Chapter (2)

Quaternions, Clifford Algebras, and Matrix Groups as Lie Groups.

Now we will discuss algebras.

Section (2.1): Quaternions, Clifford Algebras, and Matrix Groups as Lie Groups

First $\mathbb{K}$ will denote any field, although our main interest will be in the cases $\mathbb{R}, \mathbb{C}$.

Definition (2.1.1):

finite dimensional (associative and unital) algebra A is a finite dimensional $\mathbb{k}$-vector space which is an associative and unital ring such that for all r; s ∈ $\mathbb{k}$ and a; b ∈ A,

$$(ra)(sb) = (rs)(ab).$$

If A is a ring then A is a commutative $\mathbb{k}$ -algebra.

If every non-zero element $\mathbb{u}$ ∈ A is a unit, i.e., is invertible, then A is a division algebra.

In this last equation, ra and sb are scalar products in the vector space structure, while $(rs)(ab)$ is the scalar product of rs with the ring product ab.

Furthermore, if 1 ∈ $\mathbb{k}$ is the unit of A, for t ∈ $\mathbb{k}$, the element $t1$ ∈ A satisfies

$$(t1)a = ta = t(a1) = a(t1).$$

If dim $\mathbb{k}$ A > 0, then 1 ≠ 0, and the function

$$\eta: \mathbb{k} \rightarrow A; \eta(t) = T1$$

is an injective ring homomorphism; we usually just write t for η (t) = t1.

Example (2.1.2):

For $n \geq 1$, $M_n(\mathbb{k})$ is a $\mathbb{k}$-algebra. Here we have η (t) = t ,$\mathbb{C}$ is non-commutative.

Example (2.1.3):

The ring of complex numbers $\mathbb{C}$ is an $\mathbb{R}$-algebra. Here we have η (t) = t. C is commutative. Notice that $\mathbb{C}$ is a commutative division algebra.

A commutative division algebra is usually called a field while a non-commutative division algebra is called a skew field. In French corps (~field) is often used in sense of possibly non-commutative division algebra.

In any algebra, the set of units of A forms a group A^* under multiplication, and this contains $\mathbb{k}^x$.

For $A = M_n(\mathbb{k})$, $M_n(\mathbb{k})^x = GL_n(\mathbb{k})$.

Definition (2.1.4):

Let A, B be two $\mathbb{k}$-algebras. A $\mathbb{k}$-linear transformation that is also a ring homomorphism is called a $\mathbb{k}$-algebra homomorphism or homomorphism of $\mathbb{k}$ -algebras.

A homomorphism of $\mathbb{k}$-algebras φ: A → B which is also an isomorphism of rings or equivalently of $\mathbb{k}$-vector spaces is called isomorphism of $\mathbb{k}$-algebras.

Notice that the unit η: $\mathbb{k}$ → A is always a homomorphism of $\mathbb{k}$-algebras. There are obvious notions of kernel and image for such homomorphisms, and of subalgebra.

Definition (2.1.5):

Given two $\mathbb{k}$-algebras A, B, their direct product has underlying set $A \; x \; B$ with sum and product

$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$, $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$.

The zero is (0,0) while the unit is (1,1).

It is easy to see that there is an isomorphism of $\mathbb{k}$ -algebras A x B ≅ B x A.

Given a $\mathbb{k}$-algebra A, it is also possible to consider the ring $M_n(A)$ consisting of $m \; x \; m$ matrices with entries in A; this is also a $\mathbb{k}$-algebra of dimension

$Dim_k M_m(A) = m^2 dim_K A$.

It is often the case that a $\mathbb{k}$-algebra A contains a subalgebra$\mathbb{k}_1 \subseteq$ A which is also a field. In that case A can be viewed as a over $\mathbb{k}_1$ in two different ways, corresponding to left and right multiplication by elements of $\mathbb{k}_1$. Then for t $\in$ $\mathbb{k}_1$, $a \in$ A,

$$(Left\ scalar\ multiplication) \to t.a = ta;$$

$$(Right\ scalar\ multiplication) \to a.t = at.$$

These give different $\mathbb{k}_1$-vector space structures unless all elements of $\mathbb{k}_1$ commute with all elements of A, in which case $\mathbb{k}_1$ is said to be a central subfield of A. We sometimes write $\mathbb{k}_1$A and $A_{\mathbb{k}1}$ to indicate which structure is being considered. $\mathbb{k}_1$ is itself a finite dimensional commutative $\mathbb{k}$-algebra of some dimension $dim_{\mathbb{k}}\mathbb{k}_1$.

Proposition (2.1.6):

Each of the $\mathbb{k}_1$-vector spaces$_{k1}$ A and $A_{\mathbb{k}1}$ is finite dimensional and in fact

$$\dim_k A = \dim_{k1} ({}_{k1}A) \dim_k\mathbb{k}_1 = \dim_k\mathbb{k}\ A|1 \dim_k\mathbb{k}:$$

Example (2.1.7):

Let $\mathbb{k} = \mathbb{R}$ and A = $M_2(\mathbb{R})$, so $\dim_R$ A = 4. Let

$$\mathbb{k}_1 = \left\{\begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbb{R}\right\} \subseteq M_2(\mathbb{R})$$

Then $\mathbb{k}_1 \cong \mathbb{C}$ so is a subfield of $M_2(\mathbb{R})$, but it is not a central subfield. Also $\dim_{k1}$ A = 2.

Example (2.1.8):

Let $\mathbb{k} = \mathbb{R}$ and A = $M_2(\mathbb{C})$, so $\dim_R$ A = 8. Let

$$\mathbb{k}_1 = \left\{\begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbb{R}\right\} \subseteq M_2(\mathbb{C})$$

Then $\mathbb{k}_1 \cong \mathbb{c}$ so is subfield of $M_2(\mathbb{C})$, but it is not a central subfield. Here $\dim_{k1}$ A = 4.

Given a $\mathbb{k}$-algebra A and a subfield $\mathbb{k}_1 \subseteq$A containing $\mathbb{k}$ (possibly equal to $\mathbb{k}$), an element $a \in A$ acts on A by left multiplication:

$$a.u = au\ (u \in A).$$

This is always a $\mathbb{k}$-linear transformation of A, and if we view A as the $\mathbb{k}_1$vector space A $\mathbb{k}_1$, it is always a $\mathbb{k}_1$-linear transformation. Given a $\mathbb{k}_1$-basis $\{v_1,\ldots\ldots,v_m\}$ for A $\mathbb{k}_1$, there is an $m\ x\ m$ matrix ρ(a) with entries in $\mathbb{k}_1$ defined by

$$\lambda(a)v_i = \sum_{r=1}^{m} \lambda(a)_{rj} v_r$$

It is easy to check that

$$\lambda: A \to M_m(k_1); a \mapsto \lambda(a)$$

is a homomorphism of $\mathbb{k}$-algebras, called the left regular representation of A over $\mathbb{k}_1$ with respect to the basis $\{v_1,\ldots, v_m\}$.

Lemma (2.1.9):

λA $\to M_m(\mathbb{k}_1)$ has trivial kernel kerλ = 0, hence it is an injection.

Proof:

If a $\in$kerλ then λ (a)(1) = 0, giving a1 = 0, so a = 0.

Definition (2.1.10):

The $\mathbb{k}$-algebra A is simple if it has only one proper two sided ideal, namely (0), hence every non-trivial |-algebra homomorphism θ: A $\to$B is an injection.

Proposition (2.1.11):

Let $\mathbb{k}$ be a field.

i) For a division algebra $\mathbb{D}$ over $\mathbb{k}$, $\mathbb{D}$ is simple.

ii) For a simple $\mathbb{k}$ -algebra A, M_n(A) is simple. In particular, $M_n(\mathbb{k})$ is a simple $\mathbb{k}$ -algebra.

On restricting the left regular representation to the group of units of A^x, we obtain an injective group homomorphism

$$\lambda^x: A^x \to GL_m(\mathbb{k}_1); \lambda^x(a)(u) = au,$$

where $\mathbb{k}_1 \subseteq$ A is a subfield containing $\mathbb{k}$ and we have chosen a $\mathbb{k}_1$-basis of $A_{\mathbb{k}1}$ Because

$A^x \cong im\lambda^x \leq GL_m(\mathbb{k}_1)$

A^x and its subgroups give groups of matrices.

Given a $\Bbbk$-basis of A, we obtain a group homomorphism

$$p^x\colon A^X \to GL_N(k); p^x(a)(u)=va^{-1}$$

We can combine λ^x and ρ^x to obtain two further group homomorphisms

$$\lambda^x\ x\ \rho^x\colon A^x\ x\ A^x \to GL_n(\Bbbk); \lambda^x\ x\ \rho^x(a,b)(u) = aub^{-1}$$

$$\Delta\colon A^x \to GL_n(\Bbbk); \Delta(a)(u) = aua^{-1}$$

Notice that these have non-trivial kernels,

$$\text{Ker } \varphi^x\colon \rho^x = \{(1,1),(-1,-1)\}, \text{ Ker } \Delta = \{1,-1\}$$

In the following we will discuss linear algebra over a division algebra

let $\mathbb{D}$ be a finite dimensional division algebra over a field $\Bbbk$.

Definition (2.1.12):

A (right) $\mathbb{D}$-vector space V is a right $\mathbb{D}$-module, i.e., an abelian group with a right scalar multiplication by elements of $\mathbb{D}$ so that for $u;\ v \in$ V , $x;\ y \in \mathbb{D}$,

$$v(xy) = (vx)y,$$

$$v(x + y) = vx + vy,$$

$$(u + v)x = ux + vx,$$

$$v1 = v:$$

All the obvious notions of $\mathbb{D}$-linear transformations, subspaces, kernels and images make sense as do notions of spanning set and linear independence over $\mathbb{D}$.

Theorem (2.1.13):

Let V be a $\mathbb{D}$ -vector space. Then V has a $\mathbb{D}$-basis.

If V has a finite spanning set over $\mathbb{D}$ then it has a finite $\mathbb{D}$ -basis; furthermore any two such finite bases have the same number of elements.

Definition (2.1.14):

A $\mathbb{D}$ -vector space V with a finite basis is called finite dimensional and the number of elements in a basis is called the dimension of V over $\mathbb{D}$, denoted $\dim_D$ V .

For n $\geq$ 1, we can view $\mathbb{D}^n$ as the set of $n \; x \; 1$ column vectors with entries in $\mathbb{D}$ and this becomes a $\mathbb{D}$ -vector space with the obvious scalar multiplication

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} x = \begin{bmatrix} z_1 x \\ z_2 x \\ \vdots \\ z_n x \end{bmatrix}$$

Proposition (2.1.15):

Let V,W be two finite dimensional vector spaces over $\mathbb{D}$, of dimensions $\dim_D$ V = m, $\dim_D$W = n and with bases $\{v_1; : : : ; v_m\}$, $\{w_1; : : : ;w_n\}$. Then a $\mathbb{D}$-linear transformation λ: V →W is given by

$$\varphi(v_j) = \sum_{r=1}^{n} w_r a_{rj}$$

For unique elements $a_{ij} \in \mathbb{D}$ Hence if

$$\varphi\left(\sum_{s=1}^{n} v_s x_s\right) = \sum_{s=1}^{n} w_r y_r,$$

Then

$$\begin{bmatrix} y1 \\ y2 \\ \vdots \\ yn \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & x_1 \\ a_{21} & a_{22} & \cdots & x_2 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & x_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

In particular, for V=$\mathbb{D}^m$ and W=$\mathbb{D}^n$, every $\mathbb{D}$-linear transformation is obtained in this way from left multiplication by a fixed matrix.

This is of course analogous to what happens over a field except that we are careful to keep the scalar action on the right and the matrix action on the left. We will be mainly interested in linear transformations which we will identify with the corresponding matrices. If θ: $\mathbb{D}^k$→: $\mathbb{D}^k$ and $\varphi\mathbb{D}^m$→ $\mathbb{D}^n$ are $\mathbb{D}$-linear transformations with corresponding matrices $[\theta]$, $[\varphi]$, then

$$[\theta] \; [\varphi] = [\theta O \varphi], \qquad (2.1)$$

Also, the identity and zero functions Id; 0: $\mathbb{D}^m$→ $\mathbb{D}^m$ have [Id] = I_m and [0] = O_m.

