualitative value as an aid in understanding the origin of surface pressure changes, and the relationship of such changes to the horizontal divergence. This can be illustrated by considering (as one possible example) the development of a thermal cyclone. We suppose that a heat source generates a local warm anomaly in the midtroposphere (Fig. 3.11a). Then according to the hypsometric Equation (3.34), the heights of the upper level pressure surfaces are raised above the warm anomaly, resulting in a horizontal pressure gradient force

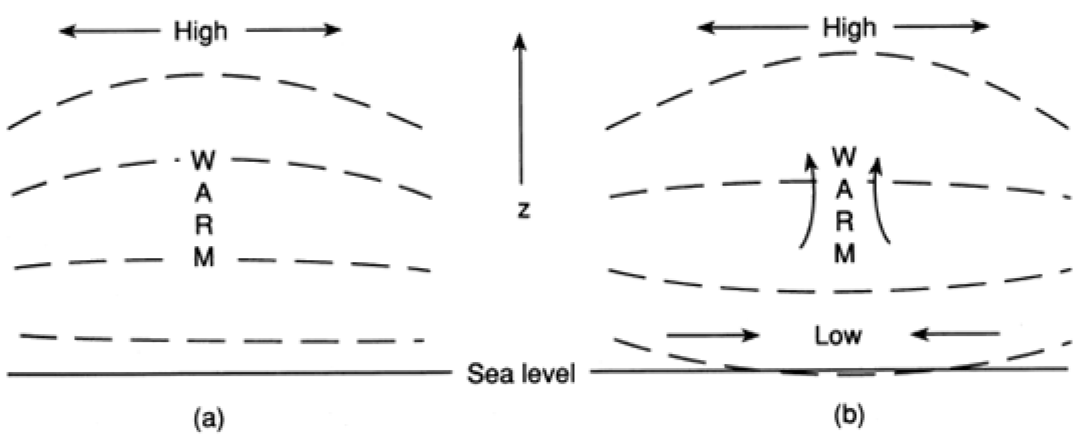


Fig.(3.11): Adjustment of surface pressure to a midtropospheric heat source. Dashed lines indicate isobars. (a) Initial height increase at upper level pressure surface. (b) Surface response to upper level divergence.

at the upper levels, which drives a divergent upper level wind. By Equation (3.44) this upper level divergence will initially cause the surface pressure to decrease, thus generating a surface low below the warm anomaly (Fig. 3.11b). The horizontal pressure gradient associated with the surface low then drives a low-level convergence and vertical circulation, which tends to compensate the upper level divergence. The degree of compensation between upper divergence and lower convergence will determine whether the surface pressure continues to fall, remains steady, or rises.

The thermally driven circulation of the above example is by no means the only type of circulation possible (e.g., cold core cyclones are important synoptic-scale features). However, it does provide insight into how dynamical processes at upper levels are communicated to the surface and how the surface and upper troposphere are dynamically connected through the divergent circulation.

Equation (3.44) is a lower boundary condition that determines the evolution of pressure at constant height. If the isobaric coordinate system of dynamical Equations (3.2), (3.5), (3.6), and (3.27) is used as the set of governing equations, the lower boundary condition should be expressed in terms of the evolution of geopotential (or geopotential height) at constant pressure. Such an expression can be obtained simply by expanding $D\Phi/Dt$ in isobaric coordinates

$$\frac{\partial \Phi}{\partial t} = -\mathbf{V}_a \cdot \nabla \Phi - \omega \frac{\partial \Phi}{\partial p}$$

and substituting from Equation (3.27) and Equation(3.43) to get

$$\frac{\partial \Phi_s}{\partial t} \approx -\frac{RT_s}{p_s} \int_0^{p_s} (\nabla \cdot \mathbf{V}) dp \qquad (3.45)$$

where we have again neglected advection by the ageostrophic wind.
In practice the boundary condition (3.45) is difficult to use because it should be applied at pressure ps , which is itself changing in time and space. In simple models it is usual to assume that ps is constant (usually 1000 hPa) and to let $\omega = 0$ at p_s, For modern forecast models, an alternative coordinate system is generally employed in which the lower boundary is always a coordinate surface.

In the following we study the Applications. Here we discuss three instructive applications in which the present approach proved useful:

(i) Semi-Geostrophic Theory

Weather fronts are examples for atmospheric flow structures that feature anisotropic horizontal scalings. When viewed from above, one may think of a front as being a narrow band of activity centered about some smooth, large scale curve. All flow variables are expected to vary substantially over relatively short distances normal to the front, while they vary on scales comparable to the characteristic

length of the curve's geometry in the tangential direction.An appropriate ansatz to capture this structural behavior in an asymptotic
analysis, borrowed from WKB theories, geometrical optics/acoustics, or the theory of thin flames in combustion, could read

$$U(t, \boldsymbol{x}, z; \varepsilon) = \sum_i U^{(i)}\left(\varepsilon^2, \frac{\Phi(\varepsilon^2 \boldsymbol{x})}{\varepsilon}, \varepsilon^2 \boldsymbol{x}, z\right) \tag{3.46}$$

Here $\Phi(\cdot)$ is a scalar function, which by the scaling of its argument, $\varepsilon^2\boldsymbol{x}$,in Equation(3.46) will have a characteristic scale comparable to the internal Rossby deformation radius L_i . The scaled coordinate $\zeta = \Phi/\varepsilon$ will then resolve rapid variations of the solution that occur between levelsets $\Phi = \Phi_0 + O(\varepsilon)$. The front will thus be centered about the level set $\Phi(\varepsilon^2\boldsymbol{x}) \equiv \Phi_0$ and have a characteristic thickness of order $l = O(\varepsilon L_i)$. According to Equations:

$$\mathrm{Ro}_{h_{sc}} = O(\varepsilon),$$

$$\mathrm{Ro}_{L_1} = O(1),$$

$$\mathrm{Ro}_{L_i} = O(\varepsilon^{\alpha})$$

$$\mathrm{Ro}_{L_e} = O(\varepsilon^2),$$

this is the characteristic length L_1 for which the Rossby number would be of order unity.

With the abbreviations

$$\tau = \varepsilon^2 t, \qquad \zeta = \frac{\Phi(\varepsilon^2\boldsymbol{x})}{\varepsilon}, \qquad \boldsymbol{X} = \varepsilon^2\boldsymbol{x}, \tag{3.47}$$

the partial derivatives in the governing Equations :

$$v_t + v\cdot\nabla v + \Omega \times v + \frac{1}{\rho}\nabla p = S_v - g\boldsymbol{k}$$

$$p_t + v\cdot\nabla p + \gamma p \nabla\cdot v = S_p \tag{$*$}$$

$$\theta_t + v\cdot\nabla\theta = S_\theta$$

Here v, p, θ are the fluid flow velocity, the (thermodynamic) pressure, and the fluid's potential temperature. γ is the isentropic exponent, assumed to be constant. Ω, g are the vector of earth rotation and the acceleration of gravity, $\boldsymbol{k}$ is a radial unit vector, pointing away from the earth's center. The source terms $S_v . S_p, S_\theta$ Su, are abbreviations for molecular or turbulent transport terms, for effective energy source terms from radiation, latent heat release from condensation of water vapor, The potential temperature is a variable, closely related to thermodynamic. Become

$$\partial_t = \varepsilon^2 \partial_\tau$$

$$\nabla = \varepsilon\, \sigma\, \boldsymbol{n} \partial_\zeta + \varepsilon^2 \nabla_X + \boldsymbol{k} \partial_z \qquad (3.48)$$

where $\sigma = |\nabla_X \Phi|$ and $\boldsymbol{n} = \nabla_X \Phi / |\nabla_X \Phi|$.

A somewhat subtle issue in semi-geostrophic theory is the scaling of the velocity components. It is assumed that they scale in proportion with the characteristic length for their associated spatial direction. Thus one allows only the flow velocity tangential to the front to be of order u_{ref}, i.e., $v \cdot t = O(u_{ref})$, where $\boldsymbol{t(X)}$ is the tangential unit vector to a level set Φ = const.. For the normal and vertical components this assumption implies $\boldsymbol{v} \cdot \boldsymbol{n} = O(\varepsilon\, u_{ref})$, and $\boldsymbol{v} \cdot \boldsymbol{k} = O(\varepsilon^2 u_{ref})$. Accordingly, the velocity field is represented as

$$\boldsymbol{v} = v^{(0)} \boldsymbol{t} + \varepsilon\left(v^{(1)} \boldsymbol{t} + u^{(1)} \boldsymbol{n}\right) + \varepsilon^2\left(v^{(2)} \boldsymbol{t} + u^{(2)} \boldsymbol{n} + w^{(2)} \boldsymbol{k}\right) + \cdots \qquad (3.49)$$

It is now straight-forward to insert the Equations (3.46)–(3.49) into the governing equations from Equations ($*$) and to collect the following leading order set of equations. Using the replacements

$$\left(\theta^{(4)}, p^{(4)} / \rho^{(0)}\right) \rightarrow (\theta, \pi)$$

$$\left(u^{(1)}, v^{(0)}, w^{(2)}\right) \rightarrow (u, v, w) \qquad (3.50)$$

$$(S_\theta^{(6)}, S_v^{(2)}) \rightarrow (S_\theta, S_v)$$

we obtain the semi-geostrophic approximation

$$-fv + \sigma \frac{\partial \pi}{\partial \zeta} = 0$$

$$\frac{Dv}{Dt} + fu^{(1)} + \frac{\partial \pi}{\partial \eta} = S_v$$

$$\frac{\partial \pi}{\partial z} - \theta = 0 \qquad (3.51)$$

$$\frac{\partial u}{\partial \zeta} + \frac{\partial v}{\partial \eta} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0,$$

Here $\rho_0(z)$ is the background density stratification corresponding to a homentropic Atmosphere

$$\frac{D}{D\tau} = \frac{\partial}{\partial \tau} + v\frac{\partial}{\partial \eta} + u\sigma\frac{\partial}{\partial \zeta} + w\frac{\partial}{\partial z}, \qquad (3.52)$$

And

$$\frac{\partial}{\partial \eta} = t \cdot \nabla_X$$

Lower order expansion functions, such as $\theta^{(2)}, \theta^{(3)}$, and $p^{(2)}, p^{(3)}$,which do not explicitly appear here, can be shown not to participate in the dynamics within a narrow front if the assumed time scaling is to be observed.

The semi-geostrophic equations are used in a range of contexts, including theories for the formation and structure of strong weather fronts.The key difference between these and the quasi-geostrophic equations is that only one of the horizontal velocity components, v, is in geostrophic balance. The velocity component normal to the front is, in contrast, the result of a balance between the Coriolis force, the pressure gradient, and the fluid particle acceleration along the front. The semi-geostrophic equations have, however, several attractive mathematical features, which are reviewed and extensively discussed .

(iii) Synoptic-Planetary Interactions in the Tropics

Here we summarize recent joint work with A.J. Majda addressing scale interactions in the tropics. A hierarchy of reduced model equations describing a range of possible flow regimes is derived by Majda and Klein [2003] using systematic multiple scales expansions. One particularly interesting regime involves interactions between the equatorial synoptic and the planetary scales. Here we review the flow regime and the key results of the analysis for this regime. For details the reader may consult the original reference.

