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Lifting Convex Approximation Properties and Cyclic Operators with Vector Lattices

رفع خصائص التقريب المحدب والمؤثرات الدورية مع شبكات المتجه

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Dedication

I dedicate my dissertation work to my family, a special feeling of gratitude to my loving parents, my sisters and brothers.
I also dedicate this dissertation to my friends and colleagues.

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First of all I thank Allah..

I wish to thank my supervisor Prof. Shawgi huSSien who was more than generous with his expertise and precious time. A special thanks to him for his countless hours of reflecting, reading, encouraging, and most of all patience throughout the entire process.

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Abstract

We demonstrate that rather weak forms of the extendable local reflexivity and of the principle of local reflexivity are needed for the lifting of bounded convex approximation properties from Banach spaces to their dual spaces. We show that certain adjoint multiplication operators are convex- cyclic and show that some are convex- cyclic but no convex polynomial of the operator is hypercyclic. Also some adjoint multiplication operators are convex- cyclic but not 1-weakly hypercyclic. We deal with two weaker forms of injectivity which turn out to have a rich structure behind: separable injectivity and universal separable injectivity. We show several structural and stability properties of these classes of Banach spaces. We provide natural examples of separably injective spaces, including $\mathcal{L}_\infty$ ultraproducts built over countably incomplete ultrafilters, in spite of the fact that these ultraproducts are never injective. We show that the Fremlin tensor product is not square mean complete when the two spaces are uncountable metrizable compact spaces.

الخـــلاصـــة

شرحنا أن الصيغ الضعيفة الأخرى للإنعكاسية الموضعية القابلة للتمديد ولمبدأ الإنعكاسية الموضعي هي مطلوبة لأجل الرفع لخصائص التقريب المحدب المحدود من فضاءات باناخ إلى فضاءاتها الثنائية. أوضحنا أن مؤثرات ضرب المرافق المعين هو محدب -دوري وأوضحنا أن بعضها محدب- دوري ولكن ليس كثيرة حدود محدبة للمؤثر هي دورية فوقية. أيضاً بعض مؤثرات الضرب هي محدبة -دورية ولكنها ليست دورية فوقية ضعيفة- 1. تفاعلنا مع صيغتين ضعيفتين شاملتين والتي تحول لتنتج تشبيهاً غنياً في الخلف: شمولية منفصلة وشمولية منفصلة عالمية. أوضحنا إنشائية متعددة وخصائص استقرار لهذه العائلات لفضاءات باناخ. أشترطنا أمثلة طبيعية للفضاءات الشاملة المنفصلة والمتضمنة الضرب الفائق ℓ_∞ المبني فوق المرشحات الفائقة غير التامة القابلة للعد وبالرغم من الحقيقة ان هذا الضرب الفائق ليس شاملاً. أوضحنا أن ضرب تنسور فزيملين ليس تاماً متوسطاً مربعاً عندما الفضائين هما فضائين متراصين قابلين للمترية غير القابلة للعد .

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Chapter 1

Banach Spaces with their Spaces and the Related Local Reflexivity

We provide a unified approach to the lifting of various bounded approximation properties, including, besides the classical ones, the approximation property of pairs properties of Banach spaces, and the positive approximation property of Banach Lattices.

Section (1.1): Lifting of the Convex Bounded Approximation Property

Approximation property is a locally convex topological vector spaces is said to have the approximation property if the identity on map can be approximated, uniformly on the compact set, by continuous linear map of finite rank [5]).

Let X be a Banach space and let A be an arbitrary subset of $\mathcal{L}(X)$. The space X has the A-approximation property if for every compact subset K of X and every $\varepsilon > 0$, there exists $S \in A$ such that $\|Sx - x\| \leq \varepsilon$ for all $x \in K$. Let $1 \leq \lambda < \infty$. The space X has the λ-bounded A-approximation property if S can be chosen with $\|S\| \leq \lambda$ (meaning that X has the $\left(A \cap \lambda B_{\mathcal{L}(X)}\right)$-approximation property). In the case when the set A is convex and contains 0, we speak about convex approximation properties.

The positive approximation property is precisely the A-approximation property where A is the cone $\mathcal{F}(X)_+$ of positive finite-rank operators. The approximation property for pairs is also a convex approximation property.

By standard arguments (e.g., that the topology of uniform convergence on compact sets coincides with the strong operator topology on bounded subsets of operators), X has the λ-bounded A-approximation property if and only if there exists a net $(S_\nu) \subset A$ with $\|S_\nu\| \leq \lambda$ for all ν, and $S_\nu \to I_X$ pointwise, i.e., in the strong operator topology. We say that X has the λ-bounded duality A-approximation property if the net (S_ν) can be chosen so that also $S_\nu^* \to I_{X^*}$ pointwise. The dual space X^* of X is said to have the A-approximation property with conjugate operators if X^* has the A^*-approximation property.

In the case of convex approximation properties, the following simple result is useful.

Proposition (1.1.1) [1] Let X be a Banach space and let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let $1 \leq \lambda < \infty$. Then the following properties are equivalent.

(a) X has the λ-bounded duality A-approximation property.

(b) X^* has the λ-bounded A-approximation property with conjugate operators.

(c) There exists a net $(S_\nu) \subset A$ such that

$$\limsup_\nu \|S_\nu\| \leq \lambda$$

and

$$x^{**}(S_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*, \forall x^{**} \in X^{**}.$$

Proof: The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are obvious. For the implication $(c) \Rightarrow (a)$, we apply the (rather straightforward) fact that the λ-bounded duality A-approximation property is equivalent to the $(\lambda + \varepsilon)$-bounded duality A-approximation property for all $\varepsilon > 0$. To show the latter property, we use that the weak and strong operator topologies on a space of the bounded linear operators yield the same dual space [1]. Hence, having a net converging in the weak operator topology on $\mathcal{L}(X)$ (or on $\mathcal{L}(X^*)$ as in (c)) means that, by passing to convex combinations, one may always assume that the net converges in the strong operator topology on $\mathcal{L}(X)$ (or on $\mathcal{L}(X^*)$) to the same element.

We introduce the following general forms of the ELR and the PLR. In Theorem (1.1.5) we shall see that these rather weak forms of the ELR and the PLR are sufficient for the lifting of different bounded approximation properties from Banach spaces to their dual spaces.

Definition (1.1.2) [1] Let X be a Banach space and let C be a subset of $\mathcal{L}(X^{**})$. Let $1 \leq \lambda < \infty$. We say that X is λ-extendably locally reflexive of type C if for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in C$ such that $T(E) \subset X$, $\|T\| \leq \lambda + \varepsilon$, and

$$|x^*(TX^{**}) - x^{**}(x^*)| \leq \varepsilon \quad \forall x^{**} \in S_E, \qquad \forall x^* \in S_F.$$

The λ-ELR of a Banach space X clearly implies the λ-ELR of type $\mathcal{L}(X^{**})$. More examples will be presented.

Definition (1.1.3) [1] Let X be a Banach space, let A and B be subsets of $\mathcal{L}(X)$ and $\mathcal{L}(X^{**})$, respectively. We say that the principle of local reflexivity of type $B \to A$ holds in X if for all $T \in B$, for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $S \in A$ such that $\|S\| \leq \|T\| + \varepsilon$ and

$$|(Tx^{**})(x^*) - x^{**}(S^*x^*)| \leq \varepsilon \quad \forall x^{**} \in S_E, \qquad \forall x^* \in S_F.$$

The PLR of type $B \to A$ means that the operators on X^{**} of "type B" are "locally" of "type A" on X.

Examples (1.1.4) [1] The following assertions are true.

(i) By the PLR, in every Banach space X, the PLR of type $\mathcal{F}(X^{**}) \to \mathcal{F}(X)$ holds.

(ii) Let X be a Banach space and let Y be a closed subspace of X. By the PLR respecting subspaces, the PLR of type $\{T \in \mathcal{F}(X^{**}) : T(Y^{\perp\perp}) \subset Y^{\perp\perp}\} \to \{S \in \mathcal{F}(X) : S(Y) \subset Y\}$ holds in X.

(iii) In every Banach lattice X, the PLR of type $\mathcal{F}(X^{**})_+ \to \mathcal{F}(X)_+$ holds.

(iv) Let A be a subset of $\mathcal{L}(X)$. Trivially, in every Banach space X, the PLR of type $A^{**} \to A$ holds.

Theorem (1.1.5) [1] Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0. Let B and C be subsets of $\mathcal{L}(X^{**})$ such that $A^{**} \circ C \subset B$. Let $1 \leq \lambda, \mu < \infty$. Assume that the principle of local reflexivity of type $B \to A$ holds in X. If X is λ-extendably locally reflexive of type C and has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.

Proof: By Proposition (1.1.1) (c), it suffices to construct a net $(R_\nu) \subset A$ such that $\limsup_{\nu} \|R_\nu\| \leq \lambda\mu$ and

$$x^{**}(R_\nu^* x^*) \to_\nu x^{**}(x^*) \quad \forall x^* \in X^*, \forall x^{**} \in X^{**}.$$

Consider the set of all $\nu = (E, F, \varepsilon)$, where $E \subset X^{**}$ and $F \subset X^*$ are finite-dimensional subspaces and $\varepsilon > 0$, directed in the natural way. Since X is λ-ELR of type C, for every ν, there exists an operator $T_\nu \in C$ such that $T_\nu(E) \subset X$, $\|T_\nu\| \leq \lambda + \varepsilon$, and

$$|x^*(T_\nu x^{**}) - x^{**}(x^*)| \leq \varepsilon \quad \forall x^{**} \in S_E, \qquad \forall x^* \in S_F.$$

The set $T_\nu(SE) \subset X$ is compact because S_E is compact. Since X has the μ-bounded A-approximation property, there exists $S_\nu \in A$ with $\|S_\nu\| \leq \mu$ such that

$$\|S_\nu^{**} T_\nu x^{**} - T_\nu x^{**}\| = \|S_\nu T_\nu x^{**} - T_\nu x^{**}\| \leq \varepsilon \quad \forall x^{**} \in S_E.$$

We have $S_\nu^{**} T_\nu x^{**} \in A \circ C \subset B$. By the PLR of type $B \to A$, there exists $R_\nu \in A$ with

$$\|R_\nu\| \leq \|S_\nu^{**} T_\nu\| + \varepsilon \leq \mu(\lambda + \varepsilon) + \varepsilon,$$

implying that

$$\limsup_{\nu} \|R_\nu\| \leq \lambda\mu,$$

and

$$|(S_\nu^{**}T_\nu x^{**})(x^*) - x^{**}(R_\nu^* x^*)| \le \varepsilon \quad \forall x^{**} \in S_E, \qquad \forall x^* \in S_F.$$

For arbitrary $x^{**} \in S_E$ and $x^* \in S_F$, we have

$$\begin{aligned}|x^{**}(R_\nu^* x^*) - x^{**}(x^*)| \\ &\le |x^{**}(R_\nu^* x^*) - (S_\nu^{**}T_\nu x^{**})\,(x^*)| \\ &+ |(S_\nu^{**}T_\nu x^{**})\,(x^*) - (T_\nu x^{**})(x^*)| + |x^*(T_\nu x^{**}) - x^{**}(x^*)| \\ &\le 3\varepsilon.\end{aligned}$$

This implies that

$$\lim_\nu x^{**}(R_\nu^* x^*) = x^{**}(x^*) \ \ \forall x^* \in X^*, \forall x^{**} \in X^{**}.$$

Indeed, let $x^* \in S_{X^*}$ and $x^{**} \in S_{X^{**}}$ be given. For $\varepsilon_0 > 0$, take $\nu_0 = (\text{span}\,\{x^{**}\}, \text{span}\{x^*\}, \varepsilon_0/3)$. If $\nu = (E, F, \varepsilon) \ge \nu_0$, then $x^{**} \in S_E, x^* \in S_F, \varepsilon \le \varepsilon_0/3$, and we have

$$|x^{**}(R_\nu^* x^*) - x^{**}(x^*)| \le 3\varepsilon \le \varepsilon_0,$$

as needed.

From Theorem (1.1.5) and Example (1.1.4) (iv), we have the following immediate corollary that will be applied in lifting results.

Corollary (1.1.6) [1] Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0 and let C be a subset of $\mathcal{L}(X^{**})$ such that $A^{**} \circ C \subset A^{**}$. Let $1 \le \lambda, \mu < \infty$. If X is λ-extendably locally reflexive of type Cand has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.

Section (1.2): Extendable Local Reflexivity Implied by Convex Approximation Properties

The lifting Theorem (1.2.1) by Johnson and Oikhberg has a strong converse, due to Rosenthal.

Theorem (1.2.1) [1] (Rosenthal). Let X be a Banach space. Let $1 \leq \lambda < \infty$. If X^* has the λ-bounded approximation property, then X is λ-extendably locally reflexive.

Recall that, by an important result, the assumption "X^* has the λ-bounded approximation property" is equivalent to "X^* has the λ-bounded approximation property with conjugate operators". We shall see that Rosenthal's Theorem (1.2.1) can be extended as follows, providing a general converse to our main Theorem (1.1.5).

Proposition (1.2.2) [1] Let X be a Banach space. Let Abe a subset of $\mathcal{W}(X)$. Let $1 \leq \lambda < \infty$. If X^* has the λ-bounded A-approximation property with conjugate operators, then X is λ-extendably locally reflexive of type A^{**}.

Proof: Let $F \subset X^*$ be a finite-dimensional subspace and let $\varepsilon > 0$. Since S_F is compact and X^* has the λ-bounded $\{S^*: S \in A\}$-approximation property, there exists $S \in A$ with $\|S\| \leq \lambda$ such that

$$\|S^*x^* - x^*\| \leq \varepsilon \ \forall x^* \in S_F.$$

Then $S^{**} \in A^{**}$ and $\|S^{**}\| \leq \lambda$. Since $S \in \mathcal{W}(X)$, we have that $S^{**}(X^{**}) \subset X$ [1]. For arbitrary $x^{**} \in S_{X^{**}}$ and $x^* \in S_F$,

$$|x^*(S^{**}x^{**}) - x^{**}(x^*)| = |x^{**}(S^*x^*) - x^{**}(x^*)| \leq \|x^{**}\|\|S^*x^* - x^*\| \leq \varepsilon.$$

Hence, for any finite-dimensional subspace E of X^{**}, the conditions of the λ-ELR of type A^{**} for X are satisfied.

In the case of Banach lattices, we have the following version of Rosenthal's Theorem (1.2.1).

Theorem (1.2.3) [1] Let X be a Banach lattice. Let $1 \leq \lambda < \infty$. If the dual lattice X^* has the λ-bounded positive approximation property, then X is positively λ-extendably locally reflexive.

Proof: Since X^* has the λ-bounded positive approximation property, it has the λ-bounded positive approximation property with conjugate operators [1]. Let $A = \mathcal{F}(X)_+$. Then $A \subset \mathcal{W}(X)$ and $A^{**} \subset \mathcal{F}(X^{**})_+ \subset \mathcal{L}(X^{**})_+$. The claim is immediate from Proposition (1.2.2).

The following result is immediate from Proposition (1.2.2) and Corollary (1.1.6). It shows that the lifting of convex approximation properties from a Banach space to its dual space is possible whenever the dual space already enjoys a weaker approximation property.

Theorem (1.2.4) [1] Let X be a Banach space. Let A be a convex subset of $\mathcal{L}(X)$ containing 0 and let B be a subset of $\mathcal{W}(X)$ such that $A \circ B \subset A$. Let $1 \leq \lambda, \mu < \infty$. If X^* has the λ-bounded B-approximation property with conjugate operators and X has the μ-bounded A-approximation property, then X has the $\lambda\mu$-bounded duality A-approximation property.

If, in Theorem (1.2.4), $A \subset \mathcal{K}(X)$ and X^* or X^{**} has the Radon–Nikodym property, then X has the metric duality A-approximation property.

Let us have general application of Theorem (1.2.4) to (positive) approximation properties of pairs.

Corollary (1.2.5) [1] Let X be a Banach space and Y a closed subspace of X. Let $\mathcal{A}$ be an operator ideal. Denote $A = \{S \in \mathcal{A}(X): S(Y) \subset Y\}$ and $B = \{T \in \mathcal{W}(X): T(Y) \subset Y\}$. Let $1 \leq \lambda, \mu < \infty$. Then the assertion of Theorem (1.2.4) holds. In the special case when X is a Banach lattice, A and B may be replaced by A_+ and B_+.

The classical cases when Corollary (1.2.5) applies are $A = \mathcal{F}(X)$ and $A = \mathcal{K}(X)$. For instance, it follows that the dual lattice X^* has the bounded (metric if X^* has the Radon–Nikodym property) positive approximation property whenever X has the bounded positive approximation property and X^* has the bounded positive weakly compact approximation property with conjugate operators.

If one adds in the definition of the λ-ELR the requirement that the operator $T \in \mathcal{L}(X^{**})$ also satisfies $T^*(X^*) \subset X^*$, then one obtains the notion of the strong λ-extendable local reflexivity. The strong λ-ELR was introduced and studied. Among others, Rosenthal's Theorem (1.2.1) was strengthened and extended in as follows.

Theorem (1.2.6) [1] (Oja). Let $\mathcal{A}$ be an operator ideal and let X be a Banach space. Let $1 \leq \lambda < \infty$. If $\mathcal{A}(X) \subset \mathcal{W}(X)$ and X^* has the λ-bounded $\mathcal{A}(X)$-approximation property with conjugate operators, then X is strongly λ-extendably locally reflexive.

It was also observed that Theorem (1.2.1) fails already for the bounded compact approximation property, i.e., $\mathcal{F}(X)$ cannot be replaced by $\mathcal{K}(X)$ in Theorem (1.2.1). This also means that the assumption "X has the λ-bounded $\mathcal{A}(X)$-approximation property with conjugate operators" is essential in Theorem (1.2.6). A "strong" example of this phenomenon was presented: there exists a strongly 1-ELR Banach space X with a monotone shrinking basis such that:

(i) its even duals X^{**}, X^{****}, ... are strongly 1-ELR, have the metric compact approximation property, but do not have the bounded weakly compact approximation property with conjugate operators;

(ii) its odd duals X^{*}, X^{***}, ... are not ELR, but have the metric compact approximation property with conjugate operators.

We shall extend Theorem (1.2.6) to convex approximation properties of pairs as follows.

Theorem (1.2.7) [1] Let X be a Banach space and Y a closed subspace of X. Let A be a linear subspace of $\mathcal{L}(X)$ containing $\mathcal{F}(X)$. Let $1 \leq \lambda < \infty$. If X^* has the λ-bounded $\{S \in A : S(Y) \subset Y\}$-approximation property with conjugate operators, then for every finite-dimensional subspace $F \subset X^*$ and for every $\varepsilon > 0$, there exists an operator $S \in A$ with $S(Y) \subset Y$ such that $\|S\| \leq \lambda + \varepsilon$ and $S^* x^* = x^*$ for all $x^* \in F$.

Moreover, if $A \subset \mathcal{W}(X)$, then the operator $T := S^{**}$ has the following properties:

$T(X^{**}) \subset X, T(Y^{\perp\perp}) \subset Y, \|T\| \leq \lambda + \varepsilon, x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in X^{**}$ and $x^* \in F$, and $T^*(X^*) \subset X^*$.

In the proof of Theorem (1.2.7), we use the lemma below, which, by standard arguments, follows from the definition.