Notice that given a $\mathbb{D}$-linear transformation φ: V→W, we can 'forget' the $\mathbb{D}$-structure and just view it as a $\mathbb{k}$-linear transformation. Given $\mathbb{D}$-bases $\{v_1\ldots, v_m\}$, $\{w_1,\ldots..,w_n\}$ and a basis $\{b_1, \ldots, b_d\}$ say for $\mathbb{D}$, the elements

$$v_r b_t \; (r = 1, \ldots. ,m \; , t = 1 \; , \ldots. \; d),$$

$$w_s b_t \; (s = 1, \ldots, n \; , t = 1 \; , \ldots \; d)$$

form $\mathbb{k}$-bases for V;W as $\mathbb{k}$-vector spaces.

We denote the set of all $m \; x \; n$ matrices with entries in $\mathbb{D}$ by $M_{m,n}(\mathbb{D})$ and $M_n(\mathbb{D}) = M_{n,n}(\mathbb{D})$. Then $M_n(\mathbb{D})$ is a $\mathbb{k}$-algebra of dimension dim $M_n(\mathbb{k}) = n^2 \dim_k \mathbb{D}$. The group of units of $M_n(\mathbb{D})$ is denoted $GL_n(\mathbb{D})$. However, for non-commutative $\mathbb{D}$ there is no determinant function so we cannot define an analogue of the special linear group. We can however use the left regular representation to overcome this problem with the aid of some algebra.

Proposition (2.1.16):

Let A be algebra over a field $\mathbb{D}$ and B ⊆ A a finite dimensional subalgebra. If u∈ B is a unit in A then u^{-1} ∈ B, hence u is a unit in B.

Proof:

Since B is finite dimensional, the powers u^k (k≥0) are linearly dependent over $\mathbb{k}$, so for some $t_r \in \mathbb{k}$ (r = 0,….,e) with $\tau_e \neq 0$ and $e \geq 1$, there is a relation

$$\sum_{r=0}^{e} t_r u^r = 0$$

If we choose k suitably and multiply by a non-zero scalar, then we can assume that

$$u^k - \sum_{r=k+1}^{e} t_r u^r = 0.$$

If v is the inverse of u in A, then multiplication by v^{k+1} gives

$$v - \sum_{r=k+1}^{e} t_r u^{r-k-1} = 0.$$

from which we obtain

$$v-\sum_{r=k+1}^{e} t_r u^{r-k-1} \in B$$

For a division algebra $\mathbb{D}$, each matrix A $\in M_n(\mathbb{D})$ acts by multiplication on the left of $\mathbb{D}^n$. For any subfield $\mathbb{k}_1 \subseteq \mathbb{D}$ containing $\mathbb{K}$, A induces a (right) $\mathbb{K}_1$-linear transformation,

$$D^n \to D^n;\ x \to Ax$$

If we choose a $\mathbb{k}_1$-basis for $\mathbb{D}$, A gives rise to a matrix $A_A \in M_{nd}(\mathbb{K}_1)$ where d = $\dim_{k1} \mathbb{D}_{\mathbb{K}1}$. It is easy to see that the function $\Lambda: M_n(\mathbb{D}) \to M_{nd}(\mathbb{K}_1)\ ;\ \Lambda(A) = \Lambda_A$. is a ring homomorphism with $\ker\Lambda = 0$. This allows us to identify $M_n(\mathbb{D})$ with the subring $\mathrm{im}\Lambda \subseteq M_{nd}(\mathbb{K}_1)$.

We see that A is invertible in $M_n(\mathbb{D})$ if and only if Λ_A is invertible in $M_{nd}(\mathbb{K}_1)$. But the latter is true if and only if $\det\Lambda_A \in 0$.

Hence to determine invertibility of A $\in M_n(\mathbb{D})$, it suffices to consider $\det \Lambda_A$ using a subfield $\mathbb{K}_1$. The resulting function

$$\mathrm{Rdet}\ \mathbb{K}_1 : M_n(\mathbb{D}) \to \mathbb{K}_1;\ \mathrm{Rdet}\ \mathbb{K}_1\ (A) = \Lambda_A.$$

is called the $\mathbb{K}_1$-reduced determinant of $M_n(\mathbb{D})$ and is a group homomorphism. It is actually true that $\det\Lambda_A \in \mathbb{K}_1$, not just in $\mathbb{K}_1$, although we will not prove this here.

Proposition (2.1.17):

A $\in M_n(\mathbb{D})$ is invertible if and only if Rdet $\mathbb{K}_1 \neq 0$ for some subfield $\mathbb{K}_1 \subseteq \mathbb{D}$ containing $\mathbb{K}_1$.

In the following we will discuss Quaternions

Proposition (2.1.18):

If A is a finite dimensional commutative $\mathbb{R}$-division algebra then either A = $\mathbb{R}$ or there is an isomorphism of $\mathbb{R}$ -algebras A $\neq$ C.

Proof:

Let α. Since A is a finite dimensional $\mathbb{R}$ -vector space, the powers $1, \alpha\ , \alpha^2, \ldots . \alpha^k \ldots$ must be linearly dependent, say

$$t_0 + t_1 \alpha + \ldots + t_m \alpha^m = 0 \qquad (2.2)$$

for some $t_j \in \mathbb{R}$ with $m \geq 1$ and $t_m \neq 0$. We can choose m to be minimal with these properties. If $t_0 = 0$, then

$$t_1 + t_2 \alpha + t_3 \alpha^2 + \ldots. + t_m \alpha^{m-1} = 0$$

contradicting minimality; so $t_0 \neq 0$. In fact, the polynomial $P(X) = t_0 + t_1X + \ldots + t_mX^m \in \mathbb{R}[X]$ is irreducible since if $P(X) = P_1(X)P_2(X)$ then since A is a division algebra, either $P_1(\alpha) = 0$ or $P_2(\alpha) = 0$, which would contradict minimality if both deg $P_1(X) > 0$ and deg $P_2(X) > 0$.

Consider the $\mathbb{R}$-subspace

$$\mathbb{R}(\alpha) = \left\{\sum_{j=0}^{k} s_j \alpha^j\right\}$$

Then $\mathbb{R}(\alpha)$ is easily seen to be a $\mathbb{R}$-subalgebra of A. The elements 1, α, α^2, α^{m-1}form a basis by Equation (2.2), hence $\dim_R \mathbb{R}(\alpha) = m$.

Let $\gamma \in C$ be any complex root of the irreducible polynomial $t_0 + t_1X + \ldots + t_mX^m \in \mathbb{R}[X]$ which certainly exists by the Fundamental Theorem of Algebra. There is an R-linear transformation which is actually an injection,

$$\varphi: \mathbb{R}(\alpha) \rightarrow \mathbb{C};\ \varphi\ \sum_{j=0}^{m-1} s_j \alpha^j) = \sum_{j=0}^{m-1} s_j \gamma^j$$

It is easy to see that this is actually an R-algebra homomorphism. Hence $\varphi\mathbb{R}(\alpha) \subseteq \mathbb{C}$ is a subalgebra.

But as $\dim_R \mathbb{C} = 2$, this implies that $m = \dim_R \mathbb{R}(\alpha) \leq 2$. If m = 1, then by Equation (2.2), $\alpha \in \mathbb{R}$. If m = 2, then $\varphi: \mathbb{R}(\alpha) = \mathbb{C}$.

So either $\dim_R$ A = 1 and A = $\mathbb{R}$, or $\dim_R$ A > 1 and we can choose an $\alpha \in A$ with $\mathbb{C} \neq \mathbb{R}(\alpha)$. This means that we can view A as a finite dimensional $\mathbb{C}$-algebra. Now for any $\beta \in A$ there is polynomial

$$q(X) = u_0 + u_1X + \ldots + u_e X^e \epsilon \mathbb{C}[X]$$

with $e \geq 1$ and $u_e \neq 0$. Again choosing e to be minimal with this property, q(X) is irreducible. But then since q(X) has a root in $\mathbb{C}$, $e = 1$ and $\beta \in \mathbb{C}$. This shows that A = $\mathbb{C}$ whenever $\dim_R$ A > 1.

The above proof actually shows that if A is a finite dimensional ℝ-division algebra, then either A = ℝ or there is a subalgebra isomorphic to ℂ. However, the question of what finite dimensional ℝ -division algebras exist is less easy to decide. In fact there is only one other up to isomorphism, the skew field of quaternions ℍ. We will now show how to construct this skew field.

Let

$$\mathbb{H} = \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} : z, w \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

It is easy to see that H is a subring of M_2(ℂ) and is in fact an ℝ-subalgebra where we view M_2(ℂ) as an ℝ-algebra of dimension 8. It also contains a copy of C, namely the ℝ-subalgebra

$$\left\{ \begin{bmatrix} z & 0 \\ 0 & \overline{z} \end{bmatrix} : z \in \mathbb{C} \right\} \subseteq \mathbb{H}$$

However, ℍ is not a C-algebra since for example

$$\begin{bmatrix} i & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

Notice that if $z, w \in \mathbb{C}$, then $z = 0 = w$ if and only if $|z|^2 + |w|^2 = 0$. We have

$$\begin{bmatrix} z & \omega \\ -\omega & \overline{z} \end{bmatrix} \begin{bmatrix} \overline{z} & -\omega \\ \overline{\omega} & z \end{bmatrix} = \begin{bmatrix} |z|^2 + |\omega|^2 & 0 \\ 0 & |z|^2 + |\omega|^2 \end{bmatrix}$$

Hence $\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix}$ is invertible if and only if $\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix} \neq 0$; furthermore in that case,

$$\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\overline{z}}{|z|^2 + |\omega|^2} & \frac{-\omega}{|z|^2 + |\omega|^2} \\ \frac{\overline{\omega}}{|z|^2 + |\omega|^2} & \frac{z}{|z|^2 + |\omega|^2} \end{bmatrix}$$

which is in ℍ. So an element of ℍ is invertible in H if and only if it is invertible as a matrix. Notice that

SU(2) = {A ∈ H : detA = 1} $\leq H^x$

It is useful to define on H a norm

$$\left\|\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix}\right\| = det\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix} = |z|^2 + |\omega|^2$$

Then

Su(2)={A∈ ℍ:$|A|$ =} ≤ $\mathbb{H}^x$

As an ℝ-basis of $\mathbb{H}^x$ we have the matrices

$$1 = I, i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

These satisfy the equations

$i^2 = j^2 = k^2 = -1$, ij = k = - k = -ij = -kj; ki = j = -ik:

This should be compared with the vector product on $\mathbb{R}^3$ From now on we will write quaternions in the form

$$q = xi + j + zk + t1\ (x, y, z, t \in R):$$

q is a pure quaternion if and only if $t = 0$, q is a real quaternion if and only if $x = y = z = 0$. We can identify the pure quaternion $xi + yj + zk$ with the element $x_{e1}+y_{e2}+z_{e3}$∈ $\mathbb{R}^3$. Using this identification we see that the scalar and vector products on $\mathbb{R}^3$ are related to quaternion multiplication by the following.

Proposition (2.1.19):

For two pure quaternions $q_1 = x_1i + y_1j + z_1k$, $q_2 = x_2i + y_2j + z_2k$,

$q_1q_2 = -(x_1i + y_1j + z_1k)\,(x_2i + y_2j + z_2k) + (x_1i + y_1j + z_1k)\_(x_2i + y_2j + z_2k)$.

In particular, q_1q_2 is a pure quaternion if and only if q_1 and q_2 are orthogonal, in which case q_1q_2 is orthogonal to each of them.

The following result summarises the general situation about solutions of $X^2 + 1 = 0$.

Proposition (2.1.20):

The quaternion q = $xi + j + zk + t1$ satisfies $q^2 + 1 = 0$ if and only if t = 0 and $x^2 + y^2 + z^2 = 1$.

Proof. This easily follows from Proposition 3.19.

There is a quaternionic analogue of complex conjugation, namely

$q = xi + j + zk + t1 \longmapsto \overline{q} = q^{*} = -xi - j - zk + t1.$

This is `almost' a ring homomorphism $\mathbb{H} \rightarrow \mathbb{H}$, in fact it satisfies

$$(\overline{q_1 + q_2}) = \overline{q}_1 + \overline{q}_2; \tag{2.3a}$$

$$(\overline{q_1 q_2}) = \overline{q}_1\overline{q}_2 ; \tag{2.3b}$$

$$\overline{q} = q \Leftrightarrow q \text{ is real quaternion;} \tag{2.3c}$$

$$\overline{q} = -q \Leftrightarrow q \text{ is a pure quaternion:} \tag{2.3d}$$

Because of Equation (2.3b) this is called a homomorphism of skew rings or anti-homomorphism of rings.

