Chapter (3)

Homogeneous Spaces and Connectivity of Matrix Groups

Now we will discuss homogeneous spaces as manifolds.

Section (3.1): Homogeneous Spaces

Let G be a Lie group of dimension dim G = n and H $\leq$G a closed subgroup, which is therefore a Lie subgroup of dimension dim H = k. The set of left cosets

$$\frac{G}{H} = \{gH: g \in G)$$

has an associated quotient map

$$\pi: G \to \frac{G}{H}: \pi(g) = gH.$$

We give G/H a topology by requiring that a subset W $\subseteq$G/H is open if and only if $\pi^{-1}W \subseteq$ G is open;this is called the quotient topology on G/H.

Lemma (3.1.1):

The projection map $\pi: C \to G/H$is an open mapping and G/H is a topological spacewhich is separable and Hausdorff.

Proof:

For U $\subseteq$G,

$$\pi^{-1}(\pi U) = \bigcup_{h \in H} Uh.$$

Where

$$Uh = \{uh \in G : u \in U\} \subseteq G$$

If U ⊆G is open, then each Uh (h ∈H) is open, implying that πU ⊆G is also open.

G/H is separable since a countable basis of G is mapped by πto a countable collection of opensubsets of G/H that is also a basis.

To see that G/H is Hausdorff, consider the continuous map

$$\theta: G \times G \to G; \theta(x, y) = x^{-1}y.$$

Then

$$\theta^{-1}H = \{(x, y) \in GxG: xH = yH\}.$$

and this is a closed subset since H ⊆G is closed. Hence,

$$\{(x; y) \in G\ xG\ ; xH = yH\} \subseteq G \times G$$

is open. By dentition of the product topology, this means that whenever x; y ∈G with xH ≠yH, thereare open subsets U; V ⊆G with x ∈U, y ∈V , U ≠V and $\pi^u \cap \pi^v = \theta$ Since $\pi^U, \pi^V \subseteq G/H$are open,this shows that G/H is Hausdorff.

The quotient map G/H has an important property which characterises it.

Proposition(3.1.2):(Universal Property of the Quotient Topology)

For any topological space X, afunction f : G/H →X is continuous if and only if f 0π:G →X is continuous.

We would like to make G/H into a smooth manifold so that π:G →G/H is smooth. Unfortunately,the construction of an atlas is rather complicated so we merely state a general result then consider someexamples where the smooth structure comes from an existing manifold which is diffeomorphic to aquotient.

Theorem (3.1.3):

*G/H*can be given the structure of a smooth manifold of dimension

$$\dim G/H = \dim G - \dim H$$

So that the projection map $\pi: G \to G/H$ is smooth and at each $g \in G$,

$$\ker\left(d\,\pi:: T_g G \to T_{gH}\frac{G}{H}\right) = dl_g \mathfrak{h}.$$

There is an atlas for G/H consisting of charts of the form $\theta: W \to \theta W \subseteq R^{n-k}$for which there is adiffeomorphism $\ominus: W \times H \to \pi^{-1}W$satisfying the conditions

$\ominus(w,h_1 h_2) = \ominus(w,h_1)\, h_2, \pi(\ominus(w,h)) = w\ (\omega \in W, h, h_1, h_2 \in H)$:

θ

W×H $\pi^{-1}w$

Proj1 π

w

The projection πlooks like proj$_1$: $\pi^{-1}w \to w$, the projection onto W, when restricted to : $\pi^{-1}w$.

For such a chart, the map $\ominus$ is said to provide a local trivialisation of πover W. An atlas consisting ofsuch charts and local trivialisations $(\theta: W \to W\,\theta; \ominus)$ provides a local trivalisation of π. This is relatedto the important notion of a principal H-bundle over G/H.

Notice that given such an atlas, an atlas for G can be obtained by taking each pair (θ: W$\to$ θW $\ominus$)and combining the map $\theta$with a chart ψ: U$\to$ ψU $\subseteq R^k$for H to get a chart

$$(\theta \times \psi) \circ \ominus^{-1}: \ominus(w\times U)\to \theta W \times \psi U \subseteq R^{n-k} \times R^k = R^n.$$

Such a manifold G/H is called a homogeneous space since each left translation map $L_g$on G gives riseto a diffeomorphism

$$\bar{L}_g:G/H \to \quad \bar{L}_g (xH) = gxH$$

for which $\pi \circ L_g=\bar{L}_g \circ \pi$.

$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ \pi \downarrow & & \pi \downarrow \\ G/H & \xrightarrow{\bar{L}_g} & G/H \end{array}$$

So each point gH has a neighbourhood diffeomorphic under $\bar{L}_g^{-1}$ to a neighbourhood of 1H; so locallyG/H is unchanged as gH is varied. This is the basic insight in Felix Klein's view of a Geometry whichis characterised as a homogeneous space G/H for some group of transformations G and subgroup H.