Proof. Let $F \subset X^*$ be a finite-dimensional subspace and let $\varepsilon > 0$. Look at X^* endowed with its weak$*$ topology and notice that $Y^{\perp}$ is weak$*$ closed. Using [1] choose a weak$*$-to-weak$*$ continuous linear projection P on X^* such that $\operatorname{ran} P = F$ and $P(Y^{\perp}) \subset Y^{\perp}$. Then there exists $Q \in \mathcal{F}(X)$ such that $P = Q^*$. Hence, $Q(Y) \subset Y$ and $F = \operatorname{ran} Q^*$.

Moreover, by assumption and Lemma (1.2.8), we have $R \in A$ with $R(Y) \subset Y$ such that $\|R^*\| \leq \lambda$ and $\|R^* x^* - x^*\| \leq (\varepsilon / \|Q\|)\|x^*\|$ for all $x^* \in F$.

Define $S = R + Q(I_X - R)$. Then, clearly, $S \in A$ and $S(Y) \subset Y$. Let us observe that

$$\|(I_X - R^*)Q^*\| = \sup_{x^* \in B_{X^*}} \|Q^* x^* - R^*(Q^* x^*)\| \leq \sup_{x^* \in B_{X^*}} \left(\frac{\varepsilon}{\|Q\|}\right)\|Q^* x^*\| = \varepsilon.$$

Hence, $\|S\| \leq \|R^*\| + \|(I_{X^*} - R^*)Q^*\| \leq \lambda + \varepsilon$. Let us also observe that

$$S^* = I_{X^*} + (I_{X^*} - R^*)(Q^* - I_{X^*}).$$

Hence, clearly, S^* is identity on $F = \operatorname{ran} Q^*$.

Assume now that $A \subset \mathcal{W}(X)$. Then $S \in \mathcal{W}(X)$ and $S^* \in \mathcal{W}(X^*)$. Therefore, $T := S^{**} \in \mathcal{W}(X^{**}, X)$ and $T^* \in \mathcal{W}(X^{***}, X^*)$. Moreover, since $S(Y) \subset Y$, we get that $T(Y^{\perp\perp}) \subset X \cap Y^{\perp\perp} = Y$. And we also have

$$x^*(Tx^{**}) = x^{**}(S^*x^*) = x^{**}(x^*) \ \forall x^{**} \in X^{**}, \forall x^* \in F.$$

We extend the strong ELR to pairs as follows.

Lemma (1.2.8) [1] Let X be a Banach space. Let A be a linear subspace of $\mathcal{L}(X)$. Let $1 \leq \lambda < \infty$. If X has the λ-bounded A-approximation property, then for every finite-dimensional subspace E of X and for every $\varepsilon > 0$ there exists an operator $S \in A$ with $\|S\| \leq \lambda$, such that $\|Sx - x\| \leq \varepsilon \|x\|$ for all $x \in E$.

Definition (1.2.9) [1] Let X be a Banach space and Y a closed subspace of X. Let $1 \leq \lambda < \infty$. We say that the pair (X, Y) is strongly λ-extendably locally reflexiveif for all finite-dimensional subspaces $E \subset X^{**}$ and $F \subset X^*$, and for all $\varepsilon > 0$, there exists $T \in L(X^{**})$ such that $T(E) \subset X, T(Y^{\perp\perp}) \subset Y, \|T\| \leq \lambda + \varepsilon, x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$, and $T^*(X^*) \subset X^*$.

It is natural to say that the pair $(X^*, Y^\perp)$ has the λ-bounded A-approximation property with conjugate operators if X^* has the λ-bounded $\{S \in A: S(Y) \subset Y\}$-approximation property with conjugate operators. Thus, the "moreover" part of Theorem (1.2.7) may be reformulated as follows.

Theorem (1.2.10) [1] Let X be a Banach space and Y a closed subspace of X. Let A be a linear subspace of $\mathcal{W}(X)$ containing $\mathcal{F}(X)$. Let $1 \leq \lambda < \infty$. If the pair $(X^*, Y^\perp)$ has the λ-bounded A-approximation property with conjugate operators, then the pair (X, Y) is strongly λ-extendably locally reflexive.

Theorem (1.2.10) contains Theorem (1.2.6) as the special case when $Y = \{0\}$ and A is the component of an arbitrary operator ideal, since the strong ELR of X coincides with the strong ELR of the pair $(X, \{0\})$.

The theorem of Johnson, mentioned in the beginning of this section, was extended from X^* to $(X^*, Y^\perp)$: if the pair $(X^*, Y^\perp)$ has the λ-bounded approximation property, then it has the λ-bounded approximation property with conjugate operators. Therefore, taking $A = \mathcal{F}(X)$, we immediately get from Theorem (1.2.10) the following version of Rosenthal's Theorem (1.2.1) for pairs.

Corollary (1.2.11) Let X be a Banach space and Y a closed subspace of X. Let $1 \leq \lambda < \infty$. If the pair $(X^*, Y^\perp)$ has the λ-bounded approximation property, then the pair (X, Y) is strongly λ-extendably locally reflexive.

Chapter 2
Convex Cyclic Operators

We give a Hahn- Banach characterization for convex-cyclic. We also obtain an example of abounded operator S on a Banach space with $\sigma_p(S^*) = \emptyset$ such that S is convex- cyclic, but S is not weakly hypercyclic and S^2 is not convex- cyclic. This solved two questions of Rezaci when $\sigma_p(S^*) = \emptyset$. We also characterize the diagnolizable normal operators that are convex- cyclic and give a condition on the eigenvalues of an arbitrary operators for it to be convex- cyclic.

Section (2.1): The Hahn-Banach Characterization for Convex-Cyclicity

The convex hull of a given set X may be defined as: (i) The (unique) minimal convex set containing X; (ii) The intersection of all convex set containing X [6].

Rezaei gave a (universality) criterion for an operator to be convex-cyclic. Using the Hahn-Banach Separation Theorem, we give a necessary and sufficient condition for a set to have a dense convex hull, as a result we get a criterion for a vector to be a convex-cyclic vector for an operator.

Proposition (2.1.1) [2] Let X be a locally convex space over the real or complex numbers and let E be a nonempty subset of X. The following are equivalent:

(i) The convex hull of E is dense in X.

(ii) For every nonzero continuous linear functional f on X we have that the convex hull of $\mathrm{Re}(f(E))$ is dense in $\mathbb{R}$.

(iii) For every nonzero continuous linear functional f on X we have that

$$\sup Re(f(E)) = \infty \text{ and } \inf Re(f(E)) = -\infty.$$

(iv) For every nonzero continuous linear functional f on X we have that

$$\sup Re(f(E)) = \infty.$$

Proof: Let $\mathbb{F}$ denote either the real or complex numbers. Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) holds. Now assume that (iv) holds and by way of contradiction, assume that $co(E)$ is not dense in X. Then there exists a point $p \in X$ that is not in the closure of $co(E)$. So, by the Hahn-Banach Separation Theorem, there exists a continuous linear functional f on X so that $Re(f(x)) < Re(f(p))$ for all $x \in co(E)$. It follows that $Re(f(E))$ is bounded from above and thus $\sup Re(f(E)) \neq \infty$. This contradicts our assumption that (iv) is true. Thus it must be the case that if (iv) holds, then (i) does also. Hence all four conditions are equivalent.

Corollary (2.1.2) [2] (The Hahn-Banach Characterization for Convex-Cyclicity). Let X be a locally convex space over the real or complex numbers, $T: X \to X$ a continuous linear operator, and $x \in X$. Then the following are equivalent:

(i) The convex hull of the orbit of x under T is dense in X.

(ii) For every non-zero continuous linear functional f on X we have

$$\sup Re\left(f\left(Orb\ (T, x)\right)\right) = \infty.$$

Below are some simple consequences of the Hahn-Banach characterization for convex-cyclic vectors.

As it was pointed the range of a cyclic operator may not be dense. For example, the range of the unilateral shift has codimension one. The closure of the range of a cyclic operator has codimension at most one. Notice that the range of hypercyclic operator is always dense. The Hahn-Banach characterization of convex-cyclicity easily shows that convex-cyclic operators must also have dense range, see the following result.

Proposition (2.1.3) [2] If T is a convex-cyclic operator on a locally convex space X, then T has dense range.

Proof: Suppose that T is a convex-cyclic operator and let x be a convex-cyclic vector for T, and by way of contradiction, suppose that T does not have dense range. Then there exists a continuous linear functional f such that $f\left(R(T)\right) = \{0\}$, where $R(T)$ denotes the range of T. By the Hahn-Banach characterization, Corollary (2.1.2), we must have that $\sup Re\ (f\left(Orb(T, x)\right) = \infty$. However, since $T^n x \in R(T)$ for all $n \geq$ it follows that $f(T^n x) = 0$ for all $n \geq 1$. So, $\sup Re(f\left(Orb(T, x)\right) = \sup Re(\{f(T0x), 0\}) < \infty$. It follows from Corollary (2.1.2) that x is not a convex-cyclic vector, a contradiction. Thus, T must have dense range.

In general, if T is hypercyclic and $c > 1$, then cT may not be hypercyclic. However, León-Saavedra and Müller proved that if T is hypercyclic and α is a unimodular complex number, then αT is hypercyclic. The same property is also true for weak hypercyclic operators. Next we present a similar result for convex-cyclic operators, that follows from the Hahn-Banach characterization of convex-cyclic vectors.

Proposition (2.1.4) [2] If T is a convex-cyclic operator on a real or complex locally convex space X, and if $c > 1$, then cT is also convex-cyclic. Furthermore, every convex-cyclic vector for T is also a convex-cyclic vector for cT.

Proof: Suppose that x is a convex-cyclic vector for T, and we will show that x is also a convex-cyclic vector for cT, by using the Hahn-Banach characterization (Corollary (2.1.2)). Let f be any non-zero continuous linear functional on X. Since x is a convex-cyclic vector for T, then $\sup Re(f(Tnx)) = \infty$. Since $c > 1$, then we have that $\sup Re[f((cT)^n x)] = \sup c^n Re[f(T^n x)] \geq \sup Re[f(T^n x)] = \infty$. So, by the Hahn-Banach characterization, x is a convex-cyclic vector for cT.

Corollary (2.1.5) If $|c| \geq 1$ and T is weakly hypercyclic, then cT is convex-cyclic.

Proof: Let $c := e^{i\theta}\beta$, where $\theta \in \mathbb{R}$ and $\beta \geq 1$. Then by dela Rosa. We obtain that $e^{i\theta}T$ is weakly hypercyclic, hence $e^{i\theta}T$ is convex-cyclic. Thus, $cT = \beta(e^{i\theta}T)$ is convex cyclic by Proposition (2.1.4).

Let us define the following convex polynomials

$$p_k^c(t); = \begin{cases} \dfrac{1 + t + \cdots + t^{k-1}}{k} & if\ c = 1 \\ \dfrac{c-1}{c^k - 1}(c^{k-1} + c^{k-2}t + \cdots + t^{k-1}) & if\ c > 1. \end{cases}$$

Definition (2.1.6) [2] Let X and Y be topological spaces. A family of continuous operators $T_i: X \to Y$ $(i \in I)$ is universal if there exists an $x \in X$ such that $\{T_i x: i \in I\}$ is dense in Y.

Let $T \in L(X)$. Denotes $M_n(T)$ the arithmetic means given by

$$M_n(T) := \frac{I + T + \cdots + T^{n-1}}{n}.$$

Recall that an operator T is Cesáro hypercyclic if there exists $x \in X$ such that $\{M_n(T)x: n \in \mathbb{N}\}$ is dense in X. See [2].

It is proved that T is Cesáro hypercyclic if and only if $\left(\frac{T^k}{k}\right)_{k=1}^{\infty}$ is universal.

Proposition (2.1.7) [2] Let X be a Banach space, $c > 1$ and $T \in L(X)$ such that $cI - T$ has dense range. Then the following are equivalent:

(i) $\frac{T}{c}$ is hypercyclic

(ii) $\left(p_k^c(T)\right)_{k\in\mathbb{N}}$ is universal.

Proof: Notice that if $c > 1$,

$$p_k^c(T)(cI-T)x = (cI-T)p_k^c(T)x = (c-1)\frac{c^k}{c^k-1}\left(x-\left(\frac{T}{c}\right)^k x\right).$$

Proposition (2.1.8) [2] If T is Cesáro hypercyclic or $\frac{T}{c}$ is hypercyclic for some $c \geq 1$, then T is convex-cyclic.

Notice that the proof of the sufficient condition for a bilateral weighted backward shift on $\ell^p(\mathbb{Z})$ to be convex-cyclic given is not correct.

Section (2.2): ε-hypercyclic operators versus convex-cyclic operators with diagonal operators and adjoint multiplication operators

Let us now exhibit the relation between ε-hypercyclic and convex-cyclic operators.

Theorem (2.2.1) [2] Every ε-hypercyclic vector is a convex-cyclic vector.

Proof: Let x be an ε-hypercyclic vector for an operator T and we will prove that for a non-zero vector $y \in X$ and $\delta > 0$, there exists a convex polynomial p such that

$$\|p(T)x - y\| < \delta .$$

Since $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that $2\varepsilon^N\|y\| < \delta$. As x is an ε-hypercyclic vector for T, there exists a positive integer k_1 such that

$$\|T^{k_1}x - Ny\| \leq \varepsilon\|Ny\| = \varepsilon N\|y\|.$$

If $T^{k_1}x - Ny = 0$, we choose l_2 such that

$$\left\|T^{l_2}x - \frac{N}{N-1}\varepsilon^N y\right\| \leq \varepsilon^{N+1}\ \frac{N}{N-1}\|y\|.$$

Thus

$$\left\|\frac{N}{N-1}T^{l_2}x - \varepsilon^N y\right\| \leq \varepsilon^{N+1}\ \|y\|.$$

Hence

$$\left\|\frac{1}{N}T^{k_1}x + \frac{N-1}{N}T^{l_2}x - y\right\| = \left\|\frac{N-1}{N}T^{l_2}x\right\| \leq 2\varepsilon^N\ \|y\| < \delta$$

and the proof ends by letting $p(z) = \frac{1}{N}z^{k_1} + \frac{N-1}{N}z^{l_2}$.

If $T^{k_1}x - Ny \neq 0$, there exists a positive integer k_2 such that

$$\|T^{k_1}x + T^{k_2}x - Ny\| = \|T^{k_2}x - (Ny - T^{k_1}x)\| \leq \varepsilon\|Ny - T^{k_1}x\|$$
$$\leq \varepsilon^2 N\|y\|.$$

If $T^{k_1}x + T^{k_2}x - Ny = 0$, analogously to the above situation we choose l_3 such that

$$\left\|\frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \frac{N-2}{N}T^{l_3}x - y\right\| = \left\|\frac{N-1}{N}T^{l_3}x\right\| \leq 2\varepsilon^N \|y\| < \delta$$

and the proof ends.

If $T^{k_1}x + T^{k_2}x - Ny \neq 0$, there exists a positive integer k_3 such that

$$\|T^{k_1}x + T^{k_2}x + T^{k_3}x - Ny\| \leq \varepsilon^3 N\|y\|.$$

By induction, in the step N, if $T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}}x - Ny = 0$, we choose l_N such
that

$$\left\|\frac{1}{N}T^{k_1}x + \frac{1}{N}T^{k_2}x + \cdots + \frac{1}{N}T^{k_{N-1}} + \frac{1}{N}T^{l_n}x - y\right\| \leq 2\varepsilon^N \|y\| < \delta$$

and the proof ends.

If $T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}}x - Ny \neq 0$, there exists a positive integer k_N such that

$$\|T^{k_1}x + T^{k_2}x + \cdots + T^{k_{N-1}} + T^{k_N}x - y\| \leq \varepsilon^N N \|y\|$$

Thus

$$\left\|\frac{T^{k_1}x + \cdots + T^{k_N}x}{N} - y\right\| \leq \varepsilon^N N \|y\| < \delta$$

Ending completely the Proof:

By a Fréchet space we mean a locally convex space that is complete with respect to a translation invariant metric.

If $\mathcal{A}$ is a nonempty collection of polynomials and T is an operator on a space X, then T is said to be $\mathcal{A}$-cyclic and $x \in X$ is said to be an $\mathcal{A}$-cyclic vector for T if $\{p(T)x : p \in \mathcal{A}\}$ is dense in X. Furthermore, T is said to be $\mathcal{A}$-transitive if for any two nonempty open sets U and V in X, there exists a $p \in \mathcal{A}$ such that $p(T)U \cap V \neq \emptyset$. Since the set of all polynomials with the topology of uniform convergence on compact sets in the complex plane forms a separable metric space, then any set of polynomials is also separable, hence we have the following result.

Proposition (2.2.2) [2]. Suppose that $T: X \to X$ is a continuous linear operator on a real or complex Fréchet space and $\mathcal{A}$ is a nonempty set of polynomials. Then the following are equivalent:

(i) T has a dense set of $\mathcal{A}$-cyclic vectors.

(ii) T is $\mathcal{A}$-transitive. That is, for any two nonempty open sets U, V in X, there is a polynomial $p \in \mathcal{A}$ such that $p(T)U \cap V \neq \emptyset$.

(iii) T has a dense G_δ set of $\mathcal{A}$-cyclic vectors.

By choosing various sets of polynomials for $\mathcal{A}$, we can get results for hypercyclic and supercyclic operators, as well as cyclic operators that have a dense set of cyclic vectors. If $\mathcal{A}$ is the set of all convex polynomials, then we get the following immediate corollary.

Corollary (2.2.3) [2]. Let $T: X \to X$ be a continuous linear operator on a real or complex Fréchet space, then the following are equivalent.

(i) T has a dense set of convex-cyclic vectors.

(ii) T is convex-transitive. That is, for any two nonempty open sets U, V in X, there is a convex polynomial p such that $p(T)U \cap V \neq \emptyset$.

(iii) T has a dense G_δ set of convex-cyclic vectors.

Proposition (2.2.4) [2] Let $\mathcal{A}$ be a nonempty set of polynomials and let $\{T_k: X_k \to X_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of linear operators on a sequence of Banach spaces $\{X_k\}_{k=1}^{\infty}$ such that for every $n \geq 1$, the operator $S_n = \otimes_{k=1}^{n} T_k$ on $X^{(n)} = \otimes_{k=1}^{n} X_k$ has a dense set of $\mathcal{A}$-cyclic vectors. Then $T = \otimes_{k=1}^{\infty} T_k$ is $\mathcal{A}$-cyclic on $X^{(\infty)} = \otimes_{k=1}^{\infty} X_k$ and T has a dense set of $\mathcal{A}$-cyclic vectors.