The inverse of a non-zero quaternion q can be written as

$$q^{-1} = \frac{1}{(g\overline{g})}\overline{g} = \frac{\overline{g}}{(g\overline{g})} \tag{2.4}$$

The real quantity $q\overline{q}$ is the square of the length of the corresponding vector,

$$|g| = \sqrt{g\overline{g}} = \sqrt{x^2 + y^2 + z^2 + t^2}$$

For z = with u, v ∈ R , z = u1- vi is the usual complex conjugation.

In terms of the matrix description of $\mathbb{H}$, quaternionic conjugation is given by hermitian conjugation,

$$\begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix} \mapsto \begin{bmatrix} z & \omega \\ -\overline{\omega} & \overline{z} \end{bmatrix}^{*} = \begin{bmatrix} \overline{z} & -\omega \\ \overline{\omega} & z \end{bmatrix}$$

From now on we will write

$$1 = 1,\ i = I,\ j = j,\ k = k.$$

Now we will discuss Quaternionic matrix groups

The above norm | | on $\mathbb{H}$ extends to a norm on $\mathbb{H}^n$, viewed as a right H-vector space. We can define an quaternionic inner product on $\mathbb{H}$ by

$$z.y = z^{*}y = \sum_{r=1}^{n} \overline{x}_r yr,$$

Where we define the quaternionic conjugate of a vector by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [\overline{x}_1\overline{x}_2 \ldots . \overline{x}_n]$$

Similarly, for any matrix $[\alpha_{ij}]$ over $\mathbb{H}$ we can define $[\alpha_{ij}]^* = [\overline{\alpha}_{ji}]$

The length of $x \in \mathbb{H}^n$ is defined to be

$$|x| = \sqrt{x^*x} = \sqrt{\sum_{r=1}^{n} |x_r|^2}$$

We can also define a norm on $M_n(\mathbb{H})$ i.e., for $A \in M_n(\mathbb{H})$,

$$\|A\| = sup\left\{\frac{|Ax|}{|X|} : 0 \neq x \in \mathbb{H}^n\right\}$$

There is also a resulting metric on $M_n(\mathbb{H})$,

$$(A, B) \mapsto \|A - B\|$$

and we can use this to do analysis on $M_n(\mathbb{H})$. The multiplication map $M_n(\mathbb{H}) \times M_n(\mathbb{H}) \to M_n(\mathbb{H})$ is again continuous, and the group of invertible elements $GL_n(\mathbb{H}) \subseteq M_n(\mathbb{H})$ is actually an open subset.

This can be proved using either of the reduced determinants

$$Rdet_{\mathbb{R}} : M_n(\mathbb{H}) \to \mathbb{R}, Rdet_C : M_n(\mathbb{H}) \to \mathbb{C},$$

each of which is continuous. By Proposition (2.1.17),

$$GL_n(\mathbb{H}) = M_n(\mathbb{H}) - Rdet_c^{-1}\ 0. \qquad (2.5a)$$

$$GL_n(\mathbb{H}) = M_n(\mathbb{H}) - Rdet_c^{-1}\ 0. \qquad (2.5b)$$

In either case we see that $GL_n(\mathbb{H})$ is an open subset of $M_n(\mathbb{H})$. It is also possible to show that the images of embeddings $GL_n(\mathbb{H}) \to GL_{4n}(\mathbb{R})$ and $GL_n(\mathbb{H}) \to GL_{2n}(\mathbb{C})$ are closed. So $GL_n(\mathbb{H})$ and its closed subgroups are real and complex matrix groups.

The $n\ x\ n$ quaternionic symplectic group is

$$Sp(n) = \{A \in GL_n(\mathbb{H}) : A^*A = I\} \leq GL_n(\mathbb{H}).$$

These are easily seen to satisfy

$$Sp(n) = \{A \in GL_n(\mathbb{H}) : \forall x.y \in \mathbb{H}^n, Ax.Ay = x.y\}.$$

These groups $Sp(n)$ form another infinite family of compact connected matrix groups along with familiar examples such as $SO(n), U(n), SU(n)$. There are further examples, the spinor groups $Spin(n)$ whose description involves the real Clifford algebras CL_n.

Now we will discuss The real Clifford algebras,

The sequence of real division algebras ℝ, ℂ, ℍ can be extended by introducing the real Clifford algebras Cl_n, where

$$Cl_0 = \mathbb{R}\ ,\ Cl_1 = \mathbb{C},\ Cl_2 = \mathbb{H}\ ,\ dim_R = 2^n$$

There are also complex Clifford algebras, but we will not discuss these. The theory of Clifford algebras and spinor groups is central in modern differential geometry and topology, particularly Index Theory. It also appears in Quantum Theory in connection with the Dirac operator. There is also a theory of Clifford Analysis in which the complex numbers are replaced by a Clifford algebra and a suitable class of analytic functions are studied; a motivation for this lies in the above applications.

We begin by describing Cl_n as an ℝ-vector space and then explain what the product looks like in terms of a particular basis. There are elements $e_1, e_2, \ldots$ $e_n \in Cl_n$ for which

$$\begin{cases} e_s e_r = -e_s e_r\ ,\ if\ s\ \neq r. \\ e_r^2\ =\ -1 \end{cases} \qquad (2.6\ a)$$

Moreover, the elements $e_{i1}e_{i2}$ for increasing sequences $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ with $0 \leq r \leq n$, form an ℝ-basis for Cl_n. Thus

$$dim_{\mathbb{R}} Cl_n\ =\ 2^n \qquad (2.6b)$$

When r = 0, the element $e_{i1}\ e_{i2}$ eir is taken to be 1.

Proposition (2.1.21):

There are isomorphisms of ℝ-algebras

$$C1_1 \cong C, Cl_2 \cong \mathbb{H}$$

Proof:

For Cl_1, the function

$$Cl_1 \to \mathbb{C}; x + ye1 \mapsto x + yi\ (x, y\ \in \mathbb{R}),$$

is an ℝ -linear ring isomorphism.

Similarly, for Cl_2, the function

$Cl_2 \rightarrow \mathbb{H}$; $t1 + x_{e1} + y_{e2} + z_{e1}e_2 \rightarrow t1 + xi + yj + zk$ $(t,x,y,z \in \mathbb{R})$;

is an R-linear ring isomorphism.

We can order the basis monomials in the er by declaring $e_{i1}e_{i2}$

to be number

$$1 + 2^{i1-1} + 2^{i2-1} + \ldots + 2^{ir-1},$$

which should be interpreted as 1 when $r = 0$. Every integer k in the range $1 \leq k$ 6 $2n$ has a unique binary expansion

$$k = k_0 + 2k_1 + \ldots + 2^j k_j + \ldots + 2^n k_n,$$

where each $k_j = 0,1$. This provides a one-one correspondence between such numbers k and the basis monomials of Cl_n. Here are the basis orderings for the first few Clifford algebras.

$Cl_1 : 1, e1; Cl_2 : 1, e1; e2; e1e2; Cl_3 : 1; e1, e2, e1e2, e3, e1e3, e2e3; e1e2e3.$

Using the left regular representation over ℝ associated with this basis of Cl_n, we can realiseCl_n as a subalgebra of $M_{2n}(\mathbb{R})$.

Example (2.1.22):

For Cl_1 we have the basis $\{1, e_1\}$ and we find that

$$\rho(0) = l_{2,}p(e1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

So the general formula is

$$\rho(x + ye1) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} (x, y \in \mathbb{R})$$

For Cl_2 the basis $\{1, e1, e2, e1e2\}$ leads to a realization in $M_4(\mathbb{R})$ for which $\rho(1) = I_4$ and

$$\rho_{(e1)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \rho_{e2}$$

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \rho_{e1e2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In all cases the matrices $\rho(e_{i1}e_{i2}\ldots e_{ir})$ are generalized permutation matrices all of whose entries are entries 0, ±and exactly on non-zero entry in each row and column. These are always orthogonal matrices of determinant 1.

These Clifford algebras have an important universal property which actually characterises them.

First notice that there is an ℝ-linear transformation

$$jn\colon \mathbb{R}^n \to Cl_n; jn\left(\sum_{r=1}^{n} x_r e_r\right) = \sum_{r=1}^{n} x_r e_r$$

By an easy calculation,

$$jn\left(\sum_{r=1}^{n} x_r e_r\right)^2 = -\sum_{r=1}^{n} x_1^2 = -\left|\sum_{r=1}^{n} x_r e_r\right|^2 \quad (2.7)$$

Theorem (2.1.23): (The Universal Property of Clifford Algebras)

Let A be a ℝ-algebra and f : $\mathbb{R}^n$→AanℝR-linear transformation for which

$$f(x)^2 = -|x|^2 1.$$

Then there is a unique homomorphism of ℝ-algebras F :Cl_n→ A for which F _j_n = *f, i.e.*, for all x ∈ $\mathbb{R}^n$,

$$F(j_n(x)) = f(x).$$

Proof:

The homomorphism F is defined by setting $F(er) = f(er)$ and showing that it extends to a ring homomorphism on Cl_n.

Example (2.1.24):

There is an ℝ-linear transformation

$$\alpha 0 : \mathbb{R}^n \to Cl_n \; ; \; \alpha 0(x) = -j_n(x) = j_n(\square x).$$

Then

$$\alpha\ 0(x)^2 = jn(-x)^2 = -|x|^2,$$

so by the Theorem there is a unique homomorphism of ℝ-algebras α: $Cl_n \to Cl_n$ for which

$$\alpha\ (j_n(x)) = \alpha_0(x).$$

Since $j_n(e_r) = e_r$, this implies

$$\alpha\ (e_r) = -e_r.$$

Notice that for $1 \leq i1 < i2 <, \ldots < i_k \leq n$,

$$\alpha\ (e_{i1}e_{i2}, \ldots e_{ik}) = (-1)^k{}_{ei1ei2} \ldots e_{ik} \begin{cases} e_{i1}e_{i2} \ldots e_{ik}\ if\ k\ is\ even \\ -e_{i1}e_{i2} \ldots e_{ik}\ if\ k\ ks\ odd \end{cases}$$

It is easy to see that α is an isomorphism and hence an automorphism.

This automorphism α: $Cl_n \to Cl_n$ is often called the canonical automorphism of Cl_n.

Clifford algebras. Consider the ℝ-algebra $M_2(\mathbb{H})$ of dimension

16. Then we can define an ℝ-linear transformation

$$\theta_4 : \mathbb{R}^2 \to M_2(\mathbb{H}) : \theta_{4(x1e1+x2e2+x3e3+x4e4)} =$$

$$\begin{bmatrix} x_1 i + x_2 j + x_3 k & x_4 k \\ x_4 k & x_1 i + x_2 j - x_3 k \end{bmatrix}"$$

Direct calculation shows that θ_4 satisfies the condition of Theorem (2.1.23) hence there is a unique ℝ-algebra homomorphism $\ominus_4 : Cl_4 \to M_2(\mathbb{H})$ with $\ominus_4\ j_4 = \theta_4$. This is in fact an isomorphism of ℝ -algebras, so

$Cl_4 \cong M_2(\mathbb{H})$:

Since $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \mathbb{R}^4$ we obtain compatible homomorphisms

$\ominus_1 : Cl_1 \to M_2(\mathbb{H})$; $\ominus_2 : Cl_2 \to M_2(\mathbb{H})$, $\ominus_3 : Cl_3 \to M_2(\mathbb{H})$;

which have images

$$\text{im}\ominus_1 = \{zI_2 : z \in C\}.$$

$$\text{im}\ominus_2 = \{qI_2 : q \in H\},$$

$$\text{im}\ominus_3 = \left\{\begin{vmatrix} q_1 & 0 \\ 0 & q_2 \end{vmatrix} : q_1 q_2 \in \mathbb{H}\right\}$$

This shows that there is an isomorphism of ℝ-algebras

$Cl_3 \neq \mathbb{H} \, x \, \mathbb{H}$,

Where the latter is the direct product of Definition (2.1.5) We also have

$$GL_5 \cong M_3(\mathbb{C}), GL_6 \cong M_8(\mathbb{R}) GL_7 \cong M_8(\mathbb{R}) x \, M_s\mathbb{R}$$

After this we have the following periodicity result, where $M_m(Cl_n)$ denotes the ring of $m \, x \, m$ matrices with entries in Cl_n.