Now we study the Expansion scheme and scaling considerations,The multiple scales expansion scheme for this regime reads

$$\Phi(t,x,z;\varepsilon) = \varepsilon^{\alpha(\Phi)} \sum_i \varepsilon^i \, \Phi^{(i)}(\varepsilon^{5/2}t, \varepsilon^{5/2}x, \varepsilon^{7/2}x, z) \qquad (3,53)$$

where $\alpha = 0$ for $\Phi \in \{p, \theta, \rho, u\}$, and $\alpha = 1/2$ for $\Phi = w, S_\theta, S_v$. Here $\boldsymbol{u}$,w are the horizontal and vertical flow velocity components, respectively, and the coordinates $\boldsymbol{x} = (x, y)$ denote the zonal (along the equator) and meridional (north-south) horizontal coordinates. As we will see shortly, this scheme merges the single scale expansion for equatorial geostrophic motions,

$$\Phi(t,x,z;\varepsilon) = \varepsilon^{\alpha(\Phi)} \sum_i \varepsilon^i \, \Phi^{(i)}(\varepsilon^{5/2}t, \varepsilon^{5/2}x, z) \qquad (3,54)$$

with a scheme that resolves planetary scale equatorial waves

$$\Phi(t,x,z;\varepsilon) = \varepsilon^{\alpha(\Phi)} \sum_i \varepsilon^i \, \Phi^{(i)}(\varepsilon^{5/2}t, \varepsilon^{7/2}x, \varepsilon^{5/2}y, z) \qquad (3,55)$$

The characteristic horizontal length scale accessed by the first scheme in Equation (3,39) is the internal Rossby deformation radius for near-equatorial flows. To verify this, we reconsider the relation $L_i \sim \frac{N h_{sc}}{\Omega}$ from Equation $L_i \sim \frac{N h_{sc}}{\Omega}$. Near the equator, the earth rotation frequency Ω must be replaced with characteristic value of the vertical component $f = \boldsymbol{k} \cdot \Omega = \Omega \sin \emptyset$ of the earth rotation vector, where $\emptyset$ is the longitude. In terms of the arclength in the meridional direction, y, we have $\emptyset = y/a$, where a is the earth's radius. Anticipating that $L_i \ll a$, which remains to be verified, we have $\sin \emptyset \sim L_i/a$, and $f \sim \beta L_i$, where $\beta = \Omega/a$. As a consequence,

$$L_i \sim \frac{N}{\beta L_i}, or \frac{L_i}{h_{sc}} \sim \sqrt{\frac{N}{\Omega}\frac{a}{h_{sc}}} \sim \varepsilon^{-5/2} \qquad (3.56)$$

The last estimate follows from Equations :

$$N = \sqrt{\frac{g}{\theta}\frac{\partial\theta}{\partial z}} \sim \frac{1}{\varepsilon^{\alpha+1}}\Omega$$

with $\alpha = 1$, and Equation :

$$L_1 \sim \frac{1}{\varepsilon}h_{sc} \text{ , } L_e \sim \frac{1}{\varepsilon^3}$$

which stated that $N/\Omega \sim \varepsilon^{-2}$ and $a/h_{sc} \sim \varepsilon^{-3}$. We verify that $L_i \ll a$, even though the difference is merely of order $\varepsilon^{1/2}$. Equation (3.56) shows that the scaling anticipated in (3.53) and Equation(3.54) in fact accesses the internal Rossby deformation radius L_i.The second expansion scheme in Equation (3.55) accesses even larger (planetary) scales via the coordinate

$$X_P = \varepsilon^{7/3}x = \frac{x^\backslash}{L_P} \quad \text{with } L_P = \varepsilon^{-7/2}h_{sc} \tag{3.57}$$

Notice that this expansion assumes anisotropic scalings along and normal to the equator. This is compatible with theories of equatorial wave motions which reveal confinement of near-tropical dynamics between about $-30^0$and 30^0degrees latitude.The non-dimensional time coordinate in Equations (3.53) and (3.55) may be re-written in two different ways:

$$\varepsilon^{5/2}t = \varepsilon^{5/2}\frac{t^\backslash}{h_{sc}/u_{ref}} = \frac{t^\backslash}{L_i/u_{ref}} = \frac{t^\backslash}{L_P/c_{iref}} \tag{3.58}$$

Where $t^\backslash$ again denotes the dimensional time variable, and ciref $c_{iref} \sim N\ Nh_{SC}$ is the characteristic speed of internal gravity waves with vertical scale comparable to h_{SC}. This dual representation shows that, on the one hand, the chosen time coordinate resolves advection with the flow velocity uref over synoptic distances comparable to L_i. On the other hand, it also resolves internal gravity waves travelling at speeds ciref over planetary distances of the order L_P . As a consequence, the multiple scales expansion from Equation (3.38) is suited to describe direct interactions of these very different phenomena on one and the same time scale.

In the following we present the Intra-Seasonal Planetary Equatorial Synoptic Dynamics (IPESD), To simplify the notation, we will use the following replacements in the rest of this Section:

$$\left(\theta^{(2)}, \theta^{(3)}, p^{(3)}/\rho^{(0)}, p^{(4)}/\rho^{(0)}\right) \rightarrow \left(\Theta_2, \theta, \pi, \pi^{\backslash}\right)$$

$$(\boldsymbol{u}^{(0)}, \boldsymbol{u}^{(1)}, w^{(2)}, w^{(3)}) \rightarrow \left(\boldsymbol{u}, \boldsymbol{u}^{\backslash}, w, w^{\backslash}\right) \tag{3.59}$$

$$(S_\theta^{(4)}, S_\theta^{(5)}, \boldsymbol{S}_u^{(1)}, \boldsymbol{S}_u^{(2)}) \rightarrow (S_\theta, S_\theta^{\backslash}, \boldsymbol{S}_u, \boldsymbol{S}_u^{\backslash})$$

$$(\varepsilon^{5/2}t, \varepsilon^{5/2}x, \varepsilon^{7/2}t) \rightarrow (T_S, \boldsymbol{X}_S, X_S),$$

where $\boldsymbol{X}_S = (X_S, Y_S)$.

Now we present the Synoptic motions With the abbreviations from Equation (3.59), the leading order set of equations describing motions on the smaller of the considered scales (which is still as large as 2000km) reads

$$\partial_z \pi = \theta$$

$$w \frac{d\Theta_2}{dz} = S_\theta$$

$$-\beta y v + \partial_x \pi = S_u \tag{3.60}$$

$$\beta y u + \partial_y \pi = S_v$$

$$\partial_x(\rho_0 u) + \partial_y(\rho_0 v) + \partial_z(\rho_0 w) = 0\,.$$

These equations describe, in this sequence, hydrostatic balance in the vertical direction, the generation of vertical motions by heat sources forcing particles to move towards their individual levels of neutral buoyancy, horizontal geostrophic balance in the zonal (x) and meridional (y) directions, and mass conservation.

The Equations in (3.60) may be considered as the three-dimensional version of the Matsuno-Webster-Gill type of models, who derived models for steady forced synoptic scale motions near the equator in the context of the shallow water approximation demonstrates that these equations reproduce typical quasi-steady large scale near-equatorial flow patterns when some physically reasonable closures for the source terms are assumed. His examples include qualitative models for the important “Hadley” and “Walker” circulations.

For given source terms S_θ, S_u, S_v the system in Equation (3.60) is linear in u, v, w, π, θ. General solutions, in this case, are superpositions of particular and homogeneous solutions. One particular solution $u^p, v^p, w^p, \pi^p, \theta^p$ is determined by,

$$w^p = \frac{S_\theta}{d\Theta_2/dz}\ , u^p = \frac{1}{\beta}\left(S_{v,x} - S_{u,y}\right) + \frac{y}{\rho_0}\left(\frac{\rho_0 S_\theta}{d\Theta_2/dz}\right)_z, \qquad (3.61)$$

$$\partial_x u^p = -\partial_y v^p - \frac{1}{\rho_0}\partial_z(\rho_0 w^p) \text{ with } \beta y\overline{u^p} = \overline{S_v}, \qquad (3.62)$$

$$\left.\begin{aligned}\partial_x \pi^p &= S_u - \beta y v^p\\ \partial_y \pi^p &= S_v + \beta y u^p\end{aligned}\right\} \quad \text{with } \overline{\pi^p}(t, 0, z) \equiv 0 \qquad (3.63)$$

$$\theta^p = \partial_z \pi^p \qquad (3.64)$$

Homogeneous solutions to Equation (3.45) satisfy,

$$v = w \equiv 0, \ (\theta, u, \pi) = (\Theta, U, P)(t, y, z), \qquad (3.65)$$

where Θ, U, P are arbitrary except for the constraints

$$\partial_y P = -\beta y U, \qquad \partial_z P = \Theta \qquad (3.66)$$

Now we discuss the Planetary waves As pointed out in conjunction with Equation (3.50), the equation system for the synoptic scales determines solutions only up to a zonal (along the equator) shear flow. In the present context of a multiple scales expansion, this zonal shear flow may still not depend on *x*, but it may well depend on the planetary scale coordinate X_P, and multiple scales asymptotic techniques should allow us to derive evolution equations for these large scale averaged mean motions. In fact, the following system is derived by Majda and Klein [2003],

$$\overline{D_t^p}U + \partial_X P - \beta y\overline{v'} = \overline{S_u'} - \overline{D_t^p u^p}$$

$$\overline{D_t^p}\Theta + \overline{w'}\frac{d\Theta_2}{dz} = \overline{S_\theta'} - \overline{D_t^p \theta^p}$$

$$\beta y U + \partial_y P = 0 \qquad (3.67)$$

$$\partial_Z P = \Theta$$

$$\partial_X U + \partial_y \overline{v'} + \frac{1}{\rho_0}\partial_z\left(\rho_0 \overline{w'}\right) = 0\ ,$$

Where

$$D_t^p = \partial_t + u^p \partial_x + v^p \partial_y + w^p \partial_z$$
$$\overline{D_t^p} = \partial_t + \overline{v^p} \partial_y + \overline{w^p} \partial_z \tag{3.68}$$

These equations are the three-dimensional analogue of the linear equatorial long wave equations, supplemented with the net large scale effects of the synoptic scale transport as represented by the terms, $\overline{D_t^p u^p}$, $\overline{D_t^p \theta^p}$ and with advection by the mean synoptic meridional velocity $\overline{v^p}$.

(iii) Balancing Numerical Methods for Nearly Hydrostatic Motions

This section summarizes a quite different development that was motivated by our asymptotic considerations for atmospheric flows. address a nagging numerical issue associated with the (asymptotically) dominant balance

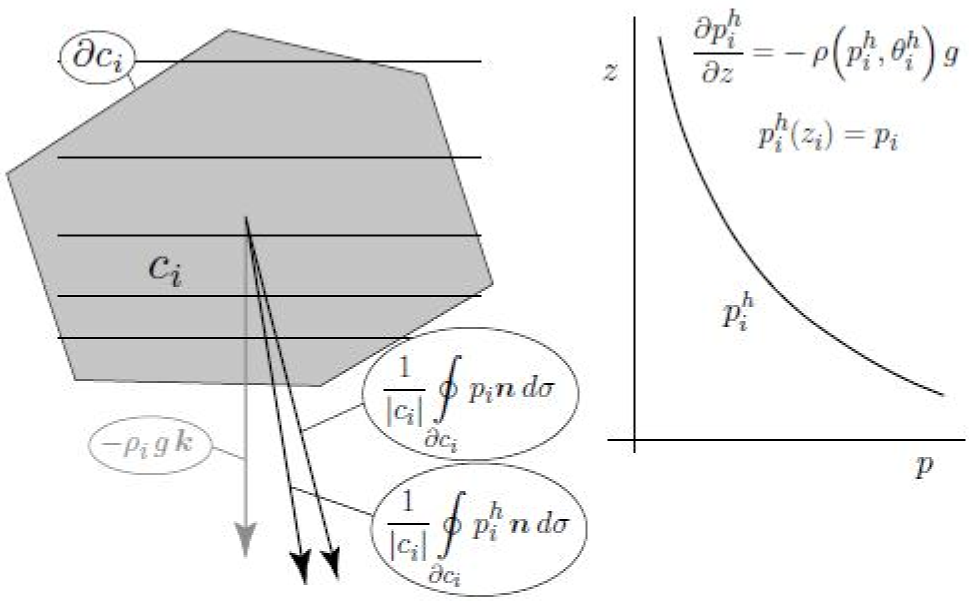


Fig.(3.12): Archimedes' principle for the gravity source term.

between pressure forces and the gravitational acceleration in most realistic atmosphere flow regimes.

Consider the vertical component of the momentum balance in the dimensionless governing equations , i.e.

$$w_t + \boldsymbol{u} \cdot \nabla w + w w_z + \varepsilon \pi_3^* \boldsymbol{k} \cdot (\Omega \times v) + \frac{1}{\varepsilon^4}\left(\frac{1}{\rho}\frac{\partial p}{\partial z} + 1\right) = S_w \qquad (3.69)$$

Suppose further that we had adopted a numerical discretization of all terms that is second order accurate in terms of the (vertical) space discretization parameter $h = \Delta z / h_{sc}$. Then the numerical truncation error induced by the two terms Equation(3.69) multiplied by $1/\varepsilon^4$ would read

$$\delta_{\mathrm{num.}} = O\left(\frac{h^2}{\varepsilon^4}\right). \qquad (3.70)$$

Current production runs with numerical weather forecasting or climate simulation codes use around 30 grid layers to resolve the vertical direction, so that $h \sim 1/30$. Our earlier estimates indicate that $\varepsilon \sim 1/7 \ldots 1/6$, so that

$$\frac{h^2}{\varepsilon^4} = O(1) \qquad (3.71)$$

under realistic conditions for up-to-date computational simulations of atmospheric flows. We conclude that the computed vertical accelerations will be highly inaccurate, unless special measures are taken to ensure that the limiting situation of hydrostatic balance is adequately captured by the numerical discretization.