Proof: Suppose that for every $n \geq 1$ the operators S_n are $\mathcal{A}$-cyclic and have a dense set of $\mathcal{A}$-cyclic vectors. We will show that T is $\mathcal{A}$-transitive. Let U and V be two nonempty open sets in $X^{(\infty)}$. Since the vectors in X with only finitely many non-zero coordinates are dense in X, then we may choose vectors $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ in $X^{(\infty)}$ such that $x_k = 0$ and $y_k = 0$ for all sufficiently large k, say $x_k = 0$ and $y_k = 0$ for all $k \geq N$, and such that $x \in U$ and $y \in V$. Since S_N is $\mathcal{A}$-cyclic and has a dense set of $\mathcal{A}$-cyclic vectors in $X^{(N)}$, there exists a vector $u = (u_1, u_2, \dots, u_N) \in X^{(N)}$ such that u is an $\mathcal{A}$-cyclic vector for S_N and so that $(u_1, u_2, \dots, u_N)$ is close enough to $(x_1, x_2, \dots, x_N)$ so that the infinite vector $\hat{u} = (u_1, u_2, \dots, u_N, 0, 0, \dots) \in U$. Since S_N is $\mathcal{A}$-cyclic, there is a polynomial $p \in \mathcal{A}$ such that $p(S_N)(u_1, u_2, \dots, u_N)$ is close enough to $(y_1, y_2, \dots, y_N)$ such that $p(T)\hat{u} \in V$. Thus, T is $\mathcal{A}$-transitive on $X^{(\infty)}$, and thus by Proposition (2.2.2) we have that T has a dense set of $\mathcal{A}$-cyclic vectors.

We next apply the previous proposition to infinite diagonal operators where $\mathcal{A}$ is the set of all convex polynomials. This extends the finite dimensional matrix result given by Rezaei to infinite dimen-sional diagonal matrices.

Theorem (2.2.5) [2] Suppose that T is a diagonalizable normal operator on a separable (real or complex) Hilbert space with eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$.

(a) If the Hilbert space is complex, then T is convex-cyclic if and only if we have that the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and for every $k \geq 1$, $|\lambda_k| > 1$ and $Im(\lambda_k) \neq 0$.

(b) If the Hilbert space is real, then T is convex-cyclic if and only if the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and for every $k \geq 1$ we have that $\lambda_k < -1$.

Proof: By the spectral theorem we may assume that $T = diag(\lambda_1, \lambda_2, \dots)$ is an infinite diagonal matrix acting on $\ell^2_{\mathbb{C}}(\mathbb{N})$ and let $\{e_k\}_{k=1}^{\infty}$ be the canonical unit vector basis where e_k has a one in its k^{th} coordinate and zeros elsewhere.

(a) If T is convex-cyclic with convex-cyclic vector $x = (x_n)_{n=1}^{\infty} \in \ell^2_{\mathbb{C}}(\mathbb{N})$, then by Corollary (2.1.2) we must have for every $k \geq 1$ that $\infty = \sup_{n\geq 1} Re(\langle T^n x, e_k\rangle) = \sup_{n\geq 1} Re(\lambda_k^n x_k)$.
This implies that $x_k \neq 0$ and that $|\lambda_k| > 1$ for each $k \geq 1$. Likewise, since the Hilbert space is complex in this case, we must have

$$\infty = \sup_{n\geq 1} Re\left(\langle T^n x, \frac{-i}{x_k} e_k\rangle\right) = \sup_{n\geq 1} Re\left(\lambda_k^n x_k \frac{i}{x_k}\right) = \sup_{n\geq 1} Re(i\lambda_k^n).$$

This implies that λ_k cannot be real, hence $Im(\lambda_k) \neq 0$ for all $k \geq 1$.

Conversely, suppose that for every $k \geq 1$ we have that $|\lambda_k| > 1$ and $Im(\lambda_k) \neq 0$. Then for $n \geq 1$, let $T_n := diag(\lambda_1, \lambda_2, \dots, \lambda_n)$ be the diagonal matrix on $\mathbb{C}^n$ where λ_k is the k^{th} diagonal entry. Since the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ are distinct and $|\lambda_k| > 1$ and $Im(\lambda_k) \neq 0$ for $1 \leq k \leq n$, then we know from Rezaei that T_n is convex-cyclic on $\mathbb{C}^n$ and that every vector all of whose coordinates are non-zero is a convex-cyclic vector for T_n. Since such vectors are dense in $\mathbb{C}^n$ for every $n \geq 1$, then it follows from Proposition (2.2.4) that T is also convex-cyclic and has a dense set of convex-cyclic vectors. (b) The proof of the real case is similar to that above.

The next theorem says that if an operator has a complete set of eigenvectors whose eigenvalues are distinct, not real, and lie outside of the closed unit disk, then the operator is convex-cyclic.

Theorem (2.2.6) [2] Let $S\colon \left\{re^{i\theta} : r > 1 \text{ and } 0 < |\theta| < \pi\right\} = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup \mathbb{R})$. Suppose that T is a bounded linear operator on a complex Banach space X and that T has a countable linearly independent set of eigenvectors with dense linear span in X such that the corresponding eigenvalues are distinct and are contained in the set S. Then T is convex-cyclic and has a dense set of convex-cyclic vectors.

Proof: Suppose that $\{v_n\}_{n=1}^{\infty}$ is a linearly independent set of eigenvectors for T that have dense linear span in X and such that the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are distinct and contained in the set S. By replacing each eigenvector v_n with a constant multiple of itself we may assume that $\sum_{n=1}^{\infty}\|v_n\|^2 < \infty$. Let D be the diagonal normal matrix on $\ell^2(\mathbb{N})$ whose n^{th} diagonal entry is λ_n. Then define a linear map $A: \ell^2(\mathbb{N}) \to X$ by $A(\{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n v_n$.

Notice that since $\{a_n\}_{n=1}^{\infty} \in \ell^2(\mathbb{N})$, then we have that

$$\|A(\{a_n\}_{n=1}^{\infty})\| = \left\|\sum_{n=1}^{\infty} a_n v_n\right\| \leq \left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}\|v_n\|^2\right)^{1/2}$$
$$= C\|\{a_n\}_{n=1}^{\infty}\|_{\ell^2(\mathbb{N})}$$

where $C \coloneqq \left(\sum_{n=1}^{\infty}\|v_n\|^2\right)^{1/2}$, which is finite. The above inequality implies that A is a well defined continuous linear map from $\ell^2(\mathbb{N})$ to X. It follows that since the eigenvectors $\{v_n\}_{n=1}^{\infty}$ have dense linear span in X, that A has dense range. Also, if $\{e_n\}_{n=1}^{\infty}$ is the standard unit vector basis in $\ell^2(\mathbb{N})$, then clearly $A(e_n) = v_n$ for all $n \geq 1$ and thus A intertwines D with T. To see this notice that $AD(e_n) = A(\lambda_n e_n) = \lambda_n v_n = T(v_n) = TA(e_n)$. Thus $AD(e_n) = TA(e_n)$ for all $n \geq 1$, thus $AD = TA$. Finally, since D has distinct eigenvalues that all lie in the set S, it follows from Proposition (2.2.5) that D is convex-cyclic and has a dense set of convex-cyclic vectors. Since A intertwines D and T and A has dense range, then A will map convex-cyclic vectors for D to convex-cyclic vectors for T. Thus, T is convex-cyclic and has a dense set of convex-cyclic vectors.

If G is an open set in the complex plane, then by a reproducing kernel Hilbert space $\mathcal{H}$ of analytic functions on G we mean a vector space of analytic functions on G that is complete with respect to a norm given by an inner product and such that point evaluations at all points in G are continuous linear functionals on $\mathcal{H}$. Naturally we also require that $f = 0$ in $\mathcal{H}$ if and only if $f(z) = 0$ for all $z \in G$. This is equivalent to the reproducing kernels having dense linear span in $\mathcal{H}$. Given such a space $\mathcal{H}$, a multiplier of $\mathcal{H}$ is an analytic function φ on G so that $\varphi f \in \mathcal{H}$ for every $f \in \mathcal{H}$. In this case, the closed graph theorem implies that the multiplication operator $M_\varphi: \mathcal{H} \to \mathcal{H}$ is a bounded linear operator.

Corollary (2.2.7) [2] Suppose that G is an open set in $\mathbb{C}$ with components $\{G_n\}_{n\in J}$ and $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions on G, and that φ is a multiplier of $\mathcal{H}$. If φ is non-constant on every component of G and $\varphi(G_n) \cap \{z \in \mathbb{C}: |z| > 1\} \neq \emptyset$ for every $n \in J$, then the operator M_φ^* is convex-cyclic on $\mathcal{H}$ and has a dense set of convex-cyclic vectors.

Proof: We will show that the eigenvectors for M_φ^* with eigenvalues in the set $S = \mathbb{C}\backslash(\overline{\mathbb{D}} \cup \mathbb{R})$ have dense linear span in $\mathcal{H}$. It will then follow from Theorem (2.2.6) that M_φ^* is convex-cyclic.

Every reproducing kernel for $\mathcal{H}$ is an eigenvector for M_φ^*. In fact, if $\lambda \in G$, then $M_\varphi^* K_\lambda = \overline{\varphi(\lambda)}\, K_\lambda$, where K_λ denotes the reproducing kernel for $\mathcal{H}$ at the point $\lambda \in G$. By assumption, for every component G_n of G, φ is non-constant on G_n, thus the set $\{\lambda \in G_n: |\varphi(\lambda)| > 1\}$ is a nonempty open subset of G_n. Also since φ is an open map on G_n, φ cannot map the open set $\{\lambda \in G_n: |\varphi(\lambda)| > 1\}$ into $\mathbb{R}$. Thus, for all $n \in J, E_n = \{\lambda \in G_n: |\varphi(\lambda)| > 1 \text{ and } \varphi(\lambda) \notin \mathbb{R}\}$ is a nonempty open subset of G_n. Let $E := \bigcup_{n\in J} E_n$. Then for every $\lambda \in E$, K_λ is an eigenvector for M_φ^* with eigenvalue $\varphi(\lambda)$ which lies in $S = \mathbb{C}\backslash(\overline{\mathbb{D}} \cup \mathbb{R})$. Since $E \cap G_n$ is a nonempty open set for every $n \in J$, then the corresponding reproducing kernels $\{K_\lambda: \lambda \in E\}$ have dense linear span in $\mathcal{H}$. Finally, since φ is non-constant on E_n for each $n \in J$, we can choose a countable set $\{\lambda_{n,k}\}_{n,k=1}^{\infty}$ in E_n that has an accumulation point in E_n in such a way that φ is one-to-one on $\{\lambda_{n,k}\}_{n,k=1}^{\infty}$. Then the countable set $\{K_{\lambda_{n,k}}\}_{n,k=1}^{\infty}$ is a set of independent eigenvectors with dense linear span in $\mathcal{H}$ and with distinct eigenvalues. It now follows from Theorem (2.2.6) that M_φ^* is convex-cyclic and has a dense set of convex-cyclic vectors.

Next we give an example of a convex-cyclic operator that is not 1-weakly hypercyclic.

Example (2.2.8) [2] Let M_{2+z}^* be the adjoint of the multiplication operator associated to the multiplier $\varphi(z) := 2 + z$ on $H^2(\mathbb{D})$. The operator $M_{2+z}^* = 2I + B$, where B is the unilateral backward shift, is not 1-weakly-hypercyclic, however M_{2+z}^* is convex-cyclic by Corollary (2.2.7).

The following result is true since powers of convex polynomials are also convex polynomials.

Proposition (2.2.9) [2] If T is an operator on a Banach space and there exists a convex polynomial p such that $p(T)$ is hypercyclic, then T is convex-cyclic.

By a region in $\mathbb{C}$ we mean an open connected set in $\mathbb{C}$. In the following theorem, we consider the operator which is the adjoint of multiplication by z, the independent variable.

Theorem (2.2.10) [2] Suppose that G is a bounded region in $\mathbb{C}$ and $G \cap \{z: |z| > 1\} \neq \emptyset$. Suppose also that $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions on G, then M_z^* is convex-cyclic on $\mathcal{H}$. In fact, there exists a convex polynomial p such that $p(M_z^*)$ is hypercyclic on $\mathcal{H}$.

Proof: Choose $n \geq 1$ such that $G^n := \{z^n : z \in G\}$ satisfies $G^n \cap \{z \in \mathbb{C}: Re(z) < 1\} \neq \emptyset$. To see how to do this, choose a polar rectangle $R = \{re^{i\theta} : r_1 < r < r_2 \text{ and } \alpha < \theta < \beta\}$ such that $R \subseteq G$. Then simply choose a positive integer n such that $n(\beta - \alpha) > 2\pi$. Then $R^n \subseteq G^n$ and R^n will contain the annulus $\{re^{i\theta}: r_1^n < r < r_2^n\}$, so certainly $G^n \cap \{z \in \mathbb{C}: Re(z) < 1\} \neq \emptyset$.

Now if $0 < a \leq 1$, then the convex polynomial $p_a(z) = az + (1 - a)$ maps the disk $B\left(\frac{a-1}{a}, \frac{1}{a}\right)$ onto the unit disk. Notice that the family of disks $\left\{B\left(\frac{a-1}{a}, \frac{1}{a}\right) : 0 < a < 1\right\}$ is the family of all disks that are centered on the negative real axis and pass through the point $z = 1$. Thus it follows that $\{z \in \mathbb{C}: Re(z) < 1\} = \bigcup_{0<a<1} B\left(\frac{a-1}{a}, \frac{1}{a}\right)$. So we can choose an $a \in (0,1)$ such that $G^n \cap \partial B\left(\frac{a-1}{a}, \frac{1}{a}\right) \neq \emptyset$. It follows that the polynomial $p(z) = p_a(z^n)$ is a convex polynomial and furthermore it satisfies $p(G) \cap \partial\mathbb{D} \neq \emptyset$.

Thus M_p^* is hypercyclic on $\mathcal{H}$. However, $M_p^* = p^{\#}(M_z^*)$ where $p^{\#}(z) = \overline{p(\bar{z})}$. Also, since p is a convex polynomial, all of its coefficients are real, thus $p^{\#} = p$. Thus, $p(M_z^*) = p^{\#}(M_z^*) = M_p^*$ is hypercyclic on $\mathcal{H}$.

We show an example of an operator that is convex-cyclic but no convex polynomial of the operator is hypercyclic. In other words, the operator is purely convexcyclic.

Example (2.2.11) [2] Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two strictly decreasing sequences of positive numbers that are interlaced and converging to zero. In other words, $0 < \alpha_{n+1} < \beta_{n+1} < \alpha_n$ for all $n \geq 1$ and $\alpha_n \to 0$ (and hence $\beta_n \to 0$). For each $n \geq 1$, let

$$G_n := \left\{ re^{i\theta} : 2 < r < 2 + \frac{1}{n} \text{ and } \alpha_n < \theta < \beta_n \right\}.$$

Let $G := \bigcup_{n=1}^{\infty} G_n$ and let $L_a^2(G)$ be the Bergman space of all analytic functions on G that are square integrable with respect to area measure on G. Then the operator M_z^* is purely convex- cyclic on $L_a^2(G)$; meaning that M_z^* is convex-cyclic on $L_a^2(G)$, but $p(M_z^*)$ is not hypercyclic on $L_a^2(G)$ for any convex polynomial p.

Proof: By Corollary (2.2.7) we know that M_z^* is convex-cyclic on $L_a^2(G)$. In order to show that no convex polynomial of M_z^* is hypercyclic, suppose, by way of contradiction, that there exists a convex polynomial p such that $p(M_z^*)$ is hypercyclic. Since p is a convex polynomial it has real coefficients thus $p^{\#}(z) = p(z)$ where $p^{\#}(z) := \overline{p(\bar{z})}$. Thus $p(M_z^*) = M_{p^{\#}}^* = M_p^*$ and it follows that M_p^* is hypercyclic on $L_a^2(G)$. Thus it follows that every component G_n of G must satisfy that $p(G_n) \cap \partial\mathbb{D} \neq \emptyset$. However since p is a convex polynomial, p is (strictly) increasing on the interval $[0, \infty)$. Thus, $p(2) > p(1) = 1$. Choose an $\varepsilon > 0$ such that $\varepsilon < p(2) - 1$. Since p is continuous at $z = 2$, and since we have an $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $|z - 2| < \delta$, then $|p(z) - p(2)| < \varepsilon$. Notice that for n sufficiently large we have that $G_n \subseteq B(2, \delta)$, thus, $p(G_n) \subseteq B(p(2), \varepsilon) \subseteq \{z \in \mathbb{C}: Re(z) > 1\}$. Thus, $p(G_n) \cap \partial\mathbb{D} \neq \emptyset$ for all large n, a contradiction. It follows that no convex polynomial of M_z^* is hypercyclic, hence M_z^* is purely convex-cyclic.

Chapter 3
Separably Injective Banach Spaces

We obtain two fundamental characterization of universally separably injective spaces: (i) A Banach space E is universally separably injective if and only if every separable subspaces is contained in a copy of ℓ_∞ inside E. (ii) A Banach space E is universally separably injective if and only if for every separable space S on has $\mathrm{Ext}\left(\frac{\ell_\infty}{S}, E\right) = 0$. We construct a consistent example of a Banach space of type $C(K)$ which is 1-separably injective but not 1-universally separably injective.

Section (3.1): Basic properties of separably injective spaces with examples

A Banach space E is separably injective if for every separable Banach space X and each separable $Y \subset X$, every operator $t: Y \to E$ extends to an operator $T: X \to E$. If some extension T exists with $\|T\| \leq \lambda\|T\|$ we say that E is λ-separably injective [7].

Definition (3.1.1) [3] A Banach space E is separably injective if for every separable Banach space X and each subspace $Y \subset X$, every operator $t: Y \to E$ extends to an operator $T: X \to E$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is λ-separably injective.

Every separably injective space E is λ-separably injective for some λ since every sequence of norm-one operators $t_n: Y_n \to E$ induces a norm-one operator $t: \ell_1(Y_n) \to E$. Separable injective spaces can be characterized as follows.

Proposition (3.1.2) [3] For a Banach space E the following properties are equivalent.

(a) E is separably injective.

(b) Every operator from a subspace of ℓ_1 into E extends to ℓ_1.

(c) For every Banach space X and each subspace Y such that X/Y is separable, every operator $t: Y \to E$ extends to X.

(d) If X is a Banach space containing E and X/E is separable, then E is complemented in X.

(e) For every separable space S one has $\mathrm{Ext}(S, E) = 0$.

Moreover,

(i) The space E is λ-complemented in every Z such that Z/E is separable if and only if every operator $t: Y \to E$ admits an extension $T: X \to E$ with $\|T\| \leq \lambda\|t\|$, whenever X/Y is separable.

(ii) If E is λ-separably injective, then for every operator $t: Y \to E$ there exists an extension $T: X \to E$ of T with $\|T\| \leq 3\lambda\|t\|$, whenever X/Y is separable.

Proof: It is clear that (c) ⇒ (a) ⇒ (b) and (c) ⇒ (d) ⇔ (e). Moreover, (i) shows that (d) ⇒ (c) and (ii) shows that (a) ⇒ (c). The remaining implication (b) ⇒ (a) follows from the proof of (ii) below.

For the sufficiency statement in (i) simply consider t as the identity on E. For the necessity statement, given an operator $t: Y \to E$ form the associated push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & Y & \stackrel{t}{\to} & X & \stackrel{\pi}{\to} & X/Y & \to & 0 \\ & & t\downarrow & & \downarrow t' & & || & & \\ 0 & \to & E & \stackrel{t'}{\to} & PO & \to & PO/E & \to & 0. \end{array}$$

Since $PO/E = X/Y$ is separable, there is a projection $p: PO \to E$ with norm at most λ, and thus, recalling that $\|t'\| \leq 1$, the composition $pt': X \to E$ yields an extension of t with norm at most λ.