In the following we will discuss Homogeneous spaces as orbits.

Just as in ordinary group theory, group actions have orbits equivalent to sets of cosets G/H, sohomogeneous spaces also arise as orbits associated to smooth groups actions of G on a manifolds.

Theorem (3.1.4):

Suppose that a Lie group G acts smoothly on a manifold M. If the element $x \in$ Mhas stabilizer $Stab_G(x) \leq G$ and the orbit $Orb_G(x) \subseteq M$ is a closed submanifold, then the function

$$f : G/Stab_G(x) \rightarrow Orb_G(x);\ f(g\, Stab_G(x)) = g\, x$$

is a diffeomorphism.

Example (3.1.5):

For $n \geq 1$, $O(n)$ acts smoothly on R^n by matrix multiplication. For any nonzerovector $v \in R^n$, the orbit $Orb_{o(n))}(v) \in R^n$,is diffeomorphic to $O(n)/O(n-1)$.

Proof:

First observe that when v is the standard basis vector e_n, for A $\in O(n)$, $Ae_n = e_n$if andonly if $e_n$is the last column of A, while all the other columns of A are orthogonal to e_n. Since thecolumns of A must be an orthonormal set of vectors, this means that each of the first (n-1) columns ofA has the form

$$\begin{bmatrix} a_{1k} \\ a_{1k} \\ \vdots \\ a_{1k} \\ 0 \end{bmatrix}$$

Where the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & a_{22} & \cdots & a_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} \end{bmatrix}$$

is orthogonal and hence in O(n-1). We identify O(n-1) with the subset of O(n) consisting of matricesof the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n-2} & 0 \\ \vdots & & \ddots & & 0 \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

and then have $Stab_{O(n)}(e_n)$ = O(n-1). The orbit of $e_n$is the whole unit sphere $S^{n-1} \subseteq R^n$since givena unit vector u we can extend it to an orthonormal basis$u_1, \ldots, u_{n-1}, u_n = u$which form the columnsof an orthogonal matrix U $\in$O(n) for which Ue_n= u. So we have a diffeomorphism

$$O(n)/Stab_{O(n)}(e_n) = O(n)/O(n-1) \rightarrow Oeb_{o(n)}(e_n) = S^{n-1}.$$

Now for a general nonzero vector v notice that $Stab_{O(n)}(\dot{v}) = Stab_{O(n)}(\hat{v})$ where $\hat{v}$= (1=/|v|)v and

$$Orb_{O(n)}(v) = S^{n-1}(|v|),$$

the sphere of radius |v|. If we choose any P $\in$O(n) with $\hat{v} = Pe_n$, we have

$$Stab_{O(n)}(v) = P\, Stab_{O(n)}(e_n)P^{-1}$$

and so there is a diffeomorphism

$$\mathrm{Orb}_{O(n)}(\mathbb{v}) \rightarrow O(n)/P\ O(n\ /1)P^{-1} \xrightarrow{x_{p-1}} O(n)/o(n-1)$$

A similar result holds for SO(n) and the homogeneous space SO(n)= SO(n-1). For the unitaryand special unitary groups we can obtain the homogeneous spaces U(n)/U(n-1) and SU(n)/SU(n-1)as orbits of non-zero vectors inC^n on which these groups act by matrix multiplication; these are alldiffeomorphic to S^{2n-1}. The action of the quaternionic symplectic group Sp(n) on H^nleads to orbits ofnon-zero vectors diffeomorphic to Sp(n)/Sp(n-1) and S^{4n-1}.

In the following we will discuss Projective spaces.

More exotic orbit spaces are obtained as follows. Let k= R,C,H and set d = $\dim_R$k. Consider k^{n+1}as a right k-vector space. Then there is an action of the group of units k^xon the subset of non-zero

vectors k_0^{n+1}=$k^{n+1} - \{0\}$:

$$z.\mathbb{x} = \mathbb{x}z^{-1}$$

The set of orbits is denoted $kP^n$and is called n-dimensional k-projective space. Projective spaces Anelement of kP^nwritten [x] is a set of the form

$$[x] = \{xz^{-1}:z \in k^x\} \subseteq k_0^{n+1}$$

Notice that [x] = [y] if and only if there is a z $\in k^x$for which y = xz^{-1}.

Remark (3.1.6):

Because of this we can identify elements $kP^n$with k-lines in k^{n+1}(i.e., 1-dimensional k-vector subspaces). $kP^n$is often viewed as the set of all such lines, particularly in the study of ProjectiveGeometry.