In the following we illusterate the Archimedes' principle for the gravity source term: Figure(3.12) displays a general control volume that might serve as the ith grid cell c_i for our finite volume method. A straight-forward second order approximation of the gravity source term for such a volume would read

$$\int_{c_i} \rho \boldsymbol{k} d^2 x = |c_i| \rho_i \boldsymbol{k} + O(\delta^2) \qquad (3.72)$$

where ρ_i is the compcomputed cell-averaged density and |ci| is the cell's volume, and $\delta = \mathrm{diam}(c_i)$ is its characteristic diameter. In the context of conservative finite volume methods the integral of the pressure gradient over the volume is discretized as

$$\int_{c_i} \nabla p d^2 x = \oint_{\partial c_i} p \boldsymbol{n} d\sigma = \sum_j A_{i,j} p_{i,j} \boldsymbol{n}_{i,j} + O(\delta^2) \qquad (3.73)$$

where $A_{i,j}$, $p_{i,j}$, $n_{i,j}$ are the length (area) of the jth section of the i th control volume's boundary, and approximations to the pressure and outward unit normal vector on that cell interface section. As indicated, these approximations are secondorder accurate in terms of δ. But this alone is insufficient to properly capture the hydrostatic limit, because the magnitude of the truncation errors is independent of whether the flow state is or is not close to hydrostatic.
To overcome this difficulty we observe that the vector $-p\boldsymbol{k}$ may be interpreted as the gradient of a virtual hydrostatic pressure distribution, i.e.,

$$p\boldsymbol{k} = -p^{hy}. \tag{3.74}$$

The gravity source term may then be rewritten as

$$\int_{c_i} \nabla p d^2 x = \oint_{\partial c_i} p^{hy} \boldsymbol{n} d\sigma = -\sum_j A_{i,j} p_{i,j}^{hy} \mathrm{n}_{i,j} + O(\delta^2) \tag{3.75}$$

As a consequence, the sum of the gravity source term and the pressure gradient is Discretizeda

$$\int_{c_i} (\nabla p + \rho \boldsymbol{k}) d^2 x = \oint_{\partial c_i} p^{hy} \boldsymbol{n} d\sigma = -\sum_j A_{i,j} \left(p_{i,j} - p_{i,j}^{hy}\right) \boldsymbol{n}_{i,j} + O\left(\delta^2 \left\| p - p_i^h \right\|\right),$$

(3.76)

where the $p_{i,j}^{hy} = p_i^{hy}(z_{j)}$ are values of a locally reconstructed virtual hydrostatic pressure distribution evaluated at the grid cell interface centers. This modification will not change the approximation order of the scheme, which is still of second order for suitable formulations of the hydrostatic pressure distribution $p_i^h(z)$. Yet, we observe that the numerical approximate expression for the sum of the two terms now vanishes identically when the $p_{i,j}$ match the $p_{i,j}^{hy}$, i.e., when the pressure is hydrostatic in the sense of p_i^h . Even if the approximate construction of p_i^h is inexact, so that $\left\| p - p_i^h \right\| \nrightarrow 0$ as the hydrostatic limit is approached, the truncation error will be $O\left(\delta^2 \left\| p - p_i^h \right\|\right)$ instead of $O(\delta^2)$, and thus reduced as much as we manage to reproduclocalvertical balance.

Chapter (4)

Circulation,Vorticity and Some Models

Section(4.1) :Circulation and Vorticity

In classical mechanics the principle of conservation of angular momentum is often invoked in the analysis of motions that involve rotation. This principle provides a powerful constraint on the behavior of rotating objects. Analogous conservation laws also apply to the rotational field of a fluid. However, it should be obvious that in a continuous medium, such as the atmosphere, the definition of "rotation" is subtler than that for rotation of a solid object.

Circulation and vorticity are the two primary measures of rotation in a fluid. Circulation, which is a scalar integral quantity, is a macroscopic measure of rotation for a finite area of the fluid. Vorticity, however, is a vector field that gives a microscopic measure of the rotation at any point in the fluid.

In the following we study the circulation theorem.The circulation, C, about a closed contour in a fluid is defined as the line integral evaluated along the contour of the component of the velocity vector that is locally tangent to the contour:

$$C \equiv \oint \underline{U} \cdot dI = \oint |\underline{U}| cos\alpha \, dl$$

Fig.(4.1): Circulation about a closed contour.

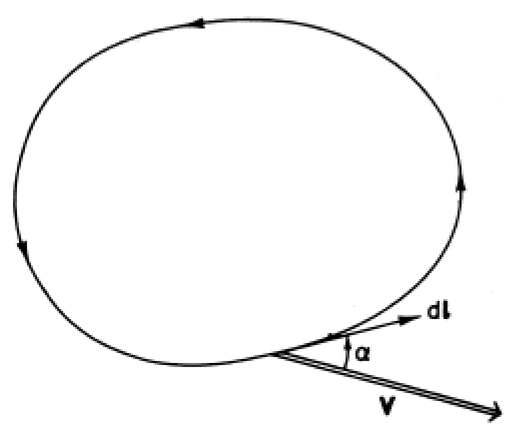
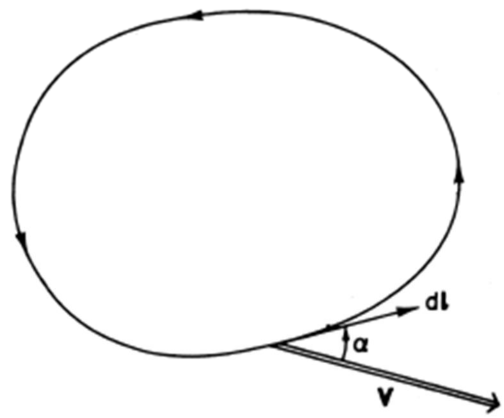


Fig. 4.1 Circulation about a closed contour.



where $I(s)$ is a position vector extending from the origin to the point $s(x, y, z)$ on the contour C, and dl represents the limit of $\delta l = l(s+\delta s)-l(s)$ as $\delta s \rightarrow 0$. Hence, as indicated in Fig.(4.1), dl is a displacement vector locally tangent to the contour. By convention the circulation is taken to be positive if $C > 0$ for counterclockwise integration around the contour.That circulation is a measure of rotation is demonstrated readily by considering a circular ring of fluid of radius R in solid-body rotation at angular velocity $\boldsymbol{\Omega}$ about the z axis. In this case, $U = \Omega \times R$, where R is the distance from the axis of rotation to the ring of fluid. Thus the circulation about the ring is given by

$$C \equiv \oint \underline{U} \cdot dI = \int_0^{2\pi} \Omega R^2 d\lambda = 2\Omega\pi R^2$$

In this case the circulation is just 2π times the angular momentum of the fluid ring about the axis of rotation. Alternatively, note that $C/(\pi R2) = 2\Omega$ so that the circulation divided by the area enclosed by the loop is just twice the angular speed of rotation of the ring. Unlike angular momentum or angular velocity, circulation can be computed without reference to an axis of rotation; it can thus be used to characterize fluid rotation in situations where “angular velocity” is not defined easily. The circulation theorem is obtained by taking the line integral of Newton’s second law for a closed chain of fluid particles. In the absolute coordinate system the result (neglecting viscous forces) is

$$\oint \frac{D_\alpha U_\alpha}{Dt} \cdot dI = -\oint \frac{\nabla_p \cdot dI}{\rho} - \oint \nabla \Phi \, dI \tag{4.1}$$

where the gravitational force g is represented as the gradient of the geopotential **Φ**, defined so that – ∇**Φ** = g = -gk. The integrand on the left-hand side can be rewritten as

$$\frac{D_\alpha U_\alpha}{Dt} \cdot dI = \frac{D}{Dt}(U_\alpha \cdot dI) - U_\alpha \cdot \frac{D_\alpha}{Dt}(dI)$$

or after observing that since I is a position vector,

$$\frac{D_\alpha I}{Dt} \equiv U_\alpha,$$

$$\frac{D_\alpha U_\alpha}{Dt} \cdot dI = \frac{D}{Dt}(U_\alpha \cdot dI) - U_\alpha \cdot dU_\alpha \qquad (4.2)$$

Substituting (4.2) into (4.1) and using the fact that the line integral about a closed loop of a perfect differential is zero, so that

$$\oint \nabla\Phi \cdot dI = \oint d\Phi = 0$$

and noting that

$$\oint U_\alpha \cdot dU_\alpha = \frac{1}{2}\oint d(U_\alpha \cdot U_\alpha) = 0$$

we obtain the circulation theorem:

$$\frac{DC_\alpha}{Dt} = \frac{D}{Dt}\oint U_\alpha \cdot dI = -\oint \rho^{-1}\, dp \qquad (4.3)$$

The term on the right-hand side in Equation(4.3) is called the solenoidal term. For abarotropic fluid, the density is a function only of pressure, and the solenoidal term is zero. Thus, in a barotropic fluid the absolute circulation is conserved following the motion. This result, called Kelvin's circulation theorem, is a fluid analog of angular momentum conservation in solid-body mechanics.

For meteorological analysis, it is more convenient to work with the relative circulation C rather than the absolute circulation, as a portion of the absolute circulationC_e, is due to the rotation of the earth about its axis. To compute C_e,we apply Stokes' theorem to the vector U_e, where $U_e = \Omega \times r$ is the velocity of the earth at the position r:

$$C_e = \oint U_e \cdot dl = \iint_A (\nabla \times U_e) . \mathbf{n} \, dA$$

where A is the area enclosed by the contour and the unit normal n is defined by the counterclockwise sense of the line integration using the "right-hand screw rule." Thus, for the contour of Fig.(4.1), n would be directed out of the page. If the line integral is computed in the horizontal plane, n is directed along the local vertical (see Fig. 4.2). Now, by a vector identity

$$\nabla \times U_e = \nabla \times (\Omega \times r) = \nabla \times (\Omega \times R)$$

$$= \Omega \nabla \cdot R = 2\Omega$$

so that $(\nabla \times U_e) \cdot n = 2\Omega \sin \varphi \equiv f$ is just the Coriolis parameter. Hence, the circulation in the horizontal plane due to the rotation of the earth is

$$C_e = 2\Omega \langle \sin\varphi \rangle A = 2\Omega A_e$$

where $\langle \sin \varphi \rangle$ denotes an average over the area element A and A_e is the projection of A in the equatorial plane as illustrated in Fig.(4.2). Thus, the relative circulation may be expressed as

$$C = C_\alpha - C_e = C_\alpha - 2\Omega A_e \qquad (4.4)$$

Differentiating (4.4) following the motion and substituting from Equation (4.3) we obtain the Bjerknes circulation theorem:

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{DA_e}{Dt} \qquad (4.5)$$

For a barotropic fluid,Equation (4.5) can be integrated following the motion from an initial state (designated by subscript 1) to a final state (designated by subscript 2), yielding the circulation change

$$C_2 - C_1 = -2\Omega \, (A_2 \sin \varphi_2 - A_1 \sin \varphi_1) \qquad (4.6)$$

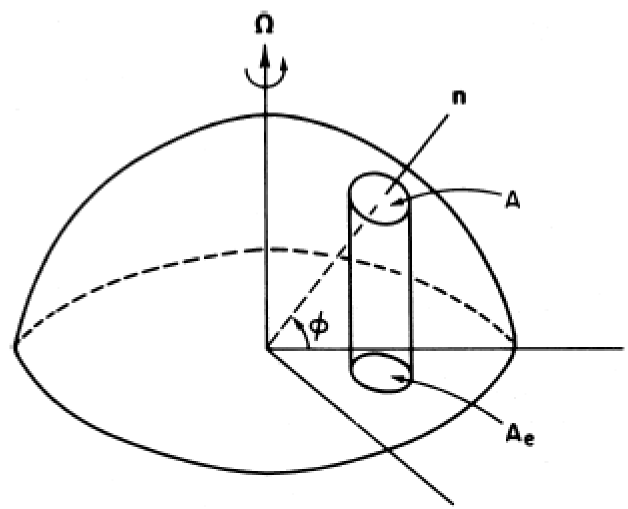


Fig.(4.2): Area A_e subtended on the equatorial plane by horizontal area A centered at latitude ϕ.

Equation (4.6) indicates that in a barotropic fluid the relative circulation for a closed chain of fluid particles will be changed if either the horizontal area enclosed by the loop changes or the latitude changes. Furthermore, a negative absolute circulation in the Northern Hemisphere can develop only if a closed chain of fluid particles is advected across the equator from the Southern Hemisphere.
Example. Suppose that the air within a circular region of radius 100 km centered at the equator is initially motionless with respect to the earth. If this circular air mass were moved to the North Pole along an isobaric surface preserving its area, the circulation about the circumference would be

$$C = -2\Omega\pi r^2[\sin(\pi/2) - \sin(0)]$$

Thus the mean tangential velocity at the radius r = 100 km would be

$$V = C/(2\pi r) = -\Omega r \approx -7\, m\, s^{-1}$$

The negative sign here indicates that the air has acquired anticyclonic relative circulation. In a baroclinic fluid, circulation may be generated by the pressure-density solenoid term in Equation(4.3). This process can be illustrated effectively by considering the development of a sea breeze circulation, as shown in Fig.(4.3). For the situation depicted, the mean temperature in the air over the ocean is colder than the mean temperature over the adjoining land. Thus, if the pressure is uniform at ground level, the isobaric surfaces above the ground will slope downward toward the

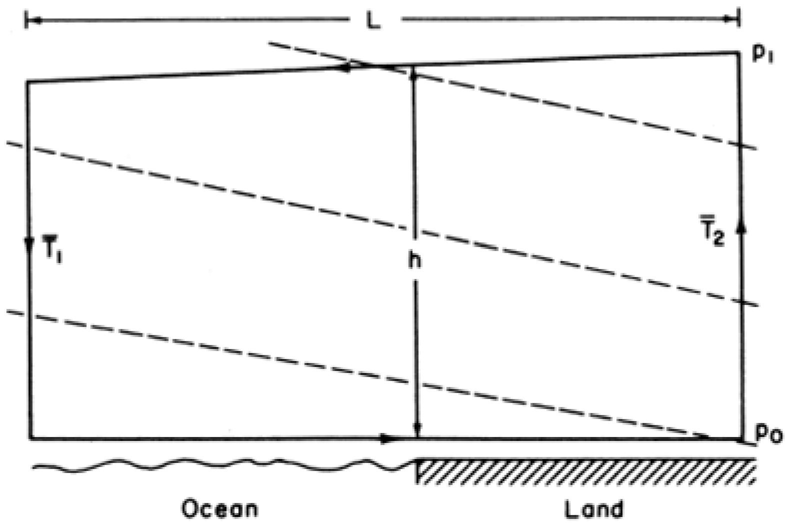


Fig.(4.3): Application of the circulation theorem to the sea breeze problem. The closed heavy solid line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces of constant density.