The proof for (ii) is a little more tricky. Let q be a surjective map from $\ell_1 \to X/Y$. The lifting property of ℓ_1 provides an operator $Q: \ell_1 \to X$. Consider thus the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \ker q & \stackrel{j}{\to} & \ell_1 & \stackrel{q}{\to} & X/Y & \to & 0 \\ & & \phi\downarrow & & Q\downarrow & & || & & \\ 0 & \to & Y & \to & X & \to & X/Y & \to & 0 \end{array}$$

Let us construct the true push-out of the couple (ϕ, j) and the corresponding complete diagram

$$\begin{array}{ccccccccc} 0 & \to & \ker q & \stackrel{j}{\to} & \ell_1 & \stackrel{q}{\to} & X/Y & \to & 0 \\ & & \phi\downarrow & & \downarrow \phi' & & || & & \\ 0 & \to & Y & \stackrel{j'}{\to} & PO & \to & X/Y & \to & 0. \end{array}$$

We can consider without loss of generality that $\|\phi\| = 1$. Let $S: \ell_1 \to E$ be an extension of $t\phi$ with $\|S\| \leq \lambda\|t\phi\| \leq \lambda\|t\|$. By the universal property of the push-out, there exists an operator $L: PO \to E$ such that $L\phi' = S$ and $\|L\| \leq \max\{\|t\|, \|S\|\} \leq \lambda\|t\|$. Again by the universal property of the push-out, there is a diagram of equivalent exact sequences

$$\begin{array}{ccccccccc} 0 & \to & Y & \stackrel{j'}{\to} & PO & \to & X/Y & \to & 0 \\ & & || & & \gamma\downarrow & & || & & \\ 0 & \to & Y & \stackrel{i}{\to} & X & \stackrel{p}{\to} & X/Y & \to & 0, \end{array}$$

where the isomorphism γ is defined as $\gamma\big((y,u) + \Delta\big) = j(y) + Q(u)$ is such that $\|\gamma\| \leq \max\{\|j\|, \|Q\|\} \leq 1$. The desired extension of t to X is $T = L\gamma^{-1}$, where γ^{-1} comes defined by

$$\gamma^{-1}(x) = \big(x - s(px), s(px)\big) + \Delta,$$

where $s: X/Y \to \ell_1$ is a homogeneous bounded selection for q with $\|s\| \leq 1$. One clearly has $\|\gamma^{-1}\| \leq 3$, and therefore $\|T\| \leq 3\lambda$.

We are especially interested in the following subclass of separably injective spaces.

Definition (3.1.3) [3] A Banach space E is said to be universally separably injective if for every Banach space X and each separable subspace $Y \subset X$, every operator $t: Y \to E$ extends to an operator $T: Y \to X$. If some extension T exists with $\|T\| \leq \lambda\|t\|$ we say that E is universally λ-separably injective.

A Banach space E is universally separably injective if and only if every E-valued operator with separable range extends to any superspace. It is also easy to show that every universally separably injective space is λ-universally separably injective for some λ.

Recall that a Banach space X has Pełczyński's property (V) if each operator defined on X is either weakly compact or it is an isomorphism on a subspace isomorphic to c_0. We will say that X has Rosenthal's property (V) if it satisfies the preceding condition with ℓ_∞ replacing c_0. It is well-known that Lindenstrauss spaces (i.e., $\mathcal{L}_{\infty+1}$-spaces) have this property.

Not all $\mathcal{L}_\infty$-spaces have Pełczyński's property (V): for example, the $\mathcal{L}_\infty$ -spaces without copies of c_0 constructed by Bourgain and Delbaen; or those that can be obtained from Bourgain-Pisier; or the space Ω constructed as a twisted sum

$$0 \to C[0,1] \to \Omega \to c_0 \to 0$$

with strictly singular quotient map. Recall that a Banach space X is said to be a Grothendieck space if every operator from X to a separable Banach space (or to c_0) is weakly compact. Clearly, a Banach space with property (V) is a Grothendieck space if and only if it has no complemented subspace isomorphic to c_0. It is well-known that ℓ_∞ is a Grothendieck space.

Proposition (3.1.4) [3]

(a) A separably injective space is of type $\mathcal{L}_\infty$, has Pełczyński's property (V) and, when it is infinite dimensional, contains copies of c_0.

(b) A universally separably injective space is a Grothendieck space of type $\mathcal{L}_\infty$, has Rosenthal's property (V) and, when it is infinite dimensional, contains ℓ_∞.

Proof: (a) Let E be a λ-separably injective space. We want to see that if Y is a subspace of any Banach space X, every operator $t: Y \to E$ extends to an operator $T: X \to E^{**}$ with $\|T\| \le \lambda\|t\|$. This implies that E^{**} is λ-injective, by an old result of Lindenstrauss. Being of infinite dimension, E^{**} is an $\mathcal{L}_{\infty,9\lambda^+}$ space and so is E. Let $t: Y \to E$ be an operator. Given a finite-dimensional subspace F of X, let $T_F: F \to E$ be any operator extending the restriction of t to $Y \cap F$. Let $\mathcal{F}$ be the set of finite-dimensional subspaces of X, ordered by inclusion, let $\mathcal{U}$ be any ultrafilter refining the Fréchet filter on $\mathcal{F}$, that is, containing every set of the form $\{G \in \mathcal{F}: F \subset G\}$ for fixed $F \in \mathcal{F}$. Then, define $T: X \to E^{**}$ taking

$$T(x) = weak^* - \lim_{\mathcal{U}(F)} T_F\left(1_{F(x)} x\right).$$

It is easily seen that T is a linear extension of t, with $\|T\| \le \lambda\|t\|$.

To show that E contains c_0 and has property (V), let $T: E \to X$ be a non-weakly compact operator (E being an infinite dimensional $\mathcal{L}_\infty$ space cannot be reflexive). Choose a bounded sequence (x_n) in E such that $\left(T_{x_n}\right)$ has no weakly convergent subsequences and let Y be the subspace spanned by (x_n) in E. As Y is separable we can regard it as a subspace of $C[0,1]$. Let $J: C[0,1] \to E$ be any operator extending the inclusion of Y into E. Since $TJ: C[0,1] \to E$ is not weakly compact, TJ is an isomorphism on some subspace isomorphic to c_0; and the same occurs to T .

(b) If, in addition to that, E is universally separably injective we may take $T: E \to Z$ and $Y \subset E$ as before but this time we consider Y as a subspace of ℓ_∞. If $J: \ell_\infty \to E$ is any extension of the inclusion of Y into E, then $TJ: \ell_\infty \to Z$ is not weakly compact. Hence it is an isomorphism on some subspace isomorphic to $\ell_\infty\infty$ and so is .

Several modifications on the proof of Ostrovskii yield

Proposition (3.1.5) A λ-separably injective space with $\lambda < 2$ is either finite-dimensional or has density character at least c.

Recall that a class of Banach spaces is said to have the 3-space property if whenever X/Y and Y belong to the class, then so X does.

Proposition (3.1.6) [3]

(i) The class of separably injective spaces has the 3-space property.
(ii) The quotient of two separably injective spaces is separably injective.
(iii) The class of universally separably injective spaces has the 3-space property.
(iv) The quotient of a universally separably injective space by a separably injective space is universally separably injective.

Proof: The simplest proof for the 3-space property (i) follows from characterization (ii) in Proposition (3.1.2): let us consider an exact sequence $0 \to F \to E \overset{\pi}{\to} G \to 0$ in which both F and G are separably injective. Let $\phi: K \to E$ be an operator from a subspace $\iota: K \to \ell_1$ of ℓ_1; then $\pi\phi$ can be extended to an operator $\Phi: \ell_1 \to G$, which can in turn be lifted to an operator $\Psi: \ell_1 \to E$. The difference $\phi - \Psi_\iota$ takes values in F and can thus be extended to an operator $e: \ell_1 \to F$. The desired operator is $\Psi + e$. A different homological proof that properties having the form $\mathrm{Ext}(X, -) = 0$ are always 3-space properties can be found.

To prove (ii) and (iv) let us consider an exact sequence $0 \to F \to E \overset{\pi}{\to} G \to 0$ in which F is separably injective and E is (universally) separably injective. Let $\phi: Y \to G$ be an operator from a separable space Y which is a subspace of a separable (arbitrary) space X. Consider the pull-back diagram

$$\begin{array}{ccccccccc} 0 & \to & F & \to & E & \overset{q}{\to} & G & \to & 0 \\ & & || & & \uparrow \phi' & & \uparrow \phi & & \\ 0 & \to & F & \to & PB & \overset{Q}{\to} & Y & \to & 0 \end{array}$$

Since F is separably injective, the lower exact sequence splits, so Q has a selection operator $s: Y \to PB$. By the injectivity assumption about E, there exists an operator $T: X \to E$ agreeing with Qs on Y. Then $qT: X \to G$ is the desired extension of ϕ.

The proof for (iii) has to wait until Theorem (3.2.1) when a suitable characterization of universally separably injective spaces will be presented.

Several variations of these results can be seen. It is obvious that if $(E_\iota)_{\iota \in I}$ is a family of λ-separably injective Banach spaces, then $\ell_\infty(I, E_\iota)$

is λ-separably injective. The non-obvious fact that also $c_0(I, E_\iota)$ is separably injective can be considered as a vector valued version of Sobczyk's theorem. Proofs for this result have been obtained by Johnson-Oikhberg, Rosenthal, Cabello and Castillo-Moreno, each with its own estimate for the constant. These are $2\lambda^2$ (implicitly), $\lambda(1+\lambda)^+$, $(3\lambda^2)^+$ and $6\lambda^+$, respectively.

Examples (3.1.7) [3]:

All injective spaces are universally separably injective. Sobczyk theorem states that c_0 –and $c_0(\Gamma)$, in general– are 2-separably injective in its natural supremum norm. They are not universally separably injective since they do not contain ℓ_∞.

(a) Twisted sums. The 3-space property yields that twisted sums of separably injective are also separably injective. In particular:

(i) Twisted sums of c_0 and $c_0(\Gamma)$: This includes the Johnson-Lindenstrauss spaces $C(\Delta_{\mathcal{M}})$ obtained taking the closure of the linear span in ℓ_∞ of the characteristic functions $\{1_n\}_{n\in\mathbb{N}}$ and $\{1_{M_\alpha}\}_{\alpha\in J}$ for an uncountable almost disjoint family $\{M_\alpha\}_{\alpha\in J}$ of subsets of $\mathbb{N}$. Marciszewski and Pol answer a question of Koszmider showing that there exist 2^c almost disjoint families $\mathcal{M}$ generating non-isomorphic $C(\Delta_{\mathcal{M}})$-spaces.

(ii) Twisted sums of two nonseparable $c_0(\Gamma)$spaces. This includes variations of the previous construction using the Sierpinski-Tarski generalization of the construction of almost disjoint families; the Ciesielski-Pol space; the WCG nontrivial twisted sums of $c_0(\Gamma)$ obtained independently by Argyros, Castillo, Granero, Jimenez and Moreno and by Marciszewski .

(iii) Twisted sums of c_0 and ℓ_∞, as those constructed.

(iv) A twisted sum of c_0 and $c_0\left(\frac{\ell_\infty}{c_0}\right)$ that is not complemented in any $C(K)$-space, as the one obtained.

(b) The space $\ell_\infty^c(\Gamma)$: A typical 1-universally separably injective space is the space $\ell_\infty^c(\Gamma)$ of countably supported bounded functions $f: \Gamma \to \mathbb{R}$, where Γ is an uncountable set. This space is isomorphic but not isometric to some $C(K)$ space, showing in this way that the theory of universally separably injective spaces does not run parallel with that of injective spaces. What makes this space universally separably injective space is the following property:

Definition (3.1.8) [3] We say that a Banach space X is ℓ_∞-upper-saturated if every separable subspace of X is contained in some (isomorphic) copy of ℓ_∞ inside X.

It is clear that an ℓ_∞-upper-saturated space is universally separably injective. We will prove later that the converse also holds.

The space ℓ_∞/c_0: Since ℓ_∞ is injective and c_0 is separably injective, it follows from Proposition (3.1.6) that ℓ_∞/c_0 is universally separably injective, although the constant is not optimal. It follows from Proposition (3.1.13) (a) that ℓ_∞/c_0 is 1-universally separably injective, hence, it is ℓ_∞- upper-saturated. This can be improved to show that every separable subspace of ℓ_∞/c_0 is contained in a subalgebra of ℓ_∞/c_0 isometrically isomorphic to ℓ_∞.

It is well-known that ℓ_∞/c_0 is not injective. The simplest proof appears in Rosenthal: an injective space containing $c_0(I)$ must also contain $\ell_\infty(I)$; it is well-known that ℓ_∞/c_0 contains $c_0(I)$ for $|I| = c$ while it cannot contain $\ell_\infty(I)$. The proof is quite rough in a sense: it says that ℓ_∞/c_0 is uncomplemented in its bidual, a huge superspace. Denoting $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$, Amir had shown that $C(\mathbb{N}^*)$ is not complemented in $\ell_\infty(2^c)$, which provides another proof that l_∞/c_0 is not injective. Amir's proof can be refined in order to get $C(\mathbb{N}^*)$ uncomplemented in a much smaller space. It can be shown that $C(\mathbb{N}^*)$ contains an uncomplemented copy Y of itself.

Proposition (3.1.9) [3]. A $C(K)$ space is 1-separably injective if and only if K is an F-space.

Simple examples show that when a $C(K)$-space is only isomorphic to a 1-separably injective then K does not need to be an F-space. It is an immediate consequence of Tietze's extension theorem that a closed subset of an F-space is an F-space. In particular, $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ is an F-space.

Given a compact space K, we write K' for its derived set, that is, the set of its accumulation points. This process can be iterated to define $K^{(n+1)}$ as $\left(K^{(n)}\right)'$ with $K^{(0)} = K$. We say that K has height n if $K^{(n)} = \phi$. We say that K has finite height if it has height n for some $n \in \mathbb{N}$.

Proposition (3.1.10) [3] If K is a compact space of height n, then $C(K)$ is $(2n-1)$-separably injective. Consequently, if K is a compact space of finite height then $C(K)$ is separably injective although it is not universally separably injective.

Proof: Let $Y \subset X$ with X separable and let $t: Y \to C(K)$ be a norm one operator. The range of t is separable and every separable subspace of a $C(K)$ is contained in an isometric copy of $C(L)$, where L is the quotient of K after identifying k and k' when $y(k) = y(k')$ for all $y \in Y$. This L is metrizable because Y is separable. Moreover, if K has height n, then L has height at most n and so it is homeomorphic to $[0, \omega^r \cdot k]$ with $r < n, k < \omega$. Since $C[0, \omega^r \cdot k]$ is $(2r + 1)$-separably injective, our operator can be extended to an operator $T: X \to C(K)$ with norm

$$\|T\| \leq (2r + 1)\|t\| \leq (2n - 1)\|t\|,$$

concluding the Proof:

When K is a metrizable compact of finite height n, Baker showed that $2n - 1$ is the best constant for separable injectivity, using arguments from Amir. There are some difficulties in generalizing those arguments for nonmetrizable compact spaces, so we do not know if it could exist a nonmetrizable compact space K of height n such that $C(K)$ is λ-separably injective for some $\lambda < 2n - 1$.

Proposition (3.1.11) [**3**] The space of all bounded Borel (respectively, Lebesgue) measurable functions on the line is 1-separably injective in the sup norm.

Proof: Clearly the given spaces are in fact Banach algebras satisfying the inequality required by Albiac- Kalton characterization. Thus they can be represented as $C(K)$ spaces. On the other hand, each measurable function can be decomposed as $f = u|f|$, with u measurable.
This clearly implies that the corresponding compacta are F-spaces.

Argyros proved that none of the spaces in the above example is injective. This is very simple in the Borel case: the characteristic functions of the singletons generate a copy of $c_0(\mathbb{R})$ in the space of bounded Borel functions. The density character of the latter space is the continuum, as there are c Borel subsets. Therefore it cannot contain a copy of $\ell_\infty(\mathbb{R})$, whose density character is 2^c.

(c) M-ideals. A closed subspace $J \subset X$ is called an M-ideal if its annihilator $J^\perp = \{x^* \in X^*: \langle x^*, x\rangle = 0\ \forall x \in J\}$ is an L-summand in X^*. This just means that there is a linear projection P on X^* whose range is $J^\perp$ and such that $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$.

The easier examples of M-ideals are just ideals in $C(K)$-spaces. In particular, if M is a closed subset of the compact space K and $L = K \setminus M$ one has that $C_0(L)$ is an M-ideal in $C(K)$ is straightforward from the Riesz representation of $C(K)^*$. A remarkable generalization of Borsuk-Dugundji theorem for M-ideals was provided by Ando and, independently, Choi and Effros. In order to state it let us recall that a Banach space Z has the λ-approximation property (λ-AP, for short) if, for every $\varepsilon > 0$ and every compact subset K of Z, there exists a finite rank operator T on Z, with $\|T\| \leq \lambda$, such that $\|T\,z - z\| < \varepsilon$, for every $z \in K$. We say that Z has the bounded approximation property (BAP for short) if it has the λ-AP, for some λ.

Theorem (3.1.12) [3]. Let J be an M-ideal in the Banach space E and $\pi\colon E \to E/J$ the natural quotient map. Let Y be a separable Banach space and $t\colon Y \to E/J$ be an operator. Assume further that one of the following conditions is satisfied:

(i) Y has the λ-AP.

(ii) J is a Lindenstrauss space.

Then t can be lifted to E, that is, there is an operator $T\colon Y \to E$ such that $\pi T = t$. Moreover one can get $\|T\| \leq \lambda\|t\|$ under the assumption (i) and $\|T\| = \|t\|$ under (ii).

One has.

Proposition (3.1.13) [3] Let J be an M-ideal in a Banach space E.

(a) If E is λ-(universally) separably injective, then E/J is λ^2-(universally) separably injective.

(b) If E is λ-separably injective, then J is $2\lambda^2$-separably injective.

When J is a Lindenstrauss space (which is always the case if E is), then the exponent 2 disappears.

In particular, if K_1 is a closed subset of the compact space K and $K_0 = K \setminus K_1$ one has:

(c) If $C(K)$ is λ-(universally) separably injective, then so is $C(K_1)$.

(d) If $C(K)$ is λ-separably injective, then $C_0(K_0)$ is 2λ-separably injective.

Proof: (a) By (the proof of) Proposition (3.1.4), E^{**} is λ-injective and so it has the λ-AP. Since $E^{**} = J^{**} \oplus_\infty (E/J)^{**}$ we see that also J^{**} and $(E/J)^{**}$ have the λ-AP. Hence both J and (E/J) have the λ-AP. Let Y be a separable subspace of X and $t: Y \to E/J$ an operator. Let S be a separable subspace of E/J containing the image of t. We may assume S has the λ-AP. Let $s: S \to E$ be the lifting provided by Theorem (3.1.12), so that $\|s\| \leq \lambda$. Now, if $T: X \to E$ is an extension of st, then $\pi T: X \to E/J$ is an extension of t, and this can be achieved with $\|\pi T\| = \|T\| \leq \lambda^2\|t\|$.

