Chapter (4)

The Maximal Tori and Application of the Morse Theory to the Topology of Lie Groups

We will describe some results on the structure of compact connected Lie groups, focusing on the important notion of a maximal torus which is central to the classification of simple compact connected Lie Groups.

Section (4.1): Compactly Connected Lie Groups and their Maximal Tori

Although we state results for arbitrary Lie groups we will often only give proofs for matrix groups. However, there is no loss in generality in assuming this because of the following important result which we will not prove (the proof uses ideas of Haar measure and integration on such compact Lie groups).

Theorem (4.1.1):

Let G be a compact Lie group. Then there are injective Lie Homeomorphisms $G \to O(m) and\ G \to U(n)$ for some m, n. Hence G is a matrix group.

In the following we will discuss Tori .

The circle group

$$T = \{z \in \mathbb{C} : |z| = 1\} \leqslant \mathbb{C}^{\times}$$

is a matrix group since $\mathbb{C}^{\times} = GL_1(\mathbb{C})$. for each r$\geqslant$ 1, the standard torus of rank r is

$$T^r = \{diag\{z_1, \dots, z_r\} \forall k,\ |z_k| = 1\} \leqslant GL_r(\mathbb{C}).$$

This is a matrix group of dimension $\mathfrak{r}$. More generally, a torus of rank $\mathfrak{r}$ is a Lie group isomorphic to T^r. We will often view elements of T^r as sequences of complex numbers $(z_1, \dots, z_r)$ with $|z_k|$=1, this corresponds to the identification.

$$T^r \cong T \times \dots \times T \leq (\mathbb{C}^\times)^r \qquad (\mathfrak{r}\text{factors}).$$

Such a torus is a compact path connected abelian Lie group.

Now Let Gbe Lie group and $T \leq G$ a closed subgroup which is a torus. Then T is maximal in G if the only Tours $T' \leq G$ for which T $\leq T'$ is T itself. Here are some examples.

For $\theta \in [0,2\pi)$, let

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2).$$

More generally, for each n≥ 1, and $\theta_i \in [0.2\pi)$ (I =1,…., n), let

$$R_{2n}(\theta_1 \dots \dots, \theta_2) = \begin{bmatrix} R(\theta)1 & 0 & \dots & \cdots & \cdots & 0 \\ 0 & R((\theta)_2) & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \dots & \dots & 0 & R(\theta_n) \end{bmatrix} \in SO(2n),$$

$$R_{2n} + 1(\theta_1 \dots \dots, \theta_n) = \begin{bmatrix} R(\theta_1) & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & R(\theta_2) & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & R(\theta_n) & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

$$\in SO(2n + 1),$$

Where each entry marked O is an appropriately sized black so that these are matrices of size $2n \times 2n$ and $(2n+1) \times (2n+1)$ respectively.

By identifying$\mathbb{C}$ with R^2 as real vector spaces using the bases $\{1, i\}$ and $\{e_1,e_2\}$, we obtain an isomorphism.

$$U(1) \to SO(2), \quad e^{\theta i} \to R_1(\theta).$$

Proposition (4.1.2):

Each of the following is a maximal tours in the stated group.

$$\{R_{2n}(\theta_1, \dots, \theta_n): \forall k, \quad \theta_k \in [0, 2\pi)\} \leq SO(2n).$$

$$\{R_{2n+1}(\theta_1, \dots, \theta_n): \forall k, \quad \theta_k \in [0, 2\pi)\} \leq SO(2n+1)$$

$$\{diag(z_1, \dots, z_n): \forall k, |z_k| = 1\} \leq U(n).$$

$$\{diag(z_1, \dots, z_n): \forall k, |z_k| = 1, \quad z_1 \dots z_n = 1\} \leq SU(n).$$

$$\{diag(z_1, \dots, z_n): \forall k \ z_k \in \mathbb{C}, |z_k| = 1\} \leq SP(n).$$

The maximal tori listed will be referred to as the standard maximal tori for these groups.

Proposition (4.1.3):

Let T be a torus. Then T is compact, path connected and abelian.

Proof:

Since the circle T is compact and abelian the same is true for T^r and hence for any torus.

If $(z_1, \dots, z_r) \in T^r$, let $z_k = e^{\theta k i}$.Then there is a continuous path

$$p: [0,1] \to T^r; \quad p(t) = \left(e^{t\theta 1 i}, \dots, e^{t\theta r i}\right),$$

With $P(0) = (1,\dots,1)$ and $p(1) = (z_1,\dots,z_r)$. So T^r and hence any torus is path connected

Theorem (4.1.4):

Let H be a compact Lie group. Then H is a torus if and only if it is connected and abelian.

Proof:

We know that H is a compact Lie group. Every torus is path connected and ablian by proposition (4.1.3) so we need to show that when H is connected and abelian it is a torus and it would be path connected. .