Theorem (2.1.25):

For n $\geq$ 0,

$$Cl_{n+8} \cong = M_{16}(Cl_n).$$

First there is a conjugation $\overline{(\ \ )}$: $Cl_n \to Cl_n$ defined by

$\overline{e_{i1}e_{i2}, \ldots e_{ik}} = (-1)k_{e_{ik}e_{ik-1}}, \ldots e_{i1}$

whenever $1 \leq i_1 < i_2 < \ldots < i_k \leq n$, and satisfying

$$\overline{x + y} = \overline{x} + \overline{y},$$

$$\overline{tx} t \overline{x},$$

for $x, y \in Cl_n$ and $t \in \mathbb{R}$. Notice that this is not a ring homomorphism $Cl_n \to Cl_n$ since for example whenever $r < s$,

$$\overline{e_r e_s} = e_s e_r = -e_r e_s = \overline{- e_r e_s} \neq e_r e_s.$$

However, it is a ring anti-homomorphism in the sense that for all

$$x, y \in Cl_n, \quad (2.8)$$

When n = 1, 2 this agrees with the conjugations already defined in ℂ and ℍ.

Second there is the canonical automorphism α:$Cl_n \to Cl_n$ defined in Example (2.1.24).

We can use α to define a $\pm$-grading on Cl_n:

$C_n^+ = \{u \in Cl_n : \alpha(u) = u\}$, $Cl_n^- n = \{u \in Cl_n : \alpha(u) = -u\}$.

Proposition (2.1.26):

i) Every element $v \in Cl_n$ can be unique expressed in the form $v = v^+ + v^-$ where $v^+ \in C_n^+$ and $v^- \in C_n^-$. Hence as an ℝ-vector space, $Cl_n = Cl_n^+ \oplus CL_n^-$.

ii) This decomposition is multiplicative in the sense that

$$uv \in C_N^+ \quad if\ u, v\ \in\ CL_N^+\ or\ u\,v\ \in C_n^-\ ,$$

$$uv, vu\ \in Cl_n^+ \text{ if } u \in\ C_n^+\ and\ v\ \in\ C_n^{-1}$$

Proof:

i) The elements

$$v^+ = \frac{1}{2}\big(v + \alpha(vv)\big)\ , v^- = \frac{1}{2}\big(v - \alpha(v)\big).$$

satisfy α $(v^+) = v^+$, $\alpha(v^-) = -v^-$ and $v = v^+ + v^-$. This expression is easily found to be the unique one with these properties and defines the stated vector space direct sum decomposition.

Notice that for bases of $Cl_n^{\pm}$ we have the monomials

$$e_{j1} \ldots e_{j2m} \in CL_n^+\ (1 \leq j1 < \ldots < j_{2m} \leq n).$$

$$e_{j1}, \ldots e^{j2m+1} \in Cl_n^1\ (1 \leq j1 < \ldots < j_{2m+1} \leq n). \qquad (2.9)$$

Finally, we introduce an inner product. and a norm | |on Cl_n by defining the distinct monomials $e_{i1}e_{i2}\, e_{ik}$ with $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ to be an orthonormal basis, i.e.

$$e_{i1}e_{i2} \ldots e_{ik} \,.\, e_{i1}e_{i2} \ldots e_{ik} = \begin{cases} 1 \quad if\ \ \ell\ = k\ and\ i_r\ =\ j_r\ for\ all\ r \\ \qquad 0\ otherwise \end{cases}$$

A more illuminating way to define is by the formula

$$u.v. = \frac{1}{2}Re(\overline{u}\ v + \overline{v}\ u\ ), \qquad (2.10)$$

Where for $\omega \in Cl_n$ we define its real part Reω to be the coefficient of 1 when w is expanded as an $\mathbb{R}$-linear combination of the basis monomials $e_{i1} \ldots e_{ir}$ with $1 \leq i_1 < \ldots < ir \leq n$ and $0 \leq r$. It can be shown that for any $u, v\ \in\ Cl_n$ and w $\in j_n R^n$,

$$(wu), (wv)\ =\ \lceil\omega\rceil^2(u\ \_\ v).. \qquad (2.11)$$

In particular, when $\lceil\omega\rceil = 1$ left multiplication by ω defines an $\mathbb{R}$-linear transformation on Cl_n which is an isometry. The norm | |gives rise to a metric onCl_n. This makes the group of units Cl_n^x into a topological group while the above embeddings of Cl_n into matrix rings are all continuous. This implies that CL_n^x is a matrix group. Unfortunately, they are not norm preserving. For example,2+$_{e1e2e3}$∈Cl^3 has $|2 + _{e1e2e3}|=\sqrt{5}$,but the corresponding matrix in $M_8(\mathbb{R})$ has norm$\sqrt{3}$. However, by defining for each ω ∈Cl_n

$$|w| = \{wx\}: x \in CL_n, |x| = 1\},$$

we obtain another equivalent norm on Cl_n for which the above embedding $Cl_n \rightarrow M_{2n}(\mathbb{R})$ does preserve norms. For $\omega \in j_n\mathbb{R}^n$ we do have $\|w\| = |w|$ and more generally, for $w_1 \ldots w_k \in j_n\mathbb{R}^n$,

$$\|w_1 \ldots w_k\| = |w_1 \ldots w_k| = |w_1| \ldots |w_k|$$

For $x, y \in Cl_n$,

$$\|xy\| \leq \|x\|\|y\|$$

without equality in general.

In the following we will study The spinor groups we will describe the compact connected spinor groups Spin(n) which are groups of units in the Clifford algebras Cl_n. Moreover, there are surjective Lie homomorphisms Spin(n)→ $SO(n)$each of whose kernels have two elements.

We begin by using the injective linear transformation $j_n : \mathbb{R}^n \rightarrow Cl_n$ to identify $\mathbb{R}^n$ with a subspace of Cl_n, i.e.,

$$\sum_{r=1}^{n} x_r e_r \leftrightarrow ij(\sum_{r=1}^{n} x_r e_r) = \sum_{r=1}^{n} x_r e_r$$

Notice that $\mathbb{R}^n \subseteq Cl_n^- C$, so for x ∈ $\mathbb{R}^n$, u ∈ C_n^+ and $v \in CL_n^-$

$$xu, ux \in C_n^-. xv, vx \in CL_n^+ \qquad (2.12)$$

Inside of $\mathbb{R}^n \subseteq Cl_n$ is the unit sphere

$$Sn^{-1} = \{x\ 2\ \mathbb{R}^n\ |x| = 1\} = \{\textstyle\sum_{r=1}^{n} x_r e_r (\sum_{r=1}^{n} x_r^2 = 1\}$$

Lemma (2.1.27):

Let u ∈ $S^{n-1} \subseteq Cl_n$. Then u is a unit in Cl_n, u ∈ Cl_n^x

Proof:

Since u ∈ $\mathbb{R}^n$

$$(-u)u = u(-u) = -u^2 = -(-|u|^2) = 1,$$

so (-u) is the inverse of u. Notice that -u ∈ $\mathbb{C}^{n-1}$

More generally, for $u_1, \ldots, u_k \in \mathbb{C}^{n-1}$ we have

$$(u_1 \ldots u_k)^{-1} = (-1)^k u_k \ldots u_1 = \overline{u_1 \ldots u_k} \qquad (2.13)$$

Definition (2.1.28):

The pinor group $\mathrm{Pin}(n)$ is th e subgroup of Cl_n^x n generated by the elements of $\mathbb{C}^{n-1}$,

$$Pin(n) = \{u_1 \ldots u_k : k \geq 0, u_r \in \mathbb{C}^{n-1}\} \leq Cl_n^x$$

Notice that Pin(n) is a topological group and is bounded as a subset of Cl_n with respect to the metric introduced in the last section. It is in fact a closed subgroup of Cl_n^x and so is a matrix group; in fact it is even compact. We will show that $\mathrm{Pin}(n)$ acts on $\mathbb{R}^n$ in an interesting fashion. We will require the following useful result.

Lemma (2.1.29):

let $u, v \in \mathbb{R}^n \subseteq Cl_n$. If $u \,.\, v = 0$, then

$$uv = -uv.$$

Proof:

Writing $u = \sum_{r=1}^{n} x_r e_r$ and $v = \sum_{r=1}^{n} y_r e_r$ with $x_r, y_s \in \mathbb{R}$, we obtain

$$vu = \sum_{s=r}^{n} \sum_{r=1}^{n} y_s x_r e_s e_r$$

$$\sum_{r=1}^{n} y_x x_r e_r^2 \sum_{r<s} (x_s y_r - x_r y_s) e_r e_s$$

$$= 1 \sum_{r=1}^{n} y_r x_r - \sum_{r<s} (x_r y_r - x_r y_s) e_r e_s$$

$$= u.v - \sum_{r<s} (x_s y_r - x_r y_s) e_r e_s$$

$$= -\sum_{r<s}(x_s y_r - x_r y_s)e_r e_s$$

$$= u.v - \sum_{r<s}(x_s y_r - x_r y_s)e_r e_s$$

$$= -\sum_{r=1}^{n}\sum_{s=1}^{n} x_r y_s e_r e_s$$

$$= -uv.$$

For u ∈ S^{n-1} and $x \in \mathbb{R}^n$,

$$\alpha(u)\overline{xu} = (\ -u)x(- \ u) = uxu.$$

If $u.x$ = 0, then by Lemma (2.1.29),

$$\alpha(u)\overline{xu} = -u^2x = -(-1)x = x. \qquad (2.14a)$$

Since $u^2 = -|u|^2 = -1$. On the other hand, if $x = tu$ for some t ∈ R, then

$$\alpha(u)x\overline{u} = tu^2u = - \ tu \qquad (2.14b)$$

So in particular $\alpha(u)x\overline{u} \in \mathbb{R}^n$. This allows us to define a function

$$\rho u: \mathbb{R}^n \to \mathbb{R}^n; \rho u(x) = \alpha(u)x\overline{u} = uxu.$$

Similarly for $u \in \mathrm{Pin}(n)$, we can consider $(u)x\overline{u}$; if $u = u^1 \dots u_r$ for $u_1 \dots u_r \in S^{n-1}$, we have

$$\alpha(u)x\overline{u} = \alpha(u_1 \dots u_r)\overline{xu_1 \dots u_r}$$

$$= ((-1)^r u_1 \dots u_r)x((-1)^r u_r \dots u_1)$$

$$= \rho_{u1} o \dots o \rho_{ur}(x) \in \mathrm{R}^n. \qquad (2.15)$$

So there is a linear transformation

$$\rho u: \mathbb{R}^n \to \mathbb{R}^n; \rho u(x) = \alpha(u)x\overline{u}$$

Proposition (2.1.30):

For u ∈ Pin(n), $\rho u : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, i.e., an element of O(n).

Since each $\rho u \in$ O(n) we actually have a continuous homomorphism

$$\rho: Pin(n) \to (n); \ \rho(u) = \rho u:$$

Proposition ρ: Pin(n) →O(n) is surjective with kernel ker ρ = {1,-1}.

follows by using the standard fact that every element

of O(n) is a composition of reections in hyperplanes.

Suppose that for some $u_1, \dots u_k \in S^{n-1}$, $u = u_1 \dots u_k \in \ker \rho$, i.e., $\rho u = I_n$. Then

$$1 = \det \rho u = \det(\rho_{u1} \dots \rho_{uk}) = \det \rho_{u1} \dots \det \rho_{uk} .$$

Each ρ_{ur} is a reection and so has $\det \rho_{ur} = -1$. These facts imply k must be even, $u \in Cl_n^+$ and then by Equation (2.13),

$$u^{-1} = u_k \dots u_1 = \overline{u}.$$

So for any $x \in \mathbb{R}^n$ we have

$$x = \rho(x) = uxu^{-1},$$

which implies that

$$xu = ux.$$

For each r = 1, … , n we can write

$$u = a_r + e_r b_r = (a_r^+ + e_r b_{\overline{r}}) + (a_{\overline{r}} + e_r),$$

where $a_r, b_r \in Cl_n$ do not involve e_r in their expansions in terms of the monomial bases of Equation (2.9). On taking $x = e_r$ we obtain

$$e_r(a_r + e_r b_r) = (a_r + e_r b_r)e_r.$$

giving

$$a_r + e_r b_r = -e_r(a_r + e_r b_r)e_r$$

$$= -e_r a_r e_r - e_2\, r\, b_r e_r$$

$$= -e_r^2 e_2\text{-}r\, a_r - e_r b_r$$

$$= a_r - e_r b_r$$

$$= (a_r^+ - e_r b_1^-) + (a_r^- - e_r b_r^-$$

$$= a_r = e_r b_r ,$$

where we use the fact that for each $e_s \neq e_r$, $e_s e_r = -e_r e_s$. Thus we have $b_r = 0$ and so $u = a_r$ does not involve e_r. But this applies for all r, so $u = t1$ for some $t \in R$. Since $\overline{u} = t1$,

$$t^2 1 = u\overline{u} = (-1)^k = 1,$$

by Equation (2.13) and the fact that k is even. This shows that t = ± and so u = ±1.