(d) –and (b)–. Let us remark that if S is a subspace of $C(K)$ containing $C_0(K_0)$ and $S/C_0(K_0)$ is separable, then there is a projection $p: S \to C_0(K_0)$ of norm at most 2. Indeed, $S/C_0(K_0)$ is a separable subspace of $C(K_1)$ and there is a lifting $s: S/C_0(K_0) \to C(K)$, with $\|s\| = 1$, and $p = \mathbf{1}_S - sr$ is the required projection. Now, let $t: Y \to C_0(K_0)$ be an operator, where Y is a subspace of a separable Banach space X. Considering t as taking values in $C(K)$, there is an extension $T: X \to C(K)$ with $\|T\| \leq \lambda\|t\|$. Let S denote the least closed subspace of $C(K)$ containing the range of T and $C_0(K_0)$ and $p: S \to C_0(K_0)$ a projection with $\|p\| \leq 2$. The composition $pT: X \to C_0(K_0)$ is an extension of t and clearly, $\|pT\| \leq 2\lambda\|t\|$.

(d) Ultraproducts of type $\mathcal{L}_\infty$: Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer to Heinrich or Sims' notes. Let I be a set, $\mathcal{U}$ be an ultrafilter on I, and $(X_i)_{i\in I}$ a family of Banach spaces. Then $\ell_\infty(X_i)$ endowed with the supremum norm, is a Banach space, and $c_0^{\mathcal{U}}(X_i)\left\{(x_i) \in \ell_\infty(X_i): \lim_{\mathcal{U}(i)}\|x_i\| = 0\right\}$ is a closed subspace of $\ell_\infty(X_i)$. The ultraproduct of the spaces $(X_i)_{i\in I}$ following $\mathcal{U}$ is defined as the quotient

$$[X_i]_{\mathcal{U}} = \frac{\ell_\infty(X_i)}{c_0^{\mathcal{U}}(X_i)}.$$

We denote by $[(x_i)]$ the element of $[X_i]_{\mathcal{U}}$ which has the family (x_i) as a representative. It is not difficult to show that $\|[(x_i)]\| = \lim_{\mathcal{U}(i)}\|x_i\|$. In the case $X_i = X$ for all i, we denote the ultraproduct by $X_{\mathcal{U}}$, and call it the ultrapower of X following $\mathcal{U}$. If $T_i: X_i \to Y_i$ is a uniformly bounded family of operators, the ultraproduct operator $[T_i]_{\mathcal{U}}: [X_i]_{\mathcal{U}} \to [Y_i]_{\mathcal{U}}$ is given by $[T_i]_{\mathcal{U}}\,[(x_i)] = [T_i(x_i)]$. Quite clearly, $\|[T_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}(i)}\|T_i\|$.

Definition (3.1.14) [3] An ultrafilter $\mathcal{U}$ on a set I is countably incomplete if there is a decreasing sequence (I_n) of subsets of I such that $I_n \in \mathcal{U}$ for all n, and $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Notice that $\mathcal{U}$ is countably incomplete if and only if there is a function $n: I \to \mathbb{N}$ such that $n(i) \to \infty$ along $\mathcal{U}$ (equivalently, there is a family $\varepsilon(i)$ of strictly positive numbers converging to zero along $\mathcal{U}$). It is obvious that any countably incomplete ultrafilter is non-principal and also that every non-principal (or free) ultrafilter on $\mathbb{N}$ is countably incomplete. Assuming all free ultrafilters countably incomplete is consistent with ZFC, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible.

It is clear that the classes of $\mathcal{L}_{p,\lambda^+}$ spaces are stable under ultraproducts. In the opposite direction, a Banach space is a $\mathcal{L}_{p,\lambda^+}$ space if and only if some (or every) ultrapower is. In particular, a Banach space is an $\mathcal{L}_\infty$ space or a Lindenstrauss space if and only if so are its ultra powers. It is possible however to produce a Lindenstrauss space out of non-even-$\mathcal{L}_\infty$-spaces: indeed, if $p(i) \to \infty$ along $\mathcal{U}$, then the ultraproduct $\left[L_{p(i)}\right]_{\mathcal{U}}$ is a Lindenstrauss space.

The following result about the structure of separable subspaces of ultraproducts of type $\mathcal{L}_\infty$ will be fundamental.

Lemma (3.1.15) [3]: Suppose $[X_i]_{\mathcal{U}}$ is an $\mathcal{L}_{\infty,\lambda^+}$ -space. Then each separable subspace of $[X_i]_{\mathcal{U}}$ is contained in a subspace of the form $[F_i]_{\mathcal{U}}$, where $F_i \subset X_i$ is finite dimensional and $\lim_{\mathcal{U}(i)} d\left(F_i, \ell_\infty^{k(i)}\right) \le \lambda$, where $k(i) = \dim F_i$.

Proof: Let us assume S is an infinite-dimensional separable subspace of $[X_i]_{\mathcal{U}}$. Let (s^n) be a linearly independent sequence spanning a dense subspace in S and, for each n, let (s_i^n) be a fixed representative of s^n in $\ell_\infty(X_i)$. Let $S^n = \operatorname{span}\{s^1, \ldots, s^n\}$. Since $[X_i]_{\mathcal{U}}$ is an $\mathcal{L}_{\infty,\lambda^+}$-space there is, for each n, a finite dimensional $F^n \subset [X_i]_{\mathcal{U}}$ containing S^n with $d\left(F^n, \ell_\infty^{\dim F^n}\right) \le \lambda + 1/n$.

For fixed n, let (f^m) be a basis for F^n containing $s^1, \ldots, s^n$. Choose representatives (f_i^m) such that $f_i^m = s_i^\ell$ if $f^m = s^\ell$. Moreover, let F_i^n be the subspace of X_i spanned by f_i^m for $1 \le m \le \dim F^n$.

Let (I_n) be a decreasing sequence of subsets $I_n \in \mathcal{U}$ such that $\bigcap_{n=1}^{\infty} I_n = \emptyset$. For each integer n put

$$J'_n = \left\{ i \in I : d\left(F^n, \ell_\infty^{\dim F^n}\right) \le \lambda + \frac{2}{n} \right\} \cap I_n$$

and $J_m = \bigcap_{n \le m} J'_n$. All these sets are in $\mathcal{U}$. Finally, set $J_\infty = J_n$. Next we define a function $k : I \to \mathbb{N}$. Set

$$k(i) = \begin{cases} 1 & i \in J_\infty \\ \sup\{n : i \in J_n\} & i \notin J_\infty \end{cases}$$

For each $i \in I$, take $F_i = F_i^{k(i)}$. This is a finite-dimensional subspace of X_i whose Banach-Mazur distance to the corresponding ℓ_∞^k is at most $\lambda + 2/k(i)$. It is clear that $[F_i]_{\mathcal{U}}$ contains S and also that $k(i) \to \infty$ along $\mathcal{U}$, which completes the Proof:

Theorem (3.1.16) [3] Let $(X_i)_{i \in I}$ be a family of Banach spaces such that $[X_i]_{\mathcal{U}}$ is a $\mathcal{L}_{\infty,\lambda^+}$-space. Then $[X_i]_{\mathcal{U}}$ is λ-universally separably injective.

Lemma (3.1.17) [3] For every function $k : I \to \mathbb{N}$, the space $\left[\ell_\infty^{k(i)}\right]_{\mathcal{U}}$ is 1-universally separably injective.

Proof: Let Γ be the disjoint union of the sets $\{1, 2, \ldots, k(i)\}$ viewed as a discrete set. Now observe that $c_0^{\mathcal{U}}\left(\ell_\infty^{k(i)}\right)$ is an ideal in $\ell_\infty\left(\ell_\infty^{k(i)}\right) = \ell_\infty(\Gamma) = C(\beta\Gamma)$ and apply Proposition (3.1.13) (a).

Corollary (3.1.18) [3] Let $(X_i)_{i \in I}$ be a family of Banach spaces. If $[X_i]_{\mathcal{U}}$ is a Lindenstrauss space, then it is 1-universally separably injective.

Section (3.2): Two characterizations of universally and 1-separably injective spaces

In Proposition (3.1.4) (b) it was proved that universally separably injective spaces contain ℓ_∞. Much more is indeed true:

Theorem (3.2.1) [3] An infinite-dimensional Banach space is universally separably injective if and only if it is ℓ_∞-upper-saturated.

Proof: The sufficiency is a consequence of the injectivity of ℓ_∞. In order to show the necessity, let Y be a separable subspace of a universally separably injective space X. We consider a subspace Y_0 of ℓ_∞ isomorphic to Y and an isomorphism $t: Y_0 \to Y$. We can find projections p on X and q on ℓ_∞ such that $Y \subset \ker p$, $Y_0 \subset \ker q$, and both p and q have range isomorphic to ℓ_∞.

Indeed, let $\pi: X \to X/Y$ be the quotient map. Since X contains ℓ_∞ and Y is separable, π is not weakly compact so, by Proposition (3.1.4) (b), there exists a subspace M of X isomorphic to ℓ_∞ where π is an isomorphism. Now $X/Y = \pi(M) \oplus N$, with N a closed subspace. Hence $X = M \oplus \pi^{-1}(N)$, and it is enough to take p as the projection with range M and kernel $\pi^{-1}(N)$.

Since $\ker p$ and $\ker q$ are universally separably injective spaces, we can take operators $u: X \to \ker q$ and $v: \ell_\infty \to \ker p$ such that $v = t$ on Y_0 and $u = t^{-1}$ on Y.

Let $w: \ell_\infty \to \operatorname{ran} p$ be an operator satisfying $\|w(x)\| \geq \|x\|$ for all $x \in \ell_\infty$. We will show that the operator

$$T = v + w\left(1_{\ell_\infty} - uv\right): \ell_\infty \to X$$

is an isomorphism (into). This suffices to end the proof since $\operatorname{ran} T$ is isomorphic to ℓ_∞ and both T and v agree with t on Y_0, so $Y \subset \operatorname{ran} T \subset X$. Since $\operatorname{ran} v \subset \ker p$ and $\operatorname{ran} w \subset \operatorname{ran} p$, there exists $C > 0$ such that

$$\|Tx\| \geq C \max\left\{\|v(x)\|, \left\|w\left(1_{\ell_\infty} - uv\right)x\right\|\right\} (x \in \ell_\infty).$$

Now, if $\|vx\| < (2\|u\|)^{-1}\|x\|$, then $\|uvx\| < \frac{1}{2}\|x\|$; hence

$$\left\|w\left(1_{\ell_\infty} - uv\right)x\right\| \geq \left\|\left(1_{\ell_\infty} - uv\right)x\right\| > \frac{1}{2}\|x\|.$$

Thus $\|Tx\| \geq C(2\|u\|)^{-1}\|x\|$ for every $x \in X$.

We can now complete the proof of Proposition (3.1.6) (iii) and show that the class of universally separably injective spaces has the 3-space property.

Proposition (3.2.2) [3] The class of universally separably injective spaces has the 3-space property.

Proof: By Theorem (3.2.1) one has to show that being ℓ_∞-upper-saturated is a 3-space property.

Let $0 \to Y \to X \overset{q}{\to} Z \to 0$ be an exact sequence in which both Y, Z are ℓ_∞-uppersaturated, and let S be a separable subspace of X. It is not hard to find separable subspaces S_0, S_0 of X such that $S \subset S_1$ and $S_1/S_0 = [q(S)]$. Let Y_∞ be a copy of ℓ_∞ inside Y containing S_0. By the injectivity of ℓ_∞, S is contained in the subspace $Y_\infty \oplus [q(S)]$ of X. And since there exists a copy Z_∞ of ℓ_∞ containing $[q(S)], S$ is therefore contained in the subspace $Y_\infty \oplus\ Z_\infty$ of X, which is isomorphic to ℓ_∞.

A homological characterization of universally separably injective spaces is also possible. We need first to show:

Proposition (3.2.3) [3]. If U is a universally separably injective space then $\mathrm{Ext}\,(\ell_\infty, U) = 0$.

Proof: James's well known distortion theorem for $\ell_1(\text{resp.}\, c_0)$ asserts that a Banach space containing a copy of $\ell_1(\text{resp.}\, c_0)$ also contains an almost isometric copy of $\ell_1(\text{resp.}\, c_0)$. Not so well known is Partington's distortion theorem for ℓ_∞: a Banach space containing ℓ_∞ contains an almost isometric copy of ℓ_∞ (see also Dowling). This last copy will therefore be, say, 2-complemented.

Let Γ denote the set of all the 2-isomorphic copies of ℓ_∞ inside ℓ_∞. For each $E \in \Gamma$ let $\iota_E: E \to \ell_\infty$ be the canonical embedding, p_E a projection onto E of norm at most 2 and $u_E: E \to \ell_\infty$ an isomorphism. Assume that a nontrivial exact sequence

$$0 \to U \to X \to \ell_\infty \to 0$$

exists. We consider, for each $E \in \Gamma$, a copy of the preceding sequence, and form the product of all these copies $0 \to \ell_\infty(\Gamma, U) \to \ell_\infty(\Gamma, X) \to \ell_\infty(\Gamma, \ell_\infty) \to 0$. Let us consider the embedding $J: \ell_\infty \to \ell_\infty(\Gamma, \ell_\infty)$ defined as $J(x)(E) = u_E p_E(x)$ and then form the pull-back sequence

$$\begin{array}{ccccccccc} 0 & \to & \ell_\infty(\Gamma, U) & \to & \ell_\infty(\Gamma, X) & \to & \ell_\infty(\Gamma, \ell_\infty) & \to & 0 \\ & & \| & & \uparrow & & \uparrow J & & \\ 0 & \to & \ell_\infty(\Gamma, U) & \to & \mathrm{PB} & \overset{q}{\to} & \ell_\infty & \to & 0 \end{array}$$

Let us show that q cannot be an isomorphism on a copy of ℓ_∞. Otherwise, it would be an isomorphism on some $E \in \Gamma$ and thus the new pull-back sequence

$$\begin{array}{ccccccccc} 0 & \to & \ell_\infty(\Gamma, U) & \to & \mathsf{PB} & \overset{q}{\to} & \ell_\infty & \to & 0 \\ & & || & & \uparrow & & \uparrow \iota_E & & \\ 0 & \to & \ell_\infty(\Gamma, U) & \to & \mathsf{PB}_E & \to & E & \to & 0 \end{array}$$

would split. And therefore the same would be true making push-out with the canonical projection $\pi_E: \ell_\infty(\Gamma, U) \to U$ onto the E-th copy of U:

$$\begin{array}{ccccccccc} 0 & \to & \ell_\infty(\Gamma, U) & \to & \mathsf{PB}_E & \to & E & \to & 0 \\ & & \pi_E \downarrow & & \downarrow & & || & & \\ 0 & \to & U & \to & \mathsf{PO}_E & \to & E & \to & 0 \end{array}$$

But it is not hard to see that new pull-back with u_E^{-1}

$$\begin{array}{ccccccccc} 0 & \to & U & \to & \mathsf{PO_E} & \to & E & \to & 0 \\ & & || & & \uparrow & & \uparrow u_E^{-1} & & \\ 0 & \to & U & \to & X & \to & \ell_\infty & \to & 0 \end{array}$$

produces exactly the starting sequence which, by assumption, was nontrivial.

However, the space PB should be universally separably injective by Proposition (3.1.6) (iii), hence it must have Rosenthal's property (V), by Proposition (3.1.4) (b). This contradiction shows that the starting nontrivial sequence cannot exist.

We are thus ready to prove:

Theorem (3.2.4) [3] A Banach space U is universally separably injective if and only if for every separable space S one has $\mathrm{Ext}\,(\ell_\infty/S, U) = 0$.

Proof: Let S be separable and let U be universally separably injective. Applying $\mathfrak{L}(-, U)$ to the sequence $0 \to S \to \ell_\infty \to \ell_\infty/S \to 0$ one gets the exact sequence

$$\dots \to \mathfrak{L}(\ell_\infty, U) \to \mathfrak{L}(S, U) \to \mathrm{Ext}(\ell_\infty/S, U) \to \mathrm{Ext}(\ell_\infty, U)$$

Since $\mathrm{Ext}(\ell_\infty, U) = 0$, one obtains that every exact sequence $0 \to U \to X \to \ell_\infty/S \to 0$ fits in a push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & S & \to & \ell_\infty & \to & \ell_\infty/S & \to & 0 \\ & & \downarrow & & \downarrow & & || & & \\ 0 & \to & U & \to & X & \to & \ell_\infty/S & \to & 0. \end{array}$$

Since U is universally separably injective, the lower sequence splits.

The converse is clear: every operator $t: S \to U$ from a separable Banach space into a space U produces a push-out diagram

$$\begin{array}{ccccccccc} 0 & \to & S & \to & \ell_\infty & \to & \ell_\infty/S & \to & 0 \\ & & t \downarrow & & \downarrow & & || & & \\ 0 & \to & U & \to & \mathsf{PO} & \to & \ell_\infty/S & \to & 0. \end{array}$$

The lower sequence splits by the assumption $\mathrm{Ext}(\ell_\infty/S, U) = 0$ and so t extends to ℓ_∞, according to the splitting criterion for push-out sequences.

Which leads to the unexpected:

Corollary (3.2.5) [3] $\mathrm{Ext}(\ell_\infty/c_0, \ell_\infty/c_0) = 0$; i.e., every short exact sequence $0 \to \ell_\infty/c_0 \to X \to \ell_\infty/c_0 \to 0$ splits.

This result provides a new solution for equation $\mathrm{Ext}(X, X) = 0$. The other three previously known types of solutions are: c_0 (by Sobczyk theorem), the injective spaces (by the very definition) and the $L_1(\mu)$-spaces (by Lindenstrauss' lifting).

Also with Proposition (3.2.2), one has:

Corollary (3.2.6) [3] Rosenthal's property (V) is not a 3-space property

Proof: With the same construction as above, start with a nontrivial exact sequence $0 \to \ell_2 \to E \to \ell_\infty \to 0$ (see [3]) and construct an exact sequence

$$0 \to \ell_\infty(\Gamma, \ell_2) \to X \overset{q}{\to} \ell_\infty \to 0,$$

where q cannot be an isomorphism on a copy of ℓ_∞. So X has not Rosenthal's property (V). The space $\ell_\infty(\Gamma, \ell_2)$ has Rosenthal's property (V) as a quotient of $\ell_\infty(\Gamma, \ell_\infty) = \ell_\infty(\mathbb{N} \times \Gamma)$, since the property obviously passes to quotients.

It is not however true that $\mathrm{Ext}(U, V) = 0$ for all universally separably injective spaces U and V as any exact sequence $0 \to U \to \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/U \to 0$ in which U is a universally separably injective non-injective space shows.

We establishe a major difference between 1-separably injective and general separably injective spaces: 1-separably injective spaces must be Grothendieck (hence they cannot be separable or WCG) while a 2-separably injective space, such as c_0, can be even separable. The following lemma due to Lindenstrauss provides a quite useful technique.

Lemma (3.2.7) [3] Let E be a 1-separably injective space and Y a separable subspace of X, with $\mathrm{dens}\, X = \aleph_1$. Then every operator $t: Y \to E$ can be extended to an operator $T: X \to E$ with the same norm.

This yields

Proposition (3.2.8) [3] Under CH every 1-separably injective Banach space is universally 1-separably injective and therefore a Grothendieck space.

Proof: Let E be 1-separably injective, X an arbitrary Banach space and $t: Y \to E$ an operator, where Y is a separable subspace of X. Let $[t(Y)]$ be the closure of the image of t. This is a separable subspace of E and so there is an isometric embedding $u: [t(Y)] \to \ell_\infty$. As ℓ_∞ is 1-injective there is an operator $T: X \to \ell_\infty$ whose restriction to Y agrees with ut. Thus it suffices to extend the inclusion of $[t(Y)]$ into E to ℓ_∞. But, under CH, the density character of ℓ_∞ is $\aleph_1$ and the preceding Lemma applies. The 'therefore' part is now a consequence of Proposition (3.1.4) (b).