Suppose that dim H = r and let $\mathfrak{h}$be the Lie algebra of H; then dim $\mathfrak{h}$ = r. from the definition of the Lie for X,Y∈ $\mathfrak{h}$.

$$[X,Y] = \frac{d}{d\,s}_{|s=0} \frac{d}{d\,t}_{|t=0} \exp(sX)\exp(tY)\exp(-sX) = 0$$

Since exp(sX), exp(tY) ∈ H and so $\exp(sX)\exp(tY)\exp(-sX)$

= exp(tY) because H is abelian. Thus all Lie brackets in $\mathfrak{h}$ are

zero. Consider the exponential map $\exp: \mathfrak{h} \to H$. For $X, Y \in \mathfrak{h}$,

$$\exp(X)\exp(Y) = \exp(X+Y), \quad \exp(-X) = \exp(X)^{-1}$$

So exp$\mathfrak{h}$ = im exp ⊆ H is a subgroup. Exp $\mathfrak{h}$ is a subgroup containing neighbor-hood of 1 hence exp $\mathfrak{h}$=H.

As exp is continuous homomorphism. its kernel k = her exp must be discrete since otherwise dim exp ($\mathfrak{h}$)<r. This means that k ⊆ $\mathfrak{h}$ is a free abelian subgroup with basis (v_1, … ,v_s} for some s≤ r.

Extending this to an R-basis $\{v_1, \dots, v_s, v_{s+1}, \dots, v_r\}$ of $\mathfrak{h}$ we obtain isomorphism'sofLie groups

$$\exp(\mathfrak{h}) \cong \mathfrak{h}/k \cong R^s/Z^s \times R^{r-s}.$$

but the right hand term is only compact if s=r, hence k contains a basic of $\mathfrak{h}$ and

$$R^r/Z^r \cong \mathfrak{h}/k \cong H.$$

Since T ≅ R/Z. this gives H the structure of a torus.

Proposition (4.1.5):

Let T be a torus of rank r. Then the exponential map exp: $t \to T$ is a surjective homomorphism of Lie groups, whose kernel is a discrete subgroup isomorphic to $\mathbb{Z}^r$. Hence there is an isomorphism of lie groups $\mathbb{R}^r/\mathbb{Z}^r \cong T$.

Definition (4.1.6):

Let *G* be a Lie group. Then an element $g \in G$ is a topological generator or just a generator of *G* if the cyclic subgroups of *G* if the cyclic subgroup $\langle g \rangle \leq G$ is dense in *G*, i.e., $\overline{\langle g \rangle} = G$.

Proposition (4.1.7):

every torus T has a generator.

Proof:

Without loss of generality we can assume $T = \mathbb{R}^r/\mathbb{Z}^r$ and will write elements in the form $[x_1, \ldots x_r] = [x_1, \ldots x_r] + \mathbb{Z}^r$. The group operation is then addition let $U_1, U_2, U_3, \ldots$ be a countable base for the topology on *T*.

A cube of side $\epsilon > 0$ in T is a subset of the form.

$$C([u_1, \ldots . u_r], \epsilon) = \{x_1, \ldots, x_r] \in T: |x_k - u_k| < \epsilon/2 \; \forall k\},$$

For some $[u_1, \ldots, u_r] \in T$.Such a cube is the image of a cube in R^r under the quotient map $R^r \to T$..

Let $C_0 \subseteq T$ be a cube of side $\varepsilon > 0$ .suppose that we have a decreasing sequence of cubes C_k of side ε_k .

$$C_o \supseteq C_1 \supseteq \cdots \supseteq C_m,$$

Where for each $0 \leq k \leq m$, there is an integer N_k satisfying $N_k \epsilon_k > 1$ and $N_k C_k \subseteq U_k$. Now choose an integer N_{m+1} large enough to quarantine that $N_{m+1} C_m = T$. Now choose a small cube $C_{m+1} \subseteq C_m$ of side ε_{m+1}so that

$N_{m+1}C_{m+1} \subseteq U_{m+1}$.Then if $z = [z_1, \ldots, z_r] \in \cap_{k \geq 1} C_k$. We have $N_k z \in C_k$ for each k, hence the power of z are dense in T, so Z is a generator of T.

Now we will discuss maximal tori in compact Lie groups

We begin to study the structure of compact Lie groups in terms of their maximal tori. Throughout the section, let G be a compact connected Lie groups and $T \leq G$ a maximal torus.

Theorem (4.1.8):

if $g \in G$,There is an $x \in G$ such that $G \in xT_{x-1}$, i.e. g is conjugate to an element of T. Equivalently.

$$G = \bigcup_{x \in G} xTx^{-1}$$

Proof:

The proof this uses the powerful Lefschetz fixed point theorem from Algebraic Topology and we only give a sketch indicating how this is used.