For $n \geq 1$, the spinor groups are defined by

$$Spin(n) = \rho^{-1}1\, SO(n) \leq Pin(n).$$

Theorem (2.1.31):

Spin(n) is a compact, path connected, closed normal subgroup of Pin(n), satisfying

$$\text{Spin(n)} = \text{Pin (n)} \cap \text{CL}_n^+ \qquad (2.16a)$$

$$\text{Pin(n)} = \text{Spin(n)} \cup_{er} \text{Spin(n)}, \qquad (2.16b)$$

for any r = 1, … , n.

Furthermore, when n ≥ 3 the fundamental group of Spin(n) is trivial, π_1 Spin(n) = 1.

Proof:

We only discuss connectivity. Recall that the sphere $S^{n-1} \subseteq R^n \subseteq Cl_n$ is path connected.

Choose a base point $u_0 \in S^{n-1}$. Now for an element $u = u_1 \dots u_k \in S^{n-1}$ we must have k even, say k = 2m. In fact, we might as well take m to be even since u = u(-w)w for any $w \in S^{n-1}$. Then there are continuous paths

$$\rho r : [0,1] \to S^{n-1}\ (r = 1, \dots 2m),$$

for which $p_r(0) = u_0$ and $p_r(1) = u_r$. Then :

$$p: [0,1] \to S^{n-1}\ \ p(t) = p_1(t) \dots p_{2m}(t)$$

is a continuous path in Pin(n) with

$$p(0) = u_0^{2m} = (-1)^m = 1, p(1) = u,$$

But t ⟼ p (p(t)) is a continuous path in O(n) with $p(p(0)) \in$ SO(n), hence $p(p(t)) \in$ SO(n) for all t. This shows that p is a path in Spin(n). So every element u ∈ Spin(n) can be connected to 1 and therefore Spin(n) is path connected.

The final statement involves homotopy theory and is not proved here. It should be compared with the fact that for n ≥ 3, π_1 SO(n) ≅ {1,-1} and in fact the map is a universal covering.

The double covering maps p: Spin(n) →SO(n) generalize the case of SU(2) → SO(3).

In fact, around each element u ∈ there is an open neighbourhood $N_u \subseteq$ Spin(n) for which p: $N_u \to N_u$ is a homeomorphism, and actually a diffeomorphism.

This implies the following.

Proposition (2.1.32):

The derivative d p: spin(n) → so(n) is an isomorphism of R-Lie algebras and

$$dim\ Spin(n) = dim\ SO(n) = \binom{n}{2}$$

In the following we will discuss The centres of spinor groups

Recall that for a group G the centre of G is

$$C(G) = \{c \in G : \forall g \in G;\ gc = cg\}.$$

Then C(G) ⊲ G. It is well known that for groups SO(n) with n ≥ 3 we have

Proposition (2.1.33):

$For\ n \geq 3,$

$$C\big(SO(n)\big) = \{tI_n : t = \pm 1, t^n = 1\} = \begin{cases} \{1_n\}\ if\ n\ is\ odd \\ \{\pm 1_n\}\ if\ n\ is\ even \end{cases}$$

Proposition (2.1.34):

For n ≥ 3

$$C\big(Spin(n)\big) = \begin{cases} \{\pm 1\}\ if\ n\ is\ odd \\ \{\pm 1, \pm e_1 \dots e_n\}\ if\ in \equiv 2\ mod\ 4. \\ \{\pm 1, \pm e_1 \dots e_n\}\ if\ n \equiv 0\ mod\ 4. \end{cases}$$

$$\begin{cases} \dfrac{z}{2}\ if\ n\ is\ odd \\ \dfrac{z}{4}\ ifn \equiv 2\ mod\ 4. \\ \dfrac{\frac{z}{2} xz}{2}\ ifn = 0\ mod\ 4. \end{cases}$$

Proof:

If g ∈ C(Spin(n)), then since ρ: Spin(n) → SO(n), ρ(g) ∈ C(SO(n)). As ±1 ∈ C(Spin(n)), this gives $|C(Spin(n))| = 2|C(SO(n))|$ and indeed

$$C(Spin(n)) = \rho^{-1}\, C(SO(n)).$$

For n even,

$$(\pm e_1 \dots e_n)^2 = e_1 \dots e_n e_1 \dots e_n = -1^{\binom{n}{2}} e_1^2 \dots e_n^2 = (-1)^{\binom{n+1}{2}}$$

Since

$$\binom{n+1}{2} = \frac{(n+1)n}{2} \equiv \begin{cases} 0 \bmod 2 \text{ if } n \equiv 2 \bmod 4, \\ 1 \bmod 2 \text{ if } n \equiv 0 \bmod 4, \end{cases}$$

this implies

$$(\pm e_1 \dots e_n)^2 = \begin{cases} 1 \text{ if } n \equiv 2 \text{ mode } 4, \\ -1 \text{ if } n \equiv 0 \text{ mode } 4, \end{cases}$$

Hence for n even, the multiplicative order of $\pm e_1 \dots . e_n$ is 1 or 2 depending on the congruence class of n modulo 4. This gives the stated groups.

We remark that Spin(1) and Spin(2) are abelian.

In the following we will discuss finite subgroups of spinor groups Each orthogonal group O(n) and SO(n) contains finite subgroups. For example, when n = 2, 3, these correspond to symmetry groups of compact plane figures and solids. Elements of SO(n) are often called direct isometries, while elements of $O(n)^-$ are called indirect isometries. The case of n = 3 is explored in the Problem Set for this chapter. Here we make some remarks about the symmetric and alternating groups.

Recall that for each n ≥ 1 the symmetric group S_n is the group of all permutations of the set n = 1, … n. The corresponding alternating group $A_n \leq S_n$ is the subgroup consisting of all even

permutations, i.e., the elements $\sigma \in S_n$ for which sign(σ)=1 where sign : $S_n \to \{\pm 1\}$ is the sign homomorphism.

For a field |, we can make S_n act on $\mathbb{K}^n$ by linear transformations:

$$\sigma \begin{vmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{vmatrix} = \begin{vmatrix} x_{\sigma-1(1)} \\ x_{\sigma-2(2)} \\ \vdots \\ x_{\sigma-1(n)} \end{vmatrix}$$

Notice that σ(er) = $e_{\sigma(r)}$. The matrix $[\sigma]$ of the linear transformation induced by σ with respect to the basis of e_r's has all its entries 0 or 1, with exactly one 1 in each row and column. For example, when n = 3,

$$[(123)] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, [(1,3)] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

When $\mathbb{K} = \mathbb{R}$ each of these matrices is orthogonal, while when $\mathbb{K} = \mathbb{C}$ it is unitary. For a given n we can view S_n as the subgroup of O(n) or U(n) consisting of all such matrices which are usually called permutation matrices.

Proposition (2.1.35):

For $\sigma \in S_n$,

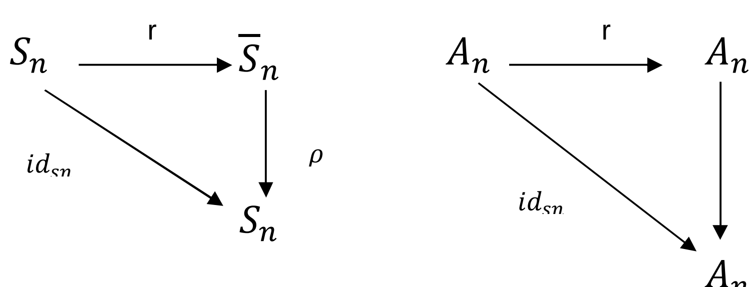
$$sign(\sigma) = det([\sigma]).$$

Hence we have

$$An = \begin{cases} SO(n) \cap S_n \; if \; k = R \\ Su(n) \cap S_n \; if \; k = C \end{cases}$$

Recall that if n $\geq$ 5, A_n is a simple group.

As ρ: Pin(n) → O(n) is onto, there are finite subgroups $\overline{S}_n = \rho^{-1} S_n = \leq$ Pin(n) and $\overline{A}_n = p^{-1A_n} \leq$Spin(n) for which there are surjective homomorphisms ρ:$\overline{S}$n →S_n and ρ: $\overline{A}_n$ →A_n whose kernels contain the two elements ± 1. Note that $|\overline{S}_n|$=2 n!, while $|\overline{A}_n| = n!$, However, for n $\geq$ 4, there are no homomorphisms r: $S_n \to S_n$, t: $A_n \to \overline{A}_n$ for which $\rho \circ \tau$ = Id.



Similar considerations apply to other finite subgroups of O(n).

In CL_n^x n we have a subgroup En consisting of all the elements

$$\pm e_{ij} \dots e_{ir} (1 \leq i_1 < \cdots < i_r \leq n, 0 \leq r)$$

The order of this group is $|E_n|$= 2^{n+1} and as it contains ± 1, its image under ρ: Pin(n) → O(n) is $\overline{E}$n = ρE_n of order $|\overline{E}_n|$= 2n. In fact, $|\{\pm 1\}|$ = C(E_n) is also the commutator subgroup since $e_i e_j e_i^{-1} e_j^{-1}$ = -1 and so $\overline{E}$n is abelian. Every non-trivial element in $\overline{E}$n has order 2 since e_i^2= -1, hence $\overline{E}_n \leq$ O(n) is an elementary 2-group, i.e., it is isomorphic to $(Z/2)^n$. Each element ρ(er) ∈ O(n) is a generalized permutation matrices with all its non-zero entries on the main diagonal. There is also a subgroup $\overline{E}_n^0 = pE_n^0 \leq SO(n)$ of order 2^{n-1} where

$$E_n^0 = E_n \cap Spin\ (n)$$

These groups E_n and E_n^0-2^n/In fact $\overline{E}_n^0$ is isomorphic to (Z).

are non-abelian and fit into exact sequences of the form

$$1 \to \frac{z}{2} \to E_n \to \left(\frac{z}{2}\right)^n \to 1, 1 \to z/2 \to E_n^0 \to (z/2)^{n-1} \to 1$$

in which each kernel Z/2 is equal to the centre of the corresponding group E_n or E_n^0 This means they are extraspecial 2-groups.

Section (2.2) : Matrix Groups as Lie Groups

Now we will discuss the basic ideas of smooth manifolds and Lie groups.

Definition (2.2.1):

A continuous map g : $V_1 \to V_2$ where each $V_k \subseteq R^{mk}$ is open, is called smooth if it is infinitely differentiable. A smooth map g is a diffeomorphism if it has smooth. inverse g^{-1} which is also smooth.

Definition (2.2.2):

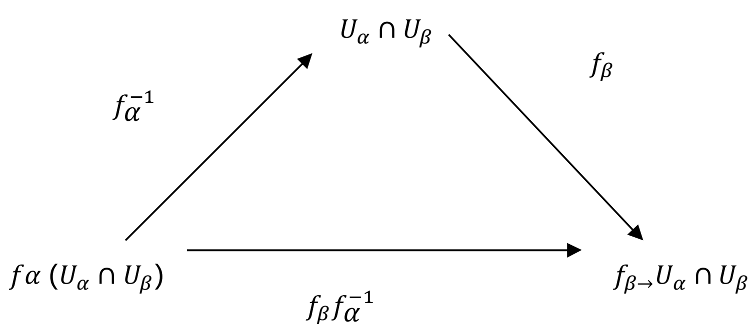
Let M be a separable Hausdorff topological space.

A homeomorphism $f : U \to V$ where $U \subseteq M$ and $V \subseteq R^n$ are open subsets, is called an n-chart for U.

If U = $\{U_\alpha U: \alpha \in A\}$ is an open covering of M and $\mathfrak{F} = \{f_\alpha \to V_\alpha\}$ is a collection of charts, then $\mathcal{F}$ is called an atlas for M if, whenever $U_\alpha \cap U_\beta U \neq 0$

$$f_\beta \circ f_\alpha^{-1}: f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism.



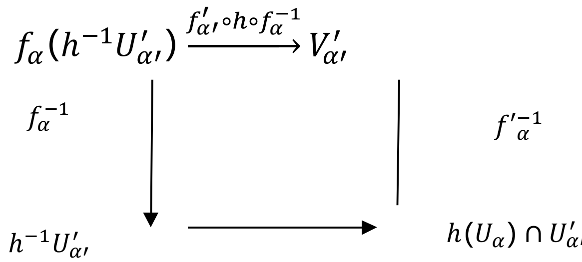
We will sometimes denote an atlas by (M,U, $\mathcal{F}$) and refer to it as a smooth manifold of dimension n or smooth n-manifold.