ocean while the isopycnic surfaces (surfaces of constant density) will slope downward toward the land. To compute the acceleration as a result of the intersection of the pressure-density surfaces, we apply the circulation theorem by integrating around a circuit in a vertical plane perpendicular to the coastline. Substituting the ideal gas law into Equation (4.3) we obtain

$$\frac{DC_\alpha}{Dt} = -\oint RT d \ln p$$

For the circuit shown in Fig.(4.3) there is a contribution to the line integral only for the vertical segments of the loop, as the horizontal segments are taken at constant pressure. The resulting rate of increase in the circulation is

$$\frac{DC_\alpha}{Dt} = R \ln\left(\frac{p_0}{p_1}\right)\left(\check{T}_2 - \check{T}_1\right) > 0$$

Letting $\langle v \rangle$be the mean tangential velocity along the circuit, we find that

$$\frac{D\langle v\rangle}{Dt} = \frac{R\,\ln(p_0/p_1)}{2(h+L)}\left(\check{T}_2 - \check{T}_1\right) \qquad (4.7)$$

If we let $p_0 = 1000\,hPa, p_1 = 900\,hPa, \check{T}_2 - \check{T}_1 = 10c^o$, $L = 20\,km$, and $h = 1$ km, Equation(4.7) yields an acceleration of about $7 \times 10^{-3}\,ms^{-1}$. In the absence of frictional retarding forces, this would produce a wind speed of 25 m s^{-1} in about $1\,h$. In reality, as the wind speed increases, the frictional force reduces the acceleration rate, and temperature advection reduces the land–sea temperature contrast so that a balance is obtained between the generation of kinetic energy by the pressure-density solenoids and frictional dissipation.

Now we discuss the vorticity ,Vorticity, the microscopic measure of rotation in a fluid, is a vector field defined as the curl of velocity. The absolute vorticity ω_α is the curl of the absolute velocity, whereas the relative vorticity ω is the curl of the relative velocity:

$$\omega_\alpha \equiv \nabla \times U_\alpha\,,\quad \omega \equiv \nabla \times U$$

so that in Cartesian coordinates,

$$\omega = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

In large-scale dynamic meteorology, we are in general concerned only with the vertical components of absolute and relative vorticity, which are designated by η and , respectively.

$$\eta \equiv k\cdot(\nabla \times U_\alpha), \zeta \equiv k\cdot(\nabla \times U)$$

In the remainder, η and ζ are referred to as absolute and relative vorticities, respectively, without adding the explicit modifier “vertical component of.” Regions of positive ζ are associated with cyclonic storms in the Northern Hemisphere; regions of negative ζ are associated with cyclonic storms in the Southern Hemisphere. Thus, the distribution of relative vorticity is an excellent diagnostic for weather analysis. Absolute vorticity tends to be conserved following the motion at midtropospheric levels.The difference between absolute and relative vorticity is planetary vorticity, which is just the local vertical component of the vorticity of the earth due to its rotation; $k\cdot\nabla\times U_e = 2\Omega sin\,\varphi \equiv f$ .Thus, $\eta = \zeta + f$ or, in Cartesian coordinates,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \eta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

The relationship between relative vorticity and relative circulation C discussed in the previous section can be clearly seen by considering an alternative approach in which the vertical component of vorticity is defined as the circulation about a closed contour in the horizontal plane divided by the area enclosed, in the limit where the area approaches zero:

$$\zeta \equiv \lim_{A \to 0} \left(\oint V \cdot dI \right) A^{-1} \qquad (4.8)$$

This latter definition makes explicit the relationship between circulation and vorticity discussed in the introduction to this chapter. The equivalence of these two definitions of ζ is shown easily by considering the circulation about a rectangular element of area δxδy in the (x, y) plane as shown in Fig.(4.4). Evaluating $V \cdot dI$ for each side of the rectangle in Fig.(4.4) yields the circulation

$$\delta C = u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta x - v\delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta x\delta y$$

Dividing through by the area $\delta A = \delta x\delta y$ gives

$$\frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \equiv \zeta$$

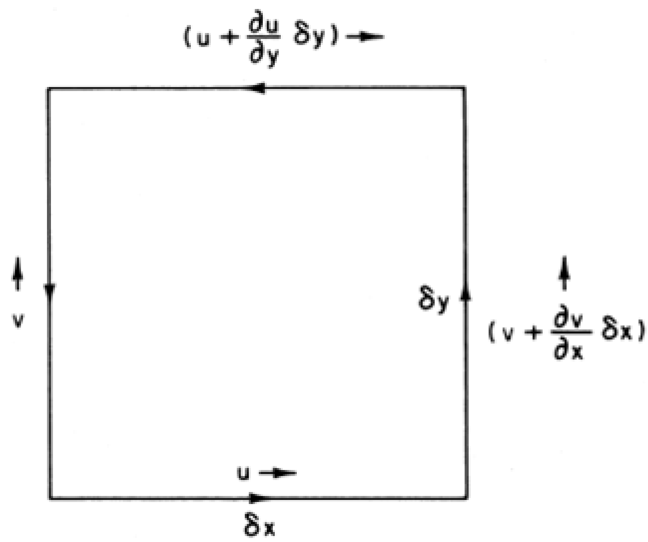


Fig.(4.4): Relationship between circulation and vorticity for an area element in the horizontal plane.

In more general terms the relationship between vorticity and circulation is given simply by Stokes' theorem applied to the velocity vector:

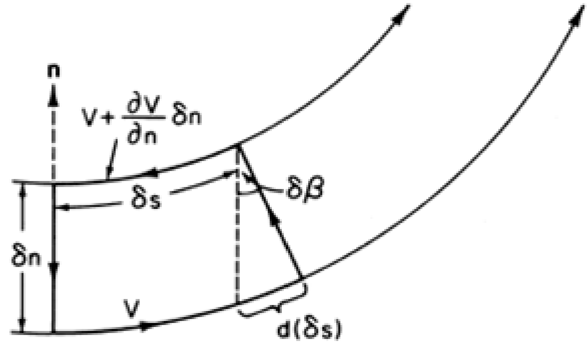
$$\oint \underline{U} \cdot dI = \iint_A (\nabla \times \underline{U}) \cdot n \, dA$$

Here A is the area enclosed by the contour and n is a unit normal to the area element dA (positive in the right-hand sense). Thus, Stokes'theorem states that the circulation about any closed loop is equal to the integral of the normal component of vorticity over the area enclosed by the contour. Hence, for a finite area, circulation divided by area gives the *average* normal component of vorticity in the region. As a consequence, the vorticity of a fluid in solid-body rotation is just twice the angular velocity of rotation. Vorticity may thus be regarded as a measure of the local angular velocity of the fluid.

In the following we study the Vorticity in Natural Coordinates, Physical interpretation of vorticity is facilitated by considering the vertical component of vorticity in the natural coordinate system . If we compute the circulation about the infinitesimal contour shown in Fig. (4.5), we obtain

$$\delta C = V[\delta s + d(\delta s)] - \left(V + \frac{\partial V}{\partial n}\delta n\right)\delta s$$

Fig.(4.5): Circulation for an infinitesimal loop in the natural coordinate system.



However, from Fig.(4.5), $d(\delta s) = \delta\beta\delta n$, where $\delta\beta$ is the angular change in the wind direction in the distance δs. Hence,

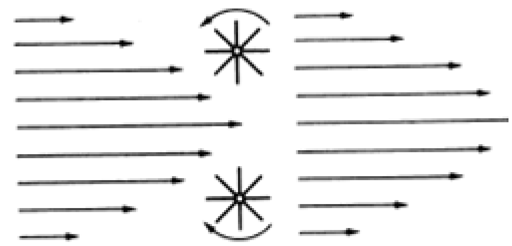
$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s}\right)\delta n\delta s$$

or, in the limit $\delta n, \delta s \rightarrow 0$

$$\zeta = \lim_{\delta n, \delta s \rightarrow 0} \frac{\delta C}{(\delta n\delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \quad (4.9)$$

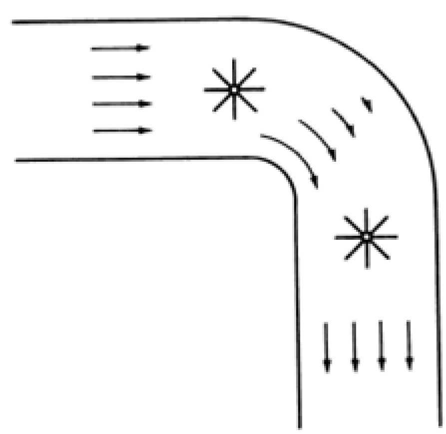
whereR_s is the radius of curvature of the streamlines [Eq. (3.20)]. It is nowapparent that the net vertical vorticity component is the result of the sum of two parts: (1) the rate of change of wind speed normal to the direction of flow $-\partial V/\partial n$, called the *shear* vorticity; and (2) the turning of the wind along a streamline V/R_s , called the *curvature* vorticity. Thus, even straight-line motion may have vorticity if the speed changes normal to the flowaxis. For example, in the jet stream shown schematically in Fig. (4.6a), there will be cyclonic relative vorticity north of the velocity maximum and anticyclonic relative vorticity to the south (Northern Hemisphere conditions) as is recognized easily when the turning of a small paddle wheel placed in the flow is considered. The lower of the two paddle wheels in Fig.4.6a will turn in aclockwise direction (anticyclonically) because the wind force on the blades north of its axis of rotation is stronger than the force on the blades to the south of the axis.The upper wheel will, of course, experience a counterclockwise (cyclonic) turning.Thus, the poleward and equatorward sides of a westerly jetstream are referred to as the cyclonic and anticyclonic shear sides, respectively.

Conversely, curved flowmay have zero vorticity provided that the shear vorticity is equal and opposite to the curvature vorticity. This is the case in the example shown in Fig.(4.6b) where a frictionless fluid with zero relative vorticity upstream flows around a bend in a canal. The fluid along the inner boundary on the curve flows faster in just the right proportion so that the paddle wheel does not turn.



(a)

Fig. (4.6): Two types of two-dimensional flow:
(a) linear shear flow with vorticity and
(b) curved flow with zero vorticity.



(b)

Now we study the potential vorticity,With the aid of the ideal gas law(1.17), the definition of potential temperature can be expressed as a relationship between pressure and density for a surface of constant θ:

$$\rho = p^{\frac{C_v}{C_p}}(R\theta)^{-1}(p_s)^{R/C_p}$$

Hence, on an isentropic surface, density is a function of pressure alone, and the solenoidal term in the circulation theorem vanishes;

$$\oint \frac{dp}{\rho} \alpha \oint dp^{(1-C_v/C_p)} = 0$$

Thus, for adiabatic flow the circulation computed for a closed chain of fluid parcels on a constant θ surface reduces to the same form as in a barotropic fluid; that is, it satisfies Kelvin's circulation theorem, which may be expressed as

$$\frac{D}{Dt}(C + 2\Omega\delta A\ sin\varphi) = 0 \qquad (4.10)$$

where C is evaluated for a closed loop encompassing the area δA on an isentropic surface. If the isentropic surface is approximately horizontal, and it is recalled from Equation(4.8) that $C \approx \zeta\delta A$, then for an infinitesimal parcel of air (4.10) implies that

$$\delta A(\zeta\theta + f) = Const \qquad (4.11)$$

where $\zeta\theta$ designates the vertical component of relative vorticity evaluated on an isentropic surface and $f = 2\Omega sin\ \varphi$ is the Coriolis parameter. Suppose that the parcel of Equation (4.11) is confined between potential temperature surfaces θ_0 and $\theta_0 + \delta\theta$, which are separated by a pressure interval $-\delta p$ as shown in Fig.(4.7). The mass of the parcel, $\delta M = (-\delta p/\mathrm{g})\delta A$, must be conserved following the motion. Therefore,

$$\delta A = -\frac{\delta M g}{\delta p} = \left(-\frac{\delta\theta}{\delta p}\right)\left(\frac{\delta M g}{\delta\theta}\right) = const \times g\left(-\frac{\delta\theta}{\delta p}\right)$$

as both δM and $\delta\theta$ are constants. Substituting into Equation (4.11) to eliminate δA and taking the $limit\ \delta p \to 0$, we obtain

$$p \equiv (\zeta_\theta + f)\left(-g\frac{\partial\theta}{\partial p}\right) = const \qquad (4.12)$$

The quantity P [units: $K\ k\mathrm{g}^{-1}\ m^2 s^{-1}$] is the isentropic coordinate form of Ertel's potential vorticity. It is defined with a minus sign so that its value is normally positive in the Northern Hemisphere.