There is a quotient map

$$q_n: k_0^{n+1} \to KP^n; qn(x) = [x].$$

and we give $kP^n$the quotient topology which is Hausdorff and separable.

Proposition (3.1.7):

$KP^n$is a smooth manifold of dimension dim kP^n= n $\dim_R$ k. Moreover, thequotient map qn $k_0^{n+1} \to KP^n$is smooth with surjective derivative at every point in k_0^{n+1}.

Proof:

For r = 1, 2, …, n, setkp_r^n= $\{|x|: x_r \neq o\}$, where as usual we write

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_{n+1} \end{bmatrix}$$ .Then$kp_r^n \subseteq KP^n is$open. There is a function

$$\sigma_r: KP_r^n \to k^n;\ \sigma_r(|x|) = \begin{bmatrix} \chi_1\chi_r^{-1} \\ \chi_2\chi_r^{-1} \\ \vdots \\ \chi_{r-1}\chi_r^{-1} \\ \chi_{r+1}\chi_r^{-1} \\ \vdots \\ \chi_{n+1}\chi_r^{-1} \end{bmatrix}$$

Which is a continuous bijection that is actually a homeomorphism. Whenever r ≠s, the induced map

$$\sigma_s^{-1} o \sigma_r: \sigma_r^{-1} kp_r^n \cap KP_s^n \to \sigma_s^{-1} kp_r^n \cap KP_s^n$$

is given by

$$\sigma_s^{-1} 0 \sigma_r(x) = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{s-1} \\ y_{s+1} \\ \vdots \\ y_{n+1} \end{bmatrix}$$

Where

$$y_i = \begin{cases} x_j x_s^{-1} if j \neq r, s, \\ x_s^{-1} if j = r. \end{cases}$$

These $(n+1)$ charts form the standard atlas for n-dimensional projective space over k.

An alternative description of KP^n is given by considering the action of the subgroup

$$k_1^x = \{z \in k^x : |z| = 1\} \leq k^x$$

on the unit sphere $S^{(n+1)d-1} \subseteq k_0^{n+1}$.Notice that every element [x] $\in KP^n$ contains elements of S^n.

Furthermore, if x,y $\in k_0^{n+1}$ have unit length $|x| = |y| = 1$, then [x] = [y] if and only if y = xz^{-1} forsome z $\in k_1^x$. This means we can also view kP^n as the orbit space of this action of k_1^x on $S^{(n+1)d-1}$, andwe also write the quotient map as $q_n: S^{(n+1)d-1} \to kP^n$; this map is also smooth.

Proposition (3.1.8):

The quotient space given by the map $q_n: S^{(n+1)d-1} \to kP^n$ is compact Hausdorff.

Proof:

This follows from the standard fact that the image of a compact space under a continuousmapping is compact.

Consider the action of O(n+ 1) on the unit sphere $S^n \subseteq R^{n+1}$. Then for A $\in O^{(n+1)}$, z = $\pm$1 andx $\in S^n$, we have

$$A(xz^{-1}) = (Ax)_z^{-1}$$

Hence there is an induced action of O(n+ 1) on RP^ngiven by

$$A.[x] = [Ax].$$

This action is transitive and also the matrices $\pm I_{n+1}$fix every point of RP^n. There is also an action ofSO(n+ 1) on RP^n; notice that $-1_{n+1} \in$SO(n+ 1) only if n is odd.

Similarly, U(n+ 1) and SU(n+ 1) act on $CP^n$with scalar matrices wl_{n+1}(w$\in$ C_1^x) fixing everyelement. Notice that if $wI_{n+1} \in$SU(n + 1) then w^{n+1}= 1, so there are exactly (n+ 1) such values.

Finally, Sp(n+ 1) acts on $HP^n$and the matrices $\pm I_{n+1}$fix everything.

There are some important new quotient Lie groups associated to these actions, the projective unitary,special unitary and quaternionic symplectic groups

$$PU(n+1) = U(n+1)/\{\omega I_{n+1}: \omega \in C_1^x\},$$

$$PSU(n+1) = SU(n+1)/\{\omega I_{n+1}: \omega^{n+1} = 1\}$$

$$PS_p(n+1) = Sp(n+1)/\{\mp I_{n+1}\}$$

Projective spaces are themselves homogeneous spaces. Consider the subgroup of O(n+ 1) consisting ofelements of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-11} & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \pm 1 \end{bmatrix}$$

We denote this subgroup of O(n+ 1) by O(n) ×O(1). There is a subgroup $\widetilde{O(n)}$ ≤SO(n+ 1) whoseelements have the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-11} & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \omega \end{bmatrix}$$

Where

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} \end{bmatrix} \in o(n), w = det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} \end{bmatrix}$$