Definition (2.2.3):

Let (M,U,$\mathcal{F}$) and (U', U', f') be atlases on topological spaces M and M'. A smooth map h: (M,U,$\mathcal{F}$)$\to$ (U', U', f') is a continuous map $h: M \to M'$ such that for each pair $\alpha, \alpha'$with $h(U_\alpha) \cap U'_\alpha \neq \theta$, the composite

$$f'_{\alpha'} \circ h \circ f_\alpha^{-1}: f_\alpha(h^{-1}U'_{\alpha'} \to V'_{\alpha'}$$

is smooth.



In the following we will discuss Tangent spaces and derivatives

Let (M,U, $\mathcal{F}$) be a smooth n-manifold and p ∈ M .γ Let : (a,b) → M be a continuous curve with α < 0 < b.

Definition (2.2.4):

is differentiable at t ∈ (a, b) if for every chart f : U → V with γ(t) ∈ U, the curve f ∘ γ : (a, b) → V is differentiable at t ∈ (a, b), i.e., (f ∘ γ)' (t) exists. γ is smooth at t ∈ (a; b) if all the derivatives of f ∘ γ exists at t.

The curve γ is differentiable if it is differentiable at all points in (a, b).

Similarly γ is smooth if it is smooth at all points in (a, b).

Lemma (2.2.5):

Let $f_0 : U_0 \to V_0$ be a chart with γ (t) ∈ U_0 and suppose that

$$f_0 \circ \gamma \ : (a, b) \cap f_0^{-1} v_0 \to v_0$$

is differentiable/smooth at t. Then for any chart $f : U \to V$ with γ (t) ∈ U

$$f \circ \gamma \ : (a, b) \cap f^{-1} V \to V$$

is differentiable/smooth at t.

Proof:

The smooth composite $f \circ \alpha$ is defined on a subinterval of (a, b) containing t and there is the usual Chain or Function of a Function Rule for the derivative of the composite

$$(f\gamma)'(t) = Jac_{f\circ f^{-1}(f\circ\gamma(t))(f\circ\gamma)'(t)} \qquad (2.19)$$

Here, for a differentiable function

$$h: w_1 \to w_2; h(x) = \begin{bmatrix} h_{1(x)} \\ \vdots \\ h_{m2(x)} \end{bmatrix}$$

with $W_1 \subseteq R^{m1}$ and $W_2 \subseteq R^{m2}$ open subsets, and $x \in W_1$, the Jacobian matrix is

$$Jac_h(x) = \left[\frac{\partial h_i}{\partial x_i}(x)\right] \in M_{m2,m1}(R)$$

If $\gamma(0) = \rho$ and γ is differentiable at 0, then for any (and hence every) chart $f_0 : U_0 \to V_0$ with $\gamma(0) \in U_0$, there is a derivative vector $v_0 = (f\gamma)'(0) \in R^n$. In passing to another chart $f : U \to V$ with $\gamma(0) \in U$ by Equation (2.19) we have

$$(f\gamma)'(0) = Jac_{ff_0^{-1}}\big(f \circ \gamma(0)\big)(f \circ \gamma)'(0).$$

In order to define the notion of the tangent space T_pM to the manifold M at p, we consider all pairs of the form

$$((f\gamma)'(0), f : U \to V)$$

where $\gamma(0) = p \in U$, and then impose an equivalence relation ~ under which

$$((f_1\gamma)'(0), f_1 : U_1 \to V_1) \sim ((f_2\gamma)'(0), f_2 : U_2 \to V_2).$$

Since

$$(f_2\gamma)'(0) = Jac_{f2f_1^{-1}}\left(f1_\gamma(0)\right)(f_1\gamma)'(0)$$

we can also write this as

$$(v, f_1 : U_1 \to V_1) \sim (Jac_{f_2f_1^{-1}(f_1(p))v,}f_2 : U_2 \to V_2),$$

whenever there is a curve α in M for which

$$\gamma(0) = p, (f_1\gamma)'(0) = v$$

The set of equivalence classes is T_pM and we will sometimes denote the equivalence class of $(v, f : U \to V)$ by $[v, f : U \to V]$.

Proposition (2.2.6):

For $p \in M$, T_pM is an R-vector space of dimension n.

Proof:

For any chart $f: U \rightarrow V$ with p ∈ U, we can identify the elements of T_pM with objects of the form $(v, f: U \rightarrow V)$. Every ∈ R^n arises as the derivative of a curve $\overline{\gamma}$: $(-\varepsilon, \varepsilon)$ → V for which $\overline{\gamma}(0) = f(p)$. For example for small enough ", we could take

$$\overline{\gamma}(t) = f(p) + tv.$$

There is an associated curve in M,

$$\gamma : (-\varepsilon, \varepsilon) \rightarrow M;\ \gamma(t) = f^{-1\overline{\gamma}}(t)$$

for which $\gamma(0)$ = p. So using such a chart we can identify T_pM with R^n by

$$[v, f : U \rightarrow V] \leftrightarrow v.$$

This shows that T_pM is a vector space and that the above correspondence is a linear isomorphism.

Let h: $(M, U, \mathcal{F}) \rightarrow (M', U', \mathcal{F}')$ be a smooth map between manifolds of dimensions n, n'. For p ∈ M, consider a pair of charts with p∈ U_α and h(p) ∈ $U'_{\alpha'}$. Since $h_{\alpha',\alpha} = f'_{\alpha'} \circ h \circ f_\alpha^{-1}$

is differentiable, the Jacobian matrix $Jac_{h\alpha'\alpha}(f_\alpha(p)$ has an associated R-linear transformation

$$d\, h_{\alpha'\alpha}: R^n \rightarrow R^{n'};\ d\, h_{\alpha'\alpha}(x) = Jach_{\alpha'\alpha}\big(f_\alpha(p)\big)x.$$

It is easy to verify that this passes to equivalence classes to give a well defined R-linear transformation

$$d\, h_p : T_pM \rightarrow T_{h(p)}M'.$$

Proposition (2.2.7):

Let $h: (M, \mathcal{U}, \mathcal{F}) \rightarrow (M'\mathcal{U}', \mathcal{F}')$ and $g: (M', \mathcal{U}', \mathcal{F}') \rightarrow (M''\mathcal{U}'', \mathcal{F}'')$ be smooth maps between manifolds M, M', M'' of dimensions n, n', n''.

a) For each p ∈ M there is an R-linear transformation dh_p: $T_pM \rightarrow T_{h(p)}M'$.

b) For each p ∈ M,

$$d_{gh(p)} \circ dhp = d(g \circ h)p$$

c) For the identity map $Id: M \rightarrow M\ and\ p \in M$

$$d\, Id_p = Id_{TpM}$$

Definition (2.2.8):

Let $(M, \mathcal{U}, \mathcal{F})$ be a manifold of dimension n. A subset N ⊆ M is a submanifold of dimension k if for every p ∈ N there is an open neighbourhood $U \in M$ of p and an n-chart $f : U \to V$such that

$$p \in f^{-1}(V \cap R^k) = N \cap U.$$

For such an N we can form k-charts of the form

$$f_\circ : N \cap U \ \ ! \ V \to R^k f_\circ(x) = f(x):$$

We will denote this manifold by $(N, U\mathcal{U}_N, \mathcal{F}_N N, FN)$. The following result is immediate.

Proposition (2.2.9):

For a submanifold $N \subseteq M$ of dimension k, the inclusion function incl : $N \to M$ is smooth and for every p ∈ N, d $incl_p$: $T_pN \to T_pM$ is an injection.

The next result allows us to recognise submanifolds as inverse images of points under smooth mappings.

Theorem (2.2.10):

(Implicit Function Theorem for manifolds). Let h: $(M, U, \mathcal{F}) \ \ ! \ (M', U', F')$ be a smooth map between manifolds of dimensions n, n'. Suppose that for some q ∈ M', $d\, hp$: $T_pM \to T_{h(p)}M'$ is surjective for every $p \in N = h^{-1}q$. Then N⊆M is submanifold of dimension $n - n'$ and the tangent space at $p \in N$ is given by $T_p\, N = \ker d\, h_p$.

Theorem (2.2.11):

(Inverse Function Theorem for manifolds). *Let h:* $(M, \mathcal{U}, \mathcal{F}) \to (M', U', \mathcal{F}')$ be a smooth map between manifolds of dimensions n, n'. Suppose that for some p ∈ M, d h_p : $T_pM \to T_{h(p)}M'$ is an isomorphism. Then there is an open neighbourhood U ⊆ M of p and an open neighbourhood V ⊆ M' of $h(p)$ such that $hU = V$ and the restriction of h to the map $h_1 : U \to V$ is diffreomorphism.

In particular, the derivative $d\ hp : T_p \rightarrow T_{h(p)}$ is an R-linear isomorphism and $n = n'$.

When this occurs we say that h is locally a diffeomorphism at p.

Example (2.2.12):

Consider the exponential function exp: $M_n(R) \rightarrow GL_n(R)$. Then

$$d\ \exp_O(X) = X:$$

Hence exp is locally a diffeomorphism at O.

In the following Lie groups

Definition (2.2.13):

Let G be a smooth manifold which is also a topological group with multiplication map mult : $G\ x\ G \rightarrow G$ and inverse map inv: $G \rightarrow G$ and view $G \rightarrow G$ as the product manifold. Then G is a Lie group if mult; inv are smooth maps.

Definition (2.2.14):

Let G be a Lie group. A closed subgroup $H \leq G$ that is also a submanifold is called a Lie subgroup of G. It is then automatic that the restrictions to H of the multiplication and inverse maps on G are smooth, hence H is also a Lie group. For a Lie group G, at each g 2 G there is a tangent space T_g G and when G is a matrix group this agrees with the tangent space. We adopt the notation g = T_1 G for the tangent space at the identity of G. A smooth homomorphism of Lie groups $G \rightarrow H$ has the properties of a Lie homomorphism.

For a Lie group G, let g ∈ G. There are following three functions are of great importance.

(Left multiplication) $L_g : G \rightarrow G;\ L_g(x) = gx.$

(Right multiplication) $R_g : G \rightarrow G;\ R_g(x) = xg.$

(Conjugation) $x_g : G \rightarrow G;\ x_g(x) = gxg^{-1}.$

Proposition (2.2.15):

For g ∈ G, the maps L_g, R_g, x_g are all diffreomorphisms with inverses

$$L_g^{-1} = L_{g-1}, R_g^{-1} = R_g^{-1}, \chi_g^{-1} = \chi_{g-1}$$

Proof:

harts for $G \: x \: G$ have the form

$$\varphi_1 \text{ x } \varphi_2 : U_1 \, x \, U_2 \to V_1 x \, V_2,$$

where $\varphi_k : U_k \to V_k$ are charts for G. Now suppose that $\mu U_1 x U_2 , \subseteq W \subseteq G$ where there is a chart θ:W →Z. By assumption, the composition

$\theta \circ \mu \circ (\varphi_1 x \varphi_2)^{-1} = \theta \circ \mu \circ (\varphi_1^{-1} x \varphi_2^{-1}) : v_1 \text{ x } v_2 \to z$

is smooth. Then $L_g(x) = \mu(g, x)$, so if $g \in U_1$ and $x \in U_2$, we have

$$L_g(x) = \theta^{-1}(\theta \circ L_g \circ \varphi_2^{-1}) \circ \varphi_2(x)$$

But then it is clear that

$$\theta \circ \varphi_2^{-1} : v_2 \to z$$

is smooth since it is obtained from $\theta \circ \mu \circ (\varphi_1 x \varphi_2)^{-1}$but treating the first variable as a constant.

A similar argument deals with R_g. For x_g, notice that

$$\mathcal{X}_g = L_g \circ R_g = R_g \circ L_g,$$

and a composite of smooth maps is smooth.

The derivatives of these maps at the identity $1 \in G$ are worth studying. Since L_g and R_g are diffeomorphisms with inverses L_{g-1} and R_{g-1}

$$d(L_g)_1, d(R_g)_1 : g = T_1 G \to T_g G$$

are R-linear isomorphisms. We can use this to identify every tangent space of G with g. The conjugation map x_g fixes 1, so it induces an R-linear isomorphism

$$Ad_g = d(x_g)_1 : g \to g.$$

This is the adjoint action of $g \in G$ on g. For G a matrix group.

There is also a natural Lie bracket [,] defined on g, making it into an R-Lie algebra. The construction follows that for matrix groups.