According to Equation (4.12), potential vorticity is conserved following the motion in adiabatic frictionless flow. The term potential vorticity is used, as shown later, in connection with several other mathematical expressions. In essence, however,

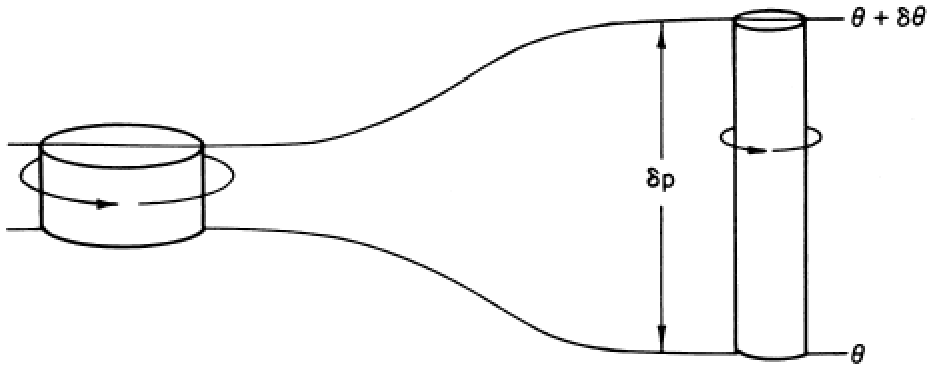


Fig. (4.7): A cylindrical column of air moving adiabatically, conserving potential vorticity.

potential vorticity is always in some sense a measure of the ratio of the absolute vorticity to the effective depth of the vortex. In Equation (4.12), for example, the effective depth is just the differential distance between potential temperature surfaces measured in pressure units $(-\partial\theta/\partial p)$.

In a homogeneous incompressible fluid, potential vorticity conservation takes a somewhat simpler form. In this case, because density is a constant, the horizontal area must be inversely proportional to the depth, h, of the fluid parcel:

$$\delta A = M(\rho h)^{-1} = const/h$$

where h is the depth of the parcel. Substituting to eliminate δA in Equation (4.11) yields

$$\frac{\zeta + f}{h} = \frac{\eta}{h} = Const \tag{4.13}$$

where ζ is here evaluated at constant height.

If the depth, h, is constant, Equation (4.13) states that absolute vorticity is conserved following the motion. Conservation of absolute vorticity following the motion provides a strong constraint on the flow, as can be shown by a simple example. Suppose that at a certain point (x_0, y_0) the flow is in the zonal direction and the relative vorticity vanishes so that $\eta(x_0, y_0) = f_0$. Then, if absolute vorticity is conserved, the motion at any point along a parcel trajectory that passes through (x_0, y_0) must satisfy $\zeta + f = f_0$. Because f increases toward the north, trajectories that curve northward in the downstream direction must have $\zeta = f_0 - f < 0$, whereas trajectories that curve southward must have $\zeta = f_0 - f > 0$. However,as indicated in Fig. (4.8), if the flow is westerly, northward curvature downstream implies$\zeta > 0$, whereas southward curvature implies $\zeta < 0$. Thus, westerly zonal flow must remain purely zonal if absolute vorticity is to be conserved following the motion. The easterly flow case, also shown in Fig. (4.8), is just the opposite. Northward and southward curvatures are associated with negative and positive relative vorticities, respectively. Hence, an easterly current can curve either to the north or to the south and still conserve absolute vorticity.When the depth of the fluid changes following the motion, it is potential vorticity that is conserved. However, again Equation (4.13) indicates that westerly and easterly flows behave differently. The situation for westerly flow impinging on an infinitely long

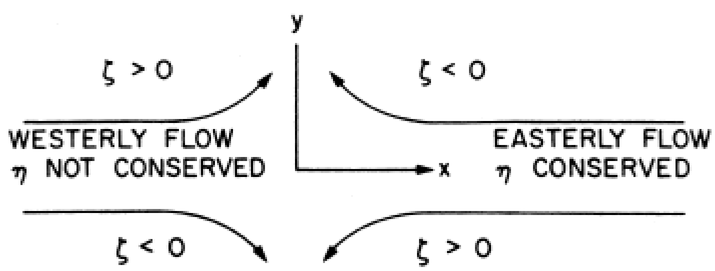


Fig. (4.8): Absolute vorticity conservation for curved flow trajectories.

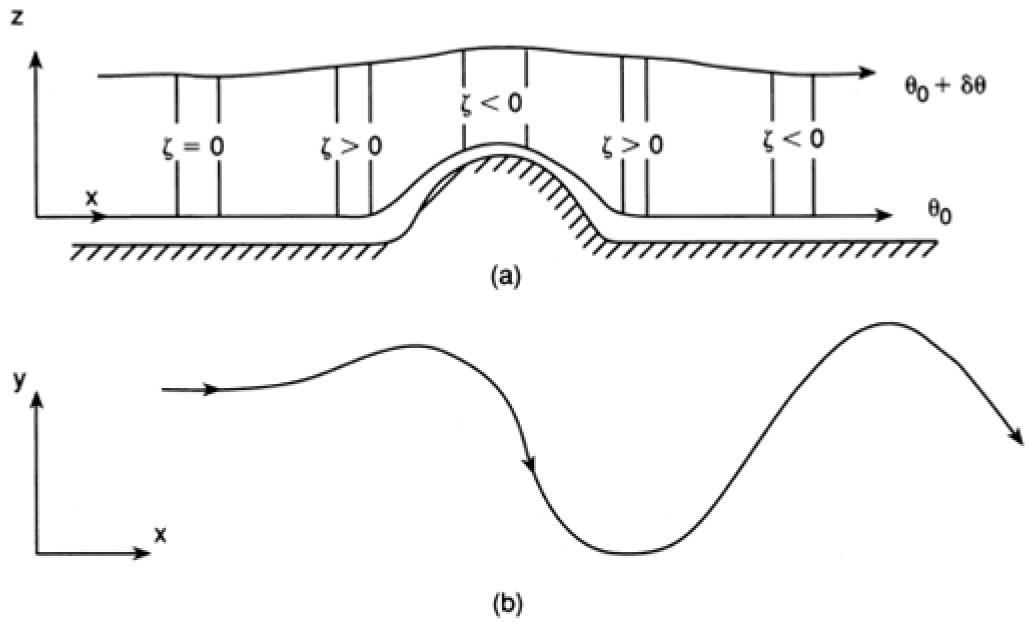


Fig. (4.9): Schematic view of westerly flow over a topographic barrier: (a) the depth of a fluid column as a function of x and (b) the trajectory of a parcel in the (x, y) plane.

topographic barrier is shown in Fig. (4.9). In Fig. (4.9a), a vertical cross section of the flow is shown. We suppose that upstream of the mountain barrier the flow is a uniform zonal flow so that $\zeta = 0$. If the flow is adiabatic, each column of air of depth h confined between the potential temperature surfaces θ_0 and $\theta_0 + \delta\theta$ remains between those surfaces as it crosses the barrier. For this reason, a potential temperature surface θ_0 near the ground must approximately follow the ground contours. A potential temperature surface $\theta_0 + \delta\theta$ several kilometers above the ground will also be deflected vertically. However, due to pressure forces produced by interaction of the flow with the topographic barrier, the vertical displacement at upper levels is spread horizontally; it extends upstream and downstream of the barrier and has smaller amplitude in the vertical than the displacement near the ground (see Figs.(4.9) and (4.10). As a result of the vertical displacement of the upper level isentropes, there is a vertical stretching of air columns upstream of the topographic barrier. (For motions of large horizontal scale, the upstream stretching is quite small.) This stretching causes h to increase, and hence from Equation(4.13) ζ must become positive in order to conserve potential vorticity.Thus, an air column turns cyclonically as it approaches the mountain barrier. This cyclonic curvature causes a poleward drift so that f also increases, which reduces the change in ζ required for potential vorticity conservation. As the column begins to cross the barrier, its vertical extent decreases; the relative vorticity must then become negative. Thus, the air column will acquire anticyclonic vorticity and move southward as shown in the x, y plane profile in Fig. (4.9b). When the air column has passed over the barrier and returned to its original depth, it will be south of its original latitude so that f will be smaller and the relative vorticity must be positive. Thus, the trajectory must have cyclonic

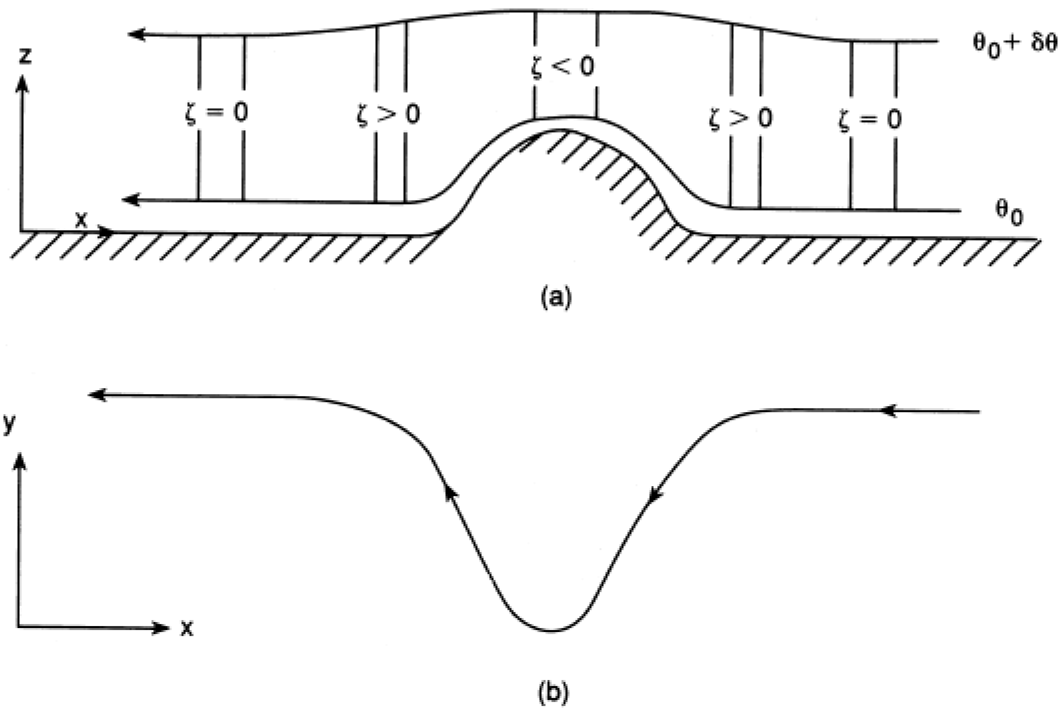
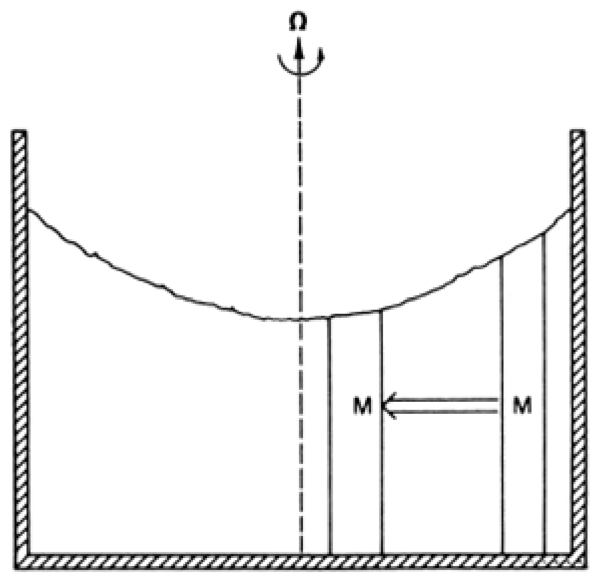


Fig. (4.10): As in Fig. (4.9), but for easterly flow.

curvature and the column will be deflected poleward. When the parcel returns to its original latitude, it will still have a poleward velocity component and will continue poleward gradually, acquiring anticyclonic curvature until its direction is again reversed. The parcel will then move downstream, conserving potential vorticity by following awave-like trajectory in the horizontal plane. Therefore, steady westerly flow over a large-scale ridge will result in a cyclonic flow pattern immediately to the east of the barrier (the lee side trough) followed by an alternating series of ridges and troughs downstream.The situation for easterly flowimpinging on a mountain barrier is quite different. As indicated schematically in Fig. (4.10b), upstream stretching leads to a cyclonic turning of the flow, as in the westerly case. For easterly flow this cyclonic turning creates an equatorward component of motion. As the column moves westward and equatorward over the barrier, its depth contracts and its absolute vorticity must then decrease so that potential vorticity can be conserved. This reduction in absolute vorticity arises both from development of anticyclonic relative vorticity and from a decrease in f due to the equatorward motion. The anticyclonic relative vorticity gradually turns the column so that when it reaches the top of the barrier it is headed westward. As it continues westward down the barrier, conserving potential vorticity, the process is simply reversed with the result that some distance downstream from the mountain barrier the air column again is moving westward at its original latitude. Thus, the dependence of the Coriolis parameter on latitude creates a dramatic difference between westerly and easterly flow over large-scale topographic barriers. In the case of a westerly wind, the barrier generates a wavelike disturbance in the streamlines that extends far downstream. However, in the

Fig. (4.11): Dependence of depth on radius in a rotating cylindrical vessel.



case of an easterly wind, the disturbance in the streamlines damps out away from the barrier.The situation described above, which assumes that the mountain barrier is infinitely long in the meridional direction, is highly idealized. In reality, because static stability tends to suppress vertical motions, large-scale flows in statically stable environments are blocked by topographic barriers and are forced to flow around mountains rather than over them. However, whether fluid columns go over or around topographic barriers, the potential vorticity conservation constraint still must be satisfied.The Rossby potential vorticity conservation law, Equation(4.13), indicates that in a barotropic fluid, a change in the depth is dynamically analogous to a change in the Coriolis parameter. This can be demonstrated easily in a rotating cylindrical vessel filled with water. For solid-body rotation the equilibrium shape of the free surface, determined by a balance between the radial pressure gradient and centrifugal forces, is parabolic. Thus, as shown in Fig. (4.11), if a column of fluid moves radially outward, it must stretch vertically. According to Equation (4.13), the relative vorticity must then increase to keep the ratio $(\zeta + f)/h$ constant. The same resultwould apply if a column of fluid on a rotating sphere were moved equatorward without a change in depth. In this case, ζ would have to increase to offset the decrease of . Therefore, in a barotropic fluid, a decrease of depth with increasing latitude has the same effect on the relative vorticity as the increase of the Coriolis force with latitude.