The quotient space G/T is a compact space and each element $g \in G$ gives rise to a continuous map

$$\mu_g : \frac{G}{T} \to \frac{G}{T}; \quad \mu_g(xT) = (gx)T = gxT.$$

Since G is path connected, there is a continuous map

$$p: [0,1] \times G/T \to G/T;$$

For which $p(0, xT) = xT$ and $P(1, xT) = gxT$, i.e., p is homotopy$Id_{G/T} \simeq \mu_g$.

The lefschetz fixed point theorem asserts that μ_g *has* a fixed point provided the Euler characteristic χ(G/T) is non-Zero. Indeed it can be shown that χ(G/T)≠0, so this tells us that there is an x∈ G such that gxT= xT, or equivalently g ∈xTx^{-1}.

Theorem (4.1.9):

If T,T'≤ G are maximal tori then they are conjugate in G, i.e., there is a y ∈ G such that T' = yTy^{-1}.

Proof:

by preposition (4.1.7)T' has a generator t say. By Theorem (4.1.8), there is a y ∈G such that t ∈, so $T' \leq yTy^{-1}$AsT' is a maximal torus and $yTy^{-1}$is a torus, and yTy^{-1} is a torus, we must have $T'=yTy^{-1}$.

The next result gives some important special cases related to the examples of proposition (4.1.2).

Notice that if A∈SO (m),$A^{-1}=A^T$ while if B ∈ U (m), $B^{-1} = B^*$.

Theorem (4.1.10):(Principle Axis Theorem)

In each of the following matrix groups every element is conjugate to one of the stated form.

- SO (2n): $R_{2n}(\theta_1, \dots, \theta_n)$, $\forall k\ \theta_k \in [1,2\pi)$;
- SO (2n+1): $R_{2n+1}(\theta_1, \dots, \theta_n)$, $\forall k\ \theta_k \in [0,2\pi)$;
- U (n): $\text{diag}(z_1, \dots, z_n)$, $\forall k\ z_k \in C, |z_k| = 1$;
- SU(n): $\text{diag}(z_1, \dots, z_n)$, $\forall k\ z_k \in C, |z_k| = 1, z_1 \dots z_n = 1$
- SP (n): $\text{diag}(z_1, \dots, z_n)$, $\forall k\ z_k \in C, |z_k| = 1$.

We can also deduce a results on the Lie algebra g of such a compact, connected matrix group G. Recall that for each g∈ G, there is a linear transformation.

$$Ad_g{:}G \rightarrow g;\ Ad_g\,(t) = gtg^{-1}.$$

Proposition (4.1.11):

suppose that g ∈ G and H,H' ≤ G are Lie subgroups with $gHg^{-1} = H'$.

Then $Ad_g\mathfrak{h}=\mathfrak{h}'$.

Proof:

By definition for $x \in \mathfrak{h}$ there is a curve $:(-\epsilon,\epsilon) \to H$ with γ (0) =1 and $\acute{\gamma}$ (0) = x.

Then

$$Ad_g(t) = \frac{d}{dt} g\gamma(t) g^{-1}_{[t=0} \in \acute{\mathfrak{h}},$$

Since t → g$\gamma(t)_g$-1 is a cure inH'.

If x,y∈g and y = $Ad_g(x)$ we will say that x is conjugate in G to y. This defines an equivalence relation on g.

For t ∈ R, let

$$R'(t) = \begin{bmatrix} 0 & -t \\ t & 0 \end{bmatrix}$$

and

$$R'_{2n}(t_1,\dots,t_n) = \begin{bmatrix} R'(t_1) & O & \cdots & \cdots & \cdots & O \\ O & R'(t_2) & O & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ O & \cdots & \cdots & \cdots & O & R'(t_n) \end{bmatrix} \in so(2n)$$

$$R'_{2n+1}(t_1,\dots,t_n) = \begin{bmatrix} R'(t_1) & O & \cdots & \cdots & \cdots & \cdots & O \\ O & R'(t_2) & O & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & O & R'(t_n) & O \\ O & \cdots & \cdots & \cdots & \cdots & O & 1 \end{bmatrix} \in so(2n+1)$$

Theorem (4.1.12): (Principle Axis Theorem for Lie Algebras)

For each of the following Lie algebras, every element x ∈g is conjugate in G to one of the stated form.

- SO (2n): $R'_{2n^n}(t_1, \dots \dots t_n)$, $\forall k \theta_k \in [0,2\pi)$;
- SO (2n+1): $R'_{2n+1}(t_1, \dots \dots t_n)$, $\forall k \theta_k \in [0,2\pi)$;
- U (n): diag$(t_{1i}, \dots \dots z_{ni})$, $\forall k t_k \in R$;
- SU (n): diag$(t_{1i}, \dots \dots z_{ni})$, $\forall k t_k \in R, t_1 + \cdots t_n = 1$;
- SP (n) : diag$(t_{1i}, \dots \dots t_{ni})$, $\forall k t_k \in R$.