Similarly, there subground U(n) × U (1) ≤ U (n+1) whose elements have the form.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega \end{bmatrix}$$

and $\breve{U}(n) \leq SU(n+1) with elements$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega \end{bmatrix}$$

Where

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} \end{bmatrix} \in U(n), \quad w = det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} \end{bmatrix}^{-1}$$

Finally we have $Sp(n) \times Sp(1) \in Sp(n+1)$ consisting of matrices of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-11} & \ddots & \ddots & a_{n-1n-1} & 0 \\ 0 & 0 & \cdots & 0 & \omega \end{bmatrix}$$

Proposition (3.1.9):

There are diffeomorphisms between

- RP^n and $O(n+1)/O(n) \times O(1)$, $SO(n+1)/\widetilde{O(n)}$;
- CP^n and $U(n+1)/U(n) \times U(1)$, $SU(n+1)/\widetilde{U(n)}$;
- HP^n and $Sp(n+1)/Sp(n) \times Sp(1)$.

There are similar homogeneous space of the general and special linear groups giving these projectivespaces. We illustrate this with one example.

SL_2© contains the matrix subgroup P consisting of its lower triangular matrices

$$\begin{bmatrix} u & o \\ w & v \end{bmatrix} \in SL_2(C)$$

This is often called a parabolic subgroup.

Proposition (3.1.10):

CP^1 is diffeomorphic to SL_2©/P.

Proof:

There is smooth map

$$\psi: SL_2© \to CP^1;\ \psi(A) = [Ae_2].$$

Notice that for B = $\begin{bmatrix} u & o \\ w & v \end{bmatrix} \in p$,

$$\begin{bmatrix} u & o \\ w & v \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix},$$

hence $[(AB)e_2] = [Ae_2]$ for any A $\in SL_2$©. This means that $\psi(A)$ only depends on the coset AP $\in SL_2(C)/P$. It is easy to see that is onto and that the induced map $SL_2(C)/P \to CP^1$ is injective.

In the following we will discuss Grassmannians.

There are some important families of homogeneous spaces directly generalizing projective spaces.

These are the real, complex and quaternionic Grassmannians which we now define:

Let O(k) × O(n- k) ≤O(n) be closed the subgroup whose elements have the form

$$\begin{bmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{bmatrix} (A \in O(k), B \in O(n-k))$$

Similarly there are closed subgroups U(k)×U(n-k) ≤U(n) and Sp(k) x Sp(n-k) ≤Sp(n) with elements

$$U(k) \times U(n-k): \begin{bmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{bmatrix} \big(A \in U(k), B \in U(n-k)\big):$$

$$Sp(k) \times Sp(n-k): \begin{bmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{bmatrix} \big(A \in Sp(k), B \in Sp(n-k)\big):$$

The associated homogeneous spaces are the Grassmannians

$Gr_{k,n}$(R) = O(n)/O(k) ×O(n-k);

$Gr_{k,n}$(ℂ) = U(n)/U(k) ×U(n-k);

$Gr_{k,n}$(H) = Sp(n)/Sp(k) ×Sp(n-k):

Proposition (3.1.11):

For k= R,ℂ,H, the Grassmannian $Gr_{k,n}$(k) can be viewed as the set of allk-dimensional |-vector subspaces in k^n .

Proof:

We describe the case k= R, the others being similar.

Associated to elementW ∈O(n) is the subspace spanned by the frst k columns ofW, say w_1, ... ,w_k;we will denote this subspace by $\langle w_1, \dots, w_k\rangle$. As the columns of W are an orthonormal set, they arelinearly independent, hence $\dim_R\langle w_1, \dots, w_k\rangle$ =k. Notice that the remaining (n-k) columns giverise to another subspace $\langle w_{k+1}, \dots, w_n\rangle$of dimension $\dim_R\langle w_{k+1}, \dots, w_n\rangle = n$-k. In fact these aremutually orthogonal in the sense that

$$\langle w_{k+1}, \dots, w_n\rangle = \langle w_1, \dots, w_k\rangle^{\perp}$$

$$= \{x \in R^n; x.w_r = 0, r = 1, \dots, k\},$$

$$\langle w_1, \dots, w_k\rangle = \langle w_{k+1}, \dots, w_n\rangle^{\perp}$$

$$= \{x \in R^n; x.w_r = 0, r = k+1, \dots, n\},$$

For a matrix

$$\begin{bmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{bmatrix} \in O(k)xO(n-k),$$

The columns in the product

$$W' = W\begin{bmatrix} A & O_{k,n-k} \\ O_{n-k,k} & B \end{bmatrix}$$

Span subspaces $\langle w'_1, \dots, w'_k\rangle$ and $\langle w'_{k+1}, \dots, w'_n\rangle$. But note that $w'_1, \dots w'_k$are orthonormal and alsolinear combinations of $w_1, \dots w_k$; similarly,$w'_{k+1}, \dots, w'_n$are linear combinations of $w_{k+1}, \dots, w_n$.