Now we stuydthe vorticity equation,The previous section discussed the time evolution of the vertical component of vorticity for the special case of adiabatic frictionless flow. This section uses the equations of motion to derive an equation for the time rate of change of vorticity without limiting the validity to adiabatic motion.

In the following we study the Cartesian Coordinate Form ,For motions of synoptic scale, the vorticity equation can be derived using the approximate horizontal momentum Equations (2.24) and (2.25). We differentiate the zonal component equation with respect to y and the meridional component Equation with respect to x:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right) \qquad (4.14)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + f = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right) \qquad (4.15)$$

Subtracting Equation (4.14) from Equation(4.15) and recalling that $\zeta = \partial v/\partial x - \partial u/\partial y$, we obtain the vorticity Equation

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial z} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$+ \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + v\frac{df}{dy}$$

$$= \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \quad (4.16)$$

Using the fact that the Coriolis parameter depends only on y so that $\frac{Df}{Dt} = \quad v(df/dy)$, Equation(4.16) may be rewritten in the form

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$+ \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \quad (4.17)$$

Equation (4.17) states that the rate of change of the absolute vorticity following the motion is given by the sum of the three terms on the right, called the divergence term, the tilting or twisting term, and the solenoidal term, respectively. The concentration or dilution of vorticity by the divergence field [the first term on the right in Equation (4.17)] is the fluid analog of the change in angular velocity resulting from a change in the moment of inertia of a solid body when angular momentum is conserved. If the horizontal flow is divergent, the area enclosed by a chain of fluid parcels will increase with time and if circulation is to be conserved, the average absolute vorticity of the enclosed fluid must decrease (i.e., the vorticity will be diluted). If, however, the flow is convergent, the area enclosed by a chain of fluid parcels will decrease with time and the vorticity will be concentrated. This mechanism for changing vorticity following the motion is very important in synoptic-scale disturbances.The second term on the right in Equation(4.17) represents vertical vorticity generated by the tilting of horizontally oriented components of vorticity into the vertical by

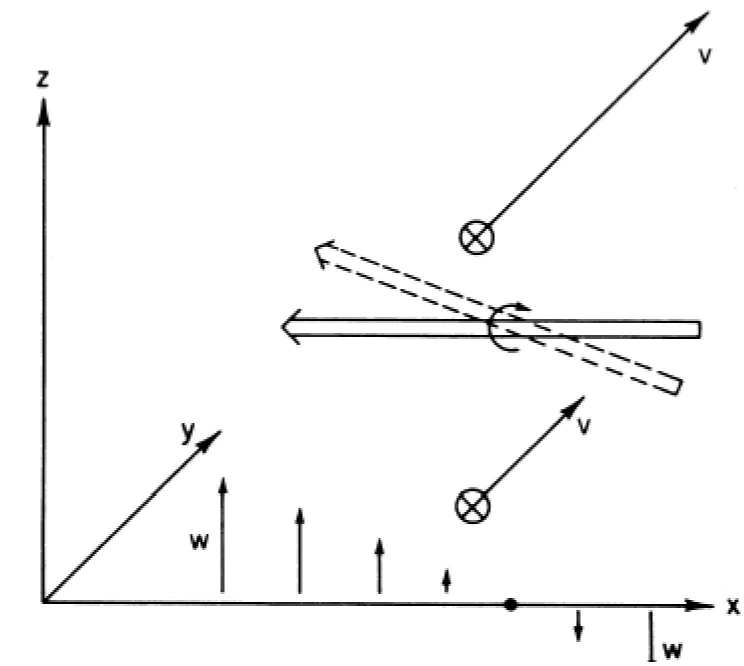


Fig.(4.12): Vorticity generation by the tilting of a horizontal vorticity vector (double arrow).

anonuniform vertical motion field. This mechanism is illustrated in Fig. (4.12), which shows a region where the y component of velocity is increasing with height so that there is a component of shear vorticity oriented in the negative x direction as indicated by the double arrow. If at the same time there is a vertical motion field in which w decreases with increasing x, advection by the vertical motion will tend to tilt the vorticity vector initially oriented parallel to x so that it has a component in the vertical. Thus, if $\partial v/\partial z > 0$ and $\partial w/\partial x < 0$, there will be a generation of positive vertical vorticity. Finally, the third term on the right in Equation(4.17) is just the microscopic equivalent of the solenoidal term in the circulation theorem (4.5). To show this equivalence,we may apply Stokes' theorem to the solenoidal term to get

$$-\oint \alpha\, dp \equiv -\oint \alpha \nabla p \cdot dI = -\iint_A \nabla \times (\alpha \nabla p \cdot k\, dA)$$

where A is the horizontal area bounded by the curve I. Applying the vector identity $\nabla \times (\alpha \nabla p) \equiv \nabla\alpha \times \nabla p$, the equation becomes

$$-\oint \alpha dp = -\iint_A (\nabla\alpha \times \nabla p) \cdot kdA$$

However, the solenoidal term in the vorticity equation can be written

$$-\left(\frac{\partial\alpha}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial\alpha}{\partial y}\frac{\partial p}{\partial x}\right)=-(\nabla\alpha\times\nabla p)\cdot k$$

Thus, the solenoidal term in the vorticity equation is just the limit of the solenoidal term in the circulation theorem divided by the area when the area goes to zero. Now we discussThe Vorticity Equation in Isobaric Coordinates, A somewhat simpler form of the vorticity Equation arises when the motion is referred to the isobaric coordinate system. This equation can be derived in vector form by operating on the momentum Equation (3.2) with the vector operator $k\cdot\nabla\times$, where ∇ now indicates the horizontal gradient on a surface of constant pressure. However, to facilitate this process it is desirable to first use the vector identity

$$(V\cdot\nabla)V=\nabla\left(\frac{V\cdot V}{2}\right)+\zeta k\times V \tag{4.18}$$

Where $\zeta=k\cdot(\nabla\times V)$, to rewrite Equation(3.2) as

$$\frac{\partial V}{\partial t}=-\nabla\left(\frac{V\cdot V}{2}+\Phi\right)-(\zeta+f)k\times V-\omega\frac{\partial V}{\partial p} \tag{4.19}$$

We now apply the operator $k\cdot\nabla\times$ to Equation(4.19). Using the facts that for any scalar $A,\nabla\times A=0$ and for any vectors a,b,

$$\nabla\times(a\times b)=(\nabla\cdot b)a-(a\cdot\nabla)b$$

$$-(\nabla\cdot a)b+(b\cdot\nabla)a \tag{4.20}$$

we can eliminate the first term on the right and simplify the second term so that the resulting vorticity equation becomes

$$\frac{\partial\zeta}{\partial t}=-V\cdot\nabla(\zeta+f)-\omega\frac{\partial\zeta}{\partial p}$$

$$-(\zeta+f)\nabla\cdot V+k\cdot\left(\frac{\partial V}{\partial p}\times\nabla\omega\right) \tag{4.21}$$

Comparing Equations(4.17) and (4.21), we see that in the isobaric system there is no vorticity generation by pressure-density solenoids. This difference arises because in the isobaric system, horizontal partial derivatives are computed with p held constant so that the vertical component of vorticity is $\zeta = (\partial v/\partial x - \partial u/\partial y)p$ whereas in height coordinates it is $= (\partial v/\partial x - \partial u/\partial y)z$. In practice the difference is generally unimportant because as shown in the next section, the solenoidalterm is usually sufficiently small so that it can be neglected for synoptic-scale motions.

In the following we study the Scale Analysis of the Vorticity Equation, the equations of motion were simplified for synoptic-scale motions by evaluating the order of magnitude of various terms. The same technique can also be applied to the vorticity equation. Characteristic scales for the field variables based on typical observed magnitudes for synoptic-scale motions are chosen as follows:

$U \sim 10 m\, s^{-1}$ horizontal scale

$W \sim 1\, cm\, s^{-1}$ vertical scale

$L \sim 106\, m$ length scale

$H \sim 104\, m$ depth scale

$\delta p \sim 10$ hPa horizontal pressure scale

$\rho \sim 1\, kg\, m^{-3}$ mean density

$\delta\rho/\rho \sim 10^{-2}$ fractional density fluctuation

$L/U \sim 105\, s$ time scale

$f_0 \sim 10^{-4}\, s^{-1}$ Coriolis parameter

$\beta \sim 10^{-11}\, m^{-1}\, s^{-1}$ “beta” parameter

Again we have chosen an advective time scale because the vorticity pattern, like the pressure pattern, tends to move at a speed comparable to the horizontal wind speed. Using these scales to evaluate the magnitude of the terms in Equation(4.16), we first note that

$$\zeta = \partial v/\partial x - \partial u/\partial y \lesssim U/L \sim 10^{-5} s^{-1}$$

where the inequality in this expression means less than or equal to in order of magnitude. Thus,

$$\frac{\zeta}{f_0} \lesssim \frac{U}{(f_0 L)} \equiv R_0 \sim 10^{-1}$$

For midlatitude synoptic-scale systems, the relative vorticity is often small (order Rossby number) compared to the planetary vorticity. For such systems, ζ may be neglected compared to f in the divergence term in the vorticity Equation

$$(\zeta + f)(\partial u/\partial x + \partial v/\partial y) \approx f(\partial u/\partial x + \partial v/\partial y)$$

This approximation does not apply near the center of intense cyclonic storms. In such systems $|\zeta/f| \sim 1$, and the relative vorticity should be retained. The magnitudes of the various terms in Equation(4.16) can now be estimated as

$$\frac{\partial\zeta}{\partial t}, u\frac{\partial\zeta}{\partial x}, v\frac{\partial\zeta}{\partial y} \sim \frac{U^2}{L^2} \sim 10^{-10} s^{-2}$$

$$w\frac{\partial\zeta}{\partial z} \sim \frac{WU}{HL} \sim 10^{-11} s^{-2}$$

$$v\frac{df}{dy} \sim U\beta \sim 10^{-10} s^{-2}$$

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \lesssim \frac{f_0 U}{L} \sim 10^{-9} s^{-2}$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) \lesssim \frac{WU}{HL} \sim 10^{-11} s^{-2}$$

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \lesssim \frac{\delta\rho\delta p}{\rho^2 L^2} \sim 10^{-11} s^{-2}$$

The inequality is used in the last three terms because in each case it is possible that the two parts of the expression might partially cancel so that the actual magnitude would be less than indicated. In fact, this must be the case for the divergence term (the fourth in the list) because if $\partial u/\partial x$ and $\partial v/\partial y$ were not nearly equal and opposite, the divergence term would be an order of magnitude greater than any other term and the equation could not be satisfied. Therefore, scale analysis of the vorticity equation indicates that synoptic-scale motions must be quasi-non divergent.The divergence term will be small enough to be balanced by the vorticity advection terms only if

$$\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right| \lesssim 10^{-6} s^{-1}$$

so that the horizontal divergence must be small compared to the vorticity in synoptic-scale systems. From the aforementioned scalings and the definition of the Rossby number, we see that

$$\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) / f_0\right| \lesssim {R_0}^2$$

and

$$\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) / \zeta\right| \lesssim R_0$$

Thus the ratio of the horizontal divergence to the relative vorticity is the same magnitude as the ratio of relative vorticity to planetary vorticity.
Retaining only the terms of order $10^{-10}\ s^{-2}$ in the vorticity equation yields the approximate form valid for synoptic-scale motions,

$$\frac{D_h(\zeta + f)}{Dt} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (4.22a)$$

Where

$$\frac{D_h}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

As mentioned earlier, Equation(4.22a) is not accurate in intense cyclonic storms. For these the relative vorticity should be retained in the divergence term:

$$\frac{D_h(\zeta + f)}{Dt} = \quad (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (4.22b)$$

Equation (4.22a) states that the change of absolute vorticity following the horizontal motion on the synoptic scale is given approximately by the concentration or dilution of planetary vorticity caused by the convergence or divergence of the horizontal flow, respectively. In Equation(4.22b), however, it is the concentration or dilution of absolute vorticity that leads to changes in absolute vorticity following the motion.The form of the vorticity equation given in Equation(4.22b) also indicates why cyclonic disturbances can be much more intense than anticyclones. For a fixed amplitude of convergence, relative vorticity will increase, and the factor $(\zeta + f)$ becomes larger, which leads to even higher rates of increase in the relative vorticity. For a fixed rate of divergence, however, relative vorticity will decrease, but when $\zeta \to -f$, the divergence term on the right approaches zero and the relative vorticity cannot become more negative no matter how strong the divergence. The approximate forms given in Equations (4.22a) and (4.22b) do not remain valid, however, in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only $\sim 100\ km$, and the vertical velocity scale is $\sim 10\ cm\ s^{-1}$. For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

Section(4.2) :Modeling Equations Of Atmosphere-Ocean

Now we study vorticity in barotropic fluids.A model that has proved useful for elucidating some aspects of the horizontal structure of large-scale atmospheric motions is the barotropic model. In the most general version of this model, the atmosphere is represented as a homogeneous incompressible fluid of variable depth, $h(x, y, t) = z_2 - z_1$, where $z_2$and z_1 are the heights of the upper and lower boundaries, respectively. In this model, a special form of potential vorticity is conserved following the motion. A simpler situation arises if the fluid depth is constant. In that case it is absolute vorticity that is conserved following the motion.