We can now give an important result which we have already seen is true for many familiar examples.

Theorem (4.1.13):

Let G be a compact, connected Lie group. Then the exponential map exp: g → G is subjective.

Proof:

Let T ≤ G be a maximal tours. By Theorem (4.1.8) every element g∈ G is conjugate to an element $xgx^{-1} \in$ T. By proposition (4.1.5)$xgx^{-1} = \exp(t)$ for some $t \in t$, hence

$$g = x^{-1} \exp(t)\, x = \exp(Ad_x(t)),$$

whereAd_x(t) ∈ g. so g ∈ exp g. Therefore expg= G .

In the following we will discuss the normalize and Wey1 group of a maximal torus.

Given Theorem (4.1.8) we can contiunue to develop the general theory for a compact connected Lie group G

Proposition (4.1.14):

Let A $\leq$G be a compact abelian Lie group and suppose that $A_1 \leq$ A is the connected component of the identify element, if A/A_1 is cyclic then A has a generator and hence A is contained in a torus in G.

Proof:

Let d =$|A/A_1|$. As A_1 is connected and Abelian, it is a torus by Theorem (4.1.4) hence it has a generator a_0 by proposition (4.1.7) Leg g $\in$ A be an element of A for which the coset gA_1 generates A/A_1.

Notice that $g^d \in A_1$ and therefore $a_o g^{-d} \in A_1$. Now choose b $\in A_1$ so that $a_o g^{-d} = b^d$. Then $a_0 = (g_d)^d$, cost g^r A_1. Hence the powers of g_b are dense in A, which shows that this element is a generator of A.

Let $T \leq G$ be a maximal tours. By Theorme (4.1.8) any generator u of A is contained in a maximal tour xTx^{-1} conjugate to T. Hence $\langle u \rangle$ and its closure A are contained in xTx^{-1} which completes the proof of the proposition.

Proposition (4.1.15):

$Let\ A\ \leq G$be a connected obelian subgroup and let $g \in G$ commute with all the elements of A. Then there is a torus $T \leq G$ containing the subgroup $(A, g) \leq G$ generated by A and g.

Proof:

By replacing A by its closure which is also connected, we can assume that A is closed in G, hence compact and so a torus, by Theorem (4.1.4)Now consider the abelian subgroup $\langle A, g \rangle \leq G$generated by A and g, whose closure B $\leq$G is again compact and abelian. If the connected component of the identity is $B_1 \leq$B then B_1has finitely many cosets by compactness, and these is of the form $g^r B_1$ (r = 0,1,….d -1) for some d. By Proposition (4.1.14),$\langle A, g\rangle$is contained in a torus.

Theorem (4.1.16):

Let T ≤G be a maximal torus and let T ≤A ≤G where A is abelian. Then A = T.

Equivalently, every maximal torus is a maximal abelian subgroup.

Proof:

For each element g ∈A, Proposition (4.1.15), implies that there is a torus containing $\langle T, g\rangle$ but by the maximality of T this must equal T. Hence A = T.

We have now established that every maximal torus is also a maximal abelian subgroup, and that any two maximal tori are conjugate in G.

Recall that for a subgroup H ≤G, the normalizer of H in G is

$$N_G(H) = \{g \in G: gHg^{-1} = H\}.$$

Then $N_G(H)$ ≤G is a closed subgroup of G, hence compact. It also contains H and its closure in G asnormal subgroups. There is a continuous left action of $N_G(H)$ on H by conjugation, i.e., for g ∈$N_G(H)$and h ∈H, the action is given by

$$g.h = ghg^{-1}.$$

If H = T is a maximal torus in G, the quotient group $N_G(T)/T$ acts on T since T acts trivially on itselfby conjugation. Notice that the connected component of the identity in $N_G(T)$ contains T, in fact itagrees with T by the following Lemma.

Lemma (4.1.17):

Let T ≤G be a torus and let Q ≤N_G(T) be a connected subgroup acting on T byconjugation. Then Q acts trivially, i.e., for g ∈Q and x ∈T,

$$g.x = gxg^{-1} = x.$$

Proof:

Recall that T ≅R^r/Z^ras Lie groups. By Proposition (4.1.5) the exponential map is a surjectivegroup homomorphism exp: $t \to T$whose kernel is a discrete subgroup. In fact, there is a commutative

diagram

kerexp→ t → T

$$\cong\downarrow\cong\downarrow \quad \cong\downarrow$$

$$Z^r \to \quad R^t \to R^r/Z^r$$

in which all the maps are the evident ones.