The "therefore" part survives in ZFC:

Theorem (3.2.9) [3] Every 1-separably injective space is a Grothendieck and a Lindenstrauss space.

Proof: The proof of Proposition (3.1.4) yields that 1-separably injective spaces are of type $\mathcal{L}_{\infty,1^+}$, that is, Lindenstrauss spaces. It remains to prove that a 1-separably injective space E must be Grothendieck. It suffices to show that c_0 is not complemented in E, so let $\mathcal{J}: c_0 \to E$ be an embedding. Consider an almost-disjoint family $\mathcal{M}$ of size $\aleph_1$ formed by infinite subsets of $\mathbb{N}$ and construct the associated Johnson-Lindenstrauss twisted sum space

$$0 \to c_0 \to C(\Delta_{\mathcal{M}}) \to c_0(\aleph_1) \to 0.$$

Observe that the space $C(\Delta_{\mathcal{M}})$ has density character $\aleph_1$, we have therefore a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & c_0 & \to & C(\Delta_{\mathcal{M}}) & \to & c_0(\aleph_1) & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & c_0 & \overset{\mathcal{J}}{\to} & E & \to & E/\mathcal{J}(c_0) & \to & 0. \end{array}$$

If c_0 was complemented in E then it would be complemented in $C(\Delta_{\mathcal{M}})$ as well, which is not.

Proposition (3.2.8) leads to the question about the necessity of the hypothesis CH. We will prove now that it cannot be dropped.

Lemma (3.2.10) [3] Let K, L, M be compact spaces and let $f: K \to M$ be a continuous map, with $\mathcal{I} = f^\circ: C(M) \to C(K)$ its induced operator, and let $\iota: C(M) \to C(L)$ be a positive norm one operator. Suppose that $S: C(L) \to C(K)$ is an operator with $\|S\| = 1$ and $S\iota = \mathcal{J}$. Then S is a positive operator.

Proof: Obviously $S \geq 0$ if and only if $S^*\delta_x \geq 0$ for all $x \in K$, where δ_x is the unit mass at x and $S^*: C(K)^* \to C(L)^*$ is the adjoint operator. Fix $x \in K$. By Riesz theorem we have that $S^*\delta_x = \mu$ is a measure of total variation $\|\mu\| \leq 1$. Let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition of μ, so that $\|\mu\| = \|\mu^+\| + \|\mu^-\|$, with $\mu^+, \mu^- \geq 0$. We have that $\delta_{f(x)} = \mathcal{J}^*\delta_x = \iota^* S^* \delta_x = \iota^*\mu$, thus

$$\delta_{f(x)} = \iota^*\mu^+ - \iota^*\mu^- \quad \text{and} \quad \|\delta_{f(x)}\| = \|\iota^*\mu^+\| - \|\iota^*\mu^-\|.$$

Since ι is a positive operator these imply that the above is the Hahn-Jordan decomposition of $\delta_{f(x)}$ and so $\iota^*\mu^- = 0$, hence $\mu^- = 0$.

Definition (3.2.11) [3] Let L be a zero-dimensional compact space. An $\aleph_2$-Lusin family on L is a family $\mathcal{F}$ of pairwise disjoint nonempty clopen subsets of L with $|\mathcal{F}| = \aleph_2$, such that whenever $\mathcal{G}$ and $\mathcal{H}$ are subfamilies of $\mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$, then

$$\overline{\bigcup\{G \in \mathcal{G}\}} \cap \overline{\bigcup\{G \in \mathcal{H}\}} \neq \phi.$$

The following lemma shows the consistency of the existence of an $\aleph_2$-Lusin family on $\mathbb{N}^*$.

Lemma (3.2.12) [3] Under MA and the assumption $\mathfrak{c} = \aleph_2$ there exists an $\aleph_2$-Lusin family on $\mathbb{N}^*$.

Proof: By Stone duality, since the Boolean algebra associated to $\mathbb{N}^*$ is $\wp(\mathbb{N})/\mathrm{fin}$, an $\aleph_2$-Lusin family on $\mathbb{N}^*$ is all the same as an almost disjoint family $\{A_\alpha\}_{\alpha<\omega_2}$ of infinite subsets of $\mathbb{N}$ such that for every $B \subset \mathbb{N}$ either $\{\alpha: |A_\alpha / B| \text{ is finite}\}$ or $\{\alpha: |A_\alpha \cap B| \text{ is finite}\}$ has cardinality $< \aleph_2$. Let $\{B_\alpha: \alpha < \omega_2\}$ be an enumeration of all infinite subsets of $\mathbb{N}$. We construct the sets A_α inductively on α. Suppose A_γ has been constructed for $\gamma < \alpha$. We define a forcing notion $\mathbb{P}$ whose conditions are pairs $p = (f_p, F_p)$ where f_p is a $\{0,1\}$-valued function on a finite subset $\mathrm{dom}\,(f_p)$ of $\mathbb{N}$ and F_p is a finite subset of α. The order relation is that $p < q$ if f_p extends $f_q, F_p \supset F_q$ and f_p vanishes in $A_\gamma \setminus \mathrm{dom}\,(f_q)$ for $\gamma \in F_q$. One checks that this forcing is ccc. Hence, by MA, using a big enough generic filter the forcing provides an infinite set $A_\alpha \subset \mathbb{N}$ such that, for all $\gamma < \alpha$,

(i) $A_\alpha \cap A_\gamma$ is finite, and

(ii) If B_γ is not contained in any finite union of A_δ's, then $A_\alpha \cap B_\gamma$ is infinite.

Theorem (3.2.13) [3] It is consistent that there exists a compact space K for which the Banach space $C(K)$ is 1-separably injective but not universally 1-separably injective.

Proof: We will suppose that $\mathfrak{c} = \aleph_2$ and that there exists an $\aleph_2$-Lusin family in $\mathbb{N}^*$. Under these hypotheses, let K be the Stone dual compact space of the Cohen-Parovičenko Boolean algebra. The definition of that

Boolean algebra implies that K is an F-space and thus $C(K)$ is 1-separably injective by Theorem (3.1.8). We show that it is not universally 1-separably injective. The argument follows the scheme, where they prove that K does not map onto $\beta\mathbb{N}$, but we use $\aleph_2$-Lusin families instead of ω_2-chains because they fit better in the functional analytic.

Let $\{U_n : n \in \mathbb{N}\}$ be a sequence of pairwise disjoint clopen subsets of K, and let $U = \bigcup_n U_n$. Let $c \subset \ell_\infty$ be the Banach space of convergent sequences, and $t: c \to C(K)$ be the operator given by $t(z)(x) z_n$ if $x \in U_n$ and $t(z)(x) = \lim_{n\to\infty} z_n$ if $x \neq U$.

If $C(K)$ were universally 1-separably injective, we should have an extension $T: \ell_\infty \to C(K)$ of t with $\|T\| = 1$. We shall derive a contradiction from the existence of such operator.

Notice that the conditions of Lemma (3.2.10) are applied, so T is positive (observe that $c = C(\mathbb{N} \cup \{\infty\})$ and $T = f^\circ$ where $f: K \to \mathbb{N} \cup \{\infty\}$ is given by $f(x) = n$ if $x \in U_n$ and $f(x) = \infty$ if $x \notin U$).

For every $A \subset \mathbb{N}$ we will denote $[A] = A^{\beta\mathbb{N}} \backslash \mathbb{N}$. The clopen subsets of $\mathbb{N}^*$ are exactly the sets of the form $[A]$, and we have that $[A] = [B]$ if and only if $(A \backslash B) \cup (B \backslash A)$ is finite.

Let $\mathcal{F}$ be an $\aleph_2$-Lusin family in $\mathbb{N}^*$. For $F = [A] \in \mathcal{F}$ and $0 < \varepsilon < \frac{1}{2}$, let

$$F_\varepsilon = \{x \in K \backslash U : T(1_A)(x) > 1 - \varepsilon\}.$$

Let us remark that F_ε depends only on F and not on the choice of A. This is because if $[A] = [B]$, then $1_A - 1_B \in c_0$, hence $T(1_A - 1_B) = t(1_A - 1_B)$ which vanishes out of U, so $T(1_A)|_{K\backslash U} = T\, 1_B|_{K\backslash U}$.

Claim 1. If $\delta < \varepsilon$ and $F \in \mathcal{F}$, then $\overline{F_\delta} \subset F_\varepsilon$.

Claim 2. $F_\varepsilon \cap G_\varepsilon = \phi$ for every $F \neq G$.

Proof of Claim 2. Since $F \cap G = \emptyset$ we can choose $A, B \subset \mathbb{N}$ such that $F = [A], G = [B]$ and $A \cap B = \emptyset$. If $x \in F_\varepsilon \cap G_\varepsilon, \bar{T}(1_A + 1_B)(x) > 2 - 2\varepsilon > 1$ which is a contradiction because $1_A + 1_B = 1_{A\cup B}$ and $\|\bar{T}(1_{A\cup B})\| \leq \|\bar{T}\| \|\bar{T}(1_{A\cup B})\| = 1$. End of the Proof of Claim 2.

For every $F \in \mathcal{F}$, let $\bar{F}$ be a clopen subset of $K \backslash U$ such that $\bar{F}_{0.2} \subset \bar{F} \subset F_{0.3}$. By the preceding claims, this is a disjoint family of clopen sets. It follows that $K \backslash U$ does not contain any $\aleph_2$-Lusin family. Therefore we can find $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$ such that

$$\overline{\bigcup\{\bar{G} : G \in \mathcal{G}\}} \cap \overline{\bigcup\{\bar{H} : G \in \mathcal{H}\}} = \phi.$$

Now, for every $n \in \mathbb{N}$ choose a point $p_n \in U_n$. Let $g: \beta\mathbb{N} \to K$ be a continuous function such that $g(n) = p_n$.

Claim 3. For $u \in \beta\mathbb{N}, A \subset \mathbb{N}, T(1_A)(g(u)) = \begin{cases} 1, & \text{if } u \in [A]; \\ 0, & \text{if } u \notin [A]. \end{cases}$

Proof of Claim 3. It is enough to check it for $u = n \in \mathbb{N}$. This is a consequence of the fact that T is positive, because if $m \in A, n \notin A$, then $0 \leq t(1_m) \leq T(1_A) \leq t(1_{\mathbb{N}\{n\}}) \leq 1$. End of the Proof of Claim 3.

The function g is one-to-one because

$$\overline{\{p_n : n \in A\}} \cap \overline{\{p_n : n \notin A\}} = \phi$$

for every $A \subset \mathbb{N}$, as the function $T(1_A)$ separates these sets. On the other hand, as a consequence of Claim 3 above, for every $F \in \mathcal{F}$ and every $\varepsilon, g^{-1}(F_\varepsilon) = F$, and also $g^{-1}(\tilde{F}) = F$. These facts make the families $\mathcal{H}$ and $\mathcal{G}$ above to contradict that $\mathcal{F}$ is an $\aleph_2$-Lusin family in $\mathbb{N}^*$.

Chapter 4
Complex Vector Lattices

We motivates the definition of Complexification of Archimedean vector lattices, the Fremlin tensor product of Archimedean complex-vector lattices, and a theory of powers of Archimedean complex vector lattices.

Section (4.1): Vector Lattices Complexifications

A vector space (V) equipped with a partial order "$\leq$" is called a vector lattices if for each pair x, y in V: (i) There is smallest element w (denoted by $x \wedge w$) for which $x \leq z$ and $y \leq z$, (ii) There is a largest element w (dented by $x \wedge w$) for which $x \leq w$ and $y \leq w$, (iii) if $x \leq y$ then $x + y \leq y + z$ for all $z \in V$, (iv) If $x \leq y$ and $c \in \mathbb{R}^{+}$ then $cx \leq cy$, the element $|x| \coloneqq x \vee (-x)$ is called the Modulus of X. The element $X^{+} \coloneqq X \vee 0$ is called the positive part of X. The $X^{-} \coloneqq (-X) \vee 0$ is called the negative part of X. If a vector lattice V is equipped with a norm $\|\cdot\|$ for which, (v) for all $x, y \in V$, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$ then V (equipped with $\leq$ and $\|\cdot\|$) is called a normed vector lattice [8].

We discuss the specific case that we will use to complexify Archimedean vector lattices over $\mathbb{R}$ and multilinear maps over $\mathbb{R}$. Using the notation, let $\mu_{2,4}(x, y) = \sqrt{\frac{|x|^2+|y|^2}{2}}$ $(x, y \in \mathbb{R})$. If E is an Archimedean vector lattice over $\mathbb{R}$ and $f, g \in E$ then $\mu_{2,4}(f, g) = \frac{1}{\sqrt{2}}(f \boxplus g)$, where $f \boxplus g := \sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\}$, as defined. Therefore, from Mittelmeyer and Wolff's Theorem, a vector space $E + iE$ over $\mathbb{C}$ is an Archimedean vector lattice over $\mathbb{C}$ if and only if E is a $\mu_{2,4}$-complete Archimedean vector lattice over $\mathbb{R}$. We refer to the $\mu_{2,4}$-completion $(E^{\mu_{2,4}}, \phi)$ of E as the square mean completion of E. Noting that $\mu_{2,4}$ is absolutely invariant. We summarize the newly found information regarding functional completions for this special case in the following corollary of.

Corollary (4.1.1) [4] If E is an Archimedean vector lattice over $\mathbb{R}$ then there exists a unique square mean completion $(E^{\mu_{2,4}}, \phi)$ of E. Moreover, if $E_1, \dots, E_s, F$ are Archimedean vector lattices over $\mathbb{R}$ with square mean completions $\left(E_k^{\mu_{2,4}}, \phi_k\right)(k \in \{1, \dots, s\})$ and F is square mean complete, then for every vector lattice s-morphism $T: \times s_{k=1}^{s} E_k \to F$ there exists a unique vector lattice s-morphism $T^{\mu_{2,4}}: \times_{k=1}^{s} E_k^{\mu_{2,4}} \to F$ such that $T^{\mu_{2,4}}(\phi_1(f_1), \dots, \phi_s(f_s)) = T(f_1, \dots, f_s)$. Furthermore, if F is uniformly complete and $T: \times_{k=1}^{s} E_k \to F$ is a positive s-linear map then there exists a unique positive s-linear map $T^{\mu_{2,4}}: \times_{k=1}^{s} E_k^{\mu_{2,4}} \to F$ such that $T^{\mu_{2,4}}(\phi_1(f_1), \dots, \phi_s(f_s)) = T(f_1, \dots, f_s)$ for every $f_k \in E_k (k \in \{1, \dots, s\})$. Here ϕ_k is the natural embedding of E_k into E_k^u.

We now turn to complexifications of Archimedean vector lattices over $\mathbb{R}$.

Definition (4.1.2) [4] For an Archimedean vector lattice E over $\mathbb{R}$ we define a pair $(E_{|\mathbb{C}|}, \phi)$ to be a vector lattice complexification of E if the following hold.

(i) $E_{|\mathbb{C}|}$ is an Archimedean vector lattice over $\mathbb{C}$.

(ii) $\phi: E \to (E_{|\mathbb{C}|})_\rho$ is an injective vector lattice $\mathbb{R}$-homomorphism.

(iii) For every Archimedean vector lattice Fover $\mathbb{C}$ as well as for every vector lattice $\mathbb{R}$- homomorphism $T: E \to F_\rho$, there exists a unique vector lattice $\mathbb{C}$-homomorphism

$$T_{|\mathbb{C}|}: E_{|\mathbb{C}|} \to F \text{ such that } T_{|\mathbb{C}|} \circ \phi = T.$$

We next prove the existence and uniqueness of vector lattice complexifications.

Theorem (4.1.3) [4] If E is an Archimedean vector lattice over $\mathbb{R}$ then there exists a vector lattice complexification of E, unique up to vector lattice isomorphism.

Proof: Let E be an Archimedean vector lattice over $\mathbb{R}$. By Corollary (4.1.1), there exists a unique square mean completion $(E^{\mu_{2,4}}, \phi)$ of E. Define $E_{|\mathbb{C}|} \coloneqq (E^{\mu_{2,4}})_{\mathbb{C}}$ and observe that $E_{|\mathbb{C}|}$ is an Archimedean vector lattice over $\mathbb{C}$ and that $(E_{|\mathbb{C}|})_\rho = E^{\mu_{2,4}}$. Next, let F be an Archimedean vector lattice over $\mathbb{C}$ and let $T: E \to F_\rho$ be a vector lattice $\mathbb{R}$-homomorphism. Since F_ρ is square mean complete, there exists a unique vector lattice $\mathbb{R}$-homomorphism $T^{\mu_{2,4}}: E^{\mu_{2,4}} \to F_\rho$ such that $T^{\mu_{2,4}} \circ \phi = T$. Define $T_{|\mathbb{C}|}: E_{|\mathbb{C}|} \to F$ by $T_{|\mathbb{C}|}(f + ig) = T^{\mu_{2,4}}(f) + iT^{\mu_{2,4}}(g)$ for every $f + ig \in E_{|\mathbb{C}|}$. Then $T_{|\mathbb{C}|} \circ \phi = T$. Moreover, for $f + ig \in E_{|\mathbb{C}|}$ we have from (see [4]) that

$$T_{|\mathbb{C}|}(|f + ig|) = T^{\mu_{2,4}}(f \boxplus g) = T^{\mu_{2,4}}(f) \boxplus T^{\mu_{2,4}}(g) = \left|T_{|\mathbb{C}|}(f + ig)\right|.$$

Thus $T_{|\mathbb{C}|}$ is a vector lattice C-homomorphism and therefore $(E_{|\mathbb{C}|}, \phi)$ is a vector lattice complexification of E. Next, we prove the uniqueness. To this end, suppose $(E_{1|\mathbb{C}|}, \phi_1)$ and $(E_{2|\mathbb{C}|}, \phi_2)$ are vector lattice complexifications of E. Then $\left(\left(E_{1|\mathbb{C}|}\right)_\rho, \phi_1\right)$ and $\left(\left(E_{2|\mathbb{C}|}\right)_\rho, \phi_2\right)$ are square mean completions of E, and hence there exists a vector lattice isomorphism $\gamma: \left(E_{1|\mathbb{C}|}\right)_\rho \to \left(E_{2|\mathbb{C}|}\right)_\rho$. Similar to $T_{|\mathbb{C}|}$ above, the map $\gamma_{\mathbb{C}}: E_{1|\mathbb{C}|} \to E_{2|\mathbb{C}|}$ defined by $\gamma_{\mathbb{C}}(f + ig) = \gamma(f) + i\gamma(g)$ is a vector lattice $\mathbb{C}$-homomorphism. The bijectivity of $\gamma_{\mathbb{C}}$ is evident.

For the square mean completion $(E^{\mu_{2,4}}, \phi)$ of E, we will from now on identify E with $\phi(E)$. Using this identification, we complexify positive $s_{\mathbb{R}}$-linear maps (respectively, vector lattice $s_{\mathbb{R}}$-morphisms) to positive $s_{\mathbb{C}}$-linear maps (respectively, vector lattice $s_{\mathbb{C}}$-morphisms) as follows. Let $E_1, \dots, E_s, F$ be Archimedean vector lattices over $\mathbb{R}$ with F square mean complete, and let $T: \times_{k=1}^{s} E_k \to F$ be a vector lattice $s_{\mathbb{R}}$-morphism.