Theorem (2.2.16):

Let G, H be Lie groups and φ: G →H a Lie homomorphism. Then the derivative is a homomorphism of Lie algebras. In particular, if G ≤ H is a Lie subgroup, the inclusion map incl : G→H induces an injection of Lie algebras d incl : g →h. Now we study Some examples of Lie groups.

Example (2.2.17):

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $GL_n(\mathbb{K})$ is a Lie group.

Proof:

$GL_n(\mathbb{K}) \subseteq M_n(\mathbb{K})$ is an open subset where as usual $M_n(\mathbb{K})$ we identify with $\mathbb{K}^{n2}$. For charts we take the open sets $U \subseteq GL_n(\mathbb{K})$ and the identity function $Id: U \rightarrow U$. The tangent space at each point $A \in GL_n(\mathbb{K})$ is just $M_n(\mathbb{K})$. So the notions of tangent space and is agree here. The multiplication and inverse maps are obviously smooth as they are defined by polynomial and rational functions between open subsets of $M_n(\mathbb{K})$.

Example (2.2.18):

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $GL_n(\mathbb{K})$ is a Lie group.

we have

$$SL_n(\mathbb{K}) = \det^{-1} 1 \subseteq GL_n(\mathbb{K})$$

Where det: $GL_n(\mathbb{K}) \rightarrow \mathbb{K}$ is continuous. $\mathbb{K}$ is a smooth manifold of dimension $\dim_R \mathbb{K}$ with tangent space $T_r\, \mathbb{R} = \mathbb{R}$ at each $r \in R$ and det is smooth. In order to apply Theorem 4.10, we will first show that the derivative $d\, \det_A : M_n(\mathbb{K}) \rightarrow \mathbb{R}$ is surjective for every $A \in GL_n(\mathbb{K})$. To do this, consider a smooth curve α: $(-\varepsilon, \varepsilon) \rightarrow GL_n(\mathbb{K})$ with $\alpha(0) = A$. We calculate the derivative on _0(0) using the formula

$$\mathrm{d}det_A(\alpha'(0)) \quad = \frac{d det_{\alpha(t)}}{dt} \Big|_{i=0}$$

The modified curve

$$\alpha_0 : (-\varepsilon,\varepsilon) \rightarrow GL_n(\mathbb{K});\ \alpha_{\circ(t)} = A^{-1}\, \alpha_{(t)}$$

satisfis $\alpha_0(0) = I$ implies

$$ddet_1(\alpha_0'(0)) = \frac{ddet\alpha_{0(t)}}{dt \quad |t = 0} = tr\alpha_0'(0)$$

Hence we have

$$d\ det_A(\alpha'(0)) = \frac{d\ det(A\alpha_0(t))}{dt \quad |t=0} = \det A \frac{d\ \det(\alpha_0(t))}{dt \quad |t=0} = \det A\ tr\alpha_0'(0)$$

So $d\ det_A$ is the $\mathbb{K}$ -linear transformation

$$d\ det_A : M_n(\mathbb{K}) \rightarrow \mathbb{K}\ d\ det_A(X) = det_A tr(A^{-1}X).$$

The kernel of this is ker $d\ det_A = Asl_n(\mathbb{K})$ and it is also surjective since tr is. In particular this is true for $A \in SL_n(\mathbb{K})$. By Theorem (2.2.10), $SL_n(\mathbb{K}) \rightarrow GL_n(\mathbb{K})$ is a submanifold and so is a Lie subgroup. Again we find that the two notions of tangent space and dimension agree.

There is a useful general principle at work in this last proof. Although we state the following two results for matrix groups, it is worth noting that they still apply when $GL_n(\mathbb{R})$ is replaced by an arbitrary Lie group.

Proposition (2.2.19):

(Left Translation Trick). Let $F : GL_n(\mathbb{R}) \rightarrow M$ be a smooth function and suppose that $B \in GL_n(\mathbb{R})$ satisfies $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. Let $A \in GL_n(\mathbb{R})$ with d FA surjective.

Then d FBA is surjective.

Proof:

Left multiplication by $B \in G$, $L_B : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$, is a diffeomorphism, and its derivative at $A \in GL_n(\mathbb{R})$ is

$$d(LB) : Mn(R) \rightarrow Mn(R);\ d\ LB(X) = BX$$

By assumption, $F \circ L_B = F$ as a function on $GL_n(\mathbb{R})$. Then

$$\begin{aligned} d\ F_{BA}(X) &= d\ F_{BA}(B(B\text{-}1X)) \\ &= d\ F_{BA} \circ d(L_B)_A(B^{-1}X) \\ &= d(F \circ L_B)A(B^{-1}X) \\ &= d\ F_A(B^{-1}X): \end{aligned}$$

Since left multiplication by B^{-1} on $M_n(R)$ is surjective, this proves the result.

Proposition (2.2.20):

(Identity Check Trick). Let $G \leq GL_n(\mathbb{R})$ be a matrix subgroup, M α smooth manifold and $F : GL_n(\mathbb{R}) \to M$ a smooth function with $F^{-1}q = G$ for some $q \in M$. Suppose that for every $B \in G$, $F(BC) = F(C)$ for all $C \in GL_n(\mathbb{R})$. If d FI is surjective then d F_A is surjective for all $A \in G$ and ker d F_A = Ag.

Example (2.2.21):

O(n) is a Lie subgroup of $GL_n(\mathbb{R})$.

Proof:

Recall that we can specify $O(n) \subseteq GL_n(\mathbb{R})$ as the solution set of a family of polynomial equations in n^2 variables arising from the matrix equation $A^TA = I$.

In fact, the following $n + \binom{n}{2} = \binom{n+1}{2}$ equations in the entries of the matrix $A = [a_{ij}]$ are sufficient:

$$\sum_{k=1}^{n} \alpha_{kr}^2 - 1 = 0 (1 \leq r \leq n), \sum_{k=1}^{n} a_{kr}a_{ks} = 0 (1 \leq r < s \leq n)$$

We combine the left hand sides of these in some order to give a function F : $GL_n(\mathbb{R}) \to \mathbb{R}^{\binom{n+1}{2}}$ for example

$$\begin{bmatrix} \sum_{k=1}^{n} \alpha_{k1}^2 - 1 \\ \vdots \\ \sum_{k=1}^{n} \alpha_{kn}^2 - 1 \\ \sum_{k=1}^{n} \alpha_{k1\alpha_{kn}} - 1 \\ \vdots \\ \sum_{k=1}^{n} a_{k(n-1)\alpha_{kn}} \end{bmatrix}$$

We need to investigate the derivative d FA : $M_n(\mathbb{R}) \to \mathbb{R}^{\binom{n+1}{2}}$

d FA is surjective for all A ∈ O(n), it is sufficient to check the case A = I. The Jacobian matrix of F at A = $[a_{ij}]$ = I is the $\binom{n+1}{2}$x n^2 matrix

$$dF_1\begin{bmatrix}2 & 0 & 0 & 0 & \cdots & 0 & 0\\ \vdots & & & \ddots & & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & 0 & 2\\ 0 & 1 & 1 & 0 & \ldots & 0 & 0\\ \vdots & & & \ddots & & \vdots & \vdots\\ 0 & 1 & 0 & \cdots & 0 & 1 & 0\\ \vdots & & & \ddots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 1 & 0\end{bmatrix}$$

Where in the top block of n rows, the r th row has a 2 corresponding to the variable a_{rr} and in the bottom block, each row has a 1 in each column corresponding to one of the pair a_{rs}, a_{sr} with r < s. The rank of this matrix is n + $\binom{n}{2} = \binom{n+1}{2}$ so dF_1 is surjective. It is also true that

$$\ker d\,F_I = \text{Sk-Sym}_n(\mathbb{R}) = o(n):$$

Hence O(n) ≤ $GL_n(\mathbb{R})$ is a Lie subgroup This example is typical of what happens for any matrix group that is a Lie subgroup of $GL_n(\mathbb{R})$.

Theorem (2.2.22):

Let G ≤ $GL_n(\mathbb{R})$ be a matrix group which is also a submanifold, hence a Lie subgroup. Then the tangent space to G at I agrees with the Lie algebra g and the dimension of the smooth manifold G is dim G; more generally, T_A G = A_g.

In the rest of this sections, our goal will be to prove the following important result.

Theorem (2.2.23):

Let G ≤ $GL_n(\mathbb{R})$ be a matrix subgroup. Then G is a Lie subgroup of $GL_n(\mathbb{R})$.

The following more general result also holds but we will not give a proof.

Theorem (2.2.24):

Let G ≤ H be a closed subgroup of a Lie group H. Then G is a Lie subgroup of H.

In the following we will discuss some useful formula in matrix groups

Let G ≤ $GL_n(\mathbb{R})$ be a closed matrix subgroup. Choose r so that

$0 < r \leq 1/2$ and if $A, B \in N_{M_n(R)}(O,r)$ then $\exp(A)\exp(B) \in \exp(N_{M_n(R)}(O; 1/2))$.

Since exp is injective on $N_{M_n(R)}(O; r)$, there is a unique $C \in M_{n(R)}$ for which

$$exp(A)\, exp(B) = exp(C) \tag{2.20}$$

We also set

$$S = C - ?A - ?\, B - \frac{1}{2}[A, B] \in M_n(R) \tag{2.21}$$

Proposition (2.2.25):

$\|S\|$ satisfies

$$\|S\| \leq 65(\|A\| + \|B\|)^3$$

Proof:

For $X \in M_n(R)$ we have

$$exp(X) = I + X + R_1(X),$$

Where the remainder term R1(X) is given by

$$R_1(x) = \sum_{k\leq 2} \frac{1}{k!} x^k$$

Hence,

$$\|R_1(x)\| \leq \|X\|^2 \sum_{k\leq 2} \frac{1}{k!} \|x\|^{k-1}$$

Since $\|C\| < 1/2$,

$$\|R_1(C)\| < \|C\|^2 \tag{2.22}$$

Similarly

$$exp(C) = exp(A)exp(B) = I + A + B + R_1(A, B),$$

Where

$$\|R_1(A,B)\| \leq \sum_{k\geq 2} \frac{1}{k!}\left(\sum_{r=0}^{k} \binom{k}{r} \|A\|^r \|B\|^{k-r}\right) = \sum_{k\geq 2} \frac{(\|A\| + \|B\|)^{k-2}}{k!}$$

$$\leq (\|A\| + \|B\|)^2$$

giving

since $\|A\| + \|B\| < 1$.

Combining the two ways of writing exp(C), we have

$$C = A + B + R_1(A,B) - R_1(C) \tag{2.23}$$

and so

$$\|C\| \leq \|A\| + \|B\| + \|R_1(A,B)\| + \|R_1 C\|$$
$$< \|A\| + \|B\| + (\|A\| + \|B\|)^2 + \|C\|^2$$
$$\leq 2\left(\|A\| + \|B\| + \frac{1}{2}\|C\|^2\right)$$

since $\|A\|, \|B\|, \|C\| \leq 1/2$. Finally this gives

$$\|C\| \leq 4(\|A\| + \|B\|).$$

Equation (2.23) Also gives

$$\|C - AcB\| \leq \|R_1(A,B)\| + \|R_1 C\|$$
$$\leq (\|A\| + \|B\|)^2 + (4\|A\| + \|B\|)^2$$

Giving

$$\|C - A = B\| = 17(\|A\|\|B\|)^2 \tag{2.24}$$

Now we will refine these estimates further. Write

$$\exp(c) = 1 + C + \frac{1}{2}C^2 + R_2(c)$$

Where

$$R_2(C) = \sum_{k\geq 3}\frac{1}{k!} \leq \frac{1}{3}\|C\|^3$$

which satisfies the estimate

$$\left\|R_2(c) \leq \frac{1}{3}\right\|^3$$

since $\|C\| \leq 1$. With the aid of Equation (2.21) we obtain

$$\exp(c) = 1 + A + B + \frac{1}{2}[A,B] + S + \frac{1}{2}C^2 + R_2(C)$$

$$= 1 + A + B + \frac{1}{2}[A, B] + \frac{1}{2}(A + B)^2 + T$$

$$= 1 + A + B + \tfrac{1}{2}(A^2 + 2AB + B^2) + T \tag{2.25}$$

Where

$$T = S + \frac{1}{2}(c^2 - (A)B)^2 + R_2(C) \tag{2.26}$$

Also

$$exo(A)\exp(B) = 1 + A + B + \frac{1}{2}(A^2 + 2AB + B^2) + R_2(A, B) \tag{2.27}$$

$$R_2(A, B) = \sum_{k\geq 3} \frac{1}{k!}\left(\sum_{r=0}^{k} \binom{k}{r} A^r B^{k-1}\right),$$

which satisfies

$$\|R_2(A, B)\| \leq \frac{1}{3}(\|A\| + \|A\|)^3$$

Since $\|A\| + \|B\| \leq 1$

Comparing Equations (2,26) and (2,27), and using(2,20) we see that

$$S=R_2(A,B)+\tfrac{1}{2}((A+B))^2- C^2) - R_2(c)$$

Taking norms we have

$$\|S\| \leq \|R_2 A, B)\| + \frac{1}{2}(A + B)(A + B - C) - (A + B - C)\| + \|\|R_2 C\|$$

$$\leq \frac{1}{3}(\|A\| + \|A\|)^3 + \frac{1}{2}(\|A\| + \|B\| + \|C\|)\left\|A + B - C + \frac{1}{3}\right\|^3$$

$$\leq \frac{1}{3}(\|A\| + \|A\|)^3 + \frac{5}{2}(\|A\| + \|B\|).17\left\|(A + B - C + \frac{1}{3}\right\|^3$$

$$\leq 65(\|A\| + \|B\|)^3.$$

yielding the estimate

$$\|S\| \leq 65(\|A\| + \|B\|)^3 \tag{2.28}$$

Theorem (2.2.26):

If $U, V \in$ M$_n$(R), then the following identities are satisfied.