Now a Lie group automorphism$\alpha\colon T \to T$lifts to homomorphism $\tilde{\alpha}\colon t \to t$restricting to anisomorphism $\breve{\alpha}_0\colon \ker \exp \to \ker exp$.Indeed, since each element of ker exp ≅ $z_r$is uniquely divisiblein t ≅$\mathbb{R}^r$, continuity implies that $\breve{\alpha}_0$determines $\breve{\alpha}_0$on t. But the automorphism group Aut(kerexp) ≅Aut($\mathbb{Z}^r$) of kerexp≅$\mathbb{Z}^r$is a discrete group.

From this we see that the action of Q on T by conjugation is determined by its restriction to theaction on ker exp. As Q is connected, every element of Q gives rise to the identity automorphism of thediscrete group Aut(kerexp). Hence the action of Q on T is trivial.

This result shows that $N_G(T)_1$, the connected component of the identity in N_G(T), acts trivially onthe torus T. In fact, if g ∈N_G(T) acts trivially on T then it commutes with all the elements of T, so byTheorem (4.1.16)$g$is in T. Thus T consists of all the elements of G with this property, i.e.,

$$T = \{g \in G: gxg^{-1} = x \forall x \in T\}. \tag{4.1}$$

In particular, we have $N_G(T)_1$= T.

The Weyl group of the maximal torus T in G is the quotient group

$$W_G(T) = \frac{N_G(T)}{T} = \pi_0 N_G(T)$$

Which is also the group of path components $\pi_0 N_G(T)$. The Weyl group $W_G(T)$ acts on T by conjugation,i.e., according to the formula.

$$gT.x = gxg^{-1}.$$

Theorem (4.1.18):

Let T $\leq$G be a maximal torus. Then the Weyl group $W_G(T)$ is fnite and actsfaithfully on T, i.e., the cosetg$_T \in N_G(T)/T$ acts trivially on T if and only if g $\in$T.

Proof:

$N_G(T)$ has fnitely many cosets of T since it is closed, hence compact, so each coset isclopen. The faithfulness of the action follows from Equation (4.1).

Proposition (4.1.19):

Let T $\leq$G be a maximal torus and x, $y \in$T. If x, $y$are conjugate in G thenthey are conjugate in $N_G(T)$, hence there is an element $w \in W_G(T)$ for which y = $w \cdot x$.

Proof:

Suppose that y $=gxg^{-1}$. Then the centralizer $C_G(y) \leq G$ of y is a closed subgroupcontaining T. It also contains the maximal torus gTg^{-1}since every element of this commutes with y.

Let H = $C_G(y)1$, the connected component of the identity in $C_G(y)$; this is a closed subgroup of G sinceit is closed in $C_G(y)$. Then as $T, gTg^{-1}$are connected subgroups of $C_G(y)$ they are both contained inH. So $T, gTg^{-1}$are tori in H and must be maximal since a torus in H containing one of these would bea torus in G where they are already maximal.

By Theorem (4.1.8) applied to the compact connected Lie group$H$, $gTg^{-1}$is conjugate to T in H, sofor some h $\in$Hwe have $gTg^{-1}= hTh^{-1}$which gives

$$(h^{-1}g)T(h^{-1}g)^{-1} = T.$$

Thus $h^{-1}g \in N_G(T) and$

$$(h^{-1}g)x(h^{-1}g)^{-1} = h^{-1}yh = y.$$

Now setting $w = h^{-1}gT \in W_G(T)$we obtain the desired result.

Section (4.2): An Application of the Morse Theory to the Topology of Lie Groups

Theorem (4.2.1):

If G is a connected or simply connected compact lie group then the spacesΩ(G) have the following properties :

a) They are free of torsion ;
b) Their odd Betti numbers vanish ;
c) Their Betti numbers can be read off form the diagram of G.

The manner in which the diagram of G determines these Betti numbers is the following one. Let D denote this diagram on the tangent space ,t, to amaximal torus $T \subset G$. we let Ɉbe fundamental chamber of D and denote by Δ the fundamental cells of D.

If $X \in t$, λ(x) shall be defined as the number of planes of D crossed by the straight line joining X to the origin of t. The function λ is constant in each fundamental cell and this constant value is denoted by $\lambda(\Delta)$. In general if s is any line segment in t, $\lambda(s)$ shall equal the number of planes of D crossed by s. we denote by Γ the lattice of G in D.

We study the space, Ω(G,N(X),p) of paths in G starting on N(X) and ending at p = exp$\tilde{p}$.

Theorem (4.2.2):

The space Ω(G, N(X),p) is free of torsion for any $X \in t$. Its odd Betti numbers vanish. The Poincare series of this space is given by

$$P(\Omega(G, N(X), p); t) = \sum t^{2\lambda(s)},$$

Where s runs over the straight. Line segments of $\tilde{s}\left(t, \widetilde{N}(X), \tilde{p}\right)$.