Hence

$$\langle w'_1, \dots, w'_k\rangle = \langle w_1, \dots, w_k\rangle, \langle w'_{k+1}, \dots, w'_n\rangle = \langle w_{k+1}, \dots, w_n\rangle.$$

So there is a well defined function

$$O(n)/O(k) \times O(n /k) \rightarrow k\text{-dimensional vector subpaces of } R^n$$

Which sends the coset of W to the subspace $(w_1, \dots, w_k)$. This is actually a bijection.

Notice also that there is another bijection

$$O(n)/O(k) \times O(n\text{-}k) \rightarrow (n\text{-}k)\text{-dimensional vector subpaces of } R^n$$

Which sends the coset of W to the subspace $\langle w_{k+1}, \dots, w_n \rangle$. This corresponds to a diffeomorphism$Gr_{k,n}$(R)→ $Gr_{n-k,n}$(R) which in turn corresponds to the obvious isomorphism O(k)×O(n-k) → O(n-k) ×O(k) induced by conjugation by a suitable element P ∈O(n).

Section (3.2):Connectivity of Matrix Groups.

In the following we will discuss connectivity of manifolds.

Definition (3.2.1):

A topological space X is connected if whenever X = U ∪V with U, V ≠ ∅, thenU ∩V ≠ ∅

Definition (3.2.2):

A topological space X is path connected if whenever x,∈X, there is a continuouspath p: [0; 1] →! X with p(0) = $x$and p(1) = y.

X is locally path connected if every point is contained in a path connected open neighbourhood.

The following result is fundamental to Real Analysis.

Proposition (3.2.3):

Every interval [,b] ,[a, b),(a,b] , (a,b) ⊆R is path connected and connected. Inparticular, R is path connected and connected.

Proposition (3.2.4):

If$X$is a path connected topological space then X is connected.

Proof:

Suppose X is not connected. Then X = U ∪V where U, V ⊆X are non-empty andU ∩V = ∅. Let x ∈U and y∈V . By path connectedness of there X, is a continuous map p: [0,1]→Xwith p(0) = $x$and p(1) = y. Then [0,1] = p^{-1}U ∪p^{-1}V expresses [0,1] as a union of open subsets withno common elements. But this contradicts the connectivity of [0,1]. So X must be connected.

Proposition (3.2.5):

Let X be a connected topological space which is locally path connected. Then Xis path connected.

Proof:

Let x∈X, and set

X_x= {∈X : ∃p: [0,1] →X continuous such that p(0) = $x$and p(1) = $\mathcal{Y}$}.

Then for each $\mathcal{Y} \in X_x$, there is a path connected open neighbourhood U_y. But for each point z∈$U_y$there is a continuous path from to z via y, hence $U_y \subseteq X_x$. This shows that

$$X_z = \bigcup_{y \in x_z} U_y \subseteq X$$

is open in X. Similarly, if w∈X -X_x, then $X_w \subseteq X - X_x$and this is also open. But then so is

$$X - X_x = \bigcup_{\omega \in X - X_z} x_\omega$$

Hence X = X∪[(X -X_x), and so by connectivity, X_x= ∅or X -X_x= ∅. So X is path connected.

Proposition (3.2.6):

If the topological spaces X and Y are path connected then their product X ×Y ispath connected.

Corollary (3.2.7):

For $n \geq 1$, $R^n$is path connected and connected.

It is also useful to record the following standard results.

Proposition (3.2.8):

Let n ≥ 2. The unit sphere $S^{n-1} \subseteq \mathbb{R}^n$ is path connected. In $\mathbb{S}^0 = \{\pm 1\} \subseteq \mathbb{R}$.

The subsets {1}and {-1}are path connected. The set of non-zerovectors $\mathbb{R}_0^n \leq \mathbb{R}^n$ is path connected.

ii) For n ≥ 1, the sets of non-zero complex and quaternionic vectors $\mathbb{C}_0^n \subseteq \mathbb{C}^n$ and $\mathbb{H}_0^n \subseteq \mathbb{H}^n$ are pathconnected.

Proposition (3.2.9):

Every manifold is locally path connected. Hence every connected manifold is pathconnected.

Proof:

Every point is contained in an open neighbourhood homeomorphic to some open subset of R^n which can be taken to be an open disc which is path connected. The second statement now followsfrom Proposition (3.2.5).

Theorem (3.2.10):

Let M be a connected manifold and N $\subseteq$M is non-empty submanifold which is alsoa closed subset. If dimN = dimM then N = M.