[Trotter Product Formula]

$$\exp(U + V) = \lim_{r\to m}(\exp\left(\left(\frac{1}{r}\right)u\right)\exp\left(\left(\frac{1}{r}\right)v\right))$$

[Commutator Formula] :

$$\exp([u, v]) = \lim_{r\to m}(\exp\left(\left(\frac{1}{r}\right)u\right)\exp\left(\left(\frac{1}{r}\right)v\right)\exp\left(-\left(\frac{1}{r}\right)u\right)\exp\left(\frac{1}{r}\right)v))$$

Proof:

For large r we may take A $=\frac{1}{r}u$ *and* B $=\frac{1}{r}v$ and apply Equation (2.21) to give

$$exp((1/r)U)\ exp((1/r)V) = exp(C_r)$$

with

$$\left\|C_r - \left(\frac{1}{r}\right)(u + v)\right\| \le \frac{17(\|U\| + \|V\|)^2}{r^2}$$

As r→ ∞

In the following we will discuss not all Lie groups are matrix groups.

For completeness we describe the simplest example of a Lie group which is not a matrix group. In fact there are finitely many related examples of such Heisenberg groups Heis_n and the example we will discuss Heis_3 is particularly important in Quantum Physics.

For $n \geq 3$, the Heisenberg group Heis_n is defined as follows. Recall the group of $n\ x\ n$ real unipotent matrices $\mathrm{SUT}_n(\mathbb{R})$, whose elements have the form

$$\begin{bmatrix} 1 & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ 0 & 1 & a_{21} & \ddots & \ddots & a_{2n} \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & a_{n-2n-1} & \vdots \\ \vdots & \vdots & \ddots & 0 & 1 & a_{n-1n} \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

with $a_{ij} \in \mathbb{R}$. The Lie algebra $\mathrm{sut}_n(\mathbb{R})$ of $\mathrm{SUT}_n(\mathbb{R})$ consists of the matrices of the form

$$\begin{bmatrix} 0 & t_{12} & \cdots & \cdots & \cdots & t_{1n} \\ 0 & 0 & t_{21} & \ddots & \ddots & a_{2n} \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & t_{n-2n-1} & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 & t_{n-1n} \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

with $t_{ij} \in \mathbb{R}$. SUT_n is a matrix subgroup of $GL_n(\mathbb{R})$ with $\dim SUT_n = \binom{n}{2}$. It is a nice algebraic exercise to show that the following hold in general.

Proposition (2.2.27):

For $n \geq 3$, the centre $C(SUT_n)$ of SUT_n consists of all the matrices $[a_{ij}] \in Heis_n$ with $a_{ij} = 0$ except when i = 1 and j = n. Furthermore, $C(SUT_n)$ is contained in the commutator subgroup of SUT_n.

Notice that there is an isomorphism of Lie groups $\mathbb{R} \cong C(SUT_n)$. Under this isomorphism, the subgroup of integers $\mathbb{Z} \subseteq \mathbb{R}$ corresponds to the matrices with $a_{1n} \in \mathbb{Z}$ and these form a discrete normal (in fact central) subgroup $\mathbb{Z}_n \lhd SUT_n$. We can form the quotient group

$$Heis_n = SUT_n / \mathbb{Z}_n.$$

This has the quotient space topology and as Zn is a discrete subgroup, the quotient map $q : SUT_n \to Heis_n$ is a local homeomorphism. This can be used to show that $Heis_n$ is also a Lie group since charts for SUT_n defined on small open sets will give rise to charts for $Heis_n$. The Lie algebra of $Heis_n$ is the same as that of SUT_n, i.e., $heis_n = sut_n$.

Proposition (2.2.28):

For $n \geq 3$, the centre $C(Heis_n)$ of $Heis_n$ consists of the image under q of $C(SUT_n)$. Furthermore, $C(Heis_n)$ is contained in the commutator subgroup of $Heis_n$.

Notice that $C(Heis_n) = C(SUT_n) = \mathbb{Z}_n$ is isomorphic to the circle group

$$T = \{z \in \mathbb{C} : |z| = 1\}$$

with the correspondence coming from the map

$$R \to T;\ t \leftrightarrow e_{\pi i t}.$$

When n = 3, there is a surjective Lie homomorphism

$$p: SUT_3 \to \mathbb{R}^2; \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix}$$

whose kernel is ker ρ = C(SUT_3). Since $Z_3 \leq$ ker ρ, there is an induced surjective Lie homomorphism p: $Heis_3 \to R^2$ for which $\overline{p}$ o $q = p$. In this case the isomorphism C($Heis_n$) ≅ T is given by

$$\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_3 \leftrightarrow e^{2\pi i t}$$

From now on we will write $[x, y, 2^{2\pi i t}]$ for the element

$$\begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} z_3 \in Heis_3$$

Thus a general element of $Heis_3$ has the form $[x, y, z]$ with $x, y \in \mathbb{R}$ and $z \in T$. The identity element is 1 = [0, 0,1]. The element 2

$$\begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$$

of the Lie algebra $heis_3$ will be denoted (x, y, t).

Proposition (2.2.29):

Multiplication, inverses and commutators in $Heis_3$ are given by

$$[x_1, y_1, z_1][x_2, y_2, z_2] = \left[x_1 + x_2 + y_1 + y_2, z_1 z_{2^{e^{2\pi i x 1 y 2}}}\right],$$

$$[x, y, z]^{-1} = \left[-x, xy, z^{-1} e^{2\pi i x y}\right]$$

$$[x_1, y_1, z_1][x_2, y_2, z_2][x_2, y_2, z_2]^{-1} = \left[0, 0, e^{2\pi i (wxy2 - y1x2)}\right]$$

The Lie bracket in heis3 is given by

$$[(x_1, y_1, t_1), (x_2, y_2, t_2)] = (0, 0, x_1 y_2 - y1x2):$$

The Lie algebra heis$_3$ is often called a Heisenberg (Lie) algebra and occurs throughout Quantum Physics. It is essentially the same as the Lie algebra of operators on differentiable functions $f : \mathbb{R} \to \mathbb{R}$ spanned by 1; q given by

$$1f(x) = f(x); pf(x) = \frac{df(x)}{dx}, gf(x) = xf(x)$$

The non-trivial commutator involving these three operators is given by the canonical commutation rela-tion

$$[p, q] = pq - qp = 1.$$

In heis$_3$ he elements (1, 0, 0) , (1 , 0, 0) , (0, 0, 1) a basis with the only non-trivial commutator [(1, 0, 0) , (1, 0, 0)] = (0, 0, 1).

Theorem (2.2.30):

There are no continuous homomorphisms φ: Heis$_3$ → GL$_n$(C) with trivial kernel ker φ = 1.

Proof:

Suppose that φ: Heis$_3$ →GL$_n$(C) is a continuous homomorphism with trivial kernel and suppose that n is minimal with this property. For each g ∈ Heis$_3$, the matrix φ (g) acts on vectors in C^n.

We will identify C(Heis$_3$) with the circle T as above. Then T has a topological generator z$_0$; this is an element whose powers form a cyclic subgroup $\langle z_0 \rangle$ 6 T whose closure is T. For now we point out that for any irrational number r ∈ R, the following is true: for any real number s ∈ R and any ε > 0, there are integers p; q ∈ Z such that

$$|s \quad pr \quad q| < \varepsilon.$$

This implies that $e^{2\pi \mathrm{ir}}$ is a topological generator of T since its powers are dense. Let λ be an eigenvalue for the matrix φ(z0), with eigenvector v. If necessary replacing z0 with z_0^{-1} , we may assume that $\lambda \geq 1$. If $|\lambda| \geq 1$, then

$$\varphi\left(z_0^k\right)v = \varphi(z0)^k v = \lambda^k v$$

and so

$$\left\|\varphi\left(z_0^k\right)\right\| \geq \|\lambda\|^k.$$

Thus $\varphi\left(z_0^k\right) \to \infty$ as $k \to \infty$, implying that φT is unbounded. But φ is continuous and T is compact hence φT is bounded. So in fact $|\|\lambda\||$ = 1.

Since φ is a homomorphism and $z_0 \in C(Heis_3)$, for any $g \in Heis_3$ we have

$\varphi(z_0)\, \varphi(g)v = \varphi(z0g)v = \varphi$ (gz0)v = φ (g) φ (z0)v = $\lambda\, \varphi$ (g)v;

which shows that φ (g) is another eigenvector of φ (z_0) for the eigenvalue λ. If we set

$$V_\lambda = \{v \in C^n : \exists k \geq 1 \text{ s.t. } (\varphi(z_0) - \lambda_n^I)^{kv} = 0\}.$$

then $V_\lambda \subseteq C_n$ is a vector subspace which is also closed under the actions of all the matrices φ (g) with $g \in Heis_3$. Choose $k_0 \geq 1$ to be the largest number for which there is a vector $v_0 \in V_\lambda$ satisfying

$(\varphi(z_0) - \lambda 1_n)^{In} v_0 = 0$, $(\varphi(z0) - \lambda 1_n \quad)^{k0-1} v_0 \neq 0$.

If $k_0 > 1$, there are vectors u, v ∈ V∈ for which

$$\varphi(z_0)u = \lambda u + v,\ \varphi(z_0)v = \lambda v.$$

Then

$\varphi(z_0^k)\, u = \varphi(z_0)^k u = \lambda^k u + k\lambda^{k-1} v$

and since $|\lambda| = 1$,

$$\left\|\varphi(z_0^k)\right\| = \left\|\varphi z o^k\right\| \geq |\lambda_u + k_v| \to \infty$$

as $k \to \infty$. This also contradicts the fact that φ T is bounded. So $k_0 = 1$ and V_λ is just the eigenspace for the eigenvalue λ. This argument actually proves the following important general result, which in particular applies to finite groups viewed as zero-dimensional compact Lie groups.

Proposition (2.2.31):

Let G be a compact Lie group and ρ: G → $GL_n(C)$ a continuous homomorphism. Then for any g ∈ G, ρ (g) is diagonalizable.

On choosing a basis for V_λ, we obtain a continuous homomorphism θ : $Heis_3 \to GL_d(C)$ for which $\theta(z_0) = \lambda I_d$. By continuity, every element of T also has the form (scalar)Id. By minimality of n, we must have d = n and we can assume $\varphi(z0) = \lambda I_n$.

By the equation for commutators in Proposition 4.34, every element z ∈ T ≤ $Heis_3$ is a commutator $z = ghg^{-1}h^{-1}$ in $Heis_3$, hence

$$\det \varphi (z) = \varphi (ghg^{-1}h^{-1}) = 1,$$

since det and $\varphi$are homomorphisms. So for every z ∈ T, $\varphi(z) = \mu(z)I_d$ and $\mu(z)_d = 1$, where the function μ: T→ C^x is continuous. But T is path connected, so $\mu(z) = 1$ for every z ∈ T. Hence for each z ∈ T, the only eigenvalue of φ (z) is 1. This shows that T ≤ ker φ, contradicting the assumption that ker φ is trivial.

A modification of this argument works for each of the Heisenberg groups $Heis_n$ $(n \geq 3)$, showing that none of them is a matrix group.