In the following we will discuss J-FIELDS.

M shall denote ac^{∞},par compact manifold in affixed complete Riemannian structure. Instead of the term, c^{∞}, we often use "smooth". Let $M_p, p \in M, stand\ for\ the\ tongent\ space$ [λ]to M at p, and if X,Y ∈ M_p, (X,Y) is their inner product.

A map

$$g: R \rightarrow M$$

of the real numbers, R , into M is called geodesic , if :

a) g satisfies the differential equations of a geodesic [10], for all t ∈ R,
b) t is proportional to arc-length form g(0).

The restriction of g to a nontrivial interval of the type[0, a], 0< a, in R, will be called a segment of g. segments will always be denoted by the symbol s.

A smooth functions:t→ Y_t, which assigns to every t ∈ R, a tangent Vector $Y_t$in$M_{g(t)}$, is by definition a "vector field along g ".

The tangent field along g is by definition the assignment t ∈ X_t, where

$$X_t f = \lim_{h \to 0}[f\{g(t+h)\} - f\{g(t)\}] / h$$

For smooth functions, f , on M.

If h ∈ R, let $h_*: M_{g(t)} \rightarrow M_{g(t+h)}$be the isomorphism, of the two spaces in question, which is defined by parallel translating the vectors of the first space, along g, into the second one. Now if t→ Y_t, is afield along g, the formula

$$Y_t' = \lim_{h \to 0}\{Y_t \rightarrow h_* Y_{t-h}\}/h$$

defines anew field: t→ Y_t' along g – the covariant of Y along g.

In the theory of Morse the vector space of "infinitesimal isometries of g" plays a crucial role. We refer to it as the space of Jacobi (or just J)-fields along g, and denote it by J_g. this vector space can be defined in terms of the notion of a $\langle\langle variation\rangle\rangle$ of g. such a variation shall be a family of maps

$$V_x: \mathrm{R} \rightarrow M$$

Indexed by α in some vicinity of o on R satisfying the following conditions:

c) $V_\propto(t)$depends smoothly on ∝ and t;
d) $V_\propto$ is a geodesic for each ∝;
e) V_0 = g.

Definition (4.2.3):

A vector field: t→ Y_t is a J-field along g, if and only if there exists a variation $V_\propto$ of g, so that

$$Y_t = \frac{d}{d_\propto} f(V_\propto(t)]_{\propto=0} \qquad (t \in R).$$

In the following we will discuss FOCAL points.

Let N be a proper, smooth, sub manifold of M.

A geodesic, g, will be said to be a geodesic of (M,N) – or of M mod N- If:

a) G starts on N;
b) The initial direction of g is perpendicular to N.

Similarly we speak of geodesic segments, s, on (M,N). for these objects the infinitesimal variations which preserve (a) and (b) are of special interest. They constitute a subspace, J_g^N, relative to N. precisely:

Definition (4.2.4):

The field t → Y_t is contained in J_g^N if and only if there exists a variation $V_\propto$ of g such that

$$Y_t = \frac{d}{d_\propto} f(V_\propto(t)]_{\propto=0} \qquad (t \in R).$$

$$V_\propto(o) \in N,$$

$$\frac{d}{d_t} V_\propto(t)\ |_{t=0} \in N\frac{1}{q} \qquad \{q = V_\propto(o)\},$$

Where $N\frac{1}{q}$ denotes the orthogonal complement of N_q in M_q.

For a segment s of g, J_s^N is again defined as the image of J_g^N in J_s under the restriction map: s_* : $J_g \rightarrow J_s$.

Definition (4.2.5):

Let s be a geodesic segment M mod N. the sub space of J_s^N consisting of the J-fields which vanish at the end-point of s shall be referred to as the focal kernel of s mod N, and will be denoted by A^N(s).

Also,

a) If dimA^N(s)> o, then s is called a focal segment of N in M;
b) The end-point of a focal segment is called a focal point of N in M;
c) The set of all focal points of N in M is called set of N in M.

Proposition (4.2.6):

An element Y ∈ J_g is in J_g^N if and only if

$$Y_o \in N_P,$$

$$Y_o' + T_g . Y_o \in N\frac{1}{P}$$

Where: p is the initial point, g(o), of g, and T_g is a self-adjoint linear Transformation of N_P, determined completely by g.the form $(T_g X, Y)$ onN_P is the second fundamental form at p relative to g.

An immediate corollary is

$$\dim J_g^N = \dim N.$$

$T_g$can be defined in several ways.