Proof:

Since N $\subseteq$M is closed, M - N $\subseteq$M is open. But N $\subseteq$M is also open since every elementis contained in an open subset of M contained in N; hence M - N $\subseteq$M is closed. Since M is connected,M -N = $\emptyset$.

Proposition (3.2.11):

Let G be a Lie group and H $\leq$G a closed subgroup. If G/H and H are connected,then so is G.

Proof:

First we remark on the following: for any g$\in$G, left translation map L_g: H $\rightarrow$gHprovides a homeomorphism between these spaces, hence gH is connected since H is.

Suppose that G is not connected, and let U, V $\subseteq$G be nonempty open subsets for which U $\cap$V =$\phi$and U $\cup$V = G. the projection _ π:G $\rightarrow$G/H is a surjective open mapping, soπ^U, π^V $\subseteq$G/H are open subsets for which π^U, π^V $\subseteq$G/H. As G/H is connected, there is an elementgH say in π^U, π^VIn G we have

$$gH = \{gH \cap U) \cup (gH \cap V),$$

Where (gH $\cap$U); (gH $\cap$V) $\subseteq$gH are open subsets in the subspace topology on gH since U; V are openin G. By connectivity of gH, this can only happen if gH $\cap$U = θor gH $\cap$V = $\emptyset$, since these are subsetsof U; V which have no common elements. As

$$\pi^{-1} gH = \{gh: h \in H\},$$

This is false, so (gH $\cap$U) \ (gH $\cap$V) $\neq \emptyset$; which implies that U $\cap$V $\emptyset$; This contradicts the originalassumption on U, V .

This result together with Proposition (3.2.9) gives a useful criterion for path connectedness of a Liegroup which may need to be applied repeatedly to show a particular example is path connected. Recallthat a closed subgroup of a Lie group is a submanifold.

Proposition (3.2.12):

Let G be a Lie group and H $\leq$G a closed subgroup. If G/H and H are connected,then G is path connected.

Example (3.2.13):

For n $\geq$1, $SL_n(R)$ is path connected.

Proof:

For the real case, we proceed by induction on n. Notice that $SL_1(R) = \{1\}$, which iscertainly connected. Now suppose that $SL_{n-1}(R)$ is path connected for some n $\geq$2.

Recall that $SL_n(R)$ acts continuously on $R^n$by matrix multiplication. Consider the continuousfunction

$$f: SL_n(R) \to R^n;\ \ f(A) = Ae_n.$$

The image of$f$is im$f$= R_0^n = R^n –{0} since every vector v $\in R_0^n$

can be extended to a basis

$$V_1 \ldots, V_{n-1}, V_n = V$$

ofR^n, and we can multiply $v_1$by a suitable scalar to ensure that the matrix $A_v$with these vectors asits columns has determinant 1. Then $A_v e_n$= v.

Notice that Pe_n= $e_n$if and only if

$$P = \begin{bmatrix} Q & 0 \\ W & 1 \end{bmatrix}$$

Where Q is (n-1)× (n-1) with detQ = 1, is the (n - 1)×1 zero vector and w is an arbitrary 1× (n - 1) vector. The set of all such matrices is the stabilizer of

e_n, $Stab_{SL_n(R)}(e_n)$, which is a closed subgroup of $SL_n(R)$. More generally, $Ae_n = v$ if and only if

$$A = A_v P \text{ for some } P \in Stab_{SL_n(R)}(e_n).$$

So the homogeneous space $SL_n(R)/Stab_{SL_n(R)}(e_n)$ is homeomorphic to R_0^n .

Since n ≥ 2, it is well known that R_0^n is path connected, hence is connected. This implies that $SL_n(R)/Stab_{SL_n(R)}(e_n)$ is connected.

The subgroup $SL_{n-1}(R)/Stab_{SL_n(R)}(e_n)$ is closed and the well defined map

$$Stab_{SL_n(R)}(e_n) = SL_{n-1}(R) \to R^{n-1}; \begin{bmatrix} Q & 0 \\ W & 1 \end{bmatrix} SL_{n-1}(R) \to (wQ^{-1})^T$$

is a homeomorphism so the homogeneous space $Stab_{SL_n(R)}(e_n)/SL_{n-1}(R)$ is homeomorphic to R^{n-1}.

Hence by Corollary (3.2.7) together with the inductive assumption, $Stab_{SL_n(R)}(e_n)$ is path connected. We can combine this with the connectivity of R_0^n to deduce that $SL_n(R)$ is path connected, demonstrating the inductive step.

Example (3.2.14):

For n ≥ 1, $GL_n^+(R)$ is path connected.