For $(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) \in \times_{k=1}^{s} E_{k|\mathbb{C}|}$, define $T_{|\mathbb{C}|}: \times_{k=1}^{s} E_{k|\mathbb{C}|} \to F_{\mathbb{C}}$ by

$$T_{|\mathbb{C}|}(f_0^1 + if_1^1, \dots, f_0^s + if_1^s) := \sum_{\epsilon_k \in \{0,1\}} T^{\mu_{2,4}}\left(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s\right) i^{\sum_{k=1}^{s} \epsilon_k}.$$

If F is uniformly complete and T above is any positive $s_{\mathbb{R}}$-linear map, we define $T_{|\mathbb{C}|}$ in a similar manner. We collect a few facts regarding this complexification in the following proposition. Statement (iii) and the statement that $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{|\mathbb{C}|}$ in (i) and (ii) are evident. The proof of (ii) follows from Corollary (4.1.1), and the proof of (i) is similar to the complexification of vector lattice homo-morphisms seen in the proof of Theorem (4.1.3).

Proposition (4.1.4) [4] Let $E_1, \dots, E_s, F$ be Archimedean vector lattices over $\mathbb{R}$ with F square mean complete.

(i) If a map $T: \times_{k=1}^{s} E_k \to F$ is a vector lattice $s_{\mathbb{R}}$-morphism then $T_{|\mathbb{C}|}$ is a vector lattice $s_{\mathbb{C}}$-morphism and $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{|\mathbb{C}|}$.

(ii) If F is uniformly complete and $T: \times_{k=1}^{s} E_k \to F$ is a positive $s_{\mathbb{R}}$-linear map then $T_{|\mathbb{C}|}$is a positive $s_{\mathbb{C}}$-linear map and $T_{|\mathbb{C}|} = (T^{\mu_{2,4}})_{|\mathbb{C}|}$.

(iii) If in (i) or (ii) all $E_1, \dots, E_s$ are square mean complete then $T_{|\mathbb{C}|} = T_{\mathbb{C}}$.

Section (4.2): The Archimedean Vector Lattice Tensor Product

We define the tensor product of Archimedean vector lattices over $\mathbb{K}$ and show the existence of the Archimedean complex tensor product by complexifying the Fremlin tensor product of Archimedean real vector lattices.

We start with the definition of these tensor products of Archimedean vector lattices over $\mathbb{K}$. For $s = 2$ and $\mathbb{K} = \mathbb{R}$ the definition coincides with Fremlin's definition of the Archimedean tensor product of Archimedean vector lattices over $\mathbb{R}$.

Definition (4.2.1) [4] Given Archimedean vector lattices $E_1, \dots, E_s$ over $\mathbb{K}$, we define a pair $\left(\overline{\otimes}_{k=1}^{s} E_k, \overline{\otimes}\right)$ to be an Archimedean vector lattice tensor product of $E_1, \dots, E_s$ if the following hold.

(i) $\overline{\otimes}_{k=1}^{s} E_k$ is an Archimedean vector lattice over $\mathbb{K}$.

(ii) $\overline{\otimes}$ is a vector lattice s-morphism.

(iii) For every Archimedean vector lattice F over $\mathbb{K}$ and for every vector lattice s- morphism $T: \times_{k=1}^{s} E_k \to F$, there exists a uniquely determined vector lattice homo- morphism $T^{\overline{\otimes}}: \overline{\otimes}_{k=1}^{s} E_k \to F$ such that $T^{\overline{\otimes}} \circ \overline{\otimes} = T$.

Below and throughout the rest of this section, $(\otimes_{k=1}^{s} V_k, \otimes)$ denotes the algebraic tensor product of vector spaces $V_1, \dots, V_s$ over $\mathbb{K}$.

Lemma (4.2.2) [4] Let $E_1, \dots, E_s$ be Archimedean vector lattices over $\mathbb{R}$.

(i) There exists an essentially unique Archimedean vector lattice $\overline{\otimes}_{k=1}^{s} E_k$ over $\mathbb{R}$ and a vector lattice s-morphism $\overline{\otimes}: \times_{k=1}^{s} E_k \to \overline{\otimes}_{k=1}^{s} E_k$ such that for every Archimedean vector lattice F over $\mathbb{R}$ and every vector lattice s-morphism $T: \times_{k=1}^{s} E_k \to F$, there exists a unique vector lattice homomorphism $T^{\overline{\otimes}}: \overline{\otimes}_{k=1}^{s} E_k \to F$ such that $T^{\overline{\otimes}} \circ \overline{\otimes} = T$.

(ii) There exists an injective linear map $S: \otimes_{k=1}^{s} E_k \to \overline{\otimes}_{k=1}^{s} E_k$ such that $S \circ \otimes = \overline{\otimes}$.

(iii) For every $w \in \overline{\otimes}_{k=1}^{s} E_k$, there exist $x_k \in E_k^{+}$ $(k \in \{1 \dots, s\})$ such that for every $\epsilon > 0$, there exists $v \in \otimes_{k=1}^{s} E_k$ such that $|w - v| \leq \epsilon(x_1 \otimes \dots \otimes x_s)$, i.e. $\otimes_{k=1}^{s} E_k$ is relatively uniformly dense in $\overline{\otimes}_{k=1}^{s} E_k$.

(iv) For every $0 < w \in \overline{\bigotimes}_{k=1}^{s} E_k$ there exist $x_k \in E_k^+$ $(k \in \{1 \dots, s\})$ such that $0 < (x_1 \otimes \dots \otimes x_s) \leq w$, i.e. $\bigotimes_{k=1}^{s} E_k$ is order dense in $\overline{\bigotimes}_{k=1}^{s} E_k$.

We deal with the existence and uniqueness of the complex Archimedean vector lattice tensor product and requires several prerequisite results. The next lemma surely is known.

Lemma (4.2.3) [4] If $V_1, \dots, V_s$ are vector spaces over $\mathbb{R}$ then $\bigotimes_{k=1}^{s} \left(V_{k^{\mathbb{C}}}\right)$ and $(\bigotimes_{k=1}^{s} V_k)_{\mathbb{C}}$ are isomorphic as vector spaces over $\mathbb{C}$.

Proof: Since the algebraic tensor product is associative, we only need to prove the result for $s = 2$, and use induction. The case $s = 2$ is the content of Theorem (see [4]), but, we provide a sketch of van Zyl's proof to correct some potential confusion caused by an accumulation of minor misprints. First let U and V be vector spaces over $\mathbb{R}$, and let $(U \otimes V, \otimes)$ and $(U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}})$ be the algebraic tensor products of U, V, respectively $U_{\mathbb{C}} V_{\mathbb{C}}$. Since $\otimes_{\mathbb{C}}: U_{\mathbb{C}} \times V_{\mathbb{C}} \to (U \otimes V)_{\mathbb{C}}$ is a bilinear map over $\mathbb{C}$, it induces a unique $\mathbb{C}$-linear map $T: U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}} \to (U \otimes V)_{\mathbb{C}}$. It is easy to see that T is surjective. To show that T is injective, let $w = \sum_{k=1}^{n} (u_k + iu'_k) \otimes_1 (v_k + iv'_k) \in U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}}$ and suppose that $T(w) = 0$. Note that $T(w) = \sum_{k=1}^{n} (u_k \otimes v_k - u'_k \otimes v'_k + iu'_k \otimes v_k + iu_k \otimes u'_k)$, and so for any $\mathbb{R}$-linear functionals ϕ on U and ψ on V we have

$$\sum_{k=1}^{n} \left(\phi\,(u_k)\psi(v_k) - \phi(u'_k)\psi(v'_k)\right)$$

$$= 0 \text{ and } \sum_{k=1}^{n} \left(\phi\,(u'_k)\psi(v_k) + \phi(u_k)\psi(v'_k)\right) = 0 \quad (*)$$

Let $\xi = \xi_r + i\xi_c$ be a $\mathbb{C}$-linear functional on $U_{\mathbb{C}}$ and let $\eta = \eta_r + i\eta_c$ be a $\mathbb{C}$-linear functional on $V_{\mathbb{C}}$, both written in their natural decompositions. Then ξ_r, ξ_c are $\mathbb{R}$-linear functional on U and η_r, η_c are $\mathbb{R}$-linear functionals on V. Now

$$\sum_{k=1}^{n}\xi(u_k+iu_k')\eta(v_k+iv_k')$$

$$\begin{aligned}
&=\sum_{k=1}^{n}\left(\xi_r(u)\eta_r(v_k)-\xi_r(u_k')\eta_r(v_k')\right)\\
&-\sum_{k=1}^{n}\left(\xi_r(u_k')\eta_c(v_k)+\xi_r(u_k)\eta_c(v_k')\right)\\
&+i\sum_{k=1}^{n}\left(\xi_r(u_k')\eta_r(v_k)+\xi_r(u_k)\eta_r(v_k')\right)\\
&+i\sum_{k=1}^{n}\left(\xi_r(u_k)\eta_c(v_k)-\xi_r(u_k')\eta_c(v_k')\right)\\
&-\sum_{k=1}^{n}\left(\xi_c(u_k')\eta_r(v_k)+\xi_c(u_k)\eta_r(v_k')\right)\\
&-\sum_{k=1}^{n}\left(\xi_c(u_k)\eta_c(v_k)-\xi_c(u_k')\eta_c(v_k')\right)\\
&+i\sum_{k=1}^{n}\left(\xi_c(u_k)\eta_r(v_k)-\xi_c(u_k')\eta_r(v_k')\right)\\
&-i\sum_{k=1}^{n}\left(\xi_c(u_k')\eta_c(v_k)+\xi_c(u_k)\eta_c(v_k')\right).
\end{aligned}$$

Applying $(*)$ again to each of these eight summands, we have that $\sum_{k=1}^{n}\xi(u_k+iu_k')\eta(v_k+iv_k')=0$. Therefore $w=0$ and T is injective. Then T is a vector space isomorphism.

In light of the previous lemma, we will from now identify $\left(\bigotimes_{k=1}^{s}V_{k\mathbb{C}}\right)_\rho$ with $\bigotimes_{k=1}^{s}V_k$ for vector spaces $V_1,\dots,V_s$ over $\mathbb{R}$.

There exists a simpler construction of the square mean completion than the construction preceding Proposition (see [4]), which was given in a more general setting. Indeed, Azouzi constructs a square mean completion of an Archimedean vector lattice E over $\mathbb{R}$ essentially as follows. Let $E_1:=E$ and for every $n\in\mathbb{N}$, define $E_{n+1}:=E_n\cup\left[\{\mu_{2,4}(f,g):f,g\in E_n\}\right]$, where $\left[\{\mu_{2,4}(f,g):f,g\in E_n\}\right]$ denotes the vector subspace of E^δ generated by $\{\mu_{2,4}(f,g):f,g\in E_n\}$. Then define $E^{\boxplus}:=\bigcup_{n\in\mathbb{N}}E_n$. To see that $E^{\boxplus}$ is a vector lattice, note that for every $f\in$

$E^{\boxplus}$ there exists $n \in \mathbb{N}$ such that $f \in E_n$. Then $|f| = \sqrt{2}\mu_{2,4}(f,0) \in E_{n+1}$. It follows that $E^{\boxplus}$ is the square mean completion of E, that is, $E^{\boxplus}$ and $E^{\mu_{2,4}}$ are isomorphic as vector lattices. In fact, from the identity $\lambda\mu_{2,4}(f,g) = \mu_{2,4}(\lambda f, \lambda g)$ for every $\lambda \in \mathbb{R}^+$ and every $f, g \in E$, we have $E_{n+1}^+ = \left\{\sum_{k=1}^{m} \mu_{2,4}(f_k, g_k): f_k, g_k \in E_n\right\}$. We use this fact in the first of the two following lemmas that are needed for Proposition (4.2.6).

Lemma (4.2.4) [4] Denote the standard sine and cosine functions on $\left[0, \frac{\pi}{2}\right]$ by $\sin$ and $\cos$, respectively. For an Archimedean vector lattice E over $\mathbb{R}$ and for every $f \in (E^{\mu_{2,4}})^+$ there exists $u_1, \dots, u_n \in E^+$ and $t_{k,1}, \dots, t_{k,p_k} \in \{\cos, \sin\}$ $(k \in \{1, \dots, n\})$ such that

$$f = \sup_{\theta_k \in \left[0, \frac{\pi}{2}\right]} \left\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_k) u_k \right\}.$$

Proof: Our proof is via mathematical induction. Let $h \in E_{n+1}^+$ and first suppose that $f = \mu_{2,4}(u, v)$ for some $u, v \in E^E$. Then $f = \sup\{u \cos\theta + v \sin\theta : \theta \in [0, \pi/2]\}$. Next, suppose that $f = \sum_{k=1}^{n} \mu_{2,4}(u_k, v_k)$. Then

$$f = \sum_{k=1}^{n} \sup_{\theta_k \in \left[0, \frac{\pi}{2}\right]} \{u_k \cos\theta_k + v_k \sin\theta_k\}$$

$$= \sup_{\theta_k \in \left[0, \frac{\pi}{2}\right]} \left\{ \sum_{k=1}^{n} (u_k \cos\theta_k + v_k \sin\theta_k) \right\}..$$

This completes the base step of the induction argument. For the inductive step, suppose that for every $f \in E_n^+$ there exists $u_1, \dots, u_n \in E^+$ and $t_1, \dots, t_{p_k} \in \{\cos, \sin\} (k \in \{1, \dots, n\})$ such that

$$f = \sup_{\theta_{k,j} \in \left[0, \frac{\pi}{2}\right]} \left\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\}.$$

Let $f \in E_{n+1}^+$. From the argument in the base step above, we may assume that $f = \mu_{2,4}(u, v)$ for some $u, v \in E_n^+$.

By the induction hypothesis there exists $u_1, \dots, u_n, v_1, \dots, v_n \in E^+$ and $t_{k,1}, \dots, t_{k,p_k}, s_{k,1}, \dots, s_{k,r_k} \in \{\cos, \sin\} (k \in \{1, \dots, s\})$ such that

$$u = \sup_{\theta_{k,j} \in \left[0, \frac{\pi}{2}\right]} \left\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \right\} \quad \text{and}$$

$$v = \sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{m}\prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j})v_k\right\}.$$

Then

$$f$$
$$= \mu_{2,4}\left(\sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k\right\}, \sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{m}\prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j})v_k\right\}\right)$$
$$= \sup_{\phi\in\left[0,\frac{\pi}{2}\right]} \left\{\sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k\right\}\cos\phi\right.$$
$$\left. + \sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{m}\prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j})v_k\right\}\sin\phi\right\}$$
$$= \sup_{\phi,\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]} \left\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})\cos\phi\, u_k + \sum_{k=1}^{m}\prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j})\sin\phi\, v_k\right\}.$$

The next lemma can be verified using mathematical induction. We do not include the Proof:

Lemma (4.2.5) [4] Let $t_1,\dots,t_n$ be Lipschitz functions on $\mathbb{R}$ with Lipschitz constant 1. Also assume that $|t_k(x)|\le 1$ for every $k\in\{1,\dots,n\}$ and every $x\in\mathbb{R}$. Then for every $x_k,y_k\in\mathbb{R}(k\in\{1,\dots,n\})$ we have $\left|\prod_{k=1}^{n} t_k(x_k)-\prod_{k=1}^{n} t_k(y_k)\right|\le\sum_{k=1}^{n}|x_k-y_k|$.

We have the following proposition.

Proposition (4.2.6) [4] If E is an Archimedean vector lattice over $\mathbb{R}$ then E is relatively uniformly dense in $E^{\mu_{2,4}}$.

Proof: Let E be an Archimedean vector lattice over $\mathbb{R}$ and first suppose that $f\in(E^{\mu_{2,4}})^+$. Say that $f = \sup_{\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]}\left\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k\right\}$ for some $u_1,\dots,u_n\in E^+$ and $t_{k,1},\dots,t_{k,p_k}\in\{\cos,\sin\}(k\in\{1,\dots,n\})$. Note that given $\theta_{k,j}\in\left[0,\frac{\pi}{2}\right]$ $m\in\mathbb{N}$ there exist $l_{k,j}\in\mathbb{N}$ such that $\left|\frac{l_{k,j}\pi}{2^m}-\theta_{k,j}\right|\le\frac{\pi}{2^m}$. Since sine and cosine are both Lipschitz functions with Lipschitz constant 1 we have from Lemma (4.2.5) that

$$\left|\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k - \sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k\right|$$

$$\leq \sum_{k=1}^{n}\left|\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) - \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)\right||u_k|$$

$$\leq \sum_{k=1}^{n}\sum_{j=1}^{p_k}\left|\theta_{k,j} - \frac{l_{k,j}\pi}{2^m}\right||u_k| \leq \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.$$

Thus,

$$\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k \leq \sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k + \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|$$

$$\leq \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k + \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.$$

Since this is true for every $\theta_{k,j} \in \left[0,\frac{\pi}{2}\right]$ $(k \in \{1,,\dots,n\}, j \in \{1,\dots,p_k\})$ we have

$$0 \leq f - \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_k\left(\frac{l_{k,j}\pi}{2^m}\right) \leq \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.$$

It follows that the sequence $\sigma_m := \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_k\left(\frac{l_{k,j}\pi}{2^m}\right)$ converges relatively uniformly to f.

Finally, for $f \in E$, there exist sequences $(a_n),(b_n)$ in E such that $a_n \overset{ru}{\rightarrow} f^+$ and $b_n \overset{ru}{\rightarrow} f^-$. Then $a_n - b_n \overset{ru}{\rightarrow} f$.

We are ready to deal with the Archimedean tensor product of Archimedean vector lattices over $\mathbb{K}$.

Theorem (4.2.7) [4] Let $E_1,\dots,E_s$ be Archimedean vector lattices over $\mathbb{K}$.

(i) There exists an essentially unique Archimedean vector lattice $\overline{\bigotimes}_{k=1}^{s} E_k$ over $\mathbb{K}$ and a vector lattice s-morphism $\overline{\otimes}: \times_{k=1}^{s} E_k \rightarrow \overline{\bigotimes}_{k=1}^{s} E_k$ such that for every Archimedean vector lattice F over $\mathbb{K}$ and every vector lattice s-morphism $T: \times_{k=1}^{s} E_k \rightarrow F$, there exists a unique vector lattice homomorphism $T^{\overline{\otimes}}: \overline{\bigotimes}_{k=1}^{s} E_k \rightarrow F$ such that $T^{\overline{\otimes}} \circ \overline{\otimes} = T$.

(ii) There exists an injective $\mathbb{K}$ linear map $S: \bigotimes_{k=1}^{s} E_k \rightarrow \overline{\bigotimes}_{k=1}^{s} E_k$ such that $S \circ \otimes = \overline{\otimes}$.

(iii) $\tau\left(\otimes_{k=1}^{s} E_k, \overline{\otimes}_{k=1}^{s} E_k\right) \leq 2$. Thus, $\otimes_{k=1}^{s} E_k$ is dense in $\overline{\otimes}_{k=1}^{s} E_k$ in the relatively uniform topology.