Suppose g starts at p ∈ N, with tangent-vector X ∈ $N\frac{1}{P}$. it is the fact that X is per-pendicular to N at p which gives rise to the transformation T_g on N_P. To see what T_g.Y, Y ∈ N_P, is, let y(t) be a curve on N, starting at p in direction Y. by parallel translation X defines a field, $\tilde{X}$, along y(t). Let t→ X_t^* be the field along y(t) which assigns to t the orthogonal projection of$\tilde{X}_t$ on $N_{y(t)}$. Now , because X_o^* vanishes, the limit

$$\lim_{t\to 0} X_t^* / t$$

Exists, and defines the vector T_g. Y in N_P.

A point of M – N which is not in the focal set of (M,N) is called a regular point of (M,N) the regular points of (M,N) are plentiful. Namely.

Proposition (4.2.7):

Let s be a geodesic segment of (M,N). then the index of s, relative to N, defined by

$$\lambda^N(s) = \sum_{S'\subset S} \dim A^N(S')$$

isalways a finite integer.

(Here the summation is to be extended over all subsegments of s.)

Proposition (4.2.8):

The regular points of (M,N) are everywhere dense in M.

In the following we will discuss the Morse series of M mod N.

Suppose that $M \supset N$ as before, and that P is a fixed regular point of (M, N). to this situation MORSE assigns a formal power series, $\mathcal{J}\mathfrak{N}$ (M, N, P; t), {or more shortly $\mathcal{J}$(t)} which we now describe. Its interest lies in the fact that although this series is determined entirely by geometric considerations, namely be the numbers dim$A^N(s)$, described in the last section, it nevertheless has topological implications.

Let S (M,N,P*)* be the set of geodesic segments of (M,N) which end at p, and along which the parameter is precisely arc-length. Because p is regular, no segment of S (M,N,P) is focal relative to N.

Definition (4.2.9):

The Morse series of (M, N, P) is defined by

$$\mathcal{J}\mathfrak{N}(\mathrm{M, N, P;\ t}) = \sum t^{\lambda^N(S)} \qquad [s \in S(M, N, P)],$$

Where $\lambda^N(S)$is the index of s relative to N,

Definition (4.2.10):

The set S(M,N,P), (P regular !) contains only a finite number of segments of length less that a given number.

In the following we will discuss the Morse Inequalities.

The topological implications of $\mathcal{J}\mathfrak{N}$(t) are contained in the "morse inequalities" which relate $\mathcal{J}\mathfrak{N}$(t)to the Poincare series of afunction space, $\Omega(M, N, P)$, constructed over M. following SEIFERT and THRELLFALL [10] rather than Morse for the time being, this space $\Omega(M, N, P)$ is defined in the following manner :

Definition (4.2.11):

$\Omega = \Omega(M, N, P)$shall denote the space of all piece-wise regular maps of the unit interval [0,I] into M which are parameterized proportion-aly to arc-length, take o into a point of N and map I onto P.

A topology is introduced into Ω by making Ω into a metric space

$$\rho(u_1, u_2) = \max_{t\in[0,1]} d\big(u_1(t), u_2(t)\big) + \mid L(u_1) - L(u_2) \mid$$

For two maps $u_i \in \Omega$, where L(u_i) is the length of u_i and d(p, q),p, q $\in M$, is the metric on M.

Let H(Ω ; K) denote the singular homology [10], of Ω with respect to coefficients k. If k is a field,

$$P(\Omega; k; t) = \sum_{n\geq 0} dim H_n(\Omega; k) t^n$$

Is the Poincare series of Ω relative to k .

Corollary (4.2.12):

If the Morse series of (M,N,P) exists and contains no odd powers of t, Then

$$\mathcal{J}\mathfrak{N}(\mathrm{M, N, P;\ t}) = P(\Omega; k; t) \text{for any k.}$$

In particular H(Ω) is then free of torsion.

Proof:

$$\mathcal{J}\mathfrak{N}(t) = \sum_{i\geq 0} m_i t^i\,; \quad P(\Omega; k; t) = \sum_{i\geq 0} p_i t^i; \quad B(t) = \sum_{i\geq 0} b_i t^i,$$

The inequalities of Morse imply that

$$m_i - p_i = b_i + b_{i-1} (i = 1,2, \dots),$$

$m_0 - p_0 = b_0$

With $b_i \geq 0$.

Hence $m_{2i+1} = 0 (i = 0,1,2, \dots) implies$

$$b_{2i+1} + b_{2i} = 0 \text{ whence} b_{2i+1} = 0, b_{2i} = 0,$$

i. e. B (t) $\equiv 0$. The rest is clear.

In the following we will discuss variational completeness.