Proof:

Since $SL_n(R) \leq GL_n^+(R)$, it suffices to show that $GL_n^+(R)/SL_n(R)$ is path connected. But for this we can use the determinant to define a continuous map.

$$det: GL_n^+(R) \to R^+ = (0.\infty),$$

Which is surjective onto a path connected space. The homogeneous space GL_n^+(R)/SL$_n$(R) is thendiffeomorphic to R$^+$and hence is path connected. So GL_n^+(R) is path connected.

This shows that

$$GL_n(R) = GL_n^+(R) \cup GL_n^-(R)$$

is the decomposition of GL$_n$(R) into two path connected components.

Example (3.2.15):

For n $\geq$1, SO(n) is path connected. Hence

$$O(n) = SO(n) \cup O(n)^-$$

is the decomposition of O(n) into two path connected components.

Proof:

For n = 1, SO(1) = {1}. So we will assume that n $\geq$2 and proceed by induction on n. SOassume that SO(n-1) is path connected.

Consider the continuous action of on $R^n$by left multiplication. The stabilizer of en is SO(n-1)≤ SO(n) thought of as the closed subgroup of matrices of the form

$$\begin{bmatrix} p & 0 \\ O^T & 1 \end{bmatrix}$$

With P ∈SO(n-1) and 0 the (n-1)×1 zero matrix. The orbit of e_n is the unit sphere S^{n-1} which ispath connected. Since the orbit space is also diffeomorphic to SO(n)/SO(n-1) we have the inductivestep.

Example (3.2.16):

For n $\geq$1, U(n) and SU(n) are path connected.

Proof:

For n = 1, U(1) is the unit circle in C while SU(1) = {1}, so both of these are pathconnected. Assume that U(n-1) and SU(n-1) are path connected for some n $\geq$2.

Then U(n) and SU(n) act on $C^n$by matrix multiplication,

$$Stab_{U(n)}(e_n) = U(n-1), Orb_{SU(n)}(e_n) = SU(n-1).$$

We also have

$$Orb_{U(n)}(e_n) = Orb_{SU(n)}(e_n) = S^{2n-1}$$

where $S^{2n-1} \subseteq C^n \cong R^{2n}$denotes the unit sphere consisting of unit vectors. Since $S^{2n-1}$is pathconnected, we can deduce that U(n) and SU(n) are too, which gives the inductive step.

In the following we will discuss the path components of a Lie group

Let G be a Lie group. We say that two elements x,y ∈G are connected by a path in G if there is acontinuous path p: [0,1] →G with p(0) = x and p(1) = y; we will then write $\underset{G}{x\sim y}$.

Lemma (3.2.17):

$\underset{G}{\sim}$Is an equivalence relation on G.

For g ∈G, we can consider the equivalence class of g, the path component of g in G,

$$G_g = \left\{x \in G: \underset{G}{x\sim g}\right\}$$

Proposition (3.2.18):

The path component of the identity is a clopen normal subgroup of G, $G_1 \lhd G$;hence it is a closed Lie subgroup of dimension dimG.

The path component $G_g$agrees with the coset of g with respect to G_1, $G_g = gG_1 = G_{1g}$and is aclosed submanifold of G.

Proof:

By Proposition (3.2.9) $G_g$contains an open neighbourhood of g in G. This shows that everycomponent is actually a submanifold of G with dimension equal to dimG. The argument used in theproof of Proposition (3.2.5)shows that each is $G_g$actually clopen in G.

Let x;y $\in G_1$. Then there are continuous paths p, q: [0, 1] →G with p(0) = 1 = q(0), p(0) = x andq(0) = y. The product path

$$r : [0,1] \to G; \quad r(t) = p(t)q(t)$$

hasr(0) = 1 and r(1) = xy. So $G_1 \leq G$. For $g \in G$, the path

$$s : [0,1] \to G; \quad s(t) = gp(t)g^{-1}$$

has s(0) = 1 and s(1) = gxg^{-1}; hence $G_1 \lhd G$. If z $\in gG_1 = G_1 g$, then $g^{-1}z \in G_1$ and so there is acontinuous path h: [0,1] → G with h(0) = 1 and h(1) = $g^{-1}z$. Then the path

$$gh : [0,1] \to G; \quad gh(t) = g(h(t))$$

has $gh(0) = g$and $gh(1) = z$. So each coset $gG_1$is path connected, hence $gG_1 \subseteq G_g$. To show equality,suppose that $g$is connected by a path k: [0,1]→G in G to w $\in G_g$. Then the path $g^{-1}k$ connects 1to $g^{-1}w$, so $g^{-1}w \in G_1$, giving w $\in gG_1$. This shows that $G_g \subseteq {}_gG_1$.