(iv) For every $w \in \left(\overline{\otimes}_{k=1}^{s} E_k\right) \backslash \{0\}$, there exist $x_1 \otimes \ldots \otimes x_s \in \otimes_{k=1}^{s} E_{k_\rho}$ with $x_k \in E_k^+ (k \in \{1 \ldots, s\})$ such that $0 < (x_1 \otimes \ldots \otimes x_s) \leq |w|$, i.e. $\otimes_{k=1}^{s} E_k$ is order dense in $\overline{\otimes}_{k=1}^{s} E_k$.

Proof: By Lemma (4.2.2), statements (i)-(iv) are valid for $\mathbb{K} = \mathbb{R}$. We assume in the proof below that $\mathbb{K} = \mathbb{C}$.

(i) Let $E_1, \ldots, E_s, F$ be Archimedean vector lattices over $\mathbb{C}$. Denote by $\left(\overline{\otimes}_{k=1}^{s} E_{k_\rho}, \overline{\otimes}\right)$ the Archimedean vector lattice tensor product of $E_{1_\rho}, \ldots, E_{s_\rho}$. We claim that the pair $\left(\left(\overline{\otimes}_{k=1}^{s} E_{k_\rho}\right)_{|\mathbb{C}|}, \overline{\otimes}_{|\mathbb{C}|}\right)$ is the unique Archimedean complex vector lattice tensor product of $E_1, \ldots, E_s$. Let $T: \times_{k=1}^{s} E_k \to F$ be a vector lattice s-morphism. From Lemma (4.2.2), the map $\overline{\otimes}$ induces a unique vector lattice homomorphism $T_\rho^{\overline{\otimes}}$ on $\overline{\otimes}_{k=1}^{s} E_{k_\rho}$ such that $T_\rho^{\overline{\otimes}} \circ \overline{\otimes} = T_\rho$. Also, the map $T_\rho^{\overline{\otimes}}$ extends uniquely to a vector lattice homomorphism $\left(T_\rho^{\overline{\otimes}}\right)^{\mu_{2,4}}$ on $\left(\overline{\otimes}_{k=1}^{s} E_{k_\rho}\right)^{\mu_{2,4}}$ (Corollary (4.1.1)). By Proposition (4.1.4) (i), the map $\overline{\otimes}_{|\mathbb{C}|}$ is a vector lattice s- morphism and $\left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|}$ is a vector lattice homomorphism. We will prove that the map $\left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|} : \left(\overline{\otimes}_{k=1}^{s} E_{k_\rho}\right)_{|\mathbb{C}|} \to F$ is the unique vector lattice homomorphism such that $\left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|} \circ \overline{\otimes}_{|\mathbb{C}|} = T$. Indeed, for every $(f_0^1 + f_1^1, \ldots, f_0^s, f_1^s) \in \times_{k=1}^{s} E_k$ we have

$$\left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|} \circ \overline{\otimes}_{|\mathbb{C}|} (f_0^1 + if_1^1, \ldots, f_0^s, if_1^s)$$

$$= \left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|} \left(\sum_{\epsilon_k \in \{0,1\}} \overline{\otimes} \left(f_{\epsilon_1}^1, \ldots, f_{\epsilon_s}^s\right) i^{\sum_{k=1}^{s} \epsilon_k} \right)$$

$$= \sum_{\epsilon_k \in \{0,1\}} T_\rho^{\overline{\otimes}} \circ \overline{\otimes} \left(f_{\epsilon_1}^1, \ldots, f_{\epsilon_s}^s\right) i^{\sum_{k=1}^{s} \epsilon_k}$$

$$= \sum_{\epsilon_k \in \{0,1\}} T_\rho \left(f_{\epsilon_1}^1, \ldots, f_{\epsilon_s}^s\right) i^{\sum_{k=1}^{s} \epsilon_k} = T(f_0^1 + if_1^1, \ldots, f_0^s + if_1^s).$$

Since every vector lattice $\mathbb{C}$-homomorphism is real, the uniqueness of $\left(T_\rho^{\overline{\otimes}}\right)_{|\mathbb{C}|}$ follows from the uniqueness of $\left(T_\rho^{\overline{\otimes}}\right)^{\mu_{2,4}}$.

The proof of uniqueness of the Archimedean complex vector lattice tensor product is not different from the real case.

(ii) Consider the newly minted tensor product $\left(\overline{\bigotimes}_{k=1}^s E_k, \overline{\otimes}\right)$ constructed in (i). By Lemma (4.2.2), there exists an Archimedean vector lattice G over $\mathbb{R}$ and a vector lattice s-morphism $T: \times_{k=1}^s E_{k_\rho} \to G$ such that the induced linear map $T^{\otimes}: \bigotimes_{k=1}^s E_{k_\rho} \to G$ is injective. By taking the square mean completion of G, if necessary, we will assume that G is square mean complete. By taking vector space complexifications, we find an injective vector lattice s-morphism $\left(T^{\otimes}\right)_{\mathbb{C}}: \left(\bigotimes_{k=1}^s E_{k_\rho}\right)_{\mathbb{C}} \to G_{\mathbb{C}}$, or equivalently by Lemma (4.2.3), $\left(T^{\otimes}\right)_{\mathbb{C}}: \bigotimes_{k=1}^s E_k \to G_{\mathbb{C}}$. Moreover, if $(T_{\mathbb{C}})^{\otimes} : \bigotimes_{k=1}^s E_k \to G$ is the unique linear map induced by $T_{\mathbb{C}}$, then for $f_0^k + i f_1^k \in E_k$ $(k \in \{1, \dots, s\})$,

$$
\begin{aligned}
(T_{\mathbb{C}})^{\otimes}(f_0^1 + i f_1^1 \otimes \dots \otimes f_0^s + i f_1^s) &= T_{\mathbb{C}}(f_0^1 + i f_1^1, \dots, f_0^s + i f_1^s) \\
&= \sum_{\epsilon_k \in \{0,1\}} T\left(f_{\epsilon_1}^1, \dots, f_{\epsilon_s}^s\right) i^{\sum_{k=1}^s \epsilon_k} \\
&= \sum_{\epsilon_k \in \{0,1\}} T^{\otimes}\left(f_{\epsilon_1}^1 \otimes \dots \otimes f_{\epsilon_s}^s\right) i^{\sum_{k=1}^s \epsilon_k}.
\end{aligned}
$$

In particular, $\left((T_{\mathbb{C}})^{\otimes}\right)$ is a real map with $\left((T_{\mathbb{C}})^{\otimes}\right)_\rho = T^{\otimes}$, and therefore we have $(T_{\mathbb{C}})^{\otimes} = \left(T^{\otimes}\right)_{\mathbb{C}}$. From part (i) of this theorem there exists a unique vector lattice $\mathbb{C}$-homomorphism $(T_{\mathbb{C}})^{\overline{\otimes}}: \overline{\bigotimes}_{k=1}^s E_k \to G_{\mathbb{C}}$ such that $(T_{\mathbb{C}})^{\overline{\otimes}} \circ \overline{\otimes} = T_{\mathbb{C}}$. Moreover, there exists a unique $\mathbb{C}$-linear map $S: \bigotimes_{k=1}^s E_k \to \overline{\bigotimes}_{k=1}^s E_k$ such that $S \circ \otimes = \overline{\otimes}$. Then $(T_{\mathbb{C}})^{\overline{\otimes}} \circ S \circ \otimes = T_{\mathbb{C}}$, and hence $(T_{\mathbb{C}})^{\overline{\otimes}} \circ S = (T_{\mathbb{C}})^{\otimes} = \left(T^{\otimes}\right)_{\mathbb{C}}$. Therefore S is injective.

(c) By Lemma (4.2.2) we know that $\bigotimes_{k=1}^s E_{k_\rho}$ is relatively uniformly dense in $\overline{\bigotimes}_{k=1}^s E_{k_\rho}$. We also know from Proposition (4.2.6) that $\bigotimes_{k=1}^s E_{k_\rho}$ is relatively uniformly dense in $\left(\bigotimes_{k=1}^s E_{k_\rho}\right)^{\mu_{2,4}}$. By taking vector space complexifications, we have $\tau\left(\bigotimes_{k=1}^s E_k, \overline{\bigotimes}_{k=1}^s E_k\right) \leq 2$.

(d) Suppose $w \in \left(\overline{\bigotimes}_{k=1}^{s} E_k\right) \backslash \{0\}$. Then $0 < |w| \in \left(\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}\right)^{\mu_{2,4}}$. Since $\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}$ is order dense in $\left(\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}\right)^{\delta}$, it is also order dense in $\left(\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}\right)^{\mu_{2,4}}$. Thus there exists $w_0 \in \overline{\bigotimes}_{k=1}^{s} E_{k_\rho}$ such that $0 < w_0 \leq |w|$. From Lemma (4.2.2), there exists $x_1 \otimes \ldots \otimes x_s \in \bigotimes_{k=1}^{s} E_{k_\rho}$ with $x_k \in E_k^+ (k \in \{1, \ldots, s\})$ such that $0 < (x_1 \otimes \ldots \otimes x_s) \leq w_0$.

In (i) it is necessary to take the vector lattice complexification of $\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}$ to ensure that $\left(\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}\right)_{|\mathbb{C}|}$ is an Archimedean vector lattice over $\mathbb{C}$. Indeed, Theorems (4.2.10) and (4.2.11) furnish examples where the vector space complexification $\left(\overline{\bigotimes}_{k=1}^{s} E_{k_\rho}\right)_{|\mathbb{C}|}$ does not suffice.

We need two lemmas first.

Lemma (4.2.8) [4] Let X and Y be nonempty subsets of $\mathbb{R}$ without isolated points. Then the function $S: (x, y) \longmapsto \sqrt{x^2 + y^2} \left((x, y) \in X \times Y\right)$ is in the square mean completion of $C(X) \overline{\otimes} C(Y)$ but for all nonempty open subsets U of X and W of Y we have $S|_{U \times W} \notin C(U) \otimes C(W)$.

Proof: For $f \in C(X)$ and $g \in C(Y)$ we identify $f \otimes g$ with the function $(x, y) \mapsto f(x)g(y)\left((x, y) \in X \times Y\right)$. Consider the element S of the square mean completion of $C(X) \overline{\otimes} C(Y)$ defined by

$$(x, y) \longmapsto \sqrt{x^2 + y^2} \left((x, y) \in X \times Y\right).$$

Let U and W be open nonempty subsets of X and Y, respectively. We will show that the vector subspace of $C(U)$ generated by $\{S(\cdot, y): y \in W\}$, whose elements are considered as functions on U, is not finite-dimensional. It follows that $S|_{U \times W} \notin C(U) \otimes C(W)$. Since W is open and nonempty and Y has no isolated points, we can choose $\alpha_k \in W (\text{for all } k \in \mathbb{N})$ for which $\alpha_i^2 \neq \alpha_j^2$ when $i \neq j$. Let $n \in \mathbb{N}$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ for which $\lambda_k \sqrt{x^2 + \alpha_k^2} = \lambda_k S(x, y_k) = 0$ for all $x \in U$. Since the function $x \longmapsto \lambda_k \sqrt{x^2 + \alpha_k^2} (x \in \mathbb{R})$ is n times differentiable at every $x \in X$, a routine calculation shows that the $n \times n$ matrix $A(x)$ defined by $A(x)_{ij} = \frac{1}{\left(x^2 + \alpha_j^2\right)^{\frac{2i-1}{2}}}$ when evaluated at the vector $(\lambda_1, \ldots, \lambda_n)$ yields the vector $(0, \ldots, 0)$ for every non-zero $x \in U$. However,

$\prod_{k=1}^{n}\sqrt{x^2+\alpha_k^2}\,\det(A(x)) = \det(B(x))$ where the $n\times n$ matrix $B(x)$ is defined by $B(x)_{ij} = \frac{1}{(x^2+\alpha_j^2)^{i-1}}$, which has (Vandermonde) determinant

$$\prod_{1\le j<k\le n}\left(\frac{1}{x^2+\alpha_j^2}-\frac{1}{x^2+\alpha_k^2}\right) = \prod_{1\le j<k\le n}\frac{\alpha_j^2-\alpha_k^2}{(x^2+\alpha_j^2)(x^2+\alpha_k^2)} \ne 0.$$

Thus $\det(A(x)) \ne 0$ for every non-zero $x\in U$, the vector subspace of $C(U)$ generated by $\{S(\cdot,y): y\in Y\}$ (as functions on U) is infinite dimensional, and $S|_{U\times W} \notin C(U)\otimes C(W)$.

Lemma (4.2.9) [4] Let X and Y be nonempty subsets of $\mathbb{R}$ without isolated points and let $f\in C(X)\,\overline{\otimes}\,C(Y)$. Then there exists a nonempty open subset V of $X\times Y$ and $g\in C(X)\otimes C(Y)$ such that $f|_V = g|_V$.

Proof: Note that $C(X)\,\overline{\otimes}\,C(Y)$ is the vector lattice generated by $C(X)\otimes C(Y)$ in $C(X\times Y)$. Every element $f\in C(X)\,\overline{\otimes}\,C(Y)$ is of the form $f = \bigwedge_{j=1}^{n}\bigvee_{k=1}^{m} f_{j,k}$ where $f_{j,k}\in C(X)\otimes C(Y)$ for each j and each k. Let $f_1, f_2\in C(X)\otimes C(Y)$. If $f_1\ne f_2$, we may assume there exists (x,y) such that $f_1(x,y) < f_2(x,y)$ and then there exists a nonempty open subset O of $X\times Y$ such that $f_1\wedge f_2 = f_1$ on O. Of course such an open subset O also exists if $f_1 = f_2$. By repeating this argument there exists a nonempty open set $U\subseteq O$ such that $\bigwedge_{j=1}^{n}\left(\bigvee_{k=1}^{m} f_{j,k}\right) = \bigvee_{k=1}^{m} f_{j_0,k}$ on U for some $j_0\in\{1\ldots,n\}$.

Similarly, there exists a nonempty open set $V\subset U$ such that $\bigvee_{k=1}^{m} f_{j_0,k_0} = f_{j_0,k_0}$ on V for some $k_0\in\{1,\ldots,m\}$.

Theorem (4.2.10) [4] Let X and Y be nonempty subsets of $\mathbb{R}$ without isolated points. Then $C(X)\,\overline{\otimes}\,C(Y)$ is not square mean complete. Therefore $\left(C(X)\,\overline{\otimes}\,C(Y)\right)_{\mathbb{C}}$ is not an Archimedean vector lattice over $\mathbb{C}$.

Proof: Assume that the element S of Lemma (4.2.8) is in $C(X)\,\overline{\otimes}\,C(Y)$. Then by Lemma (4.2.9) there exists a nonempty open set V in $X\times Y$ and an element $g\in C(X)\otimes C(Y)$ such that $g|_V = S|_V$. However the open set V contains a nonempty open subset of the form $U\times W$ with $0\notin U$. This contradicts Lemma (4.2.8).

We use Theorem (4.2.10) to prove the following.

Theorem (4.2.11) [4] If X and Y are uncountable compact metrizable spaces then $C(X)\,\overline{\otimes}\,C(Y)$ is not square mean complete. Therefore $\left(C(X)\,\overline{\otimes}\,C(Y)\right)_{\mathbb{C}}$ is not an Archimedean vector lattice over $\mathbb{C}$.

Proof: By [4], we know that both X and Y contain a closed subset homeomorphic with the Cantor set $\mathbb{D}$. Then $\mathbb{D} \times \mathbb{D}$ can be viewed as a closed subset of $X \times Y$ and the function $F_0: (x, y) \longmapsto \sqrt{x^2 + y^2}\big((x, y) \in \mathbb{D} \times \mathbb{D}\big)$is continuous. By Tietze's Extension Theorem, the function $x \longmapsto x\ (x \in \mathbb{D})$ can be extended to continuous functions f and g on X and Y, respectively. Then the function $F: (x, y) \longmapsto \sqrt{f(x)^2 + g(y)^2}\ \big((x, y) \in X \times Y\big)$ is a continuous function in the square mean completion of $C(X) \overline{\otimes} C(Y)$ that extends F_0. If F were in $C(X) \overline{\otimes} C(Y)$ itself then its restriction to $\mathbb{D} \times \mathbb{D}$ would be in $C(\mathbb{D}) \overline{\otimes} C(\mathbb{D})$ which by Theorem (4.2.10) is impossible. This proves the theorem.

It is certainly tempting to conjecture the following.

Conjecture (4.2.12) [4] If X and Y are infinite compact metrizable spaces then $C(X) \overline{\otimes} C(Y)$ is not square mean complete.

The above two theorems show that the old way of complexifying Archimedean vector lattices via vector space complexifications is inadequate for pursuing complex analysis on Archimedean complex vector lattices.

We remark that the complex Archimedean vector lattice tensor product, like its real counterpart, possesses as well a universal property with respect to positive multilinear maps and complex uniformly complete vector lattices as range. The proof of this universal property, stated in the theorem below, is similar to the proof of Theorem (4.2.7) (i).

Theorem (4.2.13) [4] Let $E_1, \ldots, E_s, F$ be Archimedean vector lattices over $\mathbb{K}$ with F uniformly complete. If $T: \times_{k=1}^{s} E_k \to F$ is a positive $s_{\mathbb{K}}$-linear map, then there exists a unique positive $\mathbb{K}$ - linear map $T^{\overline{\otimes}}: \overline{\otimes}_{k=1}^{s} E_k \to F$ such that $T^{\overline{\otimes}} \circ \overline{\otimes} = T$.

A reformulation of part (i) of Theorem (4.2.7) in terms of Archimedean real vector lattices and vector lattice complexifications is the following.s

Theorem (4.2.14) [4] Let $E_1, \ldots, E_s, F$ be Archimedean vector lattices over $\mathbb{R}$ and suppose that $T: \times_{k=1}^{s} E_k \to F$ is a vector lattice $s_{\mathbb{R}}$-morphism. There exists a unique vector lattice $s_{\mathbb{C}}$-morphism $\left(T_{|\mathbb{C}|}\right)^{\overline{\otimes}}: \overline{\otimes}_{k=1}^{s} E_{k|\mathbb{C}|} \to F_{|\mathbb{C}|}$ such that $\left(T_{|\mathbb{C}|}\right)^{\overline{\otimes}} \circ \overline{\otimes}\big|_{\times_{k=1}^{s} E_k} = T$.

Proof: Consider T to be a vector lattice $s_{\mathbb{R}}$-morphism from $\times_{k=1}^{s} E_k$ to $F^{\mu_{2,4}}$. By Proposition (4.1.4) (i) there exists a unique vector lattice $s_{\mathbb{C}}$-

morphism $T_{|\mathbb{C}|}: \times_{k=1}^{s} E_{k|\mathbb{C}|} \to F_{|\mathbb{C}|}$ such that $T_{|\mathbb{C}|}\big|_{\times_{k=1}^{s} E_k} = T$. If $\left(T_{|\mathbb{C}|}\right)^{\overline{\otimes}}$ is the unique vector lattice $\mathbb{C}$-homomorphism induced by $T_{\mathbb{C}}$ then $\left(T_{|\mathbb{C}|}\right)^{\overline{\otimes}} \circ \overline{\otimes} = T_{\mathbb{C}}$. In particular, $\left(T_{|\mathbb{C}|}\right)^{\overline{\otimes}} \circ \overline{\otimes}\big|_{\times_{k=1}^{s} E_k} = T$.

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