Suppose that k is a compact group of isometries of the Riemannian manifold M, and let N is an orbit of a point in M under the action of K.

in this case the in finitesimal motions of M determined by K, are universal J-fields mod N, i.e. they restrict to elements of J_S^N along any geodesic segment s, of (M,N). when the action of K on M is sufficiently rich, it may happen that $A^N(s)$ consists entirely of fields which can be extended to the global infinitesimal motions of K. in this very special situation, which however is the one we meet in both subsequent applications, the Morse series of (M,N,P) can be computed very simply. Here we derive this new formula for the Morse series, under this extension condition, which we call variational completeness.

Let $\pi: K \times M \rightarrow M$ be the left representation of K on M under conside-Ation. We also use

$$\pi(\sigma): M \rightarrow M$$

for the transformation determined by $\sigma \in K$. we write $\sigma(p)$ for the orbit of p in M under K.

$$o(p) = \pi(K)p.$$

The following definition enables one to treat all orbits at the same time.

Definition (4.2.13):

A geodesic segment, s, will be called transversal (properly π-transversal) if its initial direction is perpendicular to the orbit of its initial point. If p is the initial point of such a segment, J_S^N $[A^N(s)]$, N=o(p), will be denoted by $J_S^\pi[A^\pi(s)]$ respectively.

Definition (4.2.14):

The action of K on M via π is called variationally complete, if for any π-transversal geodesic segment, s,

$$A^\pi(s) \subset \dot{\pi}_s(\mathrm{k}).$$

Let c(p), p $\in M$, be the subspace of k, whose $\dot{\pi}$-image vanishes at p. Then c(s) $\subset k$, shall be the kernel of $\dot{\pi}_s$. clearly then,

$$\dim o(p) = \dim k - \dim c(p).$$

In the following we will discuss the geodesics of (G,N,P).

Proposition (4.2.15):

Let P a general point of G and T the maximal torus it determines.

If N = o(σ) is any orbit of G under π,which does not intersect P, then

$$S(G, N, P) = S(T, N \cap T, P).$$

Lemma (4.2.16):

For any $\sigma \in G$,the tangent spaces to C(σ) and o(σ), at σ,are comple-

Mentary and orthogonal.

In particular, therefore, at P, the space T_P nakes up the orthogonal

Complement of the tangent space to the orbit through P.

Proposition (4.2.17):

The exponential map exp | t, maps the straight lines of $\tilde{S}(\tilde{t}, \widetilde{N}, P)$ in a

One-to-one fashion onto the segments of S(G,N,P).

To real off the Morse series of (G,N,P) it is now sufficient to find the intersections of the segments of S(T, N ∩ T, P) with the intersection

Of the exceptional orbits with T.there points, the so-called singular points of T, form a set $\overline{D}(G)$ in T whose inverse image under exp | t

Is called the diagram of G on t. It will be denoted by D(G);

Let G be a compact connected and simply-connected Lie group, T⊂G

a maximal tours of G, N any orbit of the adjoint action of G on G. Let

P be a general point of T not on N. finally$\tilde{S}(t, \widetilde{N}, P)$ shall be the set of straight lines in t = T_{ε},

Theorem (4.2.18):

The space of paths, Ω (G, N, P), is free of torsion. Its odd Betti numbers vanish, and the Poincare series of Ω (G, N, P) coincides

With the Morse series of (G, N, P).

Theorem (4.2.19):

The Morse series of (G, N, P) is given by the formula

$$\mathcal{J}\mathfrak{N}(G, N, P; t) = \sum t^{2\lambda(s)} \qquad [s \in \tilde{S}(t, \tilde{N}, P)],$$

Where $\lambda(s)$ = number of planes of D(G) crossed by the line s.

Let $\widetilde{M}$ be an open, spherical disc about the origin in g, which is so small that exp | $\widetilde{M}$ maps $\widetilde{M}$

homeomorphically into G. the image of $\widetilde{M}$ under exp shall be M. the adjoint action, π, clearly maps M into itself.

Proposition (4.2.20):

The adjoint action of G on M is variationally complete.

Proposition (4.2.21):

The set S(M, N, P) coincides with the set S(M∩T, N∩T, P).

The inverse image of N in $\widetilde{M}$ is now unique.

It shall be denoted by $\widetilde{N}$.

The unique inverse image of P in $\widetilde{M}$ shall be $\widetilde{P}$, and $\tilde{S}(\widetilde{M} \cap t, \widetilde{N} \cap t, \widetilde{P})$

Shall stand for the set of straight lines joining points of $\widetilde{N}$ to $\widetilde{P}$ in $\widetilde{M}$.

Proof:

Let $\alpha \in \{0,1, \dots, n-1\} and\ suppose mod(\mathfrak{a}, n) = 1.$

$$Then$$

$$1 = as + nt for\ some s, t, \in Z\ or$$

$$as = tn + 1.$$

$$(as) mod\ n = a(s\ mod\ n)) mod\ n.$$

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