The quotient group $G/G_1$is the group of path components of G, which we will denote by $\pi_0 G$.

Example (3.2.19):

We have the following groups of path components:

π_0SO(n) = π_0SL$_n$(R) = π_0SU(n) =π_0U(n) = π_0SL$_n$($\mathbb{C}$) =π_0GL$_n$($\mathbb{C}$) = {1},

Example (3.2.20):

Let

$$T = \left\{\begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in R\right\} \leq SO(3)$$

And let G = $N_{SO(3)}$(T) $\leq$SO(3) be its normalize. Then T and G are Lie subgroups of SO(3) and$\pi_0 G \cong \{\pm 1\}$.

Proof:

A straightforward computation shows that

$$N_{SO(3)}(T) = T \cup \left\{\begin{bmatrix} -cos\theta & sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} : \theta \in R\right\} = T \cup \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} T.$$

Notice that T is isomorphic to the unit circle,

$$T \cong T; \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftrightarrow e^{\theta i} .$$

This implies that T is path connected and abelian since T is. The function

$$\varphi: G \to R^x; \quad \varphi([a_{ij}]) = a_{33}$$

is continuous with

$$\varphi^{-1}R^{+} = T, \varphi^{-1}R^{-} = T\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

hence these are clopen subsets. This shows that the path components of G are

$$G_1 = T, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} T.$$

Hence $\pi_0 G \cong \{\pm 1\}$.

Notice that $N_{SO(3)}(T)$ acts by conjugation on T and in fact every element of T $\lhd N_{SO(3)}(T)$ actstrivially since T is abelian. Hence $\pi_0 G$ acts on T with the action of the non-trivial coset given by

Conjugation by the matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

which corresponds to the inversion homomorphism on the unit circle T $\cong$T.

Example (3.2.21):

Let T = $\{x1 + yi : x, y \in R, x^2 + y^2 = 1\}$ $\leq$Sp(1), the group of unit quaternions. LetG = $N_{Sp(1)}(T)$ $\leq$Sp(1) be its normalizer. Then T and G are Lie subgroups of Sp(1) and $\pi_0 G \cong (\pm 1)$.

Proof:

By a straightforward calculation,

$$G = T \cup \{xj - yk : x,y \in R, x^2 + y^2 = 1\} = T \cup jT.$$

T is isomorphic to the unit circle so is path connected and abelian. The function

$$\theta\colon G \to \mathbb{R};\ \theta(t1 + xi + yj + zk) = y^2 + z^2,$$

is continuous and

$$\theta^{-1} = T, \theta^{-1}1 = jT.$$

Hence the path components of G are T, jT. So $\pi_0 G = \cong(\pm 1)$.

The conjugation action of G on T has every element of T acting trivially, so $\pi_0 G$ acts on T. Theaction of the non-trivial coset is given by conjugation with j,

$$j(x1 + yi)j^{-1} = x1 - yi,$$

corresponding to the inversion map on the unit circle $\mathbb{T} \cong T$.

The significance of such examples will become clearer whenwe discuss maximal tori and their normalizers.

In the following we will discuss another connectivity result

Proposition (3.2.22):

Let G be a connected Lie group and H $\leq$G a subgroup which contains an openneighbourhood of 1 in G. Then H = G.

Proof:

Let U $\subseteq$H be an open neighbourhood of 1 in G. Since the inverse map inv: G $\to$G is ahomeomorphism and maps H into itself, by replacing U with U$\cap$inv

U if necessary, we can assume thatinv maps the open neighbourhood into itself, i.e., inv U = U.

For k $\geq$1, consider

$$U^k = \{u_1 \ldots u_k \in G: u_j \in U\} \subseteq H.$$

Notice that inv U^k= U^k. Also, U^k ⊆G is open since for u_1; …,u_k∈U,

$$u_1 \ldots u_k \in L_{u_1 \ldots u_{k-1}} U \subseteq U^k$$

where $L_{u_1 \ldots u_{k-1}}$U = $L^{-1}_{(u_1 \ldots u_{k-1})^{-1}} U$ is an open subset of G.

Then

$$V = \bigcup_{k \geqslant 1} U^K \subset H$$

satisfies inv $V = V$.

V is closed in G since given g∈G –V , for the open set gV ⊆G, if x ∈gV ∩V there areu_1, …, u_r, v_1, …, v_s∈ U such that

$$\mathrm{G}u_1, \ldots, u_r = v_1, \ldots, v_s,$$

implying g= v_1, …, $v_s u_1{}^{-1}, \ldots\ u_r^{-1} \in V$, contradicting the assumption on g.

So $V$is a nonempty clopen subset of G, which is connected. Hence $G - V = \emptyset$, and therefore $V = G$,which also implies that $H = G$.