In the following we present The Barotropic (Rossby) Potential Vorticity Equation .For a homogeneous incompressible fluid, the continuity Equation simplifies to $\nabla \cdot U = 0$ or, in Cartesian coordinates,

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\partial w}{\partial z}$$

so that the vorticity Equation (4.22b) may be written

$$\frac{D_h(\zeta + f)}{Dt} = (\zeta + f)\left(\frac{\partial w}{\partial z}\right) \qquad (4.23)$$

Letting the vorticity in Equation (4.23) be approximated by the geostrophic vorticity ζ_g and the wind by the geostrophic wind (u_g, v_g), we can integrate vertically from z_1 to z_2 to get

$$h\frac{D_h(\zeta_g + f)}{Dt} = (\zeta_g + f)[w(z_2) - w(z_1)] \qquad (4.24)$$

However, because

$$w \equiv \frac{Dz}{Dt}$$

and

$$h \equiv h(x, y, t),$$

$$w\,(z_2) - w\,(z_1) = \frac{Dz_2}{Dt} - \frac{Dz_1}{Dt} = \frac{D_h h}{Dt} \qquad (4.25)$$

Substituting from Equation(4.25) into Equation (4.24) we get

$$\frac{1}{(\zeta_g + f)}\frac{D_h(\zeta_g + f)}{Dt} = \frac{1}{h}\frac{D_h h}{Dt}$$

Or

$$\frac{D_h \ln(\zeta_g + f)}{Dt} = \frac{D_h \ln h}{Dt}$$

which implies that

$$\frac{D_h}{Dt}\left(\frac{\zeta_g + f}{h}\right) = 0 \qquad (4.26)$$

This is just the potential vorticity conservation theorem for a barotropic fluid, The quantity conserved following the motion in Equation (4.26) is the Rossby potential vorticity.

Now we studyThe Barotropic Vorticity Equation. If the flow is purely horizontal ($w = 0$), as is the case for barotropic flow in a fluid of constant depth, the divergence term vanishes in Equation (4.23) and we obtain the barotropic vorticity equation

$$\frac{D_h(\zeta_g + f)}{Dt} = 0 \qquad (4.27)$$

which states that absolute vorticity is conserved following the horizontal motion. More generally, absolute vorticity is conserved for any fluid layer in which the divergence of the horizontal wind vanishes, without the requirement that the flow be geostrophic. For horizontal motion that is nondivergent $(\partial u/\partial x + \partial v/\partial y = 0)$, the flow field can be represented by a streamfunction $\psi\ (x, y)$ defined so that the velocity components are given as $u = -\partial\psi/\partial y, v = +\partial\psi/\partial x$. The vorticity is then given by

$$\zeta = \partial v/\partial x - \partial u/\partial y = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 \equiv \nabla^2\psi$$

Thus, the velocity field and the vorticity can both be represented in terms of the variation of the single scalar field $\psi\ (x, y)$, and Equation (4.27) can be written as a prognostic Equation for vorticity in the form:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\mathrm{V}_\psi \cdot \nabla(\nabla^2\psi + f) \qquad (4.28)$$

where $\mathrm{V}_\psi \equiv\ k \times \nabla\psi$ is a nondivergent horizontal wind. Equation (4.28) states that the local tendency of relative vorticity is given by the advection of absolute vorticity. This equation can be solved numerically to predict the evolution of the streamfunction, and hence of the vorticity and wind fields.
Because the flowin the midtroposphere is often nearly nondivergent on the synoptic scale, Equation (4.28) provides a surprisingly good model for short-term forecasts of the synoptic-scale 500-hPa flow field.

Now we discuss Equations of Motion in Isentropic Coordinates
If the atmosphere is stably stratified so that potential temperature θ is a monotonically increasing function of height, θ may be used as an independent vertical coordinate. The vertical "velocity" in this coordinate system is just $\theta \equiv D\theta/Dt$. Thus, adiabatic motions are two dimensional when viewed in an isentropic coordinate frame. An infinitesimal control volume in isentropic coordinates with crosssectional area δA and vertical extent $\delta\theta$ has a mass

$$\delta M = \rho\delta A\delta z = \delta A\left(-\frac{\delta p}{\mathrm{g}}\right) = \frac{\delta A}{\mathrm{g}}\left(-\frac{\partial p}{\partial\theta}\right)\delta\theta = \sigma\delta A\delta\theta \qquad (4.29)$$

Here the "density" in (x, y, θ) space (i.e., as shown in Fig. 4.7 the quantity that when multiplied by the "volume" element $\delta A\delta\theta$ yields the mass element δM) is defined as

$$\sigma \equiv -g^{-1}\frac{\partial p}{\partial \theta} \tag{4.30}$$

The horizontal momentum equation in isentropic coordinates may be obtained by transforming the isobaric form Equation (4.19) to yield

$$\frac{\partial \mathrm{V}}{\partial t} + \nabla_\theta\left(\frac{\mathrm{V}\cdot\mathrm{V}}{2} + \psi\right) + (\zeta_\theta + f\,)k \times \mathrm{V} = -\dot{\theta}\frac{\partial \mathrm{V}}{\partial \theta} + F_r \tag{4.31}$$

where ∇_θ is the gradient on an isentropic surface, $\zeta_\theta \equiv k \cdot \nabla_\theta \times V$ is the isentropic relative vorticity originally introduced in Equation (4.11), and $\psi \equiv c_p T + \Phi$ is the Montgomery streamfunction . We have included a frictional term F_r on the right-hand side, along with the diabatic vertical advection term. The continuity equation can be derived with the aid of Equation (4.29) .
The result is

$$\frac{\partial \sigma}{\partial t} + \nabla_\theta \cdot (\sigma \mathrm{V}) = -\frac{\partial}{\partial \theta}(\sigma\dot{\theta}) \tag{4.32}$$

The ψ and σ fields are linked through the pressure field by the hydrostatic equation, which in the isentropic system takes the form

$$\frac{\partial\,\psi}{\partial \theta} = \prod(p) \equiv c_p\left(\frac{p}{p_s}\right)^{R\backslash c_p} = c_p\frac{T}{\theta} \tag{4.33}$$

where $\prod$ is called the *Exner function*. Equations (4.30)–(4.33) form a closed set for prediction of V, σ, ψ , and p, provided that $\dot{}$ θ and F_r are known.

Now we illusterate The Potential Vorticity Equation, If we take $k \cdot \nabla_\theta \theta \times$ Equation(4.31) and rearrange the resulting terms, we obtain the isentropic vorticity equation:

$$\frac{\widetilde{D}}{Dt}(\zeta_\theta + f\,) + (\zeta_\theta + f\,)\nabla_\theta \cdot \mathrm{V} = k \cdot \nabla_\theta \times \left(F_r - \dot{\theta}\frac{\partial \mathrm{V}}{\partial \theta}\right) \tag{4.34}$$

where

$$\frac{\widetilde{D}}{Dt} = \frac{\partial}{\partial t} + V \cdot \nabla_\theta$$

is the total derivative following the horizontal motion on an isentropic surface.

Noting that $\sigma^{-2}\partial\sigma/\partial t = -\partial\sigma^{-1}/\partial t$, we can rewrite Equation (4.32) in the form

$$\frac{\widetilde{D}}{Dt}(\sigma^{-1}) - (\sigma^{-1})\nabla_\theta \cdot V = \sigma^{-2}\frac{\partial}{\partial\theta}(\sigma\dot{\theta}) \quad (4.35)$$

Multiplying each term in Equation(4.34) by $\sigma^{-1}$and in Equation (4.35) by$(\zeta_\theta + f\ )$ and adding, we obtain the desired conservation law:

$$\frac{\widetilde{DP}}{Dt} = \frac{\partial P}{\partial t} + \mathrm{V} \cdot \nabla_\theta P = \frac{P}{\sigma}\frac{\partial}{\partial\theta}(\sigma\dot{\theta}) + \sigma^{-1}k \cdot \nabla_\theta \times \left(F_r - \dot{\theta}\frac{\partial \mathrm{V}}{\partial\theta}\right) \quad (4.36)$$

where $P \equiv (\zeta_\theta + f\ )/\sigma$ is the Ertel potential vorticity defined in Equation(4.12). If the diabatic and frictional terms on the right-hand side of Equation (4.36) can be evaluated, it is possible to determine the evolution of P following the horizontal motion on an isentropic surface. When the diabatic and frictional terms are small, potential vorticity is approximately conserved following the motion on isentropic surfaces.Weather disturbances that have sharp gradients in dynamical fields, such as jets and fronts, are associated with large anomalies in the Ertel potential vorticity. In the upper troposphere such anomalies tend to be advected rapidly under nearly adiabatic conditions. Thus, the potential vorticity anomaly patterns are conserved materially on isentropic surfaces. This material conservation propertymakes potential vorticity anomalies particularly useful in identifying and tracing the evolution of meteorological disturbances.

Now we study Integral Constraints on Isentropic Vorticity.The isentropic vorticity Equation (4.34) can be written in the form

$$\frac{\partial\zeta_\theta}{\partial t} = \nabla_\theta \cdot [(\zeta_\theta + f\ )\mathrm{V}] + k \cdot \nabla_\theta \times \left(F_r - \dot{\theta}\frac{\partial \mathrm{V}}{\partial\theta}\right) \quad (4.37)$$

Using the fact that any vector A satisfies the relationship

$$\mathrm{k} \cdot (\nabla_\theta \times \mathrm{A}) = \nabla_\theta \cdot (\mathrm{A} \times \mathrm{k})$$

we can rewrite Equation (4.37) in the form

$$\frac{\partial\zeta_\theta}{\partial t} = -\nabla_\theta \cdot \left[(\zeta_\theta + f\ )\mathrm{V} - \left(F_r - \dot{\theta}\frac{\partial \mathrm{V}}{\partial\theta}\right)\right] \times k \quad (4.38)$$

Equation (4.38) expresses the remarkable fact that isentropic vorticity can only be changed by the divergence or convergence of the horizontal flux vector in brackets on the right-hand side. The vorticity cannot be changed by vertical transfer across the isentropes. Furthermore, integration of Equation(4.38) over the area of an isentropic surface and application of the divergence theorem show that for an isentrope that does not intersect the surface of the earth the global average of ζ_θ is constant. Furthermore, integration of ζ_θ over the sphere shows that the global average ζ_θ is exactly zero. Vorticity on such an isentrope is neither created nor destroyed; it is merely concentrated or diluted by horizontal fluxes along the isentropes.

In the following we study the modeling ,and we start with the primitive equations (PEs) of the atmosphere . physical laws governing the motion and states of the atmosphere and ocean can be described by the general equations hydrodynamics and thermodynamics. Using a non inertial coordinate system rotating with the earth, these equations can be written as follows:

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla_3 \vec{V} + 2\vec{\Omega} \times \vec{V} - \vec{g} + \frac{1}{\rho} grand_3 p = \vec{D}_M,$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_3(\overrightarrow{\rho V}) = 0,$$

$$c_v \frac{\partial T}{\partial t} + c_v \vec{V} \cdot \nabla_3 T + \frac{p}{\rho} \nabla_3 \vec{V} = Q + D_{H'} \tag{4.39}$$

$$\frac{\partial q}{\partial t} + \vec{V} \cdot \nabla_3 q = \frac{S}{\rho} + D_{q'}$$

$$p = R\rho T.$$

Here the first equation is the momentum equation, the second is the continuity equation, the third is the first law of thermodynamics, the fourth is the diffusion equation for the humidity, and the last is the equation of state for an ideal gas. the unknown functions are the three -dimension velocity field $\vec{V}$, the density function ρ, the pressure function p,the temperature function T,and the specific humidity function . moreover ,in the above equations, $\vec{\Omega}$ stands for the angular velocity of earth, $\vec{g}$ the gravity, R the gas conistant, c_v the specific heat at constant volume $\vec{D}_M$ the viscosity terms, D_H the temperature diffusion, Q the heat flux per unit density at the unit time interval, which includes molecule or turbulent, radiative and evaporative heating , and S the differences of the rates of the evaporation and condensation .

These equations are normally for too complicated; simplifications form both the physical and mathematical points of view are necessary.There are essentially two characteristics of both the atmosphere and ocean, which are used in simplifying the equations. The first one is that for large scale geophysical flows, the ratio between the vertical and horizontal scales is very small; this leads to the primitive equations (PEs) of both the atmosphere and the ocean, which are the basic equations for these two fluids. More precisely, the PEs are obtained from the general equations of hydrodynamics and thermodynamics of the compressible atmosphere, by approximating the momentum equation in the vertical direction with the hydrostatic equation:

$$\frac{\partial p}{\partial z} = -\rho \mathrm{g} \qquad (4.40)$$

This hydrostatic equation is based on the ratio between the vertical and horizontal scale being small. Here r is the density, g the gravitational constant, and $z = r - a$ height above the sea level, r the radial distance, and a the mean radius of the earth. Equation (4.40) expresses the fact that p is a decreasing function along the vertical so that one can use p instead of z as the vertical variable. Motivated by this hydrostatic approximation, we can introduce a generalized vertical coordinate system *s*-system given by

$$s = s(\theta, \varphi, z, t), \qquad (4.41)$$

where s is a strict monotonic function of z. Then the basic equations of the large-scale atmospheric motion in the s-system are

$$\frac{\partial v}{\partial t} + v \cdot \nabla_s v + \dot{s}\frac{\partial v}{\partial s} + fk \times v + \frac{1}{\rho}\nabla_s z = D_M,$$

$$\frac{\partial p}{\partial s}\frac{\partial s}{\partial z} + \rho \mathrm{g} = 0$$

$$\frac{\partial}{\partial s}\left(\frac{\partial p}{\partial s}\right)_s + \nabla_s \cdot \left(v\frac{\partial p}{\partial s}\right) + \frac{\partial}{\partial s}\left(\dot{s}\frac{\partial p}{\partial s}\right) = 0, \qquad (4.42)$$

$$c_v\frac{\partial T}{\partial t} + c_v v.\nabla_s T + c_v\dot{s}\frac{\partial T}{\partial s} + \frac{1}{\rho}\left(\frac{\partial p}{\partial t} + v \cdot \nabla_s p + \dot{s}\frac{\partial p}{\partial s}\right) = Q + D_H,$$

$$\frac{\partial q}{\partial t} + v \cdot \nabla_s q + \dot{s}\frac{\partial q}{\partial s} = \frac{s}{\rho} + D_q.$$

Some common *s*-systems inmeteorology are respectively the *p*-system(the pressure coordinate), the σ-system (the transformed pressure coordinate), the θ-system (the isentropic coordinate), and the z-system (the topographic coordinate or transformed height coordinate). sometimes the pressure coordinate is denoted by η, and the terrain-following by σ.

For simplicity, here we discuss only the case with the coordinate transformation from (θ, φ, z) to (θ, φ, p). The basic equations of the atmosphere are then the Primitive Equations (PEs) of the atmosphere in the p-coordinate system. As they appear in classical meteorology books ,the PEs are given by

$$\frac{\partial v}{\partial t} + v \cdot \nabla v + \omega \frac{\partial v}{\partial s} + 2\Omega \cos\theta \times v + \nabla \Phi = D_M,$$

$$\frac{\partial \Phi}{\partial p} + \frac{RT}{p} = 0$$

$$\mathrm{div} v + \frac{\partial \omega}{\partial p} = 0, \tag{4.43}$$

$$\frac{\partial T}{\partial t} + v . \nabla T + \omega \left(\frac{kT}{\partial p} - \frac{\partial T}{\partial p} \right) = \frac{\tilde{Q}_{rad}}{c_p} + \frac{\tilde{Q}_{con}}{c_p} + D_H,$$

$$\frac{\partial q}{\partial t} + v \cdot \nabla q + \omega \frac{\partial q}{\partial s} = E - C + D_q.$$

where D_M is the dissipation term for momentum and D_H and D_q are diffusion terms for heat and moisture, respectively, E and C are the rates of evaporation and condensation due to cloud processes, c_p the heat capacity, and $\tilde{Q}_{rad}$ and $\tilde{Q}_{con}$ the net radiative heating and the heating due to condensation processes, respectively. We use the pressure coordinate system$(\theta, \varphi, p)$where $\theta(0 < \theta < \pi)$ and $\varphi(0 < \varphi < 2\pi)$ are the colatitude and longitude variables, and p the pressure of the air. The nondynamical processes $\tilde{Q}_{rad}, \tilde{Q}_{con}$, E and C are called model physics. Furthermore, the unknown functions are the horizontal velocity v, the vertical velocity $\omega = dp/dt$, the geopotential Φ, the temperature T, and the specific humidity q. The operators div and ∇ are the two dimensional operators on the sphere.

Now we study the Ocean models.The sea water is almost an incompressible fluid, leading to the Boussinesq approximation, i.e., a variable density is only recognized in the buoyancy term and the equation of state. The resulting equations are called the Boussinesq equations given as follows

$$\frac{\partial v}{\partial t} + \nabla_v v + \omega\frac{\partial v}{\partial z} + \frac{1}{\rho_0}\mathrm{grad}\rho + fk \times v - \mu\Delta v - v\frac{\partial^2 v}{\partial z^2} = 0,$$

$$\frac{\partial w}{\partial t} + \nabla_v w + w\frac{\partial w}{\partial z} + \frac{1}{\rho_0}\frac{\partial \rho}{\partial z} + \frac{\rho}{\rho_0}\mathrm{g} - \mu\Delta w - v\frac{\partial^2 w}{\partial z^2} = 0,$$

$$\mathrm{div}v + \frac{\partial w}{\partial z} = 0, \qquad (4.44)$$

$$\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial s} - \mu_T\Delta T - v_T\frac{\partial^2 T}{\partial z^2} = 0,$$

$$\frac{\partial s}{\partial t} + \nabla_v s + w\frac{\partial s}{\partial z} - \mu_s\Delta s - v_s\frac{\partial^2 s}{\partial z^2} = 0,$$

$$\rho = \rho_0\big(1 - \beta_T(T - \bar{T}_0) + \beta_s(s - \bar{s}_0)\big),$$

where *v* is the horizontal velocity field, *w* the vertical velocity, and *S* the salinity. The sixth equation in (4.44) is an empirical equation for the density function based on thelinear approximation. In general, density *r* is a nonlinear function of *T*, *S*,and *p*.With higher approximations, one will encounter additional mathematical difficulties although the nonlinear equation of state is essential for some elements of ocean circulation (e.g.,cabbeling).

As in the atmospheric case, the hydrostatic assumption is usually used, leading to the PEs for the large-scale ocean:

$$\frac{\partial v}{\partial t} + \nabla_v v + \omega\frac{\partial v}{\partial z} + \frac{1}{\rho_0}\nabla p + fk \times v - \mu_v\Delta_a v - v_v\frac{\partial^2 v}{\partial z^2} = 0,$$

$$\frac{\partial p}{\partial z} = -\rho \mathrm{g},$$

$$\mathrm{div}v + \frac{\partial w}{\partial z} = 0, \qquad (4.45)$$

$$\frac{\partial T}{\partial t} + \nabla_v T + w\frac{\partial T}{\partial z} - \mu_T\Delta_a T - v_T\frac{\partial^2 T}{\partial z^2} = 0,$$

$$\frac{\partial s}{\partial t} + \nabla_v s + w\frac{\partial s}{\partial z} - \mu_s \Delta s - \nu_s \frac{\partial^2 s}{\partial z^2} = 0,$$

$$\rho = \rho_0\left(1 - \beta_T(T - \bar{T}_0) + \beta_s(s - \bar{s}_0)\right).$$

Also, we note that if the hydrostatic assumption is made first, the Boussinesq approximation is not really necessary.

In the following we illusterate the Some theoretical and computational issues for the PEs, and we start with the Dynamical systems perspective of the models From the mathematical point of view, we can put the models in the previous in the perspective of infinite-dimensional dynamical systems as follows:

$$\varphi_t + A\varphi + R(\varphi) = F,$$
$$\varphi \mid_{t=0} = \varphi_0, \tag{4.46}$$

defined on an infinite-dimensional phase space H. Here A:$H \rightarrow H$ is an unbounded linear operator, $R: H \rightarrow H$ is a nonlinear operator, F is the forcing term, and φ_0 is the initial data.

We remark here that the linear operator can usually be written as $A = A_1 + A_2$, where A_1 stands for the irreversible diabatic linear processes of energy dissipation, and $A_2$for the reversible adiabatic linear processes of energy conversation. The nonlinear term $R(\varphi)$ represents the reversible adiabatic nonlinear processes of energy conversation. The properties of these operators reflect directly the essential characteristics of two kinds of basic processes with entirely different physical meanings.

The above formulation is often achieved by (a) establishing a proper functional setting of the model, and (b) proving the existence and uniqueness of the solutions.

Here after we demonstrate the procedurewith the PEs. Due to some technical reasons, some minor and physically reasonable modifications of the PEs are made. In particular, we assume that the model physics, $\tilde{Q}_{rad}$, $\tilde{Q}_{con}$, E and C are given functions of location and time.

$$D_M = -L_1 v,$$

$$\frac{\tilde{Q}_{rad}}{c_p} + \frac{\tilde{Q}_{con}}{c_p} + D_H = -L_2 T + Q_T,$$

$$E - C + D_q = -L_{3q} + Q_q, \tag{4.47}$$

$$L_i = -\mu_i \nabla - v_i \frac{\partial}{\partial p}\left(\left(\frac{gp}{RT(p)}\right)^2 \frac{\partial}{\partial p}\right),$$

where μ_i, v_i are horizontal and vertical viscosity and diffusion coefficients, Δ is the Laplace operator on the sphere, Q_T and Q_q are treated as given functions, and $\bar{T}(p)$ a given temperature profile, which can be considered as the climate average of T. The boundary conditions for the PEs are given by

$$\frac{\partial}{\partial p}(v, T, q) = \left(\tilde{\gamma}_s(\tilde{v}_s - v), \tilde{\alpha}_s(\tilde{T}_s - T), \tilde{\alpha}_q(\tilde{q}_s - q)\right), \omega = 0 \text{ at } p = P$$

$$\frac{\partial}{\partial p}(v, T, q) = 0, \omega = 0\, atp = p_0 \tag{4.48}$$

The second and third equations (4.43) are diagnostic ones; integrating themin *p*-direction,we obtain

$$\int_{p_0}^{P} \mathrm{div} v(p') \, dp' = 0$$

$$\omega = W(v) = -\int_{p_0}^{p} \mathrm{div} v(p') \, dp' = 0 \tag{4.49}$$

$$\Phi = \Phi_s + \int_{p}^{P} \frac{RT(p')}{p'} dp'$$

Then the PEs are equivalent to the following functional formulation:

$$\frac{\partial u}{\partial t} + \Lambda(v)u + P(u) + Lu + (\nabla\Phi_s, 0,0) = (0, Q_T, Q_q)$$

$$\mathrm{div} \int_{p_0}^{P} v dp = 0, \tag{4.50}$$

where

$$u = (v, T, q), \qquad \Lambda(v)u = \nabla_v u + W(v)\frac{\partial u}{\partial p},$$

Lu corresponds to the viscosity and diffusion terms, and Pu the lower order terms. We solve then the PEs in some infinite-dimensional phase spaces H and V. In particular,we use

$$H_1 = \left\{ v \in L^2 \mid \text{div} \int_{p_0}^{P} v dz = 0 \right\}$$

as the phase space for the horizontal velocity v. Thenwe project in the phase space. Using this projection, the unknown function Φ_s plays a role as a Lagrangian multiplier, which can be recovered by the following decomposition:

$$L^2 = H_1 \oplus H_1^{\perp} ,$$

$$H_1^{\perp} = \{ v \in L^2 \mid v = \nabla \Phi_s, \Phi_s \in H^1(S_a^2) \}.$$

Then the PEs are equivalent to an infinite-dimensional dynamical system as equation (4.46).

With the above formulation, for example, we encounter the following new nonlocalStokes problem:

$$\begin{gathered} - \Delta v + \nabla \Phi_s = f , \\ \text{div} \int_{p_0}^{P} v dp = 0, \end{gathered} \tag{4.51}$$

supplemented with suitable boundary conditions. From the mathematical point of view,all techniques for the regularity of solutions are local. But our problem here is nonlocal;the regularity of the solutions for this problem can be obtained using Nirenberg's finite difference quotient method .

Othermodels such as the PEs of the ocean and the coupled atmosphere-ocean models can be viewed as infinite-dimensional dynamical systems in the form of equation(4.46)).

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